Triangulorum planorum et sphaericorum praxis arithmetica

[The Arithmetical Practice of Plane and Spherical Triangles]

(1591 (date on the manuscript; first issued 1886 — see Edition))

by Tycho Brahe (1546–1601)
Translation by Scott Weisman



Ars Astronomica
2026

"Triangulorum planorum et sphaericorum praxis arithmetica," published in 1591 (date on the manuscript; first issued 1886 — see Edition), is in the public domain. All translations, original additions, including illustrations, commentaries, and summaries, are copyright.

The source text was obtained from Bayerische Staatsbibliothek, 4 Math.p. 50 g and is available at https://mdz-nbn-resolving.de/urn:nbn:de:bvb:12-bsb11505776-7.

Translation © 2026 Scott Weisman. All rights reserved.

Licensed under CC BY-NC-ND 4.0. Share with attribution; non-commercial; no derivatives.



Published by Ars Astronomica

Translator Version 2

Upright reason dictates that the recipients of the good, of whatever type and level of beneficence it may be, must show gratitude and blessing to the beneficent person in every way possible, commensurate with the value of the beneficence. And one who has benefited all the people of the world — for example, one who invented a new instrument for the good of the world, or a good book — it is fitting for every discerning person, out of the obligation of love of fellow beings, to at least purchase it, so that the man will profit and his heart will be encouraged thereby to invent yet more good instruments in the world. And similarly, all other wise-hearted people will likewise strive and exert themselves to invent good things and instruments needed for the repair of the world and its perfection.

And therefore, whoever says: "What do I need this new instrument for?" — he does not act well towards the world. For if not for the man who invented it, where would you be? And what would your city do? And more than this, if the man did not exert himself, what would the world do?

— R. Pinchas Eliyahu Hurwitz, *Sefer HaBris HaShalem* (1797)

If you find this translation useful, consider supporting Ars Astronomica.

ArsAstronomica@protonmail.com

A note on the work

Tycho Brahe's working manual of trigonometry. In sixteen numbered rules, seven for plane triangles and nine for spherical, it shows how to find a triangle's unknown sides and angles by arithmetic, with worked examples and figures. It reveals the computing that lay behind the reductions of the Uraniborg observations. Tycho never printed it. It survives in one known manuscript at Prague, bound with a copy of Rheticus's table of trigonometric functions, and first appeared in 1886 as a facsimile edited by F. J. Studnička in Prague. Studnička took the manuscript for Tycho's autograph. J. L. E. Dreyer, in his 1913 edition (*Opera Omnia* I, pp. 315–316), argues that the hand is not Tycho's: it lacks his abbreviations and his form of M, and the name on the title differs from his signature.

This translation follows Studnička's photolithographic facsimile of the Prague manuscript, in the Bayerische Staatsbibliothek copy (4 Math.p. 50 g). His preface is dated 1 December 1885; the catalogue dates the issue 1886. In this copy a Leipzig bookseller's slip (K. F. Koehler's Antiquarium) is pasted over the Prague imprint on the title page.

A note on the translation process

This translation was produced with the translation-pipeline. See the linked repository for details on the process.

A note on the references. A scriptural or bibliographic reference given in brackets or parentheses is, in most cases, an **editorial identification of an allusion rather than a citation printed in the source**. Books of this kind quote and weave scripture constantly without naming it, and the identification is supplied here as a convenience to the reader. Where the source does print a reference of its own, it is given as printed. **An identified allusion is not evidence that the author cited that passage —** look at the original before resting an argument on one.

This text is the product of a highly refined automated translation process. It has undergone only cursory human review and should not be treated as an authoritative or scholarly rendering. It is offered solely as a reference and reading aid — a way to make an otherwise inaccessible work approachable — and any passage of real consequence should be checked against the original source.

Acknowledgements

Translated with the assistance of Claude. The translator thanks the Anthropic team for making this work possible.

Notation and abbreviations

A few conventions have been applied uniformly throughout this translation, so that one quantity is always written one way. They change how a value is *written*, never what it is; where the printed source and this translation differ in notation, the source's own form is recorded in the editorial apparatus.

Numerals are Arabic, even where the source sets Roman ones. Roman numerals are kept only in headings, running headers and signatures, which are transcribed as printed.

Dates are given as year, month, day — *1572 November 11*. Dates are Julian, as the source gives them, and have not been converted to the Gregorian calendar.

Angles are written with the conventional signs — 23° 27′ 30″. Early modern astronomy divides the degree sexagesimally and often runs to places for which modern writing has no common symbol; the orders used in this book are:

sign	order	the source's term
°	degrees	*partes*, *gradus*

Where a table in the source spreads one quantity across separate columns for degrees, minutes and seconds, those columns have been joined into a single column and the value written whole. Where a value runs to thirds and beyond, the columns are kept, since there the columns are the notation.

TYCHO BRAHE

THE ARITHMETICAL PRACTICE OF PLANE AND SPHERICAL TRIANGLES

in which their very great use, especially in astronomical matters, is concisely explained.

Now edited for the first time by

Dr. F. I. Studnička,

Imperial and Royal Full Public Professor of Mathematics at the Bohemian University, etc., etc.

Preface

If we ask who have been the most outstanding men among the Danes, we find that Tycho Brahe holds the foremost place among them: so greatly did he ennoble and bring lustre to both his own name and his country by the divine works of his genius. All the records that concern him must therefore surely be valued highly, especially any that came from him directly.

A most precious record of this kind is preserved in the Imperial and Royal Library of the University of Prague: I mean a geometrical treatise written out in Tycho's own hand. We offer an accurately made image of it both to all admirers of this distinguished man and above all to his countrymen devoted to learning, so that, since they cannot hold the codex itself in their hands, they may at least take pleasure in a photolithographic likeness of it that comes very close to the real thing. We had this likeness made with the permission of the most honourable A. Zeidler, the most careful head of the library just mentioned; I give him the greatest possible thanks for this generosity.

As is evident from the volume, splendidly bound and marked with the number XIV C 20, Tycho wrote that codex on twenty leaves. Before these leaves were filled with writing they were attached to a printed book, itself an excellent one, titled *Canon of the Doctrine of Triangles. Now published for the first time by Georg Joachim Rheticus. In the year 1551* [Canon doctrinae triangulorum. Nunc primum a Georgio Ioachimo Rhetico in lucem editus. Anno MDLI.]. By thus joining the two works, that most celebrated observer of the heavens provided an aid both most useful to the learned men of that time and, above all, most necessary to his own labours.

We can prove by many very weighty arguments that Tycho entered what he had collected and discovered on leaves already bound to Rheticus's book. We adduce only these: first, that paper of the same kind was used in the binding as in those leaves; next, that the indices of the parts of the codex fit neatly to

the margin; and finally, that on the leaf bearing the printed index of Rheticus's book these words stand written by hand: "Prague, in the year 1642. From the Tychonic library" [Pragae A⁰ 1642. Ex bibliotheca Tychoniana].

It would take long to relate what form the doctrine of triangles had at that time, and in what way Rheticus — the man we have mentioned, who was likewise a pupil and admirer of Copernicus — and Tycho rendered it service. We merely remind learned men of the relation that exists between them in that doctrine and in its use.

We therefore make that most precious codex public with no annotations added, since we do not doubt that those who in the near future write either the life of Tycho or the history of geometry will set out more fully the value of this book by the first man of that most noble triad of astronomers measuring the arcs of the heavens, to which belong those illustrious minds Brahe, Bradley [James Bradley], Bessel [Friedrich Wilhelm Bessel].

Written at Prague, 1885 December 1 [Kalendis Decembribus anni MDCCCLXXXV].

Dr. F. I. Studnička [František Josef Studnička].

THE ARITHMETICAL PRACTICE OF PLANE AND SPHERICAL TRIANGLES

[TRIANGVLORVM PLA// NORVM, ET SPHÆRICO// rum Praxis Arithmetica.]

In which their greatest use, especially in astronomical matters, is concisely explained.

[Qua maximus eorum, præsertim in Astronomicis usus compen= diose explicatur.]

Tycho Brahe

1591 January 1 [Calend. Januar. 1591]

In the triangle you will find the things that learned Mathesis busies itself with; it opens up whatever Olympus holds closed.

[In Trigono Invenies satagit quæ docta Mathesis / Ille aperit, clausum quicquid Olympus habet.][1]

[Margin: 1595 November 19 — A.C. 1595. 13 Cal. Xbris][2]

A COMPENDIUM ON PLANE TRIANGLES, CONTAINING SEVEN DOGMATA

[DE TRIANGVLIS PLA// nis Compendium, continens Dogmata septem.]

[1] *Mathesis* is kept as written. The Latin word covers mathematical learning generally and, in some authors, astrology; the verse does not say which is meant here.

[2] The note stands at the right, beneath the end of the verse. *Xbris* is the usual abbreviation for December, and 13 Cal. is taken as the thirteenth day before the Kalends of December, counted inclusively. The page does not say what this date refers to; it falls nearly five years after the date under the name.

DOGMA I OF PLANE TRIANGLES

[DOGMA Ī · PLAÑOR.]

In a right-angled triangle, given the two sides about the right angle, each of the angles is given, together with the remaining side.

[Margin: Of a right-angled triangle — TRIANGU= LI RECTĀ= GULI.]

For the angles [Pro Angulis]**:** Multiply the side opposite the angle sought by the whole [totum] [Margin: a short word, not struck, to the right of the line ending *per*, opening *si* and perhaps *sinum* [?]; beneath it a crossed sign like a barred p [?]], and divide the product by the remaining side. The result is a number which, when looked for in the body of the fecund canon [Canonis fœcundi], will show the angle sought. The remaining angle is given by this angle's complement to 90.[3]

For the side [Pro latere]**:** To find the side that subtends the right angle, multiply one side [alterum latus] by the whole and divide by the sine of the angle that this same side subtends. The result is the very side sought, the one opposite the right angle.

[Margin: Demonstration of this dogma — Demōstratio huius Dogmatis]

Diagram. A full circle with a horizontal diameter. From the point at the middle of the diameter a vertical line rises to the circumference at the top. From that top point a horizontal line runs to the right and ends at a slanting line. At the right end of the diameter a vertical line rises outside the circle and meets that slanting line at its upper end. Two lines fan out from the centre to the upper right: the upper one is the slanting line just described, and the lower one ends on the right-hand vertical. The caption stands to the right of the figure. Every label that can be seen is listed below; the right end of the diameter and its left end carry none.

- D (centre, below the diameter): D, written with a stroke running out to its left
- E (top of the circle, at the head of the vertical line): E, written in a rounded form like Ɛ
- C (right end of the horizontal line from E, just left of the slanting line): C
- B (just below C, to the right of the slanting line): B
- Apex (top of the right-hand vertical, where the slanting line ends): a small ring-shaped mark, [?]
- Right-hand vertical (where the lower line from the centre ends): a cross-shaped mark, [?] (This page is blank.)

[3] *Totum*, "the whole", is read here as the whole sine (*sinus totus*), the radius of the sine table; the line writes only *totum*, and the marginal word against this point, opening *si*, may be *sinum* [?], which would give *per sinum totum*. The fecund or "fruitful" canon (*canon fœcundus*) is a sixteenth-century name for a table of tangents; this identification is editorial, and the page does not say whose table is meant.

DOGMA II OF PLANE TRIANGLES

[DOGMA $\overline{\text{II}}$ · PLANOR.]

In a right-angled triangle, given two sides, one of them about the right angle and the other opposite it, the angles are given, together with the remaining side.

Multiply the side about the right angle by the whole and divide by the side that subtends the right angle. The result is the sine of the angle that this side subtends. The complement of that sine's arc to 90 gives the third angle. If you multiply the sine of this angle by the side that subtends the right angle and divide by the whole, the result is the other side about the right angle, which was sought.

Diagram. The figure stands in the right margin, beside the last four lines of the operation, and has no caption. A vertical line rises from C to E, and a horizontal base runs from C to D. A curved line runs from E through the point at B and down to D. From the point at B three straight lines leave: one runs horizontally left to the vertical CE, one runs diagonally down to C, and one drops vertically to the base at A. The prose on this page cites no point letters, so none of these labels has a referent in the text here. Every label that can be seen is listed below.

- E (top of the vertical line, at the start of the curve): E
- B (above and to the right of the point on the curve): B
- C (below the left end of the base): C
- A (below the base, where the vertical from the point at B lands): A
- D (right end of the base, at the foot of the curve): D
- Curve, between the point at B and D (just above the base): a short mark, [?] (This page is blank.)

DOGMA III OF PLANE TRIANGLES

[DOGMA $\overline{\text{III}}$ PLANOR]

In a right-angled triangle, given one of the acute angles together with one side, the remaining two sides are given.

$\bar{\text{I}}$. If the side that subtends the right angle is the one given, multiply the measure of that side by the sine of the angle that the side sought lies opposite, and divide by the whole. The result is the side that was sought.

$\overline{\text{II}}$. But if one of the two sides adjacent to the right angle is given, and you wish to measure the side that subtends the right angle, multiply the given side by the whole and divide by the sine of the angle opposite it. The result is the measure of the side opposite the right angle.

$\overline{\text{III}}$. If you want to find the remaining side adjacent to the right angle, multiply the number of the given side by the sine of the complement of the angle opposite it, and divide by the sine of that same angle. The result is the remaining side, which was sought.

Diagram. The figure stands below the last line of the text, toward the right of the page, with a struck-through note to its right. A circle carries a horizontal base running from B, outside the circle on the left, through E and F to C on the circle's right side. A heavier straight line runs from B up to the right, through the point at A, to D at the circle's upper right; the G label stands by this line near the place where it crosses the circle on the left. From the point at A two further straight lines leave: one drops vertically to the base at F, and one runs down to the right to C. A third stroke leaves the heavy line just short of A and runs down to the left, stopping just above the base near E, where the circle crosses it. Faint dotted and curved lines cross the circle's interior and its surroundings, and a short mark stands on the circle at its upper left; these carry no letters. The prose on this page cites no point letters, so none of these labels has a referent in the text here. Every label that can be seen is listed below.

- B (left of the left end of the base): B
- G (above the heavy line, just outside the circle where the line crosses it on the left): G
- E (below the base, near the left side of the circle): E
- F (below the base, at the foot of the vertical from A): F
- C (at the right end of the base, on the circle): C
- A (above the heavy line, at the head of the vertical): A
- D (outside the circle at the upper right, at the end of the heavy line): D

[Margin: to the right of the figure, two lines, the first struck through and the second with a stroke along its foot that may be a strike or an underscore — *Demōstratio* [?] *hui*[?] / *Dogmatis* — on sense "Demonstration of this dogma" [?].]

(This page is blank.)

DOGMA IV OF PLANE TRIANGLES

[DOGMA ĪV ·PLANOR·]

In a non-right-angled triangle, given two sides with the angle they enclose, the angles and the remaining side are given.

[Margin: TRIANGU= / LI NOM [recte: NON] / RECTAN= / GULI. — Non-right-angled triangles.]

Ī. If the triangle has two equal sides, the given angle, halved and taken away from 90, shows the remaining angles. You will get the third side if you multiply the side by half of the given angle and divide by the whole. The result is half of the remaining side; double it.[4]

ĪĪ. But if the triangle has unequal sides and the enclosed angle is ACUTE, multiply the greater given side and divide by the whole. The result is found quantity II [inventum ĪĪ]. Subtract the lesser side from it, and found quantity

[4] The leaf writes *dimidium Anguli dati*, "half of the given angle", with no word for sine at this point; left as written. Dreyer (*Opera Omnia* I, p. 317) calls this a slip of the pen for "the sine of half".

III remains. Now multiply found quantity I by the whole and divide by found quantity III. The product, in the fecund canon [Canone fœcundo], gives the complement of the angle opposite the greater side.[5]

For the remaining side, multiply found quantity I by the whole and divide by the sine of the angle just found, the one opposite the greater side. The result is the remaining side, which was sought.

But if the angle is obtuse, multiply the sine of the complement of the obtuse angle by the greater side and divide by the whole. The result is found quantity I. Next multiply the sine of the obtuse angle by the greater side and divide by the whole. The result is found quantity II. Then add found quantity I to the given lesser side, and you will have found quantity III. Now multiply found quantity II by the whole and divide by found quantity III. The result is the number of the angle adjacent to the lesser side, to be looked up in the fecund table [Tabulâ fœcundâ].

For the remaining side, multiply found quantity II by the whole and divide by the sine of the angle just found, which is adjacent to the lesser side. The result is side III, which was sought.

NB. There is a complement to 180 when that angle is obtuse. You get the complement of the obtuse angle by subtracting 90 from it. You get the sine of the obtuse angle by subtracting it from 180 and taking the sine of the remainder: the complement of the former to 90.

A shorter solution of this dogma, without dropping a perpendicular

[Compendiosior solutio huius Dogmatis absq demissâ perpendiculari.]

Ī. Take the complement of the given angle (acute or obtuse) from a semicircle, that is, from 180, and halve it. Then take the tangent of this half from the canon of tangents [Canone Tangentium].

ĪĪ. Next add the lesser given side to the greater given side and halve the sum; this will be found quantity I.

ĪĪĪ. From this first found quantity subtract the lesser side again, and the remainder will be found quantity II.

ĪV Found quantity III, moreover, will be the tangent of the angle, taken from the complement of the given angle halved earlier.

Operation [Πρᾶξις][6]

Now multiply found quantity II by found quantity III and divide by found quantity I. The result is the tangent of the angle of the difference of the angles

[5] As written, this case names no multiplier for the greater side, and no step on the leaf produces found quantity I, though the next two operations use it. Left as written. Dreyer (*Opera Omnia* I, p. 317) calls the passage badly corrupt (*valde corruptus*) and reconstructs it; his reconstruction is not adopted here.

[6] One word in Greek letters stands centred on its own line here. As written, its first letter resembles ω and its second is doubtful. Dreyer (*Opera Omnia* I, p. 287) prints Πρᾶξις (*praxis*, "working, operation"), which suits its place between the statement and the operation, and the English follows him.

subtended by the given sides.[7] So if you subtract this difference, found in the canon of tangents, from the angle found earlier through the halved complement, the angle subtended by the lesser of the given sides will be revealed. If, in turn, you add this difference to that same angle found from the complement, the angle subtended by the greater side is revealed.

Or again: given two[8] angles in plane triangles, the third becomes known through their complement to a semicircle.

DOGMA V OF PLANE TRIANGLES

[DOGMA V· PLANOR]

Given two sides of a non-right-angled triangle with the opposite angle, or two angles with a side, whether subtended or adjacent, to find the rest from these.

DOGMA VI OF PLANE TRIANGLES

[DOGMA V Ī ·PLANOR]

Given the three sides of a non-right-angled triangle, all the angles are given.

Ī [Margin: ISOPLEU=/RUM.] In a triangle of three equal sides, which they call ISOPLEURUM, the angles measure 60°.

$\bar{\text{I}}\bar{\text{I}}$. [Margin: ÆQUICRU/RIUM] In a triangle of two equal sides, which they call ÆQUICRURIUM, multiply half the base by the whole and divide by one of the equal sides. The result is the sine of half the angle contained by the two equal sides. Therefore, since the remaining angles are equal, the complement of this angle to 180 will show the halved quantity of each remaining angle.[9]

$\bar{\text{I}}\bar{\text{I}}\bar{\text{I}}$. [Margin: SCALENUM.] In a triangle of three unequal sides, take the lesser side and the next greater. Add them to each other and subtract, then multiply both remainders together and divide the product by the greatest side. You have found quantity I. Take this from that same greatest side and halve the remainder. Add this to found quantity I, and you will have found quantity II. Multiply this second found quantity by the whole and divide by the middle side, namely the one next larger than the least. The result is the sine of the complement of the angle opposite the least side. For the remaining angle, take the half of the remainder given earlier (by the half-completed

[7] The leaf says "difference" here and writes no word for half. As the operation is written, the result is the tangent of half that difference, which is also what the subtraction and addition that follow require. This is an editorial observation; the text is left as written.

[8] a short word is struck through in the hand after *datis*, "given"; under the stroke it looks like *in* or *m* [?]

[9] The leaf makes "halved" agree with "quantity", not with "complement". For the rule to work, what is halved is the complement to 180 of the whole contained angle, and "this angle" must mean that whole angle rather than the half just found. This is an editorial observation; the text is left as written.

second found quantity)[10] and the least side, as if it were opposite a right angle (which is understood), and by the second of this work[11] the angle opposite the intermediate side will be given. Therefore the third will also be known.

Diagram. The figure stands below the last line of the text, with its caption to the right: "Demonstration of this sixth dogma of Plane [triangles]" (*Plan.* is abbreviated on the leaf). A circle carries a horizontal base running from B, outside the circle on the left, through E, where the circle crosses the base, and through F, to C on the circle's right side. A straight line runs from B up to the right, through G, where it crosses the circle, and through A, to D on the circle at the upper right. From A three further straight lines leave: one drops vertically to the base at F, one runs down to the right to C, and one runs down to the left to E. The ends of the circle's stroke do not meet at the top, where the right-hand arc starts above and beside the end of the left-hand arc. The prose on this page cites no point letters, so none of these labels has a referent in the text here. Every label that can be seen is listed below.

- A (above the point where the lines meet, near the centre of the circle): A
- B (left end of the base, outside the circle): B
- C (right end of the base, on the circle): C
- D (upper right, at the end of the long line, on the circle): D
- E (below the base, where the circle crosses it on the left): E
- F (below the base, at the foot of the vertical): F
- G (on the long line, where it crosses the circle on the left): G

DOGMA VII OF PLANE TRIANGLES

[DOGMA Ⅴ $\overline{\text{II}}$· PLANOR]

Given the three angles of a triangle, to find each of its sides.

Double the sine of any angle, and the result will be the size of the side opposite this angle, in parts of the kind in which the semidiameter of the circle is the whole sine.

THERE FOLLOWS A COMPENDIUM ON SPHERICAL TRIANGLES, CONTAINING NINE DOGMATA

[SEQUITUR DE Triangulis Sphæricis Compendium continens Dogmata novem.]

DOGMA I OF SPHERICAL TRIANGLES

[DOGMA ·$\bar{\text{I}}$, SPHÆRICOR.]

[Margin: Of a right triangle — TRIANGU= / LI RECTAN= / GULI.]

In a right triangle, given two sides, to find the third, opposite the right angle.

[10] The hand sets this phrase in square brackets.

[11] The leaf has only *secundam huius*. "Work" is supplied; the noun understood with the feminine *secundam* is not written.

Operation: Add the lesser side and the complement of the greater to each other, and subtract them from each other. Add the sine of each remainder if the complement of the greater side is greater than the lesser; otherwise subtract. Take half of the difference[12], and you will have the sine of the complement of the remaining side that is sought.

DOGMA II OF SPHERICAL TRIANGLES

[DOGMA $\overline{\text{II}}$ SPHÆRICOR.]

In a right triangle, given two sides, to find any angle.

If the side subtending the right angle and the side opposite the angle to be found have not been given together, then whichever of these sides is unknown must first be found by the preceding operation, and from there one proceeds in this way. Multiply the sine of the side opposite the angle sought by the whole sine, and divide the product by the sine of the side subtending the right angle. The result is the sine of the angle to be found, and its arc measures the size of that angle.

DOGMA III OF SPHERICAL TRIANGLES

[DOGMA ·$\overline{\text{III}}$· SPHÆRICOR]

In a right triangle, given one angle together with the adjacent side, to find the remaining angle, the one opposite the given side.

Operation: Add the lesser given and the complement of the greater to each other, and subtract them from each other. Subtract the sines of the two remainders, or add them, if the angle is greater than the adjacent side. Take half of the remainder, for the sine of the complement of the angle sought, or the sine of the side opposite the given angle, when the angle is greater than the adjacent side.[13]

[Margin: NB.] This applies when the side adjacent to the right angle is given. Note too that as long as the given side is less than the given angle, both must be added and subtracted, and the sines of the remainders combined and halved, for the sine of the complement of the angle sought. But when the side is greater than the angle, you may use, as is done in the example written out[14], subtraction of the sines, taking the givens themselves, not the complement of the greater. But if both givens exceed a quadrant, then work with the complement of each given, subtracting from each other the sines of the sum and of the difference; halving these too gives the sine of the complement of the angle sought.

---

[12] The leaf names only the difference here (*differentiæ cape dimidium*), directly after *alias subtrahe*. It does not separately order the halving of the sum for the case in which the sines are added.

[13] The leaf sets the same condition, *maior latere adiacente*, against both alternatives: once after *vel adde*, and again after *vel sinus lateris oppositi*. It also names what is halved as *residui*, and does not separately name a sum for the case in which the sines are added.

[14] No example stands in the book here. Dreyer (*Opera Omnia* I, p. 317) says it is not in the manuscript.

DOGMA IV OF SPHERICAL TRIANGLES

[DOGMA ·ĪV · SPHÆRICOR.]

In a right triangle, given one angle and any side, to find the rest.

Ī. If the given side is opposite the given angle that is not right, multiply the sine of that side by the whole, and divide by the sine of the given angle. The result is the sine of the side subtending the right angle. Dogma I will give the remaining side, while Dogma II will give the angles.

ĪĪ. If the given side is opposite the right angle, multiply its sine by the sine of the given angle, and divide the product by the whole. The result is the sine of the side opposite the given angle. Find the rest by the first and second dogma, as before.

If the right angle and the given angle stand on the given side, work by the preceding one for the remaining angle, but by the following one for the remaining sides.

DOGMA V OF SPHERICAL TRIANGLES

[DOGMA V · SPHÆRICOR.]

In a right triangle, given the two remaining angles, to find any side.

Multiply the sine of the complement of the non-right angle that subtends, or is opposite, the side sought[15] by the whole, and divide the product by the remaining sine of a non-right angle. The result is the sine of the complement of the side sought.

For the remaining sides, however, multiply the sine of the side just found by the sine of the angle facing the remaining side sought, whether that angle is right or not. Divide the product by the sine of the angle that the side just found subtends. The result will be the sine of the side to be sought. Dogma I will give the third side.

DOGMA VI OF SPHERICAL TRIANGLES

[DOGMA·V Ī· SPHÆRICOR.]

[Margin: TRIANGŪLI / NON RECTAN= / GULI· — Non-right-angled triangle]

In a non-right-angled triangle, given the two sides about a given angle, to find the remaining side.

Add the lesser side and the complement of the greater to each other, and subtract them from each other. Add together the sines of the two remainders, if the supplement of the greater side is greater than the lesser side; otherwise subtract them, where the supplement is less than that side. Half of the remainder will be found quantity I. Subtracted from the first sum, it will give found quantity II. Multiply by this the sine of the complement of the given angle, if

[15] a word is struck through before "quæsito", its first letters apparently "re"; the rest is not read [?]

that angle is acute (or the sine of that angle's excess beyond a quadrant, if it is obtuse), and divide by the whole: you will have found quantity III. Add this to the first found quantity, if the given angle is acute; or subtract it from found quantity I[16] if it is obtuse. You will have the sine of the complement of the side sought.

Various observed cases of this dogma

[Vary casus observati huius dogmatis]

I. If one of the sides enclosing the given angle is greater than a quadrant, the opposite side comes out less than a quadrant. If the product in the multiplication is greater than the first found quantity:

with II [?][17] subtracted from this found quantity I, what remains is the sine of the complement of the side sought.

Demonstration of this sixth dogma

[Demōstratio huius sexti Dogmatis]

Diagram. A large circle stands below the heading "Demonstration of this sixth dogma" and spans about three-quarters of the width of the writing, set to the left. The labels are written in small letters that look Greek. Two arcs run inside the circle. One starts on the circle at α, bulges to the left through the small mark below α and through γ, then runs back to the right, ending near the label under the left-hand line of the centre; beside its lower part runs a dotted line. The other starts on the circle at β and curves down to the left through γ and ε. Heavier straight lines radiate from a thick point near the centre of the circle: one up to β, passing the small hooked mark above it; one left along a long chord that reaches the circle at far left, and right through ν to the circle beside the label at far right; and one down to the right to the circle. A line from ρ runs down to the right through γ and past the hooked mark and ν to the lower right. A line from δ runs down to the right to the thick point, and a heavy line from δ runs down to the circle at the foot, passing a chord labelled at μ and the two lower chords. A thin line from ρ runs down and meets that heavy line at τ. Three chords cross the figure from left to right at slightly rising angles: an upper one from the heavy δ line, by μ and the hooked mark, to the circle at right; the long chord through the thick point; and a lower one near λ. A dotted line drops from the hooked mark to π on the long chord. Short ticks cross the circle between α and β and cross the β arc between β and γ. The prose on this page cites no point letters, so none of these labels has a referent in the text here. Every label that can be seen is listed below.

[16] struck through at this point on the leaf: "vel ab invent[?] $\overline{\text{II}}$" ("or from found quantity II"), and then, inside parentheses left standing, "prout fieri potest" ("as can be done")

[17] The form is two strokes joined under one bar, like a small n and unlike the separate II on the previous page, so the value rests on Dreyer. Added above the line here are a sign, then *multiplicato*, "the multiplied". Dreyer (*Opera Omnia* I, pp. 290, 317) reads II, takes the sign as *sed*, and judges that words have dropped out after *multiplicato*.

- α (top of the circle, outside it): α
- β (upper right, outside the circle, at the end of the β arc): β
- ρ (upper left, at the upper end of the two lines from this point): ρ
- δ (upper left, outside the circle, at the upper end of the two lines from this point): δ
- small bowl (on the α arc, just below α): σ [?]
- γ (where the two arcs and the line from ρ meet): γ
- ε (lower end of the β arc, on the left): ε
- θ (right of the α arc, below γ): θ [?]
- small hook (directly above the point where the upper chord meets the line from β): unresolved [?]
- μ (above the left end of the upper chord): μ [?], written like a Latin u
- ν (above the long chord, right of π): ν [?]
- far right (outside the circle, at the right end of the long chord): a 5-shaped form, ς or the figure 5 [?]
- π (below the long chord, at the foot of the dotted line): π
- below the long chord (at the lower end of the α arc and the dotted line beside it): a ξ-like form [?]
- λ (above the lower chord, by the thin line from ρ): λ
- τ (on the thin line from ρ, near where it meets the heavy line from δ): τ

DOGMA VII OF SPHERICAL TRIANGLES

[DOGMA·V $\overline{\text{II}}$· SPHÆRICOR]

In a non-right-angled triangle, given two angles about a given side, to find the remaining angle.

Add the lesser angle and the complement of the greater to each other, and subtract them from each other. Add the sines of both remainders, if the complement of the greater angle exceeds the lesser angle; or subtract them, if it is less than that angle. You will then have found quantity I. Subtract this from the first sum: you will have found quantity II. Multiply by this the sine of the complement of the given side, and divide by the whole. The result is found quantity III. Add this to found quantity I, if the given side is less than a quadrant; or subtract it from the same, if it turns out greater. You have the sine of the complement of the angle sought. So the operation was similar to the previous one.[18]

DOGMA VIII OF SPHERICAL TRIANGLES

[DOGMA V $\overline{\text{III}}$ SPHÆRICOR·]

In a non-right-angled triangle, given two angles together with a side opposite one of them, to find the remaining sides and angle.

[18]Dreyer (*Opera Omnia* I, p. 317) judges this solution false, because it applies Dogma VI to the polar triangle with sides and angles exchanged.

Here it will be necessary to resolve the triangle into two right-angled triangles, by drawing a perpendicular from the unknown angle, whether it falls outside the triangle or inside it, and then to work by III and IV. For example, let ABC be a triangle whose angles at A and B are known, together with side AC. I want to find side BA. So I draw a perpendicular from the unknown angle C to BA; let it be CD. In the right-angled triangle CAD, therefore, the angle at A is given with the adjacent side AC. So by the Third, AD will be given.

Diagram. The figure stands at the right of the shortened lines of the example, with no caption. A spherical triangle has A at its apex, B at lower left and C at lower right. The left side is one curved arc from A down to B, with D on it. The right side runs from A down to C, curving outward, with short breaks in its stroke. The base BC is a slightly curved line. A dotted line runs from D down to the right to C. A single point stands inside the triangle, above the base and left of the dotted line, with no letter beside it. No right-angle mark is drawn at D. Every letter that can be seen is listed below, and all four are also cited in the text of this page.

- A (above the apex): A
- D (left of the upper end of the dotted line, outside the left arc): D
- B (left of the lower left vertex): B
- C (right of the lower right vertex): C, with a small tail below it and a small point to its right

I. Operation. [Margin: ·Ī] Add the lesser given and the complement of the greater to each other, and subtract them. Subtract the sines of both remainders from each other and halve the result, and you will have the sine of the complement of side AD.

II. Next, by I, find DC.[19] [Margin: ·I̿]

[Margin: I̿I̿·] Finally, in the other right-angled triangle BDC, for side BC, which is opposite the right angle, multiply the sine of side DC by the whole and divide by the sine of the given angle at B. The result is the sine of side BC.

[Margin: Ī V] For side BD, work by I. Add to each other the lesser side DC and the supplement of the greater, BC, and subtract them. Add the sines of both and halve; and you have the sine of the complement of side DB[20], and so BD itself.

So add side DB to AD, found earlier, and you will have the whole BA known. For angle ACB, however, find angle DCB by III, as above; add this to angle ACD, found earlier, and the whole angle ACB, which was sought, will be established. The operation is similar when the perpendicular falls outside the triangle.

[19] The first letter is written over on the leaf and can be read as B or D. Dreyer prints DC, and the next step uses DC.

[20] The first letter is written over on the leaf. Dreyer prints DB, and the following sentences use DB.

DOGMA IX OF SPHERICAL TRIANGLES

[DOGMA ĪX SPHÆRICOR·]

In a non-right-angled triangle, given the three sides, to find any angle.

Of the two sides enclosing[21] the sought angle, add the lesser side and the supplement of the greater to each other, and subtract them. Add the sines of both remainders if the supplement of the greater side exceeds the lesser side; otherwise subtract them. Half of the result [producti] is found quantity I. Subtracted from the first sum, it gives found quantity II. The third found quantity is the difference between the first found quantity and the sine of the complement of the remaining side, which is opposite the sought angle. Next, multiply found quantity III by the whole and divide by found quantity II. The result is the sine of the complement[22] of the sought angle, if that angle is acute. If it is obtuse, the arc found for that angle is to be added to a quadrant.[23]

Now this angle is acute when found quantity I is less than the sine of the complement of the third, opposite side, and obtuse when it is greater than that same sine.

[Margin: Ī·] N.[24] If the enclosing sides exceed 90°, take the complement from the south pole.

II. [Margin: ĪĪ·] [Margin: Various cases — Casus varij] But when the greater[25] side exceeds a quadrant, work with its excess over 90°. You will get found quantity III by adding found quantity I to the sine of the complement of the side subtending the sought angle, not by subtracting, as it stands in the Dogma; otherwise, according to the Dogma. And if the first sum exceeds a quadrant, you will get the true sum by subtracting from 180°.

III. [Margin: ĪĪĪ] If one of the enclosing sides exceeds 90°, while the opposite side is exactly 90°, add the whole sine [integrum sinum] to the first found quantity and divide by found quantity II. The result is the complement of the sought angle. In another way:

IV. [Margin: ĪV·] If one of the sides enclosing the sought angle, together with the side opposite it, exceeds a quadrant: of the opposite side, subtracting found quantity I from the sine of this one [?].[26]

Demonstration of Dogma IX of Spherical Triangles

[Demonstratio ĪX dogmatis Sphæricorum.]

---

[21] the letters after "amb" in this word are written over, with letters added above the line

[22] struck through at this point on the leaf: "lateris reliqui, oppositi" ("of the remaining side, opposite")

[23] the leaf sets the next sentence inside square brackets, closing one at the end of its first line and opening another at the start of its second.

[24] The leaf does not say what this letter stands for. Dreyer prints it too, without comment.

[25] struck through at this point on the leaf, by eye "est" ("is") [?]

[26] this sentence has no main verb on the leaf, and "oppositi" ("of the opposite") stands before the gerund with nothing written to govern it.

- DΓ: to be sought
- TV: found quantity I
- ƐS[27]: found quantity II

Diagram. The figure stands below the heading and the three-line key, with no caption and no prose beneath it. A large circle is drawn with a heavy line. A long horizontal line crosses it from C at the left to G at the right, and a near-vertical line runs from A at the top to the label at the foot. A heavy line runs from F at upper right, through a small circle drawn on the line, down to X at lower left. A further straight line runs from an unlettered point on the circle at upper left down to Γ at lower right, and these lines meet near the middle at the small label there. A short chord joins A and F. From the unlettered upper-left point a short heavy upright drops to H. From K a heavy upright drops to P. A thin curved line descends from A through D to L and continues below the horizontal. A broken curve leaves the unlettered upper-left point, passes D, and continues as a dotted line to Γ. A dotted upright descends from E past the group M, S to the lower part of the circle, where a dotted horizontal runs left to P. A dotted line rises from P through the small label left of L toward A, and another dotted line runs down to the label at the foot. A dotted horizontal joins the T, V group to M. Short cross-ticks mark the circle at upper left (two), between E and F, at the left (two), and between P and X, and one marks the broken curve left of D. A short heavy upright with a hook drops to the horizontal below the small circle between T and V; it could carry a letter b [?]. A solid line runs from the point below D to the heavy line at M. Every label read as a letter is listed below, with the alphabet each is taken from; that small circle could also be a letter o [?].

- A (top of the circle): A, Latin
- E (immediately right of A, at the head of the dotted upright): E [?], by eye Latin, three bars; not F, against the F next listed
- F (upper right, at the end of the chord): F, Latin
- D (above the point where the curves cross, left of centre): D, Latin
- T (left of the small circle on the dotted line): T, Latin or Greek, the forms being the same
- V (right of that small circle, above the dotted horizontal): V [?], by eye a curled form close to y or D, matched only provisionally with the V of the key; Latin or Greek ν, not settled
- Ɛ (right of the heavy line, near its crossing with the broken curve): Ɛ [?], a small round form by eye like ε, c or u; Greek or an uncial Latin E, not settled
- M (just below that, right of the heavy line): M, small, by eye an m; Latin
- S (just below M): S, small, by eye an s; Latin
- C (left end of the horizontal, outside the circle): C, Latin, written small
- H (foot of the short upright, below the horizontal): H, Latin

[27]Dreyer prints ES

- K (above the horizontal, over the upright to P): K, Latin
- small label below the horizontal, left of L: [?], by eye τ or r, with a top stroke; alphabet not settled
- L (below the horizontal, at the foot of the curve through D): L, Latin
- centre label (above the horizontal, at the crossing): I; alphabet not settled
- R (right of the centre, above the horizontal): R, Latin
- G (right end of the horizontal, outside the circle): G [?], G or C, Latin, heavily written
- Γ (lower right, outside the circle): Γ [?], by eye Greek Γ, or a Latin r with a long arm; the same form as the second letter of the key's DΓ
- P (lower left, outside the circle): P, Latin
- X (lower left, right of P, at the foot of the heavy line): X [?], with a long stroke entering from the left; Latin X or Greek χ, not settled
- label at the foot of the circle: [?], by eye J or I, with a top bar and a hook to the left, close to the capital I of this hand in ”Inventum”; alphabet not settled