Petri Nonii Salaciensis De arte atque ratione nauigandi libri duo. Eiusdem in theoricas planetarum Georgij Purbachij annotationes, & in problema mechanicum Aristotelis de motu nauigij ex remis annotatio vna. Eiusdem

de erratis Orontij Finoei liber vnus. Eiusdem de crepusculis lib. I, cum libello Allacen de causis crepusculorum

[On the Art and Method of Navigating, and Other Works]

(1573)

by Pedro Nunes / Petrus Nonius (1502–1578), Cosmógrafo-Mor of Portugal

Translation by Scott Weisman

Ars Astronomica

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Upright reason dictates that the recipients of the good, of whatever type and level of beneficence it may be, must show gratitude and blessing to the beneficent person in every way possible, commensurate with the value of the beneficence. And one who has benefited all the people of the world — for example, one who invented a new instrument for the good of the world, or a good book — it is fitting for every discerning person, out of the obligation of love of fellow beings, to at least purchase it, so that the man will profit and his heart will be encouraged thereby to invent yet more good instruments in the world. And similarly, all other wise-hearted people will likewise strive and exert themselves to invent good things and instruments needed for the repair of the world and its perfection.

And therefore, whoever says: "What do I need this new instrument for?" — he does not act well towards the world. For if not for the man who invented it, where would you be? And what would your city do? And more than this, if the man did not exert himself, what would the world do?

— R. Pinchas Eliyahu Hurwitz, *Sefer HaBris HaShalem* (1797)

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A note on the work

The collected Latin works of Pedro Nunes, royal cosmographer of Portugal and the leading mathematician of the age of discoveries, in the author's own final edition. Its centrepiece, *On the Art and Method of Navigating*, sets out the geometry of sailing a fixed compass course: Nunes showed that such a course is not a great circle but a spiral winding toward the pole — the rhumb line, or loxodrome — the problem Mercator's projection was built to solve. With it are printed his commentary on Peuerbach's planetary theory, his notes on Aristotle's problem of the rowed ship, his demolition of Oronce Fine's claims to have squared the circle, and his treatise on twilight, which solves the shortest-twilight problem and describes the *nonius*, the angle-measuring device that gave its name to the vernier scale.

A note on the translation process

This translation was produced with the translation-pipeline. See the linked repository for details on the process.

This text is the product of a highly refined automated translation process. It has undergone only cursory human review and should not be treated as an authoritative or scholarly rendering. It is offered solely as a reference and reading aid — a way to make an otherwise inaccessible work approachable — and any passage of real consequence should be checked against the original source.

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PEDRO NUNES OF ALCÁCER DO SAL [Petri Nonii Salaciensis]: ON THE ART AND METHOD OF NAVIGATING, TWO BOOKS

By the same author: annotations on the *Theoricae Planetarum* of Georgij Purbachij [Georg Peuerbach], and one annotation on Aristotle's mechanical Problem concerning the motion of a ship driven by oars.

By the same author: one book on the errors of Orontij Finœi [Oronce Fine].

By the same author: On Twilight, Book I. With the little book of Allacen [Alhazen] on the causes of twilight.

COIMBRA, at the house of Antonij à Marijs [António de Mariz], printer to the University. In the year 1573.

By permission of the Inquisitor.

TO SEBASTIAN I, OUR MOST INVINCIBLE KING AND LORD: ANTÓNIO DE MARIZ, PRINTER OF COIMBRA, WISHES LASTING HAPPINESS.

When I came upon the books on the method of navigation by that most distinguished man Pedro Nunes, I was frankly amazed at how much licence our boldness takes with the works of the most celebrated authors. The book had in fact been so corrupted that anyone who navigated by its rules would certainly have been wrecked. A good deal was missing; other things had been substituted at random; and everything had been so altered that the author himself would not have recognized his own offspring — indeed, moved by justified indignation, he would have denounced the book as swarming with errors on every side, and disowned it. Nature produces only rarely the men who do work that benefits the commonwealth, and I did not want them deterred from publishing what they have achieved by many sleepless nights and by counsel that is almost divine, for fear that the ignorance of the booksellers can easily corrupt and adulterate it. So I resolved to commit the same work to the press at my own expense, cleared of every error, fault, and disgrace, and restored to its original elegance. And to make the addition greater still, I decided to add books by the same author — *On the Errors of Oronce Fine* and *On Twilight*, published among us long ago and now very much sought after for their usefulness and their learning. In this I have spared neither care nor expense in drawing the geometrical figures, and I hope this labour of mine will not be unwelcome to all good men.

Once I saw the work was finished and understood that it needed a great patron, I scarcely hesitated to dedicate it to Your Highness. If the power of opponents were to be feared — what age has ever seen anyone stronger than you? If the boldness of those skilled in the very art the books treat is to be dreaded — who is better schooled in these arts than you? And if some reward for this labour may rightly be expected — who is more munificent than you? Add to this the author's standing and his pre-eminence among all the mathematicians of this age. And should his admirable ease in demonstration and his works full of erudition fail to prove the point, there would still be this powerful argument: your uncles, the princes of this kingdom (men whom nothing but the great ever pleased), had him for their teacher; and you at last, illustrious King, approve his teaching and gladly listen to instruction in mathematics. You should therefore neither refuse the book your protection nor judge this labour of mine to have been undertaken in vain. May God, best and greatest, long keep Your Majesty in safety. Coimbra, the day before the Ides of August [12 August], in the year 1573 from the birth of Christ our Lord.

PEDRO NUNES OF ALCÁCER DO SAL, TO THE READER.

I shall offer a few preliminary remarks on the method of navigation, kind reader, so that you may more easily grasp what this commentary contains.

Let us imagine, then, four great circles on the celestial sphere passing through the point directly overhead [punctum supra verticem — the zenith]. Let one of them be the meridian, and another the vertical circle, which cuts the meridian at right angles and passes through the points where the equator [aequinoctialis] intersects the horizon. These two circles divide the circumference of the horizon into quadrants. The remaining two are the circles that bisect those quadrants. The common sections of these same circles with the plane of the horizon are straight lines meeting at the centre. Now wherever the mariner's needle [acus nautica] is carried, it lies parallel to the horizon, and so by the power of the magnet it represents straight lines of this kind — and with them the quarters of the horizon toward which those lines point. The Spanish [Hispani] call these lines by the common name *rumbos*, rhumbs.

The middle one they call by its own name the rhumb of North and South [Septentrionis & Austri]; and the one that cuts this at right angles over the centre itself, the rhumb of *Leste* and *Oeste* — for they call Subsolanus, the east wind, *Leste*, and Favonius, the west wind, *Oeste*. Of the remaining two, the one that bisects the north-eastern quadrant and the quadrant opposite it is the rhumb of *Nordeste* and *Sudoeste*: for they call *Nordeste* the point midway between the

north and the equinoctial sunrise, and *Sudoeste* the point opposite it. And the one that finally divides the north-western quadrant, and the quadrant opposite it, into two equal parts is called the rhumb of *Noroeste* and *Sueste*.

We must further note that when sailors cast off from harbour they set their course so as to keep the ship's prow pointed continuously, voyage after voyage, at the same quarters of the horizon, with the help of the mariner's needle; and when they must, they turn to another heading. Men are said to sail from *Leste* to *Oeste* when they cover some distance with the ship's prow held toward *Oeste*; and the same judgement is to be applied to other voyages as well. We are defining regular courses, however, not irregular ones. For if the ship's prow is fixed

on *Nordeste*, and yet the ship herself is carried along the meridian — by the set of the currents, by the drive of the winds, or by some other cause — then she will be said to have sailed neither to *Nordeste* nor to the north.

These curved lines, then, which ships trace on the surface of the sea by running in this way, they likewise call rhumbs. If, for instance, the voyage has been made along a meridian toward one of the poles, the line traced will be called the rhumb of North and South; but if toward the point midway between the north and the equinoctial sunrise, the rhumb will be called that of *Nordeste* and *Sudoeste* — and so with the rest.

Some of these lines are circular, others compounded of circular ones. For if one goes toward one of the poles along a single meridian, or from the equinoctial rising to the setting along the equator itself, no one will deny that the circumferences of great circles are traced in this way on the globe of land and sea beneath. But if one sails otherwise, the lines traced must be compounded of certain small segments of certain great circles. A ship rides upon the water in such a way that along her back, or keel, she bears down with her weight toward the centre of the world. So if you conceive a plane passing through that back itself, from prow to stern, along the ship's length, then the common section of this plane and the globe of the sea will be a great circle falling upon the horizon, as is plain from the first book of the *Geometry* of Theodosius. The ship's place, accordingly, will be a small arc of that great circle — for it makes no difference if some breadth is found within so large a circuit.

Now if you drive the ship from her place by wind or by oars, the bearing her prow points to must change, because when the place changes the angles of position become unequal, as the science of triangles shows. And yet we assumed that a like heading is kept throughout the voyage. Therefore, before any

noticeable difference arises in that position or inclination, the ship turns off the first circle onto another great circle; and so the line traced will not be a single circular line, but one compounded of circular lines.

Since it was extremely difficult for sailors to draw lines like these on globes, and the work cumbersome besides, mathematicians devised a certain plane representation of the world, one not merely suited to the art of navigation they practise but very easy as well. For on it, all the straight lines laid down as rhumbs of the same name, being parallel, make equal angles with every meridian line — that is, with the rhumb of North and South. A like heading will therefore be marked on it, just as on a globe, although it seems to fall considerably short of the true principle of the planisphere, as we shall explain partly in this Commentary and partly in others that we may perhaps publish shortly.

Whenever, then, men who have stood out into the open sea want to know in the course of a voyage where they are, they find it at once from the altitude of the pole once observed together with the quality of the course — that is, from the known rhumb they have followed — or else from the quality and quantity of the course alone. The mariner's needle [acus nautica] shows the rhumb, while they reckon the length of the distance covered by certain estimates. Sometimes, too, when the quality of the course is unknown, they learn where they are from its quantity alone, the altitude of the pole having first been observed.

For in a right-angled triangle there are, besides the right angle, five things: three sides, that is, together with two acute angles. If any two of these are known, the remaining three become known. Furthermore, we always assume that the latitude of the home port [radicalis locus] they sailed from is known. And because triangles of this kind are either found already drawn on the planisphere they use, or can easily be described on it by drawing parallels, they have no need on that account of any skill in the geometrical art: with the compasses alone they work out each item, and whatever else they want from these.

Again, if the voyage is made along a single meridian or along a single parallel, it is very easy for them to find the position of the place they are in. For if they run along one meridian, they first find the distance from the equator and reckon it off on that same meridian toward the pole of the world. But if they are working along a single parallel, they measure the distance covered by estimate, and then reckon that same distance off on the same parallel from the place they set out from, toward whichever quarter of the world — east or west — they have sailed; for they hold that they have arrived at the end of such a distance. But since they make all the parallels equal, it follows that the same span takes in as many

degrees on the greater circle as on the smaller, which is absurd. Of these matters, however, elsewhere.

PRAECIPVAE SENTENCiae prioris libri.

The Chief Propositions of the First Book

- The meridian circle is the way of North and South, while the equator is the way of *Leste* and *Oeste*. The remaining ways, which the Spanish [Hispani] call rhumbs, are not circles, but consist of small segments of great circles. — In the Preface.
- Although the vertical circle which the straight line of *Leste* and *Oeste* represents in the plane of the horizon does pass through the points of equinoctial rising and setting, it is not on that account to be supposed that a man who had circled the globe of land and sea along that circle would be said to have sailed to *Leste* or to *Oeste*.
- Although we hold the ship's prow fixed continuously on the equinoctial rising or setting, it will still be impossible for us ever to reach those equinoctial points; rather, by sailing in that manner a certain circle is traced parallel to the equator.
- Furthermore, when we sail by that art we are carried along the circuits of great circles and at the same time run along a parallel of the equator — yet by certain deviations that escape all perception.
- Apart from the equator, no other of the parallels can truly be called the way of *Leste* and *Oeste*.
- How great the latitude of a place is, is shown when a star passing through the zenith rises at *Nordeste* and sets at *Noroeste*.
- Anyone who makes his course along a great circle other than the meridian and the equator must very often change the inclination of his way, because of the varying and inconstant inequality of the angles of position arising from the new meridians he comes under. Otherwise it will be impossible for him to advance on a direct course.
- Sailors, therefore, when they hold continuously toward the same quarter of the world, keeping a like heading, cannot run straight courses.
- Why are the places of the world wrongly set down on the sailors' planisphere?

PRÆCIPVAE SENTENTIAE posterioris libri.

The Chief Propositions of the Second Book

- That rectilinear planisphere which our sailors use, although it cannot present a true image of the world, is nevertheless very well suited to the art of navigation that they practise.
- I judge that Ptolemy was one and the same man, who composed both works — the astronomical, that is, and the geographical.
- Ptolemy long ago used the very same art that our sailors use, for finding how great the difference is between the meridians of two places.
- The method Ptolemy used to find the difference of longitude between Corura and Palura in the Indian Ocean is examined.
- Since Ptolemy contracts the distances of places on whatever bearing in order to take the straight line, he does so more advisedly and more cautiously than our sailors do. For these men extend into a straight line the distance they cover with many windings in sailing.
- When the line that subtends the right angle is increased,

[The entry breaks off; it is completed on the next page.]

- … it necessarily follows that the latitudes and longitudes of places are stretched beyond their limit in that same ratio as well. [The entry begun at the foot of the preceding page is completed here.]
- Why do sailors extend the interval from Spain to India beyond its proper bounds?
- The method of finding the longitudes of places from eclipses is the most certain of all.
- In what manner the longitudes of places, once known from eclipses, are to be laid down on the sailors' planisphere.
- By what art those places are to be laid down on the sailors' planisphere which present themselves to men sailing under one and the same parallel.
- The meridian is a kind of standard for all other positions.
- Not every position, or inclination of one place to another, that is found set out on the sailors' planisphere is to be taken as true, but only that one under which the passage from the one place to the other has actually been made at some time.
- Sailors are very often deceived by following those positions of places which the sea chart shows; and how they remain ignorant of the causes.

- The makers of sea charts go wrong in this, that the longitudes of places drawn from those very charts they then set down on a globe by no other art than the one by which they set down the fixed stars.
- The coasts of the Mediterranean Sea do not carry their true pole-altitudes on the sea chart itself; and from what source so great an error has arisen.
- Why does that isthmus which lies between the Mediterranean and the Arabian Gulf appear so large on the sea chart?
- A correction of Ptolemy's description of the rectilinear planisphere, and an easier demonstration of the other planisphere as well.
- If we suppose that the earth's circuit, taken along a great circle, contains 6,000 Spanish leagues, then one league will be equal to one schoenus.
- Under one and the same inclination of a great circle to the meridian, the difference of longitude will not be the same over every stretch and universally, nor will the travelling distance between two given places be the same. For if the first place stands to the second in the same position as the third stands to the fourth, and yet their distances from the manifest pole are unequal, then the travelling distances and the differences of longitude between those places will be unequal — and the rest of this kind.
- The difference of longitude of two places is on the sea chart sometimes contracted and sometimes, on the contrary, stretched out.
- How the difference of longitude of two places can be truly concluded from the sea chart.
- A table of the inclination of a great circle to the meridian, containing seven different positions.
- Since the ship's track, apart from the meridian and the equator, is angular, one must therefore sometimes lay down as certain what is uncertain — and the rest.
- The Sun's declination cannot return to the same minutes, even when the equation has been applied.
- How it can be known on what day the Sun has no declination.
- Ioannes Lucidus [Giovanni Lucido] censures Alfonso [Alfonso X] wrongly.

- Ioannes de Monteregio [Regiomontanus] subtracted one day from the span of time which is found in the Alphonsine tables between Nabonassar and Christ, and added that same day to the span between Christ and the autumnal equinox observed by Ptolemy.
- No credit is to be given to the little book *On the Significations of the Fixed Stars*, translated from the Greek by Nicolaus Leonicus [Niccolò Leonico Tomeo].
- On the day before Christ, the Redeemer of the world, was conceived, the vernal equinox fell at Rome; yet it was being celebrated on 25 March, according to Caesar's institution.
- Observations of the fixed stars made at almost the same time by Ioannes Vernerus [Johannes Werner], Copernicus, and Cardanus [Girolamo Cardano] disagree with one another.
- An error of Albertus Pighius Campensis [Albert Pigge of Kampen] in geometry is exposed.
- A certain sophism of Albert Pigge concerning the declination of the fixed ecliptic is dissolved.
- Marcus Beneuentanus [Marco Beneventano] is shown to contradict himself: he held the declination of the fixed ecliptic to be as great as the declination Ptolemy found for the movable ecliptic, and yet placed the head of Aries of the ninth sphere's ecliptic, for the year 1519, in the 28th degree and 8th minute of Pisces.
- Why we are unwilling to accept Regiomontanus's opinion about the equinoxes.
- That the head of Aries, from which the reckoning of the motions of the stars takes its beginning in the Alphonsine tables, is the vernal section.
- An observation made by us at Coimbra in the course of the year 1555 from Christ's birth, at the autumnal equinox.
- The reduction of the declination of the parts of the ecliptic onto a single plane, handed down by Vitruvius and demonstrated by us.
- The construction and use of a certain circular [?] instrument, lying in the plane of the horizon, by which the Sun's altitudes are taken.

- The construction and use of the astronomical radius; and a slip of Ioannes Schonerus [Johannes Schöner] is noted.
- An error of Hieronymus Cardanus [Girolamo Cardano] is exposed: he thought that from the known ratio of the shadow to the gnomon the altitude of any star whatever, at any hour whatever, could be found as reckoned from the centre of the earth.
- Girolamo Cardano censures Vitellio [Witelo] wrongly, and in this he is signally deceived, when he inquires to what height vapours can rise from the earth.
- The arc of the Sun's occultation on the circle of altitude cannot be equal to the arc of its distance from the point of rising, except in those places that lie under the equator, and when the Sun runs along the equinoctial circle itself.
- An explanation of a certain obscure passage in the seventh chapter of the first book of Ptolemy's *Geography*.
- That the declination of the pole star found in Hipparchus's time does not agree with Ptolemy's calculation of the motion of the fixed stars.
- The argument of Augustinus Ricius [Agostino Ricci] is refuted, who held that Ptolemy had erred by one degree and six minutes in the places of the Sun, the Moon, and the fixed stars.
- Girolamo Cardano asserts without due consideration, in his little book *On the Restitution of Times*, that exactly eight solar years elapsed between Ptolemy's two observations of the autumnal equinox.
- The canons that sailors use for finding the altitude of the pole by the altitude of the pole star when it stands off the meridian cannot be general for all climates.
- Our recent canon for finding the altitude of the pole by the meridian altitudes of the Sun and of the fixed stars.
- The method of Petrus Appianus [Peter Apian] is examined, which he used in his *Cosmography* for finding the altitude of the pole by means of a known hour.

- The method of Iacobus Zieglerus [Jacob Ziegler] for finding the altitude of the pole by the Sun's horizontal distance from the meridian is examined.
- In every place set between the equator and the tropic of Cancer, whenever the Sun is nearer to the arctic pole of the world than the zenith is, the shadows of gnomons run backwards, and there is no miracle in it.
- From the known elevation of the pole at two places, together with the position their distance holds with respect to the other meridian, it cannot be known universally how great that distance is, nor what the difference of the meridians is — although Ptolemy boasts that he found this out by the instrument called the meteoroscope, and Regiomontanus promises the same in problem 46 of the *Table of the Primum Mobile*.
- Why the altitude of the pole cannot be known from the things that either Apian or Ziegler takes as known.
- That the thirteenth proposition of the first book of Menelaus, *On Spherical Triangles*, is not universally true in the way it is propounded.
- That the latter part of the eighth proposition of chapter 14 of the first book of the *Revolutions* of Copernicus, where he treats of spherical triangles, is not true.
- And that what he teaches in the eleventh proposition is an error.
- And Copernicus likewise slipped in the sixth proposition on rectilinear triangles.
- Nor did he slip any less in the twelfth.
- On the varying relation of the Sun to the zenith in different places of the earth, before noon and after.
- An error of Ioannes Stoflerus [Johannes Stöffler] is exposed: he thought that on the day when the Sun passes through the zenith of those men who are set between the tropics, their morning shadow runs straight, cast toward the setting of the Sun on that same parallel, while the afternoon shadow extends straight eastward, to the point of the horizon over which the Sun had risen.
- How the altitude of the pole is found by the Sun's rays, when the position of the meridian is given as known.

- How the altitude of the pole is found by the Sun's rays, even if the position of the meridian is unknown.
- How the altitude of the pole is found by the Sun's rays when both the position of the meridian and the Sun's declination are unknown.
- Again, how the altitude of the pole may be found when the Sun's declination and the position of the meridian are unknown — and that within the plane of a single circle.
- The construction of a horizontal dial by which both of the Sun's distances from the meridian are ascertained, namely the one measured along the equator and the one measured along the horizon.
- That the direct shadow [umbra recta], the gnomon, and the turned shadow [umbra versa] are proportionals in continued proportion.
- The latitude of Rome, drawn out from the ratio of shadow to gnomon which Vitruvius records, does not agree with the latitude Regiomontanus found by the astrolabe.
- On the Sun's rays: which of them are parallel, which converge, and which merely appear parallel.
- The observation Eratosthenes made at Alexandria, to find how great the whole circuit of the terrestrial globe is, is examined.
- It is shown that the shadows of gnomons are not parallel but only appear so, and toward what point they converge.
- Given the latitude of two places together with the difference of longitude, a manifold method of finding the interval between them.
- How those lines are to be drawn on the surface of a globe which our sailors call rhumbs — lines like the ones a ship traces by her course on the surface of the sea when we sail.
- On the relation of these lines both to one another and to the poles of the world.
- What relation the segments of one and the same rhumb have to one another.
- On the use of that globe on which a delineation of this kind has been made.

- One Annotation on Aristotle's mechanical problem *On the Motion of a Ship under Oars*.

The chief points of what we have noted on the *Theoricae planetarum* of Georg Peuerbach

- If the arc of the zodiac which the Sun traverses by its apparent motion in a given time is bisected by the line of mean longitude, the mean motion for that time will be as great as the apparent motion.
- Given any interval of time whatever, to find the arc of the zodiac which the Sun traverses in that time by its apparent motion, and which makes the mean motion and the apparent motion equal over that same time.
- An error of Ioannes Baptista [Giovanni Battista], the old commentator, is exposed, concerning the place of the greatest equation of the Moon's centre.
- That point of the eccentric at which the equation of the centre in the Moon herself is greatest is found geometrically; and how far that point stands from the apogee.
- How great the greatest equation of the centre is, is shown numerically; and also how great, in that position, is the distance of the epicycle from the centre of the world.
- Giovanni Battista's opinion about the proportional minutes is refuted.
- When, in one and the same position of the epicycle, equal equations answer to unequal arguments, the end of the smaller argument stands farther from the end of the argument of greatest equation for that position than the end of the greater argument does.
- In Mars alone does the axis of the orb carrying the epicycle cut the axis of the zodiac — not in Jupiter, nor in Saturn. Peuerbach teaches the contrary.
- A demonstration of the greatest equation of the centre in the three superior planets, in which an error of Erasmus Reinholdus [Erasmus Reinhold] is laid open, and of another Erasmus as well, and of the old commentator.
- That the equations of the arguments in these same three superior planets have been computed for the position of mean distance of the epicycle's

centre from the earth, and not for the mean longitudes defined by Georg Peuerbach.

- Between the position of the apogee and that of the point opposite the apogee, the centre of Venus's epicycle and the Sun stand truly in the same place of the zodiac in longitude only once — namely when the distance of the epicycle's centre from the centre of the equant is equal to the radius of the deferent.
- It is demonstrated that the centre of Mercury's epicycle moves more swiftly about the apogee of the equant, that is, above the centre of the deferent, but more slowly about the point opposite the apogee.
- The equations of the argument written in the tables of Mercury are those that occur while the centre of the epicycle stands off from the centre of the world by an interval equal to the radius of the deferent; but a distance of this kind cannot be called the mean distance of the epicycle's centre from the centre of the world, unless you speak very improperly, as Georg Peuerbach does.
- The nearer the arc of the motion of the argument is to the true point opposite the apogee of the epicycle, the greater the equation of that motion of the argument becomes.
- A greater size of epicycle is not the reason why the points of station lie nearer to the true point opposite the apogee, if all else is supposed equal.
- It can indeed happen that in a smaller epicycle the points of station stand less far from the perigee of that epicycle, while in a larger one they stand farther off.
- The slowness of the motion of the argument — that is, the slower motion of the planet on its epicycle — is the true reason why the points of station approach one another more closely.
- The argument of Geber [Jābir ibn Aflaḥ] and of Ioannes de Monteregio [Regiomontanus] against Ptolemy is answered: the argument by which they contend that it can happen, in the same planets, that the arcs of station are equal at unequal distances from the centre of the world.
- The distinction which Erasmus Reinhold noted between Mercury on the one side and the three superior planets and Venus on the other, concerning

the ratios that are left over, is not sufficient to assign the causes of the difference in the stations and retrogradations of those planets.

- In the Sun's motion, by contrast, the passage is from less to greater, but not by equal steps.
- It is shown that the arcs of the ascending semicircle of the ecliptic descend directly [rectè] in the northern climates.
- What Ioannes Baptista says — that Pisces and Aries have the greatest descensions in the oblique sphere — is a delusion.
- There are certain northern places in which Sagittarius descends more directly than Aries does.
- Unless a slower descent has required a greater concealment of the Sun — even though it is bounded further on into the night — it will not be the reason why the Moon appears sooner after conjunction. For it happens that equal arcs of the zodiac have unequal descents; and moreover that a smaller concealment answers to a greater descent.
- It is demonstrated that the ninetieth degree of the ecliptic from the ascendant always lies on the great circle passing through the zenith and the poles of the ecliptic.
- It is demonstrated that the distance between the ninetieth degree of the ecliptic from the ascendant and the meridian, reckoned along the divisions of the horizon, is as great as the amplitude of rising of the ascendant.
- A clear exposition of the theory of the latitude of the three superior planets.
- The equations of the motion of access and recess of the eighth sphere increase by unequal increments.
- The remaining phenomena of the motion of the eighth sphere are demonstrated, both according to Alphonsus [Alfonso X] and according to Thebit [Thābit ibn Qurra].

FINIS.

¶ Errata.

Page 8, col. 2, line 7, read *Troglodytica*; and line 9, read *Taprobana*.

Page 33, col. 2, line 21, read *anguli ad punctum e. recti* [the angles at point e are right angles].

Page 106, col. 2, line 42, read *recessus à parallelo* [recession from the parallel].

In the book *De erratis Orontij Finaei*:

Page 26, at the other end of the semicircle drawn there, put c.; and between c. and k. put l.

Figure of the nautical instrument which the Spanish call the *acus* (the needle)

Diagram. A full-page woodcut compass rose — the card of the mariner's needle [acus nautica]. Two concentric rule-circles enclose a band divided by radial strokes into thirty-two points; within the band lies a thirty-two-point star of alternating long and short rays, the long rays ornamented with lozenges, quatrefoils and dotted grounds, converging on a small rayed hub at the centre. A fleur-de-lis marks the north point at the top of the rose. The thirty-two divisions of the band are drawn but not lettered; sixteen labels are set outside the outer circle, each aligned radially with its own point, so that all but the east and west labels read at an angle to the page. The four cardinal labels pair the Iberian name with the classical Latin wind name.

Labels, clockwise from the top:

- N (top): *Norte. Septentrio.*
- NNE: *Nornordeste.*
- NE: *Nordeste.*
- ENE: *Lesnordeste.*
- E (right): *Leste. Subsolan͡* — printed with an abbreviation stroke for *Subsolanus*
- ESE: *Lesueste.*
- SE: *Sueste.*
- SSE: *Sulsueste.*
- S (bottom): *Auster. Sul.*
- SSW: *Sulsudoeste.*
- SW: *Sudoeste.*
- WSW: *Oesudoeste.*
- W (left): *Fauonius. Oeste.*

- WNW: *Oesnoroeste.*
- NW: *Noroeste.*
- NNW: *Nornoroeste.*

Argument of the First Book

In the year of salvation 1530 that distinguished man Martinus Alphonsus à Sosa [Martim Afonso de Sousa], by order of our most invincible king, sailed with a fleet toward the winter sunset, as far as the Silver River [*argenteum fluuium*, the River Plate]. Returning to Portugal in the third year of his voyage, he told me how carefully and how diligently he had investigated the positions of places, and that he had nevertheless met with a number of things that astonished him.

First, that on the days of the equinox he had observed the sun at its rising and at its setting, and had seen that it rose at *Leste* and set at *Oeste*. He therefore asked me — and pressed me hard — why it is that, as long as we hold a course to *Leste* while at sea, we stay upon one and the same parallel and can never reach the equinoctial circle, toward which we keep the ship's prow perpetually directed as we sail in that way.

He said, besides, that he had reached a southern latitude of 35 degrees while the sun was at the beginning of Capricorn, and that on the very day of the winter solstice he had seen it rise at *Sueste* with a quarter of *Leste* (that is, one point toward the east) and set at *Sudoeste* with a quarter of *Oeste*; and he admitted that he did not know the cause of this. For the rising in southern regions, when the sun moves through the southern signs, ought to be such as it is in northern regions when it moves through the northern ones. Now at north latitude 35 degrees, when the sun is at the beginning of Cancer, it rises at *Nordeste* with a quarter of *Leste*; therefore at a southern latitude of the same 35 degrees, when the sun is at the beginning of Capricorn, it ought likewise to rise at *Nordeste* with a quarter of *Leste*.

He asked us, then, why these things should happen so. At the time I gave him the reasons in person, as well as I could; afterwards I committed them to writing — thirty years ago now — and published a single treatise on the subject in Portuguese. That treatise I have at last decided to turn into Latin, so that it may be read and understood not only by the Portuguese but by other men as well.

¶ On Two Problems concerning the Art of Navigation, by Pedro Nunes of Alcácer do Sal. One Book.

At the outset, then, we must grant that the matter stands universally as Martim Afonso says he found it in certain places: namely, that wherever we may be, the sun rises at *Leste* and sets at *Oeste* when it enters the equinoctial points.

For draw the straight meridian line through the centre of the horizon, as Vitruvius taught; then, if upon it you erect from that same centre, in the same plane, a straight line at right angles, the circle of the horizon will be divided by these two straight lines into quadrants. Of these lines the first, the meridian, is the rhumb of North and South; the second is commonly called in Spanish the rhumb of *Leste* and *Oeste*. This is what that nautical instrument represents which is commonly called the needle [*acus*, the mariner's needle], and so does every image of it drawn on the mariners' planisphere.

Now of the parallel circles the equinoctial alone is able, together with the meridian, to cut the horizon into quadrants — which must necessarily be the case with those circles that are drawn from *Leste* to *Oeste*. No parallel other than the equator, therefore, can be the rhumb of *Leste* and *Oeste*. Instead we shall understand a certain great circle of the celestial sphere, one that cuts the meridian at right angles in the zenith [*verticali puncto*] and passes through the intersections of the horizon and the equinoctial, which are called the equinoctial rising and setting. That straight line of *Leste* and *Oeste* will assuredly be the common section of the plane of this vertical circle and the plane of the horizon — as can easily be shown from Book XI of Euclid's *Elements*.

If, then, anyone follows that same line of *Leste* and *Oeste*, he will remain on that vertical circle of the equinoctial rising and setting for as long as he goes straight on; his zenith too will lie on the circumference of that same circle. But if we were treating of that true horizon which is one of the great circles of the sphere, we should say that there is only one straight line of *Leste* and *Oeste*, and that it is common to the right horizon [*recto horizonti*] — the line in which the common section of all horizons with the vertical circles certainly falls. There is, however, another horizon that we make use of, one passing through the surface of the earth and not through its centre; it is parallel to that true and central horizon, and only a little distant from it, since it too shows us very nearly half the sky.

On a horizon of this kind, therefore, every place has its own particular line of *Leste* and *Oeste,* drawn out on both sides to the equinoctial rising and setting of the sun.

But although the great vertical circle just described — the one the *Leste–Oeste* line represents — runs toward the equinoctial rising, so much so that a man who sailed round the globe of land and sea beneath it would touch that very rising-point with his zenith, it must not on that account be supposed that a man carried there in that fashion should be said to have sailed to *Leste*. For once he has covered a fair stretch, he will see the ship's prow pointing elsewhere, and not to *Leste*. That is why the helmsman at the tiller, even though he may not know the reason, guards against this error from the very outset whenever he is making eastward along a single parallel and looks to a correct passage.

Suppose in fact that in sailing we aimed the ship's prow at *Leste*; that we then braced and lashed the helm so that it could not shift at all; that the sea we used was calm and smooth; and that a following wind blew as we would have it, breathing from just where the prow was heading. If, I say, we held our course in that manner, and then, after some distance had been run, looked at the mariner's needle, we should find that the ship's prow had turned aside and was heading elsewhere, not at *Leste*.

The reason is that at the place we set out from, the meridian makes right angles with the vertical circle. But as we move away from it, carried along that same vertical circle, we straightway come upon a new horizon and a new meridian. The new meridian, then, does not make equal angles with the vertical circle of the earlier place, as it did before, but rather unequal ones. Of these, one is the exterior angle in a certain spherical triangle formed by the two meridians and the same vertical circle — the angle the geographers call the angle of position or of situation [*positionis angulus situsue*]; the other is the interior angle opposite it, the one at the zenith of the earlier place, which showed where we were heading. Whenever the great circle along which we are carried is other than the equinoctial, this exterior angle is unequal to the interior angle opposite it — sometimes greater, sometimes smaller, according to the varying designation, northern or southern, of the regions of the world toward which and through which we are carried along those great circles. For that is how the matter stands in these spherical triangles, although in rectilinear ones the exterior angle is always greater than the interior angle opposite it.

But let us return to our purpose. If, then, one had sailed in that fashion, the error would be detected, correction would be needed, and one would have to bring the course back and return to the parallel of the earlier latitude. That, however, is not how a man who is steering for *Leste* usually sails. With his eyes fixed on the mariner's needle he keeps moving and governing the tiller, and so orders his

course that the ship's prow points where the *Leste* line points. In this way he guards against the error and avoids it, so that there is no departure in latitude, or none perceptible. The ship's prow, then, is always aimed at the equinoctial rising, which is ninety degrees distant from the zenith; and yet it can never arrive at the equinoctial point itself. On the contrary, the ship stays upon one and the same parallel — which seems a thing to marvel at.

Moreover, since in this way we traverse all the places that lie upon the same parallel, it has come to be accepted that the parallels themselves are drawn from *Leste* to *Oeste*; but this is not true. For none of them except the equinoctial can be any of the rhumbs that are either already marked on the mariner's needle or can be conceived upon it. All the same, from any one place to any other of equal elevation of the pole there is a proper and perfectly certain route by which the passage ought to be made, without the waste that those necessarily incur who are carried along a parallel circle. There is a further advantage in a passage of this kind: every day we can trace out by a wholly reliable calculation the distance covered, and know plainly where we are. Those who sail from *Leste* to *Oeste*, keeping to the parallel with much perplexity and anxiety, can achieve this in no way at all. And so knowledge of the longitude of places — which is indeed the most difficult thing of all to discover — will, so far as navigation is concerned, be for the most part superfluous.

For the demonstration of what has been said above: let the circle d a p be understood as the meridian of the place whose zenith is at b; let the horizon be l c m; the equinoctial, a c p; the quadrant of the vertical circle, b c. The angle at b, therefore, is a right angle, and to it there answers in the horizon the quadrant c m — just as on the mariner's needle itself, which represents the horizon, the straight line of *Leste* and *Oeste* and the meridian line take in a single quadrant. Accordingly, if we were to set out from the place b intending to sail to *Leste*, we should direct the ship's prow, along with the *Leste* line, toward c, the equinoctial rising of the sun.

Diagram. A woodcut sphere set at the head of the left column. A large circle stands for the meridian, with a horizontal diameter running across it from left to right; over the near face are drawn a family of arcs — two nearly parallel arcs sweeping from the top of the figure down to its lower left, and beyond them a series of further arcs of the same family crowding toward the right — together with a broad arc rising from the left of the diameter toward the upper right. The lettering, as it stands on the leaf:

- d (top of the outer circle): the pole, through which the meridians pass

- l (top centre, immediately right of d) and s (just right of l): the upper ends of the two horizons l c m and s h t
- o (upper right, on the broad rising arc): the place of arrival, of the same latitude as the starting place
- b (left, on the outer circle): the zenith of the place set out from
- i (immediately right of b): a point on the track run
- g (below b): the point actually reached, in place of e
- e [?] (beside g, indistinct on the scan): the point on the vertical circle that would be reached by a straight run
- a [?] (left end of the horizontal diameter, faint on the scan): on the meridian and on the equinoctial a c p
- f (below the diameter, left of centre): on the new meridian d e f
- c (on the diameter, at the centre): the equinoctial rising for the first place
- h, then K (on the diameter, right of c): h is the equinoctial rising in the horizon s h t; K marks the next vertical circle crossed
- P (on the diameter, toward the right): the other end of the meridian and of the equinoctial
- m (lower left): the lower end of the horizon l c m
- t [?] (bottom, right of m, indistinct on the scan): the lower end of the horizon s h t

Then, if we were driven either by wind or by oars and carried along that vertical circle as far as e, the mariner's needle would at once show, at e itself, that the ship's prow was turned toward another quarter of the world and not toward *Leste*. For a new meridian, d e f, would be marked out, which with the circle b e c would make the angle of position f e c less than a right angle; and a different latitude would be obtained, smaller than the first, since the arc e f is less than a b — as has been demonstrated elsewhere.

But since the course has been set for *Leste*, it cannot come about that by sailing so we run out to e; rather we slip to g, at which place the latitude is imperceptibly less than the first. The withdrawal of the ship's prow from the straight line of *Leste* and *Oeste* is likewise imperceptible, for by turning the ship toward *Leste* from the very start we guard against any error worth noticing. From g we direct our course to i, and with the prow always aimed at *Leste* we run through the quadrant g i h, in the horizon s h t, at whose point h lies the equinoctial rising, toward which the line of *Leste* and *Oeste* inclines; for when the horizon and the meridian are changed, the rising-point must be changed too. But having advanced only a little along g i h, we straightway pass over into another vertical

circle, drawn through K, and from that we fall into yet another. And so, again and again, we pass under new horizons and new meridians by way of various vertical circles, never departing from *Leste* by anything that the senses can detect, until we put in at o, whose latitude is equal to the first.

We are therefore carried along the circuits of great circles, and at the same time we run along the parallel, by certain detours that escape all sense. That we seem to have kept exactly to the parallel is due, I think, to this: that the vertical circles along which we are led cut the meridians at right angles at those points where they touch the parallel. At the neighbouring points, then, the departure from the parallel is very slight indeed, since the vertical circle meets the nearby meridians almost at a right angle about that same point of contact. So it is not the case that, if we run along a vertical circle, we immediately leave the parallel by any perceptible difference.

Thus it comes about that, although the beginning of the sign of Cancer declines twenty-three degrees and a half from the equator, the fifth or sixth degree of that same sign — and the degrees matching them toward the end of Gemini — have a declination smaller than that greatest declination by only six or seven minutes. And I think this counts for a very great deal in explaining this knot. There is still another reason: that a circle touches a circle at one point only, when the two are understood without breadth. But those circles along which we are led are not without breadth; and so their contact will be over something divisible, not at a point. And accordingly, when we are carried across along great circles, we in a certain fashion run across a lesser one. In this way, then, I think I have resolved the first question.

Only, for a fuller explanation, we ought to recall the point that is agreed, I take it, among everyone: that no one is so ignorant, so wholly untaught in letters, as not to know that at the time of the equinox — that is, when the sun runs through the equinoctial circle — it rises at the sixth hour before noon and sets at the sixth hour after noon. Now on horizontal sundials the line of the sixth hour, which is the line of *Leste* and *Oeste*, cuts the meridian line at right angles. Therefore, just as we laid down at the beginning, there is no doubt that the sun rises at *Leste* and sets at *Oeste* when it runs through the equinoctial circle.

To dissolve the second difficulty, we must work out that same thing which we began to explain above: by what sort of path, namely, those are led who run along a parallel. I think it must be noticed, then, that although every parallel makes right angles with every meridian — as must also be the case with those rhumbs that are drawn from *Leste* to *Oeste* — nevertheless no parallel except the

equator will be said to be the rhumb of *Leste* and *Oeste*. There will perhaps be no lack of those who suspect that the cause of this is the inequality of the angles. For since the colure of the solstices, which itself performs the office of a meridian, comes from the poles of the equinoctial and also from the poles of the zodiac, [it makes] right ang—

—les with the circle of Cancer, and meets it, together with the ecliptic, at one and the same point. It is no wonder, then, if men led on by a piece of sophistry have supposed that one right angle is part of another right angle, and therefore smaller than it. But that is not so. All right angles are equal to one another, whether they arise from the meeting of great circles with great circles or with lesser ones, as I have demonstrated elsewhere.

We must hold it for certain that no parallel except the equator is the rhumb of *Leste* and *Oeste* — nor is any parallel any other of all the rhumbs which the mariner's needle [*acus nautica*] already displays or which can still be conceived upon it. You will grasp the cause and the reason once you notice that all the straight rhumbs, which mark out the courses, are drawn through the centre of the horizon, and that they are the common sections of certain great circles with the plane of the horizon — of which plane the mariner's needle, as I said above, is the image. Since, then, all parallels except the equator are lesser circles, any of them that cut the horizon will cut it into unequal parts, and away from the common centre of the horizon and of the needle itself. They can therefore in no way perform the office of any rhumb, as appears in the figure below. In that figure the circle *a b c d* represents both the horizon and the mariner's card; the straight line *a c* is the common section of the meridian and the horizon, and is the straight rhumb of North and South; the straight line

Diagram. A woodcut circle at the foot of the left column, representing at once the plane of the horizon and the face of the mariner's card. Four diameters cross at the centre, giving the eight principal rhumbs; a chord is drawn across the upper part of the figure and produced beyond the circle to the right, and a further straight line runs from the centre out to the end of that produced chord, cutting the circumference on the way. The lettering as it stands on the leaf:

- a (top of the circle): the North point; with *c* it gives the meridian, the rhumb of North and South
- b (left of the circle): with *d*, the rhumb of *Leste* and *Oeste*
- c (bottom of the circle): the South point
- d (right of the circle): the East point; the eastern quadrant is *a d*

- e (centre): the common centre of the horizon and of the needle
- f (upper right, on the circumference): the *Nordeste* point; with *g* it gives the *Nordeste–Sudoeste* rhumb, and it is the point of rising of the zenith star
- g (lower left, on the circumference): the *Sudoeste* point
- h (upper left, on the circumference): the *Noroeste* point, the point of setting of the zenith star; with *y* it gives the *Noroeste–Sueste* rhumb
- y (lower right, on the circumference): the *Sueste* point
- K (on the chord *h f*, where it crosses the meridian *a c*): the point from which *K m* is set out equal to the radius
- l (on the circumference between *f* and *d*): where the straight line *e m* cuts the circumference; the arc *d l* is the latitude sought
- m (outside the circle, to the right): the end of the chord *h f* produced

b d, again, is the common section of the horizon and of that vertical circle which stands at right angles to the meridian, and so will be called the straight rhumb of *Leste* and *Oeste*. The straight line *f g* is the common section of the horizon and of that vertical circle which bisects the quadrants *a d* and *b e* [recte: *b c*], and it is called the straight rhumb of *Nordeste* and *Sudoeste*. The remaining line *h y*, drawn in the same manner through the remaining quadrants, is the straight rhumb of *Noroeste* and *Sueste*. The intermediate positions of these rhumbs, which sailors call the half-courses, and their quarters, are to be understood in the same way.

Further: from the equinoctial circle up to 45 degrees of latitude, the parallel that passes through the zenith cuts the horizon; but of the remaining parallels, up to the visible pole, none can cut it. The parallel of 45 degrees itself touches the horizon at a single point. Zenith stars, therefore, from the equinoctial circle up to that same parallel of 45 degrees — the midpoint of the whole range of latitude — have their rising along the whole eastern quadrant *a d*. The parallel cuts the horizon upon the straight line *f h*: that is, the zenith star rises at *f* and sets at *h*, at that place where the square of the right sine of the pole's altitude is half the square of the right sine of the equator's altitude. The latitude of the place will accordingly be found by the help of proportional numbers. Geometrically it is done as follows. Let the straight line *h f* be produced as far as *m*, so that *K m* becomes equal to the radius of the circle *a b c d*; then from the centre *e* let the straight line *e m* be drawn to *m*, cutting the circumference at *l*. The arc *d l* will then be the latitude of the place at which this occurs — that is, a zenith star will rise at *Nordeste* and set at *Noroeste* wherever the distance of the zenith from the equinoctial equals that arc *d l*.

I grant, certainly, that any two places on the globe have a definite bearing one to the other, which must be kept by anyone going from the one to the other; and this holds equally for places lying under one and the same parallel. But a route of that kind is not to be laid down by any of the lesser circles. It should rather be laid down by a great circle, to be drawn through the two places in question either by the method Theodosius used or by some easier one. The arc of that great circle contained between those places is smaller than the arc of the parallel that lies between the same two places, as can be concluded by clear and necessary reasoning from geometrical principles. This advantage follows: for those who travel by it the route is shorter and quicker. But whoever sets out upon it must know that he has to change rhumbs not once only but very often indeed, and that because of the varying and inconstant inequality in the position of the angles, which arises from the new meri—

—dians. The investigation of this matter is a very subtle one, and it comes down to this: that we understand how much angles of this kind increase or decrease along that stretch. Anyone who proceeds in this way will be led straight. It is impossible for a man to travel by a direct route if he keeps for ever to one and the same rhumb — the meridian and the equinoctial excepted. On the contrary, he will have to change it as often as a direct course appears to require.

This being so, why then does the sailors' plane chart display those winding and broken rhumb lines as straight, and all at the same inclination? For these two things, as is plain from what has been said, cannot stand together. Sailors turn and bend the ship by an art that forces her, together with the needle itself, to keep making the same angles with the straight line of North and South. And they do not notice that any straight lines whatever of the chart they use are the common sections of great circles with horizons. But since they steer continually toward the same quarter of the world, keeping the same bearing, it is quite impossible for them to run along direct routes. Nevertheless they apply their reckoning to those same straight lines and look for the positions of places exactly as if they had held a direct course. So it comes about that the places of the world are wrongly set down on the chart itself. Indeed I venture to assert that not one of them is set at its true longitude, and that the error is not small but considerable. I always except, however, those places that have been met with by men sailing from north to south, or the other way about, from south to north.

As for the length of the distance run: because of the slantness and the winding of the routes they go further than they suppose, and especially where the interval between the places is great and that curved rhumb is more sharply angled — as

may be seen in the figure below. Whenever they seek the positions of places from a measurement of the route without knowing the altitude of the pole, they stretch the longitude beyond its true limit, since they project onto a straight line what is by nature bent and oblique. But when they weigh where they are from an observed altitude of the pole — which they seldom have exactly — they sometimes draw the longitude out too far and sometimes contract it. The rhumb of *Nordeste* and *Sudoeste* which they think they have followed is in this figure the line *d c e*; what they actually describe, however, is *a c b*, which is neither straight nor a single circular arc. Anyone, then, who

Diagram. A woodcut circle at the head of the right column. Several straight lines run through the centre, dividing the circle; across it, from the upper left to the lower right, lies the straight line *d c e*, and beside it a second track *a c b*, made up of two segments that meet at the centre at an angle, so that it lies close to the straight line without coinciding with it. The lettering as it stands on the leaf:

- d (upper left, outside the circumference): one end of the straight rhumb *d c e*, the *Nordeste–Sudoeste* course the sailors believe they have run
- a (upper left, just below *d*): the beginning of the track actually described
- c [?] (centre of the circle; the glyph is worn on the scan): the point at which both the straight line and the described track pass, and at which the track changes its direction
- b (lower right, inside the circumference): the far end of the described track *a c b*
- e (lower right, on the circumference): the far end of the straight line *d c e*

has looked at this and weighed it will readily see that the truth can be drawn out of the sailors' errors and false reports, even though we never visit the places ourselves. It would still be better, for learning the positions of places, to navigate by a settled art and method. That art can certainly manage the business in either of two ways. The first allows a man to hold throughout the voyage the same course once set, just as sailors do today; but then the positions of places must be traced out on a curvilinear chart, one whose rhumbs have the shape that the *Nordeste–Sudoeste* rhumb has in this figure, and not on the sailors' rectilinear chart. The second bids us follow a great circle of the sphere, with that variation of courses which the change of meridians demands; and accordingly the positions of places will have to be sought on the great circles themselves, or on some rectilinear chart that represents those same great circles otherwise than the common chart of the sailors does. On such a chart, even if the straight rhumbs are taken to be the common sections of the great vertical circles with the plane of

the horizon, they still cannot serve this purpose, because — the meridians there being equidistant — the rhumbs always make equal angles with them. And although a globe properly drawn is more suited than any chart for either of those two methods, we could nevertheless carry out the first method with the sailors' own rectilinear chart, once it had been somewhat altered.

But let us return to the point from which we digressed. Whenever, then, it is worth knowing what bearing two given places have to one another, a great circle must be drawn through both. For the arc of the horizon of the first place, contained between that great circle and the equinoctial, will show in what direction the second place lies. If, for example, that arc of the horizon comes to 45 degrees in the eastern and northern quadrant, the second place will lie from the first toward *Nor*—

—*deste*; and the case will be similar in other instances. This may be seen in the figure below, in which *b* and *d* are two northern places, the bearing of the one to the other being what we wish to know. Let the eastern horizon of the place having its zenith at *b* be *g a h*. Let the parallel of the place whose zenith is at *d* be *y d K*; and let *b d c* be the quadrant of a great circle drawn through *b* and *d*. Let the quadrant *b a* cut the meridian of the place *b* at right angles. To the angle *a b c*, then, there answers on the horizon the arc *a c*; and if that arc is found to be 45 degrees, we shall pronounce that the great circle drawn through *b* and *d* runs from *Sudoeste* to *Nordeste*. From this it is clear

Diagram. A woodcut sphere set in the left column. A large circle carries several arcs drawn across its face: one sweeping from the upper right down across the left side to the bottom (the eastern horizon of the first place), another running from the left edge across the middle toward the right (the parallel of the second place), and a third cutting across them from the upper left down to the lower right (the quadrant of the great circle through the two places); a point is marked at the centre. The lettering as it stands on the leaf:

- b (upper left, on the circumference): the zenith of the first place, from which the quadrant *b d c* and the quadrant *b a* are drawn
- g (upper right): one end of the eastern horizon *g a h*
- K (immediately below and right of *g*): the far end of the parallel *y d K*
- d (centre of the figure, on the parallel): the zenith of the second place
- c (just below *d*, at the centre of the figure): the far end of the quadrant *b d c*; the arc *a c* of the horizon answers to the angle *a b c*
- y (left edge, on the circumference): the near end of the parallel *y d K*

- [?] (left side, below *y*, indistinct on the scan): a point of the eastern horizon
- h (lower left, at the foot of the figure): the far end of the eastern horizon *g a h*
- an unlettered point marked at the centre of the circle

that of three places set under one and the same parallel, the first bears differently on the middle one than on the last, so that each of them has a different bearing to any other. As for our passing through all of them when we sail from *Leste* to *Oeste*, that is a separate matter, and I have already explained it. All places, therefore, that lie on *b c* bear toward *Nordeste*, and all that are set in the other quadrant, the one before *b*, bear toward *Sudoeste* — for all of them are being compared with *b*. But if, going back the other way, you wish to refer the position of the place *b* to *d*, understand that *b* does not bear toward *Sudoeste*, but has quite another inclination, between *Nordeste* and North, since we supposed *b* to be further north than *d*. If, on the other hand, you suppose the two to have equal altitudes of the pole, then, since *d* compared with *b* bears toward *Nordeste*, *b* referred to *d* will bear toward *Noroeste*. But suppose we make *d* the more northerly, and yet distant from the place *b* toward *Nordeste*: this can certainly happen with two places that have equal altitudes of the pole and yet are unequally distant from *b*. If, then, the same place *b* is referred to the nearer of them, it will be found to incline toward some point of the horizon between *Oeste* and *Sudoeste*; but if you make the comparison with the more distant one, you will affirm that it bears toward a corresponding point in the northern and western quadrant of the horizon, between *Oeste* and *Noroeste* — the two points, that is, being equally distant from *Oeste*. All this the science of spherical triangles teaches, and it may be tested either on a globe or on the plates of an astrolabe.

From these things you will understand that the bearing of the rising Sun is different in different places. When the Sun stands at the beginning of Cancer, for those who live at Syene, and for all who lie under the tropic of Cancer itself, it rises at *Lesnordeste* with three and a half degrees added toward *Nordeste*, since the amplitude of rising is 26 degrees. At the same season — that is, on the twelfth day of the month of June — for those who live under the equinoctial it rises at *Lesnordeste* with only one degree added, for the amplitude of rising is 23½ degrees. For those who inhabit our northern region, at a pole altitude of 35 degrees, it rises at *Lesnordeste* with nearly half of one quarter-point toward *Nordeste*, because the amplitude of rising is 29 degrees. On the horizon of Lisbon, however, where the north pole is raised almost 39 degrees, it rises at *Nordeste* with one quarter-point and two and a half degrees added toward *Leste*, for there the amplitude of rising is 31 degrees. By the amplitude of the Sun's rising [*latitudo*

ortus] astronomers mean the arc of the horizon between the equinoctial and the rising Sun itself.

From these things you will also understand at what points of the horizon the Sun sets on that same day of Cancer, and likewise where it rises and sets when it is at the winter tropic. This is clear from the fact that the right sine of the complement of the pole's altitude is to the whole sine as the sine of the Sun's declination is to the sine of the amplitude of rising. Therefore, if you have a mariner's needle that shows the position of the meridian exactly, or have determined that position in any other way, you will find the altitude of the pole above the horizon by a wholly reliable calculation from an observation of the Sun's rising. We ourselves are accustomed to find it at any hour of the day, without knowing the time and without knowing the position of the meridian.

Sailors and shipmasters, though, are so slack that although they could find the elevation of the pole in many ways, they look for it only at midday. And since it often happens that the Sun's rays are cut off at that hour, they then weigh where they are by mere estimate, which not seldom deceives them. We have seen a man who had sailed to India more than ten times and yet afterwards, when [tested] with the help of knowl—

he was left without that support, for a good many days he subtracted the Sun's declination when it should have been added, and added it when it should have been subtracted.

But to bring this discussion to a close: whether from Ptolemy's own demonstration or from the principles most proper to the science of triangles, we take it to be established that when the Sun recedes equally from the equinoctial circle, whether toward the north or toward the south, the arcs of the amplitude of rising come out equal. And in every horizon the same rhumbs belong to the same quarters. Moreover, at two places of equal altitude of the pole, one of them northern and the other southern, the Sun produces equal amplitudes of rising, and toward the same part of the horizon. Therefore, when the Sun stands at the beginning of Cancer, it will rise at those two places with an equal inclination.

Now when the Sun is in the tropic of Capricorn it rises at *Sueste*, with one quarter and nearly half a quarter added toward *Leste*, for those who have a northern altitude of 35 degrees. For that reason it will also rise, at the same time and in the same way, at *Sueste* with one quarter and nearly half a quarter added toward *Leste*, for those who have an equal altitude of the southern pole; for in either horizon the remaining arcs of the eastern and southern quadrant are equal.

Indeed, anyone who has noticed that the *Leste* of the mariner's needle points to the equinoctial rising everywhere on earth will understand at once — since the Sun declines from the equator toward the south from the autumn equinox to the spring one — that throughout the whole world, during that same period, the Sun rises on those rhumbs that belong to the eastern and southern quadrant. This is what the figure below shows.

Diagram. A single circle representing the meridian, crossed by several straight chords and by arcs. Its circumference carries the four meridian points; three chords through the centre *b* represent the equinoctial and the two horizons, and two further arcs cut across the figure for the tropics. The rising points of the Sun are marked where the tropic arcs meet the horizon lines.

- a (left edge of the circle): west end of the equinoctial
- b (centre): common intersection — the east point shared by both horizons
- c (right edge of the circle): east end of the equinoctial
- o (top of the circle): upper end of the horizon of place K
- P (bottom of the circle): lower end of the horizon of place K
- d (upper left): west end of the circle of Cancer
- e (right, above c): east end of the circle of Cancer
- f (lower left): west end of the circle of Capricorn
- g (right, below c): east end of the circle of Capricorn
- l (upper left, on the circumference): the northern place
- K (lower left, on the circumference): the southern place
- m (upper right): upper end of the horizon of place l
- n (lower left, below K): lower end of the horizon of place l
- t (centre, above b): the Sun's rising at the entry into Cancer in the horizon of the southern place
- r (right of t): the Sun's rising at the entry into Cancer in the horizon of the northern place
- s (below b, toward the bottom): the Sun's rising at the entry into Capricorn for the northern place
- z [?] (right of s): the Sun's rising at the entry into Capricorn for the southern place — the letter is faint on the leaf

In the figure, the circle *a p c o* represents the meridian of two places situated at *l* and *k*, places that have equal latitudes but in opposite parts of the world — *l* toward the north, *K* toward the south. Let *a b c* be the equinoctial, let the circle of

Cancer be *d e*, and that of Capricorn *f g*. Let the horizon of place *l* be *m b n*, and the horizon of place *K* be *o b p*.

Whenever, then, the Sun has entered Cancer, it will rise at *r* in the horizon of the northern place *l*, and at *t* in the horizon of the southern place. And since the two arcs *b r* and *b t* of the eastern and northern quadrants *b m* and *b o* are equal, the Sun will produce like risings both for those at *l* and for those at *K*. Now *b l* and *b K* are quadrants of those vertical circles that mark *Leste*, while *l r* and *K t* are quadrants of the verticals that mark the Sun's risings on the very day of Cancer. To the circumferences *b r* and *b t*, therefore, correspond equal angles *b l r* and *b K t* at the vertices *l* and *K*.

Whenever, on the other hand, the Sun has entered Capricorn, it will rise at *s* for those at *l*, and at *z* for those at *K*. And since the circumferences *b s* and *b z* are equal, in both places alike the Sun will produce like risings within those eastern and southern quadrants.

Since all these circumferences of the rhumbs are equal to one another, it is clear that the Sun rises as far above *Leste* when it is in Cancer as it rises below *Leste* when it is in Capricorn. So if the quadrant *l r* runs to *Nordeste* for those at *l*, then the quadrant *l s* will run to *Sueste*.

We have thus resolved both difficulties. One point still remains to be explained: that Martim Afonso de Sousa, as we said above, observed the Sun's rising at a certain southern place 35 degrees distant from the equinoctial, when the Sun held the beginning of Capricorn, and saw it rising at *Sueste* with one quarter added toward *Leste*, whereas our own calculation adds nearly half a quarter beyond that one quarter. There is nothing surprising in this. On the day the Sun rose there, the observation could not be made with complete exactness and without any error by the mariner's needle alone; it would have been worth adding something else to that instrument, as we have pointed out elsewhere, and for that reason nearly half a quarter was left out. From a given elevation of the pole, however, and from a known degree of the Sun, with the help of no instrument at all — indeed with the Sun itself not even in sight — we conclude by an evident and necessary argu-

—ment that 29 degrees of the horizon's circumference lie, on that very day, between the point of rising and the point of *Leste*. But *Sueste* with one quarter toward *Leste* comprehends 33 degrees 45 minutes; the difference, therefore, which contains 4 degrees 45 minutes, is nearly half a quarter — a little less, in fact. And so the Sun, when it stands at the beginning of Capricorn, rises for those

who have an altitude of the pole of 35 degrees at *Sueste* with one quarter and nearly half a quarter toward *Leste*. But since nothing smaller than a quarter of half a quadrant is marked on the usual nautical instruments, this could not be determined exactly by the mariner's needle alone.

The geometrical demonstration shows plainly that, at the winter tropic, the Sun rises exactly at *Sueste* with one quarter of *Leste* only for a place whose altitude of the pole is 44 degrees; whereas here where we live it rises at *Sueste* with one quarter of *Leste*, with 2 degrees and 45 minutes added toward *Leste*, since the amplitude of rising is 31 degrees. Everything we have written on these matters, therefore, must be accepted as free of all ambiguity, since nothing is more certain and nothing more evident than a mathematical demonstration, which no one will ever be able to contradict.

¶ PEDRO NUNES of Alcácer do Sal, on the rules and instruments for detecting the various appearances of things both maritime and celestial, drawn from the mathematical disciplines. Book II.

¶ On the sea chart, or the mariners' planisphere. Cap. 1.

That the voyages made by the Portuguese in this age are astonishing is unknown to no one. The Portuguese dared to swim across the Ocean: they found new islands wholly unknown to antiquity, new coasts, new seas, new peoples never seen before. They were not frightened off by the enormous heat of the burnt zone or by the excessive cold of the frozen one, but sailed on continuous voyages until they had passed beyond the equinoctial and rounded that great promontory of Africa which they call the Cape of Good Hope; then, turning back into the northern region and crossing the Ethiopic sea that lies off Troglodytica, and the Arabian and the Persian, they at last put in at India. From there they pushed on beyond the Ganges, beyond Taprobane, into the region of the Sinae and to the islands that face the rising Sun most directly.

They did not seek out these things recklessly, nor did they find them by chance. They carried astronomical instruments for observing the stars, and tables of the motion of the Sun and Moon, laid out by mathematical numbers and on sure

principles; and besides these that living and divine device, unknown to the men of old, which they call the mariner's needle. Its circumference, which represents the horizon, is divided into 32 equal parts and shows the cardinal points of the world. It was by the help of this instrument that they dared to leave the land and stand out far from the shore into the deep — so far that it has sometimes happened that Portuguese ships have put in at India after six months without a single island or any mainland having been sighted in the meantime. The sailors of old had no such device, and it is no wonder that they sailed only close to the coasts.

As for the rectilinear planisphere of the world that they use today, although it cannot present a true image of the globe, because of the equality it gives to the parallels, it is nevertheless very well suited to the art of navigation as they practise it. That a single island, or any stretch of land, should appear longer on it than it really is seems to matter little for the navigator's purpose, provided that the distances of places are known in parts of a great circle, or in stadia, or in miles, or in any other measures whatever. For Claudius Ptolemy, that most excellent mathematician, when in the first book of the *Geography* he wanted to investigate the distance between the promontory of Cory and the Sinae, and between certain other places on the Gangetic gulf, took equidistant straight lines for meridians and equidistant straight lines for the parallel circles as well (*Geography* I). He therefore used rectilinear triangles in place of spherical ones — as he does again in the fifth book of the *Great Composition*, when he investigates the angles formed by the meeting of the zodiac and the meridian, and the parallax of the Sun and Moon (*Almagest* V). And we do not doubt that it was one and the same Ptolemy who com-

—posed both works, the astronomical and the geographical, since in the second book of the *Great Composition* he mentions the *Geography* published by himself, and again in the eighth book of the *Geography* the astronomical work itself; and in both works the maximum declination of the Sun from the equinoctial circle is set down at almost the same quantity.

But so that it may be clear in what way and in what places straight lines are to be used in place of circular ones, we shall follow one example from the first book. The voyage from Corura [?] to Palura, he says on Marinus's [Marinus of Tyre] authority, runs toward the winter rising and is 9,450 stadia. From this he takes away a third part on account of the irregularity of the course — 3,150 stadia, that is — and 6,300 are left for the direct distance. Of these he removes a sixth, and so

5,250 stadia are left, that is, 10 degrees 30 minutes, for the distance between the meridians of those two places.

Diagram. A tall narrow right triangle set into the left column. Its base runs horizontally from the lower-left vertex to the lower-right vertex, and the third vertex stands above the base. The lettering is partly obscured by ink strike-through from the other side of the leaf.

- [?] (apex, above the base): the letter does not resolve on the scan
- c (lower left): the right angle
- b (lower right)

Let Corura be *a* and Palura *b*; let the meridian through *a* be *a c*, and the parallel through *b* be *b c*. Let the distance between *a* and *b*, with the irregularity of the voyage included, be 9,450 stadia; and when a third has been taken off, the arc *a b* will be 6,300 stadia — that is, the direct interval between *a* and *b*. The arc *a c* will be the difference of latitude of those places, and *b c* the difference of longitude on the circle parallel to the equinoctial. The angle at *c*, therefore, will be a right angle, while the acute angle *b a c* shows the bearing of place *b* with respect to *a*.

And since he supposes Palura to lie from Corura toward the winter rising, from where the wind blows, the southern and eastern quadrant [?] is divided into three equal parts, according to the ancient division of the winds; and the angle of position *b a c* will comprehend two of them.

Diagram. An equilateral triangle standing on the left side of the column, with a line drawn from the right-hand vertex to the midpoint of the left side. The left side runs vertically; the drawn line is horizontal.

- a (top of the left side)
- c (midpoint of the left side): foot of the line drawn from *b*
- b (right vertex)
- d (bottom of the left side)

For that reason, if in place of the spherical triangle we take the rectilinear triangle *a b c*, the rectilinear acute angle *c b a* will be a third part of one right angle; the straight line *a b* will be the side of the equilateral triangle *a b d*, the straight line *a c* half of it, and *b c* the perpendicular. And so the square on *a b* will have to the square on *b c* the ratio of one and a third. And since the ratio of the squares is the duplicate of the ratio of the sides, the ratio of *a b* to *b c* will therefore be nearly [?] one and a fifth. For instance, if *a b* is set at six equal parts, its square will be 36; the square on *b c* will therefore be 27, whose side will be

somewhat greater than five. By a coarser reckoning, then, Ptolemy supposes it to be five, so that the ratio of *a b* to *b c* is one and a fifth.

So, once *a b* is known, on taking away a sixth, *b c* is left known — 5,250 stadia, that is. And since the parallel of place *b* differs little, or imperceptibly, from a great circle, being very close to the equinoctial, if we reckon [?] 500 stadia for each degree of that parallel, the difference of longitude between *b* and *c* will comprehend ten degrees and half a degree.

You see, then, that this procedure differs in nothing from the one the sailors of our own day use. They, however, work far more exactly: having assumed the size of the angle of position, they extract the distance of the meridians from a certain table of numbers which they keep worked out for each several bearing. For since Ptolemy set the ratio of *a b* to *b c* as six to five, two hundred stadia and more were dropped from that computation, and to these there correspond more than two fifths of a degree. You may easily test this in the following way. Since *a c* is half of *a b*, it contains 3,150 stadia; and if you subtract its square from the square of the side *a b*, there will be left 29,767,500, the square, that is, of the side *b c*. The side *b c* itself will therefore comprehend nearly 5,456 stadia, to which nearly eleven degrees correspond.

One further point deserves notice: the distance of the journey between Corura and Palura could be known by estimation, but with the latitudes of those places unknown the angle of position of the one relative to the other could not be known — unless perhaps the situation and distances relative to some third place had been noted. For it is beyond belief that Palura can be seen from Corura.

Diagram. A triangle set into the right column, drawn with one vertex to the left, one to the right and one below. Much of the lettering is lost under ink strike-through from the other side of the leaf.

- [?] (left vertex): the letter does not resolve on the scan
- b (right vertex)
- d (lower vertex)

But if *a d* is a known arc of the meridian between *a*, Corura, and some other place *d*, and if the distance of that *d* from Palura *b*, and the distance between *a* and *b* as well, have been ascertained, [?] then the angle of position *d a b* [?] from Corura to Palura will be known. The method, however, is not very exact, especially over so great an interval and on a sea voyage.

Now if you assume that they sailed for so long toward the winter rising, keeping that same inclination throughout until they reached Palura — which is just the way the more recent sailors are accustomed to observe with the help of the mariner's needle — then it plainly appears from what we said in the earlier book that [?] and the direct distan—

—tance between those same places has a different bearing with respect to the meridian of Corura.

But if Ptolemy assumes the latitudes from the equinoctial circle to be known, he could certainly have found the difference of the meridians with the least possible trouble — and that with no regard at all to the angle of position, simply by taking the square of the difference of latitude away from the square of the direct distance between Corura and Palura. The square root of the remainder would then have to be taken as the difference of their meridians, since he chose to use straight lines in place of circular ones. If, however, you prefer to find the same thing more exactly, then in a spherical triangle — the distance of the places being known, and the complements of the latitudes known as well — you can at once determine the angle at the pole of the world that subtends the difference of the meridians. However the angle of position comes to be known, it is clear from what has been said that Ptolemy long ago used the same art for finding the longitudes of places that sailors use today.

As for his reducing the distances of places, on whatever bearing, to a straight line, he does this more judiciously and more cautiously than our own sailors do. For they take the space they cover in sailing by many windings and stretch it out straight. As a result, once the line subtending the right angle has been enlarged, the latitudes and longitudes of the places must also be stretched beyond their limit in the same proportion — as the figure below makes clear. In it, as the distance *a e* is to the distance *a b*, so is the difference of longitude *a d* to the difference of longitude *a c*; and the differences of latitude *d e* and *b c* stand in that same ratio.

Diagram. A right triangle set on a horizontal base. The vertex *a* is at the lower left; the base runs from *a* through *c* to *d* at the lower right; a vertical rises at *d* to *e* at the upper right, so that *a e* is the hypotenuse. The point *b* lies on that hypotenuse between *a* and *e*, and a shorter vertical drops from *b* to *c* on the base. The two right triangles *a c b* and *a d e* are thus similar, the smaller nested within the larger.

- a (lower left): the common vertex — the place of departure

- b (on the hypotenuse, between *a* and *e*): the far end of the true distance *a b*
- c (on the base, directly below *b*): foot of the perpendicular *b c*; *a c* is the true difference of longitude
- d (right end of the base): foot of the perpendicular *d e*; *a d* is the enlarged difference of longitude
- e (upper right): the far end of the enlarged distance *a e*

Since sailors not infrequently do this on long and protracted voyages, it is no wonder that they stretch the interval from Spain to India beyond all measure. They do the same thing indiscriminately on any bearing between places that they do when they sail under a single meridian or along a single parallel. Moreover, what Ptolemy does only for places near the equinoctial and over a moderate distance, they do everywhere across the whole globe: they boldly use rectilinear triangles in place of spherical ones even over the very greatest distances. Even so, the coastal charts of the world drawn up from their own voyages are far more reliable than those handed down by Ptolemy, who measured the length and breadth of the inhabited world partly by conjecture and partly on the false reports of certain men.

Lunar eclipses, after all, do not occur often, and when they did occur there were not mathematicians everywhere to observe them, least of all among barbarous nations. Finding the longitudes of places from eclipses is the surest method of all, yet sailors neglect it, even though they could have tables of them drawn up for many years ahead. And if one does happen to be observed by them, they lay down the place where the observation was made on the sea chart by exactly the same method they would use on the globe — that is, by degrees of longitude and latitude. In this they are certainly mistaken. For the difference of longitude taken on the parallel of the given place must first be converted into parts of a great circle, or into our customary measures, and only then is the place itself to be laid down on the sea chart by means of those.

As for the places outside the equinoctial circle that lie before sailors travelling along a single parallel, it is not easy to determine how they ought to be laid down on the sea chart. To make this plainer, let us take two places of nearly equal northern latitude between which the Portuguese sail every day: Lisbon, and that island among the western islands of Portugal which they call the Third [Terceira]. Lisbon has almost 39 degrees of latitude, the Third Island almost 40. The sea chart gives the distance between them as about 262 of our leagues — equal, that is, to fifteen degrees of a meridian; and our sailors say they have very often found it so, not only by estimating the run completed when they sail from *Leste* to *Oeste* to

that island, but by another and far more reliable reckoning. For the voyage from Lisbon to the island they call Madeira [insula Materiae] runs to *Sudoeste*, and from Madeira to the Third Island to *Noroeste*. And since from *Noroeste* to *Sudoeste*, and likewise from *Sueste* to *Nordeste*, as much space is enclosed between the meridians as between the parallels — that is, the difference of longitude is as great as the difference of latitude, because on either voyage the angle of position equals half a right angle — and since Madeira itself has a northern latitude of 32 degrees, it follows that, on the construction of the rectilinear planisphere which the sailors of our time use, the space between Lisbon and the Third Island must comprehend fif—

—teen degrees of a great circle; but of the parallel of 39 or 40 degrees of latitude there will be more degrees in that same space. This is precisely the art Ptolemy used in the first book of the *Geography* for finding the distances of places (*Geography* I).

One difficulty, however, seems to be left standing. For if the arc of the parallel of forty degrees of latitude between Lisbon and the Third Island is equal to fifteen degrees of a great circle, then — since in every parallelogram the opposite sides are equal — there will on the sea chart be fifteen degrees of the equinoctial enclosed on the equinoctial itself between the meridians of those same places; and from Theodosius, book 2, it is plain that this is impossible (Theodosius, *Sphaerica* II). You will resolve the difficulty, though, once you understand this: straight lines cutting the parallels of the equinoctial at right angles cannot be set down as meridians both on the equinoctial itself and on those parallels that are very far from it, unless the ratio of the meridian to the middle parallel is preserved — just as Ptolemy warns must be done in the tables of the provinces, so that no perceptible error is committed.

Besides, I would not have anyone troubled by my saying that the voyage from Lisbon to Madeira is made toward *Sudoeste*, and that the island lies from Lisbon in the middle of the southern and western quadrant, which plainly cannot be so. For if, setting out from Lisbon, we point the ship's prow to *Sudoeste* and sail on at that same inclination until we reach Madeira, the bearing we find will be something other than half a quadrant. You will be freed from this difficulty as well once you notice that, over distances that are not very great, it makes little or no difference whether you say that one place lies from another toward *Sudoeste*, or that for as long as we sail from the one to the other the prow is always pointed to *Sudoeste*. From all this, then, you will gather by what art those places are to be

laid down on the mariners' planisphere which lie before men sailing along a single parallel.

I think it is also established, from what we have said here and in the earlier book, that error arises not only over the position of the many places which the sea chart shows under a single meridian, but over the other bearings and inclinations of distances as well. For the meridian is a kind of standard for all other bearings; so wherever the position of the meridian has been got wrong, a mistake must follow in the inclinations of the remaining rhumbs too, and accordingly not every bearing or inclination from place to place that is set out on the sea chart is to be taken as true, but only the one along which men have actually sailed from the one to the other at some time.

For example: let those sailing from Lisbon *a* on a direct course toward the southern pole come upon the place *d*, situated under the equinoctial circle; and let those sailing toward *Sudoeste* come upon Madeira *b*, at a latitude of 32 degrees. Then on the sea chart the straight line *a d* is the latitude of place *a*; the perpendicular *b e* is the latitude of place *b*; and the perpendicular *b f* is the distance between the meridians of the two places *a* and *b*, taken on the parallel of place *b*. Now let a place *c* be marked beyond *e* on the straight line *c d*, which represents the equinoctial, standing at one and the same number of degrees from place *d* both on the globe and on the sea chart.

Diagram. A right-angled figure. The point *a* stands at the upper right, and a straight line descends from it to *d* at the lower right. The base line runs left from *d* to *c* at the lower left and represents the equinoctial. From *a* a line slopes down and to the left toward the lower-left corner, and *b* lies on it. From *b* a vertical drops to the base at *e*, which falls between *c* and *d*, and from *b* a horizontal runs right to *f* on the line *a d*.

- a (upper right): Lisbon
- b (left of centre, on the sloping line from *a*): Madeira, at 32 degrees of northern latitude
- c (left end of the base line): the place marked on the equinoctial beyond *e*
- d (lower right, where the line from *a* meets the base): the place under the equinoctial circle reached by sailing due south from *a*
- e (on the base line, below *b*): foot of the perpendicular *b e*, the latitude of *b*
- f (on the line *a d*, level with *b*): the far end of the perpendicular *b f*, the distance between the meridians of *a* and *b*

The four places *a*, *b*, *c*, *d* are therefore correctly placed on the chart. But *b* lies further west than *e* itself, as is clear from what has been said. Consequently the perpendicular *b e* does not hold the true position of the meridian, nor will the angle *c b e* be able to show, on the sea chart itself, the bearing of place *c* [?] with respect to *b*. If, on the other hand, those same places *a*, *b*, *c* and *d* were laid down on a globe by the same art — meridians being drawn through *a* and through *b*, and great circles also drawn through *a b* and through *b c* — they would beyond doubt keep their true bearings relative to one another. For on the globe, if you lay down some places by latitudes and differences of longitude and others by latitudes and angles of position, in the end all of them will agree with one another in position as they should; on the sea chart it usually turns out quite otherwise.

The same thing can be seen in the voyage our countrymen make to India. The sea chart shows that promontory of Africa with three points, at four and a half degrees of northern latitude, and the islands of Tristan da Cunha [Tristani à Cugna], which lie at 36 degrees of southern latitude, under one and the same meridian; and it makes the interval between those islands and the Cape of Good Hope come to nearly four hundred leagues. These two things cannot both stand. For if all the coasts from the Cape of Three Points as far as the Cape of Good Hope are correctly charted, and the Cape of Three Points itself lies with those same islands under the same me—

—ridian, then the distance just mentioned must be far smaller, the proportion of the degrees and the parallels being preserved. But if it is not smaller, then the islands cannot have the same meridian as the Cape of Three Points; on the contrary, they will lie further west.

The result is that sailors are very often deceived when they make from one place for another, following the bearing the sea chart shows. When that voyage fails to bring them to it, they suppose the cause of the error to be either a swift set of the water in another direction, or the deviation of the magnet's poles from the true poles of the world — whereas perhaps they went astray for this reason alone, that they did not yet know what bearings those places have with respect to one another. And they are deceived not only in supposing that the sea chart can represent the positions of all places, but in this too: whenever they wish to transfer coastlines to a globe, they do it with regard only to the numbers of degrees of longitude and latitude found on the chart, in no other way than they do when they lay down the fixed stars. So not only are those errors committed that necessarily follow from the chart because it is plane, but others as well which they could have avoided, had they first converted the distances they truly

know into degrees and then followed the longitudes and latitudes of the places themselves.

Further, in the charting of the coasts of the Mediterranean sea, since we have noticed that the latitudes of places are set down much greater than they truly are, correction is needed. Alexandria, where Ptolemy made so many observations of the stars, has a northern latitude of 30 degrees 58 minutes, yet on the sea chart it is placed at a latitude of 36 degrees. The latitude of Rhodes is only 36 degrees, but on the same chart it is placed at 42. The latitude of Rome comes to nearly 42 degrees, yet on the chart it is found at 46. Venice, set at the middle of the quadrant — where at the equinox the shadow equals the gnomon, that is, at 45 degrees of latitude — is made to have fifty; and in almost all the remaining places the latitudes have been increased in the same way.

When I once looked into the cause of this error, it occurred to me that, because the Mediterranean is narrow and voyages on it are frequent, the mutual bearings and intervals of places have been exactly explored and ascertained — so much so that those who sail there have no need of astrolabes or of any knowledge of latitude. For since sailors there have either some island or the mainland in sight every day, they can easily tell what place they are in. In earlier centuries, too, the Spanish, the French and the German seas were sailed without astronomical instruments, because sailors only skirted the shores. Afterwards, since in the more recent Portuguese voyages the greatest parts of the world have been traversed — which certainly could not have been done without the help of the mathematical arts — sailors began to observe the latitudes of places and to record them on charts. So when they wanted to fit the Mediterranean to the Ocean, to make the two hang together as one, the Mediterranean was perhaps assigned a higher position than it should have had. Or, if the two are in fact rightly joined and continued, the cause of the error may have been that the distances between the coastal places of the Mediterranean were recorded in Italian miles, while those of the Ocean coasts were recorded either in degrees or in Spanish leagues — and the makers of sea charts then converted the miles into degrees or into leagues wrongly. Or else — and this last is what I favour more — sailors did not record the bearings or the distances of the Mediterranean coasts accurately enough, and so not only the latitudes but the longitudes as well must have departed from the truth.

Suppose, then, that on the sea chart the straight line *a b* is the *Leste–Oeste* rhumb; let *a c* be any other rhumb showing a different bearing, namely the one by which one goes from place *a* to *c*; and let the straight line *b c* make right angles with *a b*

at the point *b*. The straight line *a c* will then be the interval between the two places *a* and *c*, *b c* the difference of latitude, and *a b* the difference of longitude. Next, let us conceive another bearing marked out by the angle *b a e*, under an equal interval, which is *a e*. The difference of latitude between the places *a* and *e* will be *d e*, greater than the former; but the difference of longitude will be *a d*, smaller than the former.

Diagram. Two right triangles sharing the vertex *a* and the base line. The vertex *a* is at the lower left; the base runs right from *a* through *d* to *b* at the lower right. A vertical rises at *b* to *c*, and a taller vertical rises at *d* to *e*. Two straight lines slope up from *a*, the lower to *c* and the upper, steeper one to *e*, so that the triangles *a b c* and *a d e* are right-angled at *b* and at *d* respectively.

- a (lower left): the common vertex — the place of departure
- b (right end of the base): the right angle of triangle *a b c*; *a b* is the difference of longitude on the first bearing
- c (directly above *b*): the far place on the first rhumb *a c*; *b c* is the difference of latitude
- d (on the base, between *a* and *b*): the right angle of triangle *a d e*; *a d* is the difference of longitude on the second bearing
- e (above *d*, higher than *c*): the far place on the second rhumb *a e*; *d e* is the difference of latitude

For if circles are described about the right-angled triangles *a d e* and *a b c*, the straight lines *a c* and *a e*, being equal to one another, become the diameters of the circles described; and the circles must therefore be equal. But the angle *d a e* is supposed greater than *b a c*. The arc of the angle *d a e* will therefore be greater than the arc of the angle *b a c*, and consequently the subtending straight line *d e* greater than *b c*. By the same argument, since the angle *a e d*, which is what remains from two right angles, is smaller than *a c b*, *a d* will be smaller than *a b*. These results, however, tend toward an impossi—

—bility, which you can easily demonstrate from the first book of Euclid: if the relative positions of places are kept but the distances are stretched beyond their proper limits, then both the difference of longitude and the difference of latitude will be enlarged.

By what means so great a lapse came about is doubtful; but we know for certain that the latitudes are not the true ones, and it follows that the longitudes too are for the most part false. Perhaps, even so, the whole length of the Mediterranean from the Strait of Hercules to the Gulf of Issus is truly given by the sea chart,

however much error there may be in its parts — for that could happen, provided that wherever the longitude between certain places runs over, it falls short by as much in the rest. But no one will deny that the latitudes are false, if, over and above what we have said, he looks at the isthmus that lies between the Mediterranean and the Arabian Gulf. The difference of latitude between Pelusium and the innermost part of the Arabian Gulf, where the city of the Heroes [Heroopolis] once stood, is a little more than one degree; on the sea chart, however, it is not less than five degrees. And the difference of longitude, which is practically nil, appears there far greater, because the coastal delineation of the Mediterranean has been laid down on that same chart according to the parts of a great circle. If it were instead carried over to the parts, or degrees, of its own parallel in either of Ptolemy's planispheres, Pelusium and the innermost recess of the Arabian Gulf would be seen to fall under very nearly a single meridian. On a globe this could be done most aptly of all — but not in the way our craftsmen have been accustomed to work, who transfer to the globe's parallels the same number of degrees they find in the plane delineation of the sea chart, with no regard at all to the ratio between unequal circles. Pelusium is therefore left far short of its own bounds, and the gap between the Mediterranean and the Arabian Gulf at the isthmus itself becomes enormous — unless meanwhile they choose to push the Red Sea beyond its proper limits in order to hide the defect.

We note besides (as we warned above) that there are many places which differ in longitude and yet on the sea chart appear to have one and the same meridian. On the sea chart, then, let the straight lines *a b* and *c d* be equidistant from each other and set down as meridians; let the straight lines *e f* and *g h*, perpendicular to them, represent parallels — *e f* the equinoctial, *g h* one other of the parallels — and let the straight line *a K* be a quadrant of a meridian. Suppose it has been established that two places, *y* and *K*, lie under one and the same meridian, and that two other places, *r* and *z*, stand off from them by equal intervals *y r* and *K z*.

Diagram. A rectangular grid of four straight lines. Two verticals stand for the meridians: *c d* on the left (*c* at the top, *d* at the foot) and *a b* on the right (*a* at the top, *b* at the foot). Two horizontals cross them and stand for the parallels: the upper one runs from *h* on the left to *g* on the right, the lower one from *f* on the left to *e* on the right. The upper horizontal meets the left vertical at *r* and the right vertical at *y*; the lower horizontal meets the left vertical at *z* and the right vertical at *K*.

- a (top of the right-hand vertical): the pole; *a K* is the quadrant of the meridian

- b (foot of the right-hand vertical): the far end of the meridian *a b*
- c (top of the left-hand vertical): the upper end of the meridian *c d*
- d (foot of the left-hand vertical): the lower end of the meridian *c d*
- e (right end of the lower horizontal): the eastern end of the equinoctial *e f*
- f (left end of the lower horizontal): the western end of the equinoctial *e f*
- g (right end of the upper horizontal): the eastern end of the parallel *g h*
- h (left end of the upper horizontal): the western end of the parallel *g h*
- K (lower horizontal, on the right-hand vertical): the place on the equinoctial under the meridian *a b*
- r (upper horizontal, on the left-hand vertical): the place distant from *y* by the interval *y r*
- y (upper horizontal, on the right-hand vertical): the place sharing a meridian with *K*
- z (lower horizontal, on the left-hand vertical): the place distant from *K* by the interval *K z*

r and *z* will therefore seem to be taken in by the same meridian, since they lie on the straight line *c d*. But it is not so. On the contrary: if *y* is further east than *r*, the place *z* will also be further east than that same *r*. For equal spaces are assigned to *K z* and to *y r*; but *e f* represents a larger parallel than *g h* does, and so *K z* will contain fewer degrees of its own circle than *y r* does. Yet meridian circles cut off an equal number of degrees from every parallel. Therefore *z* will stand off eastward from the meridian of the place *r* — unless the difference between the parallels is so slight that the one may be reckoned equal to the other.

But if you wish to find the place on the parallel *e f* that has a meridian in common with *r*, you must first draw out the ratio of the two parallels, either from a table of numbers drawn up for the purpose or from the instrument set out below. Then multiply the space *y r* by the number owed to the parallel *e f*, and divide the product by the number of the parallel *g h*; the division yields the distance of the place *K* from the place that has the same meridian as *r*. Reckon that distance out, then, or mark it off on the parallel *e f* with the compasses, and let it be, say, *K t*: the places *r* and *t* will then lie under the same meridian. For instance, if *g h* represents the parallel through Rhodes — of 36 degrees of latitude, that is — and *e f* the equinoctial circle, their ratio will be drawn from the table or from the instrument as roughly 5 to 4. Let the space *y r* contain 80 stadia; we shall multiply these by 5, divide the product by 4, and the division will yield

Diagram. The same grid, carried further. The two verticals are again *c d* on the left and *a b* on the right, and the two horizontals *h*–*g* (upper) and *f*–*e* (lower). Above the upper horizontal a third horizontal is drawn, running from *o* [?] on the left to *l* on the right-hand vertical; the point *m* lies on it. An oblique straight line runs from *a* at the top right down to the left, crossing the third horizontal at *m*, the upper horizontal near *r*, and reaching *t* on the lower horizontal, which lies to the left of *z*.

- a (top of the right-hand vertical): the pole
- b (foot of the right-hand vertical): the far end of the meridian *a b*
- c (top of the left-hand vertical): the upper end of the meridian *c d*
- d (foot of the left-hand vertical): the lower end of the meridian *c d*
- e (right end of the lower horizontal): the eastern end of the equinoctial *e f*
- f (left end of the lower horizontal): the western end of the equinoctial *e f*
- g (right end of the upper horizontal): the eastern end of the parallel *g h*
- h (left end of the upper horizontal): the western end of the parallel *g h*
- K (lower horizontal, on the right-hand vertical): the place under the meridian *a b*
- l (on the right-hand vertical, above the upper horizontal): the end of the line *l o*, on the quadrant *a K*
- m (on the line *l o*, where the oblique from *a* cuts it): the section of *l o* with *a t*
- o [?] (left end of the third horizontal): the far end of the line *l o*, the parallel of 60 degrees of latitude
- r (upper horizontal, on the left-hand vertical): the place sharing an interval with *y*
- t (lower horizontal, left of *z*): the point marked off so that *K t* is the computed distance
- y (upper horizontal, on the right-hand vertical): the place sharing a meridian with *K*
- z (lower horizontal, on the left-hand vertical): the place distant from *K* by *K z*

100 stadia by the division. So take from *e f* the straight line *K t* of 100 stadia, and we shall say that the two places *r* and *t* lie under the same meridian.

Even so, that is no ground for suspecting that the straight line drawn through *r* and *t* represents a meridian. For if the straight line *t r* represents a meridian, then — since the two angles at *K* and at *t* are less than two right angles — that

same line *t r*, produced in a straight line, will meet *a b*. Not before *a*, though: it cannot happen that any point other than the pole should lie on two meridians, and *a* is the pole. Nor can they meet at *a* itself, the polar point. For suppose they do meet there. Draw the straight line *l o*, representing the parallel of 60 degrees of latitude, and let its section with *a t* fall at the point *m*. Then, by the similarity of the triangles *a K t* and *a l m*, as *a K* is to *a l*, so is *K t* to *l m*. But the straight line *a K* has to the straight line *a l* the ratio of three to one; therefore *K t* is triple *l m*. Now the arc of the equinoctial taken in between two meridians is double the arc taken in between the same meridians on the parallel of 60 degrees of latitude, since the ratio of the diameters of those circles is double. Therefore *K t* has to *l m* the ratio of two to one — and it has been shown to have the ratio of three to one as well, which is impossible. Consequently, if the straight line *a t* represents a meridian, it will not meet *a K* at the polar point *a* itself.

Suppose, finally, that it is said to meet that same *a b* when produced in a straight line above *a*. It will then cut the polar line *a c*; let it cut it at *n*, as in the figure below. Now the circumferences of circles keep the same ratio as their diameters, whereas the ratio of straight lines can be increased without limit. Let us therefore take, on the spherical surface, one of the parallels to which the equinoctial has a greater ratio than *K t* has to the straight line *a n*; let the straight line *p q* represent it on the sea chart; and let the portion of *p q* taken in between the two meridians *a K* and *t n* be the straight line *s u*. The straight line *K t* will then have to the straight line *s u* the same ratio that the equinoctial circle keeps to the parallel assumed. But that ratio was posited greater than the ratio of the straight line *K t* to the straight line *a n*; therefore *K t* will have a greater ratio to *s u* than to *a n*, and consequently *s u* will be smaller than *a n*. Yet it will easily be shown to be greater: drop the perpendicular from the point *n* onto *s u*, and it must fall between *u* and *s*, because the angle *n u s* is acute.

Diagram. The same grid once more, with two additions at the top. Across the top runs the polar line, *c* at its left end and the meridian *a b* rising to meet it at the right; below it a further horizontal is drawn, ending at *q* on the right, and this is the parallel *p q*. Lower down are the two familiar horizontals, *h–r–y–g* and *f–z–K–e*, with the verticals *c d* on the left and *a b* on the right. A long oblique runs from the lower left up to the right, cutting *p q* at *u* and meeting the polar line at *n*. Several of the letters at the top and at the two lower corners are lost in the show-through on this leaf and are flagged below.

- a [?] (top right, where the meridian meets the polar line): the pole
- b (foot of the right-hand vertical): the far end of the meridian *a b*

- c (left end of the polar line): the upper end of the meridian *c d*
- d [?] (foot of the left-hand vertical): the lower end of the meridian *c d*
- e (right end of the lower horizontal): the eastern end of the equinoctial
- f (left end of the lower horizontal): the western end of the equinoctial
- g (right end of the middle horizontal): the eastern end of the parallel *g h*
- h [?] (left end of the middle horizontal): the western end of the parallel *g h*
- K (lower horizontal, on the right-hand vertical): the place under the meridian *a b*
- n [?] (on the polar line, between *c* and *a*): where the oblique *t r* produced cuts the polar line *a c*
- p [?] (left end of the upper horizontal): the western end of the assumed parallel *p q*
- q (right end of the upper horizontal): the eastern end of the assumed parallel *p q*
- r (middle horizontal, on the left-hand vertical): the place on the parallel *g h*
- s [?] (on *p q*, where it meets the meridian *a K*): one end of the segment *s u*
- t [?] (lower horizontal, left of *z*): the foot of the oblique on the equinoctial
- u (on *p q*, where the oblique *t n* cuts it): the other end of the segment *s u*
- y (middle horizontal, on the right-hand vertical): the place sharing a meridian with *K*
- z (lower horizontal, on the left-hand vertical): the place distant from *K* by *K z*

An impossibility therefore follows, and so the straight line *t r* cannot meet *a K* if it represents a meridian. But by Euclid's postulate they must meet. Therefore *t r* does not represent a meridian on the sea chart — which is what we undertook to demonstrate.

From all this you will understand that the plane delineation of the world in which straight lines are set down as meridians, handed down by Ptolemy in the first book of the *Geography* (*Geography* I), agrees poorly with the delineation made on a spherical surface. For in that plane delineation the equinoctial has to the parallel drawn through Rhodes a ratio of very nearly three to two, namely as 115 to 79, whereas it ought to be five to four. And so those straight lines will be meridians only for the places set on the equinoctial and on the parallel that passes through Thule, not for those on the parallel of Rhodes. He takes, moreover, 4 degrees of the middle meridian and makes them stand for 5 on the parallel of Rhodes, so that there at least the length of the inhabited world may keep the same

ratio to its whole breadth as it has on the spherical surface. But it is agreed that this cannot be brought off by the method he used, since the rectilinear does not fit the curvilinear.

You will therefore do it much better in the following way. Let *K m n* be the semicircle of the parallel that passes through Rhodes. We shall cut it into 22 equal parts and take *K m* of seven of them. The arc *K m* will then be equal to the radius *g K*, by what Archimedes demonstrated in *On the Measurement of the Circle* (Archimedes, *De circuli dimensione*). And so that same *K m* will contain the 79 degrees of the middle meridian which Ptolemy sets the straight line

g K to contain. From these, then, let seven be thrown out, which the arc *m z* takes in — about an eleventh part of *K m* — and there will accordingly be left the arc *K z*, of 72 degrees of the middle meridian. And since on the spherical surface 72 degrees of the meridian are equal to ninety degrees of that parallel which passes through Rhodes, we shall cut *K z* itself into six equal spaces, and each of them will be an interval of one hour on that same parallel of Rhodes.

Diagram. A large circle with its centre marked *g*. A radius is drawn from *g* straight up to *n* at the top of the circle, and the middle meridian runs from *g* straight down through *K* to *S* at the bottom of the circle. Between two further radii drawn down and outward from *g* lies a fan-shaped panel, bounded by concentric arcs about *g* and divided by straight lines drawn from *g* — the hour-meridians. The arc carrying *K* is the parallel of Rhodes; *z* lies at its left-hand end and *o* at its right-hand end, and *m* lies further round on the same arc beyond *z*. The lowest arc, passing through *S*, is the equinoctial. Letters stand at the two outer corners of the fan but are not resolvable on the scan [?] [?].

- g (centre of the circle): the common centre of the equinoctial and of all the other parallels
- K (on the middle meridian, on the parallel of Rhodes): the point from which the arcs *K m*, *K z* and *K o* are measured; the straight line *g K* is 79 degrees of the middle meridian
- m (on the parallel of Rhodes, beyond *z*): the end of the arc *K m*, which is seven of the 22 parts of the semicircle
- n (top of the circle): the far end of the radius drawn upward from *g*
- o (right-hand end of the arc of Rhodes): the end of the arc *K o*, made equal to *K z*
- S (foot of the middle meridian, on the equinoctial): the far end of the radius *g S*, which is 115 degrees of the middle meridian

- z (left-hand end of the arc of Rhodes): the end of the arc *K z*, of 72 degrees of the middle meridian, divided into the six hour-intervals

We shall then draw straight lines from the point *g* through each of the marks that divide the hour-intervals, as far as the equinoctial, and we shall (if we like) cut the hour-interval into three equal parts. We shall do the same on the arc *K o*, which we shall make equal to *K z*, and then the rest, as Ptolemy himself directs. And if you wish to draw the planisphere itself by such an art that the outermost parallels — the equinoctial, that is, and the one that passes through Thule — keep the same ratio to one another, and to the meridians, that they have on the spherical surface, then you must do for the equinoctial the very thing we have just done for the parallel of Rhodes. For the semicircle of the equinoctial will have to be cut into 22 equal parts, seven of which will be equal to the radius *g S*, that is, to 115 degrees of the middle meridian. Throw out 25 degrees, then, and ninety are left at last — the interval, that is, of six hours. This will have to be cut into six spaces, straight lines drawn from the centre *g*, and the rest carried through as before.

In the other plane delineation of the world, in that same first book, he searches out with many syllogisms how great the straight line *f g* is in the figure below. For *g* is the common centre of the equinoctial and of all the other parallels. Yet he could have found it by a very easy calculation and demonstration. For *e f* is 23 parts and five sixths of the parts of which *b e* is 90, and *g* is the centre of the circle *b f d*. Let the semicircle *f b r* therefore be completed, and let *b r* be joined. The angle *f b r*, standing on the diameter in the circumference, will therefore be a right angle, and so, as *e f* is to *e b*, so is *e b* itself to *e r*, by the 9th proposition of the sixth book of Euclid's *Elements* (Euclid, *Elements* VI.9). We shall therefore multiply *e b*, that is 90 parts, into itself, and divide the product, which is 8100, by *e f*, which has 23 parts and five sixths; the division will yield the parts that *e r* has. To these we shall add the 23 parts and five sixths that are in *e f*, and *f r* will be made up, of which the half is *f g*. However Ptolemy may have laid the world out on a plane,

Diagram. A large circle with its horizontal diameter drawn from *b* at the left to *d* at the right, the centre marked *e*. A vertical line runs through *e*: downward it meets the circle at *f*, upward it passes *g* at the top of the circle and is carried on far above the circle to *r*. A heavy arc of much larger radius springs from *b*, sweeps up and over to the right, and closes on *r*; the straight line *b r* is drawn as its chord. A lighter arc crosses the circle obliquely from near *b* toward the upper

right, and a faint vertical is ruled down the right-hand side of the figure. The letter *c* stands at the foot of the whole figure, below *f*.

- b (left end of the horizontal diameter): the end of the diameter *b d*, and the point from which the great arc rises to *r*
- d (right end of the horizontal diameter): the other end of the diameter *b d*
- e (centre of the circle, on *b d*): the point of section of the diameter and the vertical
- f (foot of the circle, on the vertical through *e*): the lower end of the line *f r*
- g (top of the circle, on the same vertical): the point between *e* and *r* on that line
- r (topmost point of the figure): the upper end of the vertical through *g*, *e* and *f*, joined to *b*
- c (below *f*, at the foot of the figure): the lowest point marked

— we approve more highly, for the art of navigation, the method Ptolemy himself used for drawing maps of particular provinces: those, that is, in which the ratio of the meridian to the middle parallel is preserved. For in such maps, since the meridians are equidistant, any straight line falling across them makes equal angles with them throughout. The outermost parallels, however, must not lie very far apart from one another. Every such map should carry the whole longitude of the world, but the latitude only by climates, as it were. Even if a whole province is not found entire on one map but comes out divided, that does not much matter for the purpose in hand.

We do give this warning, though: few places, or practically none, ought to be transferred to these maps from the customary sea chart, because the longitudes of the places laid down on it are uncertain — and still less from Ptolemy's maps. Maps of this kind will serve only those who intend to travel the world afresh and to examine the true positions of places. Yet the surest method of all will be to draw on the surface of a globe those winding and broken rhumb lines which we defined in the earlier book. Then, from the altitude of the pole observed at each end of the run, and from the character of the course, the difference of longitude and the distance between the places will be known.

But if you want to test this from the length of the voyage actually completed, you must first subtract the surplus that arises from the crookedness of the tracks — which our sailors do not do. Determining longitude from eclipses, moreover, is a method every computation confirms. There is besides the method by the Moon's motion, or by her conjunction with some fixed star; we have discussed that line of

inquiry in the book on the errors of Oronce Fine. Let this be enough said about the sailors' planisphere.

¶ On that table of numbers which sailors use, for finding how great the direct interval is, and also the difference of longitude, between any two places laid down on the sea chart.

Cap. 2.

Sailors have besides a certain table of numbers drawn up by mathematicians, from which they can learn how great is the direct interval [*directum interuallum*] answering, at each inclination of the course, to each degree of the difference of latitude, and also how great the difference of the meridians is under that same inclination. From the same table, again, if the direct interval of the course between two places and the difference of latitude are supplied as known, they learn the distance between the meridians, and the inclination itself as well.

For in the rectilinear right-angled triangle *a b c*, let *a b*, a portion of a meridian, be the difference of latitude of the two places *a* and *c*; let *b c* be the difference of longitude of those same places on the parallel of the place *c*; and let the straight line *a c* be the direct interval between those same two places.

Diagram. A plain right-angled rectilinear triangle. The vertex *a* stands at the top left, *b* at the foot of the left side, and *c* at the right end of the base; the right angle is at *b*. The three sides are the vertical *a b*, the horizontal *b c*, and the hypotenuse *a c*.

- a (top left): the first place; *a b* is the difference of latitude
- b (lower left, the right angle): the section of the meridian *a b* with the parallel *b c*
- c (lower right): the second place; *b c* is the difference of longitude on the parallel of *c*, and *a c* the direct interval

I say that if, besides the right angle, one of the two acute angles is known, or if the ratio of two of the sides is supposed known, everything else will become known. For the right sines of the angles and the sides subtending them are proportional in the same order — as you will grasp at once if you describe a circle to the measure of *a c* about either of its endpoints. So if the angle *b a c* is supplied as known, the ratio of the whole sine to the right sine of that same angle will be known, and therefore the ratio of *a c* to *b c* will be known too. And from the known angle *b a c* the remaining angle *a c b* becomes known immediately, so that

the ratio of *a c* to *a b* cannot escape us either. Once the proportion of the triangle's three sides is known, therefore, if any one of them is known in parts of a great circle, or in stadia, or in any other customary measure, the remaining sides will be disclosed in that same measure.

But suppose no angle except the right angle is supposed known, and yet two sides are known: the remaining side becomes known at once by the 47. proposition of the first book of Euclid. And from the known sides each acute angle will be found by the rule of proportional numbers together with the table of right sines. If one of the two sides supposed known is the side subtending the right angle, you can find the third side without extracting a square root, provided you use a table of right sines. For if the ratio of *a c* to *a b* is known, let *a c* be taken as the whole sine; then, by the rule of proportional numbers, the straight line *a b* will come out known in parts of the radius. The arc to which *a b* belongs as its right sine will therefore be known, and when that arc is subtracted from the quadrant the arc will be left known whose right sine *b c* is. So once the ratio of *a c* to *b c* has been found by this device, *b c* too will be known in whatever measure *a c* is known.

It is in this way that we examined that little table of numbers, and made it far more exact. Now one degree of a great circle on the earth's surface contains 17 leagues and a half, as the Portuguese say. Among them, however, there are some who reckon that it takes in only sixteen, together with two thirds of one league, so that the whole circuit comes to 6000 leagues. And since, on the view of Ptolemy and Marinus [Marinus of Tyre], five hundred stadia answer to one degree of a great circle, while thirty stadia make one schoenus [*schœnus*], there will accordingly be 16 schoeni and two thirds in one degree. One league will therefore be equal to one schoenus.

Inclination to the meridian, by quarter-points [*Inclinatio ad meridianū per quartas.*]	Direct interval [*Directū interuallū.*] — Leagues [*Leucœ.*]	Difference of longitude [*Differentia longitudinis.*] — Leagues [*Leucœ.*]
1	17 with five eighths	3 with a half
2	19 with three eighths	7 with one quarter
3	21	11 with two thirds
4	24 with three quarters	17 with a half
5	31 with a half	26 with one fifth
6	45 with three quarters	42 with one quarter
7	89 with three quarters	88

Now if Ptolemy himself was permitted — as he writes in the first book of the *Geography* — to measure the difference of longitude in a rectilinear triangle from the known bearing of one place with respect to another together with the travelled distance between those places, I do not see why it should not equally be permitted, on the same foundations, to find the difference of latitude and everything else, over any stretch and universally. If you do so, however, your results will conflict with what we have next to demonstrate.

All navigation follows the circumferences of great circles in certain very small segments, as we explained in the preface [*Prælatione*] to the first book. On the globe, therefore, matters will fall out very differently for those who make their voyage along great circles. For if the same difference of latitude is preserved, and likewise the same inclination of the great circle to the meridian, then, as we approach the pole, the travelled distance will be found smaller — and the difference of longitude smaller in the same way — between places farther from the visible pole [*manifestus polus*] than between places nearer to that same pole.

Let there be on the globe two places *a* and *f*, farther from the visible pole *c* than two others *b* and *d*, but let the differences of latitude be posited equal. Let the great circles drawn through *a* and *f*, and through *b* and *d*, make equal inclinations to the meridians *a c* and *b c*, at the acute angles *c a f* and *c b d*. Let the one coming from *a*, along the circumference of the great circle *a f*, meet the parallel of the place *f* at *f* itself; and likewise let the one coming from *b*, along the circumference of the great circle *b d*, meet the parallel of the place *d* at *d* itself. I say, then, that the travelled interval [*interuallum viatorium*] *b d*, between the places *b* and *d*, which are nearer the visible pole *c*, is greater than *a f*; and that the difference of longitude between those same places *b* and *d* is greater than the difference of longitude of the two places *a* and *f*.

Diagram. A figure of the sphere set out at the foot of the column, with the pole at the top. The letter *c* stands at the apex; from it lines run down to *a* at the lower left corner of the figure and to *b* at the foot, toward the centre. Two curvilinear panels stand side by side beneath: the left one is lettered *g* at its left and *f* below its middle, the right one is lettered *d* at its right. Several further letters are set in the crowded upper part of the figure; on the scan they will not resolve.

- a (lower left corner): the more remote of the two places on the left-hand meridian
- b (foot of the figure, toward the centre): the place on the other meridian
- c (apex): the pole
- d (right side): the place nearer the pole

- f (below the middle of the left-hand panel): the place remote from the pole
- g (left side): the section of the parallel of *f* with the meridian *a c*
- e [?] (a small letter in the middle of the figure, not resolved on the scan)
- i [?] (upper middle, not resolved on the scan)
- K [?] (upper left, not resolved on the scan)
- l [?] (upper middle, not resolved on the scan)

For about the pole *c* let a parallel be described through *d*, cutting the meridian *b c* at the point *e*; and likewise a parallel through *f*, cutting the meridian *a c* at *g*. Since *c g* [?] is greater than *c e* by hypothesis, let there be taken on *g c* an arc *g K* equal to *c e*, that is, to *c d*; and about *K* as pole, to the measure of *K g*, let a circle be described through *g*. By the sixth proposition of the second book of Theodosius it will touch the parallel *f g* at *g* itself. On that same circle let there be taken an arc *g i* equal to the arc *d e*; for the circles described through *d* and through *g*, about the poles *c* and *K*, are equal. Consequently, if a great circle is drawn through *K* and *i*, and one is also described through *c* and *d*, the two angles *a K i* and *b c d* will be equal to one another.

We shall therefore draw a great circle through *a* and *i*, and it will be no other than the one that comes through *a* and *f*. For suppose it fell inside the triangle *a c f*, dividing the angle *c a f*: it would then make with *a K*, at the point *a*, an angle equal to the angle *c b d* — by the proposition analogous to the fourth of Euclid's first book, which Menelaus demonstrates in the first book *On Spherical Triangles* — and therefore equal to the angle *f a K*; so that, by the common notion, a part would equal the whole, which is impossible. The same absurdity would follow if it fell outside that same triangle. And so the great circle described through *a* and *i* passes through *f*.

Thus the interval *a f* will be less than the interval *a i*. But *a i* is equal to *b d*. Therefore the travelled interval *b d*, between the places *b* and *d*, which are nearer the visible pole, is greater than the travelled interval *a f* between the places *a* and *f*, which are farther from the visible pole and have equal differences of latitude. That is what we had to demonstrate.

Further, that the difference of longitude is greater as well we shall show by drawing a great circle through *c* and *f*, which cuts *K i* at the point *l*. Since the two places *d* and *f* have *c* as their visible pole, the arcs *c d* and *c f* are less than quadrants, and therefore *c l* and *K l* will be less than quadrants; and so in the triangle *K l c* the exterior angle *a K l* is greater than the interior angle *K c l*. But *a K l* and *b c d* are equal to one another, in those two equiangular triangles *a K i*

and *b c d*. The angle *b c d* will therefore be greater than the angle *K c l*. And proportional to these are two arcs of the equinoctial circle, of which one is the difference of longitude of the two places *b* and *d*, the other that of the two places *a* and *f*. Greater, therefore, will be

the difference of longitude of the two places *b* and *d* than that of the two places *a* and *f* — which we had likewise undertaken to demonstrate.

This demonstration also shows that it makes no difference whether the two places *a* and *b* have the pole *c* visible or hidden, provided that the same pole *c* is visible at the place *d*, and at the place *f* is at any rate not hidden, but either plainly in view there or set on the horizon. We assumed, however, that the circle *g i* cannot cut, between *a* and *f*, the circle that comes through *a* and *f*; otherwise an impossibility would follow — a part greater than its whole — once the great circle *a K c* is extended until it cuts the circles *g i* and *g f* again.

Suppose now that the size of the angle of position is kept the same from the first place to the second and from the third to the fourth, and the difference of longitude the same as well; that the first and second places are farther from the visible pole than the third and fourth; and that the first is farther than the second, and the third than the fourth. Then the travelled distance between the first and second will be greater than between the third and fourth, and the difference of latitude greater too.

For let the first place *a* and the second *b* be farther from the pole *c*, which is visible to them, than the third place *d* and the fourth *e*; and let the angle of position *c a b* be posited equal to the angle of position *c d e*. The difference of longitude, moreover, is the same, since *a* and *d* lie on the same meridian *a c*, and likewise *b* and *e* on the same meridian *b c*. Let the latitude of the place *b* exceed the latitude of the place *a* by the difference *a g*, and the latitude of the place *e* exceed the latitude of the place *d* by the difference *d K*. I say that *a b*, the travelled interval between *a* and *b*, will be greater than *d e*, the travelled interval between *d* and *e*, and that the difference of latitude *a g* is greater than the difference *d K*.

Diagram. A figure of the sphere set into the foot of the left-hand column, with the pole *c* at the top. Two meridians run down from *c*. The left-hand one passes through *K*, then *d*, then *g*, and ends at *a* in the lower left corner of the figure; the right-hand one passes through *e* and ends at *b* near the foot, toward the centre. Two great-circle arcs are drawn out toward the right and meet at *f* on the right-hand edge: one rises from *a* through *b*, the other from *d* through *e*. Short cross-arcs meet the left-hand meridian at *K* and at *g*.

- c (apex): the pole visible to all four places
- a (lower left corner): the first place, on the meridian *a c*
- b (foot of the figure, toward the centre): the second place, on the meridian *b c*
- d (on the left-hand meridian, below *K*): the third place, on the same meridian as *a*
- e (on the right-hand meridian, above *b*): the fourth place, on the same meridian as *b*
- f (right-hand edge): the meeting point of the great circles *a b* and *d e*
- g (on the left-hand meridian, between *d* and *a*): the section of the parallel of *b* with the meridian *a c*; *a g* is the difference of latitude of *a* and *b*
- K (on the left-hand meridian, between *c* and *d*): the section of the parallel of *e* with that same meridian; *d K* is the difference of latitude of *d* and *e*

For let the great circles *a b* and *d e* be drawn out toward the parts *b* and *e*, and let them meet at *f*. Since the two acute angles *c a b* and *c d e* are posited equal, the two arcs *d f* and *a f* taken together will be equal to a semicircle. But in the triangle *d f a* the side *a f* is greater than the side *d f*, because it is subtended by the obtuse angle *a d f*. The side *d f* will therefore be less than a quadrant, and *d e*, the travelled distance between *d* and *e*, much less than a quadrant.

Now in the triangle *c e d*, as the right sine of the angle *c d e* is to the right sine of the angle *d c e*, so is the right sine of the side *e c* to the right sine of the side *d e*; and likewise in the triangle *a b c*, as the right sine of the angle *b a c* is to the right sine of the angle *a c b*, so is the right sine of the side *b c* to the right sine of the side *a b*. Further, the right sines of the angles *c d e* and *b a c*, which are equal to one another, have the same ratio to the right sine of the angle *d c e*. Therefore the right sine of the side *c e* will have to the right sine of the side *d e* the same ratio that the right sine of the side *b c* has to the right sine of the side *a b*. And so, by alternation, as the right sine of *c e* is to the right sine of *b c*, so is the right sine of *d e* to the right sine of *a b*. But the right sine of *e c* is less than the right sine of *b c*, since the arc *b c* is posited less than a quadrant. Therefore the right sine of *d e* is less than the right sine of *a b*. And it was shown that the arc *d e* is less than a quadrant; therefore the arc *d e* is itself less than the arc *a b* — which was the first thing to be demonstrated.

Further, that *a g*, the difference of latitude of the places *a* and *b*, is greater than *d K*, the difference of the two places *d* and *e*, you will demonstrate by the very easy reduction to the impossible used in the preceding demonstration. For if they are equal, then the difference of longitude of the two places *d* and *e* will be greater

than that of the two places *a* and *b*, and *d e* will likewise be greater than *a b*. But we posited the difference of longitude to be the same, and we showed *a b* to be greater than *d e*; therefore it is impossible. And if you assert *d K* to be the greater, you will see a far greater absurdity follow if you take a point before *K*, as far distant from *d* as *g* is from *a*, and draw a circle equidistant from the equinoctial which cuts *d e* between *d* and *e*.

It is worth demonstrating the same thing by a direct demonstration [*ostensoria demonstratio*] as well, in this manner. Since in the spherical triangle *a c b* the side *a c* is greater than the side *b c*, the angle *a b c* will be greater than the angle *b a c*. But the angle *c b f* together with that angle *a b c* is equal to two right angles; therefore that same angle *c b f* together with the angle *b a c* will be less than two right angles. Yet the angle *c b f* is itself greater than the angle *c a b*, since the two sides *a c* and *b c* taken together are less than a semicircle — for by hypothesis the place *a* has the pole *c* visible. Therefore the right sine of the angle *c b f* will be greater than the right sine of the angle *c a b*. Consequently the right sine of the angle *a f d* has a greater ratio to the right sine of the angle *d a f* than it has to the right sine of the angle *f b e*. But as the right sine of the angle *a f d* is to the right sine of the angle *d a f*, so is the right sine of the side *a d* to the sine of the side *d f*, in the spherical triangle *a d f*; and again, as the right sine of the same angle *a f d* is to the right sine of the angle *f b e*, so is the right sine

of the side *b e* to the sine of the side *e f*, in the triangle *b e f*. Therefore the sine of the side *a d* will also have a greater ratio to the sine of the side *d f* than the sine of the side *b e* has to the sine of the side *e f*. Consequently the right sine of the arc *a d* will have a greater ratio to the right sine of the arc *b e* than the right sine of the arc *d f* has to the right sine of the arc *e f*, by the twenty-seventh proposition of the fifth book of Euclid added by Campanus [Campanus of Novara]. Now the arc *d f* is, as was demonstrated above, less than a quadrant. Therefore the right sine of *d f* will be greater than the right sine of the arc *e f*, and consequently the right sine of the arc *a d* much greater than the right sine of the arc *b e*; and therefore the arc *a d* is greater than the arc *b e*. But the arcs *b e* and *g K* are equal, being contained between two parallels. Therefore *a d* is greater than *g K*. Consequently, when the common part *d g* is taken away, *a g* will be left greater than *d K*. So it is clear that the difference of latitude between *a*, the first place, and *b*, the second, is greater than between *d*, the third, and *e*, the fourth — which was the last thing to be demonstrated.

Finally, suppose the first place to the second and the third to the fourth have the same position, and the travelled intervals are equal as well — whether that pole

of the world toward which we are approaching is visible at those places or hidden — and suppose the first place is farther from that pole than the third. Then the difference of latitude between the first and second will be greater than between the third and fourth.

And if the distances of the second place and of the fourth from that same pole, taken together, are equal to a semicircle, the difference of longitude between the first and second will be just as great as between the third and fourth. This will come about if, as we go toward the parts of the northern pole, the southern latitude of the second place is as great as the northern latitude of the fourth. But if those distances taken together are greater than a semicircle, the difference of longitude between the first and second will be greater than between the third and fourth; and if they are less than a semicircle, it will be less.

For let the first place *a* have toward the second place *b* the position that the acute angle *e a b* shows, and let the third place *c* have an equal position with *d*, the fourth; and let the travelled distances *a b* and *c d* be equal. Let *e* be that pole of the world toward which we approach as we go, and let it be posited that the place *a* is farther from that pole *e* than *c* is. I say that the difference of latitude between *a* and *b* is greater than the difference of latitude between *c* and *d* — whether the pole *e* toward which we approach is visible at those places or hidden, or visible at some of them and hidden at others.

Diagram. A figure of the sphere occupying the middle of the right-hand column, with the pole *e* at the upper right. Arcs run down from *e* toward the lower left, where *a* stands at the corner, and toward the foot of the figure at *b* and *c*, and out to *d* at the right-hand edge. On the arc from *e* to *a* the letter *K* is marked high up; below it, on the left-hand side of the figure, stand *g* and, beneath *g*, *i*. The letter *f* is set in the middle of the right-hand half of the figure, between *c* and *d*. Two cross-arcs run from the left-hand side across to the right, and a further arc joins *b* to *K*.

- e (upper right): the pole of the world approached
- a (lower left corner): the first place
- b (foot of the figure, centre): the second place
- c (foot of the figure, right of *b*): the third place
- d (right-hand edge): the fourth place
- f (middle right): the section of the parallel of *d* with the meridian of *c*; *c f* is the difference of latitude of *c* and *d*
- g (left side, below *K*): the section of the parallel of *b* with the meridian of *a*; *a g* is the difference of latitude of *a* and *b*

- i (left side, below *g*): where the circle described about *K* to the measure *K b* cuts the meridian *a e*
- K (on the meridian *a e*, high on the left): the point taken so that *a K* equals *c e*, and the pole of the circle drawn through *b*

For let the parallel of the place *d* pass through *f*, cutting the meridian of the place *c* at that point; and let the parallel of the place *b* pass through *g*, cutting the meridian of the place *a* at that point. Since the arc *a e* is posited greater than the arc *c e*, we shall cut off from *a e* an arc *a K* equal to *c e*, and through the points *b* and *K* we shall describe a great circle *b K*.

Since, then, the angles of position *b a K* and *d c e* are posited equal, and the travelled distances *a b* and *c d* are equal to one another, therefore *d e* and *b K*, the bases of the spherical triangles *a b K* and *c d e*, will be equal, and the angles *d c e* and *a K b* will also be equal to one another. But the arc *b K* will be greater than *K g*, because the two sides *b K* and *K e* of the spherical triangle *e b K*, taken together, are greater than *b e*, and consequently greater than *e g*; so that *b K* is left greater than *K g*, by the common notion — or you may demonstrate the same by the 25 proposition of the second book of Theodosius. About the point *K*, then, as pole, to the measure *K b*, we shall describe a circle, which will cut the meridian *a e* between *a* and *g*; let it cut it at *i*. Then *a i* will be equal to the arc *c f*; and therefore *c f*, the difference of latitude of the two places *c* and *d*, will be less than *a g*, the difference of latitude of the places *a* and *b* — which was the first thing to be demonstrated.

The latter part will be demonstrated in the same figure as follows. The arc *b K* is equal to *d e*, the distance of the fourth place from the pole *e*. But *b e* is the arc of the meridian by which the second place is distant from that same pole. So in the spherical triangle *e b K*, if the two sides *b e* and *b K* taken together are equal to a semicircle, the exterior angle *a K b* will be equal to the interior angle *b e K*. And therefore the difference of longitude of the places *c* and *d* is equal to the difference

of longitude of the places *a* and *b*. But if they are greater than a semicircle, the angle *a K b* will be less than the angle *b e K*; and consequently the difference of longitude between the first and second is greater than the difference of longitude between the third and fourth. And if they are less than a semicircle, the angle *a K b* will be greater than the angle *b e K*, and therefore the difference of longitude between the first and second will be less than the difference of longitude between the second [recte: third] and fourth.

Add this too: if two men set out from two places lying under one meridian, at an equal or like inclination of a great circle to that meridian — the more northerly toward the southern region, the more southerly toward the north — and each holds on until they reach the middle parallel, provided this is not the equinoctial circle, then the one whose course lies toward the pole nearer to that middle parallel will cover the greater distance, and will end farther from the root meridian [*radicalis meridianus*], than the one who goes toward the other pole.

For let *a* and *b* be the poles of the world, and *a b* the semimeridian on which the two places *c* and *d* lie; and let *e f g* be the middle parallel, which is not the equinoctial. Let the journey from the place *d* to that parallel be *d f*, following the inclination of the acute angle *c d f* — toward the parts, that is, of the pole *a*, which is the nearer to that middle parallel *e f g*. I say that if a man setting out from the place *d* under such an inclination comes to *f*, he will cover a greater distance, and will end farther from the root meridian *a b*,

Diagram. A figure of the sphere at the foot of the left-hand column. A long meridian arc runs from *a* at the top down through *c*, then *g*, then *d*, to *b* at the very foot of the figure. Crossing it near the middle, a broad, shallow arc runs from *e* at the far left across through *g* and on to *f* at the right; the letter *y* is set on this arc a little to the left of *f*. A second arc rises from *h* at the lower left and crosses the meridian at *g*, running up to the right past *K*, which is marked to the right of *g*. Further arcs descend from *c* and from *d* toward the right-hand part of the figure.

- a (top): the pole nearer the middle parallel
- b (foot): the other pole of the world
- c (on the meridian, below *a*): the more northerly of the two places on the meridian *a b*
- d (on the meridian, below *g*): the other place on that same meridian
- e (far left): the left-hand end of the middle parallel *e f g*
- f (right): the point of the middle parallel reached by the course from *d*
- g (on the meridian, between *c* and *d*): the section of the middle parallel with the meridian
- h (lower left): the end of the great circle *h g K* erected at right angles to the meridian
- K (right of *g*): the intersection of that great circle with the course *d f*
- y (on the middle parallel, left of *f*): the point at which the great circle drawn through *c* and *K* cuts the parallel

than a man who sets out from the place *c* under an equal inclination and comes to the same parallel. For from the point *g* we shall erect a great circle *h g K*

at right angles to the meridian *a g b*, and let its intersection with *d f* be at *K*. That circle will therefore touch the parallel *e f g* at the point *g* itself, by the fourth proposition of the second book of Theodosius. Through the two points *c* and *K* we shall describe a great circle cutting that same parallel at *y*.

Now since the two sides *c g* and *g K* are equal to the two sides *d g* and *g K*, and the angles at the point *g* are equal — for they are right angles — the bases *c K* and *d K* of the spherical triangles *c g K* and *d g K* will be equal to one another, and the angles *g c K* and *g d K* equal to each other. Consequently the great circles *c K* and *d K* will make equal inclinations with the root meridian at those same places *c* and *d*. And since *c y* is less than *c K*, it will be much less than *d f*. But it is agreed that a man who has set out from the place *c* and comes to the place *y*, which is nearer the meridian than *f* is, has covered the space *c y*. The difference of longitude, therefore, will be greater between *d* and *f* than between *c* and *y*, and the travelled distance greater too; which was to be demonstrated.

Add also that a man who goes out, and returns under the same inclination to a great circle, does not travel the same way. He therefore does not come back to the place he set out from: he will come back to the same meridian, but on a different parallel, and to the same parallel, but on a different meridian.

For let there be two places *b* and *c*, on the meridians *a b d* and *a c e*; let *a* be the visible pole; and let the great circle *b c f* make with the meridian *a b d*, at the point *b*, the inclination of the acute angle *a b c*, and with the meridian *a c e*, at the point *c*, the inclination of the acute angle *b c e*. Now since the two sides *a b* and *a c* taken together are less than a semicircle, the angle *a c f* will be greater than the angle *a b c*. Consequently the vertically opposite angle *b c e* will also be greater than that angle *a b c*. We shall therefore make at the point *c* an angle *d c e*, a great circle being described through *d* and *c*, this angle being equal to *a b c*; and therefore the man who set out from the place

Diagram. A second figure of the sphere, set into the foot of the right-hand column and wrapped about by the text. The visible pole *a* stands at the top. From *a* two long arcs descend: the left-hand one runs down to *b* at the left and on to *d* at the lower left corner; the right-hand one runs down through *c* in the middle of the figure and on toward *e* at the lower right. A further arc crosses the figure from *b* through *c* and out to *f* at the right, above *g*. The letter *K* is set at the foot of the figure, toward the centre, and *g* on the right-hand side below *f*.

- a (top): the visible pole
- b (left): the first place, on the meridian *a b d*
- c (centre): the second place, on the meridian *a c e*
- d (lower left corner): the lower point of the meridian *a b d*
- e (lower right) [?]: the lower point of the meridian *a c e*
- f (right, above *g*): the far point of the great circle *b c f*
- g (right side, below *f*): a point of the figure not yet named in the text on this page
- K (foot, toward the centre): a point of the figure not yet named in the text on this page

the place *b* to *c* along the circumference of a great circle, and then returns from there under the same inclination to a great circle, he will not arrive at *b* but at *d* — and on a different parallel at that. Let the intersection of the circle *c d* with *b g*, the parallel of the place *b*, fall at the point *k*. It is clear, then, that under that same inclination of the great circle *c d* he will arrive at *K*, on the same parallel as the place *b* but on a different meridian. Which was to be demonstrated.

The same must happen if the pole *a* is hidden from those same places *b* and *c*, as in the next figure. For since the angle *a c f* is less than the angle *a b c*, let a great circle *c i h* be described which makes the angle *g c h* [?] equal to the angle *a b c*, and let it cut the meridian *a b* at the point *i* and the parallel *b g* at *h*. You will then prove that a man who has gone from the place *b* to *c*, when he returns under the same inclination, will not go to *b* but to *i*, on a different parallel, and to *h*, on a different meridian. The inequality among the distances travelled is easy to grasp from the same figures.

Diagram. A spherical figure for the case in which the pole is hidden from both places. Two arcs run down from the upper point across a lower arc, meeting it at the base line; the labels are:

- *a* (top): the hidden pole
- *f* (right of centre): on the great circle carried through *c*
- *c* (centre): the further place
- *i* (left, on the meridian): where the great circle *c i h* cuts the meridian *a b*
- *h* (bottom left): where the same circle cuts the parallel of *b*
- *b* (bottom centre): the place set out from
- *g* (bottom right): on the parallel of *b*

Since this is so, it is no wonder that sailors go astray again and again while at sea, and, because they do not know the causes, fall repeatedly into serious errors. For let the distance run on a voyage from *a* to *b*, under the inclination of the acute angle *c a d*, be the broken line *a d e b*; and let the meridians *c a*, *c d*, *c e*, and *c b*, coming from the visible pole *c*, make equal angles with it at the points *a*, *d*, *e*, *b* — at the intermediate points somewhat larger ones, but by a very small difference, one that escapes the helmsman's notice.

Through *a* and *b* let the segment *a f b* of a great circle be drawn. That this is shorter than the broken line *a d e b* is agreed. For if the segment *a e* of a great circle is drawn through *a* and *e*, then *a d* and *d e* taken together will be greater than that segment *a e*; and again *a e* and *e b* joined are longer than *a f b*. Therefore *a d*, *d e*, and *e b* are much greater than the segment *a f b*; while the angle of departure, or of the traverse, *c a d* will be greater than the angle of position *c a b*.

Diagram. A lens-shaped spherical figure with the visible pole at the top, from which four meridians descend to the points of the track; a single arc cuts across them from the top-most point of the track to the last. The labels are:

- *c* (top): the visible pole
- *b* (right): the place arrived at
- *e* (below *b*): the last intermediate point of the track
- *f* (centre left): on the great-circle segment *a f b*
- *d* (lower centre): the first intermediate point of the track
- *a* (bottom): the place set out from

On the sea chart, then, we shall set the straight line *g K* in place of the broken and curved line *a d e b*, giving it the same inclination to the meridian *g l* that *a d* has, in the world, to the meridian *a c*. And for the segment *a f b* let the straight line *g m* be cut off from *g K* in the same proportion. *K m* will then be the amount that runs over on account of the obliquities, once *a f b* is subtracted from *a d e b*. Further, from the point *m* let the straight line *m o* be erected at right angles upon *g l*.

In the right-angled rectilinear triangle *g m o*, then, following Ptolemy's practice, the line *m o* will show us the difference of longitude of the two places *a* and *b*, and the line *g o* the difference of latitude. But by the sailors' rules, if *l K* is drawn parallel to *m o*, that same *l K* will be the difference of longitude, and the straight line *g l* that of latitude.

Diagram. The corresponding plane-chart figure: a long right-angled triangle with its right angle at the top left, the hypotenuse running from the bottom vertex up to the top right, and two horizontals drawn across it. The labels are:

- *l* (top left): on the meridian *g l* of the chart
- *K* (top right): the far end of the straight course-line *g K*
- *o* (left, below *l*): the foot of the perpendicular on the meridian
- *m* (right of centre): the point on *g K* cut off in proportion to the great-circle segment
- *g* (bottom): the place set out from

It is true that if the line *g k* is divided into spaces proportional to *a d*, *d e*, and *e b*, and meridians and parallels are drawn besides in each of the two figures, the differences of longitude and of latitude do appear equal to one another in the small spherical triangles and in the rectilinear ones. Even so, it will still not be permissible to gather the whole distance in longitude and in latitude from the parts. For what is neglected in each single case because it is so slight becomes noticeable once it is accumulated over many.

Suppose besides that in the world, on a voyage from *a* to *b*, the angle of inclination is *c a d*, or *c d b* — angles which those made at the intermediate points exceed, though by an imperceptible difference, between *a* and

and *d*, and between *d* and *b*. Let the visible pole be *c*, let *e b* be the parallel of the place *b*, let the difference of latitude *a e* be assumed known, and the angle of inclination known as well. On the sea chart, further, let *f* and *g* stand for *a* and *b*, let *K* stand for *e*, and let *k f g* stand for the angle *c a d*. I say that on the sea chart the difference of longitude of the places *a* and *b* is drawn out beyond its proper bounds.

For let the great circle that passes through *a* and *d* cut the parallel *b e* at *l*. The point *l* will then fall beyond *b*, because the exterior angle *c d l* is greater than the interior angle *c a d*, that is, than *c d b* [?]. The rectilinear triangle *f g K* is therefore set proportionally in place of the spherical triangle *a l e*, and the difference of longitude *k g* must be taken for *e l*. But *e b* is less than *e l*, and so on the sea chart the difference of longitude of the places *a* and *b* is stretched beyond its due figure.

Diagram. A spherical figure for the case of the visible pole, drawn against a faint circle: two meridians descend from the pole to the base arc, with a further arc cut across them and carried past the place arrived at. The labels are:

- *c* (top): the visible pole

- *e* (left): on the parallel of *b*, on the meridian through *a*
- *b* (right of centre): the place arrived at
- *d* (bottom centre): the intermediate point
- *a* (bottom left): the place set out from
- *L* (right): where the great circle through *a* and *d* cuts the parallel *e b*, beyond *b*

Diagram. The plane-chart triangle standing for the spherical triangle: a right-angled triangle with the right angle at the top left. The labels are:

- *K* (top left): standing for the point *e*
- *g* (right): standing for the place *b*
- *f* (bottom left): standing for the place *a*

Now suppose again that in the world the difference of latitude of the two places *a* and *b* is found to be *a e*; that the pole *c* is hidden; that the angle of inclination, or of departure, *c a d* is equal to the angle *c d b*; and that the great circle drawn through *a* and *d* cuts the parallel *b e* at *f*. The point *f* will then fall short of *b*, because the angle *c d f* is less than the angle *c a d*, and therefore *e f* is less than *e b*.

Diagram. A spherical figure for the case of the hidden pole: two arcs descend from the upper point to the base arc, with a third arc cut across them, meeting the parallel short of the place arrived at. The labels are:

- *c* (top): the hidden pole
- *e* (left): on the parallel of *b*
- *f* (centre): where the great circle through *a* and *d* cuts the parallel *b e*
- *b* (right): the place arrived at
- *d* (bottom centre): the intermediate point
- *a* (bottom left): the place set out from

In the rectilinear triangle *g h K*, however, let the line *g h* of the sea chart be put for *a e*; and let the inclination of the acute angle *c a d* be set equal to the angle *h g K*. The line *h K* is therefore put for *e f* of the spherical triangle *e a f*. But *e b* is greater than *e f*; on the sea chart, then, the difference of longitude has been contracted. It will therefore be worth showing by what method the true longitudes of places are to be drawn out of the sea chart itself.

Diagram. The second plane-chart triangle: a right-angled triangle with the right angle at the top left and the hypotenuse running down to the bottom left vertex. The labels are:

- *h* (top left): standing for the point *e*
- *K* (top right): standing for the point *f*
- *g* (bottom left): standing for the place *a*

On finding the difference of longitude of two places from the sea chart. Cap. 3.

Although the places of the world are wrongly set down on the sea chart, their true longitudes and the intervals between them can nevertheless be inferred from it — provided one knows by what method they were found and laid down on that chart. Otherwise it is altogether impossible.

So, in order to accomplish this, we shall show first that between the equinoctial and one of the world's poles the inclinations of great circles to the meridians increase less toward that same pole in places nearer the equinoctial circle itself than in places further from it. For let *a* be the pole of the world, and let the great circles *b c f* and *d e g* make equal inclinations to the meridians *a b* and *a c*. Let the points *b* and *c* be nearer

to the equinoctial circle than *d* and *e*, but let *a b* exceed *a c* by exactly as much as *a d* exceeds *a e*. Now the angle of inclination *a c f*, which the great circle *b c f* makes with the meridian *a c*, is greater than the angle of inclination *a b c*, which the same circle *b c f* makes with the meridian *a b*, because *a b* and *a c* joined are less than a semicircle. By the same argument the angle of inclination *a e g*, which the great circle *d e g* makes with the meridian *a e*, is greater than the angle of inclination *a d e*, which that same great circle makes with the meridian *a d*.

Diagram. The figure for this chapter: the pole at the top, with two meridians running down to the lower arc, and two further great circles cut across them, crossing each other at an interior point. The labels are:

- *a* (top): the pole of the world
- *d* (centre left): on the great circle *d e g*
- *e* (centre): on the same circle, further along
- *g* (right): the far end of the circle *d e g*
- *i* (below *d*): where the two cutting circles meet
- *f* (right, below *g*): the far end of the circle *b c f*
- *c* (lower centre): on the circle *b c f*, on the meridian *a c*
- *z* (right of *c*): on the circle *b c f*
- *b* (bottom left): the foot of the meridian *a b*

- *b* [?] (left, above the preceding): a second *b*-shaped label on the meridian *a b*

I say, then, that the acute angle *a c f* exceeds *a b c* by less than the acute angle *a e g* exceeds *a d e*. For since the arc *d e* is greater than the arc *b c*, by what we demonstrated above in the preceding chapter, we shall take an arc *b z* equal to *d e*, cut off from *a b* an arc *b h* equal to the arc *a d*, and through the points *z* and *h* describe a great circle, which cuts *a c* at *i*. Consequently, in the two triangles *b h z* and *d a e*, the angle *a e d* will be equal to the angle *b z h*, and so the two exterior angles *h z f* and *a e g* are left equal. But the angle *h z f* is itself greater than the angle *a c f*, because the two sides *c i* and *z i* of the triangle *c i z* [?], joined, are less than a semicircle. The angle *a e g* is therefore greater than *a c f*; and by hypothesis the two angles *a b c* and *a d g* [recte: *a d e*] are equal to one another. Therefore the angle *a c f* exceeds the angle *a b c* by less than the angle *a e g* exceeds the angle *a d g* [recte: *a d e*]. And accordingly, between the equinoctial and the pole of the world, the inclinations of great circles to the meridians increase less in places nearer the equinoctial itself than in places further off — which was the first thing we had to show.

You may prove the same thing another way, namely as follows, by the proportions of sines. On the meridian on which *a b* lies, take *a K*, *a l*, and *a m* equal to *a c*, *a d*, and *a e*; let *n* be the centre of the sphere; and to the semidiameter *a n* draw at right angles *b o*, *k p*, *l q*, and *m r* — that is, the right sines of those arcs. Besides this, from the points *K* and *m* draw perpendiculars, *K s* upon *b o* and *m t* upon *l q*, and join the lines *b K* and *l m*. Now since the arc *b K* is by hypothesis equal to the arc *l m*, *K s* will be greater than *m t*; we demonstrated this in our annotation on the motion of the eighth sphere (*Annotationes in Theoricas planetarum Georgii Purbachii*). And since the line *b K* is equal to the line *l m*, the square on *b s* will be less than the square on *l t*, and so *b s* itself is less than *l t*. Therefore *l t* will have a greater ratio to *q t* than it has to *s o*.

Diagram. A circle with a vertical semidiameter drawn from the top of the circumference down to the centre, and a horizontal radius from the centre to the right. Four horizontal chords run from the left-hand arc to the vertical semidiameter, and two short perpendiculars are dropped between them. The labels are:

- *a* (top of the circumference): the pole-end of the meridian
- *m* (upper left, on the arc): the end of the arc *a m*
- *r* (right of *m*, on the semidiameter): the foot of the right sine *m r*
- *l* (left, below *m*): the end of the arc *a l*

- *q* (right of *l*, on the semidiameter): the foot of the right sine *l q*
- *t* (on the line *l q*): the foot of the perpendicular *m t*
- *K* (left, below *l*): the end of the arc *a K*
- *p* (right of *K*, on the semidiameter): the foot of the right sine *k p*
- *b* (left, below *K*): the end of the arc *a b*
- *o* (right of *b*, on the semidiameter): the foot of the right sine *b o*
- *s* (on the line *b o*): the foot of the perpendicular *K s*
- *n* (bottom of the semidiameter): the centre of the sphere

But that same *l t* has a greater ratio to *s o* than *b s* has to *s o*; therefore *l t* has a greater ratio to *t q* than *b s* has to *s o*. By composition, then, *l q* will have a greater ratio to *t q* than *b o* has to *s o*. But *t q* is equal to the line *m r*, and *s o* to the line *K p*. Therefore the right sine of the arc *a l* has a greater ratio to the right sine of the arc *a m* than the right sine of *a b* has to the right sine of *a k*. And accordingly, in the earlier figure, the right sine of *a d* has a greater ratio to the right sine of *a e* than the right sine of *a b* has to the right sine of *a c*.

But as the right sine of the angle *a e g* is to the right sine of the angle *a d e*, so is the right sine of the arc *a d* to the right sine of the arc *a e*. Likewise, as the right sine of the angle *a c f* is to the right sine of the angle *a b c*, so is the right sine of the arc *a b* to the right sine of the arc *a c*. Therefore the sine of the angle *a e g* has a greater ratio to the sine of the angle *a d e* than the sine of the angle *a c f* has to the sine of the angle *a b c*; and by hypothesis the two angles *a d e* and *a b c* are equal. For that reason the right sine of the arc of the angle *a e g* will be greater than the sine of the angle *a c f*; and since each of them is taken as acute, the angle *a e g* will therefore be greater than the angle *a c f*. So the angle *a c f* exceeds the angle *a b c* by less than *a e g* exceeds *a d e* — which again was to be demonstrated.

From this you will conclude that if equal inclinations of great circles to the meridians are equally increased, the difference of latitude will be greater between places nearer the equinoctial circle than between places further off.

We shall show besides that if, between the equinoctial and one of its poles, two great circles are unequally inclined to the meridians toward that same pole while the sections of the meridians are equal, the difference will be greater between the greater inclinations than between the smaller. For let *a* be one of the poles of the world, and let two [segments] of the meridians

a b and *a d* be equal, neither of them greater than a quadrant, and let two others, *a c* and *a e*, be smaller than these but equal to each other. Let the great circle *b c f* be inclined to *a b* and *a c*; let a further great circle *d e g* be inclined to *a d* and *a e*;

and let the angle of inclination *a b c* be greater than the angle of inclination *a d e*. I say that the acute angle *a e g* — the inclination of the circle *d e g* upon *a e* — exceeds the acute angle *a d g*, the inclination of that same *d e g* upon *a d*, by less than the acute angle *a c f* exceeds the acute angle *a b c*.

That the angle *a e g* is greater than the angle *a d e*, and likewise the angle *a c f* greater than *a b c*, is clear from this: by hypothesis none of the given segments of the meridians is greater than a quadrant.

Diagram. An uncaptioned woodcut in the left column. A sheaf of lines springs from a single point *a* at the top of the figure and descends to the left and downward; two long arcs cross the sheaf from the lower left toward the upper right and end at the right-hand side of the block. The labels, with their positions on the leaf:

- *a* (apex, top centre-right): the pole of the world, from which the meridian segments run.
- *b* (foot of the figure, far left): the end of one meridian segment *a b*.
- *c* (above and to the right of *b*): the end of the meridian segment *a c*.
- *d* (foot of the figure, centre): the end of the meridian segment *a d*.
- *e* (above and to the right of *d*): the end of the meridian segment *a e*.
- *f* (right-hand edge, lower): the far end of the great circle drawn through *b* and *c*.
- *g* (right-hand edge, above *f*): the far end of the great circle drawn through *d* and *e*.

That the angle *a c f* is greater than the angle *a e g* follows in this way. In the triangle *a b c*, as the sine of the side *a b* is to the sine of the side *a c*, so is the sine of the angle *a c f* to the sine of the angle *a b c*. Again, in the triangle *a d e*, as the sine of the side *a d* is to the sine of the side *a e*, so is the sine of the angle *a e g* to the sine of the angle *a d e*. But *a b* and *a c* are equal to *a d* and *a e* respectively, each to each. Therefore, as the sine of the angle *a c f* is to the sine of the angle *a b c*, so is the sine of the angle *a e g* to the sine of the angle *a d e*; and so, by alternation, as the sine of the angle *a c f* is to the sine of the angle *a e g*, so is the sine of the angle *a b c* to the sine of the angle *a d e*. But the sine of the angle *a b c* is greater than the sine of the angle *a d e*, and therefore the sine of the angle *a c f* will be greater than the sine of the angle *a e g*. Since each of them is acute, the angle *a c f* will accordingly be greater than the angle *a e g*.

That this same angle *a c f* exceeds the angle *a b c* by a greater difference than *a e g* exceeds *a d e*, we shall show in another figure.

Diagram. An uncaptioned woodcut at the head of the right column: a circle with a vertical diameter running from *h* at the top down to *n* at the foot, crossed by four horizontal chords drawn from the diameter out to the right-hand arc. Two short perpendiculars stand on the lower chords. Labels as read on the leaf, from the top down:

- *h* (top of the circle, on the circumference): the point from which every arc is reckoned.
- *t* [?] (on the vertical diameter, uppermost of the marked points): the foot of the right sine *s t*.
- *s* (right-hand arc, level with *t*): the end of the arc *h s*, the arc of the angle *a d e*.
- *r* (on the vertical diameter, below *t*): the foot of the right sine *q r*.
- *q* (right-hand arc, below *s*): the end of the arc *h q*, the arc of the angle *a e g*.
- *u* (on the chord *q r*, toward its right end): where the perpendicular dropped from *s* meets *q r*.
- *y* (on the vertical diameter, below *r*): the foot of the right sine *z y*.
- *z* (right-hand arc, level with *y*): the end of the arc *h z*.
- *p* (on the vertical diameter, below *y*): the foot of the right sine *o p*.
- *o* [?] (right-hand arc, level with *p*): the end of the arc *h o*, the arc of the angle *a b c*.
- *n* (foot of the vertical diameter): the foot of the right sine *m n*.
- *m* (right-hand arc, lowest of the marked points): the end of the arc *h m*, the arc of the angle *a c f*.
- *l* (on the chord *m n*, near its right end): where the perpendicular dropped from *o* meets *m n*.
- *x* [?]: the text places it on *m n*, where the perpendicular from *z* falls; the letter itself will not resolve on the scan.
- *i* and *K* [?]: the text names the circle *h i K*, but neither letter is legible on the scan.

In the circle *h i K*, then, let *h m* be the arc of the angle *a c f* and *m n* its right sine, and let *h o* be the arc of the angle *a b c* and *o p* its right sine. Let *h q* further be the arc of the angle *a e g*, with right sine *q r*, and let *h s* be the arc of the angle *a d e*, with right sine *s t*. From the point *o* let the straight line *o l* be erected at right angles upon *m n*, and from *s* let *s u* be erected at right angles upon *q r*; and from *o* to *m*, and from *s* to *q*, let straight lines be drawn.

Now if the circumference *o m* is not greater than the circumference *q s*, it will be either equal to it or less. Suppose it equal: then the two straight lines *o m* and *s q* will be equal; but *o l* is greater than *s u*, so the remainder *m l* will be less than *q u*. And *l n* is greater than *u r*. Therefore *q u* will have a greater ratio to *u r* than *m l* has to *l n*, and for that reason the whole *q r* will have a greater ratio to *u r* than the whole *m n* has to *l n*. It follows that the right sine of the angle *a e g* will have a greater ratio to the sine of the angle *a d e* than the sine of the angle *a c f* has to the sine of the angle *a b c* — which is impossible, since we have demonstrated the ratio to be one and the same.

The circumference *o m* is therefore not equal to the circumference *q s*; nor is it less. For suppose it were less. Take the circumference *m z* equal to that same *q s*, and let *z y* be the right sine of the segment *h z*. From the point *z* to *m* let the straight line *m z* be drawn, and from that same *z* the straight line *z x* at right angles upon *m n*. By this you will show, by the same procedure, that *q r* has a greater ratio to *u r* than *m n* has to *x n*. But *m n* has a greater ratio to *x n* than it has to *l n*, since *l n* is greater than *x n*. Therefore *q r* will have a far greater ratio to *u r* than *m n* has to *l n*. Consequently the sine of the angle *a e g* will have a greater ratio to the sine of the angle *a d e* than the sine of the angle *a c f* has to the sine of the angle *a b c* — which is again impossible, since the contrary was shown above.

The difference *m o*, by which the angle *a c f* exceeds the angle *a b c*, is therefore greater than the difference *q s*, by which the angle *a e g* exceeds the angle *a d e*; and so the difference of the greater angles is greater than the difference of the smaller — which is what we undertook to demonstrate.

All this may be seen in the figure that follows and in the table of numbers we have drawn up, in which *a b* and *a c* are the segments of the meridians of the places *b*

We take the circumference of the horizon, and likewise the graduated face of the mariner's instrument, to be divided into 32 equal parts: that is, into 8 rhumbs, 8 half-rhumbs (which they call the half-inclinations or half-courses), and sixteen quarter-rhumbs.

Since — as it is fair to believe — the man at the helm does not feel an inclination increased or diminished by about two degrees, the amount being so small, we shall accordingly assume that the ship keeps to one and the same great circle until the original inclination has grown by two degrees, when the sailing is toward the parts of the visible pole; and that it then passes onto another great circle which makes good that small slippage of inclination, supposing we mean to

hold the same inclination and the same course throughout the voyage. For the ship's track is necessarily an angular one, and an inequality is necessarily found in the angles themselves.

We grant that an inequality of this kind is variable and inconstant. Still, while a matter remains unsettled, one must sometimes lay down the uncertain as though it were certain, and the gain is this: that what is altogether unknown becomes known in some part. Now the positions of the places set down on the sea chart are unknown, although their latitudes are known and the angles of departure are known. For the longitudes are unknown, and the angles of position between any two places are unknown as well, even though the inclinations of the tracks have been ascertained. Our table, however, will help us greatly toward finding the true longitudes of places and the angles of position.

Suppose, for example, that on the globe of land and sea the broken line *b c d e f g* has an inclination of one quarter-point to the segments of the meridians at the points *b c d e f g* themselves, and let the place *b* be set under the equinoctial. The angle of departure from the place *b* to *c* will then be 11 degrees 15 minutes, which is less by two degrees (as we supposed) than the angle *a c k*. So if the complement of the latitude of the second place is found to be 58 degrees 20 minutes, we shall be certain that the second place lies where *c* is.

Diagram. An uncaptioned woodcut filling the foot of the left column: a sheaf of curves converging on *a* at the upper right, with a nearly straight base line running away to the left. The labels, as read on the leaf:

- *a* (upper right, where all the curves meet): the pole.
- *b* (left end of the base line): the place set under the equinoctial, the start of the broken line.
- *c* (on the base line, to the right of *b*): the second point of the broken line.
- *d*, *e*, *f* (in order, rising to the right along the middle of the figure): the further points of the broken line *b c d e f g*.
- *g* (right-hand edge, above *f*): the last point of the broken line.
- *K* (below and to the right of *c*, at the foot of the figure): the point defining the angle *a c k*.

In that case the angle of departure *a b c* will also be the angle of position, and the direct interval will be *b c*. And so in the spherical triangle *a b c*, from *a b* and *a c* known, together with the acute angle *a b c* — the angle *a c b* being obtuse — the remaining angle *b a c*, which is the difference of longitude between those two places, will be known, and the direct interval *b c* itself will be known as well.

But if the complement of the latitude of the second place is found greater than 58 degrees 20 minutes, then the second place lies between *b* and *c*, and by the same method the difference of longitude and the travelled interval will be found. And if the complement of the latitude of the second place is found less than 58 degrees 20 minutes, then the second place lies beyond *c*.

The right sines of the segments *a b*, *a c*, *a d* and the rest are proportionals in continued proportion — that is, as the right sine of *a b* is to the right sine of *a c*, so is the right sine of *a c* to the right sine of *a d*, and so on — because the angles at the bases of the triangles are equal. We shall therefore multiply the right sine of the segment *a c*, of 58 degrees 20 minutes, into itself, and divide the product by the sine of the segment *a b*, namely 100000 parts; and there will come out in the quotient the right sine of the segment *a d*, from which the segment *a d* itself will be known at once from the table of sines.

If that segment is found equal to the complement of the latitude of the second place, the second place is where *d* is. Then in the spherical triangle *a c d*, from the two sides *a c* and *a d* known, together with the angle *a c d* — the angle at *c* being obtuse — the remaining angle *c a d*, which is the difference of longitude of the two places *c* and *d*, will be found. But the angle *b a c* had already been found by the same reasoning; so the whole angle *b a d*, that of the difference of longitude of the two places *b* and *d*, will be disclosed, and with it the circumference *c d*. The oblique interval of the voyage *b c d* will therefore be known. And if one wishes to know the direct interval, let a great circle be drawn through *b* and *d*: then in the spherical triangle *a b d*, from the two sides and the angle *b a d* known, the base *b d* will be found, and with it the angle of position *a b d*, which is other than the angle of departure.

But if the segment *a d* is found less than the complement of the latitude of the second place, the second place lies between *c* and *d*. In that case we shall determine the difference of longitude between it and the place *c* just as we said when it lay between *b* and *c*, and to that we shall add the difference of longitude of the two places *b* and *c*. The whole difference of longitude of the first and second place will thus be known.

The oblique interval too, and the direct, will be found in the manner already described. Nor shall we proceed any differently when the complement of the latitude of the second place exceeds the segment *a d*.

From all this you will understand how the difference of longitude of two places is to be investigated when *a b*, the complement of the latitude of the first place, is 80

degrees, or 70, and so on; and likewise when the angle of departure is other than the one of a single quarter-point that we assumed in this example. The table we have drawn up would be far more convenient if it were carried out by fives, or by threes, or by twos; or if it were built in such a way that, the segment *a b* being taken as 90 degrees, the remaining segments *a c*, *a d*, *a e*, *a f*, *a g* and so on — which are proportionals in continued proportion — were entered in the same table. And this for every inclination of the broken line, or magnitude of the angle of departure.

A table of that kind could be made with no more labour than the one we have drawn up. For at every inclination, or angle of departure, the common multiplier will be the right sine of that inclination, while the common divisor will be the right sine of the angle exceeding the angle of the given inclination by two degrees — if one chooses to subdivide in that way — or by one degree only, if you wish to handle the matter more exactly. For example, at the inclination of *Nordeste* and *Sudoeste*, or of *Noroeste* and *Sueste*, the common multiplier will be the sine of 45 degrees, and the common divisor the right sine of 47 degrees, or of 46 if you prefer.

Beginning then from the equinoctial, the whole sine will be the first number to be multiplied by the common multiplier; the product is then divided by the common divisor, and there comes out in the quotient the right sine of the segment *a c*. That in turn we shall multiply by the common multiplier and divide the product by the common divisor, and there will come out in the quotient the right sine of the segment *a d*. This right sine we shall then multiply by the common multiplier and divide the product by the common divisor, and there will come out in the quotient the right sine of the segment *a e*; and so we must work with the rest. Once the right sines of the several segments are known in this way, the segments themselves — which are the complements of the latitudes — will be known from the table of right sines. But since segments of this kind are innumerable, for the least of the proportionals cannot be assigned, it will be enough to extend such a table as far as 60 degrees of latitude.

And if, at every inclination, you set down beside the number of degrees and minutes of the complement of latitude the number of degrees and minutes of the angle *b a c* — that is, the difference of longitude between *b* and *c* — and the magnitude of the direct interval *b c* as well; and if you do the same beside the remaining segments of the meridians for the differences of longitude and the intervals between the angles of the broken line *b c d e f g*: this will be of great use

to us, not only for drawing the true longitudes out of the sea chart, but also for tracing on the globe lines like those a ship describes on the surface of the sea.

When, however, the complement of the latitude — either of the place you set out from or of the place you make — is not found exactly in the said table under your own angle of departure, you must work the proportion in no other way than if you were using astronomical tables.

Suppose, for the sake of an example, that the voyage has been made from the place *c* to the place *l*, which lies between *c* and *d*, under the given inclination of the angle *a b c*; and that in the table just described you have the segment *a c* as 72 degrees, *a d* as 63 degrees, the angle *c a d* of the difference of longitude between *c* and *d* as 6 degrees, and the interval *c d* as 10 degrees; and further that the complement of the latitude of the place *l*, found by observation, is 69 degrees.

Diagram. An uncaptioned woodcut in the right column, of the same design as the one on the preceding leaf: a sheaf of curves converging on *a* at the upper right above a base line running off to the left. The labels, as read on the leaf:

- *a* (upper right, where the curves meet): the pole.
- *b* (left end of the base line): the first place.
- *c* (on the base line, to the right of *b*): the place sailed from in the worked example.
- *l* (above *c*, the first of the row of letters across the middle of the figure): the place made, lying between *c* and *d*.
- *d*, *e*, *f* (in order to the right of *l*): the further points of the broken line.
- *g* (right-hand edge, below *a*): the last point of the broken line.

It will then be worth the trouble to find, by that same table, the difference of longitude between *c* and *l*, and also the direct interval *c l*. To do so, we shall set the difference of the two segments *a c* and *a d* — that is, 9 degrees — as the first term of the proportion. The second term will be the difference of longitude of those places *c* and *d*, namely 6 degrees. As the third term we shall put 3 degrees, the difference of the two segments *a c* and *a l*. We shall then multiply the third into the second and divide the product by the first, and there will come out from the division 2 degrees for the difference of longitude of the places *c* and *l*; and we shall find the interval *c l* by the same method.

— we shall find 3 degrees 20 minutes. For the first and third terms are the same as in the earlier operation, but as the second we shall set 10 degrees, the amount the interval *c d* contains. If, however, you prefer to work by an exact method, you

will call on the science of spherical triangles, just as you have been accustomed to do in constructing that same table.

Suppose, then, that two places have been laid down on the sea chart and that the difference of longitude between them has to be found. It can be gathered from sailors' reports, by the method we have set out. For either a voyage has at some time been made from the one to the other, or no one has ever sailed from the one to the other, but rather from some other single place to both of them. Now if a voyage has been made from the one to the other, and either from north to south or, conversely, from south to north, it is certain that those two places do not differ in longitude. But if the voyage was other than one made under a single meridian or along a single parallel, it will not be difficult — by what we have already taught — to find the difference of longitude from the angle of departure and from the latitudes of those same places. If, on the other hand, no one has ever sailed from one of the given places to the other, but rather from some one third place to the two given places, or from them to that third place, we shall then investigate by the same art the differences of longitude between that third place and the two places proposed. From these the difference of longitude of the two given places set down on the sea chart will become plain.

And so that you may learn the differences of longitude of a good many places with less trouble, one should take as the root place — the place with which all the rest are to be compared — one of the coastal places, or better one of the island places lying far out from the mainland, from which it is customary to sail to many provinces of the world. We assume, in operations of this kind, that the angles of departure are known. For either that traveller's instrument which the Spaniards call the mariner's needle [*acus nautica*] points out the cardinal points of the world correctly, and consequently the other directions as well; or, if it wavers — as experience has taught — then let it first of all be ascertained how great its deviation [*nutatio*] from the poles of the world is at every place.

¶ On the Declination of the Sun. Cap. 4.

In the table of the Sun's declination which they use for finding the latitude, the greatest declination ought not to exceed 23 degrees 30 minutes; so the table stands in need of correction. They go wrong besides in this, that they look up in that same table the Sun's declination by the day of the month, without knowing the degree of the zodiac the Sun is in. And it is agreed that their table cannot be true for several years running, since the declination cannot return to the same minutes, even when the equation is applied. They would do better, therefore, to find the Sun's true place first, by a table computed for four years, at the end of

which the equation will have to be used; and then, once the Sun's place is known, the declination should be drawn from the table of declinations. In that enquiry we judge that the difference of meridians may be neglected, unless it exceeds a span of six hours, or unless they are making the enquiry on those days on which the declination increases or decreases by a marked difference — that is, near the equinoctial points.

But however they choose to compute the Sun's declination, there is a far greater ambiguity in another matter. For in the tables sailors use it is laid down that in our own age, in a common year, the Sun has no declination on the eleventh day of March; and I see that this is not at all well agreed among learned mathematicians. Those who hold that the eighth sphere is moved with the motion of trepidation must necessarily grant — since the table of the Sun's motion is constructed to the ecliptic of the first movable, whose beginning is the immovable section — that the Sun, when set at the beginning of Aries and of Libra, very often has a declination from the equinoctial of the first movable, and consequently that it does not have its greatest declination at the beginning of Cancer. This is what Georg Peuerbach infers; and yet it must be denied by those who refuse to accept that motion of trepidation.

A difficulty of this kind could easily be resolved if at the summer solstice we observed the Sun's least distance from the zenith, and in the same place its greatest distance about the winter solstice, so that the exact distance between the tropics would be left known. If half of that distance — which is the greatest declination — be subtracted from the Sun's greatest altitude, the altitude of the equinoctial above the horizon of the place where the observation was made will be left known. Once that is known, you will indeed be able to work out easily on what day the Sun has no declination. For if, about the times of the equi-

noxes, you observe the Sun's meridian altitude, and do so day after day until it is found equal to the altitude of the equinoctial above the horizon, there will be no doubt that on that very day the Sun has no declination. Once the Sun's true place for that same day has been found, the very degree of the ecliptic of the first movable in which the Sun in our age has no declination will be known.

But for ease of teaching we shall call the vernal section of the ecliptic of the eighth sphere the beginning of Aries; and from it let the computation of the Sun's true place take its start, for finding the Sun's declination in our present age. On these assumptions, then, the latitudes of places can be truly concluded from the meridian altitude together with the Sun's declination.

Observations of this kind ought to be made no less by those who posit the said motion of trepidation than by those who deny that it exists in nature. For both parties use the tables and the computation of King Alfonso to find the true place of the Sun, of the Moon, and of the planets on any given day. That computation certainly could not have been so exact that nothing worth notice has been lost in these three hundred years that have run down to our own times.

A certain man, most diligent about the correction of chronology, gave too little heed to this. He had gathered from the Alphonsine tables that the Sun's entry into Aries in our time falls on the tenth day of March, whereas the vernal equinox was marked by Julius Caesar on the twenty-fifth day of the same month; and he therefore concluded that the length of the year assumed by Alfonso was false, on the ground that the fifteen days which fall between the two vernal equinoxes could not be completed except in 2000 years. He approves al-Battānī's opinion on the same matter, because it fills out those very 15 days. But he failed to notice that Campanus, in the year 1200 from the birth of Christ, had attacked that very opinion of al-Battānī's about the anticipation of the equinoxes in his *Great Computation*, by an exactly similar argument: that it would follow from it that the day of the winter solstice preceded the day of Christ's birth by two days. Besides, he did not see that the Sun's entry into Aries is referred to the immovable section on the ecliptic of the first movable, whereas the vernal equinox is referred to the movable section of the ecliptic of the eighth sphere. Since therefore he does not take the same terms in that computation, nothing can be concluded from it.

But suppose he now wishes to maintain that in Alfonso there is no movable section, and that the true vernal equinox falls at the point which is found by the tables at the beginning of Aries — though if account had to be taken of the motion of trepidation one would have to judge otherwise. Then the Alphonsine tables are true for showing the equinoxes, and consequently the length of the year which those same tables set down is true. And the point he takes as most certain — that in the age of Julius Caesar, namely 45 years before Christ, the vernal equinox fell on 25 March, in a leap year — is of greater doubt.

For if we believe Ptolemy, that most exact observation of the autumn equinox which he made in the seventeenth year of Hadrian took place 879 Egyptian years, 66 days and 2 hours after the beginning of the years of Nabonassar. But from the very beginning of Nabonassar's reign to the beginning of the years of Christ there elapsed (as Alfonso writes) 746 Roman years and 310 days. The said autumn equinox therefore fell in the year 152 [recte: 132] from the birth of Christ; for 131 Roman years, 268 days and 2 hours intervened, and that year was a leap year.

Accordingly, when the reckoning is made through the days of the months, it follows that that autumn equinox occurred on 24 September.

If, however, you follow the computation of Georg Peuerbach and Regiomontanus in the third book of the *Epitome*, you will find that it fell on the following day, that is, on the 25th of the same month. For they subtracted one day from the space of time which the Alphonsine tables make to have elapsed between Nabonassar and Christ, and added that same day to the interval between Christ and the said autumn equinox — which indeed agrees with what Georgius Valla [Giorgio Valla] reports out of Ptolemy on the rising and setting of the signs. For he writes that the autumn equinox was completed on 25 September, and the vernal on 22 March. Johannes Stöffler affirms the same in his *Calendar*.

We do find, however, in a certain little book on the significations of the fixed stars, translated from the Greek by Nicolaus Leonicus [Niccolò Leonico Tomeo], which he says is Ptolemy's, that the vernal equinox falls on 26 March in a common year. No credit is to be given to that book in this matter, since it puts the autumn equinox on 21 September — statements which cannot hang together and which contradict the observations. For Ptolemy showed that there are 187 days between the vernal equinox and the autumn. So if the vernal fell on 26 March, the autumn ought accordingly to have fallen on 29 September, not on the 21st.

It is plain, then, from what has been said, that in the year 132 from Christ's birth the vernal equinox fell either on 21 or on 22 March. In the year of the Conception, therefore, which was likewise a leap year, it must have fallen either on the 22nd or on the 23rd. And so, even if (as Giovanni Lucido himself says) in the year of our Lord 1545 the vernal equinox fell on the tenth day of March, still the anticipation of the equi-

noxes from the 45th year before Christ's birth cannot amount to 15 days.

Campanus, however, since he had embraced Thābit ibn Qurra's opinion about the length of the year and the motion of the fixed stars, asserts in his *Great Computation* that the vernal equinox fell on the day before Christ the Redeemer of the world was conceived in the Virgin's womb; yet at Rome it was celebrated on the very day of the Conception, that is on 25 March, according to Caesar's institution. For since Hipparchus and other astronomers had defined the length of the year as 365 days and a quarter, Caesar accordingly set aside the quarters of three years and added one day to the fourth, which he called the bissextile; and he reckoned that in those four years the Sun completed its course exactly. And because it had at some time been observed by the older astronomers that the

vernal equinox fell on a certain day of March which, by the form of the calendar he instituted, was the 8th before the Kalends of April in a bissextile year, he judged that it had from then on a fixed and unvarying seat — not that the Sun's entry into the vernal section had become known to Caesar by an observation made at the time.

As for his saying that King Alfonso had not read al-Battānī's work because it had not yet been translated into Latin, that is false. For Alfonso made full use of Arabic books, of which all Spain was at that time most full; and, helped by certain Moors of Toledo, he constructed his tables of the celestial motions. Indeed, in that great work written by him in Spanish, which is extant in the library at Alcalá, he set down the very tables that are in circulation, and the tables of Ptolemy and of al-Battānī as well, so that anyone might use whichever tables he pleased. But these things are better known than that we need to labour them.

I see that a good many astronomers of our own time go astray in much the same way. Although they follow the Alphonsine position on the motion of the starry sphere, they nevertheless derive a star's declination from the Sun's greatest declination in this age, from the star's latitude, and from its true place found by the tables; and conversely they seek the true place from the known declination — this in order to learn how far the fixed stars have advanced from the times of Ptolemy, or of Alfonso, or of others, down to the present. But they fail to notice that Ptolemy referred the beginning of the motion of the fixed stars to the movable section of the ecliptic, which he nevertheless supposed to be immovable. Therefore, whether in the Alphonsine tables their computation begins its reckoning from the movable section in which the vernal equinox falls, or from the immovable one, the same terms are not kept.

Yet it is agreed that these same authors compute the motions of the fixed stars from the vernal section, the angle of longitude of the spherical triangle being set up at the pole of the ecliptic of the eighth sphere, just as the *Tables of Directions* of Regiomontanus lay down. For if you place the Greater Dog in the seventh degree and 18 minutes of the sign Cancer, and give it a southern latitude of 39 degrees 10 minutes, then — the greatest declination of the Sun in our age being supposed 23 degrees 30 minutes, which is also that of the ecliptic of the eighth sphere — you will conclude that the declination of that same star is 15 degrees with 49 minutes, just as our own calculation showed in the book *On Twilights*, and as I also find in the common Ephemeris of Johannes Stöffler. And accordingly they do not refer the motion of the fixed stars to the beginning of Aries of the first movable, but to the section of the equinoctial and of the ecliptic of the eighth sphere.

al-Battānī did indeed find the motions of the fixed stars by that same art, but he was unaware of the said motion of trepidation — if it exists in the heaven at all. Johannes Werner of Nuremberg posited a double trepidation in the motion of the eighth sphere, so that what he had found by observation might agree with what had been found by Alfonso, by al-Battānī and by Ptolemy, and with what other more ancient astronomers had found. Most recently Nicolaus Copernicus of Toruń [*Torinœus*] devised another scheme to achieve the same end, but he makes no mention of what had been found by Alfonso. Which of the two is to be followed we frankly do not know. For they observed the fixed stars at very nearly the same time, and laid down the same greatest declination of the Sun, namely 23 degrees 28 minutes 30 seconds; and yet — whether because of the deceptiveness of their instruments, or because the latitudes of the places in which they made their observations were not sufficiently established — they disagree with each other. For Werner found the Spike of Virgo in 16 degrees 54 minutes of Libra, whereas Copernicus, using the same method, found it in 17 degrees 14 minutes of the same sign; and Girolamo Cardano says that he found that same star again in Italy, twenty-two years later, in 16 degrees 18 minutes, from eleven observations made by him.

We, meanwhile, although we make constant observations of the stars, do not yet have instruments we could use with confidence; and so, affirming nothing for certain, we side with al-Battānī. The writings of Marcus Beneuentanus [Marco Beneventano] have not come into our hands, but we have read the book on the equinoxes and solstices and the *Apologia* of Albert Pigge, who does not win as often as he thinks he wins. And since some have persuaded themselves that

he had clearly demonstrated, from the Alphonsine position, that in our own age the vernal equinox precedes the Sun's entry into the head of Aries of the Alphonsine tables by five days, it will now be worth the trouble to examine that very claim.

He tries first of all to show that the motion of the fixed stars found by Alfonso's tables does not agree with Ptolemy's observations — a point which Nicolaus Cusanus [Nikolaus von Kues] was the first to note. For if (he says) you subtract the motion of the eighth sphere between Ptolemy and Alfonso from the place of the star the Heart of the Lion [*cor Leonis*, Regulus] as observed by Alfonso, there will be left 4 degrees 20 minutes of that same sign; yet Ptolemy found that star in 2 degrees 30 minutes. But since he holds that Alfonso's computation begins from the beginning of Aries of the first movable on the fixed ecliptic, whereas Ptolemy began his calculations from the movable section of the ecliptic of the eighth

sphere, the only conclusion I see following is this: that in Ptolemy's time that same star stood in 4 degrees 20 minutes of Leo on the fixed ecliptic, and consequently that the vernal section then lay in the first degree and 50 minutes of Aries. For that reason the heads of Aries of the ninth sphere and of the first movable were far from conjunction at the time of Christ's nativity; and the vernal section is not in our own age, nor was it for many centuries before, in the sign of Pisces. Certain other consequences follow in their turn, in which there is perhaps an absurdity — though not the one he alleges, about one motion failing altogether to agree with another.

He says next that the compilers of Alfonso's tables determined some place for the head of Aries of the ninth; that the heads of Aries of the ninth sphere and of the first movable were in conjunction in the year of the Lord's incarnation; and that this is clear from Peuerbach and from all those who came after Alfonso. I cannot gather any such thing from Peuerbach himself. Indeed I think it obvious that, at whatever place we understand the head of the ninth to be, the motions of the fixed stars can be computed none the less, and that this is why no mention of the matter was made in the tables. As for the declination of the fixed ecliptic, which is in fact unknown, he takes it as known to himself at 23 degrees 51 minutes — yet lower down he makes it smaller. Since, then, it was from these and from other no less doubtful hypotheses about the intersection of the two ecliptics — the point at which he departs from Peuerbach — that he concluded from the Alphonsine position that the vernal section at that time lay at the beginning of the 26th degree of Pisces, what he was contending for was therefore not demonstrated by him.

In the matters he gathers by reasoning, moreover, he shows himself insufficiently practised in geometry. For he thinks that in spherical triangles the same ratio is not preserved between the right sines of the angles and those of the opposite sides, unless those opposite sides taken together are less than a semicircle. Besides, when he had set himself to find by demonstration how great the arc of the equator was between the two sections of the ecliptics in the year 16 from the Virgin's delivery — the head of Aries of the eighth being placed at the summit of the small circle — he assumed in that calculation that the angles of the two ecliptics with the equinoctial were equal to each other, namely 23 degrees 51 minutes, and he drew out the said arc as 21 degrees 10 minutes or thereabouts. But it does not seem to follow from this that the two sides of the triangle he conceived, which contain the angle opposite that same arc, are together equal to one semicircle — whereas he had concluded that they were less than a semicircle. And this he does more than once.

For he next enquires into the declination of the head of Aries of the ecliptic of the eighth for the year 263 from Christ's nativity, on the supposition that the declination of the fixed ecliptic is 23 degrees 51 minutes. Then again, from the declination so found, by Ptolemy's table of declinations — which supposes that same obliquity of the ecliptic — he investigates the arc of the ecliptic of that same eighth sphere between the same point and the movable section. In this way, then, he makes the two angles of the ecliptics with the equinoctial equal; and consequently two sides of the triangle taken together will be equal to one semicircle, though he had earlier demonstrated them to be less.

Oronce Fine fell into the same error. In canon 56 of the second book of his *On the Calculation of the Celestial Motions*, having set himself to find the distance of the vernal section of the movable ecliptic from the section of the fixed ecliptic, he seeks the declination of that same head of Aries from the true place and latitude of the head of Aries of the movable ecliptic, once these are known, by Problem 2 of Regiomontanus's *Tables of Directions*. Then, from the declination so found, he directs one to find the corresponding arc of that same movable ecliptic by entering the table of the Sun's declination by the arc. But since the declination tables themselves are constructed for the obliquity of one ecliptic only, namely 23 degrees 30 minutes, he seems on that account to suppose the obliquities of the ecliptics equal: he thinks, that is, that the angle *d b c* of the obliquity of the fixed ecliptic is equal to the angle *f a c* of the obliquity of the movable ecliptic — the exterior to the interior, in the figure he drew. From this it is inferred that the two arcs of the ecliptics which run from the sections *a* and *b* themselves as far as their western meeting-point are equal to one semicircle, which is impossible. For they are parts of two quadrants, terminated at that great circle which

passes through the poles of the ecliptics.

Albert further denies that the latitude of a region can be known otherwise than through the Sun's place or its declination, and hence that the time of the vernal equinox can be known from the Sun's meridian altitude while the Sun's place is unknown — exactly as Marco Beneventano maintained. But surely no method can be better suited to determining the equinoxes. For from the greatest and least altitude of the Sun found in a region, the distance between the two tropics becomes known; and if half of that distance be subtracted from the greatest altitude, or added to the least, you will know the altitude of the equator above the horizon, which is the complement of the region's latitude. Accordingly, when the Sun has that much meridian altitude above the horizon, you will conclude that it

is on the equinoctial circle. This is how Regiomontanus, in the third book of the *Epitome*, directs one to observe the equinoxes.

The demonstration, moreover, which the same Albert brought forward out of Marco Beneventano, to show that the equations of the motion of the eighth sphere written in Alfonso's own tables are arcs of the ecliptic of the eighth, is entirely certain — provided only that we understand the theory of that motion as Peuerbach handed it down: namely, that the great circle passing through the poles of the two ecliptics always passes through the head of Aries of the ninth. Johannes Werner demonstrated the same in his book *On the Motion of the Eighth Sphere*, and it was noted by Regiomontanus in problem 62 of the *Tables of the First Movable*. Albert nevertheless thinks that the poles of the ecliptics and the head of Aries of the eighth are always on the same great circle, and that this appears at once if, with one sphere enclosed within another, the head of Aries of the eighth is carried round on a small circle; and in this way he reckons that Marco's demonstration is overthrown.

But by that very same instrument all the phenomena can be shown which, on Peuerbach's exposition, follow upon this motion of access and recess — and others besides, which agree with neither position. For suppose you understand the eighth sphere to be moved in such a way that its ecliptic always touches the small circle at the very beginning of Aries which is carried round about that same small circle, and not only when that same beginning of Aries has been placed at the northernmost or at the southernmost point: you will then behold another figure of the motion, one that agrees with neither position. But if, while the head of Aries of the eighth is carried round on the small circle, you require the ecliptic of the eighth to intersect the ecliptic of the ninth at the beginnings of Cancer and Capricorn of that same eighth, then one and the same great circle will certainly pass through the head of Aries of the eighth and through the poles of the ecliptics, and the figure of the motion obtained will be the one Albert handed down. If, on the other hand, the intersection be made at the beginnings of Cancer and Capricorn of the ninth, then the poles of the ecliptics will always lie on the great circle that comes through the beginning of Aries of the ninth, exactly as Peuerbach handed it down. His theory of the motion of access and recess of the starry orb seems to agree better with the tables themselves.

Let *a*, then, in the figure below, be the head of Aries of the ecliptic of the ninth, and *b* the head of Aries of the eighth, which is to be understood as placed in the first quadrant of the small circle; and let *h* be the northernmost point on that same circle. On the ecliptic of the ninth let *c* be the beginning of Capricorn and *K*

that of Cancer. Let a great circle come through *b* and *c*, cutting the arc *a h* at *e*. Then by Theodosius's demonstrations in the first book *On Spheres*, the arc *b c* will be greater than a quadrant, and the angles at the point *e* will be right angles. Accordingly, by Peuerbach's theory, the ecliptic of the eighth will have the position *b e c*. Now from the point *b* let an arc of a great circle *b g* descend at right angles upon the ecliptic of the ninth; let *d g* be a quadrant; and through the points *b* and *d* let a great circle come cutting the arc *a h* at *f*. The arc *b d* will therefore be a quadrant, and the angle *d b g* will be a right angle; and consequently, on Albert's imagining, the ecliptic of the eighth will have the position *b f d*. For when the head of Aries is at *b*, the head of Capricorn will be at *d*.

Diagram. A small circle drawn about the point *a*, crossed by two long, gently curved arcs that run the full width of the page from *c* on the left to *K* on the right. The nearly straight of the two long arcs — the ecliptic of the ninth — passes through *d*, *a* and *g*; the other cuts across the small circle through *b*. Within and about the small circle the lettering runs as follows.

- *h* (above the small circle, at its top): the northernmost point of the small circle.
- *b* (on the circumference of the small circle, upper right): the head of Aries of the eighth, in the first quadrant.
- *e* (inside the small circle, upper left of *b*): where the great circle through *b* and *c* cuts the arc *a h*.
- *f* (inside the small circle, below *e*): where the great circle through *b* and *d* cuts the arc *a h*.
- *a* (at the centre of the small circle): the head of Aries of the ecliptic of the ninth.
- *g* (just to the right of *a*, on the long arc): the foot of the perpendicular arc *b g*.
- *z* (immediately right of and below *g*): the eastern point of the small circle.
- *c* (far left, on the long arc): the beginning of Capricorn on the ecliptic of the ninth.
- *d* (on the long arc, left of the small circle): the point such that *d g* is a quadrant.
- *K* (far right, on the long arc): the beginning of Cancer on the ecliptic of the ninth.

The equation, then, which in the tables answers to the arc *b h* is either *b e*, or *b f*, or finally *a g*. But it is plain that the Alphonsine table [*Abacus Alphonsinus*] agrees

with the quantity of the arc *b e*; the other two are greater. For the angle *b f e* is acute, and therefore *b f* will be greater than *b e*; the angle *g b K* is likewise acute, and therefore *g K* will be less than *b K*. When these are subtracted from the quadrants *a K* and *e K*, *b e* will be left less than *a g*. And so the position of the ecliptic *b e c*, on Peuerbach's teaching, agrees better with Alfonso's tables than the position of the ecliptic *b f d* which Albert devised.

In this, however, Peuerbach departed from Alfonso: he set down the arc *a g* as the equation which is written in the tables, whereas it is rather *b e*. And we do not suppose that he was unaware of this. Perhaps, rather, he noticed that the true equation of the motion of the eighth sphere is an arc of the ecliptic of the ninth — since it is on that ecliptic that the mean motion of the apogees and of the fixed stars is computed — but that the difference between it and the arc of the ecliptic of the eighth is extremely small; and that the compilers of the tables therefore calculated the equations on the ecliptic of the eighth itself, because they could do so with less labour.

For as the whole sine is to the right sine of the angle *b a e*, which subtends the mean motion, so is the right sine of the arc *a b* to the right sine of the arc *b e*. We shall therefore multiply the right sine of the arc *a b* — that is, of nine degrees — into the right sine of the arc of the angle of the mean motion of access and recess; and from the product we shall throw away the last five figures, if we use a table that supposes the radius to be 100000 equal parts; and the right sine of the arc *b e* will come out in the quotient. By the table of right sines, then, the arc *b e* itself will be known. It was assuredly by this art that the aforementioned table of equations was composed, and from it you will be able to draw out how great the arc *b g* is, the latitude of the head of Aries of the eighth.

Indeed, if you understand the point *h* to be the northernmost and *z* the eastern, then the arc *b e* will be the equation of *h b*, while *b g* is the latitude of the point *b*. If on the contrary you conceive *h* as the eastern point and *z* as the northernmost, then *b g* will be the equation of the arc *b z*, and *b e* the latitude of that same point *b*. When, therefore, *h* is supposed to be the northernmost point, you should enter the table of the equation with the number of degrees which *b z* contains — that is, with the complement of the quadrant — and the equation answering to it will be the latitude of the point *b*.

This rule, however, will not be able to serve those who define the equations of the eighth sphere as arcs of the ecliptic of the ninth; yet Albert Pigge used it none the less. You will see his slip at once if you place the point *b*, the head of Aries of the eighth, in the middle of the quadrant, between *h* and *z*. Then *h b* and *b z* will be

equal; and the arc *a g* is in the tables (as he himself supposes) the equation of the arc *h b*. If, therefore, you enter the table of equations with the number of degrees which *b z* contains, you will meet with the equation *a g*; and consequently the arc *b g*, the latitude of the point *b*, will on Albert's view be equal to *a g*. But you will conclude that they are unequal on this ground: that in every spherical triangle made up of arcs of great circles, its three angles are greater than two right angles. Now the angle *g* of the triangle *a g b* is right, and the angle *g a b* is half a right angle; the remaining angle *a b g* will therefore be greater than half a right angle, and for that reason *a g* will be greater than *b g*. They are therefore not equal.

That he did not sufficiently understand the very demonstration he brought forward from Marco appears from this: he thinks that the right sine of that arc of the ecliptic of the ninth which, according to Peuerbach, is the equation in Alfonso's tables, is equal to the right sine of the argument of the motion of the eighth sphere. But that same sine of the argument is the right sine of the arc which Beneventano holds to be the equation in those same tables. The sines of the equations of Beneventano and of Peuerbach will therefore be equal to each other; and since each arc is less than a quadrant, the arcs themselves will therefore be equal — although they were shown to be unequal by that aforementioned demonstration of Beneventano's.

Albert, then, was misled by his want of skill in geometry. In the quadrant of the small circle *a b c d*, whose centre is *a* and whose pole is *g*, let *d* (he says) be the point of northern latitude, and *a f b* the semidiameter, which is the right sine of the arc *b h g* of nine degrees of the fixed ecliptic. The head of Aries of the eighth being therefore placed at *c*, *c d* will be the argument of the motion of the eighth sphere, whose sine *c e* is perpendicular to the semidiameter *a e d*.

Diagram. A quadrant of a circle drawn in perspective, its arc running from *d* at the upper left through *c* at the top down to *b* at the lower right, with straight radii *a e d* and *a f b* meeting at the corner *a*. Inner lines join *c* to *e* and to *f*, so that *a e c f* stands out as a quadrilateral; further lines run out to the left to *g* and *z*. The lettering reads:

- *d* (upper left, at the end of the arc): the point of northern latitude.
- *c* (top, on the arc): the head of Aries of the eighth.
- *e* (left, on the radius *a d*, below *d*): the foot of the sine *c e*.
- *h* (in the middle of the figure): the point through which the arc of the great circle passes.
- *g* (far left, low): the pole.

- *z* (below and right of *g*).
- *a* (bottom left): the centre.
- *f* (bottom centre, on the radius *a b*): the foot of the sine *c f*.
- *b* (bottom right, at the end of the arc).

Therefore *c e* is parallel to the semidiameter *a f b*. Moreover *c f*, the sine of the arc *b c*, is perpendicular to the semidiameter *a f b*. Accordingly the quadrilateral *a e c f* is a parallelogram, and indeed a rectangle, and *a f* is equal to *c e*. But the sine *c f* is also the right sine of the arc *c h* of the great circle passing through the poles of the fixed ecliptic and the head of Aries of the eighth — which arc is the latitude of the head of Aries of the eighth from the fixed ecliptic.

So far Albert takes true premises and reasons correctly; but let us look at what follows. "Therefore," he says, "from the point *h* of the fixed ecliptic, through which the arc of the said circle passes, the straight line *h f* descending to the point *f* is perpendicular both to *c f* and to *a f*, those straight lines." For so he judges. And since the straight line *a g*, coming from the pole *g* to the centre *a*, is also perpendicular to *a f*, he concludes that *a g* and *f h* are parallel. But the straight line *a f* is parallel to *h z*, the right sine of the arc *g h*. It follows, therefore, he says, that *a z h f* is a parallelogram; and so he concludes that it is equal

to *a f*; and consequently that the sines *h z* and *c e* are equal to each other, by the common notion.

But this is just where Albert goes wrong: he supposes that *h f* is perpendicular to *a f*, or else parallel to the straight line *a g*. For the straight line *h f* itself lies on the common section of the plane of the great circle *c h* and the plane of the ecliptic *g h b*; produced in a straight line, therefore, it will pass through the centre of the sphere. In the same way, the straight line *a g* lies on the common section of the plane of the ecliptic and of the great circle coming through *d* and *g* — or else, because it joins the centre of the small circle to the pole of that same circle, it too, produced in a straight line, will pass through the centre of the sphere itself. Hence *f h* and *a g* meet in that same centre, and for that reason they are not parallel; nor is the angle *a f h* a right angle, but rather obtuse. It is in fact equal to the one right angle at *a* taken together with the one acute angle at the centre of the sphere which, because the two lines *a g* and *f h* meet there, subtends the arc *g h*. The sine *h z* is accordingly shown to be greater than *a f*, and therefore greater than *c e*; and consequently the equation on the ecliptic of the ninth is greater than the equation on the ecliptic of the eighth, exactly as Beneventano had demonstrated.

Albert, however, took the sine of the equation assigned by Alfonso to be the sine of the argument to which that equation answers, while he supposed that the sine of the equation defined by Peuerbach is equal to the sine of the argument. Yet whether you compute the equations to the ecliptic of the ninth or to the ecliptic of the eighth, you will find a very slight difference, one that will perhaps never exceed a single whole minute. The reason is this: as the right sine of the arc *d e* — the equation, that is, conceived on the fixed ecliptic — is to the sine of the arc *b t*, the equation on the ecliptic of the eighth (for let us use the diagram which Albert took from Marco), so is the whole sine to the sine of the arc *b l*, that is, of the complement of the latitude of the head of Aries of the eighth. But this ratio is always less than the one the whole sine has to the sine of 81 degrees; and that difference is nevertheless very small, for *b l* is greater than *l g*. Should you nevertheless wish to compute to the fixed ecliptic, you will learn from the argument *b g* the arc *a b*, which is left over from the quadrant; and if with it you enter

Diagram. The figure Nunes says Albert took over from Marco Beneventano, cut small at the foot of the left-hand column with the text wrapping round it. A long arc sweeps in from the upper left and curves down to the lower right; a second, shorter arc crosses it, and from the crossing-point at *b* several short lines radiate. The lettering runs as follows.

- *a* (top, on the long arc): the endpoint of the arc *a b* left over from the quadrant.
- *e* (immediately right of *a*): the other endpoint of the arc *b e*, the latitude of the head of Aries of the eighth.
- *d* (upper right, above the crossing): the endpoint of the arc *d e*, the equation conceived on the fixed ecliptic.
- *b* (centre, where the arcs cross).
- *t* (right of *b*, on a short radiating line): the endpoint of the arc *b t*, the equation on the ecliptic of the eighth.
- *g* (below and right of *t*).
- *l* (at the foot of the figure, on the long arc).
- one further letter at the far left, on the long arc: does not resolve on the scan [?].

the Alphonsine table of the equation, you will learn the arc *b e*, the latitude of the head of Aries of the eighth.

Next you will multiply the right sine of 81 degrees into the whole sine by adding five ciphers [*ziphrae*], if you are using a table that supposes the radius to be of 100000 parts. Divide the product by the sine of the arc *b l*, that is, of the complement of the latitude *b* [?], and there will come out in the quotient the right sine of the complement of the arc *d e*; and so *d e* itself will be known. The demonstration of this operation is that, in the spherical triangle *b d e*, since the angle *b e d* is a right angle and each of its sides is less than a quadrant, it will follow that as the whole sine is to the right sine of the complement of the arc *b e*, so is the sine of the complement of *d e* to the sine of the complement of the arc *b d*. Therefore, if the product of the first into the fourth be divided by the second, the third will come out of the division — the right sine, that is, of the complement of the arc *d e*. The arc will accordingly be known by the table of right sines, and *d e*, which is left over from the quadrant, will be known as well.

The same Albert sets the declination of the fixed ecliptic at 22 degrees 45 minutes, and this is his argument. Having supposed the declination of that same fixed ecliptic to be 23 degrees 51 minutes, he gathers by many and various arguments that the declination of the movable ecliptic for the year 1519 is 24 degrees 56 minutes [?]. Then he reasons in this fashion: of such parts as the declination of the fixed is set at 23 degrees 51 minutes, the declination of the movable is shown to be 24 degrees 36 minutes in the said year; therefore, of such parts as the declination of the movable was 23 degrees 28 minutes, the declination of the fixed is 22 degrees 45 minutes, by the sixteenth proposition of the sixth book of Euclid (*Elements* VI.16), with the help of the table of the right sine. Now at that same time the declination of the movable was 23 degrees 28 minutes; the declination of the fixed therefore contains 22 degrees 45 minutes.

But in the first syllogism he assumes two things he has not demonstrated: that the heads of Aries of the eighth and of the ninth sphere were in conjunction at the time of Christ's nativity, and that the figure of the motion of the eighth sphere according to Alfonso is other than the one handed down by Peuerbach. Besides, in that very same syllogism he takes as known to himself the declination of the movable ecliptic — the very thing proposed as unknown — at 23 degrees 30 minutes, since that is the amount Regiomontanus's table of declinations gives; and so he goes wrong. The second syllogism, moreover, is sophistical, for that sixteenth proposition of Euclid's sixth book cannot be fitted to the arcs of the angles of a triangle. Suppose you conceive, for the year 1519, such a constitution of the sphere as the figure he drew represents, so that

f b d is the fixed half-ecliptic and *f a d* the movable, and let the arc of the equinoctial *a b g* cut the movable at *a* and the fixed at *b*. The angle *d* will therefore be known by Alfonso's tables, and the side *b d* will be made clear as well by what Albert himself supposes. If, then, the angle *d b g* of the declination of the fixed ecliptic be brought in as known, the remaining sides of the triangle *a b d*, together with the remaining angles, will become known, and the triangle itself will be entirely given. Consequently, if — keeping the angle *d* and the side *b d* — you set the angle of the declination of the fixed smaller than *d b g* itself, the angle of the declination of the movable must necessarily become smaller too.

Diagram. The first of two woodcuts in the left-hand column, the text wrapping round it. Two long arcs run down the page and meet at top and bottom, enclosing a narrow lune; a third arc crosses them.

- *d* (top, where the two long arcs meet): the pole-side meeting-point of the two half-ecliptics.
- *g* (upper right, on the crossing arc): the far end of the equinoctial arc *a b g*.
- *b* (middle, on the right-hand arc): where the equinoctial cuts the fixed half-ecliptic *f b d*.
- *a* (middle, on the left-hand arc, below *b*): where the equinoctial cuts the movable half-ecliptic *f a d*.
- *f* (at the foot, where the two long arcs meet again).

The equinoctial, however, will have another position, *c b k*, and another triangle will be got, *c b d*. Now if the arcs of the four angles are proportional — so that as the arc of the angle *d b g*, of the declination of the fixed, is to the arc of the angle *d a b*, of the declination of the movable, in the former arrangement, so in the latter is the arc of the angle *d b K*, of the fixed, to the arc of the angle *d c b*, of the movable — then, three of them being known, the fourth arc will become known, by that same sixteenth proposition of the sixth book.

Diagram. The second woodcut in the left-hand column, repeating the first figure with the equinoctial in its second position. Two long arcs meet at top and bottom; two further arcs cross them, one running out to *K* and one to *g*.

- *d* (top, where the two long arcs meet).
- *K* (upper right): the far end of the equinoctial in its second position, *c b k*.
- *g* (right, below *K*): the far end of the equinoctial in its first position, *a b g*.
- *b* (middle, where both crossing arcs meet the fixed half-ecliptic).
- two further single letters low on the left-hand side, close together on the movable half-ecliptic: the upper reads *a*, the lower does not resolve on the

scan [?]. By the argument they are *a* and *c*, the cuts made by the equinoctial in its first and second position respectively.

- *f* (at the foot, where the two long arcs meet again).

But that the said arcs are proportional cannot be shown from that same sixteenth proposition. Albert therefore reasons wrongly upon the differing angles of two triangles: of such parts as the declination of the fixed is set at 23 degrees 51 minutes, the declination of the movable was found in the year 1519 to be 24 degrees 36 minutes; therefore of such parts as the declination of the movable was 23 degrees 28 minutes, the declination of the fixed is 22 degrees 45 minutes, by that sixteenth proposition of the sixth book.

The table of the right sine, moreover, can be of no use for inferring this — indeed it is impossible that the right sines of those same angles should be proportional. For as the right sine of the angle *d b g*, of the declination of the fixed, is to the sine of the angle *d a b*, of the declination of the movable, in the former arrangement, so is the sine of *a d* to the sine of *b d*; and again, in the latter arrangement, as the sine of the angle *d b K*, of the declination of the fixed, is to the sine of the angle *d c b*, of the declination of the movable, so is the sine of *c d* to the sine of *b d*. But the sine of *a d* has a greater ratio to the sine of *b d* than the sine of *c d* has to that same *b d*, because, since each of the arcs *a d* and *c d* is greater than a quadrant, the sine of *a d* will be greater than the sine of *c d*. The sine of the angle *d b g* will therefore have a greater ratio to the sine of the angle *d a b* than the sine of the angle *d b K* has to the sine of the angle *d c b*; they are therefore not proportional.

Now if, with no change made in the triangle *a b d* itself, Albert should wish to reason in this fashion — the angle *d b g* having 23 degrees 51 minutes, the angle *d a b* will be 24 degrees 36 minutes — then if, with no change made in the sides and angles, that same angle *d a b* be conceived as 23 degrees 28 minutes, the first angle *d b g* will be understood as 22 degrees 45 minutes. Besides the manifest impossibility which such an argument involves, another absurdity follows: that those proportional arcs of the four angles have proportional right sines, and in the very ratio which holds between the sines *a d* and *b d*. Yet the same table of right sines shows the opposite.

Besides, why will it not be permissible to argue in the same way about two interior angles of the same triangle? Of such parts, namely, as the angle *a b d* is set at 156 degrees 9 minutes — for that is what is left when the angle of the declination of the fixed is subtracted from two right angles — of such parts was the angle *d a b*, of the declination of the movable ecliptic, found in the year 1519 to be 24 degrees 36 minutes. Therefore, of such parts as the declination of the

movable was 23 degrees 28 minutes, that same angle *a b d* is 148 degrees 57 minutes, by that same sixteenth proposition of the sixth book of Euclid. But the angle of the declination of the movable ecliptic was 23 degrees 28 minutes, from Peuerbach's observations. Therefore the angle *a b d* is 148 degrees 57 minutes, and consequently the declination of the fixed contains 31 degrees 3 minutes — which by a like argument he concludes to be 22 degrees 45 minutes. A contradiction, then.

Albert's sophism will be plainly dissolved, and the fallacy of the argument laid open, once its sense has been made plain; and its sense is certainly this. If the arc of the declination of the fixed ecliptic has 23 degrees 51 minutes, then the arc of the declination of the movable ecliptic in the year 1519 was 24 degrees 36 minutes. Accordingly,

this arc of the declination of the movable ecliptic being divided into parts somewhat larger, and for that reason fewer — so that they are, namely, 23 with 28 sixtieths of one part — there will be in the arc of the declination of the fixed twenty-two of those same parts with 45 sixtieths, by the common rule of proportional numbers. This much is rightly inferred from what has been laid down. But what he says further — that the declination of the movable ecliptic was found to be smaller, since it is 23 degrees 28 minutes, and that the declination of the fixed therefore has only 22 with 45 minutes — this cannot be concluded from the foregoing; only that it is of 22 parts with 45 sixtieths, though those parts are a little larger than degrees. And if in the year 1519 the declination of the movable ecliptic was found to be 23 degrees 28 minutes, the only thing he could conclude was that the declination of the fixed is not 23 degrees 51 minutes.

Its quantity is in fact easy to find from the very things the same Albert supposed. For if in the figure above an arc of a great circle *b n* be drawn from the point *b* at right angles upon *a d*, then in the right-angled triangle *b n f* the side *b f* will be known: for when the arc *b l* of 19 degrees 56 minutes is subtracted from the quadrant *l f* — as Albert computed by Alfonso's tables for the year 1519 — *b f* will be left known. The angle *f*, too, is known, because the arc *h l*, the latitude of the head of Aries, divides the half-ecliptics into equal parts, according to the same Albert, and it is 1 degree 59 minutes [?]. The side *b n* will therefore become known by a single syllogism. Next, from the complement of *b n* itself and the complement of the angle *f*, the angle *f b n* will be made clear. In exactly the same way, from the same complement of the side *b n* and the complement of the angle *n a b*, of the known declination of the movable ecliptic — namely 23 degrees 28 minutes — the angle *n b a* will be known; and this we shall subtract from the

known angle *f b n*, and the angle *a b f*, of the declination of the fixed ecliptic, will be left known for the year in question, namely

Diagram. A woodcut in the left-hand column, the text wrapping round it: a long, gently curved arc running the width of the figure from *d* at the lower left to *f* at the right, with a second arc descending steeply and cutting it, and a third short arc above.

- *g* (at the top of the figure, above the crossing).
- two further letters immediately below *g*, set close together: neither resolves on the scan [?] [?].
- *b* (on the long arc, left of centre): the point from which the perpendicular arc *b n* is drawn.
- *n* (on the long arc, right of *b*): the foot of that perpendicular.
- *a* (on the long arc, immediately right of *n*).
- *d* (lower left, at the left end of the long arc).
- *f* (far right, at the right end of the long arc).

22 degrees 44 minutes — a declination of the fixed which comes very close (I grant it) to the one Albert found, but which has been found by more reliable syllogisms.

Now if for Ptolemy's time the arc *b l* be put (as he himself judges) at 2 degrees 2 minutes of Aries of the first movable, and the arc *h l*, the latitude of the head of Aries of the eighth, at 8 degrees 56 minutes 28 seconds, then the arc *b f*, which is left over from the quadrant, will be 87 degrees 58 minutes, and the angle *f* will be 8 degrees 56 minutes 28 seconds. The angle *n a b*, further, of the declination of the eighth, was at that same time 23 degrees 51 minutes 20 seconds. The angle *a b f*, therefore, of the declination of the fixed ecliptic, will be found by like syllogisms to be 21 degrees 51 minutes 40 seconds — the quantity which we found earlier by the same method to be 22 degrees 44 minutes, and Albert 22 degrees 45 minutes.

And since no greater trust is to be placed in Peuerbach's observations than in Ptolemy's when the Sun's greatest declination is under investigation, it is plainly rash of Albert, in his account of the Alphonsine position on the motion of the eighth sphere, to have set the declination of the fixed ecliptic at 22 degrees 45 minutes. For from the hypotheses he accepted about the meeting of the head of Aries of the ninth and of the tenth sphere in the year of the Lord's incarnation, it follows no less that the declination of the fixed is 21 degrees 51 minutes 40 seconds than that it is 22 degrees 44 or 45 minutes.

Beneventano, moreover — who (as Albert says) thinks the declination of the fixed ecliptic to be as great as the declination of the movable ecliptic which Ptolemy found, and who yet placed the head of Aries of the ninth in the year 1519 at 28 degrees 8 minutes of Pisces — is in open conflict with himself, as we shall show presently. For let *a b c* be the northern half-ecliptic of the first movable, cutting the equinoctial at the point *a*, the beginning of Aries, and at *c*, the beginning of Libra. Let the northern half-ecliptic of the eighth sphere, in Ptolemy's time — that is, in the year 140 after the birth of Christ the Redeemer — have had the position *d b e*: the vernal section, then, was *d*, the autumnal *e*. The angle *d b a* had 8 degrees and about 56 minutes, for so great was at that time the latitude of the head of Aries of the eighth, than which the arc of that same angle *d b a* — the arc, that is, which cuts the half-ecliptics through the middle between *b* and the opposite point — was insensibly greater. The angle *a b e*, therefore, is left at 171 degrees 4 minutes. And since, according to Marco Beneventano, the two angles *b d e* [?] and *b a e* [?] were equal to each other, while the angle *b e a* is equal to *b d e* [?] itself, the two angles *b a e* and *b e a* will therefore be equal to each other by the common notion. Moreover the arc of the great circle *b f*, at right

Diagram. A wide, shallow figure. A nearly straight base line runs from left to right, carrying the labelled points *d*, *a*, *f* and *c* in that order. From the apex *b* at the top of the figure an outer arc springs upward from the left, near *d*, passes through *b*, and descends to the right, ending beyond *c*; an inner arc joins *a* to *b*; and a short straight line drops from *b* to *f* on the base. The base line is the equinoctial, *a b c* the half-ecliptic of the first movable, the outer arc through *b* the half-ecliptic of the eighth sphere, and *b f* the perpendicular let fall on the equinoctial.

- *b* (apex): the common point of the two half-ecliptics
- *d* (base, far left): vernal section of the eighth sphere
- *a* (base, left of centre): beginning of Aries of the first movable
- *f* (base, centre): foot of the perpendicular from *b*
- *c* (base, right): beginning of Libra
- [?] (right-hand end of the outer arc): a single letter, not resolved on the scan

— angles, meets the equinoctial at the point *f*. The two angles *a b f* and *e b f* will therefore be equal to each other, and so the angle *a b f* will be 85 degrees 32 minutes. The angle *b a f*, on Beneventano's hypothesis stated above, has 23 degrees 51 minutes 20 seconds — the value Ptolemy found for the Sun's greatest declination — so its complement will be 66 degrees 8 minutes 40 seconds.

Now the whole sine is to the sine of the complement of the side *b f* as the sine of the angle *a b f* is to the sine of the complement of the angle *b a f*. By the rule of proportional numbers, then, and by the table of right sines, the complement of the arc *b f* is found to be 66 degrees 32 minutes 30 seconds; the arc *b f* itself is therefore 23 degrees 27 minutes 30 seconds.

Since the side *b f* is now known, together with the angle *b a f* opposite it, and with the whole sine intervening, the sine of the side *a b* will be disclosed by that same common rule of proportional numbers: namely 98430 parts, where the semidiameter is set at 100000. But *a b* is less than a quadrant, because *a f* is less than a quadrant, and *b f* likewise less than a quadrant. By the table of right sines, therefore, the arc *a b* is found to be 79 degrees 50 minutes.

Now the beginning of Cancer of the ninth ecliptic lies at the point *b*, the common intersection of that ecliptic with *d b*, the ecliptic of the eighth sphere. In Ptolemy's time, then, the head of Aries of that same ninth sphere stood 10 degrees 10 minutes before *a*, the beginning of Aries of the first movable — that is, at 19 degrees 50 minutes of Pisces. And since the motion of the ninth sphere from the year 140 to the year 1519 is 10 degrees and nearly 8 minutes, the head of Aries of the ninth stood in that year 1519 at nearly 29 degrees 58 minutes of the same sign, a mere 2 minutes before the head of Aries of the first movable. It was therefore not at 28 degrees 8 minutes of Pisces, which is what Albert Pigge says Marco Beneventano maintained.

Nor could Pigge say without absurdity that in the said year 1519 the head of Aries of the ninth sphere stood two minutes before the head of Aries of the first movable, since it follows that after a few more years the two heads of Aries must have coincided. The arrangement of the spheres at that time must therefore have been as follows.

Diagram. A second figure, set in the right-hand column. A line descends from *c* at the top through *e* to *a* below: this is the great circle *a c* drawn through the poles. A long, gently curving line runs from *d* at the lower left through *a* and on down to *f* at the lower right — the equinoctial *d a f*. A second curve rises from *d*, crosses the first line at *e*, and continues to *b* at the upper right: this is the ecliptic of the eighth sphere, *d e b*.

- *c* (top): pole-ward end of the great circle *a c*
- *e* (centre): intersection of the eighth ecliptic with the great circle *a c*
- *d* (lower left): point where the eighth ecliptic cuts the equinoctial

- *a* (lower centre): beginning of Aries of the ninth ecliptic and of the first movable
- *b* (upper right): point on the eighth ecliptic
- *f* (lower right): point on the equinoctial

Take *a* as the beginning of Aries of both ecliptics, the ninth and that of the first movable; let *a b* be a quadrant; and let the great circle *a c* pass through the poles of the eighth ecliptic and of the first movable. The eighth ecliptic then had necessarily to lie along *d e b*, so that *d* is the point at which it cuts the equinoctial *d a f*, and *e* the point where it intersects *a c*.

e b is therefore a quadrant, and for that reason the two sides *a b* and *d b* of the triangle *a b d* are together greater than a semicircle. The angle *b d a*, which is the declination of the eighth ecliptic, is consequently greater than *b a f*, which is the declination of the fixed ecliptic. And so, if Beneventano himself set the declination of the fixed ecliptic at 23 degrees 51 minutes, the declination of the eighth sphere at that time was greater than 23 degrees 51 minutes — which conflicts with the observations.

Pigge, moreover, is the more to be blamed on this count: even if one granted him everything he assumed before his demonstration about the meeting of the heads of Aries and about the figure of the eighth sphere's motion, he still could not find by demonstration what he was contending for in the *Apologia* and in the tenth proposition of his book *On the Equinoxes* — namely, how great an arc of the eighth ecliptic is intercepted between the point of the vernal equinox and that point of the eighth ecliptic which lies in the same longitude as the head of Aries of the first movable. And so I shall now show this very thing, at least by an example, together with certain other results, by a demonstration of the utmost rigour and clarity.

Let the ecliptic of the first movable, *a b c*, cut the ecliptic of the eighth sphere, *a g f*, at *a*. Let *d* be the head of Aries of the ninth sphere; *e c* [?] the quadrant of the small circle; *f* the head of Aries of the eighth sphere, and *f i* its latitude, in the year 1465 from the birth of Christ, when the Sun was found by the tables at the beginning of Aries. Let the arc *e d* cut the arc *a f* at the point *K*; and let the equinoctial *g b h* cut the eighth ecliptic at *g*, and the ecliptic of the first

Diagram. The figure for the demonstration that follows, set at the head of the left-hand column. *e* stands at the top; from it a line runs down and to the right past *k* toward *d*. *f* is at the upper right, with a short arc *f i* dropping from it toward *i*, and *c* lies to the right of *i*. *m* stands just above the centre, with *k* to its right. *g* is

at the left, *b* at the centre, *d* below and right of *b*, *h* at the lower right, and *a* at the lower left. Straight and curved lines join *b* to *m*, to *d*, to *g* and to *f*, and a long shallow curve sweeps from *a* through *g* and *b* toward *h* on the right.

- *a* (lower left): common intersection of the ecliptic of the first movable and the ecliptic of the eighth sphere
- *b* (centre): intersection of the equinoctial with the ecliptic of the first movable
- *c* (right)
- *d* (right of centre): head of Aries of the ninth sphere
- *e* (top)
- *f* (upper right): head of Aries of the eighth sphere
- *g* (left): intersection of the equinoctial with the eighth ecliptic
- *h* (lower right): further point on the equinoctial *g b h*
- *i* (right): endpoint of the latitude *f i*
- *k* (upper centre): intersection of the arc *e d* with the arc *a f*
- *m* (upper centre, left of *k*): point where the perpendicular from *b* meets the ecliptic *a f*
- The point *l*, which the argument uses throughout, could not be located on the scan [?]

movable at *b*. Accordingly, if we hold to the figure of the motion of trepidation that Pigge has handed down, *a f* and *a i* will be quadrants. And since at the time of Christ's birth the points *b* and *d* coincided — as Pigge himself supposes — the arc *b d* will be known by reckoning. The arc *e f*, the motion of access and recess, is likewise known. We shall therefore make known the arc *d i*, which he calls the equation, either by taking from the tables, in place of that arc, the equation proper to *e f* itself, or by finding the same arc *d i* in the spherical triangle *d f i* from *d f* and the angle *f d i*.

The arc *b i*, which they call the common apogee, will therefore be known; and subtracting it from the quadrant *a i* leaves the arc *a b* known. Next, from the point *b* we shall draw the arc *b l* of a great circle at right angles upon *g K*. And since the arc *f i*, the latitude of the head of Aries of the eighth sphere, which fixes the size of the angle *a*, is known, the angle *a* itself will be known.

Now in the right-angled spherical triangle *a b l*, as the whole sine is to the right sine of the angle *a*, so is the right sine of the side *a b* to the right sine of the side *b l*. Three of these are known, so the fourth will be disclosed — that is, the right sine

of the arc *b l*; and the arc *b l* itself will therefore be known from the table of right sines.

By exactly the same reasoning, in the triangle *g b l*, from the whole sine and the sine of *b l* together with the sine of the angle *b g l* — which at that time was 23 degrees 28 minutes — the sine of the side *b g* will be disclosed, and by the same table of right sines the arc *b g* itself will be laid open: that is, the distance between the vernal section and the beginning of Aries of the first movable, measured on the equator.

Then again, as the whole sine is to the sine of the complement of the arc *b l*, so is the sine of the angle *a b l* to the sine of the complement of the angle *a*. Of these the first, the second and the fourth are known, so the third will be disclosed — that is, the right sine of the angle *a b l*. By similar reasoning, in the triangle *b l g* the right sine of the angle *g b l* will be disclosed. Both angles *a b l* and *g b l* will therefore be laid open by the same table of sines. Subtracting the smaller from the greater leaves the angle *a b g* known, the declination of the fixed ecliptic.

Finally, from the point *b* we shall erect the arc *b m* of a great circle at right angles upon *a b*, cutting the ecliptic *a f* at the point *m*. This *m* will fall between *l* and *K*, because the arc *a l* is less than a quadrant, and consequently the angle *a b l* is acute. Subtracting that angle from the right angle *a b m* leaves the angle *l b m* known.

In the right-angled triangle *b l m*, then, as the whole sine is to the sine of the complement of the side *b l*, so is the sine of the angle *l b m* to the sine of the complement of the angle *b m l*. The first, second and third are known, so the fourth will be disclosed. By the table of right sines, therefore, the arc of the complement of the angle *b m l* will be known; and if it is subtracted from 90 degrees, the arc of the angle *b m l* itself is left known.

From the right angle *a b m* we shall next subtract the angle *a b g*, which has already been disclosed, and the angle *g b m* is left known. And since in the triangle *b g m* the sine of the angle *g m b* is to the sine of the angle *m b g* as the right sine of the side *b g* is to the sine of the side *g m*, and three of these are known, the fourth will be disclosed; so that by the table of sines the arc *g m* itself will be laid open.

The point *m* lies in the same longitude as *b*, because *b m* passes through the poles of the ecliptic of the first movable, by the seventeenth proposition of the first book of Theodosius (*Sphaerica* I.17). And so, for the said year 1465, when the Sun was at the beginning of Aries of the first movable, the arc *g m* of the Sun's path —

that is, of the eighth ecliptic — which lay between the vernal section and that beginning of Aries of the first movable, will be known; which is what I undertook to demonstrate.

If you calculate that arc correctly you will find it to be 5 degrees 14 minutes 20 seconds; the arc *b g* on the equinoctial, 5 degrees 40 minutes 52 seconds; and the angle *a b g*, the declination of the fixed ecliptic, about 22 degrees 36 minutes.

But if you hold instead to the figure of the motion of trepidation as Georg Peuerbach shaped it, *a d* and *a K* will be quadrants, while the arc *d k* will be a little greater than *f i* — though you can still determine it in the right-angled triangle *d f K* from the known *d f* and *k f*. The angle *a* will accordingly be a little greater. The arc *b d*, the motion of the ninth sphere, will be known by reckoning; subtracting it from the quadrant leaves *a b* known. You will then reason as before, and will find very nearly the same distance of the point *m* from the vernal section.

Either way, you will find the declination of the fixed ecliptic in the said year unequal to the one that is found by similar reasoning for the year 140 from the birth of Christ. Nor can any other place be assigned to the head of Aries of the ninth sphere on the ecliptic of the first

movable without absurdity. I therefore judge that no place has been assigned to it in the tables, and that the radices of the motion of the apogees and of the fixed stars are not set to the eras — especially since, even without knowing them, the places of the fixed stars themselves can be got from those same tables. In any case it is established, from what I have just demonstrated, that if the eighth sphere is agitated by some motion of trepidation, it still cannot be the motion ascribed to Alfonso.

Ioannes Schonerus [Johannes Schöner] quotes a fragment of a letter of Regiomontanus, in which Regiomontanus says he found, from Alfonso's principles, that in the year 1465, when the Sun was found by the common calculation at the head of Aries, the arc of the ecliptic contained between its true place and the equinoctial was then nearly six degrees, and that the Sun was therefore not entirely without declination. "And since this bears on the yearly judgments," he says, "how will the astrologer avoid error if he is wholly ignorant of the head of the year, the radix of his own prediction?" — and so on.

That great man's authority certainly counts for much with me. But since the conclusion cannot be drawn except on the supposition that the heads of Aries of the ninth sphere and of the first movable coincided at the time of Christ's birth —

which does not follow from Alfonso — I decline to accept the opinion. All the more so because the same author, in his *Calendar*, having found the Sun's degree in the table (and it is none other than the one drawn from the Alphonsine tables), directs the reader to go straight into the table of the length of the day without any correction. Besides, in the year 1462, on the third day of January, when he had investigated the latitude of the city of Rome from an observation of the Sun, he added to the meridian altitude the declination answering to the Sun's true place as drawn from the Alphonsine tables.

From these facts it is plain that for Regiomontanus the starting point for reckoning the motions of the stars in the Alphonsine tables is the vernal section of the ecliptic of the eighth sphere, not the head of Aries of the ecliptic of the first movable — even though the contrary is to be gathered from the letter just cited. Whatever his own view may have been, I hold it entirely certain that the head of Aries from which the reckoning of the motions of the stars begins in the Alphonsine tables is the vernal section of the ecliptic of the eighth sphere itself; and you may establish this by the following argument.

Ptolemy, in the seventeenth year of Hadrian, observed the Sun at the autumnal section on the 7th day of the Egyptian month Athir, 2 hours after noon. Now from the beginning of the years of Christ there had elapsed 131 Roman years, 268 days and 2 hours. The radix of the Sun's mean motion at that same beginning of the years of Christ contains, according to Alfonso, 4 signs 38 degrees 21 minutes, for the meridian of Toledo. And since Alexandria lies further east, the difference of the meridians being nearly two hours and two thirds of an hour, we shall accordingly subtract from the radix itself 6 minutes 36 seconds of the Sun's mean motion in that interval, and there will remain 4 signs 38 degrees 14 minutes 24 seconds for the meridian of Alexandria.

To this we shall add the Sun's mean motion drawn from the Alphonsine tables for 131 years, 268 days and 2 hours; and, whole revolutions being cast out, there will remain 3 signs 2 degrees 42 minutes. At the autumnal section, therefore, the Sun stood by the Alphonsine reckoning 182 degrees 42 minutes from the beginning of Aries by mean motion, whereas by Ptolemy it then stood 182 degrees 10 minutes [?] from the vernal section.

For the Sun's distance from the greatest longitude of the eccentric is 116 degrees 40 minutes; and since Ptolemy put the greatest longitude at 5 degrees 30 minutes of Gemini, the Sun's distance by mean motion from the vernal section was therefore 182 degrees 10 minutes. You will find exactly as much if, on the same radix, you reckon the Sun's mean motion by Ptolemy's own tables.

The difference, then, is only 32 minutes, by which the Sun's mean motion in Alfonso exactly exceeds the Sun's mean motion in Ptolemy over that interval. And so the vernal section in Ptolemy is the head of Aries to which the motions of the stars are referred in the Alphonsine tables.

You can show the same thing again by another example. From the beginning of the years of Nabonassar to the beginning of the years of Christ there elapsed, according to Alfonso, 746 Roman years and 310 days. In that interval the Sun moves — whole revolutions being cast out — 307 degrees 30 minutes 18 seconds, by Ptolemy's tables. The radix for Christ according to Alfonso contains 4 signs 38 degrees 21 minutes. To this we shall add a whole circle, and from the total sum we shall subtract 307 degrees 30 minutes 18 seconds; there will remain 330 degrees 50 minutes 42 seconds.

In the first year of Nabonassar, then, on the first day of the month Thoth by Egyptian reckoning, at noon, the Sun stood 330 degrees 50 minutes 42 seconds from the head of Aries of the Alphonsine tables. It was therefore holding 50 minutes 42 seconds of the first degree of Pisces by mean motion for the meridian of Toledo, and 44 minutes 6 seconds for the meridian of Alexandria. And since Ptolemy, in the third book, chapter 8 (*Almagest* III.8), put it at 45 minutes of the first degree of Pisces, it is established that the head of Aries in the Alphonsine tables is the vernal section of the eighth ecliptic — that is, the beginning of the signs in Pto-

—lemy's. From all this you will understand that Georg Peuerbach was wrong, in the *Epitome*, to subtract one day from the interval between Nabonassar and Christ and to add that same day to the interval between Christ and Ptolemy's observation. For by following Alfonso we have shown that everything agrees with everything else. And it will be right to bear witness to what we have also found by many observations.

We came into possession of a well-made astrolabe two palms in diameter, and with it we observed the Sun over a good many years in the summer season, finding the least distance from the zenith at Coimbra to be precisely 17 degrees. And since the Sun's greatest declination in our age is about 23 degrees 30 minutes, we concluded from this that the latitude of Coimbra is about 40 degrees 30 minutes. Later, in the course of the year 1555 from the birth of Christ, on 14 September, we found the Sun's least distance from the zenith to be 40 degrees 40 minutes. At noon on that day, therefore, it was declining 10 minutes to the south; and since near the equinoctial points the Sun's declination changes by one minute in one hour, the autumnal section fell on 14 September, 10 hours before

noon — the very time at which the tables put the Sun almost exactly at the beginning of the sign Libra. The beginning of Libra in the tables is therefore nothing other than the autumnal section, and by the same token the beginning of Aries is nothing other than the vernal section. That was the point worth our attesting.

Since, then, the Sun's declination has to be found, it will not be necessary to add five degrees to the Sun's true place as drawn from Alfonso's tables, as Albert Pigge, Johannes Schöner, and certain others hold one must. We shall instead enter the table set out below. In it the side numbers running downward belong to the signs whose names are written at the head of the table, and the side numbers running upward to those whose names are at the foot. In the body of that same table, under the sign in which the Sun stands, we shall find the declination sought.

If, however, some minutes over and above whole degrees attach to the Sun's true motion, then — by a double entry, as is usually done with other astronomical tables — the proportional part must be sought according to the ratio of those minutes to 60, taken from the difference between the two entries. That proportional part is to be added to the number of degrees and minutes of the first entry if the sign under which the Sun is carried is found at the head of the table, or subtracted if it is found at the foot. The number obtained by this method will be the declination sought.

And if by some recent observation the Sun's entry into the vernal or the autumnal section has been ascertained, and the length of the year found with the utmost exactness, you will then be able, from the Sun's true place once known, to find its declination by this table with a wholly reliable calculation.

¶ TABLE OF THE SUN'S DECLINATION, assuming a greatest declination of 23 degrees 30 minutes.

Column heads at the top of the table: **Aries** / **Libra** · **Taurus** / **Scorpius** · **Gemini** / **Sagittarius** — read with the argument descending (left-hand column). Column heads at the foot: **Virgo** / **Pisces** · **Leo** / **Aquarius** · **Cancer** / **Capricorn** — read with the argument ascending (right-hand column). `gr.` = degrees of the argument; `Gr.` = degrees of declination; `m.` = minutes of declination.

gr.	Gr.	m.	Gr.	m.	Gr.	m.	*(gr.)*
0			11	30	20	12	30

gr.	Gr.	m.	Gr.	m.	Gr.	m.	*(gr.)*
1		24	11	51	20	25	29
2		48	12	12	20	37	28
3	1	12	13 [?]	33	20	49	27
4	1	36	12	53	21	0	26
5	2	0	13	13	21	11	25
6	2	23	13	33	21	22	24
7	2	47	13	53	21	32	23
8	3	11	14	13	21	42	22
9	3	35	14	32	21	51	21
10	3	58	14	51	22	0	20
11	4	22	15	10	22	9	19
12	4	45	15	28	22	17	18
13	5	9	15	47	22	25	17
14	5	32	16	5	22	32	16
15	5	55	16	23	22	39	15
16	6	19	16	40	22	46	14
17	6	42	16	57	22	52	13
18	7	5	17	14	23 [?]	57	12
19	7	28	17	31	23	3	11
20	7	50	17	47	23	7	10
21	8	13	18	3	23	12	9
22	8	35	18	19	23	15	8
23	8	58	18	34	23	19	7
24	9	20	18	49	23	22	6
25	9	42	19	4	23	24	5
26	10	4	19	18	23	26	4
27	10	26	19	32	23	28	3
28	10	47	19	46	23	29	2
29	11	9	19	59	23	30	1
30	11	30	20	12	23	30	0

¶ On the declination of the parts of the ecliptic by means of an instrument. Cap. 5.

The declinations of the parts of the ecliptic can be found not only from globed instruments but from flat ones as well. On the flat surface of the back of an astrolabe, let the circle *a b c d*, described about the centre *e*, be the circle that represents the ecliptic. Let *a* be the point of the beginning of Aries, *b* of Cancer, *c* of Libra, and *d* of Capricorn. Let the given point be *f*, whose declination is to be found.

In the quadrant *b c*, then, take the arc *c g* equal to *a f*. Lay a rule across the two points *f* and *g*, or stretch a thread between them, and mark where it cuts the radius *e b*; in this example let that point be *K*. Next take from the quadrant *a b* the arc *b l* of the ecliptic's greatest declination, and to its end *l* apply the astrolabe's rule, which turns about the centre *e*; let its position be *l e y*. Then, with a second rule, or with a thread stretched perfectly straight, apply it to the point *K* in such a way that it becomes parallel to *l e y*. You will know that it is parallel when it cuts off equal arcs on either side. Let that position be *m K n*. I say that the arc *l m*, or *y n*, is the declination of the point *f*. Since, then, the circle *a b c d* is itself divided into degrees, the declination *l m* is made known in the manner just described once the arc *a f* is known.

Diagram. A single circle, cut by two perpendicular diameters meeting at the centre, representing the ecliptic laid out on the flat back of an astrolabe. Two further straight lines are drawn across the upper part of the circle: a chord running from the upper left down to the lower right, and a second line parallel to it a little further in, so that the two form the rule *l e y* and its parallel *m K n* of the demonstration. Readable labels:

- *b* (top of the circle, at the upper end of the vertical diameter): Cancer.
- *l* (on the circumference to the left of *b*): the end of the arc of greatest declination.
- *m* (on the circumference between *l* and *b*).
- *o* (just inside the circumference, below and right of *l*).
- *r* (just inside the circumference, to the right of *o*).
- *f* (on the circumference at the left, in the quadrant above the horizontal diameter): the given point.
- *K* (in the interior, on the radius running up to *b*): the intersection of the line *f g* with that radius.

- *g* (on the circumference at the right, above the horizontal diameter).
- *e* (at the centre).
- *c* (at the right end of the horizontal diameter): Libra.
- *y* (in the lower part of the figure, on the prolongation of the rule through *l* and *e*).
- *n* (toward the lower right).
- *d* (at the foot of the circle): Capricorn.
- *a* [?] (expected at the left end of the horizontal diameter; the letter does not resolve on the scan).

The operation is easy, and the demonstration will not be hard either. The diameter *a e* [?] is the common section of the plane of the equinoctial and the plane of the ecliptic, while the diameter *b d* is the common section of the plane of the solstitial colure and the plane of the ecliptic. The straight line *f g* is the common section of the plane of the ecliptic and the plane of the circle parallel to the equinoctial that is drawn through *f* — as Euclid XI.16 shows.

Now imagine, in the plane of the solstitial colure itself, a sector whose base is the arc of the ecliptic's greatest declination. One of its two sides is *e b*; the other is a straight line equal to *e b* lying in the common section of the plane of the equinoctial and the plane of that same colure. The plane of the parallel circle drawn through *f*, then, while it cuts the plane of the ecliptic along *f K g*, will at the same time cut that sector along a certain straight line — a line which cuts the side *e b* at *K* and is parallel to the remaining side of the same sector, as you will show by Euclid XI.16. And from the arc of greatest declination, which is the base of the sector, it will cut off an arc equal to the declination of the point *f,* as you will conclude from the definition of a pole and from a common notion.

Now the sector *b l e*, which we have imagined described in the plane of the ecliptic, is similar and equal to that first sector, and has the side *e b* in common with it, on which the same point *K* remains. The straight line *K m* is therefore parallel to the side *e l*, and will likewise cut off *l m* equal to the declination of the point *f* — which was to be demonstrated.

And if from the point *b* you erect the straight line *b o* at right angles upon *e l*, then by Euclid VI.2, taking the ratio compounded and alternated, you will conclude that as *e b*, the whole sine, is to *b o*, the right sine of the greatest declination, so *e K*, the right sine of the arc *a f*, is to *o r* [?], the right sine of the declination of the point *f* — which we demonstrated by another route in the book *On Twilights*.

The declinations of the parts of the ecliptic can also be brought down onto a single plane, by the very art Vitruvius used in his ninth book. Let the circle *a b c d*, described about the centre *e* and divided into quadrants, represent the solstitial colure, and let *a c* be its common section with the equinoctial. In the quadrants *a b* and *a d* take the two greatest declinations of the Sun, *a f* and *a g*; draw the straight line *f g*; and on its midpoint *h*, with the interval *f h* or *h g*, describe in its plane the circle *f y g K*, which will represent the ecliptic — *y* the beginning of Aries, *f* of Cancer, *K* of Libra, *g* of Capricorn. Now just as, on the sphere, the parallel circles that cut the ecliptic cut off from the arc of greatest declination arcs equal to the declinations of those points of the ecliptic through which the same parallels are drawn,

— so exactly does the straight line *l m*, drawn parallel to the diameter *a c*, cut off from *a f*, the arc of the greatest northern declination, an arc *a r* equal to the declination of the point *l* in the first quadrant, or of the point *m* in the second. In the same way the straight line *n o*, parallel to that same diameter, cuts off the arc *a t*, which is equal to the declination of the point *o* in the third quadrant, or of the point *n* in the fourth.

Whenever, then, you wish to find the declination of the point *l*, take the arc *K m* equal to the arc *y l*, and lay the rule across the two points *l* and *m* so that it is parallel to the straight line *a c*. Its intersection with the arc of the colure will show the declination sought at once.

Diagram. A large circle cut by two perpendicular diameters meeting at its centre, and within it a smaller circle whose centre lies on the horizontal diameter, offset toward the left-hand end. The large circle represents the solstitial colure, the small one the ecliptic brought down into the colure's plane. Two short chords are drawn across the small circle parallel to the horizontal diameter. Readable labels:

- *b* (at the top of the large circle, upper end of the vertical diameter).
- *c* (at the right end of the horizontal diameter).
- *d* (at the foot of the large circle).
- *e* (at the centre of the large circle).
- *K* (in the upper interior of the small circle): by the construction of the preceding page, the beginning of Libra.
- *g* (at the lower left of the small circle): the beginning of Capricorn.
- *t* [?] and *o* [?] (on the short chord below the horizontal diameter).
- The remaining letters of the demonstration — *a*, *f*, *y*, *l*, *m*, *n*, *r* — are present in the woodcut but do not resolve on the scan.

The geometrical demonstration of this instrument and of its use is as follows. In the figure below let *a b c* be a semicircle of the ecliptic, either the northern or the southern, and let *a e c* be the semicircle of the equinoctial that rises with it. Divide both into quadrants at the marks *b* and *e*, and let *b e* be the arc of the solstitial colure lying between the equinoctial and one of the tropics — that is, one greatest declination of the Sun. From *d*, the centre of the world, draw the straight lines *d b* and *d e* to the points *b* and *e*. Then from the point *b* drop the perpendicular *b f* onto the semidiameter *d e*; produce *d e* in a straight line; and about the centre *f*, with the interval *b f*, in the plane of that same solstitial colure, describe the semicircle *g b k*, whose parts *g b* and *b k* must necessarily be quadrants.

Diagram. A semicircle standing on a horizontal base line, with a second, flatter arc drawn within it on the same base; a large arc struck from a centre on the base line sweeps across the whole figure, and several straight lines are drawn as chords and as radii from the base line. Readable labels:

- *a* (at the left end of the base line).
- *d* (on the base line, at its middle).
- *K* (on the base line, to the right of *d*).
- *q* [?] (on the base line between *a* and *d*).
- *b* (at the summit of the figure).
- *g* (at the upper left).
- *x* (just below and right of *g*).
- *f* (in the interior, right of centre).
- *m* (to the right of *f*).
- *e* (in the interior, left of *f*).
- *z* [?] (between *e* and *f*).
- *p* (at the right of the figure, outside the inner arc).
- *n* [?] (at the left, below *g*).
- The letters *l*, *o*, *r* and *t* of the demonstration are present in the woodcut but do not resolve on the scan; the right-hand end of the base line, which the text names *c*, likewise does not resolve.

Now take any arc *g l* on *g b*, and from the point *l* erect the straight line *l m* parallel to *g k*; let it cut *b e*, the arc of greatest declination, at the point *o*, and the straight line *b f* at *t*. I say that the arc *e o* is equal to the declination of the point that terminates that arc of the ecliptic which begins from the other equinoctial section — the arc to which *g l* in the quadrant *g b* is proportional.

For let a plane parallel to the equinoctial pass through the straight line *l m*, and let the common section of that plane with the plane of the ecliptic be the straight line *n p*. The common section of these two straight lines will then be a single point — call it *r* — lying in both planes, that of the ecliptic and that of the colure. But the common section of the plane so drawn and of the sphere will be the circle *n o p*, by the first proposition of Theodosius's first book (*Sphaerica* I.1). Now from the pole of the world draw a great circle through *n*, cutting the equinoctial at *z*. The arc *n z* will therefore be the declination of the arc *a n* of the ecliptic, and equal to the arc *e o* which the straight line *l m* marks off from *b e*. This will be established from the definition of a pole and from common opinion.

That the arcs *g l* and *a n* are similar and proportional will appear as follows: as *d b*, the semidiameter of the ecliptic, is to *b f*, the semidiameter of the circle *g b k*, so is *d r* to *t r* — by the second proposition of the sixth book of Euclid (*Elements* VI.2), taking the ratio compounded and alternated. But the straight line *d r* is equal to *n q*, the right sine of the arc *a n*, because the straight line *n p* is parallel to *a c*, by the sixteenth proposition of the eleventh book of Euclid (*Elements* XI.16); and the straight line *t r* is equal to *l x*, that is, to the right sine of the arc *g l*. The ecliptic and the circle *g b k* are therefore proportional to the arcs *a n* and *g l*. For we assume here that if in two circles the semidiameters and the right sines of two arcs are proportional, then the arcs themselves will also be proportional to their own circles. This is easily demonstrated: the angles set up at the centres of those same circles and subtending those very arcs are equal, by the seventh proposition of the sixth book of Euclid (*Elements* VI.7); the arcs are therefore similar, which was what we assumed.

In this demonstration, as you see in the one above, as the whole sine *b d* is to *b f*, the sine of the greatest declination, so is *d r*, the sine of the arc *a n*, to *t r*, the sine of the declination *e o*, which corresponds to the point *n*. And so, in the right-angled spherical triangle *a n z*, once the angle *a* and the side *a n* are known, the side *n z* becomes known by the art already described, within the plane of a single circle. It is shown also that the right sines of the angles and of the opposite sides are proportional.

The advantage of an instrument of this kind is that the degrees of declination come out much larger than the degrees of the ecliptic. For if you set the arc of greatest declination at 23 degrees 30 minutes, the ratio between the two sorts of degree will be nearly two and a half to one [*dupla sesquialtera*], so that two degrees of the colure are nearly equal to five degrees of the ecliptic; and therefore

one degree of the ecliptic will be equal to twenty-four minutes on the arc of greatest declination.

You can also construct the same instrument far more easily by drawing the ecliptic first and the arc of greatest declination afterwards. In the semicircle *a b c*, let *a* be set as the beginning of Aries, *b* of Cancer, *c* of Libra — and this same semicircle will be able to serve for the southern semicircle as well. Then take the arc *b e*, double the greatest declination, and through the two points *b* and *e* draw a straight line; its intersection with *a c* produced in a straight line will be the centre of the circle to be described through *b*, the circle that represents the solstitial colure. The arc *b f* will therefore be one greatest declination of the Sun.

This once gave me occasion to draw a circular planisphere that achieves exactly what the *tabula primi mobilis* of Ioannes de Monteregio [Regiomontanus] achieves. In the field of a planisphere of that kind 89 arcs are drawn. All of them have one endpoint in common, at the point *b*, while their remaining endpoints lie on the diameter *a c*. The arc nearest the centre is of one degree only, the next after it of two degrees, and so with the rest in their order; so that the one farthest from the centre contains 89 degrees of its own circle. A rule laid parallel to the diameter *a c*, in whatever position, therefore shows the numbers of the arcs answering to a single transversal. For it cuts off the lateral divisions [*laterales*] from *a b*, but in the present figure the arc divisions [*arcales*] from *f b* — *f b* itself being the transversal. But of these matters more fully elsewhere.

Diagram. A semicircle standing on a vertical diameter at the left of the figure, its arc bulging to the right. A nearly straight, very flat arc runs from the left across to the right-hand extremity of the semicircle; it carries a graduated scale of tick marks. Below and to the left, a slanting straight line runs off the figure toward the far-distant centre from which the flat arc is struck, and is labelled with the word *centrū.* (*centrum*, "centre"). Readable labels:

- *a* (at the top of the vertical diameter): the beginning of Aries.
- *b* (at the right-hand extremity of the arc): the beginning of Cancer.
- *c* (at the foot of the vertical diameter): the beginning of Libra.
- *d* (on the vertical diameter at its middle): the centre of the semicircle.
- *e* (on the arc below *b*): the end of the arc *b e*, double the greatest declination.
- *f* (at the left-hand end of the flat graduated arc, just above *d*).
- Scale numerals along the graduated arc, left to right: *5*, *10*, *15*, *20*, and *23 ½* at *b*.

- *centrū.* (below left, at the end of the slanting line): the direction of the arc's centre.

¶ On the instruments by which the altitudes and distances of the stars are taken. Cap. 6.

Sailors use hanging astrolabes, because at sea they cannot have a quiet or steady horizon. The ancient astronomers, by contrast, set up every instrument with which they observed the stars upon a levelled horizontal face; for in that way the instrument's plumb line could not lean to either side. In hanging astrolabes, on the other hand, the half of the alidade that stands higher — and is for that reason the heavier, as Iordanus [Jordanus Nemorarius] proved in the case of balances — will, at the point where it is attached to the instrument, pull the instrument itself somewhat out of true.

You will therefore build a hanging astrolabe without sight-vane or alidade, in this way. Let a circular ring of moderate size be made of metal, with square faces, after the fashion of the circles of a solid sphere, its breadth and thickness equal, one digit each. On its concave face, running lengthwise down the middle, let the circle *a b c* be drawn, and let *d* be understood as its centre. Answering to this on the curved outer face let there be the circumference of the circle *f K l*. Take the point *f* on that outer circumference above *a*, in line with the diameter *a e*, and let the suspension ring from which the astrolabe hangs be fastened to *f* by a small pin. Then from the circumference *a b e* take the arc *a g*, half of one quadrant, and an arc *a b* equal to it in the other semicircle. Let *c* be the point diametrically opposite *b*, and let the semicircle *b e c* be divided into ninety equal parts, with the proper numbers written against them, the reckoning starting at *b*. Next, along the middle of its width, cut away from the thickness of the instrument itself a piece in the shape of the obtuse angle *h g K*, and at the point *g* leave a very narrow hole to admit the Sun's rays.

Diagram. The hanging astrolabe seen face on: two concentric circles, the outer standing for the ring's outer face and the inner for the circle drawn on its concave face. At the top a small ring is attached; at the upper left a wedge has been cut out of the metal, forming an obtuse angle whose apex carries the pinhole. Two diameters cross at the centre, and the lower right half of the inner circle is graduated with tick marks and numerals. Readable labels:

- *f* (at the top, on the outer circle, where the suspension ring is fixed).
- *h* (at the upper left, on the outer edge of the cut-away wedge).
- *K* (at the left, the other arm of the wedge).

- *g* (within the wedge): the pinhole.
- *a* (at the top of the inner circle, directly below *f*).
- *b* (on the inner circle at the upper right, half a quadrant from *a*).
- *d* (at the centre).
- *e* (at the foot of the inner circle).
- *c* [?] (on the inner circle at the lower left, diametrically opposite *b*).
- *90* (the last numeral of the graduation, beside the lower-left end of the scale); further scale numerals run along the right-hand side of the graduated semicircle but do not resolve on the scan.

Since the removal of that small piece leaves the instrument the lighter on that side, the diameter *a e* will necessarily depart from the plumb line, and an equal weight of metal must therefore be taken away on the other side. When the instrument is finished and you wish to learn the Sun's altitude above the horizon, hang the astrolabe by its ring and turn the side in which the hole is toward the shining Sun: its ray will at once show, on the semicircumference *b e c*, the altitude sought above the horizon. The degrees of altitude on this instrument are twice as large as those that would be given if an alidade were turned about the centre, as we see on the ordinary astrolabes; for of two equal angles, the one made at the circumference of a circle takes in twice the arc of the one made at the centre.

The Sun's altitude can be found not only from instruments set upright above the plane of the horizon, but also from instruments lying flat, provided they lie parallel to it. Let a circular plate *a b c d*, its circumference divided as usual into 360 degrees, be laid parallel to the horizon; and out of any hard material let a right-angled isosceles triangle *f g h* be made, whose two sides *f g* and *g h*, which contain the right angle, are equal to the semidiameter of the circle described. Set this same triangle upright on that circular plate, and fit it so that the side *g h* falls exactly along *a e*, a semidiameter of the circle, *g* coinciding with *a* and the point *h* with *e*; the point *f* will then stand aloft. Further, set up a small rod at right angles to the same plane, over any point you please of the diameter *b d*.

Diagram. The flat-lying instrument seen from above: a large circle cut by two perpendicular diameters, with a quadrant arc drawn across its upper left and the erected triangle's cast shadow laid out as straight lines running to a point outside the circle at the left. Readable labels:

- *a* (at the left end of the horizontal diameter).
- *b* (at the top, upper end of the vertical diameter).
- *c* (at the right end of the horizontal diameter).

- *d* (at the foot of the vertical diameter).
- *f* (outside the circle at the left, where the several straight lines converge).
- *K* (above and left of *f*, outside the circle): the shadow point.
- *l* (on the arc at the top, between *f* and *b*).
- The letter at the centre of the circle, which the text requires to be *e*, is blotted and does not resolve [?].

Diagram. The right-angled isosceles triangle drawn on its own below the preceding figure, with the right angle at the lower left. Readable labels:

- *f* (at the apex, top).
- *g* (at the lower left, the vertex of the right angle).
- *h* (at the lower right).

When you wish to find the Sun's altitude above the horizon, turn the instrument about until the rod's shadow lies stretched along the straight line *b d*. Then the shadow of the side *f h* — that is, of *f e* — will mark on the quadrant *a b* the altitude sought, reckoned from the point *b* toward *a*; and the remaining part of the quadrant, as far as *a*, will be the distance between the Sun and the zenith point.

The demonstration of this operation is as follows. Let the plane surface of the circle *a b c d*, which is laid parallel to the horizon, be understood as extended straight out toward the side on which the shadows are cast; and let the shadow of the right-angled triangle *a f e*, which stands perpendicular to that plane, extended in the same plane, be the triangle *a K e*. Let the straight line *a f* cast its shadow along *a K*, and let the shadow of the straight line *e f* be *e K*, which cuts the quadrant *a b* at *l*. Now since the Sun's rays are reckoned parallel by those same authorities [?], the straight line *a K* and the rod's shadow, stretched out along the length of the straight line *e b*, will be parallel. But the angle *a e b* is a right angle; therefore the angle *e a K* is a right angle as well. And *e a f* is a right angle; therefore the angle *f a K* [?] will be a right angle, by the third definition of the eleventh

book of Euclid. Now in the two triangles *a K e* and *a f K*: the side *a e* of the one equals the side *a f* of the other, the side *a K* is common to both, and the two angles contained by those equal sides are equal, being right angles. Therefore the two angles *a f K* and *a e K* will be equal to one another, by the fourth proposition of the first book of Euclid. But the angle *a f K* is vertically opposite the angle at the point *f* which subtends the arc of the distance between the Sun and the zenith; and therefore the angle *a e K* will likewise subtend the arc *a l* in the quadrant *a b*.

The remaining arc *b l* will then be similar to the arc of the Sun's altitude above the horizon — which was to be demonstrated.

From this demonstration you have the following. If an instrument of this kind is given a square form, so that in it the straight line *a k* can be drawn touching the circle itself at the point *a*, there will be no need of the pin or little rod whose shadow is cast along the straight line *b d*. Instead we shall turn the instrument itself round until the shadow of the straight line *a f* falls along the straight line *a K*; for in that way the shadow of the straight line *e f* will show the arc of the Sun's altitude above the horizon. And if you make the three sides of the triangle *f g h* twice as long, so that the side *g h* is equal to the diameter *a c* and fits it exactly, then you will divide the semicircle *a b c* into ninety equal parts, and the degrees of the Sun's altitude will accordingly be twice as large.

But suppose you take this same instrument, built in that fashion, set it upright on the plane of the horizon, and turn it to the Sun so that the shadow of the straight line *a f* — a shadow that will now be not a right shadow [*umbra recta*] but a turned one [*umbra versa*] — is extended along the straight line *a K*. Then the arc *a l* will be the Sun's altitude above the horizon, and the remaining arc *b l* will be the arc of the distance between the Sun and the zenith. For it is in this way that the right shadow and the turned shadow are interchanged: take two altitudes of the Sun that together make 90 degrees, and the right shadow of one and the same gnomon under one of those altitudes will be exactly as long as the turned shadow answering to the other. Moreover, under one and the same altitude of the Sun above the horizon, whether the gnomons set up are equal or unequal, the right shadow stands to its own gnomon as any other gnomon stands to its own turned shadow. This is easily demonstrated by the fourth proposition of the sixth book of Euclid. So by the common rule of proportional numbers you will find the turned shadow from the right shadow, and, conversely, the right shadow from the turned.

The ordinary quadrant that sailors use is very well suited to taking the altitudes of the Sun and of the other stars. But in place of the thread with its plumb, a rule should be fitted with a weight attached at one end, contrived so that the face directed toward the centre of the instrument always stays upright over the plane of the horizon. For the thread jumps about, and now and then sticks in one place even while the observer turns the quadrant itself; and that is why the altitudes taken with quadrants come out unreliable.

It does happen sometimes, though, that the instrument has been correctly made and the stars carefully observed, and yet the altitudes obtained are still not exact.

The reason is simply this: because the instrument is small, its parts cannot be subdivided any further into finer divisions, so that beyond a whole number of degrees you cannot estimate how much altitude is to be added. It will help, then, to describe forty-four circles about the same centre within the compass of the instrument, on its face. Let the quadrant of the outermost be cut into 90 equal parts, the one next to it into 89, the one following that into 88, and so on in order, as I taught in the book *On Twilights*.

This, I believe, is what Claudius Ptolemy did. For he says that he found the greatest declination of the Sun to be 23 degrees 51 minutes 20 seconds for this very reason: because the ratio discovered between the whole circle and the arc between the tropics was the one that 83 has to 11. It is clear, then, that some quadrant described within the compass of his instrument had been divided into those same 83 equal parts, of which the arc between the tropics contained 44. For the instrument Ptolemy used was not of such a size that first and second minutes could be marked on it.

Sailors use the astronomer's cross-staff [*radius astronomicus*] to find how great the altitude of the pole star above the horizon is. But it is very difficult to find a reliable altitude in that way. The cross-staff is, however, a most apt instrument for finding the distance between two stars whose interval is less than a quadrant of a great circle. Regiomontanus set out its construction and its use in his book on the comet.

The length of the staff [*fustis*] is to be divided into any number of equal parts you please, and the length of the sliding cross-piece [*pinacidium*] must be taken in those same parts. A table of numbers must also be constructed, by which, from a given ratio between the two sides of a right-angled triangle that contain the right angle, the size of the angle opposite the shorter side may be found. Such is the table that Georg Peuerbach, that most excellent mathematician, composed for use with the geometric square. So when two

stars have been sighted through the ends of the cross-piece, let the number of parts in half the length of the cross-piece be multiplied by 1200 — for the side of the aforementioned square is supposed to have that many parts. Let the product be divided by the number of parts on the staff between the position of the cross-piece and the observer's eye; and with the quotient we shall enter the table of the geometric square itself. For the number found opposite it in that table will be the arc of half the distance between the observed stars, and when this is doubled their whole interval will be revealed.

An example. On 17 October in the year of Christ 1475, Bernardus Vualther [Bernhard Walther] observed the distance of Mars and Saturn with the astronomer's cross-staff. In the parts of which the length of the movable cross-piece was 210, the length of the staff between the eye and the position of the cross-piece was found to be 807. We shall find the distance of those planets in this way. We shall multiply the number 105 — that is, half the length of the cross-piece — by 1200, the side of the geometric square, and 12600 will result. This number we shall then divide by 807, and the division will yield 156 108/807. Or else we shall multiply 210, the length of the cross-piece, by 1200, divide the product by 807, and take half the quotient, which is 156 108/807. With this, then, we shall enter Georg Peuerbach's table and draw from it an arc of 7 degrees 24 minutes 47 seconds. This we shall double, and so at last gather 14 degrees 49 minutes 34 seconds of a great circle for the distance between Mars and Saturn at the time of the observation just mentioned.

The demonstration of this operation is easy. Let the straight line *a b* be the length of the staff, let the observer's eye be at *a*, and let the cross-piece *c d* be at the position *e*, so that it exactly fills the distance between Mars and Saturn. Let *a e* be found to be 807 of the parts of which *c d* is 210, and its half *c e* 105. In the parts, then, in which that same *a e* is 1200, *c e* will be 156 108/807, by the common rule of proportional numbers. And therefore, by Georg Peuerbach's table, the arc of the angle *c a e* will be found to be 7 degrees 24 minutes 47 seconds; so the doubled arc, which answers to the angle *c a d*, will have 14 degrees 49 minutes 34 seconds.

Johannes Schöner, however, found a smaller value in this same example, because he entered that same table of Georg Peuerbach with 312 216/807 as the length of the cross-piece. And so the angle he found by that method is not

Diagram. The cross-staff demonstration. Lines radiate upward from a single vertex at the foot of the figure, spreading into a fan that is closed at the top by a broad arc; a vertical line runs from the vertex up to the middle of that arc. Across the fan, a little below the arc, a straight transversal is drawn — the cross-piece — and beyond each of its ends stands an engraved star of six points, representing the two observed planets. Readable labels:

- *b* (at the top, on the vertical line where it meets the arc): the far end of the staff.
- *f* (at the left-hand end of the transversal).
- *c* (on the transversal, to the right of *f*).

- *e* (on the transversal, at the vertical line): the position of the cross-piece on the staff.
- *d* (at the right-hand end of the transversal).
- The letter at the bottom vertex — the observer's eye, which the demonstration calls *a* — does not resolve on the leaf; the inked glyph reads as *d* [?].

the angle *c a d*, which subtends the arc of the distance of Mars and Saturn, but a different and smaller one. For let the straight line *c d* be produced in a straight line, let *c f* be taken along it equal to *c e* or *e d*, and let *a f* be joined. Then *e f* will be 312 216/807 of the parts of which *a e* is 1200. And so the angle that Schöner drew from the table cited above is *e a f*, which we shall show to be smaller than *c a d* itself. For the side *a f* is greater than *a e*; and therefore, if the angle *e a f* is bisected, the straight line dividing the angle will cut the base *e f* between *c* and *e*, so that no impossibility follows contrary to the third proposition of the sixth book of Euclid. Hence, by the common axiom, the angle *f a c* will be much smaller than the angle *c a e*. But the two angles *c a e* and *d a e* are equal to one another; therefore the whole angle *e a f* will be smaller than the angle *c a d*, that is, than the distance of Mars and Saturn — which was to be demonstrated.

It must be noted, however, that the intervals of Mars, Jupiter, and Saturn, and of the fixed stars, from the zenith, as they are obtained with instruments, subtend nearly equal angles at the centre of the terrestrial globe and at the observer's eye on the surface of that same globe — for their vast distances from the earth, compared with the earth's own semidiameter, make the difference imperceptible. With the Moon, though, and with the Sun, it is otherwise.

Ptolemy observed the Moon's distance from the zenith with the instrument of rules [*instrumentum regularum*]; and from her true place and her latitude, found by computation, he found her declination. Then, from the declination he had found and from the Moon's known distance from the meridian, with the days equated, he found the true interval between the body of the Moon and the zenith. This he subtracted from what the observation had given, and so concluded how great the parallax [*aspectus diuersitas*] was at the time of the observation in question. Then from these data he found the distance of the centre of the Moon's

body from the centre of the earth, in the parts of which the earth's semidiameter is one, by a geometrical argument; and from the same observation he found the ratio to the earth's semidiameter of the semidiameters of the Moon's eccentric and epicycle and of the eccentricity. Since he could not find the visual diameters of the Sun and the Moon exactly enough with any instrument, he investigated

them very ingeniously from two lunar eclipses. It was therefore not difficult to exhibit the ratio of the earth's semidiameter to the semidiameter of the Moon's body. From these results, then, he determined the diameter of the Sun and the distance of its centre from the centre of the earth in the parts of which the earth's semidiameter is one, and also the ratios of the three bodies — Sun, earth, and Moon — to one another.

Consequently, to find the parallax thereafter in any position whatever, it was not necessary to observe the stars themselves. Instead he taught how to investigate the parallax on the circle of altitude by a geometrical argument, from the given distance of the Sun or Moon from the centre of the earth and its elongation from the pole of the horizon. The greatest of these parallaxes is in the Moon, 1 degree 43 minutes; and he constructed tables of the parallaxes. We may, all the same, find the Moon's parallax itself by observation and computation together, as Ptolemy himself did. But the Sun's parallax is much smaller — by Ptolemy's numbers the greatest contains only two minutes together with 51 seconds — and for that reason it could not be found by observation, as the Moon's was. And so he could not conclude how great the Sun's distance from the earth is from the parallax, since the parallax was extremely difficult to find with instruments on account of its smallness. On the contrary, he found the parallax from the Sun's distance from the centre of the earth, which he had come to know by the method described above and which he took to be invariable.

From all this it is plain that the altitudes of the stars above the Sun, as obtained with instruments, are to be taken as true. In the Sun itself the parallax, so far as it bears on the latitudes of places, is to be reckoned as nothing; but in the Moon it is on no account to be neglected, unless she is placed near the zenith.

From this it also appears that Girolamo Cardano did not sufficiently consider what he wrote in the fourth book of *De subtilitate* about what can be known from the rays of the stars: namely, that the altitude of any star whatever, at any hour whatever, reckoned from the centre of the earth, can be found from the known ratio of the shadow to the gnomon. As if, indeed, all the stars could light up opaque bodies set before them in such a way that plain shadows are cast on the far side — something granted to no star except the Sun and the Moon. He puts *a* for the centre of the earth; let *a d* be its semidiameter, *b c* a plane parallel to the horizon, and let the rod *d e* stand perpendicular to that plane. Let the star *f* send out the ray *f e h*, and let the shadow *d h* have, at that moment, a known ratio to *d e*. The angle *d e h* will therefore be known; and hence *a e f*, which is what remains from two right angles, will be known as well.

Diagram. Cardano's construction as Nunes sets it out. A circle stands for the earth, with its centre marked; a long straight line — the plane parallel to the horizon — runs across the figure above the circle and touches it near the middle. From the point where that line meets the circle a short upright rod rises, and from the top of the rod one line runs steeply up to the right, to the star, while another runs down to the left to meet the horizontal line at the shadow's end. A further long straight line is drawn from the centre of the circle up to the star, and the vertical through the centre and the rod is carried on above. Readable labels:

- *a* (at the centre of the circle): the centre of the earth.
- *c* (at the left-hand end of the horizontal line).
- *b* (at the right-hand end of the horizontal line).
- *d* (on the horizontal line, where the earth's semidiameter meets it): the foot of the rod.
- *e* (just above *d*): the top of the rod.
- *h* (on the horizontal line, a little to the left of *d*): the end of the shadow.
- *f* (at the upper right): the star.
- *K* (on the upright line above *e*): the zenith direction from the observer.

"Let the star's altitude above the horizon be taken with the planisphere," he says; "its difference from 90 degrees will be the arc of the angle *f a e*" — so he supposes — "and therefore the remaining angle *a f e* cannot be unknown. Now, then, in the triangle *a e f*, from the known angles together with the side *a e*, the remaining sides will be revealed, and accordingly the ratio of *a f* to *a e* will be known." Such, in substance, is Cardano.

But from what I have said I think it clear that the star's distance from the zenith, taken with the astrolabe, cannot subtend the angle *f a e*, but some other angle, equal to the angle *d e h* — the angle whose size is found from the ratio of the shadow to the gnomon. And the exterior angle *d e h* is greater than the interior angle *f a e*. So Cardano proves nothing by that demonstration of his. Indeed, in the case of the Sun the ratio of *a f* to the earth's semidiameter is not learned from the shadow, but either by Ptolemy's method or by Regiomontanus's in the *Epitome*. Nor in the Moon's case either, since the boundary of her shadow is less certain than the boundary of the Sun's shadow. There the angle *K e f* will have to be found with Ptolemy's rules or with any other instrument fit for the purpose, and the interior angle *e a f* by computation, from the Moon's distance from the meridian and her declination, both known. When that interior angle is subtracted from *K e f* itself, the angle *a f e*, the parallax, is left known — as

—and from that the ratio of *a f* to *a e*, that is, to the earth's semidiameter, becomes clear at once. But if the height above the earth cannot be found from the shadow either of the Sun or of the Moon, then still less can the heights of the remaining stars be found: around opaque bodies their illumination barely distinguishes light from shadow. And even if the light of the superior planets and of the fixed stars surpassed the light of the Sun, the ratio of the height to the earth's semidiameter would still not be learned from the angles, because the angle *a f e* itself would be reckoned a quantity too small to perceive.

He goes just as far astray when, in the same book, he undertakes to show to what height above the earth vapours can rise; and there he faults Vitellio [Witelo] quite wrongly. "Let us observe," he says, "the Sun standing under the equinoctial circle. It begins the twilight at 19 degrees before its rising — that is, roughly an hour and a quarter before the Sun itself comes up — and it is plain that at that moment the first ray of the Sun which lights the air touches the earth. For if it did not touch it, a line drawn from the topmost place of the vapours so as to touch the earth would reach a place lower than the first, and so the twilight would begin earlier than has been said. On this supposition, then, let a circle be set out representing the earth, with centre *c*; let *a d* be the tangent line, *e* the topmost part of the vapours, *f* the place of the Sun's ray, and where it cuts *a d* let *g* be set. Since the Sun's distance is very great in comparison with the earth, the angle *f g d* is as it would be if it stood at *c*, the earth's centre; and so it is of 19 parts, and therefore so likewise is *e g a,* as if at the centre of the circle. But the angles at *a* and at *b* are right angles; therefore, since [the angle at *e*] is common to the two triangles *c b e* and *a e g*, those triangles will be similar, and so the ratio of their sides is known. Now *b c* is, as has been said, five thousand thousand paces; therefore *a e* is 298 thousand paces, and to that height the vapours rise. Do you see how far the subtlety of human wit can reach? I am well aware that Witelo was mistaken when he taught that they rise only to 52 thousand paces, since they rise more than five and a half times as far as he said. And although he contracts the earth's circumference and on that account makes the paces somewhat larger, even so he cannot bring the figure down to a quarter of the height that is due. But if the twilight is taken at its maximum, so that it occurs two hours before daybreak, then the angle *c* at the circumference — which is equal to *g* — will be of 60 parts, and *e* of 120; and so the line *a e*, which is the height of the vapours, will be 772 thousand paces. This is the greatest space to which vapours can rise from the earth."

Diagram. A small woodcut set into the left column, illustrating the argument just quoted from Cardano. A circle stands at the right of the figure — the earth — and

a straight horizontal line runs in from the left and meets it. A second, longer straight line crosses the first obliquely, running from upper right down to lower left, so that the two lines cut one another close to the circle. Readable labels:

- *d* (at the left-hand end of the horizontal line).
- *g* (on the horizontal line, at the crossing).
- *a* (on the horizontal line, to the right of *g*, where it meets the circle).
- *e* (at the upper end of the oblique line).
- *b* (just below *g*, inside the angle at the crossing).
- *f* (at the lower left end of the oblique line).
- *c* (within the circle, at its lower right): the earth's centre.

So far Cardano — whom I shall show at once to have been grossly deceived, and not Witelo, who borrowed that fine demonstration of the height of the uppermost vapours from Alhazen. Alhazen's little book *On Twilights* I published in print twenty years ago, together with another on the same subject written by myself. The cause of Cardano's error was this: he supposed that the uppermost vapours which produce the twilight lie at *e*, whereas they are not there. For the first ray at the beginning of morning twilight that shows us reflected light is *f g e*, and the reflection itself takes place on the horizon at *g*, and therefore not at *e*. For who has ever seen the twilight light stand overhead at the zenith? *a* is in fact the centre of the sensible horizon. The distance of the uppermost vapours from the earth is therefore far less than *a e*.

But so that all this may be grasped more easily, I shall go over the demonstration of the height of the uppermost vapours in the form in which I gave it in the book *On Twilights* just mentioned. In the figure below, let the sphere whose centre is *a* be the body of the Sun, and the sphere whose centre is *b* the globe of the earth. Now conceive a great circle *a p R q* described about *b*, the centre of the world, at the interval *a b*, drawn through the pole of the horizon and through the Sun's centre at the beginning of morning twilight. Let the common section of the plane of this conceived circle with the Sun be the circle *c d e*, and with the earth the circle *f g h*; and from the arc *e c* let the solar rays *c l* and *e l* fall, touching the earth at the points *g* and *h*. Under the arc *g f h*, then, is comprised the part of the earthly globe lit by the Sun's rays, and under the remaining arc *g h* the part that is darkened by the shadow. Let the point *R* be, besides, the pole of the horizon, and let *b R* be joined, cutting the circle *f g h* at the point *t*, at which the centre of sight is placed. Next let the straight line *p q*, passing through the centre of the world, be the common section of the horizon and of the described circle *a p R q*; and let the straight line *z t u* be the common section of that same circle and of another circle

in which the sensible horizon lies, parallel to the conceived horizon that passes through the centre of the world. The two straight lines *p q* and *z u* are therefore parallel, by the sixteenth proposition of the eleventh book of Eucli-

d. And the angle *R b p* is a right angle, because *R p* is a quadrant; therefore the angle *b t u* is a right angle too, which could equally well be concluded from the first book of Theodosius. The straight line *z u* accordingly touches the circle at the point *t*, by the corollary to the sixteenth proposition of the third book of Euclid.

Now since no reflection of light takes place from pure, thin air, let us conceive in the mind a sphere of the vapours rising from land and sea, which thicken and condense the air to the point where it can produce a reflection of the Sun's light. For whatever lies beyond this sphere toward the sky, although it is lit by the Sun during the night, is not visible for want of reflection. Let *y r s* be an arc of a great circle of such a sphere, described about the centre *b* and lying in the same plane as the greatest circle of the earth *f g h*; and let the straight line *z u* cut it at the point *r*. Now even though the Sun's light was reflected from every point of the arc *r s* before the morning twilight, no ray could nevertheless reach *t*, the centre of sight, because below the straight line *t u* no other straight line can be taken that does not cut the circle *f g h*, as is demonstrated on the sixteenth proposition of the third book of Euclid. The earth's roundness was therefore an obstacle preventing whatever lay below that same straight line *r u* from being seen. And whatever falls within the earth's twin shadow *g l h* cannot be looked upon either. The first point, then, that appears illuminated at the beginning of morning twilight, when it grows light, is *t*. For it lies neither in that thinnest and clearest air which gives us back none of the Sun's light, nor within the earth's shadow, nor beneath the plane of the sensible horizon.

So let the straight line *b r* be joined, cutting the earth's circle at the point *o*. The straight line *o r* will then be the greatest height of the vapours that are lifted aloft from the earth, and we shall investigate its length as follows. The angle *p b t* is a right angle; the angle *a b p*, the Sun's depression below the horizon, is supposed known from observation — 19 degrees, namely, according to Alhazen and Witelo. The whole angle *a b t* will therefore be known. From it we shall subtract the angle *a b g*, which we take as known, since it subtends half of the greatest circle of the earth lit by the Sun; and so the angle *g b t* will be left known. Further, the angle that *b g* makes with the straight line *g l*, which touches the circle at the point *g*, is a right angle, by the eighteenth proposition of the third book of Euclid; the angle at *t* is a right angle too. Therefore the two triangles *b r g* and *b t r* have equal sides, by the forty-seventh proposition of the first book and the common notion.

Those triangles are accordingly equiangular, by the eighth proposition of the first book, and the angle *t b r* is half the angle *t b g*. But that angle *t b g* has already been made known; therefore *t b r* will be known too, and so the remaining angle *t r b* of the triangle *b r t* will be known.

Now as the right sine of the angle *t r b* is to the whole sine, so is the straight line *b t* to the straight line *b r*. Of these four quantities the first two are known; and as for the third, the straight line *b t*, how many stadia it contains is known once the number of stadia in the whole circumference *f g h* is supposed from Ptolemy or from Eratosthenes, and the ratio of that same circle to its diameter is supposed from Archimedes. Therefore, by the common rule of proportional numbers, the number of stadia in the straight line *b r* will be known; and from it we shall take away the number of stadia in the semidiameter, leaving *o r* known — that is, the distance at which the highest vapours stand from the earth, which is what we set out to investigate.

Three things are premised by me, and demonstrated, in the book on twilights just mentioned. First: if a luminous spherical body lights another spherical body, the outermost luminous rays must touch both spheres; and if rays falling upon them touch both bodies, those rays must be the outermost and the longest. Second: a luminous sphere lights more than half of a smaller sphere, and the lit part is comprised within a single cone whose vertex falls in the smaller sphere. Witelo demonstrated this in his second book, but Aristarchus of Samos did so far better in his book *On the Sizes and Distances of the Sun and Moon*. Third: given the known distance between the centres of the said spheres and the ratio of the semidiameters — both to one another and to that distance itself — to state in definite numbers the arc of the greatest circle of the smaller sphere under which the lit part is comprised. This last is concluded from the earlier two for the proposed spheres of the Sun and the Moon [?].

Let the straight lines *a c* and *a e* be joined; from *a e* let *e i* be cut off equal to *b h*, the earth's semidiameter, and let *b i* be joined; likewise from *a c* let the straight line *c k* be cut off equal to the semidiameter *b g*, and let *b K* be joined. Since the two angles at the points *e* and *h* are right angles, by the eighteenth proposition of the third book of Euclid, the quadrilateral *b e* is therefore right-angled and a parallelogram; and by the same reasoning you will conclude that the quadrilateral *b c* is right-angled. The angles at the points *i* and *K* are therefore right angles, and so, by the eighth and the forty-seventh propositio-

n of the first book of Euclid, the two angles *a b i* and *a b K* will be equal. Now the two arcs *h m* and *g n* are quadrants, because the angles *h b m* and *g b n* are right

angles. The arc *n m* is therefore the difference by which the semicircle of the earth is exceeded by the arc under which the lit part is comprised, and the arc *f m*, or *f n*, is half of that difference — a quantity that it will be easy to state in definite numbers. For *b h* and *e i*, opposite sides of a parallelogram, are equal to one another; and the ratio of the straight line *a b* both to *a e* and to *b h* is supposed known; therefore the ratio of that same *a b* to *a i* will be known. And in the right-angled triangle *a i b*, as the straight line *a b* is to the straight line *a i*, so is the whole sine to the right sine of the angle *a b i*. The right sine of the arc of the angle *a b i* will therefore be known, and from the table of right sines the arc of that same angle, which is *m f*, will be discovered, and consequently the whole arc *m n* will be revealed.

For example, let the greater sphere be the Sun and the smaller the earth. Then, according to al-Battānī's view, in the parts of which the earth's semidiameter is 1, *a e* is five and a half and *a b* is 1108 at mean distance; therefore *a i* will be four and a half of those same parts. Accordingly we shall multiply 4 1/2 by 100000, the whole sine, and divide the product by 1108; and the division will yield 406 parts of the right sine, to which nearly 14 minutes answer in the table. At mean distances, then, the Sun lights the earth under an arc of a great circle containing nearly 180 degrees 28 minutes.

Diagram. A large woodcut filling the lower three-quarters of the leaf, showing the demonstration just given. A great circle is drawn about the earth's centre; the earth appears as a small circle at its middle, and the Sun as a larger circle at the foot of the great circle, so that the great circle passes through the Sun's centre. Two long straight lines run up from the left and right edges of the Sun's disc, graze the little circle of the earth on either side, and converge at a point high above it — the vertex of the shadow cone. Two nearly horizontal chords cross the figure: the lower one passes through the earth's centre (the true horizon), the upper one just above it, touching the top of the earth's circle (the sensible horizon). A second small arc, concentric with the earth, is drawn a little outside it: the sphere of the vapours. Readable labels:

- *L* (high up, at the meeting of the two long tangent lines).
- *R* (at the top of the great circle, right of *L*): the pole of the horizon.
- *z* (left-hand end of the upper chord) and *u* (right-hand end of the upper chord).
- *q* (left-hand end of the lower chord) and *p* (right-hand end of the lower chord).
- *b* (at the centre of the small circle): the centre of the earth and of the world.

- *t* (on the earth's circle at the top, where the upper chord touches it).
- *r* (just right of *t*, on the arc of the vapour sphere, where the upper chord cuts it).
- *o* (between *b* and *r*, on the earth's circle).
- *y* (on the vapour arc to the left) and *s* (on the vapour arc to the lower right).
- *h* (on the earth's circle to the right) and, on the left of the earth's circle, a letter that reads as *b* on the leaf but which the demonstration requires to be *g* [?].
- *m* and *n* (below the earth's circle).
- On the Sun's disc: *e* (at its left) and *c* (at its right), the ends of the arc from which the rays fall; *K* (on the line *e c*, near *c*); *a* (within the disc, at the top): the Sun's centre; *d* (at the foot of the disc).
- Two further letters stand on the line *e c*, between *e* and *K*; the first reads as *i*, the second as *x*, but neither resolves securely on the leaf [?].

Moreover, so that the height of the topmost vapours above the earth may be computed more easily, we must notice the following. Suppose the Sun did not begin to light us before it had, below the horizon, an arc of concealment equal or similar to the difference between the quadrant of a great circle of the earth and half the illuminated arc: then it would produce no morning twilight at all, for its highest ray would merely graze the horizon. But it does in fact produce morning twilight. Therefore it begins to light us before it is concealed under an arc equal to that same difference between the quadrant and half the illuminated arc. Accordingly the arc of the Sun's concealment below the horizon, at the beginning of morning twilight or at the end of evening twilight, is always greater than the difference between the quadrant and half the illuminated arc.

That difference, then, subtracted from the arc of concealment, leaves an arc equal to the one lying between the point where the Sun's ray touches the terrestrial globe and the centre of the sensible horizon — as one may observe in the figure itself. For the two angles *n b g* and *p b t* are right angles; take away from them the common angle *p b g*, and the two angles *n b p* and *g b t* are left equal. The same angle *n b p* is also what remains when the angle *a b n* — the angle of the difference between the quadrant and half the illuminated arc — is subtracted from the angle *a b p* of the Sun's concealment at the beginning of morning twilight; for the same judgment holds of angles as of arcs.

Whenever, then, one wishes to measure the greatest height of the vapours, we shall multiply the difference of the semidiameters of the Sun and the earth into the whole sine, appending five zeros, and divide the product by the distance

between the centres; and the right sine of the difference between the quadrant and half the illuminated arc will result. We shall subtract the arc belonging to that sine from the arc of the Sun's depression, and there will remain the arc between the centre of the sensible horizon and the point at which the Sun's ray touches the terrestrial globe. Then we shall take the complement of half this arc, and by the right sine of that complement we shall divide the number produced by multiplying the whole sine into the number of stadia of the earth's semidiameter. From the division there will indeed result the distance of the topmost vapours from the centre of the earth; and so, once the number of stadia of the semidiameter is taken away, the greatest height to which the vapours rise is left known.

The difference between the quadrant and half the illuminated arc was found to be 14 minutes. We shall therefore subtract it from the 19 degrees of the Sun's concealment, and there will remain 18 degrees 46 minutes. Half of this arc is 9 degrees 23 minutes, and its complement, 80 degrees 37 minutes, has a right sine of 98661. Now let the 40090 stadia which the semidiameter contains — if with Cardano we accept Eratosthenes' view of the earth's circumference — be multiplied into the whole sine, and the product will be 4009000000. Let this in turn be divided by 98661, and the division will yield 40634 stadia. From these we shall take away 40090, and there is left as the total height of the vapours 544 stadia, or 68 miles. But by Alhazen's reckoning you will find only 52 miles, because he set the earth's circumference at 24000 miles — a result that agrees exceptionally well with sailors' observations.

Cardano further judges the angle *f g d* to be of 19 parts, as though it were at the centre of the earth, and holds that this comes about because of the Sun's very great distance in comparison with the earth. But from what we have demonstrated it plainly appears to be 18 degrees 46 minutes: the 14 minutes of the difference between the quadrant and half the illuminated arc are wanting. The magnitude of the Sun's distance in comparison with the earth produces a greatest parallax which, as we said above, is 2 minutes 51 seconds on Ptolemy's view, but 3 minutes 13 seconds according to al-Battānī. The angle *f g d* in this figure of Cardano's is the angle *c r u* in our own figure; and the angle *C S P* is equal to *c r u*, because the lines *z u* and *p q* are parallel. The angle *k b p*, again, is equal to *c s p*, since *b K* and *c s* are parallel. Therefore the two angles *K b s* and *c r u* are equal, by the common notion.

Diagram. A small woodcut set into the column beside the text. A straight line runs across the figure from left to right through the points *d*, *g* and *a*, with *a* at its

right end. At the lower right stands a circle, with *c* marked within it near the centre; *b* is marked just above and to the left of that circle. The point *e* stands high at the upper right and *f* low at the left, and lines are drawn from them across the horizontal line, crossing it near *g*. The exact course of two of the lines is obscured by show-through on the leaf.

- *d* (left, on the horizontal line): endpoint of the horizontal line
- *g* (centre, on the horizontal line): vertex of the angle *f g d*
- *a* (right, on the horizontal line): endpoint of the horizontal line
- *b* (left, just outside the circle)
- *c* (within the circle, at its centre)
- *e* (upper right)
- *f* (lower left)

Furthermore, the angle *a b p* of the Sun's concealment is greater than *K b s*, for their difference is *a b K*, that is, 14 minutes. Therefore the angle *c r u* is less than *a b p*; and so, if *a b p* be put at 19 degrees, *c r u* will be 18 degrees 46 minutes. Nor will that same angle *f g d* in Cardano's own figure be greater or less — which was to be shown.

He will indeed be able to plead no excuse, unless he says that the Sun does not begin morning twilight before the ray from its centre, falling upon the sphere of the vapours, produces a reflection — as though the other rays could not light the air. Witelo concludes the contrary in Book 2, proposition 17, from the theory of shadows, and

Cardano himself shows in the same fourth book that rays go out from the whole Sun in every direction, arguing from eclipses: for the part (he says) that lies opposite the centre of the Sun is taken up by the Moon, and even then the air and the walls are lit. Besides, if it is the ray from the centre that can produce the reflection, and no other, then in the evening, once the Sun's centre stands on the horizon, evening twilight will begin — yet it will not begin until the solar body has first been hidden below the horizon, since before that the Sun was lighting us with its primary light, that is, with direct rays. And for that reason, at the beginning of morning twilight when day breaks, the rays that produce the reflection of light are other rays, not the central ones.

Cardano supposes besides — so far as I can understand from what he writes — that the arc of the Sun's concealment below the horizon, taken on the altitude circle, is equal to the arc of the Sun's distance from the point of rising, when the Sun runs along the equinoctial. For he says the Sun begins twilight 19 parts

before its rising, nearly an hour and a quarter before the Sun's own ascent; and if (he says) twilight is drawn out to its maximum, so that it falls two hours before day, the angle of the Sun's concealment will be of 60 parts on the circumference. So if it has 60 parts on the circumference, it has 30 at the centre; and accordingly the arc of concealment on the altitude circle will be equal to the arc of the length of twilight on the equinoctial. We shall show that this happens only for those who live under the equinoctial, and for them only on the very day of the equinox, since for them on that day the equinoctial circle becomes an altitude circle.

For in the heavens let *a b c* be the horizon, *b e d* the meridian, *a c f* the equinoctial; and let the point *e* be the zenith of those placed off the equinoctial. Let the Sun be set at the point *f* of the equinoctial below the horizon, at the beginning of morning twilight, and let the vertical circle be *e g f*, of which *e g* is a quadrant and *g f* the arc of the Sun's concealment. I say that *c f* is greater than *g f*. For the angle *c g f* of the triangle *c f g* is a right angle, and the angle *g c f*, that of the complement of the pole's altitude, is acute; therefore the arc *c f* is greater than *g f*.

But now let the circle *a c f* be not the equinoctial but a circle parallel to it. I say again that the arc *g f* is less than the equinoctial arc proportional to *f c* which rises together with it. For through the two points *c* and *f* let a great circle be drawn, and let its segment between *c* and *f* be *c h f*. Since the arc *f g* is less than a quadrant — for it contains the 19 degrees of the Sun's concealment below the horizon at the beginning of morning twilight — the angle opposite it, the one which *c g* makes at the point *c* with the arc *c h f*, is therefore acute. And for that reason, in the right-angled triangle *c f g*, formed of segments of great circles, the side *f g* will be less than *c h f*. But the equinoctial arc which rises together with the arc *c f* of the parallel circle, being proportional to it, is greater than that same *c h f*. Therefore the arc *f g* of the Sun's concealment on the altitude circle is less than the equinoctial arc which rises from the beginning of morning twilight up to the Sun's rising. You will be able to demonstrate by the same method that the same thing happens for those who live under the equinoctial, when the Sun has been placed off the equinoctial itself.

We shall now at once demonstrate the two things we assumed. First, that the arc *c f* of the parallel circle rises together with the equinoctial arc proportional to it which is terminated at the section of the horizon. For let *i m* be an arc of the equinoctial circle *k i l* proportional to *c f*, and let meridians be drawn through *c* and *f*, whose segments between the points *c* and *f* and the equinoctial are *c n* and *f l*. Then the arc *n l* will be proportional to *c f*, by the 14th proposition of the second

book of Theodosius. But the arc *i m* is also proportional to that same *c f*, by hypothesis; therefore the two arcs *i m* and *n l* will be equal to each other. And since the motion of the equinoctial is uniform at every time, then, as the sphere turns,

Diagram. A woodcut at the foot of the left-hand column: a circle representing the sphere, crossed by two great-circle arcs that meet within it. *b* stands at the top of the circle and *d* at the bottom, on the meridian; *a* lies at the left on the horizontal arc. Within the circle *c* is set at the upper left and *g* at the centre, with *h* immediately to the right of *g*; *e* lies below and to the right of *g*. The point *f* is set outside the circle at the right, and lines run to it from within, closing a narrow lens-shaped figure with the arc through *h*.

- *b* (top of the circle)
- *d* (bottom of the circle)
- *a* (left, on the horizontal arc)
- *c* (upper left, within the circle)
- *g* (centre of the circle)
- *h* (immediately right of *g*)
- *e* (below and right of *g*)
- *f* (outside the circle, at the right)

Diagram. A companion woodcut at the foot of the right-hand column, of the same construction: a circle with *b* at the top and *a* within it at the upper left; *K* stands at the left on a chord running across the circle, and *c* at the right, with *f* just outside the circle beyond *c* and *l* below *f*, outside the circle. Further letters on the two lower arcs are too faint to read on the leaf [?].

- *b* (top of the circle)
- *a* (upper left, within the circle)
- *K* (left, on the lower arc)
- *c* (right, within the circle)
- *f* (just outside the circle, at the right)
- *l* (outside the circle, below *f*)
- further letters on the lower arcs: [?]

when *l* is where *n* is, *m* will be where *i* is. But when *l* is where *n* is, the meridian *f l* will have the position *c n*, and *f* will be where *c* is. Therefore, when *m* is where *i* is, *f* will be where *c* is; and accordingly the equinoctial arc which has its terminus

at *i*, the section of the horizon, being proportional to *c f*, rises together with it — which was to be shown.

We shall demonstrate besides the other thing we assumed: namely, that the arc of the equinoctial proportional to *c f* is greater than the arc *c h f*. For in the plane of the circle *a c f*, whose centre is *o*, let a great circle be drawn through *c* and *f*, and let its arc between the same points *c* and *f* be called (as before) *c h f*; for these are equal, even though they lie in different planes, because they have the same straight line subtended under them. Then let *o c* and *o f* be drawn out as straight lines to the measure of the semidiameter of a great circle — let that be *o p* or *o q* — and with that same interval *o p* or *o q*, about the centre *o*, let a great circle *p q r* be described. The circle so described will therefore stand in for the equinoctial, and its arc *p q* will be similar, or proportional, to the arc *c f* of the lesser circle, by the last definition of Euclid's third book. Now let *c f* and *p q* be joined by straight lines. Since the rectilinear triangles *o p q* and *o c f* are similar, *p q* will for that reason be greater than *c f*; therefore the arc *p q* will be greater than the arc *c h f* — which was to be shown.

Diagram. A woodcut set into the left-hand column. A large outer circle carries *p* at the upper left and *q* at the upper right; from each of these a straight line descends towards the lower centre, the two converging in a narrow V. Between the two lines, near their upper part, lies a narrow lens-shaped figure closed by a chord and an arc, with *c* at its left end and *h* marked just below it; the letter at its right end is not legible on the leaf [?]. Below the convergence stands a smaller circle, with *o* marked within it towards the top and *a* marked within it below *o*.

- *p* (upper left, on the outer circle)
- *q* (upper right, on the outer circle)
- *c* (left end of the lens-shaped figure)
- *h* (below the chord, between the two lines)
- right end of the lens-shaped figure: [?]
- *o* (within the inner circle, towards its top)
- *a* (within the inner circle, below *o*)

On the distance of the pole star from the arctic pole of the world, and on its true place. The method sailors use for finding the altitude of the pole above the horizon by the stars of the Lesser Bear is also examined.

Caput. 7.

The star set at the very tip of the tail of the Lesser Bear is called the pole star because in our own time it is the nearest of all to the arctic pole: sailors of our age affirm that it stands only 3 degrees 30 minutes from that pole. But if the motion of the fixed stars found by Johannes Werner's computation from the Alphonsine Tables is true, that distance amounts in our time — that is, in the year 1500 — to four degrees and about 9 minutes. If we accept al-Battānī's view, you will make the said distance somewhat less than if you follow Alfonso; yet the time will come when this same star withdraws to about half a degree from the pole of the world, namely when it has completed the sign of Gemini, in which it now stands. For its latitude is 66 degrees, the least of all the remaining stars of the same constellation, and so its distance from the northern pole of the zodiac, 24 degrees, is the greatest.

It was therefore not without reason that Marinus, following Hipparchus (Ptolemy reports this in chapter 7 of the first book of the *Geography*), called that last star of the tail the southernmost — especially since in that age it was also the farthest from the pole of the world, standing 12 degrees and two fifths away, although now it is the nearest of all. Failing to notice this, certain translators of Ptolemy rendered it "northernmost", though the Greek codex protests against it. In Werner's translation, however, and in Bilibald's [Willibald Pirckheimer] earlier edition, you will find "southernmost".

There is besides, in the same passage, another notable error of all the translators: for five hundred stadia they have put five thousand five hundred. To make this easier for you to understand, I shall set out the author's meaning. For those who live under the equinoctial circle, he says, all the stars rise and set; but as one moves on from there towards the arctic pole of the world, certain stars of the Lesser Bear are left above the horizon without setting — though not the whole figure — until one has come to places 500 stadia further north than Ocelis. There the whole Lesser Bear, and it alone, will first appear above the horizon without rising or setting, while the last star of the tail will be seen to touch the horizon. For since at Ocelis the north pole is raised above the horizon 11 degrees and two fifths, at 500 stadia — that is, one degree beyond Ocelis — the same pole will be raised 12 degrees and two fifths. And since he found so great a

distance for the last star of the Lesser Bear's tail from the pole itself, then that last star of the tail will trace a complete circle above the horizon — a circle which must nevertheless touch the horizon at one point.

Now in Hipparchus's day that same star was, of all the others in the same constellation, the farthest from the pole of the world. The whole figure, therefore,

was first left above the horizon without setting precisely in those places 500 stadia further north than Ocelis. For the remaining constellations this could not yet happen, since they are further from the pole than the Lesser Bear is. From all this it is plain that five thousand stadia were added by the translators of Ptolemy, and that no more than five hundred are found in the Greek codex.

One must notice, however, the following. Suppose the latitude of the star that stands last in the tail of the Lesser Bear is put at 66 degrees, as much as Hipparchus and Ptolemy found; and suppose besides that its distance from the beginning of Cancer in Hipparchus's time is put at 32° 30′ — for that is what you will find if you count back 2° 40′ from the tenth minute of the first degree of Gemini, where the same star stood in Ptolemy's time, that being the amount by which the fixed stars had advanced from Hipparchus to Ptolemy. On these assumptions it is impossible that the star's distance from the pole of the world in Hipparchus's time was twelve degrees and two fifths — and this holds even if you suppose the Sun's maximum declination to have been other than they themselves set down.

Diagram. A small woodcut set into the left column, showing a figure bounded by curved sides — the spherical triangle of the argument, made of arcs of great circles. The print here is faint and partly rubbed on the scan.

- b (left vertex): `b` — the star at the end of the tail
- c (lower right): `c` — the point on the solstitial colure
- third vertex (upper right): label not legible `[?]` — by the text it is the point *a*

For in the spherical triangle *a b c*, made up of segments of great circles, let *a* be the pole of the zodiac, [Illegible: roughly a line and a half where the woodcut is set into the column; the words still visible are "…ei Boreus"], and let *b* be that star which stands at the end of the tail. Arc *a b* is 24 degrees, angle *a* is 32° 30′; and let arc *b c* fall at right angles upon the solstitial colure *a n c*. That arc *b c* will therefore be the shortest distance of the star *b* from the colure, and it is found to be twelve degrees and thirty-seven minutes. Consequently, from any other point of the same colure — whether above *c* or below *c* — the same star will stand off by a still greater arc than 12° 37′.

Now further: in the triangle *d e f*, let *d* be the pole of the zodiac and *f* the pole of the world, and let arc *d f*, the distance of the poles, be 23° 51′, as much as Hipparchus found (Ptolemy testifies to this); and let the angle *d* of 32° 30′ be kept. Then, if *d e* is the arc of the great circle passing through the pole of the zo-

Diagram. A woodcut at the head of the right column: a curvilinear triangle with its apex at the top and a base drawn between the two lower corners.

- d (apex, top): `d` — the pole of the zodiac
- e (lower left): `e` — the foot of the perpendicular on arc *d e*
- f (lower right): `f` — the pole of the world

diac and the star, and if arc *e f* falls at right angles upon *d e*, then arc *e f* will be the shortest distance of the pole of the world from the circle *d e*, and it is found to be 12° 33′. And therefore, if you place the star itself either above *e* or below *e*, it will stand off from the north pole of the world by a still greater distance than 12° 33′.

In the first arrangement, if you make the point *c* the north pole of the world, the distance of the poles left over will be much less than 23° 51′.

In the second, if you place the star itself at *e*, you will find the arc *d e* much less than twenty-four degrees. And if you wish to vary both distances — that is, the Sun's maximum declination and the complement of the star's latitude — so that arc *e f*, or *b c*, is left at 12° 24′, you will have to make the Sun's maximum declination much smaller, and the complement of that same star's latitude will also come out less than Hipparchus and Ptolemy set down. And therefore, unless the star's distance from the beginning of Cancer is cut back — that is, unless you make the angle *d* less than 32° 30′ — all these things cannot stand together at once.

We shall therefore set the star's distance from the pole of the equinoctial at twelve degrees and two fifths, the Sun's maximum declination at 23° 51′, and the complement of the star's latitude at 24 degrees; for these three were so set down by Hipparchus. And by the sixth proposition of our book *On Twilights* the angle *d* will be found — that is, the distance of the last star of the Lesser Bear's tail from the beginning of Cancer — to be 30° 53′. In Hipparchus's day, then, that same star was at 29° 7′ of the sign Taurus. Add the 2° 40′ by which the fixed stars advanced in the 265 years down to Ptolemy's time, and the star's place in Ptolemy's time was therefore 1° 47′ of Gemini. Yet in the seventh book of the *Great Composition of the Stars* [*Almagest*] the same star is placed at the tenth minute of the first degree of that same sign; the difference is therefore one degree and thirty-seven minutes. So if the case stands thus, the star in question has advanced further than the astronomers' calculation shows, by that very difference of one degree 37 minutes; for everyone's reckoning assumes Ptolemy's figures, and consequently

the star is nearer the arctic pole in our own age than they suppose. This uncertainty could be resolved at once if the same star were observed when

it has its greatest altitude and when its least — or if either the greatest alone or the least alone is taken, provided the elevation of the pole above the horizon is known beforehand from observations of the Sun at midday. And although a tiny error in the declination of the parts of the ecliptic near the tropical points produces a great variation in longitude, that cannot happen with stars that have a large latitude. For if you set the said star's distance from the beginning of Cancer at 32° 30′ in Hipparchus's time — as you necessarily must if you have used Ptolemy's calculation — you will still find its distance from the north pole of the world to be no less than thirteen degrees with two minutes besides.

The difference from 12° 24′ is therefore thirty-eight minutes, and these correspond to the one degree and 37 minutes of difference in longitude between 30° 53′ and 32° 30′. So in the end the difference of declination amounts to about two fifths of the difference of longitude; and from Hipparchus's times down to our own age the difference of declination keeps roughly the same proportion to the difference of longitude, and will keep it perpetually in future, until it reaches the point closest to the pole.

Augustinus Ricius [Agostino Ricci] thinks otherwise, however. Arguing against Ptolemy, he maintains that fixed longitudes cannot be determined from the declinations of the stars from the equinoctial, because (he says) the smallest particle of declination produces a great and notable variation in longitude — which is not altogether true. Still less probable is it that Ptolemy erred by one degree and six minutes in the places of the Sun and the Moon and of the fixed stars, which the same Ricci tried to show by a very slight and misleading argument. Its substance is this. Ptolemy (he says) believed the Sun's motion to be slower than experience afterwards revealed it to be; for he set the length of the year at 365 days and a quarter, less the 300th part of a day, while later astronomers such as Alfonso and others set it, more accurately, at the same number of days with a quarter less the 136th part of a day. The difference between the motions of Ptolemy's calculation and Alfonso's (if you reckon it correctly) will therefore be, over 265 years, one degree and six minutes. And since no method can be more certain for learning the motions of the fixed stars than a conjunction of the Moon with one of the fixed stars, or the distance between Moon and star measured with instruments — for from the Moon's place the star's place becomes known — it was indeed by that art that Ptolemy

determined the place of the star in the Heart of the Lion at the middle of the third degree of Leo. Wherever, then, an error has been made in the Moon's place, there too there will be an error in the place of the star observed. And however great the slip in the Sun's place, just so great will it be in the Moon's; for the Moon's place could be determined only from its distance from the Sun. Ptolemy therefore [?] ... the years by which he himself [?] Hipp[archus] ... [?] of the Sun 1 degree [?] minutes ... of the fixed stars committed just as much. [Illegible: the outer ends of four lines here are stained and lost on the scan.] But the answer to this argument is that Ptolemy observed the Sun's entries into the equinoctial points with the greatest care; and unless later astronomers assumed his observations and the roots of the motions to be true, they could not establish any fixed length for the year. With the help of instruments, then, he found most exactly the time at which the Sun occupied the beginning of Libra. And since his examinations of the fixed stars were made at very nearly the same period, then even though he believed the Sun's motion somewhat slower than the moderns have made it, he could not on that account fall into any perceptible error in computing the Sun's motion over those few years, so little removed from his root. To make this clearer still, we shall report the observation Ptolemy made on the star in the Heart of the Lion — as Ricci himself does.

In the second year of Antoninus, on the ninth day of the Egyptian month Pharmouthi, at Alexandria, as the Sun was setting, five and a half hours after midday, he examined the Sun and the Moon with an instrument, and the Moon's apparent distance from the Sun was 92° 7′ 30″. Half an hour later, when the Sun had already set, he saw that the star in the Heart of the Lion was 57° 10′ from the Moon in the order of the signs, along a circle drawn through the middle of the zodiac. Now the Sun was at about 3° 3′ of the sign Pisces; so the Moon appeared to be at about 5° 10′ of Gemini. Adding then 15 minutes for its motion in half an hour, and subtracting five for parallax, the Moon's place is left at last at 5° 20′ of Gemini, when the Sun was already below the horizon. Therefore the star in the Heart of the Lion, being at that moment 57° 10′ from the Moon in the order of the signs, held 2° 30′ of Leo. But Ricci maintains that the Sun was then at 4° 9′ of Pisces, the Moon (given the said distance from the Sun) at 6° 26′ of Gemini, and the Heart of the Lion at 3° 36′ of Leo; for he asserts that the Sun had at that time advanced further by one degree and six minutes. We, however, shall show most plainly

that the place of the Sun found by Ptolemy at the time of that observation was correctly determined; and this will be established from other most careful observations made by him. Among his other observations of the equinoxes, he

says he made a most exact one in the autumn of the 17th year of Hadrian, on the 7th day of the month [?], [Illegible: the ends of four lines are damaged; the words still readable are "...according to the Egyptians, after midday ...equal, He gathers moreover ...of Nabonassar down to"] the autumn equinox set out above: 879 Egyptian years, 66 days, and 2 equal hours.

And since in the second year of Antoninus 885 Egyptian years had run since the beginning of Nabonassar's reign — as he says in the fifth book — there were therefore, from Nabonassar's reign down to the time of the examination of the star in the Heart of the Lion mentioned above, 885 Egyptian years, 218 days, 5½ hours. Subtract from these 879 years, 66 days, 2 hours, and there remain 6 years, 152 days, 3½ hours, by which the observation of the star in the Heart of the Lion was later than the observation of the equinox. If, then, you compute the Sun's mean motion for that interval of time by Ptolemy's tables, you will find that beyond whole revolutions the Sun had run through 148° 30′. And since at the time of that autumn equinox the Sun was distant from the apogee [aux] by mean motion 116° 40′ — for the apogee was at 5° 30′ of Gemini — and the difference between the true and the mean motion is 2° 10′, it follows that in the second year of Antoninus, when the star in the Heart of the Lion was being observed, the Sun was distant from the apogee by mean motion 265° 10′. To these must be added the 2° 23′ of the equation, or difference, and there results an arc of 267° 33′ of true motion, reckoned from the apogee. The Sun was therefore at 3° 3′ of the sign Pisces — the very place in which Ptolemy found it at the time of the observation.

But if you compute the Sun's mean motion by Alfonso's tables for the six years, 152 days and 3½ hours that fell between those two observations, you will find 148° 31′ 40″, whereas by Ptolemy's tables 148° 30′ was found before; the difference is therefore 1′ 40″. And so, by Alfonso's calculation, the Sun should have been found at 3° 4′ 40″ of Pisces. In the same way the Moon and the star in the Heart of the Lion had advanced further by 1′ 40″ — not by one degree and six minutes, as Ricci has it.

The same thing can be shown again in another way. The second year of Antoninus was the 462nd from the death of Alexander, as is clear from the second chapter of the third book of Ptolemy's *Great Composition*. When, therefore, he was observing the star

in the Heart of the Lion, there had passed since the death of Alexander 461 Egyptian years, 218 days, 5½ hours; while the observation of the autumn equinox that we recalled fell 455 years, 66 days and 2 hours after Alexander's death, as is gathered from chapter 8 of that same third book. So if the smaller number is

subtracted from the greater, there still remain six years, 152 days, and 3½ hours; and therefore the same place of the Sun will be obtained as in the earlier example.

You will find that this agrees with another most exact observation which Ptolemy made of the autumn equinox, on the ninth day of the month Athyr, about one hour after sunrise, in the third year of Antoninus, the 463rd from the death of Alexander; for 462 years, 67 days and 19 hours had elapsed. The difference, then, between the time of the observation made on the star in the Heart of the Lion, mentioned above, and this autumn equinox, is 214 days and 13½ hours. The Sun's mean motion in that interval, by Ptolemy's tables, is 211° 29′. To these we shall add 148° 30′ — that is, the mean motion between the first observation of the autumn equinox and the time at which he made his examination of the star in the Heart of the Lion — and they will make up 359° 39′ [?]; so that for complete revolutions of the Sun between the two equinoxes mentioned only one minute is wanting. Ptolemy's observations therefore square exactly with the place of the Sun he found.

But suppose you now wish, using Ptolemy's tables, to find the Sun's true place for the second year of Antoninus, the ninth day of the month Pharmouthi, and 5½ hours after midday, working upward from the root [radix] of Nabonassar and reckoning from the beginning of his years forward in the order of the signs down to the time set out: you will light on exactly the same place, namely 3° 3′ of the sign Pisces. For although Ptolemy made the Sun's motion slower than the younger astronomers have found it, and although that root cannot be true which he set at the 45th minute of the first degree of Pisces for the beginning of Nabonassar's reign — arriving at it by counting back through parts of the circle of the signs from the autumn equinox of Hadrian's 17th year — and although the slip is a notable one, it is nevertheless certain that if you add to that same root a uniform motion answering to the difference of the times, you will light again on the same place of the zodiac which he found to be 116° 40′ from the apogee. And going forward from here, and computing the uniform motion by the same tables down to the second year of Antoninus, and to the very day and hour of the observation of the Heart of the Lion, the true

place again at 3° 3′ of the sign Pisces.

But here is what settles the whole controversy: Ptolemy found the Sun's place for that moment not by computation but by instrument and observation, and that is why he added 3 minutes beyond the 3 degrees — on account of parallax [*aspectus diuersitas*], which near the horizon was not to be neglected. For he was able to

determine the Sun's distance from the meridian in degrees of the horizon from the shadow of the gnomon, and at the same time its distance from the zenith [*verticale punctum*]. The altitude of the pole at Alexandria he already knew; and so, in a spherical triangle, once two sides and the angle they contain are known, the third side and the remaining angles become known. The Sun's distance from the meridian in degrees of the equator, then, and its declination at the same moment, cannot escape him. From the declination it was easy to find the Sun's place. And all of this he could learn from the armillary instrument alone, without any multiplication or division of numbers.

Now between those two observations of the autumn equinox there elapsed — as is perfectly clear from the evidence I have set out — seven Egyptian years, one day, and 17 hours. If you compute the Sun's uniform motion for that interval, again and again, by Ptolemy's tables, you will find that only one minute is wanting to complete the revolutions exactly.

Girolamo Cardano therefore wrote carelessly, in his little book *On the Restitution of Times*, that the one observation stands distant from the other by exactly eight solar — not Egyptian — years. For he had gathered from chapter 8 of Book 3 that the earlier observation was made 455 Egyptian years, 66 days and 2 hours from the death of Alexander — that is, on the seventh day of the month Athyr, at the second hour; and since the later one fell in the year 463 from Alexander's death, on the ninth day of the same month, he subtracted the smaller number of years from the larger. Because eight are left, he supposed that eight Egyptian years had intervened, out of which — together with the two days' difference between the seventh and the ninth of Athyr — eight solar or Roman years would be made up. But he failed to notice that when the earlier observation was made, 455 years had elapsed since Alexander's death and the 456th was running, whereas at the later one the 463rd year was running and 462 had elapsed. So seven years are left as the difference. And even if eight years had intervened, they could not have been solar years, since in eight years the return could not be made from the second hour after midday to one hour after sunrise. Noticing this himself, he says: "let us suppose that those observations were not exact as to the hours" — for it could not have happened that within a span of eight years the second observation should have run ahead of the first by seven hours. But it is strange that Ptolemy should declare both observations perfectly exact, when so great a slip is found in eight years. Let Cardano see, then, how the conclusions he draws can follow; and let us return to the point we digressed from.

We must observe, then, that just as the declination of any star standing on the meridian is disclosed from the known altitude of the pole above the horizon, so conversely the altitude of the pole becomes known from the star's declination. Sailors, however, since they know very few stars, look for the altitude of the arctic pole only by the star at the tip of the Lesser Bear's tail and by the two stars of the last side of the quadrilateral of that same figure, which through almost the whole of this northern region are visible all night long. And because the same stars do not come to the meridian on every night, they have for that purpose certain canons — got, perhaps, from some unskilled mathematician — from which they draw off how much the pole star's altitude, in any position it may hold, is greater or less than the elevation of the north pole. So they suppose that on any night, and not once only but repeatedly, the pole's elevation is made plain from the observed altitude of the pole star together with its known distance from the meridian position. But they are deceived very often. For when the star stands off the meridian, it is not depressed or raised by one and the same difference under every horizon.

Let *d g* be a segment of the meridian less than a quadrant, in which *d* is the north pole of the world and *g* the zenith of one place. From the point *d* draw the arc *d b* of a great circle at right angles to *d g* itself, and let the pole star be placed at *a*, between *d* and *b*. Then, with a great circle drawn through *a* and *g*, and about *g* as pole of the horizon, let a circle be described on the surface of the sphere with the interval *a g*, cutting the meridian at *c*. Then *d g* will be the comple-

Diagram. A woodcut at the foot of the right column: a set of arcs of great circles meeting at a common point on the right. Along the meridian on the left, the labels stand one below another; three arcs run from those points to the star's position, and a horizontal arc runs from the pole to a point on the right.

- e (left edge, uppermost): `e` — where the circle drawn about *f* cuts the meridian
- c (left edge, below *e*): `c` — where the circle drawn about *g* cuts the meridian
- d (left edge, below *c*): `d` — the north pole of the world
- a (centre, where the arcs converge): `a` — the pole star
- b (right end of the horizontal arc): `b` — the end of the arc *d b*, at right angles to *d g*
- f (left edge, below *d*): `f` — the zenith of the second place
- g (left edge, lowest): `g` — the zenith of the first place

ment of the pole's altitude, and *a g* the complement of the altitude of the star *a*. So *d c* will be the difference between the altitude of the pole *d* and the altitude of the pole star *a* itself — and I shall show that this difference necessarily varies under every horizon.

For let *f* be the zenith of another place, lying between *g* and the same pole; and, with a great circle drawn through *a* and *f,* let the circle *a e* be described about *f* as pole of the horizon, with the interval *a f*. Then the arc *d e* will be the difference between the altitude of the pole and the altitude of the pole star *a*. But *d e* is greater than *d c*. So although the position of the stars is the same, and the same relation to the meridian's position is preserved, nevertheless the same difference between the altitude of the pole and that of the pole star will not be preserved in every climate — which is what I meant to show.

That the point *e* stands farther from the pole *d* than *c* does is clear from this: the two arcs *a f* and *f g* taken together are greater than *a g* itself, and therefore *e f* and *f g* will be greater than *c g*. When the common *f g* is taken away, then, *e f* will be left greater than *c f,* and so the point *e* will stand farther from the pole *d* than *c* — which was assumed in the demonstration.

Below, therefore, I shall hand down a surer method, by which we can find the altitude of the pole at whatever time we please.

¶ On finding the altitude of the pole by the meridian altitudes of the Sun and of the fixed stars. Cap. 8.

The canons that sailors are accustomed to use for finding the pole's altitude above the horizon at the time of meridian passage I have summarized more clearly and more surely in the following way.

The declination the Sun has on the very day of the observation is to be subtracted from a quadrant if it is found northern, but added to the quadrant if southern; for the number left by the subtraction, or made up by the addition, will be the Sun's distance from the north pole of the world. Then on the same day of observation you will determine, by an astrolabe or by any other instrument fit for the purpose, the Sun's least distance from the zenith. Subtract this from the Sun's distance from the arctic pole already found, if the zenith has been placed between the Sun and that arctic pole; but add it, if the Sun is found to stand

between that same pole and the zenith. For the number of degrees and minutes that comes out of such a subtraction or addition will be the distance of the zenith from the north pole of the world, and from it the latitude of the place at once

becomes known, since the altitude of the visible pole [*manifestus polus*] above the horizon is equal to it. For if such a distance is found equal to a quadrant, the zenith will lie on the equator. If unequal, its difference from a quadrant will be the latitude of the place — northern if the distance found is less than a quadrant, southern if greater.

How you may know whether the Sun lies between the north pole of the world and the zenith, or on the contrary the zenith lies between the Sun and that same pole, will not be difficult. For if, with your face turned to the Sun at the very time of observation, when it is nearest the zenith, you see it carried round with the world from left to right, you may be certain that the zenith has been placed between the Sun itself and the arctic pole. But if from right to left, then you will not doubt that the Sun stands between the zenith and that same arctic pole. Sailors learn the same thing from shadows and from the mariner's instrument; but my way is simpler and easier, and needs no instrument at all.

This further point remained to be said: if the Sun is found overhead, the latitude of the place will be of the same kind and the same size as the Sun's declination. Note besides that in the most northerly places — those lying between the north pole of the world and the circle described by the pole of the zodiac in the daily motion — when the Sun is in the northern signs it neither rises nor sets for some days, but within twenty-four hours has two meridian altitudes, one greatest and the other least. You will therefore be able to find the latitude of the place not only by the greatest, as has been said, but also by the least, in another way. For you will subtract the Sun's distance from the pole from the greatest distance between the zenith and the Sun — that is, from the complement of the least altitude — and there will be left the arc of the distance between the zenith itself and that same pole of the world; and so the latitude of the place cannot remain unknown. The same is to be done in the most southerly places, those lying between the other circle described by the pole of the zodiac and the south pole. For you will subtract the Sun's distance from the south pole of the world itself from the complement of the least altitude, and there will be left the distance between the zenith and that same south pole. And wherever it happens

over some interval of time that the Sun's altitude above the horizon neither increases nor decreases, know that the pole of the world is overhead.

The demonstrations of all this are very easy: from the common notions, whatever I have set down in this place can be concluded at once. In observations of this kind I judge that the Sun's parallax in the circle of altitude is to be neglected. And by exactly the same art by which the altitude of the pole above the horizon is

found from the Sun's meridian altitudes, whether greatest or least — as I have taught — you will be able at night as well to determine that same elevation of the pole from the meridian altitudes of the stars. For the method is entirely the same, and so is the procedure.

¶ Our old canon for finding the latitude of a place by the meridian ray. Cap. 9.

We shall observe the Sun when it has its greatest altitude above the horizon, which we shall do at the time of meridian passage. Then, if the shadows of upright bodies on the plane of the horizon are cast toward the same side to which the Sun has declined on the day of the observation, we shall add the complement of the greatest altitude to the declination, and there will be made up the number of degrees and minutes of the latitude of the place, of the same denomination as the Sun's declination — northern or southern. But if the shadows are cast toward the opposite side, then the Sun's declination must be compared with the complement of its own greatest altitude. If they are found equal, the zenith of the place will lie under the equator. If unequal, let the less be taken from the greater, and the latitude of the place will be left: of the same denomination as the declination, if the declination itself is found the greater; but of the opposite denomination, if the less. When the Sun has no declination, the complement of the greatest altitude is itself the latitude of the place, that is, the distance of the zenith from the equator — and toward the side to which the shadows are cast. Whether the shadows are cast to the north or rather to the south you will learn from the mariner's needle [*acus nautica*]. And finally, when the Sun is over-

head, the Sun's declination, if it has any at that time, will be the latitude of the place.

¶ The method of Petrus Appianus [Peter Apian], which he used in his *Cosmographia* to find the altitude of the pole on any day by means of a known hour, is examined. Cap. 10.

That teaching of Apian's, for finding the altitude of the pole through knowledge of the hour, can have no use at all. For anyone who does not know the altitude of the pole will necessarily not know the hour either. This is clear to anyone who understands the construction of sundials and the use of the astrolabe. But if the time elapsed since noon has in fact been learned by other clocks — whether of moving wheels, or of running sand, or of water — it follows that the instant of noon was known exactly from the Sun's ray, and consequently that the latitude of the place, which is equal to the altitude of the pole, could have been known far

more exactly by the Sun's meridian ray, as I taught in the preceding chapter. Indeed, even granting that the hour has been exactly known, and the Sun's degree known too, and its altitude above the horizon determined, I shall nevertheless show by a wholly certain demonstration that the altitude of the pole cannot yet be known universally from these three.

For let *a* be the north pole in the world, *a b* a quadrant of the meridian, *a c* a quadrant of the circle of the Sun's declination, the arc *d c* the Sun's declination, and *d* the Sun itself. Let the arc *b c* of the equator show the hours before or after midday, and let it be put less than a quadrant, so that the angle *a* is acute. From the point *d* let the arc *d f* of a great circle be drawn at right angles to the meridian *a b*. The arc *a d* will therefore be greater than the arc *a f*. Let *d e* be the segment of the diurnal parallel between the meridian and the Sun, and between *e* and *f* let any point *K* be taken; and

Diagram. A woodcut at the foot of the right column: a curvilinear figure with its apex at the top and a curved base, the meridian running down the left side and a series of arcs drawn across to the right-hand point.

- a (apex, top): `a` — the north pole of the world
- (left edge, uppermost of the four stacked labels, immediately below the apex): label not legible on the scan `[?]`
- f (left edge, second): `f` — the foot of the perpendicular arc *d f* on the meridian
- K (left edge, third): `K` — the point taken between *e* and *f*
- e (left edge, fourth): `e` — the end of the segment *d e* of the diurnal parallel, on the meridian
- m (interior, below the stacked labels): `m`
- n (interior, upper right of the convergence): `n`
- d (interior, where the arcs converge): `d` — the Sun
- b (foot of the meridian, lower left): `b`
- C (base, right of *b*): `c` — on the equator, the arc *b c* measuring the hours
- l (right, where the arcs meet): `l`

and above *f* let there be a point *g*, standing off from the perpendicular *d f* by an equal interval, so that the arc *f g* is equal to the arc *f K*; and let great circles be drawn through *d* and *K*, and likewise through *d* and *g*. It is clear, then — by a proposition like the fourth of the first book of Euclid's *Elements* — that the arcs *d K* and *d g* are equal to each other.

And so, with the Sun placed in this way, let us set the zenith [*verticale punctum*] of one place at *K*, and that of the other place, the more northerly one, at *g*. The Sun's altitudes above the horizon will be equal in both places, and the hour will be the same as well — that is, the distance from midday, which the arc *b c*, the Sun's distance from the meridian measured along the equator, shows. Yet the latitude *b g* will be greater than the latitude *b K*, and so the altitudes of the pole will be unequal. Consequently it will be uncertain where the zenith of the place at which such an observation was made actually lies — whether at *K* or at *g*.

Now the interior angles at *g* and at *K* are equal to each other, and both are acute. The circle of altitude *g d* therefore runs into the southern quadrant of the horizon while *K d* runs into the northern, each departing on opposite sides by an equal amount from the intersection of the two horizons with the equator. So if the line of equinoctial rising and setting is exactly known in the plane of the horizon, you will be able to learn the Sun's horizontal distance from the shadow at the very time of observation, and then it will be plain where the zenith lies. But this does not follow from what has been assumed.

Ioannes de Monteregio [Regiomontanus], in Problem 19 of the *Table of the Primum Mobile*, assumes only those three things in order to find the Sun's horizontal distance from the vertical circle. By one train of reasoning he brings out the arc *d f*, by another the angle *f g d* or *f K d*, which he subtracts from 90 degrees so as to leave the Sun's horizontal distance from the vertical circle that runs through the equinoctial east and west. But since what has been assumed cannot settle whether the distance found is northern or southern — for with the zenith at *K* it is northern, with the zenith at *g* southern — he therefore directs that the Sun's altitude in the vertical circle be made known by the preceding problem of the same *Table of the Primum Mobile*. If that altitude is found greater than the Sun's given altitude, the one it has at *d*, the distance in question will be northern; if less, southern.

For let the vertical circles *g l* and *K l* be drawn through *g* and *K*, and let the vertical *K l* cut the parallel described by the Sun at *m*, while the vertical *g l* cuts the same parallel at *n*. It is clear, then, that if the Sun stands at *d* before midday and is referred to the zenith *g*, it will have a greater altitude above that horizon than it had when it was at the point *n* of the vertical circle *g l*, and so the horizontal distance found will be southern. But if it is referred to *K*, it will have a smaller altitude above the horizon than when it has reached the point *m* of the vertical circle *K l*, and the horizontal distance will be northern. And therefore,

once the altitude the Sun has in the vertical circle is known, there is no way of failing to tell whether the Sun's distance as found is northern or southern.

But in order to learn how great the Sun's altitude in the vertical circle is, he takes the altitude of the pole above the horizon as known. He therefore supposes four things known in order to find the distance in question: the altitude of the pole, the Sun's declination, the Sun's altitude above the horizon, and the hour. Yet it would have been enough to know only three: the altitude of the pole, the Sun's declination, and either the hour or the Sun's altitude above the horizon. We conclude, then, that neither by the instrument whose use Appianus sets out in the *Cosmographia*, nor by any other instrument, can the altitude of the pole above the horizon be found universally from those three things he assumes.

¶ Jacob Ziegler's method of finding the altitude of the pole by the Sun's horizontal distance from the meridian is examined. Cap. 11.

Iacobus Zieglerus [Jacob Ziegler], in the commentary he published on the second book of Pliny's *Natural History*, in the chapter on the canonical working of the sphere from the planets by observations of the sky, teaches in his first canon how to find the position of the meridian by means of the altitude of the pole, which is known from the length of the day as the clocks indicate it. Then in his sixth canon he looks for the elevation of the pole from the known position of the meridian, by means of the Sun's degree and the Sun's altitude above the horizon. But Ziegler's method too can be of no use. For if he teaches no other way of finding the position of the meridian than by the elevation of the pole known from the length of the day, how will a man who does not know the pole's altitude come to know the position of the meridian? And what he teaches in the seventh canon — namely, by what art both may be found when the position of the meridian and the altitude of the pole are alike unknown, from the Sun's altitude

alone together with its declination — is a great delusion.

For in almost infinitely many places on the earth, on one and the same day — that is, under the same declination of the Sun's degree — the Sun's altitudes above the horizons of those places are equal, and equal at one and the same instant of time; and yet the altitudes of the pole of the world will differ from one another, and be very unequal among themselves. Likewise the Sun's distances from the meridians of those same places, both those taken on the equator and those taken on the horizon, will differ from one another.

Failing to notice this, Ziegler says: "let us turn the whole machine on its foot, to a certain likeness of mid-heaven; let us also raise the pole of the world out of the

horizon; and while this is being done let us likewise turn round the sphere of the fixed stars, with a motion on the poles of the meridian of declinations, meaning to catch the rays against the Sun through the openings of the sight; and let us keep trying these motions until the ray is caught. Where it is caught, there the meridian will stand in the position of the celestial meridian, and the pole of the world at the altitude that the place of observation requires."

But he is badly mistaken. For suppose the altitude of the pole and the position of the meridian have been found in that way, as he supposes they have. Since he assumes neither the one distance of the Sun from the meridian nor the other as known, we shall be free to turn that same sphere of the fixed stars about the Sun's degree and the point opposite to it as poles. Once that motion has been made, the Sun's rays will be caught through the openings of the sight none the less; and yet the earlier position of the meridian and the earlier altitude of the pole will both have changed. So it will not be settled at all which position of the meridian, or which altitude of the pole, we are to choose for ourselves.

You will understand this more easily from the figure set below. In it let the circle *a b c* be the ring of the horizon; let the Sun's degree on the globe itself be *f*; let the position of the meridian found by Ziegler's art be *a f g*, in which the zenith is *d* and the north pole of the world *e*. The arc *a e* is therefore the altitude of the pole, and *e f* the complement of the declination of the point *f* — for we place the Sun's degree in the northern semicircle of the ecliptic — and *f g* will be the Sun's altitude, which follows to be its meridian altitude. Now let us turn that same sphere of the fixed stars about *f*, the Sun's degree, and the point opposite it, as about poles. All the points of that sphere will therefore change except *f* and the opposite point of the ecliptic. The north pole *e* will describe the circle *b e c*, and what was the zenith will describe the circle *d h* [?] *k*. The Sun's altitude *f g*, however, will remain what it was before, because the ring of the horizon *a b c* is unmoved and the Sun's degree *f* is unmoved as well. Let us understand, then, that the pole of the world *e* has by a motion of this kind

now arrived at *l*. The circle of declination of the point *f* will accordingly have the position *f l*. And if a great circle is drawn through *d* and *l* cutting the horizon at *m* and *n*, that will be the position of the meridian — if Ziegler is working correctly — while the arc *l n* will be the altitude of the north pole above the horizon. And since the arc *d l* is greater than the arc *d e*, by proposition 28 of the second book of Theodosius, *l n* will be left less than *a e* itself, by the common notion. So you will have not only a different position of the meridian, but a different elevation of the pole.

But if you then take the same arctic pole of the world to have arrived at *o*, and draw a great circle through *d* and *o* cutting the horizon at *p* and *r*, you will conclude by a like argument that the position of the circle of declination of the Sun's degree is *f o*; that the Sun's altitude and declination are not changed at all; that the position of the meridian, however, is *r d o p*; and that the altitude of the pole of the world above the horizon is the arc *o p*, smaller indeed than *l n*. It is clear, then, that nothing certain can be found by the method of Ziegler's just described.

And in exactly the same way we shall show that even if the position of the meridian is granted as known — as he himself assumes in the sixth canon — the altitude of the pole still cannot be found universally. "Let the pole *B*," he says, "be raised from *S*, the point of the horizon, until the appointed degree of the Sun comes beneath the appointed intersection of the circle of altitude and the vertical. And the altitude of the pole *B* is caught by the arc *B S*." But we shall show that this appointed intersection of the circle of altitude and the vertical belongs in common to unequal elevations of the pole.

For let the ring of the horizon be the circle *A S C*, the position of the meridian *S D G*, the

Diagram. A woodcut occupying the head of the right column: a large circle (the ring of the horizon) with the meridian drawn as an arc down through it, and two further great-circle arcs struck across it from the left-hand side to the right, together with a smaller circle crossing the lower part of the figure. Straight chords run from the point where the arcs converge, near the foot of the figure, up to the labels ranged along the upper left.

- a (top of the outer circle): `a` — one end of the meridian *a f g*, on the ring of the horizon
- n (just right of *a*, on the outer arc): `n` — where the great circle through *d* and *l* cuts the horizon
- p (further right and lower on the outer arc): `p` — where the great circle through *d* and *o* cuts the horizon
- e (left, at mid-height): `e` — the north pole of the world in its first position
- l (centre, right of *e*): `l` — the pole after the first turning
- o (centre right of *l*): `o` — the pole after the second turning
- d (left of centre, below *e*): `d` — the zenith
- f (lower centre, where the chords converge): `f` — the Sun's degree on the globe

- (immediately right of *f*): the glyph does not resolve on the scan `[?]` — the text names the circle *d h k*, so an `h` is what the construction calls for at or near this place
- c (right edge of the outer circle): `c` — on the ring of the horizon; also one end of the circle *b e c*
- b (lower left of the outer circle): `b` — on the ring of the horizon; the other end of the circle *b e c*
- K (bottom, left of centre): `K` — on the circle described by the zenith
- r (bottom left): `r` — where the great circle through *d* and *o* cuts the horizon
- m (bottom centre): `m` — where the great circle through *d* and *l* cuts the horizon
- g (bottom, right of *m*): `g` — the lower end of the meridian *a f g*

pole of the horizon *D*, and let the quadrant of the vertical circle running through the Sun at the very time the observation is considered be *D E F*; and let the point *F* lie in the northern quadrant of the horizon. The arc *F S* will therefore be known from the Sun's ray. Let the Sun's altitude *E F* be put less than the declination, but let the complement of that declination be put greater than the arc *E O*, which meets the meridian at right angles at the point *O*. With these things so laid down, let the arctic pole of the world be raised from *S*, the intersection of the horizon and the meridian — as Ziegler directs — until the appointed degree of the Sun comes beneath *E*, and let the pole of the world then stand at *B*. The arc *B S* will therefore be the elevation of the north pole above the horizon. But since the Sun's altitude has been put less than the declination, *D E*, the complement of the altitude, will be greater than

Diagram. A woodcut set into the middle of the left column: a plain circle with a vertical diameter running from top to bottom, a chord drawn from a point on that diameter out to the upper right of the circumference, and a small arc and a short cross-line in the upper interior.

- S (top of the circle): `S` — the intersection of the horizon and the meridian
- B (inside, below *S*, on the vertical line): `B` — the north pole of the world in its first position
- F (upper right, on the circumference): `F` — the end of the vertical quadrant *D E F*
- o (inside, small, below and right of *B*): `o` — where the arc *E O* meets the meridian at right angles

- E (inside, right of the vertical line): E — the point beneath which the Sun's appointed degree is brought
- A (left of the circle, at mid-height): A — on the ring of the horizon
- K (inside, on the vertical line below *o*): K — the pole's second position
- D (inside, on the vertical line below *K*): D — the pole of the horizon, that is, the zenith
- C (right of the circle, at mid-height): C — on the ring of the horizon
- G (bottom of the circle): G — the lower end of the meridian *S D G*

B E, the complement of the declination. Moreover *B E* itself has been put greater than *E O*. Therefore, if — with the meridian standing fast, and the horizon too, and the vertical *D E F* — the sphere of the fixed stars is turned by a motion about the Sun's degree and the point opposite it as poles, the pole *B* will describe a circle cutting the meridian between *O* and *D*. Let it cut it at *K*. So when the pole *B* has come to lie at *K*, the meridian will be the same, the vertical *D E F* the same, the Sun's altitude *E F* the same, and the complement of the declination *K E* will be the same as well, since *B* and *K* stand at equal intervals from *E* itself. But the altitude of the pole *K S* will be greater than the former one, and the Sun's distance from midday on the equator will be greater too. For the two angles on the base *B K* of the isosceles triangle *B E K* are equal to each other and acute; the angle *D K E* will therefore be obtuse, and so the two angles *K B E* and *D K E* taken together will be equal to two right angles. Suppose, for example, that the angle *K B E* is one of five hours: the angle *D K E* will then be one of seven hours. And so, if over and above what has been laid down it were established that the interval of time before or after midday is less than six hours, it would be certain that the altitude of the pole at the place where such an observation is made is the arc *B S*; but if greater than six hours, that it is the arc *K S*. From what has been assumed, however, neither of the two can be made clear, and so the elevation of the pole itself will be left unknown.

You will grasp this still more plainly in the following way. On the globe let the arc *c n* be a segment of the meridian and the point *c* the north pole of the world. Let the arc *c m* be put equal to the arc *K D* of the figure above, and *c n* equal to *B D*. At the point *c* let the angle *m c p* be made — the angle which the great circle *c p* makes with *c n* — equal to the angle *D K E*, and the angle *n c q* equal to the angle *D B E*. Let the arcs *c p* and *c q* be equal to each other and equal to *B E* and *K E*; and let great circles be understood as drawn through *m* and *p*, and through *n* and *q*. Each of the arcs *m p* and *n q* will therefore be equal to *D E*, and so those same arcs

m p and *n q* will be equal to each other; while the angles *c m p* and *c n q* will be equal to each other and equal to the angle *K D E*.

Diagram. A woodcut inset into the right column, with the text running round it: a curvilinear spherical figure, its labelled point at the top with arcs radiating downward and outward from it, and two further arcs drawn across between the points on the lower boundary.

- c (top): `c` — the north pole of the world
- P (right): `p` — the point to which the Sun is first put
- m (centre left): `m` — the first zenith
- q (bottom centre): `q` — the point to which the Sun comes by the motion of the first sphere
- (left edge, at mid-height): the glyph does not resolve on the scan `[?]`
- (lower left corner): the glyph does not resolve on the scan `[?]`

So if we put the Sun at *p* and the zenith at *m*, the Sun itself will stand in the northern quadrant, under the complement of altitude *m p* and the complement of declination *c p*, and distant from midday by as great an arc of the equator as the angle *m c p*. But when, by the motion of the first sphere, it has come to *q*, it will be seen by those whose zenith is at *n* under the same vertical circle and with the same complement of altitude, while its distance from midday will be the one that the angle *n c q* shows. And if at the pole *c* you make, with the meridian *c z*, the angle *z c q* equal to the angle *m c p*, put the arc *c z* equal to *c m*, draw a great circle through *z* and *q*, and take the Sun to have now arrived at *q*, then at that very instant it will be seen from the two places on the earth that lie under *z* and *n* in the same vertical circle and at the same altitude above the horizon, even though it is unequally distant from their meridians along the equator.

Petrus Appianus too, in proposition 69, tries to find the elevation of the pole from the Sun's altitude and the azimuth by means of propositions 39, 40 and 41; but this is begging the question. For in 39 and 40 he requires the hour, and in 41 the elevation of the pole itself.

Besides this, we think it worth noting a property that belongs to every place lying between the equinoctial and the tropic of Cancer. Whenever the Sun is nearer to the arctic pole of the world than the zenith [*verticale punctum*] is, the Sun stands in one and the same vertical circle twice before noon, and in the same way twice after noon — so that it sends its rays a second time before noon from the very point of the horizon at which it rose. The shadows of gnomons in those places must therefore go backwards, and no miracle is involved.

For let there be in the heavens the circle of Cancer, or any other northern parallel of the Sun, *a b c*; let the segment *a b* on it be less than a quadrant; and through the points *a* and *b* let a great circle be drawn. Its segment between those same points *a* and *b* will likewise be less than a quadrant, since this was shown above, in chapter 6, *On the instruments by which the altitudes of the stars are taken*, at the end of that chapter. Let the arc *d a b e* be, furthermore, a quadrant of that same great circle just drawn; let the point *f* be the north pole of the world; and through *d* and *f* let a great circle be drawn. The arc *d f* will then be less than a quadrant.

Diagram. A woodcut filling the lower half of the left column: a large ovoid outline (the sphere seen from outside) crossed by several great-circle arcs. One arc runs from the upper left down across the figure to the right; a second, more steeply inclined arc rises to the top of the figure; and a shallow, nearly horizontal arc — the Sun's parallel — is drawn across the middle, meeting the outline on the left. A sheaf of straight lines converges on a point low at the left, from which chords run up to the labelled points on the arcs.

- K (top centre): `K` — the point at a quadrant's distance from *d* along the great circle through *d* and *f*
- f (upper left, below and left of *K*): `f` — the north pole of the world
- m (upper right, on the outline): `m` — where the horizon meets the Sun's parallel, the point of rising
- c (far left, at the end of the shallow arc): `c [?]` — the western end of the Sun's parallel *a b c*
- (four small letters ranged left to right along the parallel, between the left-hand outline and *m*): read as `n [?]`, `a`, `g`, `h [?]` — the points named in the text as lying on the parallel *a b c*
- o (below the parallel, at the centre of the figure): `o` — the point at which the vertical circle of greatest azimuth touches the parallel
- p (right edge): `p` — the end of the quadrant *d o p*
- d (low left, where the straight lines converge): `d` — the zenith of the place
- l (foot of the figure): `l` — the southern point of the horizon, on the meridian *f d l*

For if the arc *a b* is understood to be bisected at the point *g*, and from the pole *f* arcs of great circles are drawn to *a*, to *b* and to *g*, the angles at *g* will be right angles. But *a f* is less than a quadrant, and *a g* likewise less than a quadrant; therefore *f g* will be less than a quadrant. And *d g* is less than a quadrant; therefore the arc *d f* will be less than a quadrant. Again, since *f g* is less than a quadrant, the angle *f a g* will be acute, and for that reason the angle *d a f* obtuse.

But the angle *a d f* is acute, because *f g* is less than a quadrant. The arc *d f* is therefore greater than *a f*, and for that reason the arc *d f* cuts the parallel *a b c* at the point *h*, between *d* and *f*.

Now let *d f K* be a quadrant of a great circle, and about the pole *d*, with the distance *d K*, let the semicircle *K e l* be described, and let its intersection with the Sun's parallel *a b c* fall at the point *m*. We shall take the point *d* to be the zenith of some northern place, at which the altitude of the pole above the horizon is the arc *f K*, and the complement of that altitude *d f*; the eastern semicircle of the horizon is *k e l*, and the meridian is *f d l*. When the Sun describes the parallel *a b c*, the point of noon is *h*; the point at which it rises is *m*, lying on the vertical circle *d m*, which cuts that same parallel again at the point *n*, between *a* and *h*.

But if from the zenith *d* a great circle is drawn touching the parallel *a b c* at the point *o*, in the way Theodosius teaches, its quadrant *d o p* will be the vertical circle that departs furthest from the meridian, while the remaining vertical circles will cut the semidiurnal arc *h b m* in two places each. At its rising, then, and again at the point *n* before noon, the Sun will be seen under one and the same vertical circle; but at *n* it will have the altitude *m n*. At *a* and at *b* it will lie under the vertical circle *d e*, but the altitudes will be unequal, for *b e* is less than *a e*.

The Sun's horizontal distance from the meridian therefore increases continuously from its rising up to *o* before noon, but from *o* itself up to *n* it decreases. Consequently, if a gnomon is set perpendicular to the plane of the horizon: when the Sun is at its rising, the shadow it casts — which is then reckoned infinite — will stand off from the meridian line by an arc equal or similar to *K m*; when the Sun is at *b*, the shadow cast will stand off from that same meridian line by an arc equal or similar to *K e*; and further, when the Sun is at *o*, the shadow will stand off from the meridian line by the greatest amount, namely by an arc equal or similar to *K p*. After that it will begin to approach the meridian again, for at *a* it will stand off just as far as at *b*, and at the point *n* by the same interval as at *m*; and from that place it will reach the meridian without any further going back. After noon the same order of advance and retreat will be kept.

There is nothing absurd, then, in shadows advancing and going back in those places. In this northern region of ours, however, which lies on this side of the tropic of Cancer, it could not happen without a miracle — as we read that it did happen at God's command, as a sign of King Hezekiah's recovery.

From this you may also see that knowing the altitude of the pole, together with the Sun's declination and its distance from the meridian measured along the horizon, is not enough to determine the hour. For at *a* and at *b* the Sun is equally distant from the meridian along the horizon —

by the arc *e K*, that is — but unequally along the equinoctial; for the angle *b f d* is much greater than the angle *a f d*. Sundials are nonetheless true instruments. In the horizontal ones we generally use, the shadow of the world's axis, which shows the hour, never goes back; but in those whose style stands perpendicular to the plane of the horizon we tell the hour not from the shadow's distance from the meridian line alone, but from the size of the shadow itself.

From this figure you will also understand that when the elevations of the pole at two places are known, together with the bearing their line of distance keeps to one of the meridians, it is still not possible in general to determine that distance itself, nor the difference of the meridians — for all that Ptolemy boasts of having found these by the meteoroscope. For let us put the zenith of one of the two places at *d*, and that of the other somewhere on the parallel *m o h*, and let the altitudes of the pole be given as known. Let the bearing which the distance keeps to the meridian *d f* be supposed known as well, namely the one shown by the angle *e d f*. Then the interval between those places will be either *d a*, with a difference of longitude as great as the angle *a f d* shows, or perhaps *d b*, which is in fact greater than *d a*, with the difference of longitude that the angle *b f d* indicates. For that reason it will remain uncertain where the zenith of the more northerly place lies — whether at *a* or at *b*. For in a spherical triangle made up of arcs of great circles, and equally in a rectilinear one, even though two sides are given as known together with the acute angle contained by the third side and by the greater of the two given sides, the third side itself and the remaining angles are still not determined by these data. This is a further reason why the altitude of the pole cannot be found from what Appianus, or what Ziegler, assumes as known.

Ioannes de Monteregio [Regiomontanus], in Problem 46 of the *tabula primi mobilis*, proposes to investigate the difference of longitude from the latitudes of two places and the angle of position. In the course of the working, however, he takes the interval between the given places as known to him, and from that interval, together with the latitude of the first place and the angle of position, he seeks the latitude of the second place and the difference of longitude. That these can be found from the things assumed is what the geometrical art teaches — although the same author chose a method not well enough suited to finding what

was sought. For before he can find the latitude of the second place he is forced to determine whether that second place is the more northerly or the more southerly, and to do so from the character of the angle of position. Yet it is clear from the figure set out above that *g* is a more northerly place than *a*, while *b* is of the same northern latitude; and any point taken between *b* and *e* will be more southerly — all with the angle of position *f a e* remaining the same.

From these considerations you will also understand that the third proposition of the first book of Menelaus, *On Spherical Triangles*, is not true universally, in the form in which it is put forward. It runs thus: when two angles of two triangles formed from arcs of great circles upon the surface of a sphere are equal, and the arcs containing the other two angles of each are equal — every arc, that is, to its correspondent — and neither of the two remaining angles is a right angle, then the remaining arc of the one of the two triangles is equal to the remaining arc of the other, and the two remaining angles are equal to the two remaining angles, every angle to its correspondent.

"An example of this," he says, "is: let there be two triangles *a b g* and *d e r* upon the surface of a sphere; let the angle *a* be equal to the angle *d*, the arc *b g* equal to the arc *e r*, and the arc *g a* equal to the arc *d r*; and these are the arcs containing the two angles *g* and *r*; and let neither of the two angles *b* and *e* be a right angle." And he asserts that the arc *a b* is equal to the arc *d e*, the angle *g* equal to the angle *r*, and the angle *b* equal to the angle *e*.

But since in the demonstration he takes the equal angles *a* and *d* to be, in the first place, not right angles, let us therefore put them acute. It will then be possible for one of the two angles *b* and *e* to be acute and the other obtuse:

Diagram. Two spherical triangles drawn side by side across the right column, each a curvilinear triangle with a rounded apex and a base drawn as a shallow arc.

- e (apex of the left-hand triangle): `e`
- d (foot of the left-hand triangle, left): `d`
- r (foot of the left-hand triangle, right): `r`
- b (apex of the right-hand triangle): `b`
- a (foot of the right-hand triangle, left): `a`
- g (foot of the right-hand triangle, right): `g`

The conclusion therefore does not follow unless we lay it down that each of the angles *b* and *e* is greater than a right angle, or that each is less.

Nicolaus Copernicus Turinensis [Copernicus] paid too little attention to this relation between the sides and the angles of a triangle, being taken up above all with the question of how to bring back to light the ancient and almost forgotten astronomy of Aristarchus of Samos — the mobility of the earth and the rest of the Sun and of the eighth sphere, which Archimedes records in his book on the number of the sand — by Ptolemy's method, first principles and demonstrations. For the eighth proposition of chapter 14 of the first book of the *Revolutions*, in which he treats of spherical triangles, runs thus: "If two triangles have two sides equal to two sides, each to each, and an angle equal to an angle — whether the angle the equal sides contain or the one at the base — they will also have base equal to base, and the remaining angles equal to the remaining angles."

But we shall show by an easy demonstration that the latter part of this is not true. For in the spherical triangle *a b c* let the two sides *a b* and *a c* be equal, and let us produce the base *b c* to *d*, the arc *c d* being nevertheless less than a semicircle; and through the points *a* and *d* let us draw the arc of a great circle *a d*. Then in the two spherical triangles *a b d* and *a c d*, the two sides *a b* and *a d* of the triangle *a b d* are equal to the two sides *a c* and *a d* of the triangle *a c d*, and the angle *a d b* is common to both, being at the base of each triangle.

Diagram. A woodcut set into the left column: a curvilinear triangle with its apex at the top and its base drawn as a nearly straight line, with a further arc drawn from the apex down to the left-hand end of the base, so that the base is divided into two unequal parts.

- a (apex): `a`
- d (left end of the base): `d`
- b (on the base, between *d* and *c*): `b`
- c (right end of the base): `c`

Therefore the base *b d* of the triangle *a b d* will be equal to the base *c d* of the triangle *a c d* — the part to the whole, by Nicolaus Copernicus's own eighth proposition — which is impossible. The same absurdity follows for the two angles *b a d* and *c a d*, for the one is a part of the other. The angle *d b a*, moreover, will always be unequal to the angle *d c a*, unless the sides *a b* and *a c*, which have been put equal, are quadrants. Let us therefore make them less than quadrants: the angle *d c a* will accordingly be acute, the angle *d b a* obtuse, and the angle *a d b* acute.

What he teaches in the eleventh proposition, then — that every triangle two of whose sides are given together with some angle has its angles and sides

determined — is a delusion. He has gone wrong in the same way in proposition 6, on rectilinear triangles. For in a triangle two of whose sides are given together with one angle only, and that at the base, the remaining side and the remaining angles cannot be found, unless the given angle is either right or obtuse, or, if it is acute, subtends the greater of the given sides. For if the case is put otherwise, it will not be settled from what has been laid down whether the remaining angle at the base is the acute one, or the obtuse one that makes up two right angles with it; and accordingly the base itself will be left unknown as well.

He has gone wrong no less in proposition 12, which runs thus: "Further, if any two angles whatever are given together with some side, the same results will follow." For let a spherical triangle *b c g* be constructed in which the two sides *b c* and *c g*, taken together, are equal to a semicircle; let the side *b g* be extended to *a*, and let a great circle be drawn through *a* and *c*; and let the two angles *c a b* and *c b a* of the triangle *a b c* be given as known, together with the side *a c*, which is opposite the angle *c b a*. Even so, the remaining angle and the remaining sides will not yet be known from these things supposed known. For since the two sides *c b* and *c g* taken together are equal to a semicircle, the angle *a b c* will be equal to the angle *b g c*.

Diagram. A woodcut at the head of the right column: a curvilinear figure like a spherical quadrant, with an apex at the top, a curved left side, and a base drawn as a shallow arc, divided by a straight line running from the apex down to a point on the base.

- c (apex): `c`
- a (left, at the end of the base): `a`
- b (on the base, where the line from the apex meets it): `b`
- g (right end of the base): `g`

Therefore in the triangle *a c g* as well, the two angles *c a g* and *a g c* are supposed known, and the side *a c*, opposite the angle *a g c*, is taken as known; for in both triangles, *a b c* and *a g c*, the hypotheses are the same. From the things taken as known, then, it will still not be possible to tell whether the remaining angle, the one that was unknown, is *a c b* or *a c g*, nor whether the remaining sides, which were unknown, are *c b* and *a b* or *c g* and *a g*.

Whether Copernicus gives a satisfactory answer to the arguments Ptolemy used to show that the earth is in no way moved in a circle — when he says that not only the earth but earthy bodies too, and all heavy things wherever they are placed, are carried by a natural motion from west to east; that rectilinear motion

comes upon them when they are in any way wandering outside their natural places; and that the motion no more remains circular alongside the rectilinear than a living creature remains alongside sickness [?] — is a matter for the philosophers to dispute. For he will admit that nothing moves either from the centre or towards the centre without being carried about that same centre as well. He devised all this so as to be able to give a reason why, if the earth is carried round in a circle, heavy bodies thrown upwards nonetheless come back at right angles to the places lying beneath them.

As for astronomy, he exchanges the places of the Sun and the earth; and in order to make the Sun and the fixed stars motionless, he assigns to the earth a threefold motion on an eccentric orb, together with two librations, so that in every age the observations of the fixed stars may agree with one another — after the manner of the two trepidations that Johannes Werner invented for the same purpose. He places the Moon, not without reason, on an epicycle of an epicycle, the centre of the smaller lying on the circumference of the greater. I note, however, that the whole of the smaller must be enclosed within the greater, so that the heaven is not torn apart — if he thinks that an inconvenience. And since he posits eccentric orbs, it will be necessary to posit others as well, to fill out the planets' spheres as concentric with the world. In my judgment, therefore, he ought to have contended for this alone: in what way, namely, from his own observations and those of others, tables

of the celestial motions could be made more exact. And this he could indeed have achieved with the eighth sphere in motion and the Sun also in motion, while the earth stayed motionless at the centre of the world, as in the common astronomy. But of these matters elsewhere; let us return to our subject.

Suppose you wish to learn from the figure drawn above — for a given altitude of the place, and a given northerly declination of the Sun — how far the shadows go back on a surface parallel to the horizon, and in how much time. Then let a great circle be drawn through the two points f and m, and likewise one through f and the point of contact o. In the spherical triangle f m k the angle at k, formed by the meeting of the meridian and the horizon, is right; f K, the altitude of the pole, is given as known; and f m, the complement of the declination, is given as well. The remaining side and the remaining angles, therefore, cannot fail to be known. So the arc K m — the Sun's distance from the meridian measured along the horizon, that is, the complement of the amplitude of rising [*latitudo ortus*] — will come to light, and with it the angle k f m opposite to it, which gives the size of the semi-

nocturnal arc; and consequently the remaining angle d f m, that of the semidiurnal arc, is left known.

Diagram. A sphere drawn in perspective as a large closed oval, drawn out to a point at the bottom. A great-circle arc runs from the apex at the top down the left side; a second, flatter set of arcs crosses the middle of the figure horizontally; and a lens-shaped pair of arcs runs from the left-hand side across to the right. Straight and curved lines join the lettered points to one another.

- K (top, outside the bounding curve, at the apex): the meridian's meeting with the horizon
- f (inside, upper left, on the top arc): the pole
- m (inside, upper right): the Sun on its parallel
- h (left, at the left end of the middle line)
- n (just to the right of h, on the middle line)
- o (centre of the figure, the point ringed by a small circle): the Sun at its greatest departure from the meridian
- p (right, at the right end of the middle line)
- d (lower left, on the bounding curve): the zenith
- L (bottom, at the point of the figure, in bold capital)

In the triangle d f o the angle d o f is right; and so, since d f (the complement of the pole's altitude) and f o (the complement of the declination) are known, the remaining side and the remaining angles become known too. In this way we obtain d o, the complement of the Sun's altitude when it stands at the point o, where it departs furthest from the meridian; and likewise the angle f d o, which gives that same distance from the meridian measured along the horizon, and the angle o f d, which shows the distance measured along the equinoctial. None of these can remain unknown. From the angle f d o we then subtract the angle f d m, which is known because the arc K m is known; the angle o d m is left known, and the arc m p — the regression of the shadows — is subtended by it.

For example: let the arc f K be 12 degrees, which is the elevation of the north pole above the horizon in the city of Cananor, in India within the Ganges, under the kings of Portugal. Let the arc h o m be a segment of the parallel of the first point of Cancer. The complement of the arc K m — that is, the amplitude of rising of the first point of Cancer — will then be 24 degrees 3 minutes, and K m itself 65 degrees 57 minutes; while the angle K f m, the semi-nocturnal arc, is 84 degrees 44 minutes 20 seconds, so that the semidiurnal arc is very nearly 95 degrees 16 minutes. The Sun's altitude o p is 31 degrees 26 minutes, and the arc K p, which is

the magnitude of the angle f d o, is 69 degrees 38 minutes. From this we subtract k m, and 3 degrees 41 minutes are left for p m, the regression of the shadows.

How long the shadows take to go back, and how high the Sun stands above the horizon at the other limit of the regression, will be easy to learn from the same figure. For in the right triangle f k m, from the known f k and f m the angle K m f will be found. We subtract it from the right angle d m K, formed by the meeting of the vertical circle d m with the horizon, and the angle f m n is left known. Now in the isosceles triangle m f n the base angles are known, together with the two equal sides; so the base itself — which is the Sun's altitude above the horizon — and the angle n f m come to light. And therefore the angle d f n, left over from m f d, will be known, and with it the time before noon.

I admit, however, that I questioned men who often travel to those parts of the world lying between the equinoctial and the tropic of Cancer: whether in those places, when the Sun is in Cancer, they had seen the shadows of upright bodies go back a little on the ground, morning and evening. They answered that they had seen nothing of the kind. No wonder: the regression of the shadows is exceedingly small, and so they failed to notice it. But this at least should not have escaped them — that in the space of four hours before noon the length of the shadows is drawn in very sharply, while after noon it is drawn out to its greatest, and no circular motion is meanwhile perceived about the foot of the gnomon. For by the demonstration just given the angle d f o contains 60

degrees 44 minutes; therefore the angle o f m is found to be 34 degrees 32 minutes; m n, the Sun's altitude, [?] 55 degrees; the angle n f m, further, 60 degrees 28 minutes; and therefore the angle d f n, 34 degrees 48 minutes.

On the varying relation of the Sun to the zenith in different places of the earth, before noon and after — what they call the Sun's Zenith. Cap. 12.

We judge it no small help, when the altitude of the pole is to be found by the Sun's ray, to understand the relation the Sun itself bears to the Zenith overhead, before noon and after, for different places in the zodiac and for different altitudes of the pole above the horizon. This will be easy to grasp from what we are now about to say.

When the Sun has a northern declination, then for those who stand further from the north pole it will move all day long in the northern vertical circles, whether the latitude of the place is southern or northern. For it cannot happen on that day

that the Sun reaches the vertical circle of the equinoctial rising and setting — the prime vertical, which separates the northern verticals from the southern. For those who are placed under the Sun's own parallel it likewise moves all day in the northern verticals, though at the instant of noon it stands directly overhead. But for those whose zenith is nearer to the north pole, so long as the Sun's altitude above the horizon is equal to the declination or less than it, the Sun will be in a northern azimuth.

Diagram. A circle representing the sphere. A straight line runs down from the top of the circle through the interior to its foot; three lettered points lie along it below the topmost. On the right-hand side an arc runs from the top of the circle down to the foot, carrying three further lettered points, and straight lines join the point on the left-hand part of the vertical line to the points on that arc.

- e (top of the circle, at the head of the vertical line)
- f (on the vertical line, just below e)
- a (on the vertical line, below f): the north pole of the world
- b (below and to the left, at the lower end of the vertical line): the zenith of the place
- g (on the right-hand arc, upper part)
- c (on the right-hand arc, below g)
- K (on the right-hand arc, lower right)
- d (foot of the circle, at the bottom of the vertical line)

Let a be the north pole of the world, b the zenith of the place where we are, c d e the Sun's parallel, and e a d the meridian. With b taken as pole and with the interval b f equal to the arc a d, let a circle be described on the surface of the sphere. It will cut the parallel c d e, because b f is greater than b d. Let one of their intersections be at c, and let great circles be drawn through a and c, and through b and c; then a c and b c will be equal. And so, when the Sun in the motion of the first sphere reaches c, its altitude above the horizon will be equal to the declination. [Margin: *altitudo* — correcting the *latitudo* printed in the text.] And since in the isosceles triangle a b c the two equal sides a c and b c are less than quadrants, the angles at a and b on the base will be acute; and therefore the vertical circle b c, in which the Sun lies, will be a northern one.

Now let g be a point between e and c on the Sun's parallel, and k a point between c and d. When great circles have been drawn through b and g, and through b and K, b g will be greater than b c and b k less, by the 25th proposition of the second book of Theodosius (*Spherics* II.25). Therefore, so long as the Sun's altitude is less

than the declination, it will lie between e and c, as at g; and so the angle a b g will be acute, and the vertical b g in which the Sun lies northern — which was to be demonstrated.

Now let the Sun again have a northern declination, and let the zenith of the place be nearer to the north pole, but let the Sun's altitude above the horizon be greater than the declination. I say that from these data it cannot be settled in which vertical the Sun lies: whether in the vertical of the equinoctial rising and setting, or in a northern one, or in a southern. For since the angle c b d is obtuse, let a great circle b k be drawn falling on the meridian at right angles at the point b. The angle d b K will then be right, and b k will itself be the vertical of the equinoctial rising and setting. So when the Sun is at K it will be in that same vertical; when it is between c and k, in the northern verticals; when between k and d, in the southern — which was to be shown. And the Sun will be in the vertical of the equinoctial rising and setting when it has an altitude above the horizon such that the right sine of that altitude keeps the same ratio to the sine of the declination as the whole sine keeps to the sine of the pole's altitude. So when its altitude is less than this it will be in the northern verticals; when greater, in the southern.

For in the spherical triangle a b K the angle K b a is right, and its sides are

less than quadrants. Therefore, as the right sine of the complement of the arc b K is to the sine of the complement of the arc a K, so is the whole sine to the sine of the complement of the arc a b. But the complement of the arc b K is the Sun's elevation above the horizon when it is in the vertical b K; the complement of the arc a K is the Sun's declination from the equinoctial; and the complement of the arc a b is the latitude of the place. Hence, when the Sun has the altitude just described it will be in the vertical circle of the equinoctial rising and setting; when it has less, in the northern verticals; when more, in the southern.

From this demonstration it follows that if the Sun is in the northern signs and stands either in the vertical of the equinoctial rising and setting or in one of the southern verticals, then in those places which are nearer to the north pole it will have a greater altitude above the horizon than its declination on that day. It also follows that, wherever the zenith may be placed, when the Sun is in the northern signs, so long as its altitude above the horizon is less than the declination or equal to it, the Sun will be in a northern azimuth. Further, it follows that when the Sun is in the northern signs and in a southern azimuth, its altitude above the horizon will be greater than the declination, and the north pole will be less distant from the zenith than from the Sun.

When the Sun moves through the southern signs, what relations it bears to the zenith will be easy to understand from what has been said. For those who are further from the south pole, it will move all day in the southern verticals; and for those placed under the Sun's own parallel it likewise moves all day in the southern verticals, except that at the instant of noon it stands directly overhead. Again, for those whose zenith is nearer to the south pole, so long as the Sun's elevation is less than the declination or equal to it, the Sun will be in a southern azimuth; but if the Sun's elevation is greater than the declination, it may be in the vertical circle of the equinoctial rising and setting, or in the northern verticals, or in the southern. It will be in the vertical of the equinoctial rising and setting when it has an altitude above the horizon such that the right sine of that altitude keeps the same ratio to the sine of the declination as the whole sine has to the sine of the pole's altitude in that place. So when its altitude is less than this, it will be in a southern azimuth; when greater, in a northern.

From all this you will conclude likewise that if the Sun is in the southern signs and stands either in the vertical of the equinoctial rising and setting or in one of the northern verticals, then in those places which are nearer to the south pole it will have a greater altitude above the horizon than its declination on that day. It also follows that if the Sun is in the southern signs, so long as its altitude above the horizon is less than the declination or equal to it, it will be — wherever we ourselves may be — in a southern azimuth. And it follows from all the above that if the Sun is in the southern signs and in a northern azimuth, its altitude above the horizon will be greater than the declination, and the south pole will be less distant from the zenith than from the Sun.

When the Sun runs along the equinoctial circle, it rises and sets for everyone in the vertical of the equinoctial rising and setting; but through the rest of the day it becomes southern for those in the north, and northern for those in the south. For those placed under the equinoctial circle itself it sends out its rays all day long along that one straight line of the equinoctial rising and setting, which the Portuguese call the rhumb of *Leste* and *Oeste* (east and west); and at midday it comes directly overhead.

For let the circle a b c d be the right [?] horizon of those who have their zenith at the point e; let b e d be the equinoctial and a e c the meridian; let the circle b f d be the vertical of those who are toward the northern region at f, and b g d the vertical of those who are toward the southern region at g. Now since the angles at f and g are right, if great circles are drawn from those zenith points f and g to any point of the equinoctial between d and e — let it be K — or between e and b, these

great circles will make acute angles with the meridian. So the Sun at d rises in the vertical circle of the equinoctial rising and setting; but when it has been raised to K, for those who are at

Diagram. A circle with a vertical diameter and a horizontal diameter crossing at the centre, and two lens-shaped pairs of arcs drawn from the left-hand end of the horizontal diameter to its right-hand end, one bulging above the centre and one below. Straight lines run from the two points on the vertical diameter to the point marked on the horizontal diameter at the right of centre.

- a (top of the circle): on the meridian
- f (on the vertical diameter, between a and the centre): the zenith of those in the northern region
- e (at the centre): the zenith of those under the equinoctial
- K (on the horizontal diameter, between the centre and d)
- d (right end of the horizontal diameter)
- b (left end of the horizontal diameter)
- g (on the vertical diameter, between the centre and c): the zenith of those in the southern region
- c (bottom of the circle): on the meridian

f, it is in the southern azimuth *f K*; but for those who are at *g*, it is in the northern azimuth *g K*. For the rest — those who live under the equinoctial — the Sun will move all day long in the equinoctial vertical, and so it sends out its rays along the straight line that is the common section of the equinoctial and the horizon.

Now, once the position of the meridian is known, the Sun's position with respect to the zenith [*verticale punctum*] — that is, its distance from the meridian measured along the horizon — is known from the shadows of gnomons. Take care, then, not to be misled by what Johannes Stöffler writes in his commentary on Proclus's *Sphere*, in the chapter on the circles of the sphere. He supposes that there is this difference between the shadows of those who live in the temperate zones and the shadows of those placed within the tropics. For us, who are placed outside the tropics, when the Sun rises at the beginning of Cancer, a body set up in its light casts its shadow toward the winter setting of the Sun; but when the Sun rises in Capricorn, the shadow is cast toward the summer setting of the Sun — and the same judgment will hold for the Sun's setting. For those placed between the tropics, by contrast, on the day (he says) on which the Sun passes through their zenith, the morning shadow runs straight, cast toward the setting of

the Sun on that same parallel, just as the afternoon shadow runs straight toward the rising, extended to the point of the horizon over which the Sun had risen.

But in truth no such difference between the shadows of the one group and those of the other is found anywhere. On the contrary, it is common to every habitation that, when the Sun rises, the straight shadow of the gnomon is extended to the opposite point of the ecliptic. So when the Sun rises with the beginning of Cancer, for those placed under the tropic of Cancer itself the shadow is cast toward the winter setting of the Sun — not toward the setting of Cancer itself, that is, into the northern quarter, as Stöffler supposes. For the straight shadow of the gnomon lies on the common section of the plane of the horizon and of the vertical that passes through the Sun; and great circles of the sphere bisect one another. It is therefore necessary that, when the Sun rises with the beginning of Cancer, the gnomon's shadow be cast to the opposite point of the sphere — the point that is common to that vertical circle, to the horizon, and to the ecliptic as well.

Nor, again, is the style's shadow on the horizontal sundial of those placed under Cancer cast toward the setting of Cancer. For on that day the Sun rises for them before the sixth hour, and so the morning shadow will be extended into the southern and western quadrant of the horizon.

¶ *To find the altitude of the pole by the Sun's rays, when the position of the meridian is given as known. Cap. 13.*

On a globe of perfect roundness let a great circle *a b c d* be described. This circle we intend to serve as the horizon, and let its pole be *e*. Let a meridian *a c e* be described besides on the surface of the same globe, and let each circle be divided into 360 equal parts. Then out of any hard material let a single great circle, or circular ring [*armilla*], be made, to turn upon that pole *e* and upon the pole opposite it, lying flush against the convexity of the globe; and let the face of it that is directed toward the poles of the horizon likewise be divided into degrees in the usual way. A circular ring of this kind will represent both the meridian and any vertical circle whatever.

Diagram. A large circle — the horizon *a b c d* — with a horizontal diameter and a second, smaller circle drawn within it, crossing the large one in the upper part and running down toward the lower right. A straight line runs from *a* at the left end of the horizontal diameter through the centre *e* to *c* at the right end: this is the meridian. A second straight line runs from *f* on the lower-left circumference up through *K* and *e* to *g* at the upper-right circumference: this is the movable vertical *f e g*. Readable labels:

- *d* (top of the large circle)
- *g* (upper right, on the circumference, where the movable vertical meets it)
- *a* (left end of the horizontal diameter)
- *l* (on the meridian, between *a* and the centre *e*)
- *e* (centre of the large circle)
- *o* (on the meridian, just to the right of *e*)
- *c* (right end of the horizontal diameter)
- *K* (below *e*, on the line running down to *f*)
- *f* (lower left, on the circumference)
- *b* (bottom of the large circle)

So when you wish to find the altitude of the pole above the horizon by the Sun's rays, and the position of the meridian is known, the thing will be very easy to find from what was said in the preceding chapter. Take some flat circular plate lying parallel to the horizon, at whose centre a pin stands at right angles to cast a shadow, whose circumference is likewise divided into degrees, and on which a straight meridian line has been marked. From the distance of the shadow from that meridian line at the moment of observation we shall determine how far the Sun stands from the meridian along the horizon. By an astrolabe, however, or by a quadrant,

we shall determine how many degrees it is seen raised above the horizon. We shall therefore reckon the Sun's distance from the meridian on the horizon of the globe, from *a* toward *b*; let it be, for example, the arc *a f*. Next we shall bring the movable great circle, or circular ring, to the point *f*, in the position *f e g*. Then we shall lay off the Sun's altitude, once found, on that movable vertical itself, from *f* toward *e*, and mark it on the globe's surface with the point *K*. By this device the point *K* will be set on the surface of the globe exactly as the Sun is set in the world.

Now, so that we may see at what point of the meridian *a c e* the visible pole [*manifestus polus*] of the world lies, we shall take between the legs of a compass the complement of the Sun's declination at that same time of observation, known from the table of declinations; and keeping one leg of the compass on *K* as on a pole, we shall carry the other round, so that a circle is described on the surface of the globe itself.

Now if at the moment of observation the Sun was in the northern signs but in a southern azimuth, then the Sun will be less distant from the zenith than from the north pole, and the pole too will be less distant from the zenith than from the

Sun, by the sixth rule [*documentum*]. Accordingly the circle described about *K* will cut the meridian in two places: above *e*, as at *o*, and below *e*, as at *l*. The north pole will therefore be at *o*, namely on that side of the meridian on which the angle made with the vertical *e K* is obtuse; and so the arc *o c*, the elevation of the arctic pole, will be known.

But if the Sun is in the northern signs and in the vertical circle of the equinoctial rising and setting, then the north pole will be less distant from the zenith than from the Sun, and the Sun in turn will be less distant from the zenith than from the pole. Accordingly the circle described about *K* will cut the meridian in two places standing at equal intervals from the zenith, and the north pole can be placed at either of them. So when that interval is subtracted from 90 degrees, the arc of the elevation of the arctic pole above the horizon is left known.

Again, with the Sun still in the northern signs, if it is found in a northern azimuth and moreover stands at equal intervals from the zenith and from the north pole, then the circle described about *K* will cut the meridian in two places, one of which will be the north pole, the other the zenith of the place at which the observation is made; and therefore the distance between the zenith and the north pole will be known. If that distance is found to be a quadrant, the zenith will lie on the equinoctial; if greater than a quadrant, the excess above the quadrant will be the altitude of the south pole; but if it is less than a quadrant, what remains of the quadrant will be the altitude of the north pole.

But if the Sun stands in the northern signs and in a northern azimuth, and yet at the moment of observation is less distant from the zenith than from the north pole, then the circle described about the point *K*, which represents the Sun, will cut the meridian in two places; and the zenith will lie between those two intersections — as you will easily show from what we said in the preceding chapter — while the place of the arctic pole will be that intersection which lies on the side where the Sun's azimuth makes an acute angle with the meridian. So once the distance between the zenith and the north pole is known, the altitude of the visible pole above the horizon cannot fail to be known.

But if the Sun has northern declination and is found set in a northern azimuth, yet is less distant from the north pole than from the zenith, then the circle described about *K* must either touch the meridian or cut it in two places. If it touches, the place of the north pole will be at the point of contact itself; and therefore, once the distance between the zenith and the north pole — which is indeed less than a quadrant — is known, the arc left over from 90 degrees will be the elevation of the arctic pole above the horizon, and the Sun itself will stand six

hours from noon. For let *a f* be the Sun's distance from the meridian along the horizon at the time of observation, and let the circle described about the point *K*, which represents the Sun, touch the meridian at *r*: the place of the north pole will then be at *r* itself. Now since *K r* comes from the poles of the meridian, by the 6th proposition of the 2nd book of Theodosius, the angles at *r* will be right, by the 19th of the first book. But the arc *e K* is less than a quadrant, and *K r* too is less than a quadrant; therefore the remaining side *e r* of the triangle *e K r* will likewise be less than a quadrant. So the arc *c r* will be the elevation of the north pole; and since the angle *e r K* is right, the Sun's distance

Diagram. A large circle with a horizontal line — the meridian — running across it from *a* at the left through the centre *e* and on through *r* to *c* at the right. A smaller circle lies inside the large one, below and to the right of *e*, touching the meridian line at *r*; its centre region is marked *K*. A straight line runs from *e* down to the right through *K* and on to the circumference, and a second line runs from *r* down to *K*. Readable labels: *a* (left), *e* (centre), *r* (on the meridian between *e* and *c*, at the point of contact), *c* (right), *K* (within the smaller circle, below the meridian).

from noon will be six hours. But if the circle described about *K* cuts the meridian, it will cut it in two places, as at *i* and *l*; so the north pole will be either at *i* or at *l*. And therefore, if it has been ascertained that the place where an observation of this kind is made lies in the southern region, while we do not know how great the elevation of the south pole there is, this can be learned from the same observation. For the north pole can be in no other place than at *l*. Let great circles be understood to be drawn through *K* and *i*, and likewise through *K* and *l*: then in the isosceles triangle *i K l*, made up of segments of great circles, the two angles on the base *i l* will be acute, and therefore the angle *e i K* is obtuse.

Diagram. A portion of a large circle, drawn as an arc down the left side, with the meridian running up from *e* at the bottom through a point below *l* to *l* itself at the top left. Straight lines run from *e*, from that intermediate point, and from *l* to *K* at the right, forming the triangle described in the text. Readable labels: *l* (upper left), *K* (right), *i* (on the meridian below *l*), *e* (bottom).

And since, while the Sun moves through the northern signs, it sets before the sixth hour and rises after the sixth hour for those who are in the southern region, they cannot have the north pole at *i*, but rather at *l*, where the angle *e l K*, the angle of the Sun's distance from noon, is acute. So when a quadrant is subtracted from the arc *e l*, which lies between the zenith and the north pole, the distance from the equinoctial toward the south pole is left known, and consequently the elevation of the south pole at that place will be known.

If, however, we are wholly ignorant of where the place is at which the observation was made, the altitude of the pole cannot be found in the way just described. Indeed, even if it has been established that the place lies in the northern region, it still cannot be known from the data how great the altitude of the arctic pole is. This much, however, will be certain: that the north pole is either at *i* or at *l*. It will be at *i* if the Sun's distance from noon is greater than six hours, but at *l* if it is less than six hours.

But when both are proposed as unknown — the altitude of the pole and the Sun's distance from noon — then, so that both may be established, after the first observation has been made, in which the Sun stands at *K* under the known vertical *e K*, we shall observe the Sun again after a short interval of time; let it be found, for example, raised higher, on the vertical *e o*. Then about the point *o*, which represents the Sun in its later position, we shall describe a circle of the same size as the first, that is, with an interval equal to the complement of the declination.

This later circle will therefore cut the meridian either at *y* or at *l*, and at some other point besides. For it cannot cut it at both *y* and *l*, or the impossibility of the 7th proposition of Euclid's first book would follow. Yet it must cut it at one of the two, since the arctic pole is placed either at *y* or at *l*. Let it then cut at *y* and at *m*; and therefore the arctic pole itself will be at *y*. So once the distance *e y* between the zenith and the north pole is known, the altitude of the visible pole above the horizon cannot fail to be known. As for the time before noon, it will be learned from the angle *e y o* at the pole of the world in the later observation, and in the earlier one from the angle *e y K*; and so that short interval of time will likewise become known.

Diagram. An arc of the meridian running down the left, with *l* at the top, then *y*, then *m*, and *e* at the bottom. To the right stand *K* (upper) and *o* (lower), each joined by straight lines to *l*, *y*, *m* and *e*, so that two sets of triangles are formed on the meridian — the first observation at *K*, the second at *o*. Readable labels: *l* (top), *K* (upper right), *y* (on the meridian, middle left), *o* (right, below *K*), *m* (on the meridian below *y*), *e* (bottom).

Further, how one should proceed when the Sun moves through the southern signs you will gather from the same rules. For if at the moment of observation it stood in a northern azimuth, then, making a pole at the point that represents the Sun, and with an interval equal to the complement of the declination, we shall describe a circle on the surface of the globe itself, and the place of the south pole will be found as in the first canon. But if it is in the azimuth of the equinoctial

rising and setting, the place of the south pole can be found as in the second. If it is found set in a southern azimuth and stands at equal intervals from the zenith and from the south pole, we shall find, as in the third, the distance of the zenith from the south pole itself, and from that the altitude of the visible pole will become known. If it is in a southern azimuth, yet less distant from the zenith than from the south pole, we shall find, as in the fourth, the distance of the zenith from the south pole, from which the altitude of the visible pole will be made plain. If again it is in a southern azimuth, but less distant from the south pole than from the zenith, and the circle described touches the meridian, the place of the south pole will be at the point of contact itself; and the Sun's distance from noon will amount to 90 degrees, to which six hours correspond on the equinoctial circle. And when the interval between the zenith and the south pole itself is subtracted from one quadrant, the altitude of that same south pole is left known.

But if it does not touch, but cuts, then in two places

— it will cut the meridian itself. So if it has been ascertained that the place where the observation is made lies in the northern region, while we do not know how great the elevation of the north pole is, this can be discovered from the same observation. For the place of the south pole on the globe will be that intersection which is the farther from the zenith; and therefore, once the place of the south pole has been found, the elevation of the north pole will be made plain by the method of the sixth canon.

If, however, we are entirely ignorant of where the place is at which an observation of this kind is made, the altitude of the pole cannot be known in the way just described. Indeed, even if it has been established that the place lies in the southern region, you still cannot determine from the data how great the altitude of the south pole is. For that reason we shall observe the Sun again after a short interval of time, and, as in the eighth canon, the altitude of the visible pole above the horizon will become known.

When the Sun has no declination from the equinoctial circle, it is very easy to find the altitude of the pole. For if it is found in a southern azimuth, the visible pole will be the northern one; but if in a northern azimuth, the visible pole will be the southern one. We shall therefore describe a great circle on the surface of the globe, making a pole at the point that represents the Sun; and the intersection nearer the zenith will show the place of the visible pole.

¶ *Ad inueniendum altitudinem poli per radios Solis, etiam si meridiani situs ignoretur. Cap.14.*

To find the altitude of the pole from the Sun's rays, even when the position of the meridian is unknown. Cap. 14.

On that flat circular plate which we used in the preceding chapter — even though no straight meridian line is marked out on it — let the position of the gnomon's shadow be noted while the Sun is shining, and at the same moment let the Sun's altitude above the horizon be taken with the astrolabe. Then, after a further short interval of time, we shall make a like observation: again we shall note the position of the shadow and take the Sun's altitude above the horizon. For from these two elevations of the Sun, together with the shadow's advance along the circumference of the circular plate, it will not be difficult to find the altitude of the pole.

We shall reckon the difference between the shadows at the two observations on the horizon of the globe, beginning from whatever point we please. Let the arc be *h m*, for example. Next we shall carry the movable vertical across to the point *h*, into the position *h e i*, and we shall count the Sun's altitude at the first observation from *h* toward *e*; let the end of it be marked by the point *n*. By the same method, with the vertical moved across to *m* into the position *m e t*, and the Sun's altitude at the second observation counted off, we shall mark its end with the point *p*. The points *n* and *p* will thus be placed on the globe with respect to the point *e* exactly as the Sun stood in the world with respect to the zenith.

Diagram. A single large circle representing the globe, crossed by several arcs. A nearly horizontal line runs across it from the left edge, through *h* and *n*, to the centre *e*, and on to *t* at the right edge; a second arc through *e* rises to *z* just above *t*. From the upper left an arc descends through *x* and past the centre toward *i* on the right-hand circumference; another arc runs from *s* at the lower left up through *p* and *e*. A further arc runs from *m* on the left, low down, through the interior to the right side, and *u* stands at the bottom of the circumference. Two small compass crossings are drawn — one at the lower right, beside *q*, and one on the horizontal line outside the circle at the far left. Readable labels: *x* (upper left, inside), *z* (right, above *t*), *t* (right circumference), *i* (right circumference, below *t*), *e* (centre), *h* (left, on the horizontal line), *n* (right of *h*, slightly below), *m* (lower left), *p* (lower centre-left), *s* (lower left circumference), *q* (lower right, at a crossing), *u* (bottom circumference). One further label, at the far left crossing outside the circle, does not resolve on the leaf [?].

Therefore, in order to learn the position of one of the world's poles relative to the zenith itself, we shall take between the legs of the compasses the arc of the complement of the Sun's declination on the day of observation; and, making the points *n* and *p* poles, we shall describe two circles whose intersections fall at the points *q* and *r*. That pole of the world from which the Sun's declination takes its name — the one nearer the Sun at the time of the observation — will lie either at *q* or at *r*. If it is at *q*, the Sun's parallel will be *p z n*, and the arc of the meridian between *e*, the zenith, and that pole of the world will be *e q*. But if it is at *r*, the Sun's parallel will be *p x n*, and the arc of the meridian between the zenith and that same pole of the world will be *e r*.

At which of the two points it lies we shall learn by this method. If, when we turn to face the Sun, it is seen to move from left to right, we shall say that the zenith is placed between the north pole of the world and the Sun's parallel, and the Sun's parallel between the zenith and the south pole — or else the zenith will lie on the Sun's parallel itself. If it moves from right to left, we shall pronounce the contrary: for when this happens, the Sun's parallel will be placed between the north

pole and the zenith, and that same zenith between the Sun's parallel and the south pole, or on the Sun's parallel itself. So if the pole of the world which at that time is nearer the Sun is the northern one, and the Sun itself turns from left to right, it will be certain that the place of the north pole is at *q*; and thus the arc *e q* becomes known, from which the elevation of the visible pole is at once made plain. But if it is seen to turn from right to left — which you will easily recognize from the circuit of the shadows — the north pole will be at *t*, and the segment of the meridian between that same pole and the zenith will be *e t*, from which the elevation of the visible pole and the position of the meridian become known. One must reason in the same way when the pole nearer the Sun is the southern one.

Once the position of the meridian has been learned by this method, the horizontal distance of the Sun from the meridian itself at each of the two observations cannot remain unknown. And from this you will know at once how far the meridian line of the mariner's instrument departs from the true meridian, if you set up a style at right angles over the middle of it. This is not merely useful to seamen but absolutely necessary, so that they may understand where they are heading as they sail, and the true positions of places.

In those places, however, where the shadows of gnomons advance before and after noon and then go back — which I mentioned above — this rule of ours about the relation of the zenith to the Sun's parallel and to the poles of the world

may deceive the operator. For the Sun, rising at *m* in the figure of chapter XI, is seen by those who are at *d* in the vertical *d n m*; but when it has reached *o*, it will be seen in the vertical *d o p*. Now things looked at by rays farther to the right appear farther to the right, and those looked at by rays farther to the left appear farther to the left, by the postulates of Euclid's optics. From *m*, then, as far as *o*, the Sun will be seen to turn from left to right, while the shadows of the gnomons move oppositely, from right to left; but from *o* as far as *h*, the section of the meridian, it will be seen to revolve from right to left — for on reaching *n* [?] it will return to the vertical *d n m* [?] — and so the shadow moves from left to right.

Accordingly, so that no error or ambiguity may be left in this investigation of ours into the pole of the world, it will be necessary to make a third observation, in which the Sun's altitude is noted together with the difference between the last two shadows. And by the same method we used before, we shall mark a point on the globe to represent the Sun at this last observation; and making that point a pole, we shall describe a circle to the same measure of the complement of the declination. This circle will cut the two earlier circles at one of the two intersections, namely either at *q* or at *r*; that it should cut them at both is impossible, unless the Sun has no declination. Where it does cut, there will be the place of that pole which was the nearer to the Sun at the time of the observation.

When, therefore, the Sun travels along the equinoctial, a third observation is not needed: for to those in the north it turns from left to right all day long, and to those in the south from right to left; while for those placed under the equinoctial itself it turns neither from right to left nor from left to right, since the shadows of the gnomons are thrown along one straight line.

¶ *Ad inueniendum altitudinem poli per radios Solis situ meridiani & declinatione Solis ignoratis. Cap.15.*

To find the altitude of the pole from the Sun's rays when the position of the meridian and the Sun's declination are both unknown. Cap. 15.

When not only the position of the meridian but also the Sun's declination is unknown, it will not be difficult, from what we have taught, to establish both. For we shall make three observations of the Sun, separated by an interval of time long enough for the Sun's own altitudes to increase or decrease by a perceptible difference, and for the shadow's advance along the circumference of the circular plate to be evident.

Then, working just as we did in the preceding chapter, we shall reckon the differences of the three shadows on the horizon of the globe; and by carrying the movable vertical across to the three positions of those shadows, and counting the three altitudes of the Sun on that same movable vertical, we shall mark three points on the globe, which will represent the three positions of the Sun with respect to the zenith. And since three such points must necessarily lie on the Sun's parallel, we shall find the poles of the circle that passes through those same three points according to the precepts of the geometrical art: for those will be the two poles of the world, the northern and the southern.

For example, let the movable vertical be set in whatever position you please, and let it be *a e b*; let *a c* be the Sun's altitude found at the first observation, and let *a d*, on the horizon of the globe, be that arc which the gnomon's shadow traversed along the circumference of the plane instrument between the first and the second observation. Then, with the movable vertical carried across to *d*, let *d f* be the Sun's altitude

found at the second observation. Then again, with the same vertical carried across to *g*, let *d g* be the arc traversed by the gnomon's shadow between the second and the third observation, and let the arc *g h* be the Sun's altitude found at that third observation. The three points *c*, *f* and *h* will then be placed on the surface of the globe with respect to the point *e* exactly as the Sun was found in the world at those three observations.

So, in order to find the poles of the world and to describe the circle that passes through those three points, we must work by no other method than the one commonly used for finding, in a single plane, the centre of a circle passing through three given points that do not lie on one straight line; and the demonstration of the one will be like the demonstration of the other. For in this case arcs of great circles are to be drawn through each pair of points, but in that case straight lines. There we argue by propositions 8 and 4 of the first book of Euclid; here by the similar propositions 4 and 8 which Menelaus demonstrated in the first book of his *Spherics* [*triangulorum sphæricorum*].

On the points *c* [?] and *f*, then, with an interval greater than half of *c f* but nevertheless less than a quadrant, we shall make crossings at *i* and *k*; and fitting to those points *k* and *i* some circular ring [*armilla*] like the movable vertical, we shall describe along it a great circle on the surface of the globe itself, *l k i*. In the same way, on *f* and *h*, with an interval greater than half of *f h*, we shall make two other crossings at *m* and *n*; and with the same circular ring fitted to those points *m* and *n*, we shall describe the great circle *l n m*.

Diagram. A large circle representing the globe, with *p* standing above it and *q* below, at the ends of the great circle drawn through *e* — the horizon points of that circle. The movable vertical runs from *a*, on the upper left of the circumference, through the centre *e* to *b* on the lower right. A long arc sweeps in from *L*, far outside the circle at the left, rises through *k* (marked by a compass crossing, also outside the circle), enters the globe at *d*, and continues up through *c* and *i* (likewise marked by a crossing) near the top. A second long arc runs from the lower left through *n* (a crossing, outside the circle) and *g*, into the globe past *h*, and up toward *m*, high on the right. Further arcs cross at the centre *e*. Readable labels: *p* (top, outside), *a* (upper left, inside), *c* (right of *a*), *i* (upper centre, at a crossing), *m* (upper right of centre), *e* (centre), *d* (left, inside), *f* (right of *d*), *h* (below *f*), *k* (left, outside, at a crossing), *n* (lower left, outside, at a crossing), *g* (right of *n*), *L* (far left, outside), *b* (lower right circumference), *q* (bottom, outside). One further label, immediately to the right of *i*, does not resolve on the leaf [?].

Now let one intersection of these two great circles be at the point *o* above the horizon, and the other at *l* below the horizon. I say, then, that those points *l* and *o* are the two poles of the world, the arctic and the antarctic — so that a circle described on *o* or on *l* through *c* will pass also through *f* and *h*. Whichever pole is found nearer to the vertical point *e* will be the visible one; the farther, beneath the horizon, will be the hidden one. The arc *e o* will therefore be the complement of the pole's altitude, a great circle having been described through the points *e* and *o*, which cuts the horizon at *p* and *q*.

If the arc of the great circle between *e* and *o* is found equal to a quadrant, the Sun will be moving that day on the equinoctial; but if it is found less or greater than a quadrant, the difference from the quadrant will be the Sun's declination. When in this way it has become known how great the elevation of the visible pole is and how great the Sun's declination, then, if it is known in which half of the zodiac the Sun stands at that time, not only will these show what the declination itself is, but also which pole of the world it is that is seen elevated, the northern or the southern. As for the position of the meridian, you will learn it from the distance of the shadow from the point *p* or *q*, just as in the preceding chapter.

¶ *Rursus declinatione Solis & meridiani situ ignoratis, altitudinem poli in plano vtrius circuli inuenire. Cap.16.*

Again, with the Sun's declination and the position of the meridian unknown, to find the altitude of the pole in the plane of either [?] circle. Cap. 16.

First of all we must recall the arrangement and relation of certain circles of the sphere, of their poles, and of the centres of those same circles, in the common planisphere of Ptolemy. For the centre of the equinoctial is put in place of the visible pole. Straight lines coming from that centre serve in place of great circles, which

are drawn through the poles of the world. The right horizon, then, is simply a straight line. For the remaining circles, circles are set down — but not all of one and the same size, and their centres lie elsewhere than their poles do. The centres of the oblique horizons and of the circles parallel to them, together with their zeniths, lie on the meridian line. Yet although the horizon and the circles parallel to it have the same pole, their centres are not the same. Once the position of the pole (or of the centre of the horizon) is known, however, the centres and diameters of the parallel circles — which in Arabic are called almucantarats [*Almicantarath*] — will be known too; and conversely, from the known diameter of any one of those parallels, the relation and the distance of the pole of the horizon from the pole of the world will be brought to light.

Since this is so, we may, whenever the work requires it, interchange the poles of the world with the poles of the horizon, the equinoctial with the oblique horizon, the circles parallel to the equinoctial with those parallel to the horizon itself, and the meridians with the vertical circles — so that what was the right horizon becomes the vertical circle of the rising and setting of the equinoctial. In this exchange only one straight line keeps its own office: the one that serves as the meridian, passing through the poles both of the world and of the horizon. From this it will be easy to see by what art we can find, in the planisphere, the distance of the pole of the world from the pole of the horizon, given the diameter and position of any one of the circles parallel to the equinoctial.

In the planisphere, then, let the equinoctial *a b c* be put in place of the horizon and cut into 360 equal parts; and let the centre *d*, which was the pole of the world, now be the pole of that horizon, that is, the zenith. Through this centre draw two hidden diameters cutting one another at right angles; and from the end of one of them — call it the beginning of the first quadrant of that circle — draw straight lines to the single degrees of the second quadrant, marking with points the places where these lines cut the other diameter. This is what we regularly do when, in Ptolemy's common planisphere, we derive the circle of Cancer and any other parallels from the equinoctial. One of the semidiameters having been divided in this way into 90 parts, we shall divide the index or pointer of the astrolabe to the same measure and into the same parts, set out plainly, and shall

write against them their proper numbers. The pointer will then serve in place of a movable vertical circle, and by its help the Sun's altitudes may be marked on the planisphere.

So when it is worth our while to learn the altitude of the pole above the horizon from three altitudes of the Sun and two differences of the shadow, we shall lay off on the circle *a b c*, starting from whatever point we please — say from *e* — the shadow's run between the first observation and the second; let that be *e f*. From the point *f* we shall in the same way lay off the shadow's run between the second observation and the third, that is, the last; let that be *f g*. Then, with the movable vertical (the pointer) set in the position *d e*, we shall lay off the Sun's first altitude from *e* toward *d*; let it be *e h*. In the position *d f* we shall lay off the second altitude, *f i*, and in the position *d g* the third, *g k*. These three points *h*, *i*, *K* will therefore stand in the planisphere in the same relation to the point *d* as the Sun in the heavens stands to the zenith.

By the rules of geometry, then, let a centre be found about which a circle may be described passing through those same three points; let that centre be *l*. Accordingly, if with the measure *l h* (or *l i*, or *l K*) the circle *K m h* is described, it will be the Sun's parallel on the day on which the observations just mentioned were made. Now join *d* and *l* by a straight line, and let it, produced both ways, cut the circumference of the horizon at the points *o* and *n*. The straight line *n o* will therefore stand for the meridian, and so the position of the meridian will be known — for the gnomon's shadow at the last observation must stand off from the meridian by as many degrees and minutes as there are in the arc *o g*.

From the point *d*, the centre of the planisphere, erect on *n o* a straight line at right angles in the same plane, and produce it both ways to the length of the diameter; let it be *p q*. Then *p q* will stand for the vertical circle of the rising and setting of the equinoctial. From the point *p* draw straight lines to *r* and *s*, the intersections of the Sun's parallel with the meridian line; these cut the horizon at *t* and *u*. Let the arc *t u* be bisected at *z*; and from *p* to *z* draw a straight line cutting the meridian at *x*. Then *q z* will be the distance between the zenith and the visible pole of the world, and the point *x* itself will represent that pole in the planisphere. So if the circle *a b e* [?] is taken as the meridian, *q* will be the zenith and *z* the visible pole of the world; and the arc *z u* will be the distance, from that same pole of the world, of the parallel which the Sun's degree describes — and thus the Sun's declination will be known. The Sun's meridian altitude will be *o u*. Let the eastern intersection of that same parallel of the Sun with the horizon be at

y: the arc of the horizon *q y* will then be the amplitude of rising [latitudo ortus]. Let the two arcs *z A* and *z B* be qua-

drants. In this example, then, *A u* will be the declination of the Sun's place toward the visible pole; and it will be evident even if the greatest obliquity of the zodiac is unknown. From the same point *p* draw straight lines to *A* and *B*, cutting the line *n o*, produced further, at *C* and *D*. *C D* will therefore be the diameter of the equinoctial. So if a circle is described about the midpoint *E*, it will pass through *p* and *q*, and will discharge in the planisphere the office of the equinoctial. The arc *A q*, moreover, is the latitude of the place, and *z n* the altitude of the pole above the horizon. Likewise, if you set the index or pointer that stands for the movable vertical circle in the position *n o*, the number of parts between *n* and *x* will also show the altitude of the pole above the horizon, and *C d* the latitude of the place.

None of this is difficult to carry out; and what has to be taken in order to find the thing sought is very little, and lies ready to everyone's hand — namely that, while the Sun shines, its altitude above the horizon can be caught, and the run of the gnomon's shadow in a plane parallel to the horizon. The many things found in this way, most worth knowing, are the foundations of astronomy and cosmography.

Diagram. A large circle — the planisphere of this chapter — filling the whole left-hand column and running under the right-hand column. A vertical straight line is drawn from *o* at the top of the circumference downward past the centre and out beyond the circle to *D* at the foot: this is the meridian *n o* produced. On it, reading downward, stand *C* (just inside the circumference at the top), *r*, the centre *d*, *x*, *l*, *E*, and *s*. A horizontal straight line runs from *p* at the left-hand circumference across to *q* at the right, and from *p* a sheaf of straight lines fans out to the right-hand and lower parts of the figure. Two further circles are drawn within and across the principal circle: the Sun's parallel, cutting the vertical line at *r* and *s*, and a second large circle passing through *p* and *q*. Readable labels: *o* (top of the circumference), *a* (upper left, outside the circumference), *C* (inside, below *o*), *r* (on the vertical line, upper middle), *d* (centre), *x* (on the vertical line, below the centre), *l* (below *x*), *E* (below *l*), *s* (on the vertical line, low), *D* (foot, outside the circle), *p* (left circumference), *q* (right side), *g* (right, high), *f* (right, below *g*), *e* (right, below *f*), *h* (just inside the circumference beside *e*), *i* (interior, left of *f*), *k* (interior, above *i*), *z* (right, low), *y* (right, low, inside), *c* (lower right), *b* (left, middle), *B* (lower left), *t* (lower left, beside *B*), *m* (twice, in the lower left interior). Two further marks in the upper right of the figure do not resolve on the leaf [?] [?].

¶ *Nocturno tempore altitudinem poli supra horizontem inuenire. Cap.17.*

To find the altitude of the pole above the horizon by night. Cap. 17.

If some star of known declination is found on the meridian — that is, at its greatest or least altitude — you will be able to find the altitude of the pole from it in exactly the same way as from the Sun's rays. If not, take the altitudes of two known stars that stand on different vertical circles; and on the star-globe that astronomers use, describe two circles about those same stars as poles, with the complements of their altitudes. These circles will have two intersections; and since the zenith of the place at which the observation is made lies at one of them, you will learn which of the two is to be taken from the turning of the stars, just as we said above, in chapter 14, about the Sun. In this way the distance of that zenith from the equinoctial — a distance equal to the altitude of the pole — will become known.

¶ *De instrumento, quo vtraq; Solis distantia à meridiano per aequinoctialem videlicet & per horizontem inueniuntur, & de vmbrarum ratione ad gnomonem. Cap.18.*

On an instrument by which both distances of the Sun from the meridian are found — namely along the equinoctial and along the horizon — and on the relation of the shadows to the gnomon. Cap. 18.

Sailors seldom use sundials, because in sailing they do not stay long under one and the same elevation of the pole of the world. More often they observe the Sun in order to learn on which vertical circle, or azimuth, it stands; and this they determine by estimation alone, with the help of the mariner's instrument, not from the Sun's ray nor from the shadows of gnomons. It will therefore not be useless to construct a sundial by which both distances of the Sun from the me-

ridian may be caught, namely along the equinoctial and along the horizon. Let the circle of a horizontal dial be divided into the hour spaces in the usual way; then about *a*, the point of noon, and to the same measure, let one circle be described and divided into 32 equal parts, drawing lines from the centre to the points of division. A circle of this kind will stand in place of the mariner's instrument, which the Spanish call the needle [*acus*]. Next, at *a* itself let the stylus *c d* be erected at right angles to the plane of the dial, and of such a height that the thread which must be fastened to the centre *b* and to the top *d* makes with *a b*, at

the point *b*, the angle of the pole's altitude in the given region. With all this prepared, if you set the instrument itself in some plane parallel to the horizon, and the straight line *a b* has besides been laid in the position of the meridian, then the shadow of the stylus *c* [?] *d* falling in the circle whose centre is *a* will show the Sun's azimuth, while the shadow falling on the dial will show the hour of the day.

Sailors suppose that the Sun's distances from the meridian, reckoned along the horizon and along the equinoctial, are always equal to one another. They are mistaken. These are equal only once if they are reckoned from the same side of the meridian — namely when the Sun's altitude above the horizon is found to be as great as its declination toward the hidden pole. They are equal once again if they are reckoned from opposite sides, namely when the Sun's altitude above the horizon is as great as its declination toward the visible pole.

For let *a b c* be the meridian, *e c* the semicircle of the equinoctial, and *d f* that of the horizon; let *a* be the visible pole and *b* the zenith, and let the Sun, set at *g*, be in the same part of the world as the zenith, whether before noon or after. Let the circle of declination *a i* be drawn through the Sun, and the circle of altitude *b k*. Then the two arcs *a g* and *b g* taken together are less than a semicircle, and therefore, in the triangle *a g b*, the exterior angle *g b d* will be greater than the interior angle *b a g*. Consequently *d K*, the Sun's distance from the meridian along the horizon, will be greater than *e i*, its distance along the equinoctial. You will conclude the same thing, and by the same art, if the Sun stands on the equinoctial circle.

Diagram. The horizontal sundial described in this chapter, occupying the lower half of the left-hand column. A broad graduated fan fills the upper part: a double-ringed border subdivided into many small cells, with straight hour-lines radiating inward and converging on the dial centre. From that centre a horizontal line runs to the right, and a short upright at the right-hand edge carries the top of the style; a thread is drawn from the dial centre up to that top. Below the fan, on the vertical line through the centre, a smaller double circle is drawn containing a dark, roughly diamond-shaped pointer — the 32-point circle that stands in place of the mariner's needle. Readable labels: *a* (at the convergence of the hour-lines), *b* (right, on the horizontal line), *c* (right-hand edge of the fan), *d* (upper right, at the top of the upright). One or two further small marks inside the fan do not resolve on the leaf [?].

Diagram. The first of two spherical figures in the right-hand column: a circle standing for the meridian, crossed by three arcs. One arc runs from *e* on the left

of the circumference across to *f* on the right; a second, lower, runs from *d* on the left to *c* on the right; a third slants from the upper part of the circle down toward the lower left. Two short arcs meet in the upper middle of the figure at *g*, with *i* just below and to the left, and *K* lower still, near the centre-left. Readable labels: *b* and *a* (side by side at the top of the circumference, *b* to the left), *e* (left circumference), *f* (right circumference), *c* (right circumference, below *f*), *d* (left circumference, below *e*), *g* (upper middle, interior), *i* (interior, left of *g*), *K* (interior, below *i*).

Further, with the same circles drawn, let us suppose that the Sun declines toward the parts of the hidden pole, and that the arc of altitude *g K* is equal to the arc of declination *g i*. Then the two arcs *b g* and *a g* taken together are equal to one semicircle; and therefore the exterior angle *e b g* [?] will be equal to the interior angle *b a g* in the same triangle *g b* [?], and so the distance *d K* along the horizon will be equal to the distance *e i* along the equinoctial. But let us now suppose the arc *g K* of the Sun's altitude to be less than *g i*, the arc of declination. Then the two arcs *b g* and *a g* taken together are greater than one semicircle; so the exterior angle will be less than the interior, and therefore *d K* will be less than *e i*. And let us suppose *g K* to be greater than the declination *g i*: then the two arcs *b g* and *a g* taken together will be less than one semicircle, and for that reason the exterior angle will be greater than the interior

Diagram. The second spherical figure of the right-hand column, drawn to the same pattern as the first and standing below it: a circle for the meridian crossed by arcs, with a faint second circumference sketched outside the lower right. Readable labels: *b* (top of the circumference), *a* (to the right of *b*, inside), *e* (upper left circumference), *d* (left circumference), *f* (right circumference), *c* (lower right circumference), *K* (interior, left of centre), *i* (interior, right of centre), with a small interior mark below *e* [?]. A further short label at the right of the interior does not resolve on the leaf [?].

will be greater, in the same triangle *a g b*. The distance *d k*, therefore, will be greater than *e i* itself.

Finally, let us suppose that *g k*, the Sun's altitude above the horizon, is equal to *g i*, its declination toward the visible pole *a*. Then in the spherical triangle *a g b* the two sides *a g* and *b g* will be equal to one another; therefore the two angles *g b a* and *b a g* will be equal to one another as well; and consequently the arc of the horizon *f K*, by which the Sun stands off from the midnight point, will be equal to the arc of the equinoctial *e i*, by which it is distant from the meridian on the opposite side.

Further, if you wish to compare these two distances — the one taken along the horizon, the other along the equinoctial — when the Sun is at its rising or setting, this will be easy to see in the figure below.

Diagram. A single circle representing the meridian, crossed by several chords. The circumference carries *b* at the top and *a* just below and to its right; *e* on the upper left and *d* on the left; *f* on the right and *c* at the lower right. In the interior are *i* (left of centre), *g* (centre), and *K* below *g*. A chord runs from *e* down across to *c*; a second nearly horizontal chord runs from *d* across to *f*; a short line descends from *b* through *g* to *K*, and another from *a* down to *i*.

- b (top of the circumference): pole point of the demonstration
- a (upper right, next to *b*): the visible pole
- e (upper left of the circumference)
- d (left of the circumference)
- f (right of the circumference)
- c (lower right of the circumference)
- i (interior, left of centre)
- g (interior, centre)
- K (interior, below *g*)

For let the Sun, having declination toward the visible pole, be placed at the point *l* of the horizon — that is, at its rising or at its setting. Then the arc *b l* will be a quadrant, while *a l* is less than a quadrant; and so the two arcs *b l* and *a l* taken together are less than one semicircle. But at the point *m* of the horizon, when the Sun declines toward the other pole, the two arcs *b m* and *a m* taken together are greater than one semicircle. Therefore the angle *d b l*, of the distance along the horizon, will be greater than the angle *b a l*, of the distance along the equinoctial toward the meridian point. And consequently the angle *l b a*, of the remaining distance along the horizon, will be less than the angle *l a f*, of the distance along the equinoctial toward the midnight point.

Diagram. A second circle of the same construction, more densely ruled: *b* stands at the top of the circumference, *e* on the upper left and *a* on the upper right, *d* on the left and *f* on the right. Chords run from *b* down to the lower left and lower right, and a network of further chords crosses the interior, meeting near the centre. Two interior labels sit near the middle of the figure.

- b (top of the circumference)
- e (upper left of the circumference)

- a (upper right of the circumference)
- d (left of the circumference)
- f (right of the circumference)
- m [?] (interior, at centre-left)
- [?] (a second interior label, right of centre, not resolvable on the scan)

The contrary of this happens when the Sun is at the point *m* [?]. But if it is placed at the point *n* — the equinoctial rising or setting — then the distances *e n* and *d n* will be equal to one another, since both are quadrants.

At this point, however, we shall demonstrate what we mentioned earlier about the ratio of the shadows to the gnomon: namely, that these three lengths — the direct shadow [*umbra recta*], the gnomon, and the turned shadow [*umbra versa*] — are proportional; for as the direct shadow is to its own gnomon, so is any gnomon to its turned shadow.

Let *b d* be a straight line lying in a plane parallel to the horizon; let the straight line *a b* be a gnomon standing perpendicular to that same plane, and *b c* the shadow it casts. Let *d e* be, besides, the turned shadow on the face of a wall standing at right angles to the plane of the horizon, and let the straight line *d c* be the gnomon that casts it. Let the Sun's ray be *a e*. It makes no difference whether the gnomons *a b* and *d c* are equal or unequal. I say that *a b* is to *b c* as *d e* is to *d c*. For the two triangles *a b c* and *d c e* are equiangular; therefore they have proportional sides about their equal angles, so that as *a b* is to *b c*, so *d e* is to *d c*, by proposition 4 of Euclid VI.

Diagram. The gnomon-and-shadow figure, in the right column. A horizontal line runs from *d* on the left to *b* on the right, with *c* on it between them; from *b* a vertical line rises to *a*; from *d* a vertical line descends to *e* at the lower left; and a long straight ray runs from *e* up through *c* to *a*.

- a (top right): top of the upright gnomon *a b*
- b (right, foot of the upright gnomon): the direct shadow runs from *b* to *c*
- c (on the horizontal line, between *d* and *b*): where the Sun's ray *a e* crosses
- d (left end of the horizontal line): foot of the horizontal gnomon *d c* on the wall face
- e (lower left): end of the turned shadow *d e*, and the point the ray *a e* is drawn from

And although the shadows are marked out not by the same ray *a e* but by different rays at the same instant of time, the demonstration will nevertheless hold, because the triangles are similar.

The sailors of our own day, however, know little of shadows, and do not derive from them the distance of the zenith from the equinoctial. The mathematicians of antiquity, by contrast (as we find in Vitruvius, book 9), marked the obliquity of horizons simply by the ratio of the noon shadows to the gnomons at the time of the equinox. For once the ratio of the gnomon *a b* to the shadow *b c* is known, the side *a c* subtending the right angle becomes known by proposition 47 of Euclid, book I. And as *a c* is to *b c*, so is the whole sine to the right sine of the angle *b a c*; therefore, by the common rule of proportional numbers,

the right sine of the angle *b a c* itself will become known, and by the table of right sines the arc of that same angle will be disclosed — which is the Sun's distance from the zenith; and so the latitude of the place will be known.

By the table of the geometric square of Georg Peuerbach you will find the same result without extracting a square root, in this way. If the shadow is less than the gnomon, multiply the parts contained in it by 1200 and divide the product by the number of parts of the gnomon; then enter the table just named with the quotient, and you will find the magnitude of the arc of the angle *b a c*.

An example. At Rome the ratio of the gnomon to its direct shadow at noon at the equinox is as 9 to 8, as Vitruvius states. We shall therefore multiply 8 by 1200 and divide the product by 9, which gives 1066 and two-thirds. With this I draw from the table 41 degrees 38 minutes for the latitude of the city of Rome — a latitude that Ioannes de Monteregio [Regiomontanus], working from the meridian altitude and the Sun's declination, found to be 42 degrees 4 minutes, or 8 [?].

But if the direct shadow is greater than the gnomon, multiply the parts of the gnomon by 1200 and divide the product by *b c*; then with the quotient we shall draw from the same table the arc of the angle *a c b*, the Sun's altitude above the horizon: and so its distance from the zenith will be known. And when the shadow is equal to the gnomon, the Sun's altitude above the horizon will be as great as its distance from the zenith — 45 degrees, that is.

Besides this, we judge it worth noting that geometers accept that the Sun's rays appear parallel at the earth, and likewise the shadows of gnomons — yet not any rays whatever, but only those that converge at the greatest distance from the earth. Georgius Valla [Giorgio Valla] and Iacobus Zieglerus, however, hold the opposite, along with Pliny: for they say that the rays which either cast the

shadows from gnomons or enter through the apertures in the sighting-plates of the astrolabe's alidade [*tabellae dioptrae*] not only seem parallel but are parallel, and that the shadows of gnomons are truly equidistant as well. It will therefore not be foreign to our present undertaking to treat and examine these matters point by point.

Note, then, that countless solar rays are sent to the earth parallel, and that they stand apart from one another on the earth's surface at whatever interval you please. For of such parts as the Sun's semidiameter contains five and a half, the Earth's semidiameter is only one. And so, if two parallel lines embracing the earth itself are drawn as far as the Sun, within the circuit of the illuminated part, neither of them will touch the Sun's body, but will rather cut it.

It is established from optics [*perspectiua*], moreover, that the Sun's light is diffused along straight luminous lines, which they call rays; and therefore there is no doubt that countless rays sent down from the Sun to the earth are parallel. Countless solar rays also converge at the earth's surface and near the earth. For if from any point of the earth's surface two straight lines are drawn touching the Sun on opposite sides, then however many straight lines are drawn from that same point toward the Sun between those two, they will cut the Sun's body; and it is plain that by these the Sun's light is carried to one and the same point of coincidence.

But since the geometers called the Sun's rays parallel not absolutely, but at the earth, it is clear that they had in mind neither those rays that are truly parallel nor those that converge at the earth. It will not then be difficult to understand which rays they supposed to appear parallel at the earth. For it is established from optics that when we observe the Sun's altitude with astrolabes, the rays sent down from the segment of the Sun facing us to the aperture in the sighting-plates come together somewhat before that aperture into the shape of a point; and then, past that meeting, passing through [?] the aperture set before them with the cone now inverted, they shine out on a wider base. In this way the ray of the centre, which is also the axis of the cones, is what discloses the Sun's altitude.

And since different cones of the Sun's rays, and different axes, are formed toward different parts of the earth, it is clear that rays are transmitted from different parts of the Sun to different parts of the earth as the means of determining the Sun's altitude — rays which nevertheless converge at one common point of coincidence, and that point is the Sun's centre. This is what we wished to show first.

We shall now show, in the following way, that those rays which cast the shadows of gnomons converge at the greatest distance from the earth. Let *a* be the centre of the earth and *b* the centre of the Sun, and let the straight line *a b* be drawn joining them; and through it let a plane be passed cutting the body of the Sun and the body of the earth. The common sections of this conceived plane with the bodies of the Sun and of the earth will therefore be great circles, by the first and the sixth propositions of Theodosius, book I (Theodosius, *Spherics* I.1 and I.6); let these be *c d e* and *f g h*. Let the extreme solar rays illuminating the earth be *c i* and *d i*, which must of necessity touch both bodies, the Sun's and the earth's, by what Aristarchus of Samos, Allacen [Alhazen], and a good many others have demonstrated. For the earth is illuminated by the Sun not only by those rays that set out from its centre, but also by those sent from its circumference. Let the rays *c i* and *d i*, then, touch the solar body at *c* and *d*, and the earthly body at *f* and *g*; and when the straight line *a b* has been extended,

it will meet those same rays at *i*. The earth, therefore, will be illuminated along *f h g*, a segment of a great circle.

From any point *K* you please between *a* and *i*, let a straight line be drawn touching the Sun's circle *c d e* at a point *l* on the near side of *c* — for it cannot touch it beyond *c*, lest an impossibility follow, contrary to the last of the common notions, that two straight lines alone do not enclose a space — and let this same line *k l* cut the earth's circle at *m*. For that reason *K l* will meet *c i*, the straight line touching the circles of the Sun and of the earth, before the point *c* at the Sun. And by the same method you will show that from any other point besides *K* that lies between *a* and *i*, a straight line drawn so as to touch that greatest circle of the Sun meets those same lines *c i* and *k l* at the Sun.

From the point *o*, which lies near the earth on the straight line *m l*, let a straight line be drawn as far as *a*, the centre of the earthly globe, cutting the circle *f g h* at the point *n*. We shall understand this point *n* to be some place on the earth's surface at which the Sun is seen raised above the horizon, and the straight line *n o* to be a gnomon, over whose top *o* the Sun's ray *l o* arrives, marking off the shadow *m n* on the earth's surface. The angle *m o n*, then — or the angle vertically opposite to it, which *n o* produced in a straight line makes with the ray *l o* — will subtend the Sun's distance from the zenith of the place *n*.

But for those who are at *h*, the Sun's ray *b h*, falling perpendicularly toward the centre of the earth, will cast no shadows toward any quarter of the horizon, but shadows hidden beneath the feet of the gnomons. This perpendicular ray *b h*,

therefore, will meet the ray *l o* at the point *K* beneath the earth's centre, and not at the Sun.

Let the same be understood to happen, and the same accounts of the shadows will hold, in all places that stand off from the place *h* by intervals equal to *h n* or to *h m*. You will grasp this readily if from the point *l* you draw the straight line *l p*, cutting the straight line *b K* at right angles at the point *r*, and then conceive the right-angled triangle *K r l* to be carried around while *K r* stays fixed. By that means a certain cone will be described, whose axis will be *K r*; the triangle through the axis will be *k p l*, and the base a circle whose diameter is *l p* and whose semicircumference is *l q p*. One part of this cone will be another cone, having for its base a circle on the earthly globe whose diameter is the straight …

Diagram. A tall figure occupying the whole right column: the Sun's great circle above, the earth's small globe below, and the pair of extreme rays running down from the Sun past the earth to a point of convergence at the foot. In the upper circle, *e* stands at the top, *b* at the centre, *c* on the left circumference and *d* on the right. Below *b*, on the vertical axis, is *q*; a long lens-shaped construction crosses the lower part of the solar circle, with *l* at its left end and *p* at its right, and *r* on the axis between them. Far below, the small circle of the earth carries *a* at its centre, *m* at the upper left, *n* and *h* at the top, *s* at the upper right, *f* at the lower left and *g* at the lower right; *o* stands on the left-hand ray just above the earth. The axis continues downward through *K*, just under the earth, and the two outer rays meet at *i* at the very bottom.

- e (top of the solar circle)
- b (centre of the solar circle)
- c (left of the solar circle): point of contact of one extreme ray
- d (right of the solar circle): point of contact of the other extreme ray
- q (on the axis, below *b*): on the semicircumference *l q p*
- l (left end of the chord *l p*): point of contact of the line drawn from *K*
- p (right end of the chord *l p*)
- r (on the axis, between *q* and the chord): foot of the perpendicular from *l* on *b K*
- o (on the left-hand ray, just above the earth): top of the gnomon *n o*
- n (top of the earth's circle, left of *h*): foot of the gnomon
- h (top of the earth's circle): the place where the ray falls perpendicular
- m (upper left of the earth's circle): end of the shadow *m n*
- s (upper right of the earth's circle)

- a (centre of the earth's circle)
- f (lower left of the earth's circle): contact of the extreme ray *c i*
- g (lower right of the earth's circle): contact of the extreme ray *d i*
- K (on the axis, below the earth): meeting of the perpendicular ray *b h* with the ray *l o*
- i (foot of the figure): convergence of the extreme rays *c i* and *d i*

line *m s*, which cuts at right angles the straight line *a h*, the semidiameter of the earth [?]; and its semicircumference will be *m t s*. And therefore, however many straight lines are drawn from the cone's vertex *K* to the circumference *l q p*, they will touch the sun's body at points of that same circumference, but they will cut the earthly globe at points of the circumference *m t s*.

Now at the sun let the points of contact themselves be joined to the sun's centre. The triangles so set up will then be equilateral and equiangular with the right-angled triangle *K b l*, by proposition 8 of Euclid I; and the side *b K* will be common to all of them. From the remaining sides — those that are equal to the radius *K l* — let parts be cut off equal to the straight line *K o*, and from their endpoints let straight lines be drawn to the point *a*. The triangles constructed by this device will be equilateral and equiangular with the triangle *a K o* itself.

For this reason, in every place situated on the semicircle *m t s*, the solar rays that mark off the shadows of the gnomons will exhibit equal distances of the sun from the zeniths, and they will meet at *K*, the common point of coincidence — which must necessarily happen for the remaining places of the other semicircle as well. So it is clear that the solar rays which determine the shadows in those places whose zeniths have been set on one and the same great circle passing through the sun's centre, whether before the sun itself or beyond it, will converge on the sun's side, but not in the sun itself. But in those places whose zenith points stand off from the sun by equal arcs, the rays will coincide below the earth's centre. From this it necessarily follows that with any ray whatever that marks off a shadow, countless other rays converge at the sun, and countless others below the earth's centre. Accordingly, rays that belong to neither the first kind nor the second — since they do not lie in one plane — are neither parallel nor will they converge.

That the sun's rays appear parallel at the earth we find demonstrated by Witelo, and in the book *On the Construction of Various Mirrors* by an uncertain author. We shall nevertheless show the same thing far more exactly, in this way. Let two equal solar rays *a b* and *a c*, arriving at the earth's surface, meet at the point *a* — whether at the sun, or near the sun, or below the earth's centre — and let their

equal portions *b d* and *c e* at the earth be of a magnitude imperceptible in comparison with the length of those same rays *a b* and *a c*; and let *b c* and *d e* be joined. The two straight lines *b c* and *d e* will then be equidistant, by proposition 2 of Euclid VI;

Diagram. A tall, narrow triangle set into the right column. Its apex is *a* at the top; the two long sides descend from *a* to the foot of the figure. Near the bottom a short horizontal segment runs from *d* on the left side to *e* on the right side, and just below it a slightly longer horizontal segment runs from *b* on the left side to *c* on the right side.

- a (apex, top): the point where the two solar rays meet
- d (on the left side, just above *b*): upper end of the portion *b d*
- e (on the right side, just above *c*): upper end of the portion *c e*
- b (lower left, foot of the left side): where the ray *a b* reaches the earth
- c (lower right, foot of the right side): where the ray *a c* reaches the earth

and therefore the two triangles *a b c* and *a d e* will be equiangular and similar. Consequently, as *a b* is to *a d*, so *b c* is to *d e*; and, by conversion of the ratio, as *a b* is to *b d*, so *b c* is to the excess by which that same *b c* exceeds the straight line *d e*. But *b d* is of imperceptible magnitude if it is compared with *a b* or with *a d*; therefore the difference in magnitude between the straight lines *b c* and *d e* will be imperceptible, if it is compared with either of them. And so *b c* and *d e* appear equal; and since they are equidistant, the two lines *b d* and *c e*, which were laid down as equal, will appear equidistant.

For equidistant straight lines seem to nod toward one another and, as it were, to converge whenever the interval between them appears to grow smaller and they seem to approach one another more closely — as was shown by Euclid in proposition 6 of the *Optics*, and by Witelo in book four. And therefore, when the intervals between converging lines appear equal, those converging lines will seem neither to nod toward one another nor to draw away, and they will appear altogether parallel. For that reason the straight lines *b d* and *c e* will be seen as equidistant at the earth. It makes no difference at all whether you take *b c* and *d e* themselves for the intervals between the straight lines *b d* and *c e*, or the perpendiculars drawn from their endpoints. You may also, if you like, draw the conclusion by proposition 33 of Euclid I, as Witelo does and as is done in the book *On Mirrors* itself.

The same thing is proved in another way, by experiment, in that same book. The ray *a b*, on an astrolabe whose centre is *b*, shows the sun's altitude *c d*. Let the

horizon line *b d* be produced in a straight line, and in the same plane in which the astrolabe's own face lies let a second astrolabe be hung, having its centre *e* on that same straight line. Then, with the sun's ray *e f* coming through the centre *e*, at the same instant an altitude arc *g K* will appear equal to *c d*; therefore the angles *a b d* and *f e g* are equal. Consequently the two rays *a b* and *e f* will appear parallel, by proposition 28 of Euclid I — which was to be demonstrated.

This second demonstration, however, we do not approve. For let those two rays meet at the point *l*, the sun's

centre, and on the ray *b l* let two equal portions *l m* and *m n* be taken; and from the points *m* and *n* let two lines *m o* and *n p* be erected equidistant from the straight line *b e* [?].

Diagram. At the head of the left column, two long straight lines descend from the point *l* at the top and diverge as they fall toward two small ruled circles at the foot of the figure. On the left-hand line stand *m* (higher) and *n* (lower); on the right-hand line stand *o* and *p*, at the same heights respectively. Two horizontal lines cross the figure: the upper runs from *m* through *o* and on to *r* [?] at the right; the lower runs from *n* through *p* and on to *q*. Lower down, the letter *a* stands on the left-hand line and *f* on the right-hand line. At the foot are the two astrolabes, each drawn as a circle ruled with crossing diameters, and a single horizontal line passes through both of them.

- l (top): the point at which the two rays meet, taken as the sun's centre
- m (upper point on the left line), n (lower point on the left line): the two equal portions *l m*, *m n*
- o (upper point on the right line), p (lower point on the right line)
- r [?] (right end of the upper horizontal line, beyond *o*)
- q (right end of the lower horizontal line, beyond *p*)
- a (on the left line, below *n*): the ray *a b*
- f (on the right line, below *p*): the ray *e f*
- b [?] (centre of the left circle), c (on its rim above), d (on it at the horizon line): the first astrolabe, with the altitude arc *c d*
- e (centre of the right circle), K (on its rim above), g (on it at the horizon line, to the right): the second astrolabe, with the altitude arc *g K*

n l will therefore be double *m l*, and so, because the triangles *l n p* and *l m o* are similar, *n p* will be double *m o*; and consequently the two will appear unequal. *n p* and *m o* will therefore not be seen as equidistant. Yet if *n p* and *m o* are produced as far as *q* and *r* [?], the angle *l p q* will be found by the astrolabe to be equal to

the angle *l n p*, and the angle *l o r* equal to the angle *l m o*, because the difference is imperceptible: for the angles *l n p* and *l m o* are equal to the angle *a b d*, and the angles *l o r* and *l p q* are likewise equal to *f e g*.

Nor, conversely, does it follow — if two straight lines have been seen as equidistant — that the alternate angles, or the exterior angle and the interior, will on that account be found equal by the astrolabe. For in the equilateral triangle *a b c*, whose sides are very long, let equal portions *b d* and *c e* be taken, of a magnitude nevertheless imperceptible if it is compared with *a b* and *a c*, and let *d e* be joined.

Diagram. At the head of the right column, a triangle with its apex *a* at the top, its base running from *b* at the lower left to *c* at the lower right, and the base produced beyond *c* to *h* at the right. A straight line joins *d*, on the left side above *b*, to *e*, on the right side above *c*.

- a (apex, top)
- b (lower left, end of the base)
- c (lower right, end of the base)
- h (right, on the base produced beyond *c*)
- d (on the side *a b*, above *b*): upper end of the portion *b d*
- e (on the side *a c*, above *c*): upper end of the portion *c e*

The difference between the two straight lines *b c* and *d e* will therefore be imperceptible if it is compared with either of them, and so those same straight lines *b c* and *d e* will appear equal; and since they are equidistant, you will conclude that the two lines *b d* and *c e* appear equidistant, just as in the first figure. Yet it is certain that if *b c* is produced to *h*, and the astrolabe's centre is placed first at *b* and then at *c*, the exterior angle *a c h* will be found far greater than the interior angle *a b c* itself — for it is double it. It is therefore not because the solar rays appear equidistant that they are seen to make equal angles at the astrolabe's centre, the exterior equal to the interior with the horizon line; nor conversely, because such angles are found equal by the astrolabe, will the solar rays themselves appear parallel.

Rather, the angles appear equal on astrolabes or on sundial instruments [*sciotherica instrumenta*], even though they are unequal, because the angle which those same rays make either at the sun or near the sun — the amount by which the exterior exceeds the interior — is imperceptible on account of its smallness.

Now although Eratosthenes assumed the sun's rays to be equidistant, and therefore concluded that the alternate angles at the gnomon's top and at the earth's centre are equal — in that observation which he made at Alexandria to find how great the whole circuit of the earthly globe is along a great circle — our demonstration is nonetheless true. From it we gather that in Eratosthenes' own observation the sun's rays coincide below the earth's centre, and that the angle made at the gnomon's top is smaller by about a quarter of one degree than the alternate angle at the earth's centre. Accordingly the arc of that angle which stands at the earth's centre will be 7 degrees with 27 minutes. And so, if there are 5,000 stadia between Syene and Alexandria, then in the whole circuit of the earthly globe there will be only 241,610 stadia, not 250,000.

For let it be noon exactly for those who are at Syene and at Alexandria — these two places lie under one and the same meridian. Let the common sections of that meridian with the sun's body, and likewise with the earth's, be the circles *a b c* and *d e f*; let the sun's centre be *g*, the earth's *h*, and let *g h* be joined. At Syene let the gnomon be *d i*, standing perpendicular to the horizon, and the zenith point *a*. Let Alexandria be where *K* is, and let a straight line be drawn through *h* and *K* as far as the meridian, where the point *l* is, which stands above the zenith there; and let the gnomon standing perpendicular to the horizon be *K m*. From the magnitude of the angle *d h K*, then, the ratio of the similar arcs *d K* and *a l* to their own circles is concluded; and the magnitude of that angle is ascertained from two solar rays. One of these, *g i*, sent from the sun's centre, makes one straight line with the gnomon *d i* and with the earth's semidiameter *d h*, for it falls perpendicularly; and

Diagram. A full-page figure for the Eratosthenes demonstration, drawn in three parts. Across the upper part of the page runs a wide, shallow arc — the meridian — rising from the lower left to its summit at *a*, where a vertical straight line meets it, and then descending to the right. The point *l* stands on this arc at the upper left, and from *l* a long straight line is drawn down and to the right toward the earth's globe below. Beneath *a*, the sun's disc is drawn as a circle whose centre *g* lies on the vertical line; a radius is drawn from *t* on its left rim to *g*, and *o* stands on its lower-left rim, *p* on its lower-right rim, with a further letter on its right rim read as *c* [?]. Three long lines descend from the sun's circle — from *o*, from *g*, and from *p* — narrowing toward the earth. At the lower part of the page the earth is drawn as a small circle with its centre at *h*: *e* stands at its left and *f* at its right, and along its upper left and top run *m*, with *n* just below it, then *K*, then two further letters flanking *d*, with *i* set above and to the right of *d*. Below the earth's globe two of the descending lines cross and meet at a point.

- a (summit of the great arc, on the vertical line): the zenith point of Syene
- l (on the great arc, upper left): the point of the meridian above the zenith of Alexandria, reached by the line drawn from *h* through *K*
- g (centre of the sun's circle)
- t (left rim of the sun's circle, joined to *g* by a radius)
- o (lower-left rim of the sun's circle), p (lower-right rim of the sun's circle): the two extreme descending rays
- c [?] (right rim of the sun's circle)
- h (centre of the earth's circle)
- d (on the earth's circumference, right of centre-top): Syene, foot of the gnomon *d i*
- i (above and right of *d*): top of the gnomon at Syene
- K (on the earth's circumference, left of *d*): Alexandria, foot of the gnomon *K m*
- m (outside the circumference, above and left of *K*): top of the gnomon at Alexandria
- n (just below *m*)
- e (left of the earth's circle), f (right of the earth's circle)
- [?] a further letter between *K* and *d*, read as *s* [?]
- [?] a further letter immediately to the right of *d*, not resolvable on the scan

and therefore that same gnomon admits no shadow at midday. The other solar ray is the one sent to Alexandria: it passes through the point *m*, which is the tip of the gnomon, and marks off the shadow *k n* in the hollow of the hemicycle; it touches the sun's body at the point *b*, while it meets the straight line *g h* at the point *r*. They must meet, because the angle *b g h* at the sun's centre is acute.

Further, in the triangle *m g h* the interior angle *g h m* is reckoned equal to the exterior angle *g m l*, because the angle *h g m* — the sun's parallax [*diuersitas aspectus*], which is the difference between those two angles — is of imperceptible size in that position. But this same exterior angle *g m l* exceeds the angle *b m l* by the angle *g m b*; the angle *g h m*, therefore, will be greater than that same *b m l* by the difference *g m b* itself. Now the angle *K m n* is equal to *b m l*, being vertically opposite to it; so the angle *g h m* will exceed the angle *K m n* by that same difference, which is the angle *g m b*.

Eratosthenes found the angle *K m n* to subtend a fiftieth part of its circle, that is 7 degrees 12 minutes; whereas the angle *g m b* is about a quarter of one degree, by what Ptolemy demonstrated in Book V of the *Almagest*. You can also conclude this

at once in the right-angled triangle *b g m*, from the known ratio of *b g* to *g m*, namely as 5½ to 1210 [?]. And therefore this same angle *g h m* exceeds the angle *k m n* by that quarter of a degree; the angle *g h m* will consequently contain seven degrees together with 27 minutes, and there will be just as many in the arc *d K*, or in *a l*. And since, as Eratosthenes says, the distance *d K* is five thousand stadia, the whole terrestrial circuit will therefore be 241,610 stadia — which was to be shown.

But suppose Eratosthenes had explored the distance of the zenith from the sun not from the shadow of a gnomon but from the sun's ray entering the pinholes of the sighting-vane [*dioptra*] of an astrolabe or a quadrant. I grant that he would have found such a distance greater than the fiftieth part of his circle; even so, he would have demanded in vain, for his demonstration, that the rays sent out from different parts of the sun be parallel. For the rays by which we take the sun's altitude on astrolabes come from the sun's centre, not from different parts of it.

Now let two straight lines be drawn from the earth's centre, touching the sun at the points *o* and *p* and cutting the earth at *s* and *t*. The angle *p h o* will then contain about half of one degree, the apparent diameter of the sun; and so, over the whole stretch of earth *s t*, gnomons will be seen at midday without shadows. That, I believe, is why Eratosthenes said — as Cleomedes reports — that when the sun stands directly overhead at Syene, gnomons are free of shadow for three hundred stadia.

From these things it is also plain that the sun's altitude taken by an astrolabe is smaller than the altitude inferred from the ratio of the shadow to its gnomon, and that the difference between the two altitudes is exactly what remains when the sun's parallax is subtracted from its apparent semidiameter. I am genuinely surprised that Ptolemy gave us no warning of this, when in Book II of the *Almagest* [*magna compositio astrorum*] he taught how to find the sun's altitude from the ratio of the shadow to the gnomon.

Now, having treated of rays, I shall show that the shadows of gnomons cast on the surface of the terrestrial globe are not parallel but only appear so. Take two shadows of two gnomons set perpendicular. If the solar rays that bound those shadows are of the first kind — that is, if the zenith points of the gnomons lie in one plane of some great circle passing through the sun's centre — then the shadows of those gnomons will neither meet nor be parallel; rather, when produced lengthwise they will form only one line, and a circular one, not two. But if the solar rays bounding the given shadows are of the second kind, I shall show that they do meet, yet appear parallel.

Let there be, set perpendicular on the surface of the terrestrial globe, two equal gnomons *a b* and *c d*, whose zenith points are separated from the sun by equal intervals. Let the solar rays that bound the shadows be *a e* and *c f*, and let the shadows cast on the surface of the terrestrial globe be *b e* and *d f*. I say that these shadows *b e* and *d f*, produced further in both directions, do meet, but nevertheless appear parallel.

For since the rays *a e* and *c f* are of the second kind, they will lie in the planes of the great circles that pass through the zenith points of the gnomons, through the centre of the sun's body, and through the centre of the earth. The shadows *b e* and *d f* will therefore lie in the common sections of those same planes with the globe of the earth; and so the shadows *b e* and *d f* will be arcs of great circles of the terrestrial globe, by the first and the sixth propositions of Theodosius, *Spherics* I. And therefore, if the same shadows

Diagram. A plane quadrilateral drawn in perspective, in the right column. The quadrilateral is *e b d f*: *e* at the left, *b* at the right, *f* at the lower left and *d* at the lower right, with *e b*, *b d*, *d f* and *f e* joined. Above the figure stands *a*, joined by a short line to *b* and by a long line to *e*; within the quadrilateral stands *c*, joined by a short line to *d* and by a long line to *f*. The lines *a c*, *e d* and *f b* also run across the figure.

- a (apex, above the right of the figure): tip of the first gnomon *a b*
- b (upper right): foot of the first gnomon
- c (interior, centre): tip of the second gnomon *c d*
- d (lower right): foot of the second gnomon
- e (left): end of the shadow *b e*, bounded by the ray *a e*
- f (lower left): end of the shadow *d f*, bounded by the ray *c f*

b e and *d f* are produced continuously, they will meet in both directions — which was the first thing to be shown.

As to their appearing parallel, we shall demonstrate it in this way. The shadows of the gnomons and the intervals between them, compared with the extent of the terrestrial globe's surface, appear as straight lines lying in a plane surface. Let *e b* and *d f*, then, be taken for straight lines, and let *b d*, *e f* and *a c* be joined. Now the gnomons *a b* and *c d* are equal by hypothesis, and when produced they meet at the earth's centre. Therefore, by the equality of the angles and the similarity of the triangles, which have a common angle at that same centre of the earth, the straight line *a c*, the base of the one, will exceed *b d*, the base of the other, imperceptibly — just as we demonstrated above for the sun's rays.

Again, in the two triangles *a e b* and *c f d* the two angles *a b e* and *c d f* are equal to one another; and the two angles *b a e* and *d c f*, opposite to them, are equal as well. Therefore the remaining sides of those triangles, each to each, together with the remaining angles, will be equal, by proposition 26 of Euclid, Book I. Consequently the rays *a e* and *c f* are equal; and if they are produced further, they will meet below the earth's centre at the vertex of a certain cone, as was shown above. Therefore, by the equality of the angles and the similarity of the triangles that have a common angle at that same point, the vertex of the cone, the straight line *a c*, the base of the one, will imperceptibly exceed the straight line *e f*, the base of the other. So *b d* and *e f* will appear equal, by the common notion: for they are exceeded by the straight line *a c* by an imperceptible difference.

Now let *d e* be joined, and by proposition 8 and proposition 27 of Euclid, Book I, show that *e b* and *f d* appear parallel. In my own judgement, however, you will grasp it better from this: that the equal straight lines *e b* and *f d* are seen to be separated by equal intervals, namely *e f* and *b d*, just as we concluded above about the sun's rays.

Diagram. The same figure repeated at the foot of the left column: the quadrilateral *e b d f* drawn in perspective, with *e* at the left, *b* at the right, *f* at the lower left and *d* at the lower right; *a* stands above, joined to *b* and to *e*, and *c* stands within the figure, joined to *d* and to *f*. The lines *a c*, *e d* and *f b* cross the figure.

- a (apex, above the right of the figure): tip of the gnomon *a b*
- b (upper right): foot of the gnomon *a b*
- c (interior, centre): tip of the gnomon *c d*
- d (lower right): foot of the gnomon *c d*
- e (left): end of the shadow *b e*
- f (lower left): end of the shadow *d f*

And not only will the shadows of gnomons extended on the curved surface of the terrestrial globe look parallel; so will those cast on a single plane surface, provided only that the gnomons themselves depart little from the perpendicular. For since, when produced in a straight line, they coincide at the earth's centre, both of them cannot stand at right angles to one and the same plane.

Let the preceding figure, then, be repeated, and let us set the gnomon *a b* at right angles on a surface parallel to the horizon of the place *b*, while the gnomon *c d* rests on that same plane but departs from the perpendicular by an imperceptible

difference; and let the tips of those same gnomons stand off from the sun by equal intervals in the universe.

The straight line *c d*, extended as far as the earth's centre, will therefore be greater than *a b* carried to the same centre, though by an imperceptible difference; but the straight lines *a b* and *c d* have been laid down as equal. The two straight lines *a c* and *b d* will therefore be taken as parallel, by proposition 2 of Euclid, Book VI. And now, reasoning in the similar triangles just as we reasoned about the solar rays, we shall conclude that *a c* exceeds *b d* by an imperceptible difference.

Further, in the two triangles *a b e* and *c d f*: the angles at the points *b* and *d* are assumed equal, since the gnomon *c d* departs imperceptibly from the perpendicular; likewise the two angles *b a e* and *d c f*, opposite to them, which subtend equal distances between the tips and the sun, are equal; and the gnomons *a b* and *c d* themselves are laid down as equal. Therefore the remaining sides of those triangles will be equal to the remaining sides, each to each, by proposition 26 of Euclid, Book I; and so the two rays *a e* and *c f* will be equal.

But it was shown above that these meet below the earth's centre, at the vertex of a certain cone having its base on the surface of the sun's body. Therefore, because of the immense length of the intervals, *a e* and *c f*, when compared with those intervals, will be reckoned of imperceptible size; and so, in the similar triangles whose bases are *a c* and *e f*, we shall conclude, as before, that *a c* exceeds the straight line *e f* by an imperceptible difference. It was shown, moreover, that *a c* likewise exceeds the straight line *b d* imperceptibly. The two lines *b d* and *e f* will therefore be taken as equal; and consequently the two lines *b e* and *d f*, since they are separated by equal intervals, will appear parallel.

Or, if you prefer to infer this from the geometrical elements of Euclid, Book I: let *b f*, or else *e d*, be joined; and since *b e* and *d f* have been shown equal, then by proposition 8 and by proposition 27 of that same first book you will conclude that the two straight lines *b e* and *d f* appear parallel.

But they must meet on the side of *b d* if they are produced in a straight line — which will not be difficult to demonstrate. For if the ray *a e* is understood to be produced as far as the vertex of the cone conceived, it will have a greater ratio to *a e* than the straight line *a b*, extended as far as the earth's centre, has to *a b* itself. And so, in those two triangles whose common base is *a c*, while the opposite angle lies in the one at the vertex of the cone and in the other at the earth's centre, *a c* will have a greater ratio to the excess by which it exceeds *e f* than to the excess by

which it exceeds *b d*, by conversion of the ratio and by proposition 13 of Book V. Consequently this same *a c* will exceed the straight line *e f* by a smaller difference than it exceeds the straight line *b d*, and therefore *e f* will be greater than *b d*. From this you will at once conclude that *b e* and *d f* meet on the side of *b d*.

For let *d e* be joined. In the two triangles *e f d* and *e b d*, the two sides *b e* and *d f* have been shown equal; the side *d e* is common to both triangles; but the base *e f* of the triangle *e f d* is greater than the base *b d* of the triangle *e b d*. Therefore the angle *e d f* of the triangle *e f d* will be greater than the angle *b e d* of the triangle *e b d*, by proposition 25 of Euclid, Book I.

At the point *e*, then, the end of the straight line *e d*, we shall make with *e d* itself an angle *d e g* equal to the angle *e d f*, by proposition 23 of that same first book of Euclid. And therefore the two straight lines *d f* and *e g* will be parallel, by proposition 27 of the same first book of Euclid; and consequently the two angles *e f d* and *f e g* will be equal to two right angles, by proposition 29. But the angle *f e b* is less than the angle *f e g*; the two angles *e f d* and *f e b* will therefore be less than two right angles, and so the two straight lines *f d* and *e b* will meet on the side of *b d*, by the fifth postulate — which is what we undertook to demonstrate.

Diagram. A rectangle in the left column, drawn with *e* [?] at the upper left, *b* at the upper right, *f* at the lower left and *d* at the lower right. A dashed line runs from the upper-left corner along the top to the point *g*, which lies on the upper side between the left corner and *b*. A solid diagonal runs from the upper-left corner down to *d*.

- e [?] (upper left corner): end of the shadow *b e*
- g (on the upper side, between *e* and *b*): the point taken so that the angle *d e g* equals the angle *e d f*
- b (upper right corner): foot of the gnomon *a b*
- f (lower left corner): end of the shadow *d f*
- d (lower right corner): foot of the gnomon *c d*

Now, granting Eratosthenes' account of the circuit of the terrestrial globe and Archimedes' demonstration on the measurement of the circle: when the gnomon *c d* has departed from the perpendicular by a tenth part of one degree, that is by 6 minutes, the interval *b d* between the two shadows *b e* and *d f* will contain nearly nine thousand paces; but when it has declined by one minute only, that interval will be about 1500 paces. For the angle which the two gnomons *a b* and *c d* make where they coincide at the earth's centre is equal to the declination of the gnomon *c d* itself, and it subtends the distance between the shadows.

Lemma.

In order to show that the shadows *b e* and *d f* meet on the side of *b d*, we assumed that the straight line *a e*, extended as far as the vertex of the cone conceived, has a greater ratio to the ray *a e* than the straight line *a b*, carried as far as the earth's centre, has to the gnomon *a b*. We shall now show this in the figure below.

Let *K* be the centre of the earth and *l* the centre of the sun; let *a b* be the gnomon; and let *K l* be joined and produced in a straight line toward [?]. Let the ray *a e*, produced in both directions, touch the sun at *m* and coincide with the straight line *K l* below the earth's centre at *n*; let it bound the shadow *b e* on a plane surface parallel to the horizon of the place *b*; and let the length of the gnomon *a b* be understood to be produced as far as *K*.

I say that *a n* has a greater ratio to *a e* than *a K* has to *a b*. For since the angle *b k l* subtends an arc that is the complement of the sun's altitude above the horizon, it is acute; and therefore the remaining angle *b k n* will be obtuse.

From the point *K*, then, let the straight line *k i* be erected at right angles upon *a K*, in the plane of the triangle *a k n*. Therefore, by the equality of the angles and the similarity of the triangles *a K i* and *a b e*, just as

Diagram. A tall figure filling the right column. At the top is the disc of the sun, a circle with a straight line drawn across it; *m* stands at its left-hand rim and *l* at its centre. From the sun two long straight lines descend, converging as they fall, and meet at a heavy point at the very foot of the figure. Lower down, and much smaller, is the circle of the earth, its centre marked *K*; a short line runs from *K* to *i*, set to the left of the centre inside the earth's circle, and near the upper left of the earth's circle stand *a* [?] and *e*. The vertical line drawn through the sun's disc continues down through the earth's centre to the point of convergence below.

- m (left rim of the sun's disc): the point at which the ray *a e* touches the sun
- l (centre of the sun's disc): the sun's centre
- K (centre of the small circle below): the earth's centre
- i (inside the earth's circle, left of *K*): the end of the perpendicular *k i*
- a [?] (upper left of the earth's circle): the tip of the gnomon *a b*
- e (left of the earth's circle): the end of the shadow *b e*
- [?] (the point of convergence at the foot of the figure): not legibly lettered on the leaf; the text names it *n*

a K is to *a b*, so *a i* will be to *a e*. But *a n* is greater than *a i*, and therefore *a n* has a greater ratio to *a e* than *a i* has to that same *a e*; and so *a n* has a greater ratio to *a e* than *a K* has to *a b*, by the 12th proposition of the fifth book of Euclid in Campanus's version — which is what we had assumed.

In this figure too it can be shown by another route that the straight line *b d* is less than the straight line *e f*. For as *a b* is to *b K*, so is *a e* to *e i*; but *a e* has a greater ratio to *e i* than it has to *e n*. Therefore *a b* has a greater ratio to *b K* than *a e* has to *e n*, and consequently the whole *a K* will have a greater ratio to *b K* than *a n* has to *e n*. Now as *a K* is to *b K*, so, in the similar triangles, is *a c* to *b d*; and as *a n* is to *e n*, so, in the other similar triangles, is *a c* to *e f*. Therefore *a c* will have a greater ratio to *b d* than it has to *e f*, and consequently *b d* will be less than *e f*.

By the same method we shall show that the remaining shadows — those marked off by the other rays, which are neither of the first kind nor of the second — are likewise not parallel, but only appear so. At the places *a* and *b*, then, let the gnomons *a c* and *b d* cast the shadows *a e* and *b f*; and let the rays *c e* and *d f* which mark those shadows off be neither of the first kind nor of the second. That is, let them not lie in the plane of a single great circle passing through the zeniths of those same places, the centre of the sun's body, and the centre of the earthly globe, nor mark equal distances from the zeniths; but let the angle *a c e*, which the ray *c e* makes with the gnomon *a c*, be less than the angle *b d f*, which the ray *d f* makes with the gnomon *b d*. I say that the shadows *a e* and *b f* are not parallel, but merely appear to be.

For it was shown above that *a e* is a segment of a great circle of the earth. Let it therefore be extended toward the side opposite the sun, and on that same great circle let a place *g* be conceived, from whose zenith the sun is as far distant as it is removed from the zenith of the place *b*. Let the gnomon *g h* at that place *g* cast the shadow *g K* at the same instant; the sun's ray marking off that shadow will be *h K*. But from what we have shown above it is established that the rays *d f* and *h K* are of the second kind. The two shadows *b f* and *g K*, therefore, will appear parallel, and yet if they are produced continuously they will meet. Now the shadow *a e* lies on the same great circle as *g K*; it will therefore meet *b f*, and will appear parallel to it — which was to be shown.

Note, however, that what we have shown of the shadows of equal gnomons can also be demonstrated of the shadows of unequal gnomons, since the shadows of a larger gnomon and of a smaller are extended along one and the same line.

Diagram. Three right triangles drawn in outline, uncaptioned. The two upper ones stand on a single continuous horizontal base line; the third is set below and to the right.

- *h* (apex of the left-hand triangle): the top of the gnomon at the place *g*
- *K* (left end of the base line): the end of the shadow *g K*
- *g* (foot of the perpendicular of the left-hand triangle, on the base line): the place itself
- *c* (apex of the second triangle): the top of the gnomon at the place *a*
- *e* (on the base line, left of the second triangle's perpendicular): the end of the shadow *a e*
- *a* (foot of the second triangle's perpendicular, right end of the base line): the place itself
- *d* (apex of the lower triangle): the top of the gnomon at the place *b*
- *f* (left end of the lower triangle's base): the end of the shadow *b f*
- *b* (foot of the lower triangle's perpendicular): the place itself

Each triangle has the gnomon as its vertical side, the shadow as its horizontal side, and the sun's ray as its hypotenuse.

Instrumentum fabricare, quo absque numerorum tabulis cordas atque sinus datorum arcuum, necnon & rationem aequinoctialis ad quemuis aequidistantium inuenire possis, & quaedam alia. Cap. 19.

To construct an instrument by which, without tables of numbers, you may find the chords and the right sines of given arcs, and also the ratio of the equinoctial to any of the circles parallel to it, together with certain other things. Cap. 19.

On any flat board let a semicircle *a b c* be described and divided into ninety equal parts. On the point *c*, the end of the diameter *a c*, let a ruler be pivoted, equal in length to that diameter *a c*, and let the edge of it which is directed toward the point *c* be divided into 60 equal parts.

So when you wish to find the right sine of a given arc, we shall count off on the semicircle the number of degrees the arc contains, beginning at the point *c* and running toward *a*. Suppose, for example, that the end falls at *b*. We shall accordingly bring the ruler across to *b*, into the position *c f*; for however many sixtieths there are in *c b*, so many will the right sine of the given arc have.

The demonstration of this is easy. About the point *a*, with radius *a c*, conceive a circle described, and let it be *c f g*; and from that centre *a* let a straight line be drawn through *b* to meet the circumference of the circle so conceived at the point *h*. Since the angle *a b c* stands in a semicircle it is right, and therefore the straight line *b c* will be the right sine of the arc *c h*. Now as the right angle *a b c* is to the acute angle *b a c*, so is the semicircle *a b c* to the arc *b c*. Likewise, as a right angle set up at the centre *a* is to that same acute angle *b a c*, so is the quadrant of the circle *c f g* to the arc *c h*; and all right angles are equal to one another. Therefore, as the semicircle *a b c* is to the arc *b c*, so is the quadrant of the circle *c f g* to the arc *c h*. And accordingly, as many ninetieths of the semicircle *a b c* as there are in the arc *b c*, so many ninetieths of the quadrant of the circle *c f g* will there be in the arc *c h*.

Diagram. A large circle with its centre marked, drawn in outline at the head of the left column. From the centre a straight line runs to the right-hand point of the circumference; on that line as diameter a smaller semicircle is drawn, bulging upward. A second straight line runs from the centre up to the right, crossing the semicircle and meeting the large circumference; and a further straight line descends from the top of the large circle to the right-hand point, cutting the semicircle at the same crossing.

- *a* (centre of the large circle): the centre, and one end of the diameter of the semicircle
- *f* (top of the large circumference): the point marking off the quadrant from the right-hand end of the diameter
- *b* (on the semicircle, upper right): the point to which the ruler is brought
- *h* (on the large circumference, just beyond *b*): where the line drawn from *a* through *b* meets the circle *c f g*
- *g* (lower left, within the large circle): the third letter naming the circle *c f g*
- [?] (right-hand end of the diameter, where the lines converge): the point the text calls *c*; the letter itself does not read on the leaf

So many did we count off on the semicircle as there were in the proposed arc; therefore the right sine of the given arc has been found by that method — which was to be shown.

[Margin: a manuscript annotation in an early hand, mostly illegible [?]]

You may also conclude that the two arcs *c b* and *c h* are equal. For as circle is to circle, so is diameter to diameter; therefore the quadrant of the circle *c f g* and the semicircle *a b c* will be equal. But it has been shown that the semicircle *a b c* is to

the arc *b c*, and the quadrant of the circle *c f g* to the arc *c h*, in the same ratio. Therefore, as the semicircle is to the quadrant, so is *b c* to *c h*, by alternation; and consequently the arcs *c b* and *c h* are equal, which is what we wished to show.

Note, however, that by this method the right sine is found of a given arc smaller than one quadrant. If the proposed arc is greater than a quadrant, it must be subtracted from 180, and we shall find the right sine of the remainder; for one and the same straight line is the right sine both of the arc taken away and of the arc left.

And since the semidiameter of any circle parallel to the equinoctial is the sine of that circle's distance from the nearer pole of the world: whenever it is worth finding the ratio of the equinoctial circle to any of the circles parallel to it, we shall find the right sine of the arc by which the given circle parallel to the equinoctial is distant from the nearer pole. For as the number of parts found in the sine so obtained is to 60, so will the given parallel be to the equinoctial.

Moreover, since any two circles and their similar parts stand in the same ratio, it will not be difficult to discover how the degrees of circles parallel to the equinoctial are to be converted into degrees of a great circle. For we shall lay down that the diameter *a c* is one degree of the equinoctial, and accordingly each sixtieth of that diameter will be one minute. Therefore, however many sixtieths are found in the right sine of the given parallel's distance from the nearer pole — that is, however many sixtieths the semidiameter of the given parallel has — so many minutes of one degree of the equinoctial will one degree of the given parallel contain. Then, by multiplying those same minutes by the number of degrees contained in the proposed arc of the given parallel, we shall collect the sum of the degrees and minutes of a great circle contained in that arc of the given parallel.

By exactly the same method one can also find how many Italian miles on the earth's surface answer to one degree of a given circle parallel to the equinoctial. For we shall lay down (as above) that the diameter *a c* is one of the degrees of a great circle, and accordingly one sixtieth of it will be one Italian mile, so that there are 60 miles in one degree — for that is the figure we see accepted. Therefore, however many sixtieths are found in the semidiameter of the given parallel, so many Italian miles will one degree of that same parallel contain. But if you prefer to use measures other than miles, the diameter *a c*, or the length of the ruler, must be divided into the number of parts that answers to one degree of a great circle in the given measure; and then we shall work as before.

Now if the right sine is given as known, but the arc it answers to is unknown, we shall count off the number of sixtieths of the given right sine along the ruler of the instrument, taking the beginning from the point *c* itself and marking the end with some mark; and then we shall carry the ruler round until the mark set upon it comes to the circumference of the semicircle. For the arc between that mark and the point *c* will show us how many degrees the arc sought contains.

Further, if the arc is given as known but the versed sine is unknown: if the arc is less than a quadrant, you will find the right sine of its complement, and this you will subtract from 60, and the versed sine of the given arc will be left known. But if the arc is greater than a quadrant, you will find the right sine of the arc by which it exceeds the quadrant, and you will add this to the 60 parts, and the number of parts of the versed sine answering to the proposed arc will be made up.

So too, if the versed sine is given as known but the arc it answers to is unknown, you will subtract that versed sine from 60, provided the number of sixtieths in it is less than 60; for the right sine will be left which answers to the complement of the arc sought.

When, therefore, the arc of that right sine has been found in the manner set out above, we shall subtract it from 90 degrees, and the arc answering to the given versed sine will be left known. But if the given versed sine is greater than 60, let 60 be subtracted from it, and there will be left the right sine of a certain arc by which the arc sought exceeds the quadrant. Let the arc answering to that right sine be found, then, and added to the quadrant, and the arc that was sought will be made up.

If, on the other hand, the arc is known but the chord unknown, we shall look for the right sine of half the proposed arc; and when this is doubled, the chord of the proposed arc will be evident.

But if the chord is known and the arc unknown, we shall find the arc to which half the given chord answers as its right sine. When this arc is doubled, the arc that was sought becomes known. One and the same chord, however, answers to two circumferences, of which one is less than a semicircle and the other greater, the two together completing the circle.

The length of the ruler — that is, the semidiameter of the great circle — we have divided in this instrument into 60 equal parts, following Ptolemy's practice. But since more recent mathematicians solve a good many problems far more easily than Ptolemy does, namely by mere multiplication and division of four

proportional quantities, one of which is always the whole sine, you will carry out those multiplications and divisions more quickly if you divide the semidiameter of the circle, or the length of the ruler, into 100 parts or into a thousand.

Datis latitudinibus & longitudinibus, duorum locorum eorum intercapedinem metiri. Cap. 20.

Given the latitudes and longitudes of two places, to measure the distance between them. Cap. 20.

This can be found in two ways, either by numbers or by an instrument. By numbers, in the following way. The two given places either lie on one meridian, or on one parallel, or on different meridians and parallels. If they lie on one meridian, and both are northern or both southern, then when the lesser latitude has been subtracted from the greater, the arc of the meridian that remains will be the travelling distance [*distantia viatoria*] between the two given places. But if they lie on the same meridian while one is southern and the other northern, we shall add the two latitudes together into one sum, and the travelling distance will come out known.

If, however, they lie on one parallel but differ in meridian, let the chord of the difference of longitude between those places be multiplied by the right sine of the complement of the altitude of the pole, and let the product be divided by 60; and there will come out in the quotient the number of parts contained by the chord of the arc of the great circle passing through the two given places. For we take the semidiameter of a great circle to be of 60 equal parts. And since the arc cannot be unknown once the chord is known, the arc of the great circle described through those places will accordingly be known.

The demonstration of this is easy. For as the semidiameter of the equinoctial is to the semidiameter of the proposed parallel, so is the straight line subtending the arc of the difference of longitude on the equinoctial to the straight line subtending the arc of the difference of longitude on that same parallel — as you will conclude by the 14th of the second book of Theodosius and the fourth of the sixth of Euclid. But the right sine of the complement of the altitude of the pole — that is, of the complement of the declination of the given parallel — is the semidiameter of that same parallel. Therefore, if you multiply the second of these four proportional quantities into the third and divide the product by the first, the fourth will come out known at once.

And since one and the same straight line which subtends the arc of the difference of longitude on the given parallel also subtends an arc of the great circle passing through those same places, it follows that once that line is known in parts of the semidiameter of a great circle, the arc answering to it cannot be unknown, and so the travelling distance between those places will be revealed.

Petrus Appianus [Peter Apian] and Ioannes Stoflerus [Johannes Stöffler] and certain others suppose they have settled this when they have converted the degrees of the difference of longitude lying on the parallel between the given places into degrees of a great circle, and then those degrees of the great circle into miles, or stadia, or any other measures whatever. But they fail to notice that in this way they do not find the travelling distance — which ought to be a segment of a great circle — but only how many miles or stadia the arc of the parallel between those same places contains.

When, however, the two given places have different meridians and different parallels, the present problem is solved with greater labour. For some measure the distance between the given places in a spherical right-angled triangle exactly as they would in a rectilinear one — that is, by taking the roots

— of the squares of the two sides that contain the right angle. Others investigate the same thing by the same method, but more exactly: they first convert the one side of the triangle that is a segment of a parallel into parts of a great circle. We can use both of these procedures over short distances without any perceptible error, but over long ones another technique will be required. That is why Johannes Werner and Ioannes de Monteregio [Regiomontanus], in order to find the distances between places in thoroughly reliable numbers, treated the matter in a very different way.

Werner's method is this. Let two given places lie on the different meridians a b c and a d c. Let the vertex of the place farther from the equinoctial be b, and the vertex of the place that departs less from the equinoctial be d. Let the segment of the parallel of place b lying between those meridians be b e, and the segment of the parallel of place d lying between the same meridians be d f. Let the arc of the great circle between b and d — the size of which we wish to know — be b g d, and let b d be its chord [recta subtensa]. The straight lines b e and d f, then, subtend the segments of the given parallels; while the two straight lines b f and e d subtend two equal arcs of the meridians between the same parallels.

Now since the straight lines b f and e d are equal and subtend known arcs, they will become known from the table of arcs and chords. The parallels themselves

are also known, and so are their segments enclosed between the meridians. Accordingly the two straight lines b e and f d will be known — in the units in which the diameter of the equinoctial or of a meridian is 120 — by the technique set out a little earlier.

Next, from the points b and e let b h and e i be drawn perpendicular to the straight line f d. Then in the parallelogram b e i h the straight line b e will be equal to the straight line i h, and the straight line b h equal to the straight line e i, by Euclid I.34. Accordingly, in the two right-angled triangles b f h and e i d the two sides f h and i d will be equal, by proposition 47 of that same first book, together with the common notion that if equals are taken from equals the remainders are equal.

Diagram. A sphere drawn as an ovoid outline with the pole a at the top and the pole c at the bottom, joined by the two meridian arcs a b c and a d c. Across the upper part run the two chords of the parallel segments and the meridian chords, forming the quadrilateral b — e — d — f with the diagonal b d passing through g; the perpendicular b h drops from b onto the line f d.

- a (top of the figure): the pole a
- b (upper left, on the left-hand meridian arc): vertex of the place farther from the equinoctial
- e [?] (upper right, on the right-hand meridian arc): the end of the parallel segment b e
- g (centre, on the diagonal): the point marking the great-circle arc b g d
- f (lower left): the end of the parallel segment d f on the left-hand meridian
- h (on the line f d, right of f): foot of the perpendicular from b
- d (lower right): vertex of the place nearer the equinoctial
- c (bottom of the figure): the pole opposite a

Therefore each of the lines f h and i d will be half the difference of the two straight lines d f and b e. But d f and b e are known; so the half-difference will be known, and when it is subtracted from the straight line f d, the straight line d h is left known.

Again, in the right-angled triangle b f h, when the square on f h — now known — is taken away from the square on b f, the square on b h is left known. In the same way the square on d h stands known. Therefore in the right-angled triangle b d h the square on the side b d, which subtends the right angle, will be known, since by Euclid I.47 it equals those same two squares; and so the side b d itself cannot fail to be known. Hence by Ptolemy's table of arcs and chords the great-circle

segment b g d contained between the given places will be disclosed — which was to be shown.

In this demonstration, however, the chief point — the one that most of all needed showing, and without which the rest cannot stand — has been passed over by Werner. It was well worth demonstrating that the two straight lines b e and f d are parallel; you would conclude that at once from proposition 16 of Euclid's eleventh book, provided it had first been shown that these same straight lines b e and f d lie in one plane. But that is not evident.

To demonstrate this very thing, then, let us set k as the centre of the sphere, and let the common section of the meridians be the straight line a c, the axis of the world. In the plane of the meridian a b c let the straight line K l make right angles with the axis a c; and likewise let the straight line K m, in the plane of the meridian a d c, make right angles with a c as well. Let the zenith [verticale punctum] b lie toward the side of the pole a, and the zenith d toward the opposite pole, which is c; and let b be farther from the equinoctial point l than d is from m — for that is how we are positing it here.

Further, let the arc l n be equal to f l or to d m, and let f n be joined, cutting the straight line K l at o; let f K and d K be joined as well. Consequently the rectilineal angle n o K will be right, and a K o is right too; therefore the two straight lines f n and a k are parallel. The straight line f K falls across them. Therefore the two angles K f n and a K f will together equal two right angles, by proposition 29 of Euclid's first book. So the two angles b f k and a k f will be less than two right angles, and for that reason the two straight lines b f and a K will meet on the side of a b, by the fifth postulate. It will be shown in the same way that the two straight lines d e and a K meet on the side of a e. Now let b f and a k meet

in the point r. I say that the two straight lines d e and a K meet at that same point r as well. For the two arcs a f and a d are equal to one another, so the two angles a K f and a K d will be equal. And the two acute angles b f K and e d K are equal to each other. For the angle b f K lies in that smaller segment which is left when the circumference b f is taken away from the semicircle; and b f and e d have been shown to be equal; so the two angles b f k and e d k lie in equal segments, and therefore they will be equal, by proposition 27 of Euclid's third book. Since, then, on the equal straight lines f K and d k the two straight lines b f and d e make equal angles at the points f and d, and since the straight line a K makes equal angles with them at the point K, then unless you admit that b f and d e meet at one and the same point — namely r — you will run into an impossibility, by

proposition 26 of Euclid's first book: for the whole and the part would be equal. They therefore run together at that very point r.

Diagram. The fuller version of the same construction, occupying most of the outer column. The sphere is again drawn as an ovoid with the pole a at the top and c at the bottom; the centre k lies at the middle, with the equinoctial radii K l running to the left and K m to the right. The quadrilateral b — e — d — f is drawn as before, with f n cutting K l at o. Two long straight lines — b f produced and the axis a k produced — run upward beyond the top of the sphere and converge on a point marked above the figure.

- r [?] (apex, above the sphere): the point at which b f and a k meet
- a (top of the sphere): the pole a
- b (upper left, on the left-hand meridian): vertex of the more distant place
- e (upper right, on the right-hand meridian)
- l (left, on the equinoctial line): the equinoctial point in the plane of meridian a b c
- o (just right of l): where f n cuts K l
- k (centre): the centre of the sphere
- m (right, on the equinoctial line): the equinoctial point in the plane of meridian a d c
- f (lower left)
- d (lower right)
- c (bottom of the sphere): the pole opposite a

But when two straight lines cut one another, they lie in one plane; and every triangle lies in one plane, by proposition 2 of Euclid's eleventh book. Therefore the straight line d f is the base of the triangle f d r, and the straight line b e lies in the same plane — which was to be demonstrated. At the same time you will conclude by Euclid VI.2 that b e and f d are parallel. We shall show the same thing, by an entirely similar argument, if both places lie toward the same pole, whether the northern or the southern; for the lines will meet in one and the same point.

Werner's method of finding the interval between two places is therefore more reliable than the others, but the work is extremely long-winded, since it involves a good many multiplications, divisions and subtractions — although a square root is extracted only once. If, however, you consult the second book of Euclid's *Elements*, you will settle this same problem by a far shorter calculation.

For once you have converted the straight lines b e and f d into parts of the diameter of the great circle, multiply the one into the other, and to the product add the square of the straight line b f (or e d); the square root of the sum will be the straight line b d itself. The arc b g d will then be known from the table of arcs and chords.

The demonstration is very easy. The difference of the two straight lines f d and b e is divided into two equal lines, f h and d i, and you are to understand h i as the added segment. Hence what is produced by multiplying the whole d f into the added segment, together with the square on f h (or on d i), will equal the square on d h, by proposition 6 of Euclid's second book. And in the right-angled triangle b h d the square on the straight line b d equals the squares that arise from d h and b h, by proposition 47 of the first book. Therefore the square on b d will equal the rectangle contained by d f and h i, together with the squares on f h and b h. But the square on b f equals those two squares on f h and b h. So the square on b d equals the rectangle contained by d f and h i — that is, by b e — together with the square on b f. Accordingly you will multiply b f into itself, and to the product add what is produced by d f into e b: the square root of the sum will be the straight line b d.

One thing is left to investigate: in what way, that is, the travelling distance [distantia viatoria] is to be found

when the given places have different meridians and opposite parallels — which is the easiest case of all. For we shall convert into parts of the diameter of the great circle the straight line subtending the arc of the parallel that lies between the meridians of the given places, and multiply it into itself; and to the product we shall add the square of the straight line subtending the arc of the meridian intercepted between those same parallels. The square root of the sum will be the straight line that subtends the great-circle arc between those two places.

Let the meridian of place o be a o c, and let the meridian of place p, which is set on the opposite parallel, be a p c. Let the segment of the parallel of place o, between those meridians, be o q s, with the chord o s. Let the segment of the parallel of place p, between the same meridians, be p z u, with the chord p u. Let the arc o u, between those same parallels, have the straight line o u as its chord. And let the axis of the sphere be the straight line a c, the common section of the meridians.

Now the axis a c stands at right angles to the planes of all the parallels and passes through their centres, by proposition 12 of the first book of Theodosius. So let the

point t be the centre of the parallel that lies toward the pole a, and the point x the centre of the parallel that lies toward the pole c; and let o t and u x be the common sections of the meridian a o c with those same parallels, as far as the centres t and x. Consequently the straight lines o t and u x will be parallel, by proposition 16 of Euclid's eleventh book.

And since straight lines joining equal and parallel straight lines are themselves equal and parallel, by proposition 33 of Euclid's first book, the two lines o u and t x are equal and parallel. But when one of two parallel straight lines stands at right angles to some plane, the other will stand at right angles to that same plane too, by proposition 8 of Euclid's eleventh book. Now the axis a c is perpendicular to the planes of the parallels; therefore the straight line o u will be at right angles to the plane of the parallel whose centre is at the point x. And for that reason the rectilineal angle o u p will be right, by definition 3 of the eleventh book. Accordingly the square on the straight line o p will equal the two squares on o u and u p; and those squares are known. Therefore the square on o p will be known, and consequently the straight line itself known; and the arc o p will likewise be known from Ptolemy's table of arcs and chords — which was the thing to be found.

Diagram. A sphere drawn as an ovoid with the pole a at the top, showing the two opposite parallels as slanting ellipse-arcs. The upper parallel carries the chord o — s with its centre t; the lower parallel carries a chord with its centre x and its left end u. The chords o u and t x are drawn, together with the meridian arcs and a further arc through q.

- a (top of the figure): the pole a
- t (upper centre): the centre of the parallel lying toward the pole a
- o (left end of the upper chord)
- s (right end of the upper chord)
- q (below the upper chord, on the arc)
- x (lower centre): the centre of the parallel lying toward the opposite pole
- u (left end of the lower chord)
- [?] (right end of the lower chord — a single letter, not resolvable on the leaf)
- [?] (far left, on the upper chord line — a single letter, not resolvable)
- [?] (centre, between q and x — a single letter, not resolvable)
- [?] (foot of the figure — a single letter, presumably the pole opposite a, not resolvable)

In problem 45 of his *tabula primi mobilis*, Regiomontanus determines the interval between given places from the proportion of the sines in spherical triangles. For every entry, whether lateral or areal, embraces four proportional numbers, of which the greatest — the whole sine — bears the same ratio to the right sine of the arc of the transverse number as the sine of the arc of the lateral number bears to the sine of the arc of that areal number which is placed to the right alongside the same lateral. It therefore makes no difference whether you multiply and divide the numbers by the rule of proportional numbers and then draw the arc of the quotient out of the table of right sines, or instead enter the table of the *primum mobile* itself.

So let there be two places whose vertices are a and b, of unequal latitudes, on the meridians c a d and c b d [?]; and let both be northern or both southern. Let the difference of their longitude be the arc f g of the equinoctial, and let c be the visible pole [manifestus polus]. Since, then, the difference of longitude is supposed known, the angle a c b will be given as known. Likewise, since the latitudes are given as known, their complements a c [?] and b c will be known.

Diagram. A lens-shaped figure formed by two meridian arcs running from the pole c at the top to d at the bottom, with the equinoctial drawn across from f on the left to g on the right. The vertex a lies on the left-hand meridian, and the great-circle segment a e is drawn from a to meet the right-hand meridian at e. A solid dot is inked at the centre of the figure.

- c (top): the visible pole
- b [?] (upper right, on the right-hand meridian): the vertex of the second place
- a (left, on the left-hand meridian): the vertex of the place of lesser latitude
- e (right of a, on the right-hand meridian): the foot of the perpendicular great-circle segment a e
- f (left, on the equinoctial)
- g (right, on the equinoctial)
- d (bottom): the pole opposite c

So in the triangle a b c the base a b — the interval between the places — will be disclosed in the following way. From the point a, the place of lesser latitude, let a great-circle segment a e come at right angles to the meridian c b d. Then, as the whole sine is to the right sine of the angle c, the difference of longitude, so is the right sine of the arc a c, the complement of the lesser latitude, to the right sine of the arc a e. And alternately: as the whole sine is to the right sine of a c, the

complement of the lesser latitude, so is the sine of the angle c, the difference of longitude of the given places, to the right sine of the arc a e. Consequently the arc a e itself will come out known in the area of the table — the quantity he calls the first found. For it will be found alongside the lateral a c, if you have taken as the transverse the difference of longitude itself. But alongside the lateral ar-

— the arc that is the difference of longitude, if you take the complement of the lesser latitude as the transverse. Either of the two may be used as the transverse, since the same author warns that of two numbers the greater must always be looked for at the head of the table.

Now, as the whole sine is to the sine of the complement of the arc *a e*, so is the sine of the complement of the arc *e c* [?] to the sine of the complement of the arc *a c*; and here the third proportional term is the unknown. We shall therefore enter the table by its area with the complement of the first found and with the lesser latitude, for the table will show in its side the complement of the arc *e c*, which is *e g*. Then, when *e g* is subtracted from *b g*, the greater latitude, *b e* is left known — the quantity he names the second found.

So if that number, found in the descending [?] of the table, turns out equal to the greater latitude, you may take it that the first found is the travelling distance [distantia viatoria] between the two given places, and that the arc drawn at right angles from *a* to the meridian *c b d* has fallen on *b*, the vertex of the place of greater latitude, and not on *e* between *b* and *g*. It will also happen sometimes that it falls between *b* and *c*; in that case what is found in the side of the table is greater than the greater latitude. The smaller must therefore always be taken from the greater, so that the second found is left.

But — however that arc making right angles with *c b d* may fall, whether above *b* or below it — as the whole sine is to the sine of the complement of the first found, so is the sine of the complement of the second found to the sine of the complement of the arc *a b*; and here the fourth term of the proportion is the unknown. We shall therefore enter those complements laterally into the table, and in its area, alongside the lateral number, we shall display the complement of that same arc *a b*. When that is subtracted from 90 degrees, *a b* is left known — the interval between the given places — provided the difference of longitude is less than a quadrant. That is how the author's rule must be understood.

If you prefer, you may use a simpler method, as follows. From the point *b*, that of the greater latitude, let a great-circle arc *b h* be drawn at right angles to *a c*. Then in the right-angled spherical triangle *b h c*: as the whole sine is to the right sine of

the acute angle *c*, the difference of longitude, so is the right sine of the arc *b c*, the complement of the greater latitude, to the right sine of the arc *b h*. We shall therefore enter the table laterally with the difference of longitude and with the complement of the greater latitude, and in the area of that table we shall find the arc *b h*, which we shall call the first found. And since in the same triangle, as the whole sine is to the sine of the complement of the first found, so is the sine of the complement of the arc *c h* to the sine of the complement of *b c*, the third term of the proportion is the unknown and the other three are known.

Diagram. A sphere drawn as an ovoid outline, occupying the upper part of the outer column, with the two meridian arcs forming the sides of the oval and the equinoctial drawn across the middle. The visible pole *c* stands at the top and the pole *d* at the bottom; the equinoctial runs from *f* at the left to *g* at the right. The two given places are marked on the two meridians above the equinoctial, and a further great-circle arc is drawn between them. The woodcut is heavily overlaid by show-through from the other side of the leaf, and several of the interior labels do not resolve.

- c (top of the figure): the visible pole
- b (upper right, on the right-hand meridian): the vertex of the place of greater latitude
- a (left, on the left-hand meridian above f): the vertex of the place of lesser latitude
- h [?] (upper left, between a and c): the foot of the perpendicular arc *b h* on *a c*
- f (left end of the equinoctial): intersection of the left-hand meridian with the equinoctial
- g (right end of the equinoctial): intersection of the right-hand meridian with the equinoctial
- d (bottom of the figure): the pole opposite *c*
- [?] the point *e*, named in the text as lying on *c b d* between *b* and *g*, carries no label I can read on this leaf

We shall therefore enter that same table by its area with the second and fourth terms, and find the third in the side of the table; and when it is subtracted from the quadrant, the arc *c h* is left known. Then we shall take *c h* away from *a c*, the complement of the lesser latitude, and the arc *a h* is left, which we call the second found. Finally, in the right-angled spherical triangle *a b h*, enter the table laterally with the complements of the first and of the second found, and you will find in

the area of that table the complement of the arc *a b*; subtract it from 90, and the arc *a b* itself is left known.

From this you have it that if both places are northern, or both southern, and the difference of longitude is less than a quadrant, the interval between the given places will be less than a quadrant.

But let us suppose once more that the difference of longitude is less than a quadrant, that the place whose zenith is at *a* is northern and the one at *b* southern, and that the arc *a k* meets *c b* at right angles. We shall then enter the table laterally with the difference of longitude and with the arc *a c*, the complement of the northern latitude, as the author directs, and in the area of the table the arc *a k* will be found, which he calls the first found. The complement of this found, together with the complement of the arc *a c* — that is, together with the northern latitude — we shall put into the table by its area;

Diagram. The second woodcut on this page, filling the lower part of the outer column: a larger construction in which the arcs run well beyond the body of the sphere, drawn for the case in which one place is northern and the other southern. Two long great-circle arcs cross each other, and a further arc falls at right angles on one of them. As with the figure above, the impression is much obscured by show-through and only part of the lettering can be made out on the scan.

- c (middle left of the crossing arcs): the pole
- k (immediately right of *c*, on the arc *c b*): the foot of the perpendicular arc *a k*
- a (right): the vertex of the northern place
- b (lower right): the vertex of the southern place
- f (left): the point at which the drawn arcs run out to the left
- d (bottom of the figure): the pole opposite *c*
- [?] a further label stands at the right of the figure between *k* and *a*; it does not resolve on the leaf

for the number that turns up in the side of the table, which is *k g*, added to the southern latitude, which is *b g*, will be called the second found. So if in the right-angled triangle *a k b* the complements of the two given sides *a k* and *b k* are put laterally into the table, the number standing at the cell common to both, taken from the quadrant, will leave known the circumference *a b*, the interval between the given places — provided the second found has come out less than a quadrant.

For if it is a quadrant, *a b* must be a quadrant too; and if it is greater than a quadrant, *a b* will likewise be greater than a quadrant.

For that reason, when the second found is greater than a quadrant, we shall prolong the segments *b c* and *a b* until they meet at *i*; and when the second found is subtracted from the semicircle *b k i*, the arc *k i* is left known. Then we shall enter the table laterally with the complements of the two arcs *a k* and *k i*, and in the area we shall find the complement of the arc *a i*; add a quadrant to it, and the arc *a b* comes out known.

We have set all this down so that people may understand that no one may use the *tabula primi mobilis* without the demonstrations of the problems.

But let us go on, and suppose the difference of longitude greater than a quadrant yet less than a semicircle — whether the two places *a* and *b* depart from the equinoctial toward the same part of the world, northern or southern, or toward opposite parts. Let *a* stand farther than *b* from one of the poles, which is *c*. We shall then prolong the two arcs *a c* and *a b* until they meet at *f*, and from the point *b* we shall draw the great-circle arc *b e* at right angles onto *c f*. So in the right-angled triangle *b e c*: as the whole sine is to the sine of the arc of the acute angle *b c e* — the arc that is left when the difference of longitude is taken from the semicircle — so is the sine of the distance *b c*, by which *b* stands from the pole *c*, to the sine of the arc *b e*. We shall therefore enter the table laterally with what is left when the difference of longitude is subtracted from the semicircle, and with the arc *b c* itself; for in the area of that same table we shall display the arc *b e*, which will be called the first found.

Diagram. A woodcut at the foot of the inner column showing the case in which the two arcs *a c* and *a b* are prolonged until they run together. The two arcs cross and are carried up to the right, where they meet; the perpendicular arc *b e* is dropped from *b* onto the arc that runs from *c* through *e* to that meeting point.

- e (upper part of the figure, on the arc running to *f*): the foot of the perpendicular *b e*
- f (upper right, at the apex where the prolonged arcs meet): the point at which *a c* and *a b* run together
- c (left): the pole from which *a* stands farther than *b*
- b (centre): the vertex of the place nearer the pole *c*
- a (lower left): the vertex of the place farther from the pole *c*
- d (lower right, at the foot of the outline): the pole opposite *c*

Next, with the complements of the arcs *b c* and *b e* we shall enter that same table by its area, and in the side we shall find the complement of the arc *c e*; and when that is subtracted from the quadrant, the arc *c e* is left known. We shall add it to the arc *a c*, and the arc *a e* is made up — the second found.

Now if the places proposed, *a* and *b*, are both northern or both southern, and the second found has come out equal to a quadrant, the arc *a b* — the interval between the given places — will be a quadrant as well; but if it is less than a quadrant, *a b* will likewise be less than a quadrant. We shall therefore enter the table laterally with the complements of the first and of the second found, for in the area we shall find the complement of the distance sought; and when that is taken from 90 degrees, the distance itself is left known.

If, on the other hand, the second found comes out greater than a quadrant even with the two places set in the same part of the world, or if the same places lie toward opposite parts of the world, there is a single route to follow. We shall take the second found away from the semicircle, and the arc *e f* is left known. With the complement of that arc and the complement of the first found we shall enter the table laterally, and in the area we shall display the complement of the arc *b f*. That we shall add to the quadrant, and the whole arc *a b*, the interval between the given places, will be disclosed.

Let us suppose once again that the given places have unequal latitudes, but that the difference of longitude is equal to a quadrant; the angle *a c b* will then be a right angle. We shall accordingly enter the table laterally with the latitudes themselves, and take the number found in the area of the table away from the quadrant; and the distance sought is left, if the two places are set in the same part of the world, southern or northern. But we shall add that same number to the quadrant if one of them is northern and the other southern, and the arc of the distance sought is made up.

For let the two places *a* and *b* be set in the same part of the world, either northern or southern, with *a f* the latitude of the one and *b g* the latitude of the other. Then as the whole sine is to the sine of the complement of the arc *a c*, which is *a f*, so is the sine of the complement of the arc *b c*, which is *b g*, to the sine of the com-

Diagram. The second woodcut on this page, at the foot of the outer column: a sphere drawn as an ovoid outline with the two meridians as its sides and the equinoctial across the middle, drawn for the case in which the difference of longitude is a quadrant.

- c (top of the figure): the pole

- b (upper right, on the right-hand meridian): the vertex of the one place
- a (left, on the left-hand meridian): the vertex of the other place
- f (left, where the left-hand meridian meets the equinoctial): the foot of the latitude *a f*
- g (right, where the right-hand meridian meets the equinoctial): the foot of the latitude *b g*
- d (bottom of the figure): the pole opposite *c*

plement of the arc *a b*. So we shall put the latitudes of the places themselves laterally into the table, and we shall come upon the complement of the arc *a b* in its area; and when that complement is taken from the quadrant, the distance *a b* itself is left known.

But let one place be northern and the other southern. We shall then prolong the arcs *a c* and *a b* until they meet at *i*. Accordingly, in the right-angled triangle *b c i*: as the whole sine is to the sine of the complement of the arc *b c*, so is the sine of the complement of the arc *c i* to the sine of the complement of the arc *b i*. Now the latitude *b g* is the complement of the arc *b c*; and since *a c i* is a semicircle and the arc *f c* a quadrant, the latitude *a f* together with *c i* will make up the other quadrant. We shall therefore enter the table laterally with the latitudes themselves, and in the area we shall come upon the complement of the arc *b i*, which we shall add to the quadrant; and *a b*, the interval between the given places, is made up.

Diagram. A woodcut in the inner column: the sphere drawn as an ovoid outline with the equinoctial across the middle, and with the arcs *a c* and *a b* carried up beyond the outline until they run together at a point above and to the right of the figure.

- i (top right, outside the outline): the point at which the prolonged arcs *a c* and *a b* meet
- c (upper left, on the outline): the pole
- b (centre): the vertex of the one place
- f (left, on the equinoctial): the foot of the latitude *a f*
- g (right, on the equinoctial): the foot of the latitude *b g*
- a (lower left): the vertex of the other place
- d (bottom of the figure): the pole opposite *c*

But when the given places have equal latitudes and lie in the same part of the world, northern or southern, and the difference of longitude is less than a

semicircle, we shall enter the *tabula primi mobilis* itself laterally with the complement of the latitude and with half the difference of longitude. For the number found in the area will be half the interval between those same places; double it, and you will have the whole interval between them.

For let *a m b* be the arc of the parallel between the two places *a* and *b*, and let the segment of the great circle between the same places be *a n b*. From the pole *c* let the great-circle arc *c n* come down, cutting the segment *a n b* at right angles at the point *n*. Accordingly the acute angle *a c n* is half the angle *a c b*, and shows half the difference of longitude of the given places; while the arc *a n* is half the arc *a n b*. So in the triangle *a n c*: as the whole sine is to the sine of the angle *a c n*, half the difference of longitude, so is the sine of the arc *a c*, which is the complement of the latitude, to the sine of the arc *a n*. And so by a lateral entry we shall find the arc *a n*, of which *a n b* is double.

Diagram. At the head of the outer column, the sphere drawn as an ovoid outline with the equinoctial across the middle. Two arcs are drawn between the two given places: the upper one, cut at right angles by the arc dropped from the pole, and the lower one, the segment of their common parallel.

- c (top of the figure): the pole
- n (below *c*, where the arc from the pole cuts the upper arc): the point of the right-angled section on *a n b*
- a [?] (left end of the two arcs): the vertex of the one place
- b (right end of the two arcs): the vertex of the other place
- m (below *n*, on the lower arc): the midpoint of the parallel segment *a m b*
- f (left, on the equinoctial): the left-hand end of the equinoctial
- g (right, on the equinoctial): the right-hand end of the equinoctial
- d (bottom of the figure): the pole opposite *c*

But if one place is northern and the other southern, and the latitudes are nevertheless equal, a lateral entry with the complement of the latitude and the complement of half the difference of longitude will yield the complement of half the interval. For the two right-angled triangles *a f o* and *b g o* are equiangular: the angles at *o* are vertically opposite; those at *f* and at *g* are right; and the angles *f a o* and *g b o* are equal for this reason, that the two arcs *a c* and *c b* taken together are equal to one semicircle. Therefore the arc *a o* is equal to *b o*, and *f o* is equal to *o g*. So *f o* is half the difference of longitude, while *a o* is half the interval between the places *a* and *b* themselves.

Now since, as the whole sine is to the sine of the complement of *a f*, so is the sine of the complement of *f o* to the sine of the complement of the arc *a o*, we shall enter the table laterally with the complement of the latitude and the complement of half the difference of longitude, and in the area of that table we shall find the complement of half the interval. So half the interval itself will be known, and the whole interval is thereby disclosed.

Diagram. The lowest woodcut on the page, in the outer column: the sphere drawn as an ovoid outline with the equinoctial across the middle and the two arcs from the pole crossing it, drawn for the case in which one place is northern and the other southern while the latitudes are equal.

- c (top of the figure): the pole
- a (upper left, on the left-hand arc): the vertex of the northern place
- f (left, on the equinoctial): the foot of the perpendicular from *a*
- o (centre, where the arcs cross the equinoctial): the point of intersection
- g (right, on the equinoctial): the foot of the perpendicular from *b*
- b (lower right, on the right-hand arc): the vertex of the southern place
- d (bottom of the figure): the pole opposite *c*

Finally, let the place *a* have the latitude *a f*, and let the place *b* be set on the equator, and let it be required to find the distance *a b*. Now if the difference of longitude *b f* is a quadrant, the point *b* will be the pole of the circle *c a f*; the distance *a b* is therefore a quadrant, and so is known. But if that difference of longitude is less than a quadrant, *a b* will likewise be less than a quadrant. So, as the whole sine is to the sine of the complement of the arc *b f*, so is the sine of the complement of *a f* — which is *a c* — to the sine of the complement of *a b*. A lateral entry into the table with the complement of the difference of longitude and the complement of the latitude of the place *a* will accordingly give the complement of the arc *a b*; take that from a quadrant, and the distance *a b* itself is left known.

Diagram. A woodcut in the inner column: a curvilinear (spherical) triangle drawn as a quadrantal sector, with two arcs running down from the vertex *c* at the top to the base, the base itself a curved line from *f* at the lower left to the right-hand corner. The chord-like arc *a*–*b* crosses the interior, cut at *p*.

- c (top of the figure): the pole of the circle *c a f*
- a (left, on the left-hand arc): the vertex of the given place *a*
- b (right, where the interior arc meets the right-hand arc): the vertex of the place *b*, on the equator

- p (interior, on the arc *a b*): the point where the great circle through *c* and *l* cuts *a b*
- f (lower left corner): the foot of the latitude *a f* on the equator
- l (base, to the right of *f*): the point a quadrant *b l* from *b*

But suppose the difference of longitude is greater than a quadrant, yet less than a semicircle. Then we shall take from it the quadrant *b l*, and we shall understand a great circle to be drawn through *c* and *l*, cutting the arc *a b* at *p*. The arc *b p* is therefore a quadrant, and since the angles at *p* are right, in the triangle *a p c* we have: as the whole sine is to the sine of the arc of the angle *a c p*, so is the sine of the arc *a c* to the sine of the arc *a p*. Now the arc of the angle *a c p* is *f l*, which is the amount by which the difference of longitude of the given places exceeds the quadrant *b l*; the arc *a c* is the complement of the latitude of the place *a*; and *a p* itself is the excess of its distance over a quadrant. A lateral entry into the table, then, with the complement of the latitude and with that same excess of the difference of longitude over a quadrant, will indicate the arc *a p*, which we shall add to the quadrant, and the distance *a b* itself will come out known.

Nor can the intervals between places be found with any less trouble by following the method we demonstrated in the book *On Twilights*, proposition 6. For when both places are northern, or both southern, then as the square of the whole sine is to the rectangle contained by the right sines of the complements of the latitudes of the given places, so is the versed sine of the difference of longitude of those places to a certain straight line, which we shall not unreasonably call the argument of the interval [*argumentum intercapedinis*]. For if that line is found equal to the right sine of the complement of the difference of latitude of those places, the interval sought will be a quadrant. But if it is unequal, then the difference of the two lines is of course the right sine of some arc, which will have to be subtracted from the quadrant — if the straight line we have found, the one we call the argument, is the smaller — so as to leave the interval between the given places known; whereas it is to be added when that same straight line is found to be the greater, and the interval between those places will then come out known. When, however, one place is northern and the other southern, we shall work with the one place and with the antipode of the other, and with what is left when the difference of longitude of the given places is subtracted from 180 degrees; and we shall then take the interval so found away from a semicircle, and the interval between the given places will be left known.

For let the circle *a b c d*, described about the centre *e*, be the meridian of the first place, having its vertex at *b*; let *a* be the visible pole [*manifestus polus*]; let the

straight line *d f* [?] be the section of the equinoctial, and the straight line *g h* the section of the horizon of that first place. Through the zenith of the second place let two circles be understood to be drawn: one about the pole of the world, whose section with the meridian is *k i*; the other about *b*, whose common section with the meridian *a b c d* is *l m*, cutting the straight line *i K* at the point *n*. From the point *i* [?], the end of the straight line *i K*, let the perpendicular *i o* be dropped onto the straight line *g h*, and let it cut the straight line *l m* at *p*. Therefore, by what we demonstrated in the book already named: since the common section of the circles just conceived stands at right angles to the plane of the meridian *a b c d* at the point *n*, while the straight line *k i* is the diameter of the circle described through the zenith of the second place about the pole *a*, the straight line *i n* will be the versed sine of the difference of longitude of the given places, in that same circle whose diameter is *K i*.

Diagram. The lower woodcut in the outer column: a full circle — the meridian of the first place — about its centre *e*, crossed by the horizontal section of the equinoctial, by the chord *i K*, by the slanting diameter *g h*, and by the slanting line *l m*, with the perpendicular dropped from the left-hand end of *i K* onto the slanting diameter. Heavy show-through from the other side of the leaf crosses the whole woodcut with ghost lines of type, and the letters standing on the lower arc will not separate from the bleed.

- b (upper left, on the circumference): the vertex of the first place
- m (top of the figure): the upper end of the slanting line *l m*
- h (upper right, on the circumference): the upper end of the slanting diameter *g h*
- K (right, on the circumference): the right-hand end of the chord *i K*
- l (left, on the circumference): the lower end of the slanting line *l m*
- e (centre): the centre of the meridian
- c (left, on the horizontal section): the left-hand end of the horizontal section
- f (right, on the horizontal section): the right-hand end of the horizontal section
- p (interior, upper left): the point where the perpendicular cuts *l m*
- o (interior, below *e*): the foot of the perpendicular on *g h*
- d (lower right, on the circumference): the point named with *a b c* in the lettering of the meridian
- [?] (top of the figure, above *m*): a label I cannot separate from the show-through; the text names *a* as the visible pole

- [?] (lower left, on the circumference): a label I cannot resolve; the text names *g* as the lower end of the slanting diameter

Now in the right-angled triangle *p i n*, the acute angle *i n p* is equal to the angle *f e h*, the angle of the complement of the altitude of the pole of the first place; and the arc *c i* is equal to the latitude of the second place. So *b i* will be the difference of the two latitudes *c b* and *c i*, and the arc *g i* is therefore the complement of the difference of latitude of the given places, whose right sine will be *o i*.

Now if the straight line *l m* cuts the meridian between *b* and the straight line *g h*, as in this first figure, then the straight line *i p*, which subtends the angle *i n p* and which we call the argument of the interval, will be found smaller than *i o* itself. The difference will be the straight line *o p*, equal to the right sine of the arc *g l*; and *b l*, the remainder of the quadrant, will be equal to the interval between the given places — seeing that the point *b* is the pole of the circle passing through the zenith of the second place. That interval will be brought to light in this way. As the whole sine is to the right sine of the complement of the latitude of the second place, so is the versed sine of the difference of longitude of those places in the equinoctial circle to the versed sine of the difference of longitude in the parallel of the second place. For the right sine of the complement of the latitude of the second place is the radius of that parallel; and the arcs of parallel circles intercepted between two meridians are not only proportional, but their right and versed sines are also proportional to the radii of those parallels. Furthermore, in the right-angled triangle *n i p*, as the whole sine is to the right sine of the angle *i n p* — that is, of the complement of the latitude of the first place — so is the straight line *i n* to the straight line *i p*. The whole sine is therefore twice an antecedent. And accordingly, as the square of the whole sine is to the rectangle contained by the right sines of the complements of the latitudes of the given places, so is the versed sine of the difference of longitude of those places in the equinoctial to the straight line *i p*; for this ratio, which the versed sine of the difference of longitude of the given places has to *i p*, is compounded of two ratios — the one which that versed sine has to *i n*, and the one which the same *i n* has to *i p*.

Now when three of four proportional magnitudes are given as known, the fourth cannot remain unknown. The first magnitude is known, namely the square of the whole sine; the second is known too, the rectangle contained by the right sines of the complements of the latitudes; and the third is likewise known, namely the versed sine of the difference of longitude. We shall therefore multiply the second into the third and divide the product by the first — a division that can be carried out simply by striking off the last ten figures, if you assume the whole sine to

contain one hundred thousand equal parts — and the fourth magnitude will come out known in the quotient, namely the straight line *i p*, the argument of the interval. And since *g i*, the complement of the difference of latitude, is left known once the difference of latitude has been taken from a quadrant, *i o*, the right sine of that complement, will be known from the table of right sines. We shall therefore compare the known line *i p* with the known line *i o*. If *i p* is found smaller than *i o*, as in the figure drawn here, their difference *o p* will come out known, and so the arc *g l* will be known from the table of right sines. We shall take that arc from the quadrant *b g*, and the arc *b l* will at last be left known, equal to the interval between the given places. But if *i p* is found greater than *i o*, this will be the reason: that the straight line *l m* cuts the meridian between the straight line *g h* and the point opposite to *b*, as in the second figure. We shall therefore add the arc *g l* to the quadrant *b g*, and the arc *b l* will come out known, equal to the interval between the given places. And if that same straight line *i p* is found equal to the straight line *i o*, then we must grant that the circle drawn through the zenith of the second place, whose pole is *b*, cuts the meridian upon the straight line *g h*. In that case the fourth term of the proportion just mentioned, which is the argument of the interval between the given places, will be the right sine of the arc *g i*, and accordingly the quadrant *b g* will be equal to the interval between those places.

Diagram. The woodcut at the foot of the outer column — the "first figure" of the demonstration: a full circle about the centre *e*, with the horizontal chord *i K* above the horizontal section *c f*, the slanting diameter *g h* running from the lower left to the upper right, and the slanting line *l m* crossing them from the lower left to the top, cut at *n* on *i K*; the perpendicular from *i* meets *g h* at *o* and *l m* at *p*.

- a (top of the figure): the pole of the world
- b (upper left, on the circumference): the vertex of the first place
- h (upper right, on the circumference): the upper end of the slanting diameter
- m (right of *h*, on the circumference): the upper end of the slanting line *l m*
- K (right, on the circumference): the right-hand end of the chord *i K*
- i (left, on the circumference): the left-hand end of the chord *i K*
- n (interior, on *i K*): where *l m* cuts *i K*
- e (centre): the centre of the meridian
- c (left, on the horizontal section): the left-hand end of the horizontal section
- f (right, on the horizontal section): the right-hand end of the horizontal section

- o (interior, below and left of *e*): the foot of the perpendicular from *i* on *g h*
- p (interior, below *o*): where that perpendicular cuts *l m*
- g (bottom left, on the circumference): the lower end of the slanting diameter
- l (bottom, right of *g*): the lower end of the slanting line *l m*

But so as to settle the present problem completely on every side, in the figure set below we shall put the point *a* to be the northern pole and *b* the southern. Let the first place have its vertex at *c*, on the meridian *a c b*, and a northern latitude. Let the second place have its vertex at *d*, on the meridian *a d b*, under a southern latitude. Then, when a great circle has been drawn through *c* and *d* cutting the meridian of the first place at *e*, the interval between the given places will be *c d*.

Diagram. A woodcut in the inner column: a full circle with two arcs drawn across it from the upper part down to the bottom, meeting at the foot of the figure, one of them crossing the other in the interior of the circle.

- c (upper left, on the circumference): the vertex of the first place, in northern latitude
- a (top of the figure): the northern pole
- d (interior, where the two arcs cross): the vertex of the second place, in southern latitude
- b (bottom of the figure): the southern pole
- e (right, on the circumference): where the great circle through *c* and *d* cuts the meridian of the first place

And since the two semicircles *a c b* [?] and *c b e* are equal to one another, when the common segment *c b* is taken away the two remaining segments *a c* and *b e* are left equal. Therefore those who live under *e* are the antipodes of those who live under *c*, having an equal latitude, but southern. We shall accordingly find the interval *d e* of the two southern places *d* and *e* in the way we have taught, and take it away from the semicircle *c d e* [?], and the interval *c d* [?] between the given places *c* and *d* will be left known.

Moreover, if you prefer to find distances of this kind by an instrument, the very figure of the demonstration, together with a ruler and a pair of compasses, will serve you as one. Let the circumference of the circle be divided into degrees, as usual, and let the number of degrees of the difference of longitude of the given places be counted off from *c* toward *a*; let this arc be, for example, *c q*. From *e* to *q* we shall draw a hidden straight line *e q*, and from it we shall take *e r* equal to the radius of the parallel of the second place. To that point *r* we shall fit the ruler, and

let it be carried round upon that same point until it lies parallel to the diameter *a d*. It will lie parallel when it has cut off equal arcs on either side out of the two quadrants; and we shall mark its intersection with *i K*, which let be at *n*. So the straight line *i n* will be the versed sine of the difference of longitude of the given places in the parallel of the second place. We shall therefore fit the ruler to *n*, and carry it round until it lies parallel to the diameter *g h*, in the position *l m*; and when *g l* has been taken from the quadrant, the interval between the given places will be left known.

That the straight line *i n* really is the versed sine of the difference of longitude in the parallel of the second place will not be hard to see. For let the ruler passing through *r* and *n*, parallel to the axis *a d*, cut the straight line *e c* at *t*; and about the centre *e*, with the distance *e r*, let a circle be described, cutting the radius *e c* [?] at *u*. Then, since the angle *r t u* [?] is right, the straight line *t u* will be the versed sine of the arc *r u*. But the two straight lines *e u* and *s i* are equal; so when *t e* and *s n*, which are equal, are taken away from them, the two straight lines *t u* and *n i* are left equal, by the common notion. The straight line *i n*, therefore, is the versed sine of the difference of longitude in the parallel of the second place. But when the versed sine is greater [?] than the radius, it can be found [?] much more easily, as you already know.

Diagram. The woodcut at the head of the outer column — the "second figure": a full circle about the centre *e*, with the vertical axis *a d*, the horizontal chord *i K* above the horizontal section *c f*, the slanting diameter *g h* from the lower left to the upper right, the slanting line *l m*, the arc drawn from *q* down through the interior, and the smaller arc about *e* cutting the radius toward *c*.

- q (upper left, on the circumference): the end of the arc *c q*, the difference of longitude counted from *c*
- a (top of the figure): the northern pole
- m (upper right, on the circumference): the upper end of the slanting line *l m*
- r (interior, upper left): the point taken on *e q* at the radius of the parallel of the second place
- h (right, on the circumference): the upper end of the slanting diameter
- i (left, on the circumference): the left-hand end of the chord *i K*
- n (interior, on *i K*): the intersection of the ruler with *i K*
- s (interior, right of *n*): the point on *i K* used in the proof
- K (right, on the circumference): the right-hand end of the chord *i K*

- c (left, on the horizontal section): the vertex of the first place
- u (interior, below *c*): where the small circle about *e* cuts the radius toward *c*
- t (interior, left of *e*): where the ruler through *r* and *n* cuts *e c*
- e (centre): the centre of the meridian
- f (right, on the horizontal section): the right-hand end of the horizontal section
- l (lower left, on the circumference): the lower end of the slanting line *l m*
- g (lower left, below *l*): the lower end of the slanting diameter
- d (bottom of the figure): the southern pole

Besides this, following the demonstration of Johannes Werner, the interval between the given places can be found in a single plane, if you construct a rectilinear quadrilateral of given sides in which two opposite and equal sides are the straight lines subtending the arcs of the meridians between the two parallels, while the two remaining sides, which are parallel to one another, are the straight lines subtending the arcs of the parallels between those meridians. For the straight line between the opposite angles will subtend the arc of the interval sought.

Likewise, the same interval can be found on the plate or table of the general astrolabe [*Astrolabium generale*], by the same art by which, from a star's known distance from the meridian, its declination

— its declination being known — the distance of that star from the zenith is found.

It will be worth the trouble, however, to extend the same plate beyond the tropic of Capricorn, on account of places lying further south. The construction and use of the general plate itself were written up long ago, and put into print, by Ioannes Vasurtus [?], astronomer of Salamanca. Afterwards, so that we might use it without ambiguity, we investigated by demonstration the principle both of its construction and of its use. Then, some years later, we found the same plate engraved on many Arabic astrolabes built centuries before, which the most illustrious prince Ludovicus [Luís], infante of Portugal, brought back among the spoils of the city of Tunis.

The easiest method of all, though, will be this. Set the two given places on a globe according to the rules of the art, and then take the distance between them with the legs of a pair of compasses. Transfer that opening at once to the meridian or to the equinoctial, and you will find how many degrees of a great circle the interval sought contains.

¶ *On what must first be laid down for drawing on a globe those lines which sailors call rhumbs.*

Cap. 21.

Near the beginning of the earlier book we showed that the line which a ship describes on its course, when that course is neither the meridian nor the equinoctial, is not circular, but is made up of certain very small segments of great circles. We remarked, however, that it can be said with some reason to be a curved line of a different form, produced like a helix out of two motions. For someone may perhaps suspect that a ship's motion, so long as it holds a course other than the meridian or the equinoctial, arises from two motions, or from two movers. The first motion is that by which the ship itself, set lengthwise in the plane of the great circle that looks toward the desired quarter of the horizon, is carried forward by the wind or by the drive of oars. The second is sideways, or oblique: the helmsman, holding the tiller with the mariner's needle to guide him, meanwhile twists the ship aside and deflects it toward the point the prow was facing when a course of that kind was set.

That is to say: since by changing place the ship falls upon new meridians, and consequently upon new horizons, it is by that art that it directs its course toward the corresponding quarters of those horizons. So, if the matter stands thus, the line described — the line we call a rhumb — will be neither circular nor compounded of circular arcs. To us, however, it appears otherwise. For we have observed that a ship is carried straight ahead for some time before it is deflected sideways; and so we judge that a line of this kind is made up of very small segments of great circles. Why, after all, should the ship be compelled to be carried sideways without interruption, if — even though it is travelling on the great circle along which the wind blows, and only for a short run — the helmsman does not in the least perceive the prow pointing elsewhere? All the same, a man skilled in geometry establishes by sure and unquestionable reasoning that, however slight the change made, unequal angles are formed with the new meridians the ship enters, and that the ship's prow therefore does point elsewhere; but that error escapes the senses.

To see the cause and reason of this plainly, we must first understand the following. Let a spherical triangle a b c be given, formed of segments of great circles, in which the angle c is right, the angle a acute, and the side a b subtending the right angle not greater than a quadrant. Let there also be given an acute angle d, greater than a itself. It will not be difficult to draw from the point b, to the

opposite side a c, a segment of a great circle which shall make, at some point between a and c, an angle with that a c equal to the given angle d.

For at the point a, the end of the side a c, let us construct an acute angle c a e equal to the angle d, by the first proposition of the first book of Menelaus; and, the side b c being produced, let it meet the segment a e at the point e. Next, three straight lines being given — the first the right sine of the segment c e, the second the right sine of a e, the third the right sine of b c — let a fourth proportional be found in the plane of the circle c b e, by the 12th of the sixth book of Euclid; and let it be f g. We shall show that this f g is greater than the right sine of the segment b c, but less than the whole sine. For since the angle b a c is set down as acute, and the side a b as not greater than a quadrant, the side b c will be less than a quadrant; and the side a c will be not greater than a quadrant, by the eleventh proposition of the first

book of Geber. Again, in the triangle a e c: since the angle c a e is acute, the side subtending it will be less than a quadrant, by that same eleventh proposition. Moreover, the side a c has been shown to be not greater than a quadrant; therefore the side a e will not be greater than a quadrant, by the same 11th of the first book of Geber. But c e is less than a e, by the seventh proposition of the first book of Menelaus, because it subtends the smaller angle; therefore the right sine of the segment c e will be less than the right sine of the segment a e. But as the right sine of c e is to the right sine of a e, so we have set the right sine of b c to the straight line f g; therefore the right sine of b c is less than that straight line f g.

That the same f g is less than the whole sine will be easy to demonstrate. For since, as the right sine of the segment c e is to the right sine of a e, so stands the right sine of b c to the straight line f g, it follows that as the sine of c e is to the sine of b c, so is the sine of a e to the straight line f g, by alternation of the proportion. But the sine of c e is greater than the sine of b c; therefore the right sine of the segment a e will be greater than the straight line f g. And the right sine of the segment a e does not exceed the whole sine; therefore the straight line f g will be less than the whole sine.

We shall therefore take a straight line f h, double f g, and fit into the circle e b c a line equal to it, namely b i, subtending a circumference less than a semicircle.

Diagram. A large circle occupies the upper left of the leaf, standing for the great circle of the construction, with several arcs drawn across its interior. A nearly vertical arc descends from the top of the circle to its lower part, and further arcs spring from the left side of the circle and run across toward its right side, crossing

one another near the centre. To the right of the circle stand two long, nearly straight lines lettered at their upper and lower ends; below them, a separate narrow V-shaped pair of lines represents the given acute angle drawn apart from the sphere. Readable labels:

- l (top of the circle, on the circumference): the upper end of the near-vertical arc
- n (left of the circle, on the circumference)
- a (far left, on the circumference)
- e (interior, upper middle) [?]
- b (interior, middle)
- o (left interior, between a and c)
- c (interior, below b)
- K (lower part of the near-vertical arc)
- m (right interior, toward the circumference)
- f (upper end of the outer straight lines, at the right)
- g (upper end of the second outer straight line)
- h (lower end, at the right)
- i (lower end, at the left of the pair)
- d (at the separate V-shaped angle, below the outer lines)

The pairing of the upper endpoints f and g with the lower endpoints h and i is not resolvable on the scan, since the two lines cross.

Let b K be half of b i. The right sine of b K will therefore be equal to the straight line f g, by the definition of the right sine and by the common notion; and accordingly the segment b K will be greater than the segment b c. About the pole b, then, and with the interval b K, we shall describe a circle, and this circle must cut the great circle a c l in two places. Let one intersection be before c, at the point m. I say that the other intersection will fall between c and a.

For it will not fall at a. The right sine of the acute angle c a e has a greater ratio to the whole sine than the right sine of the acute angle b a c has to that same whole sine. But as the right sine of the angle c a e is to the whole sine, so is the sine of the segment c e to the sine of the segment a e; and as the sine of the angle b a c is to the whole sine, so is the sine of the segment b c to the sine of a b, by the 13th proposition of the first book of Geber. Therefore the sine of c e has a greater ratio to the sine of a e than the sine of c b has to the sine of a b. But as the sine of c e is to the sine of a e, so is the sine of c b to the sine of b K [?]; therefore the sine of c b will have a greater ratio to the sine of b K than the sine of that same c b has to the

sine of a b, and accordingly the sine of the segment b K is less than the sine of the segment a b.

And since the segment b k was shown to be less than a quadrant, while the segment a b was laid down as not greater than a quadrant, b k will be less than a b. Accordingly the circle described through k cannot cut the great circle a c at a. For if it cut it at a, the two segments a b and b k would be equal to one another; but a b is greater than b k.

Nor can it cut it at any other point beyond a, such as at n. For since b c is less than a quadrant, in the triangle n b c the angle c n b will be acute; but the angle b a n is obtuse, and therefore in the triangle a b n the side b n will be greater than the side a b, by the 9th of the first of Menelaus, and so much greater than the segment b K. The circle described therefore cannot cut the great circle a c m beyond a, nor at a itself.

Let it then cut it at o, between c and a. We shall accordingly describe a great circle through the two points b and o, which makes at o the angle b o c. I say that this angle b o c is acute, and equal to the given angle d. For as the right sine of c e is to the right sine of a e, so is the right sine

of b c to the right sine of b o. But as the right sine of c e is to the right sine of a e, so is the right sine of the arc of the angle c a e to the whole sine. And as the right sine of b c is to the sine of b o, so is the right sine of the angle b o c to the whole sine. Therefore, as the right sine of the angle c a e is to the whole sine, so is the right sine of the angle b o c to that same whole sine; and on that account the two right sines of the angles c a e and b o c are equal to one another. But c a e is acute by hypothesis, and b o c is acute likewise, because in the right-angled triangle b c o the opposite side b c is less than a quadrant. Those same angles c a e and b o c will therefore be equal to one another. But c a e is equal to the angle d; therefore b o c will be equal to that same d.

And so, in the triangle a b c formed of segments of great circles, in which the angle c is right, the angle a acute and less than the given angle d, and the side a b not greater than a quadrant, we have drawn from the remaining angle b, to the opposite side a c, the segment b o of a great circle, which at the point o constitutes the angle b o c equal to that same given angle d — which was what had to be done.

And since the difference between the acute angle a and a right angle can be divided into two equal parts, and the half again into two equal parts, and so on to infinity, we can draw from the remaining angle b a segment of a great circle

which shall make, at some point of the side a c, an acute angle exceeding a by so slight a difference that to the judgement of the senses it appears equal to it — so slight, indeed, that their inequality cannot be detected by any instrument.

Diagram. At the top right of the leaf stands a pointed-oval figure formed by two great-circle arcs that meet at a point above and converge again below. A straight line descends from the apex toward the right, crossing the right-hand arc; a further arc crosses the oval horizontally at about mid-height; and below, two short arcs run down to the base of the figure. A small separate angle is drawn at the right. Readable labels:

- a (apex, at the top where the two arcs meet)
- d (upper right, at the end of the line descending from the apex)
- c (right, where that line crosses the right-hand arc)
- e (lower right, at the separate small angle)
- b (lower left, on the left arc)
- h (just to the right of b, inside the figure)
- g (bottom left)
- f (bottom centre, where the arcs converge)

By the same art of demonstration you will also conclude the following. In a spherical triangle a b c formed of segments of great circles, if the side a b is greater than a quadrant, a c is a quadrant, and the angle a b c is acute, then, the side b c being produced, the exterior angle a c d will be less than the acute interior angle a b c — and this because the two sides a b and a c taken together are greater than a semicircle, by hypothesis.

Therefore, another acute angle e being given, still less than a b c but greater than a c d, I say that it is possible to draw from the angle a, to the opposite side b c, a segment of a great circle which shall make with that b c an angle equal to e itself, on the side toward c. For let the sides a b and a c be extended until they meet at f, and from f let a segment of a great circle f g be drawn at right angles upon b c. This segment must fall outside the triangle b f c, because the angle c b f is obtuse, while f g is less than a quadrant. Then, since a c f is a semicircle and a c is a quadrant, the segment c f will be a quadrant as well.

The angle b c f, moreover, is acute, being equal to the vertical angle a c d. Accordingly, in the right-angled triangle c g f — in which the side c f is not greater than a quadrant and the angle f c g is acute — we can draw from the remaining angle c f g, to the opposite side c g, a segment of a great circle, by the art delivered a little earlier, such that with c g toward g it makes an acute angle equal to the

given angle e. Let such a segment be f h, which at the point h makes the angle f h g equal to e itself, and let that same f h be produced as far as a. The vertical angle a h c will then be equal to that same e.

Therefore, in the given triangle a b c, in which the side a c is a quadrant, a b greater than a quadrant, and the angle a b c acute, we have drawn from the remaining angle a, to the opposite side b c, the segment a h, which on the side toward c makes the angle a h c equal to the given acute angle e — the angle that was set down as less than the acute angle a b c and greater than the exterior angle a c d. This is what we proposed to do.

But f h cannot fall at the point b. For the angle a b c is equal to its vertical angle f b g, and therefore the angle f b g would be greater than the angle e, by hypothesis and by the common notion; so it is not equal to it. Nor will it fall between b and g. For b f would then be greater than f h, since b f subtends an obtuse angle while f h subtends an acute one; and so the two sides b f and f h taken together would be less than a semicircle, and consequently the angle f h g would be far greater than that same angle a b c. The angle e would therefore be far greater than a b c — again contrary to the hypothesis.

From this you will likewise conclude that from the point a,

— there can be drawn a segment of a great circle upon the subtended side *b c*, one that the acute angle *a b c* exceeds by so slight a difference that it escapes perception altogether, so much so that no instrument can detect an excess of that kind.

Therefore, consider those who follow the rules of the art of navigation and set a course that lies neither along a meridian nor along the equator. Even though for some time they are moving on one and the same great circle, and for that reason turn a little aside from the course they set and head elsewhere, they will not be able to detect a deviation of this sort by sense. Let the great circle *a b c* be the meridian of the place *b*, and let *a* be the visible pole. For those who put out from the place *b*, let the course be laid down according to the size of the acute angle of departure *a b d*, the angle that the great-circle segment *b d* makes with the meridian at the point *b*. From *a* let there be drawn the great-circle segment *a e*, at right angles upon *b d*; and let another acute angle *f* be proposed, one that exceeds *a b d* by an insensible difference, and by a difference smaller than that by which *a b d* falls short of a right angle.

Diagram. A large circle carrying the construction of the preceding argument. The visible pole *a* stands near the top of the circle; from it lines run down to the

points *b*, *g* and *e*, which lie in order along the arc that continues to *d* on the circle at the right. Outside the circle to the left, two straight lines meet at *f*, forming the given acute angle that opens toward *b* and *g*.

- a (top of the circle): the visible pole
- b (left, on the arc toward *d*): the place of departure
- g (between *b* and *e*): the point at which the segment *a g* makes the required angle
- e (right of *g*): the foot of the perpendicular segment *a e*
- d (right, on the circle): the far end of the great-circle segment *b d*
- f (outside the circle, left): the vertex of the proposed acute angle

Now in the spherical triangle *a b e* the side *a b* is not greater than a quadrant, while the angle *a b e* is acute and smaller than the angle *f*. A point can therefore be found on the side *b e* — let it be *g* — at which the great-circle segment *a g* makes the angle *a g e* equal to *f*. So the angle *a g e* will exceed the angle of departure *a b e* by an insensible difference, and *a g* will be the meridian of the place *g*. Moreover, at any point between *b* and *g* the angles made with the circles coming from *a* are still smaller than *a g e*, yet greater than *a b e* — for the exterior angle at the base of a triangle is greater than the interior and opposite angle, whenever the two sides taken together are less than a semicircle. Those same angles will therefore exceed the angle *a b e* by a smaller difference.

Now the angle of departure *a b e* is proportional to the rectilinear angle that, on the mariner's instrument, the straight rhumb makes with the straight meridian line. These spherical angles will therefore differ from the magnitude of the rectilinear angle by an imperceptible difference. Accordingly, as long as the ship is on *b g*, a segment of a great circle, it is constantly heading off in another direction, even though that deviation cannot be perceived by sense; for the prow will appear to point exactly where the straight rhumb runs. You will show the same for voyages made toward the hidden pole, if you use the preceding figure. By meridian circles and poles on the underlying globe of sea and land we mean circles and poles corresponding to those found on the celestial sphere.

We assumed, further, that the curvilinear angle of departure is proportional to the rectilinear one. You may show this by an easy demonstration, in the following way. On the surface of the sea let *a b* be a quadrant of a meridian, and let the point *a* be the place we depart from. Let the quadrant *a b*, together with the quadrant *a c*, make the curvilinear angle of departure *b a c*; let the straight line *a d* touch the circle *a b* at *a*, and likewise the straight line *a e* touch *a c* at that same

a; let *f* be the centre of the globe; and let *a f*, *b f* and *c f* be joined. The two straight lines *a d* and *b f* will then be parallel, and likewise the two lines *a e* and *f c*, by proposition 28 of Euclid's first book. The plane containing the angle *d a e* will therefore be parallel to the plane containing the angle *b f c*. But *b f c* lies in the plane of the horizon, and so the surface in which the angle *d a e* lies is

Diagram. A quadrant figure demonstrating the proportionality of the curvilinear and rectilinear angles of departure. From *a* at the upper left a straight tangent runs right to *d*; a second straight line from *a* runs to *e*, a little below *d*. Two arcs descend from *a*, one meeting the circle at *b* on the right, the other at *c* at the foot; the centre *f* stands at the lower left, joined by straight lines to *b* and to *c*.

- a (upper left): the place of departure, the common end of both quadrants
- d (upper right): the end of the straight line touching the circle *a b* at *a*
- e (below *d*): the end of the straight line touching *a c* at *a*
- b (right): the far end of the meridian quadrant
- c (foot): the far end of the quadrant *a c*
- f (lower left): the centre of the globe

parallel to the horizon; and *a d* is the straight meridian line, while *a e* is the straight rhumb, which makes with *a d* the acute angle *d a e*. That angle, I say, is proportional and similar to the spherical angle *b a c*. For the two angles *d a e* and *b f c* are equal to one another by proposition 10 of Euclid's eleventh book, and the angle *b f c* defines the magnitude of the spherical angle *b a c*. The rectilinear angle *d a e* and the spherical angle *b a c* are therefore proportional: that is, as *d*[?] *a e* stands to a rectilinear right angle, so *b a c* stands to a spherical right angle contained by circumferences of great circles. And that is what had to be demonstrated.

These points, then, we judged had to be set down first, so that the character of the routes by which we approach one or the other of the world's poles may be rightly understood. But since in sailing one sometimes keeps the same distance from one and the same pole, it will be worth investigating the character of this route as well, the one that runs parallel to the equator. That voyages can be made not only on the great circles of the sphere but on lesser circles too, no one will ever doubt, once he observes the following: straight lines drawn from the centre of the sphere of the sea — which we take to be the centre of the world — to the individual points on the circumference of a lesser circle run off into the sky if you extend them further, and heavy bodies fall downward along them. So if someone stands on the circumference of a lesser circle in such a way that his feet are

below and his head above, along the length of the line just conceived, he will indeed be able to walk forward on that same circumference without any natural discomfort. The mathematicians, however, advise that voyages ought to be made along the circumferences of great circles, because the distance taken on a great circle is the shortest. For one and the same straight line subtends two arcs, one of a great circle and the other of a lesser; and so, if you conceive the two circles as set in one plane, the segment of the great circle will be shown to be contained within the segment of the lesser. Therefore, by that postulate of Archimedes in the first book of *On the Sphere and Cylinder* — that the containing line is greater than the contained — the distance taken on a great circle will be shorter than the distance taken on a lesser one. Johannes Werner demonstrated this far more clearly in his annotations on Ptolemy's *Geography*.

But whether, in sailing by the help of the mariner's needle, we describe a circle exactly equidistant from the equator, as sailors believe, is not easy to settle. Suppose the ship is set at the place *a*, directing its prow toward *d*, the equinoctial west; let it have *b a c* as its meridian; let *b d e* be the equator, and *d a e* the vertical circle; let *f* be one of the world's poles; and let the vertical circle itself be carried along with the ship by the motion of the first heaven. It will then be plain that the points *d* and *e* run along the equator, while the ship runs along the parallel *a g h*. Yet although by that motion the ship heads continually toward the equinoctial west and describes a parallel circle, it will still not be said to have sailed by the driving force of wind or of oars, according to the rules of the art of navigation, or by the help of the mariner's needle. It will have done so no more than those who, while trying to sail toward the northern pole, were nevertheless brought along a circle equidistant from the equator because the violence of the wind was driving the ship elsewhere.

Diagram. Two concentric circles seen about the pole. A straight line runs from *b* at the top, through the centre *f*, to *c* at the foot, cutting the inner circle at *a*. The outer circle carries *d* at the left and *e* at the right; the inner circle carries *g* at the left and *h* at the right. A slanting arc crosses from *d* through *a* to *e*.

- b (top, outer circle): one end of the meridian *b a c*
- c (foot, outer circle): the other end of the meridian
- f (centre): one of the poles of the world
- a (inner circle, on the meridian line): the ship's place
- d (left, outer circle): the equinoctial west, toward which the prow is directed

- e (right, outer circle): the opposite point of the equator on the vertical circle *d a e*
- g (left, inner circle): a point of the parallel *a g h*
- h (right, inner circle): a further point of the same parallel

Besides, why shall we say that a voyage of this kind was made from *Leste* to *Oeste*, if no advance at all was made with respect to the equator? Or why will those who wish to keep to the same parallel have to look for the east wind [*Solanus*]? For a voyage goes favourably when the wind blows in the direction in which the helmsman points the prow — and for no other reason than that, with the wind blowing so, the ship runs faster. But if they mean not to depart from the parallel at all, they will not be able to give a reason why, with the mariner's needle to guide them and the rudder to help, they themselves keep turning the ship continually toward the equinoctial west. Our own view of the matter will therefore be different (as we said in the earlier book). It is this: those who sail from *Leste* to *Oeste* short of the equator, following the rules of the art of navigation, describe on the surface of the sea not a parallel but a certain line made up of very small segments of great circles. And although they suppose that they are heading exactly and continually toward *Oeste*,

they very often turn aside. That deviation from a straight course, however, and likewise the withdrawal from the parallel, cannot be perceived by sense, because it is so slight. Let *a* be the place we depart from, having *b* as its visible pole, and lying on the parallel *a c d*. Let the course laid down for the given voyage be from *Leste* to *Oeste*, that is, toward the equinoctial west. To show, then, what sort of line those who sail in that way describe on the surface of the sea: from the point *a*, the end of the meridian *a b*, we shall draw the great-circle segment *a e* at right angles upon *a b*; and about the pole *b*, at an interval exceeding *a b* by an imperceptible difference, let a parallel be understood to be drawn, whose intersection with *a e* falls at the point *f*. We call a difference imperceptible when it is neglected in astronomical calculations on account of its smallness, or when it cannot be perceived by sense as we observe the stars. Let *b f* be the segment of the great circle passing through the points *b* and *f*. In the right-angled triangle *a b f*, then, the side *a b* will be less than a quadrant, so the angle *a f b* will be acute. And therefore a great-circle segment can come from the point *b* which, at some point between *a* and *f*, makes with *a f* an angle equal to any acute angle you please that is greater than *a f b*. So let the segment *b g* make with *a g* the angle *b g a*, smaller than a right angle by an imperceptible difference, but greater than *a f b*. Then *b g* will be still less than *b f*. And since the meridians that fall between *a* and *g* make angles greater than *b g a* — of which the more distant is greater than

the nearer — it follows that those who put out from the place *a* and, with the mariner's needle pointing out the quarters of the sky, try to hold continually toward the equinoctial west will appear, so long as they are on *a g*, not to diverge in any way from the course they set.

Diagram. A many-sided broken line closed about the pole. From the centre *b* radii run out to every vertex of the polygonal figure, and arcs of the parallel are drawn between the vertices. Toward the foot the letters *i*, *K*, *g* and *a* lie close together on successive segments, with *e* set outside the figure at the lower left.

- b (centre): the visible pole
- a (foot): the place of departure
- g (beside *a*): the end of the first segment of the broken line
- K (between *g* and *i*): the intersection of the segment *b g* with the parallel
- i (left of *K*): the point at which the next great circle touches the parallel
- c (left): a point of the parallel *a c d*
- d (upper right): a further point of the same parallel
- e (outside, lower left): the end of the great-circle segment *a e*

And because the segment *b g* exceeds *a b* by an insensible difference, they will think they have been carried along the parallel, although in fact they have been on *a g*. Now let *K* be the intersection of the segment *b g* with that same parallel, and on the parallel let an arc *K i* be taken equal to *a K*. If, then, a great circle is drawn through *g* and *i*, and another great circle through *b* and *i*, you will show that the angle *b i g* is equal to the right angle *b a g*, and that the segment *a g* is equal to the segment *g i*, by a proposition similar to Euclid I.4. The great circle drawn through *g* and *i* will therefore touch the parallel *a c d* at *i* itself. So if the ship is carried along the segment *g i*, it will seem to hold the very course that was set at the outset, that is, from *Leste* to *Oeste*; and the latitudes of places along the whole segment will appear equal to the latitude of the place *a*. Hence, just as we showed by the earlier argument, the ship will be thought to have withdrawn not at all from the parallel of the place *a*. And since it follows that the ship's course behaves in this same way round the remaining circuit of the parallel — and since it makes no difference whether we set the remaining segments equal to *a g* or smaller, provided only that they touch the parallel — it is clear that the line the ship describes on the surface of the sea, when we sail from *Leste* to *Oeste* short of the equator, is not a parallel. It departs from the parallel insensibly, and the difference is smaller the more angles that line has. It is therefore reasonable that, in place of lines of this kind, circles equidistant from the equator should be drawn on the globe. For if we were to draw broken lines in that manner with

some determinate number of angles, however large — the kind we have shown ships to run over from *Leste* to *Oeste* — we should be justly reproached: why do we not draw others that come closer to the parallel, or with more angles, so that the withdrawal from the parallel and from the course set may turn out smaller? Those, finally, who are at a place without latitude and sail toward *Leste* or *Oeste* travel on the equinoctial circle itself, because the meridians make right angles with the equator everywhere.

¶ *That it is possible to delineate a given globe with rhumbs.* Cap. 22.

So it is clear from what has been said that lines of the same kind we describe when we sail on the surface of the sea can be drawn on any globe. Lines of this sort we call by their common name *rhumbs*. They are: the rhumb of North and South; of *Leste* and *Oeste*; of *Nordeste* and *Sudoeste*; of *Noroeste* and *Sueste*; those that lie midway between these; and again the others that lie between the one set and the other. Of these, the rhumbs of North and South are great circles, namely the meridians; those of *Leste* and *Oeste* are the equinoctial together with its parallels, as we have demonstrated. The rest are curved lines put together out of segments of great circles.

Now such segments must, at any of their points, make angles equal — so far as sense can tell — with the successive meridians, the exterior angle equal to the interior one, which is the angle of departure; and we have demonstrated that this can be done. Therefore, given any proposed rhumb, proceed as follows. From the point where the given meridian cuts the equinoctial, draw a great circle making with that meridian an acute angle proportional to the rectilinear angle which the given rhumb, drawn as a straight line, makes with the meridian. Take a segment of that great circle such that at any of its points it makes the exterior angles greater by an imperceptible difference. Then from the end of that same segment draw two great circles: one through the poles of the world, the other making with it an angle equal to the one first made at the point on the equinoctial. From this second circle take a further segment, one that at any of its points makes the exterior angles equal — so far as sense can tell — to the interior angle; and so on, around the curvature of the globe, toward the one pole and the other. The broken line so produced will be entirely like the line a ship describes on the surface of the sea when the voyage is made along the proposed rhumb. And since the remaining rhumbs can be drawn by exactly the same method, it is possible to draw on any proposed globe those lines that sailors call rhumbs.

¶ *To publish a certain table of numbers, by whose help we may describe any rhumbs whatever on a given globe.* Cap.23.

The segments of great circles out of which a given rhumb is to be constituted must be of such a size that the two angles, exterior and interior, which they make with the meridians at their own ends, are treated as equal even though they are unequal. We shall keep the difference between those angles below one degree of the circumference of the horizon, since it is credible that smaller differences escape the helmsman's perception. The description of the rhumbs will begin on the equinoctial circle.

Now, to make known numerically the segments of the meridians between the nearer pole and the ends of the segments that constitute the given rhumb: in the figure below let the point a be one of the poles of the world, let a b be a quadrant of a meridian, and let the segments b c, c e, e g, g i constitute the rhumb of the given angle of departure a b c. Let b c be produced further to d, c e to f, e g to h, and g i to K. From the pole a itself let meridians run to the points c, e, g, i — namely a c, a e, a g, and a i.

I say that the meridian segments a b, a c, a e, a g, and a i have right sines proportional in continued proportion, and that this ratio is the one which the sine of the exterior angle a c d has to the sine of the interior and opposite angle a b c in the spherical triangle b a c. For in that spherical triangle, as the right sine of the side a b is to the right sine of the side a c, so is the right sine of the angle a c b to the sine of the angle a b c, by the 13th proposition of the first book of Geber [Jābir ibn Aflaḥ]. But the two angles a c b and a c d have one and the same right sine. Therefore as the sine of a b is to the sine of a c, so is the sine of the angle a c d to the sine of the angle a b c. By the same reasoning you will show that as the sine of a c stands to the sine of a e, so does the sine of the angle a e f stand to the sine of the angle a c e. And since in the two triangles b a c and c a e the interior angles are assumed equal to one another, the two exterior angles a c d and a e f are equal to each other as well. Therefore as the sine of the angle a c d is to the sine of the angle a b c, so is the sine of the angle a e f to the sine of the angle a c e.

Diagram. A spherical construction drawn as a fan. The point a stands at the upper right, and from it five curved lines (the meridians) sweep down to the left and downward. Along the lower boundary a broken line runs from b at the lower left up toward the right through the successive vertices c, e, g, i; each of its straight portions is prolonged beyond the following vertex to a further lettered point.

- a (upper right, apex of the fan): the pole of the world; all the meridian arcs meet here.

- b (lower left, on the base line): the starting point on the equinoctial; a b is a quadrant of the meridian.
- c (on the base line, right of b): end of the first rhumb segment b c.
- d (on the base line, right of c): the point to which b c is produced.
- e (on the base line, right of c): end of the second segment c e.
- f (right of e, level with the base line): the point to which c e is produced.
- g (above and right of e): end of the third segment e g.
- h (upper right, above f): the point to which e g is produced.
- i [?] (small mark between g and h): end of the fourth segment g i.
- K (upper right, above h): the point to which g i is produced.

And accordingly, as the sine of the segment a b is to the sine of the segment a c, so is the sine of that segment a c to the sine of the segment a e. In the same way you will show that as the sine of a c is to the sine of a e, so is the sine of a e to the sine of a g, and that the sine of a g stands to the sine of a i in the same ratio. So it is plain that the meridian segments which run to the points b, c, e, g, i have their right sines proportional under one and the same ratio — the ratio which the sine of the angle a c d keeps to the sine of the angle a b c.

This being so, it will not be difficult to make those meridian segments themselves known. We shall multiply the whole sine — which is the sine of the quadrant a b — by the sine of the angle a b c of the given rhumb, adding five ciphers [*ziphrae*, zeros]; then divide the product by the sine of the angle a c d, and the quotient will give the sine of the segment a c. This we shall multiply by itself, and divide the product by the whole sine, casting away the five last figures; and the right sine of the segment a e will come out. This in turn we shall multiply by the sine of the segment a c, and divide the product by the whole sine in the manner just described; the division will give the sine of the segment a g. Finally we shall multiply the sine of the segment a g itself by the sine of a c, then divide the product by the whole sine, and the sine of the segment a i will come out. And so with the rest. For once the right sine of a c is known, it will be the common multiplier for finding the sines of the remaining segments, while the whole sine will be the common divisor. Once the sines are known, the required arcs will be disclosed by the table of right sines. These arcs are the meridian segments between the pole of the world and the ends of the segments that constitute the given rhumb.

Diagram. A small spherical triangle figure at the head of the right column. The point a stands at the top; two great-circle arcs descend from it, one curving down to the left to b, the other running down to the right to d. A short line runs along

the bottom from b to e, and from e a line rises to c, which lies on the line drawn from b through c to d.

- a (top): the pole of the world nearer to the rhumb.
- b (lower left): the beginning of the rhumb segment b c, on the equinoctial; a b is a quadrant of the meridian.
- c (middle, on the line b–d): the end of the rhumb segment b c; a c is less than a quadrant.
- d (right): the point where the great circle drawn from the pole a cuts b c produced at right angles.
- e (bottom, right of b): the point on the equinoctial to which a c is produced; the angle b e c is right.

Next it will be worth the trouble to measure the lengths of the segments that constitute the given rhumb, and to determine the angles subtended by those same segments at the poles of the world — which does call for longer chains of reasoning. Let b c be the first arc of the given rhumb, beginning at the equinoctial; let the point a be the nearer pole of the world; let a b be a quadrant of a meridian, and a c less than a quadrant. To make b c and the angle b a c known numerically, we shall produce the segment a c as far as e, a point on the equinoctial; and there c e makes a right angle b e c with b e, a segment of the equinoctial.

So in the triangle c e b the angle e b c is known: it is what remains when the angle a b c of the given rhumb is subtracted from the right angle a b e. The side c e subtending that angle is known as well, because a c — the complement of the quadrant — has already been found. Therefore the remaining sides of the triangle, b c and b e, will be known. For as the whole sine is to the sine of the angle e b c, so is the sine of the side b c to the sine of the side c e. Since, then, of the four proportional quantities the first, second and fourth are known, the third cannot remain unknown; and once the sine is known, the arc is disclosed at once. Furthermore, as the whole sine is to the sine of the complement of c e, so is the sine of the complement of b e to the sine of the complement of b c. By the rule of proportional numbers, then, the right sine of the complement of the arc b e will be disclosed; and for that reason the arc b e itself, which subtends the angle b a c, can be known immediately.

By this method, then, you will find the length of the first segment of the given rhumb, the segment that takes its start from the equinoctial; and the difference of the meridians that pass through its ends; and the arc of the equinoctial that

answers to it. But now let us suppose that b c is a segment of the given rhumb, though not the first, and that its length has to be measured, along with the difference of the meridians between its ends. From the pole a, then, let a great circle be drawn cutting the segment b c, produced further, at right angles at d. In the right triangle a d b, the acute angle a b d is assumed known, while a b — the meridian segment between the pole and the beginning of the arc b c — has already been found. Therefore, by the same reasoning we used a little earlier, the right sines a d and b d will be found, and the sides themselves will be known from the table of right sines. Similarly, in the triangle a c d, since the side a d is known and the meridian segment a c is assumed known, the remaining side c d will be found. When c d is subtracted from b d, the segment b c of the given rhumb must necessarily be left known. From this, by one further step of reasoning, you will at once conclude that the angle b a c is known — the angle contained by the two meridians a b,

a c, which defines the difference of longitude between b and c. For in the triangle b c a, as the sine of the side a c is to the sine of the side b c, so is the sine of the angle a b c to the sine of the angle b a c, by the 13th proposition of the first book of Geber. If, then, you multiply the sine of the angle a b c by the sine of the side b c — that is, the sine of the angle of the given rhumb by the sine of the proposed segment — and divide the product by the sine of a c, you will find in the quotient the sine of the angle b a c. And that angle is acute, because the whole angle b a d is acute. So by the table of right sines the arc belonging to the angle b a c will come out known. But if, for ease of working, you would rather have the whole sine always enter in, then what we conclude as known in four steps of reasoning will have to be brought out in five; and for that we shall use the fourteenth proposition of the first book of Geber.

With these matters demonstrated in this way, a table of numbers will have to be drawn up, divided into seven columns, each column into three spaces. The first column will be that of the first rhumb — or rather of the first quarter-point of the rhumb — which in the common tongue we call *Norte quarta de Nordeste*, with its opposite *Sul quarta de Sudoeste*; and on the other side *Norte quarta de Noroeste*, *Sul quarta de Sueste*. The first space of this column contains the arcs of the meridian that terminate at the ends of the segments of the given rhumb. The second space comprises the travelling lengths of the segments of that same rhumb — that is, it shows how great each single segment of the rhumb is. In the third, the differences of longitude between the ends of each segment of the same rhumb must be written. The second column, divided in the same way into three spaces, will contain the numbers belonging to the middle course, which they call

Nornordeste / *Sufudoeste*, and on the other side *Nornoroeste* / *Sufueste*. In the third are to be placed the numbers of the third quarter-point, which they call *Nordeste quarta de Norte* and *Noroeste quarta de Norte*, with their opposites. The remaining columns are to be drawn up by the same method. On the left-hand side of the table the number and order of the segments of each rhumb must be written out one by one.

So in the first column the angle of departure of the first quarter-point contains 11 degrees, 15 minutes. In the second, which is that of the middle courses, it takes in 22 degrees, 30 minutes. In the third the angle of departure is 33 degrees, 45 minutes. In the fourth, 45 degrees. In the fifth, 56 degrees, 15 minutes. In the sixth, 67 degrees, 30 minutes. In the seventh and last, 78 degrees, 45 minutes. Each column, then, is to be filled with its proper numbers, answering to the numbers in the side margin.

One point must be noted, however: the meridian arc set in the first space is the one that comes to the *end* of the segment, not the one that comes to its beginning. For since the description of every rhumb must take its start from the equinoctial circle, the meridian arc reaching the beginning of the first segment is a quadrant. So the number of degrees and minutes written in the first space will be that of the meridian arc which terminates at the end of the first segment, and so with the rest. And since the beginning of the following segment is the end of the preceding one, if the later arcs are known the earlier ones cannot remain unknown.

¶ *There follows the layout of the table, divided into seven parts. As for the numbers that are to be written inside the area of the table, diligent young men will find them out for themselves according to the preceding demonstrations, and will extend them as far as they please.*

[The remainder of the page is blank; the ruled table itself begins on the following leaf.]

[Printed page 110]

The table of the rhumbs

The whole leaf is given over to the ruled table announced at the end of Cap. 23. It is printed sideways, turned a quarter-turn on the leaf, so that the seven columns run the long way of the page; the page number stands upright in the outer corner in the ordinary way.

Head of the left-hand margin: *Numerus & ordo segmento-|rum in quolibet rumbo.* — "The number and order of the segments in any given rhumb."

The margin is ruled off into nineteen numbered spaces, 1 to 19, running down the side of the grid.

The seven column headings, verbatim:

- **I.** *Prima quarta norte quarta de nordeste, norte quarta de noroeste cū oppositis.* — First quarter-point: *norte quarta de nordeste*, *norte quarta de noroeste*, with their opposites.
- **II.** *Secunda quarta nornordeste, nornoroeste cum oppositis.* — Second quarter-point: *nornordeste*, *nornoroeste*, with their opposites.
- **III.** *Tertia quarta nordeste quarta de norte, noroeste quarta de norte cū oppositis.* — Third quarter-point: *nordeste quarta de norte*, *noroeste quarta de norte*, with their opposites.
- **IV.** *Quarta quarta, nordeste, noroeste, cum oppositis.* — Fourth quarter-point: *nordeste*, *noroeste*, with their opposites.
- **V.** *Quinta quarta, nordeste quarta d'leste, noroeste quarta d'oeste cum oppositis.* — Fifth quarter-point: *nordeste quarta de leste*, *noroeste quarta de oeste*, with their opposites.
- **VI.** *Sexta quarta lesnordeste, oesnoroeste, cum oppositis.* — Sixth quarter-point: *lesnordeste*, *oesnoroeste*, with their opposites.
- **VII.** *Septima quarta, leste quarta de nordeste, oeste quarta de noroeste cū oppositis.* — Seventh quarter-point: *leste quarta de nordeste*, *oeste quarta de noroeste*, with their opposites.

Each of the seven columns is ruled into three spaces, headed in turn:

1. *arcus meridiani.* — the arc of the meridian
2. *longitudo itineris.* — the travelling length of the segment
3. *differentia longitudinis.* — the difference of longitude

and every one of the twenty-one spaces carries the sub-heading *Gr.* | *m̄* (degrees, minutes).

The interior of the grid is blank. Nothing whatever is inked in the twenty-one spaces on any of the nineteen lines: the sheet carries only the marginal numbers, the headings, and the column rules. This is exactly what the closing note of Cap. 23 promised — the values were deliberately left for the reader to compute.

I·	I·	I·	II	II·	II	II·	III·	III	IV·	IV	IV·	V	V·	V	V	V	V	V	VII
ari	lon	diff	·	lon	·	lon	diff	·	lon	diff	·	diff	·	diff	I·	diff	·	lon	·
rcu	gitu	eren	diff	gitu	ara	gitu	eren	ara	gitu	eren	ara	eren	ara	eren	lon	eren	ara	gitu	diff
sme	doit	tial	eren	doit	ric	doit	tial	ric	doit	tial	ric	tial	ric	tial	gitu	tial	ric	doit	eren
ridi	inud	osmo	tial	inud	udi	inud	osmo	udi	inud	osmo	udi	osmo	udi	osmo	doit	osmo	udi	inud	tial
ani	erid	nit	osmo	erid	ani	erid	nit	ani	erid	nit	ani	nit	ani	nit	inud	nit	ani	erid	osmo
ni	nis	i	nit	nis	ni	nis	i	ni	nis	i	ni	i	ni	i	erid	i	ni	nis	nit
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r.	r.	r.	(G	r.	r.	r.	r.	r.	r.	r.	r.	r.	r.	r.	(G	r.	r.	r.	(G
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					II			II			IV			V			V			V		
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		di			di			di		I	di			di			di		V	di		
		ff		II	ff		II	ff	I	V	ff		V	ff		V	ff	V	II	ff		
	I·	e	II	·	e	II	I·	e	V	·	e	V	·	e	V	I·	e	II	·	e		
I·	lo	r	·	lo	r	I·	lo	r	·	lo	r	·	lo	r	I·	lo	r	·	lo	r		
a	n	e	a	n	e	a	n	e	a	n	e	a	n	e	a	n	e	a	n	e		
r	gi	n	r	gi	n	r	gi	n	r	gi	n	r	gi	n	r	gi	n	r	gi	n		
c	t	ti	c	t	ti	c	t	ti	c	t	ti	c	t	ti	c	t	ti	c	t	ti		
u	u	a	u	u	a	u	u	a	u	u	a	u	u	a	u	u	a	u	u	a		
s	d	lo	s	d	lo	s	d	lo	s	d	lo	s	d	lo	s	d	lo	s	d	lo		
m	o	n	m	o	n	m	o	n	m	o	n	m	o	n	m	o	n	m	o	n		
e	it	gi	e	it	gi	e	it	gi	e	it	gi	e	it	gi	e	it	gi	e	it	gi		
ri	i	t	ri	i	t	ri	i	t	ri	i	t	ri	i	t	ri	i	t	ri	i	t		
di	n	u	di	n	u	di	n	u	di	n	u	di	n	u	di	n	u	di	n	u		
a	e	di	a	e	di	a	e	di	a	e	di	a	e	di	a	e	di	a	e	di		
n	ri	n	n	ri	n	n	ri	n	n	ri	n	n	ri	n	n	ri	n	n	ri	n		
i	s	is	i	s	is	i	s	is	i	s	is	i	s	is	i	s	is	i	s	is		
(	(	(	(	(	(	(	(	(	(	(	(	(	(	(	(	(	(	(	(	(		
G	G	G	G	G	G	G	G	G	G	G	G	G	G	G	G	G	G	G	G	G		
r.	r.	r.	r.	r.	r.	r.	r.	r.	r.	r.	r.	r.	r.	r.	r.	r.	r.	r.	r.	r.		
m̄	m̄	m̄	m̄	m̄	m̄	m̄	m̄	m̄	m̄	m̄	m̄	m̄	m̄	m̄	m̄	m̄	m̄	m̄	m̄	m̄		
)	)	)	)	)	)	)	)	)	)	)	)	)	)	)	)	)	)	)	)	)		

[Printed page 111]

¶ *On the relation of the rhumbs both to the poles of the world and to one another.*

Cap. 24.

The rhumbs of North and South are meridians, and so they pass through the poles of the equinoctial — the poles we call the poles of the world. Those of *Leste* and *Oeste*, by contrast, are parallels, of which the greatest is the equinoctial itself, and these cannot pass through those poles. All the rest are understood to be built up out of segments of great circles that meet the meridians at acute angles. They therefore neither pass through the poles of the world nor keep an equal distance from them; they can be extended without limit, and the further they are extended the nearer they come to the poles — but enter them they cannot.

For just as, in the figure drawn above, the rhumb b c e g makes acute and equal angles with the meridians at the ends of its segments — the angles at the intermediate points being unequal, though the difference is imperceptible — so it can be prolonged from the point g to i, and from i again, and so on as far as one pleases. For at any point at all of a given meridian an angle equal to the given angle of the rhumb conceived can be made by that meridian together with some other great circle. Moreover, from what we have demonstrated it is clear enough that the segments of the meridians grow continually smaller. So the further the rhumb is extended, the nearer it will come to that same pole; yet it will never be able to enter it.

Suppose it does enter. Since we have set a as the nearer pole of the world, let that entry be along the segment l m a of the given rhumb, and through a and l let the meridian a n l be drawn. Then two great circles will cut one another into unequal parts at the very points a and l — which is impossible, since a m l and a n l are segments smaller than semicircles. But if you insist that the meridian drawn through the points a and l is the very circle that has l m a as the given rhumb's segment, then l m a will itself be a rhumb of North and South, contrary to the hypothesis. And so a rhumb cannot possibly enter the poles of the world — which was to be demonstrated.

Diagram. At the foot of the left column, an uncaptioned woodcut showing a single long curve — the rhumb — sweeping from the lower left round to the upper right and closing on a point at the top, with a short subsidiary arc (the meridian) meeting it there. The lettered points, in order along the curve from its lower-left end:

- b (far left, on the curve): the starting point of the rhumb
- c (left, just inside b): end of the first segment
- e (lower middle): end of a further segment
- g (lower right of centre): the point named in the text as the one the rhumb is prolonged from

- i (right, rising): the point the rhumb is prolonged to
- l (upper right, on the curve): the point through which the meridian a n l is drawn
- m (upper right, between l and a): the intermediate point of the segment l m a
- a (top, at the convergence of the curve and the short arc): the nearer pole of the world

The point n, which the text requires for the meridian a n l, is not legible among the engraved letters [?].

The rhumbs that share one and the same name will stand in this relation to one another: they enclose proportional segments of the equinoctial and of the parallels, exactly as the meridians do. The further they are extended, the nearer they come to one another; yet they can never meet. Extended rhumbs are said to approach one another more closely when, at a greater distance from the equinoctial, they intercept a smaller space between the points of the parallels.

Let us conceive two rhumbs of one name, b c d e f and g h i K l, beginning from the points b and g on the equinoctial, and let quadrants of meridians be drawn through the ends of the segments of those rhumbs, as appears in the figure below. Let the first segments be b c and g h. In the two triangles a b c and a g h, the angle c b a is assumed equal to the angle h g a, and the angle a c b likewise equal to the angle a h g, while the side a b is equal to the side a g — for both are quadrants. The remaining sides, then, being less than quadrants, will be equal to one another, and the remaining angles equal, by Menelaus, *Spherics* I.16. So the meridian segment a c will be equal to the meridian segment a h.

Now let a parallel to the equinoctial be drawn through c and h, and let its segment be c h. I say that b g and c h, the segments of the equinoctial and of the parallel, are similar — that is, proportional: as the equinoctial is to the parallel, so is b g to c h. For since the two angles b a c and g a h have been shown equal, the arcs of the equinoctial b m and g n will be equal to one another; and if we add g m to each, then b g and m n will be equal, by the common notion. Therefore, as b g is to c h, so is m n to that same segment c h. But the two arcs m n and c h are themselves similar, that is proportional, by Theodosius, *Spherics* II.14. So b g and c h will be proportional. This you could in fact demonstrate by proposition 17 of that same second book of Theodosius alone.

[Printed page 112]

Diagram. A large uncaptioned woodcut filling the upper half of the leaf and spanning both columns. A set of concentric arcs — the equinoctial outermost and four parallels within it — is drawn about a centre a, which stands for the pole; from a a fan of straight radii (the meridian quadrants) runs down to the outermost arc, and two curves cross the radii obliquely, spiralling in toward a: these are the two rhumbs of one name. Readable letters, by position:

- a (centre of the fan, at the top): the pole
- b (far left, on the outermost arc): the start of the first rhumb
- m (below and left of b, on the outermost arc): the foot of the meridian through c
- c (left of centre, on the first parallel in from the outermost)
- e (centre of the figure, on an inner parallel)
- f (right of centre, on the innermost drawn parallel)
- h (right of centre, below e)
- i (right edge, on the outer part of the fan)
- o, y, g (along the bottom arc, reading left to right)
- n (bottom, right of centre)
- p (bottom right, on the outermost arc)

Four further engraved letters could not be resolved: the mark standing between g and n along the bottom arc [?]; the mark standing between n and p [?] (the argument requires x there); a letter on the right-hand side at mid-height, read as either r or z [?]; and the letter set between c and e on the upper rhumb, read as d [?]. The points d, u, z, K and l named in the demonstration cannot all be matched to legible engraved letters on the leaf.

The same you will demonstrate in the same way of the segments of the remaining parallels enclosed between those same rhumbs. For since in the two triangles a c d and a h i the sides a c and a h have been shown equal, and the angles standing on the bases c d and h i are put as equal, the remaining angles at a will be equal, and the remaining sides too — being less than quadrants — will be equal; and so a d and a i will be equal. In the same way the two segments of the equinoctial m o and n p will be equal to one another, and so the whole b o will be equal to the whole g p, by the common notion that adding equals to equals gives equals. To each we shall add g o; and therefore b g and o p will be equal.

Further, let the segment of the parallel drawn through d and i be d i. Then o p and d i, enclosed by the meridians a o and a p, will be proportional; and so b g and d i will be proportional as well. That the two segments c h and d i are

proportional to their own circles you will conclude by equality of ratio [*per aequam proportionem*], with b g interposed.

But suppose the segment u z belongs to the parallel drawn through the points that lie above d and i and below c and h: we shall show none the less that b g and u z are similar segments. For if from the point a the quadrants a y and a x are drawn through the points u and z themselves, the two sides a d and a u of the triangle a d u will be equal to the two sides a i and a z of the triangle a i z. The acute angle u d a is equal to the angle z i a, and the two angles a u d and a z i are obtuse, by what we demonstrated above; therefore the two remaining angles d a u and i a z will be equal. So the two segments o i[?] and p x will be equal to one another; and from this you will conclude that b y and g x are equal, and therefore that b g and y x are equal to one another, by the common notion. But y x and u z are similar; so b g and u z will be similar too. You will conclude, then, that it is entirely true in general that the segments of the parallels between rhumbs of one name — that is, of the same inclination — are proportional, which is what we undertook to demonstrate.

Now we shall show that the further they are extended, the closer they come to one another. Since the arcs of the parallels enclosed between the rhumbs b f and g l are proportional to those circles themselves, the straight lines subtending those arcs will be proportional to the radii of those same circles; and this you can easily demonstrate by Euclid, Book VI. So the straight line subtending the circumference b g will be greater than the line subtending c h, and this in turn greater than the line subtending d i, and so on. Accordingly, if you conceive a great circle drawn through the points c and h, and likewise a great circle through d and i, you will conclude that b g is greater than the circumference of the great circle lying between c and h, and that this in turn is greater than the circumference contained between d and i, and so with the rest. And therefore, by the definition we laid down, the further the rhumbs themselves are extended, the closer they will come to one another — which was to be demonstrated.

We know, of course, that the distance between two lines ought to be measured along the perpendiculars drawn from the points of the one to the other. But in broken lines of this kind we judged it better to reckon instead by the intervals of the points inter-

— between corresponding rhumbs on each parallel. For suppose that, of two ships, one sets out from the place *b* to run the distance of the rhumb *b f*, while the other sets out from *g* to run the distance of the corresponding rhumb *g l*, and that both are carried at the same speed. Then it is clear from what we have

demonstrated that, so long as they are borne along in this fashion, they fall upon the same parallels at the same moment, and that the further they advance the closer they come to one another.

That they never meet, however, even if they were produced to infinity, we shall show by a demonstration leading to an impossibility. They will not meet over the poles, since it has been demonstrated that they cannot enter them. So suppose they meet elsewhere. Let the two rhumbs of the same name, *b f* and *g f*, meet at *f*, the end-point of the corresponding segments *e f* and *k f*. The meridian arcs *a e* and *a k* will therefore be equal, by what we demonstrated a little earlier in the preceding figure; and moreover the angles *e a f* and *f a k* will be equal to one another — a part to the whole — which is impossible.

But suppose you say that these corresponding segments meet not at *f* but at some point between *e* and *f*. Let such an intersection then be at *r*, and let *K r* be produced as far as *i*, on the parallel of the point *f*. Now it has been shown that the end-points of corresponding segments are distant by equal intervals from the same pole, and we have set the point *f* as the end of the segment *e f*; the point *i*, therefore, will be the end of the segment *k i*. Let *a i* be the arc of the meridian between the pole *a* and *i*. Then the two angles *e a f* and *i a k* will be equal to one another, which is again impossible. And therefore they do not meet — which was the last thing to be demonstrated.

Diagram. At the foot of the left-hand column: a spherical figure drawn in free curves. From a point *a* at the top, two long lines descend and diverge toward the lower left; two curved arcs sweep across them from the lower left toward the right, crossing them and each other. The readable labels, with their positions:

- *a* (apex, top): the pole of the world.
- *e* [?] (left, at the crossing of the lower arc with the left-hand line).
- *i* (centre, on the right-hand descending line).
- *r* [?] (immediately left of *i*, at a crossing).
- *f* (right of *i*, at the crossing of the upper arc).
- *K* (below centre, at the crossing of the lower arc).
- *b* (lower left, at the start of the lower arc).
- *g* [?] (bottom centre, below and right of *b*).

¶ *What relation the segments of one and the same rhumb bear to one another. Cap. 25.*

The segments of great circles out of which the rhumbs are understood to be composed — those rhumbs, that is, which are neither North-and-South nor *Leste*-and-*Oeste* — stand to one another in this relation: in any one of those rhumbs, begun at the equinoctial and prolonged toward either pole of the world, the segments nearer the equinoctial are greater than the more remote ones, and the differences of longitude and of latitude between the end-points of those same segments are greater as well. So much so that, the further from the equinoctial any rhumb is carried, the smaller its segments become, and the smaller too the differences of longitude and of latitude between their end-points.

For let two segments be conceived in a rhumb begun at the equinoctial and produced toward the pole *a*: *b c*, the nearer to the equinoctial, and *c d*, the more remote; and let the meridian arcs *a b*, *a c* and *a d* be drawn to their ends. I say that *b c* is greater than *c d*; that the difference of longitude between the two points *b* and *c* is greater than the difference of longitude between *c* and *d*; and likewise that the arc *a b* exceeds the arc *a c* by a greater difference than *a c* exceeds *a d*.

The demonstration of the first point will proceed in this way. Since the arc *a c* is less than *a b*, let it be produced on the side of *a*, and let *c e* be equal to that same *a b*. Then from the point *e*, the end of *c e*, let a great circle be drawn which makes with *c e* itself, at that same point *e*, an angle equal to the angle *b a c* — by the first proposition of the first book of Menelaus.

Diagram. At the foot of the right-hand column: a fan of great-circle arcs springing from a vertex at the top of the figure and spreading toward the lower right, cut across by two arcs that run from the lower left to a point at the far right. The readable labels, with their positions:

- *e* [?] (topmost, at the apex of the fan).
- *a* [?] (just below the apex, at the head of the innermost arcs).
- *i* (upper left, on the second arc of the fan).
- *h* [?] (centre right, at a crossing).
- *g* (right of *h*, on the transversal).
- *m* (far right, at the end of the transversal).
- *k* (below centre, at a crossing).
- *f* (below and right of *k*).
- *c* (bottom centre-left, at the foot of an arc).
- *l* (far right, at the lower end of the sweeping arc).

Now a circle of this kind will cut the segment *c d* produced lengthwise, but not at *d*, nor between *c* and *d*. For if it cuts at *d*: since in the two triangles *a b c* and *e c d* the two sides *a b* and *e c* are equal, and the two angles of the one that stand upon those sides *a b* and *e c* are equal, each to each, to the two angles of the other, the remaining angle *a c b* will therefore be equal to the remaining angle *e d c*, by Menelaus, *Spherics* I.4 [?]. The exterior angle *e d g* will consequently be equal to the exterior angle *a c f*. But *a d g* is equal to that same exterior angle *a c f*, because we assumed that the angle *a c f* exceeds the angle *a b c* by just as great a difference as the angle *a d g* exceeds *a c d*, which is equal to *a b c*. The angle *e d g*, then, would be equal to the angle *a d g* — a part to the whole — which is impossible. And therefore it does not cut at *d*.

Nor will it be able to cut between *c* and *d*. For if it does cut between *c* and *d*, let such an intersection be at *k*. Then by similar arguments we shall show that the two angles *e k g* and *a d g* are equal to one another. Let the arc *e K* cut the arc *a d* at *i*. In the triangle *K d i*, then, the exterior angle *i d g* will be equal to the interior and opposite angle *i K d*. Therefore the two sides *d i* and *K i* taken together will be equal to a semicircle. But *d i* is far less than a quadrant, since the whole arc *a d* is less than a quadrant; and *K i* likewise is far less than a quadrant, since the arc *e k*, being equal to *a c*, is less than a quadrant. This is therefore impossible.

And so it does not cut between *c* and *d*. Let it then cut at *g*. In the triangle *e c g*, the side *c g* will be equal to the side *b c* of the triangle *a b c*. But *c d* is less than *c g*; therefore this same *c d* will be less than *b c* — which was the first thing to be shown.

The second point will be demonstrated in the same figure. Let the two arcs *a d* and *e g*, produced on the sides of *d* and *g*, meet at *l*, and let *d g* be produced to *m* on the side of *g*. The two angles *e g m* and *a d m* will then be equal to the angle *a c f*, and therefore equal to one another, by the common notion. It follows necessarily that the two angles *m g l* and *m d l* are equal; and therefore that in the triangle *d g l* the two sides *g l* and *d l* together will be equal to a semicircle, and consequently that in the triangle *e a l* the two sides *a l* and *e l* together will be greater than a semicircle. The exterior angle *c a l* will therefore be less than the interior and opposite angle *a e l*. But *b a c* is equal to that same *a e l*; therefore *c a d*, that is *c a l*, will be less than that same angle *b a c*. Now the angle *c a d* marks out on the equinoctial circle the amount of the difference of longitude between *c* and *d*, while the angle *b a c* marks out the amount of the difference of longitude between *b* and *c*. Therefore the difference of longitude between *b* and *c* is greater than the difference between *c* and *d* — which was to be shown.

The last point, moreover, we shall show by an easy demonstration. Let parallels be drawn through *c* and *d*. Since the arc *a b* is greater than *a c*, and *a c* itself greater than *a d*, the parallels so described will cut the meridians *a b* and *a c*. Let them cut at *n* and *o*. Then *b n* will be the difference of latitude between *b* and *c*, and *c o* the difference of latitude between *c* and *d*. I say, then, that the difference *b n* is greater than the difference *c o*.

For since we have demonstrated that the segment *b c* is greater than *c d*, let there be taken from that same *b c* a segment *b p* equal to *c d*, and from *a b* an arc *b r* equal to the arc *a c*.

Diagram. In the right-hand column: a curvilinear triangle with its apex at *a* (upper right) and its base running along the bottom from *b* (lower left) to *d* (right). Several arcs descend from *a* to the base, and a transversal crosses them. The readable labels, with their positions:

- *a* (apex, upper right): the pole of the world.
- *r* (upper left, on the leftmost descending arc).
- *n* (left, below *r*).
- *o* (centre, on an inner arc).
- *q* (far left, on the outer arc).
- *d* (right, on the base).
- *b* (lower left, at the foot of the outer arc).
- *p* (bottom centre, on the base).
- *c* (bottom centre, immediately right of *p*).

Then let a great circle be drawn through the two points *p* and *r*, and another great circle through *a* and *p*. In the triangle *a p r*, therefore, the two sides *a r* and *r p* together are greater than the third side *a p*. But *a p* is greater than *a c*, because in the triangle *c a p* the angle *a c p* is obtuse while *a p c* is acute. And so those two sides *a r* and *r p* together are far greater than *a c*. But the meridian segment *a n* is equal to that same *a c*; therefore *a r* and *r p* are greater than *a n*. Take away the common *a r*: what is left, *r p*, is accordingly greater than *r n*, by the common notion. And since *r p* and *a d* are equal to one another — by a proposition like the fourth of the first book of Euclid — while *a d* is less than *a c*, *r p* will therefore be less than *b r*.

So if we understand the point *r* as a pole, and describe a circle through the point *p* with the interval *r p*, it will cut *b r* between *b* and *n*. Let it cut at *q*. Then the meridian segment *b q* will be equal to the segment *c o*. But *b q* is less than *b n*; therefore *c o* too will be less than that same *b n*. So the difference of latitude

between *c* and *d* will be less than the difference of latitude between *b* and *c* — which was to be shown. And it makes no difference whether *b c* and *c d* are taken as adjoining or as separate.

But if you wish to compare with one another the segments of different rhumbs, you will easily be able to show, from what has been demonstrated in this place and earlier, that the difference of latitude between the end-points of the first segment of the first rhumb — that is, of the first quarter-point, whose inclination to the meridian is 11 degrees 15 minutes — is greater than the difference of latitude of the end-points of any other segment whatever, whether of that first rhumb itself or of the others; and that the difference of longitude between the end-points of the first segment of the seventh rhumb — that is, of seven quarter-points — is greater than that of any other.

For let the point *a* be the pole of the world, and let the arc *b c* be the first segment of the first rhumb, that is of the first quarter-point, begun at the point *b* of the equinoctial; and let the arc *d e* be the first segment of any other rhumb, whose beginning *d* is likewise a point of the equinoctial. We shall show first that the difference of latitude between *b* and *c* — or rather the latitude of *c* itself — is greater than the difference of latitude between *d* and *e*; but that the difference of longitude between those same *b* and *c* is less than the difference of longitude between *d* and *e*.

Let the quadrants *a b*, *a c f*, *a d* and *a e g* be drawn, and let *b c* be produced to *h* and *d e* to *i*. Then the angle *a c h* will exceed the angle *a b c* by one degree, by what we have assumed; and likewise the angle *a e i* will exceed the angle *a d e* by one degree. But the two angles *a c h* and *a b c* are less than the two angles *a e i* [?] and *a d e*, by hypothesis. For the angle *a b c* is 11 degrees 15 minutes, since that is the inclination of the first quarter-point to the meridian, while the angle *a d e* is posited greater. Consequently the ratio between the right sines of the arcs of the angles *a c h* and *a b c* will be greater than the ratio between the right sines of the arcs of the angles *a e i* and *a d e*. That is to say, the right sine of the arc of the angle *a c h* has a greater ratio to the right sine of the arc of the angle *a b c* than the right sine of the arc of the angle *a e i* has to the right sine of the arc of the angle *a d e* — by what we demonstrated earlier, in chapter 3 [?], on finding the longitude of places from the sea chart.

Now as the right sine of the angle *a c h* is to the sine of the angle *a b c*, so is the sine of the quadrant *a b* to the sine of the arc *a c*, in the spherical triangle *a c b*; for the exterior and the interior angle at *c* have the same sine. Likewise, as the right sine of the angle *a e i* is to the right sine of the angle *a d e*, so is the sine of

the quadrant *a d* to the sine of the arc *a e*, in the triangle *a e d*. Therefore the sine of the quadrant *a b* will have a greater ratio to the sine of the arc *a c* than the sine of the quadrant *a d* has to the sine of the arc *a e*. And accordingly the arc *a c* will be less than *a e*; so the arc *c f*, the difference of latitude between *b* and *c*, is left greater than *e g*, the difference of latitude between *d* and *e*.

Now the difference of latitude between *b* and *c* has been shown to be greater than the difference of latitude of the end-points of any other segment of the same rhumb; and likewise the difference of latitude between *d* and *e* is greater than the difference of latitude of the end-points of the other segments of the rhumb begun at the point *d*, whose inclination *a d e* is supposed greater than the inclination *a b c*. Therefore the difference of latitude between the end-points of the first segment of the first rhumb, that is of the first quarter-point, is greater than the difference of latitude between the end-points of all the remaining segments, whether of that first rhumb itself or of the others — which was to be shown.

Diagram. At the foot of the left-hand column, printed over heavy show-through from the reverse of the leaf: a fan of arcs descending from a vertex at the top to a curved base that runs from the lower left to the right, with a transversal arc cutting the fan. The readable labels, with their positions:

- *a* (apex, top; the ink is blotted): the pole of the world.
- *b* [?] (centre, on an inner arc).
- *r* [?] (left of *b*).
- *e* (centre right, at a crossing).
- *i* [?] (far right, at the head of the outer arc).
- *b* (lower left, on the base).
- *f* (bottom centre, on the base).
- *d* [?] (bottom centre-right, on the base).
- *g* (right, at the lower end of the outer arc).

The remaining point we shall demonstrate by the same method. Since the angle *a c h* is equal to its vertical angle *b c f*, and the angle *a e i* likewise equal to its vertical angle *d e g*, by hypothesis the angle *d e g* will exceed *a d e* by just as much as the angle *b c f* exceeds *a b c*.

Conversely, then, by as much as the complement of the angle *a b c*, namely *c b f*, exceeds the complement of the angle *b c f*, by so much will the complement of the angle *a d e*, namely *e d g*, exceed the complement of the angle *d e g*; for this has been demonstrated in the arithmetical books [*in Arithmeticis*]. But the angle *e d g* is less than the angle *c b f*, and the complement of the angle *d e g* is less than the

complement of the angle *b c f*. Therefore the sine of the angle *e d g* will have a greater ratio to the sine of the complement of *d e g* than the sine of the angle *c b f* has to the sine of the complement of the angle *b c f*.

And since, as the whole sine is to the sine of the complement of *b f*, so is the sine of the angle *c b f* [?] to the sine of the complement of *b c f*; and likewise in the triangle *d g e* [?], as the whole sine is to the sine of the complement of *d g*, so is the sine of the angle *e d g* to the sine of the comple-

— ment of *d e*. The whole sine will therefore have a greater ratio to the sine of the complement of *d g* than to the sine of the complement of *b f*; and for that reason the complement of *d g* will be less than the complement of *b f*, and consequently the arc *d g* will be left greater than *b f* itself. Let us then set *d e* as the first segment of the seventh rhumb, that is of seven quarter-points, whose inclination to the meridian is 78 degrees 45 minutes, and *b c* as the first segment of any other rhumb whatever; and we shall conclude, as before, that *d g* is the greatest difference of longitude.

¶ *To draw the rhumbs on a proposed globe. Cap. 26.*

Let the proposed globe be set within a movable meridian, one semicircle of which is cut by the poles into two quadrants. Let the quadrants be divided into 90 degrees, with the proper numbers written against them, beginning at the poles themselves and ending at the section made by the equinoctial. The equinoctial circle itself is to be divided into degrees in the same way, but these are to be marked off by points and lines only, without numerals, as appears in the figure below.

In that figure *a b c d* represents the inner circle of the surface of the movable meridian, or circular ring, which passes through the poles of the world — *a* the northern and *c* the southern. Let the point *b* be one of the two intersections of that circle with the equinoctial, and *d* the other. On the globe as proposed, then, the semicircle *b d* is one half of the equinoctial, while *a b* and *b c* are the two quadrants of the meridian. Let those quadrants be divided into degrees, beginning at *a* and *c* and ending at *b*, where the number 90 is written. The equinoctial is to be divided into 360 degrees — that is, the semicircle *b d* into 180, and the other semicircle left on the opposite side likewise into 180. The degrees of the equinoctial are to be distinguished by points and lines, but no numerals are to be written against them.

Since, on the present plan, all the rhumbs are to be drawn from the equinoctial, let the point *b* be the beginning of one description; and let there first be

described, toward the right-hand side (which we take to be the north-east), the rhumb commonly called *Norte quarta de Nordeste*, in this way. The number of degrees and minutes of the difference of longitude found in the body of the table given above, opposite the first segment, we shall count off from *b* toward *d* along the equinoctial circle. Let the end of that count be the point *e*. We then shift the semicircle *a b c* of the movable meridian to the position *a e c*, and in it count from *a* toward *e* the number of degrees and minutes written in the same table opposite that first segment under the heading "arc of the meridian"; and we mark the end on the surface of the globe with the letter *f*. From the same table we next take the number of degrees and minutes of the difference of longitude and of the arc of the meridian opposite the second segment, and count that difference of longitude on the equi-

Diagram. Occupying the foot of the left-hand column and running under the right-hand column: a globe enclosed by the graduated ring of the movable meridian. The ring is drawn as a broad band divided into cells, each cell carrying a degree number; the numbering runs 10, 20, 30 … up to 90, the 90 standing at the point where the equinoctial meets the ring. A straight graduated line, ticked but unnumbered, crosses the globe horizontally — the equinoctial. A separate curved band lies outside the globe at the upper right, detached from the ring; it carries an engraved caption. The readable labels, with their positions:

- *a* (top of the globe): the north pole.
- *c* (bottom of the globe): the south pole.
- *b* [?] (left-hand end of the horizontal graduated line, where the 90 of both quadrants stands; the letter itself will not resolve on the scan).
- *d* (right-hand end of the horizontal graduated line).
- *e* (on the horizontal line, a short way in from its left-hand end).
- *g* (on the horizontal line, to the right of *e*).
- *f* (left of centre, above the horizontal line).
- *h* [?] (above and to the right of *f*).
- *Sphæricum regulamentum* ("the spherical ruler") — engraved along the detached curved band at the upper right.

— noctial from *e* toward *d*. To the end of it, say *g*, we shift the semicircle of the movable meridian, setting it in the position *a g c*; in it we count (as before) the number of degrees and minutes of the arc of the meridian from *a* toward *g*, and mark the end on the surface of the globe with the letter *h*. So we must go on, imprinting on the globe by the same procedure as many points as there are segments written in the table; for it is established from what we demonstrated

above that the table itself can be extended indefinitely. Still, it will be enough to carry it as far as a latitude of about 60 degrees — that is, until the arc of the meridian on every rhumb takes in nearly 30 [?] degrees.

Once these points have been marked on the given globe, we shall prepare another circular ring equal to the movable meridian, and cut from it a segment spanning a number of degrees not less than the greatest found anywhere in the whole table of rhumbs under the heading "travelling distance". We shall use this segment of the ring for drawing the arc of a great circle from point to point on the surface of the globe, exactly as we use flat rulers for drawing a straight line from one point to another in a plane. So with this spherical ruler properly fitted to the points *b* and *f*, we shall draw the arc of a great circle *b f*; and from the point *f* to the point *h* we shall draw the arc *f h* by the same means; and in the same way we shall join every other one of the points imprinted on the globe to its neighbour, until at last we have the rhumb described that they call *Norte quarta de Nordeste*.

Next we shall take from the table given above the first and third numbers of the second column, and by the same procedure draw that rhumb from the same point *b*, taking the beginning that belongs to the middle course. The remaining rhumbs are to be drawn across the convexity of the globe into the northern and eastern quarter in just the same way. Then, starting again from that same point *b*, we shall draw toward the northern and western quarter those rhumbs whose inclinations to the meridians are equal to these. In common speech they are called *Norte quarta de Noroeste*, *Nornoroeste*, *Noroeste quarta de norte*, *Noroeste*, *Noroeste quarta de Oeste*, *Oesnoroeste*, *Oeste quarta de Noroeste*. When that description is finished we shall make another description of the rhumbs, taking its beginning from the point *d*; and besides these, two more from the points midway between *b* and *d* — this on globes of middling size.

On larger globes not four descriptions only but eight, or more, are to be made; for the more there are, the readier the route that will be found for any given course. When the descriptions of the rhumbs for the northern hemisphere have been completed, we shall make just as many again in the southern, likewise begun from the same points of the equinoctial. The meridians drawn through those points are to be in black, and the equinoctial itself as well; and so too are those rhumbs that lie midway between them, namely the *Nordestes* and *Sudoestes*, the *Noroestes* and *Suestos* [?]. The rhumbs of the intermediate courses are to be painted green, and the rest red, as on the mariners' planisphere. Circles parallel to the equinoctial may besides be drawn in whatever number and at whatever

spacing one pleases, but in black, since they commonly serve for the rhumbs of *Leste* and *Oeste*.

The rhumbs can also be described in certain other ways. One of these is as follows: about *b* as pole, and with an interval equal to the first segment of the given rhumb, let the circumference of a small circle be described on the surface of the globe. Then the semicircle of the movable meridian is to be set in the position *a e c*; and since it cuts that circumference twice, the more northerly of the two sections is to be taken as the end of the first segment. Very like this is another way: neglecting the difference of longitude between the ends of the first segment, carry the semicircle of the movable meridian round until the end of the arc of the meridian belonging to that first segment reaches the circumference of the small circle. Where it reaches it, there will be the other extremity of that same first segment; and the points of the remaining segments are to be marked by the same means.

A most apt method will be this one as well. Out of a thin plate of copper or iron, or of some other material, make a spherical quadrant *K l m* whose concave face is shaped to the convex surface of the globe as proposed. Let its sides *K m* and *l m*, which contain the right angle *K m l*, be a little longer than the largest rhumb segment found in the table given above, and let them be divided into degrees of a great circle of the given globe. Let the circumference *K l* be cut into eight equal parts, and from the point *m* let arcs of great circles be drawn to the points of division: *m n*, *m o*, *m p*, *m q*, *m r*, *m s*, *m t*. The acute angle *l m t* will then be of one quarter-point; *l m s* of two quarter-points; *l m r* of three;

Diagram. At the head of the left-hand column: the spherical quadrant, drawn as a fan. All the lines spring from a single vertex at the bottom and spread upward and to the right, closed off by an outer arc that runs from the upper left down to the right. Each of the radiating lines carries a number, 1 to 8, the 8 standing next to the left-hand side of the fan and the 1 next to its right-hand side. The readable labels, with their positions:

- *m* (bottom): the vertex from which all the arcs are drawn.
- *K* (upper left, at the outer end of the left-hand side).
- *n* [?] (on the outer arc, immediately right of *K*).
- *o* (on the outer arc, right of *n*).
- *p* [?], *q* [?], *r* [?], *s* [?] (the successive points of division along the outer arc; these letters will not resolve on the scan).
- *t* [?] (near the lower right end of the outer arc).

- *l* [?] (the lower right corner, at the end of the outer arc).

l m q of four, *l m p* of five, *l m o* of six, *l m n* of seven; but the right angle *K m l* will take in eight quarter-points.

With these things thus prepared, let *i* in the figure below be a point on the equinoctial circle from which the drawing of the rhumbs is to begin. And let the rhumb of *Nordeste* and *Sudoeste* be the first proposed for description. The spherical quadrant is accordingly laid on and fitted to the convexity of the globe so that the point *m* coincides with *i*; and the semicircle of the movable meridian is set in the position *a e c*, under which the spherical quadrant is turned about *m*, or *i*, until the circumference *m r q* coincides with *a i*.

Then from the table given above we take the number of degrees and minutes of the size of the first segment of that rhumb of *Nordeste* and *Sudoeste*, and count it from *m* toward *l*, marking the end. We imprint it on the globe, where *z* stands; *z* will therefore be the end of the first segment. To find the end of the second segment, exactly the same means must be used. We shift the movable semicircle to the position *a z c*, and under it the spherical quadrant is to be fitted to the globe so that *m* lies where *z* is; and about that same *m*, that is *z*, it is to be turned until the circumference *m q*, or *z q*, coincides with *z a*. Then, having counted the number of degrees and minutes of the size of the second segment from *m*, that is from *z*, toward *l*, we shall mark on the globe the end of the second segment, which let be *u*. The remaining points are to be found by the same operation, and are at last to be joined as we taught above; for the side of the spherical quadrant will serve as a ruler.

But if ease of working should please anyone more than exact calculation, he will be able, setting the table of numbers aside, to describe the rhumbs on the given globe in this way. With the circumference *m q* placed under *a i*, let the circumference *i l* be drawn on the surface of the globe from the point *i*, or *m*, along the side *m l* of the quadrant; for the side of the quadrant will serve as a spherical ruler, and accordingly the end of the first segment of the given rhumb will fall on that same *i l* — which we shall find as follows.

Draw the spherical quadrant across the surface of the globe, but in such a way that its side *m l* runs along the circumference *i l*; and let the semicircle of the movable meridian be carried round so as to pass through *m* in every position. Let the semicircle and the spherical quadrant be carried along together until only one degree of the circumference *K l* lies between the circumference *m q* and the

semicircle of the movable meridian. For when that happens, wherever *m* is, there will be the end

Diagram. At the foot of the left-hand column: a large circle representing the globe, crossed by a horizontal line and by a vertical line through the centre. Near the centre, above the crossing, sits a small fan-shaped figure — the spherical quadrant shown in two successive positions, each with its corner arc drawn. The readable labels, with their positions:

- *a* [?] (top of the circle, where the lines converge): the pole.
- *c* (bottom of the circle): the opposite pole.
- *b* (left, on the horizontal line).
- *d* (right, on the horizontal line).
- *m* and *i* (centre, written one above the other at the vertex of the small fan, the two points coinciding).
- *z* [?] (immediately above *m i*).
- *K* (twice: once above and once to the left of the small fan — the quadrant's corner in each of its two positions).
- *q* (twice: once beside each *K*).
- *u* (centre right, below the second *q*).
- *f* [?] (right of *u*).
- *l* (upper right, at the outer end of the fan).

of the first segment. Let us suppose, then, that *m* has been carried over to *x* [?], and together with it the semicircle of the movable meridian into the position *a x c*, so that a single degree of the circumference *K l* lies between the circumference *q m* (or *q x*) and the position of the semicircle. The angle *a x l* will then exceed the angle *a i l* — the inclination of the given rhumb — by one degree, and therefore the point *x* will be the end

Diagram. The globe drawn as a large circle: the pole *a* at the top and the opposite pole *c* at the bottom, with *b* at the left end of the horizontal diameter and *d* at the right; the horizontal is drawn as a broken line, and the lower left arc of the circle is likewise broken. Near the centre stand two successive positions of the spherical quadrant, drawn as narrow sectors sharing their vertex at the centre, where *m* and *x* are lettered close together; the meridian arc runs from *a* down through this centre to *c*.

- *a* (top of the circle): the pole
- *c* (bottom of the circle): the opposite pole

- *b* (left end of the horizontal): unlabelled beyond the letter
- *d* (right end of the horizontal): unlabelled beyond the letter
- *K* (upper part of the inner figure, on the left-hand sector): corner of the quadrant in its first position
- *K* (repeated, slightly higher and further right, on the second sector): corner of the quadrant in its second position
- *q* (inside the inner sector, below the two *K*'s): on the circumference *q m* / *q x*
- *L* (right of *q*, at the outer edge of the sector): end of the quadrant's side
- *m*, *x* (at the shared vertex, centre of the figure): the vertex of the quadrant in its two positions
- *y* [?] (upper right, at the outer edge of the second sector, followed by a second character that will not resolve on the leaf)

of the first segment, by what we have assumed. Accordingly, if we now turn the spherical quadrant about *x* until the circumference *q x* lies under the movable meridian in the position *a x*, while the side *m l* comes into the position *x y*, and if a circumference is then drawn on the surface of the globe from *x* to *y*, the end of the second segment will lie on that same *x y*; and it must be sought by the same method we have just used. The same procedure of finding is to be kept for all the rest.

Once this is done, the outline of the world's coasts must be laid down on the globe itself. For leagues, miles, and the other customary measures, let scales be drawn from arcs of great circles. Among the Spaniards some assign 17½ leagues to one degree of a great circle on the earth's circumference, while others assign 16⅔. So if you prefer the first opinion, divide an arc of four degrees of a great circle into seven equal parts: each of them will take in ten leagues, and in this way you will be able to extend a scale of leagues as far as you please. But if the second opinion suits you better, divide three degrees into five equal parts, and each part will likewise be of ten leagues — though these parts will be larger than the others.

¶ *On the use of a globe on which the rhumbs have been drawn. Cap. 27.*

Once a globe has been so prepared that it carries the rhumbs drawn by the method described, in the northern hemisphere and likewise in the southern, it can be of great service to seamen — as we shall show by a set of rules.

1. Suppose no rhumb is found drawn through two given places set on the globe, and yet the route by which one must go from the one to the other has to be tracked down. Then we shall bring the movable meridian to one of them. If in that position it takes in the other place as well, the two places

> undoubtedly lie on one and the same rhumb of North and South. But if they have different meridians, we shall ask what the latitudes of those places are, and toward which part of the world they lie away from the equinoctial. For if the latitudes are found equal, and toward the same part of the world, whether north or south, it will be certain that the places lie under one and the same parallel, and therefore on the rhumb of *Leste* and *Oeste*. If, however, they have neither a common meridian nor a common parallel, the method of finding it will be another. Of the two given places, let the one further from the arctic pole be called the first, for the convenience of the exposition, and the one nearer that pole the second. Now if the second place is further east than the first, the rhumb running from the first to the second will be one of those tending into the eastern and northern quadrant of the horizon. So that it can be determined which of them it is, each must be tried, in the following way. Carrying the movable meridian round, we shall mark on each rhumb the two points at which the parallels of the given places cut those rhumbs. Then we shall take the interval of the given places themselves

between the points of the compasses, and compare it at once with the intervals found between the marked points. For the rhumb to be chosen is the one that shows the route from the first place to the second — the one, that is, on which the distance between the marked points is found equal to the interval of the given places. But if none is found equal to it, we shall know for certain that no rhumb can be drawn from the first place to the second, and the nearest must therefore be taken. By "nearest" I mean the one whose distance between the marked points differs by the smallest amount from the interval of the given places just mentioned. Sailing on that nearest rhumb, one will go from the first place to some other place lying on the parallel of the second place: further east than the second place itself, if the interval of the given places is found to be the smaller, and further west if it is the greater. From there it will not be hard to reach the intended place by sailing along that same parallel. By a like method the rhumb from the first place to the second must be investigated when the second place is further west than the first. And the same procedure of finding will be kept when one has to go from the second to the first.

The rhumb between two given places can be tracked down not only from the intervals, but also from the differences of longitude — that is, by comparing the difference found between the meridians of those places with the differences enclosed between the meridians of the marked points on each rhumb. How this is to be carried out will easily be clear from what we have just said.

2. Suppose it is worth measuring the travelling interval between two given places set on the globe — the interval taken along the rhumb by which one goes from the one to the other. The way of finding such a distance will not be one and the same. For if the given places lie on one meridian, we shall multiply the number of degrees found between those places by the number of leagues answering to one degree; the product will make the travelling interval known. The same procedure will be kept when the given places lie under the equinoctial circle. But if they are found under one parallel outside the equinoctial, we shall convert the degrees of the difference of longitude reckoned on that parallel into degrees of a great circle, by the method set out above, and multiply these by the number of leagues due to a degree of a great circle. In this way the distance sought along that parallel will come out known. If, on the other hand, some other rhumb is found drawn through the two given places — not the rhumb of North and South, nor that of *Leste* and *Oeste* — and you want to find the interval along that rhumb, you will find it with the compasses: take a small space of ten leagues between the points of the compasses, and with it measure the interval of the rhumb between the given places; the number of leagues sought will then not fail you.

3. If the rhumb of the voyage made is known, together with the position of the home port [radicalis locus] we sailed from, and if the latitude of the place we are now in has been determined, its position on the globe will not be hard to find. For if that rhumb is found drawn through the home port, we shall carry the movable meridian round until it cuts that rhumb at the point marking the latitude of the place we have reached by sailing. Where it cuts it, there will lie the place we are in. But if no such rhumb is drawn through the home port on your globe, then on that rhumb, wherever it is found drawn, let two points be marked, as far from the equinoctial as the home port and as the place we are in. Whatever difference of longitude is found between those two points must be the difference between the home port and the place we are in. And since the place we are in has a known latitude, it must therefore have a known position on the globe.

4. If the position of the home port we sailed from is known, together with the rhumb, and if the distance run along that rhumb is known as well, the position of the place we are in cannot fail to be found. For if on your globe the rhumb of the voyage made passes through the home port, the small space of ten leagues taken between the points of the compasses will be the measure of the course run. So when you have measured that space off with

the compasses, the position of the place we are in will appear at once. But if the rhumb of the voyage made does not pass through the home port, let a single point be marked on it, wherever it is found drawn, as far from the equinoctial as the home port itself and toward the same part. Taking that point as the start of the reckoning, we shall lay off along the rhumb as much space as has been run; and at the end we shall press a mark on the surface of the globe. For whatever the distance of that impressed mark from the equinoctial, so great will be the latitude

of the place we are in; and the difference of longitude between it and the home port will be as great as the difference found between the point we took for the home port and the impressed mark.

5. If the position of the home port we sailed from is known, together with the distance of the course run, and if the latitude of the place we have reached has been determined, both its position and the rhumb we followed will be discovered. For the distance run is either the straight interval between the two places, or else a slanting and winding one along the track of some rhumb. If it is straight, then, taking it between the points of the compasses, we shall describe with that measure the circumference of a circle about the home port; and at the same time we shall carry the movable meridian round. For where the point of the movable meridian that shows the latitude of the place we have reached touches the circumference described, there will be the position of that place. Sometimes it will touch it at one point only — namely when the one place lies to the north and the other to the south, under one and the same meridian; sometimes at two, namely when one place lies to the east and the other to the west. But which of the two we are in we shall easily know from the turning of the world itself and from the voyage we have made. The rhumb between the two places will then appear from the first rule. If, however, the distance run has been covered along the track of some rhumb, take the small space of ten leagues between the points of the compasses, begin the reckoning from the home port, and try each rhumb in turn. For the place we have reached by sailing will lie on the rhumb on which the end of the measured-off distance is found to have the same distance from the equinoctial as the latitude found, and toward the same part. But since the several rhumbs are not drawn through every single place set on the globe, the reckoning must be begun sometimes from the home port and sometimes from other places having latitudes equal to the home port's. For the difference of longitude found by reckoning in this way will be equal to the difference between the home

port and the place we have reached by sailing, and so its position cannot fail to be found.

We think this should also be recorded: for those going from the equinoctial toward the poles of the world off the meridian, and returning by the usual precepts of the art of navigation, the route cannot be exactly the same. The difference, however, will be small. But if anyone wants to keep the most exact reckoning, let him add further sets of rhumb lines, beginning from latitude 60 degrees and ending at the equinoctial.

¶ *A single annotation on Aristotle's mechanical problem on the motion of a ship by oars.*

When we were once expounding Aristotle's *Mechanical Questions* to our pupils, we made a number of notes on that problem: why the ship goes forward more than the blade of the oar [palmula] goes the opposite way. Aristotle's reasoning is obscure; still, so that it might have some light, we explained it in the following way, and because of the kinship of the subject we have appended it to these our books on the art of navigation. The author assumes that the blade of the oar moves backward whenever the ship advances forward, and that the thole [scalmus] — the point about which the oar turns with a circular motion — is set at the middle of the oar, so that it stands as far from the handle [manubrium] as from the blade.

So let two equal straight lines *a b* and *d e* be laid down, cutting one another at the midpoint *c*, and let *d a* and *b e* be joined. At the beginning of one stroke let the oar have the position of the straight line *a b*, with *a* the handle, *b* the blade, and *c* the thole. When, therefore, *a*, the head of the oar, has at the end of that stroke been carried over to where *d* is, *b* will not be where *e* is. For if it were there, the oar would have the position of the straight line *d e*; and since the vertically opposite angles at *c*

Diagram. Two long straight lines cross near the middle of the figure, forming a bow-tie of two triangles set point to point. Along the top runs a horizontal line lettered *b* at its left end and *e* at its right, with *z* marked on it between them; along the bottom runs a horizontal line lettered *d* at its left end and *a* at its right. The line from *b* (upper left) descends to *a* (lower right), and the line from *e* (upper right) descends to *d* (lower left); they cut one another at *c*, marked just below the crossing.

- *b* (upper left): end of the upper horizontal
- *z* (upper middle, on the line *b e*)

- *e* (upper right): end of the upper horizontal
- *c* (centre, at the intersection of the two crossing lines): the thole
- *d* (lower left): end of the lower horizontal
- *a* (lower right): end of the lower horizontal
- one further small letter stands just above and left of *c*, at the crossing; it will not resolve on the leaf [?]

Diagram. A small woodcut standing in the outer margin at the head of the page, repeating the crossing-lines construction of the preceding leaf: two long straight lines cross one another in a narrow X, and short lines close the figure at top and bottom, so that the whole reads as a flattened bow-tie. The upper closing line runs to the right, the lower to the left; the crossing point is lettered near the middle. The woodcut is small and the letters are printed with thin strokes, so most of them will not resolve on the leaf.

- *e* (right end of the upper line): the end of the upper closing line
- *c* (at the crossing of the two long lines): the thole
- (upper line, left of *e*): a letter that will not resolve [?]
- (lower left): a letter that will not resolve [?]
- (lower right): a letter that will not resolve [?]

are equal, and the two sides *a c* and *d c* of triangle *a d c* are likewise equal to the two sides *b c* and *c e* of triangle *b e c*. The remaining angles, then, and the bases of those triangles, will be equal, by proposition 4 of the first book of Euclid [Euclid, *Elements* I.4]. On that showing *b* would travel exactly as much space as *a*; the thole *c* would not move at all; and so the ship in which that same thole is fixed would not move either — which contradicts the hypothesis. For the question assumes that the ship moves forward under that stroke of the oars while the blade of the oar [*palmula*] goes backward. And the thole [*scalmus*], though it has no share in the oar's circular motion, is nevertheless carried along by the motion of the ship.

So let the oar at the end of the stroke have the position of the straight line *d z*, which cuts the straight line *a b* between *b* and *c*, and cuts the straight line *b e* at *z*. Now the two alternate angles *c a d* and *c b e* have been shown to be equal, and the angle *a t d* is equal to its vertically opposite angle *b t z*. The two triangles *a t d* and *b z t* are therefore equiangular, by 32 of the first book together with the common notion. They are accordingly similar, and will have their sides proportional, by the fourth of the sixth book: as *a t* is to *b t*, so is *d a* to *b z*. But *a t* is greater than *b*

t; therefore *d a* will be greater than *b z* — a result that can also be reached by the common notion alone, setting the similarity of the triangles aside.

The handle [*manubrium*], then, runs through a greater distance than the blade of the oar does, and the ship will be carried to the place to which the grip of the oar has been carried. Moving in the opposite direction, therefore, the ship passes over more space than the blade of the oar does. In this we are following the translation and the demonstration-figure of Victor Faustus [Vittore Fausto].

One point must be noticed, however: when the oar has the position *d z*, the blade of the oar will lie beyond *z*. For the two sides *a c* and *d c* of triangle *a d c* were laid down as equal, so the two angles at *d* and at *a* will be equal; the angle *a d t* will therefore be greater than the angle *d a t*, and for that reason the side *a t* of triangle *a t d* will be greater than the side *d t*, by the nineteenth of the first book. The angle *b z t* has further been shown equal to the angle *a d t*, and the angle *d a t* equal to the angle *t b z*; the angle *b z t* will therefore be greater than the angle *t b z*, and for that reason the side *b t* of triangle *b t z* will be greater than the side *t z*. The whole straight line *a b* will therefore be greater than the whole of *d z*; and so, when the oar has the position of the straight line *d z*, the blade will lie beyond *z*. Let it be at *K*, then, and let the straight lines *b d* and *b K* be joined. The distance run through by the blade itself will accordingly not be *b z* but *b K* — which we shall show to be smaller still than…

Diagram. A quadrilateral standing at the head of the right-hand column, drawn with heavy rules. *b* is at the upper left and *e* at the upper right, joined by the top line; *d* is at the lower left and *a* at the lower right, joined by the bottom line. The two diagonals *b*–*a* and *d*–*e* cross at *c*, near the middle of the figure. A further heavy line rises from the lower left, cutting the diagonal *a b* at *t* and the top line at *z*, and running on to *K*, which stands above the top line; the joining lines *b d* and *b K* are drawn in.

- *b* (upper left): the blade at the beginning of the stroke
- *e* (upper right): the point the blade would reach if the oar took the position *d e*
- *d* (lower left): the point to which the handle *a* is carried
- *a* (lower right): the handle at the beginning of the stroke
- *c* (centre, at the crossing of the diagonals): the thole
- *t* (on *a b*, between *b* and *c*): where the line from *d* cuts *a b*
- *z* (on the top line *b e*): where the line from *d* cuts *b e*

- *K* (above the top line): the actual position of the blade at the end of the stroke

d a itself. For the two sides *b d* and *d K* of triangle *b d K* are equal to the two sides *b d* and *d e* of triangle *b e d*, while the angle *b d K* is less than the angle *b d e*; the base *b K* will therefore be less than the base *b e*, by the twenty-fourth of the first book — which was to be demonstrated.

There is a further point. What Aristotle takes for granted in the course of his reasoning — that the ship covers as much distance as the handle of the oar — is ambiguous. For the handle of the oar is carried by two motions: one its own and circular, about the thole; the other that by which it is carried along together with the ship itself. The distance run through by the handle of the oar taken as a whole will therefore be greater than the distance covered by the ship. But if he means instead that the handle in its own motion and the ship run through equal distances, that reading is not free of difficulty either. For the ship will sometimes traverse a greater distance and sometimes a smaller, according to the strength of the rowers and according to how far the blade of the oar has been immersed in the sea; whereas the handle of the oar, even when it is moved by a slight force, will still describe no smaller a circuit than if it were moved by much greater power. To examine this opinion of Aristotle's, therefore, we have demonstrated the theorems that follow.

Proposition One.

¶ *If the rowers are able to move the ship, the handle of the oar always traverses a greater distance than the ship does.*

Let the oar be *a c*, the handle *a*, and the thole *b*, which because of the ship's motion runs through the distance from *b* to *d*; and in that position let the oar *a c* itself have the straight position *e f*. Let the distance which *a* covers be the curved line *a e*, to which corresponds the straight line *a z*, perpendicular to the straight line *e f*. The ship will cover the same distance as the thole *b* does. I say, then, that the straight line *a z* is greater than the straight line *b d*.

Diagram. The oar in its two positions, drawn from a common vertex at the left. *a* stands at the left; from it two straight lines open out, the upper running up to *e* and the lower running down to the right past *b* and *g* to *f* and *c*. A curved arc joins the two ends at the right, and a vertical line drops from *e* through *z* and *d* to *f,* cutting the lower line at *g*.

- *a* (left, at the vertex): the handle at the beginning of the stroke

- *e* (upper right): the handle at the end of the stroke
- *z* (on the vertical below *e*): foot of the perpendicular from *a* on *e f*
- *b* (lower middle): the thole at the beginning of the stroke
- *d* (right of *b*, on the vertical): the thole at the end of the stroke
- *g* (below *b*, where the vertical cuts the oar's line): the crossing of *a c* and *e f*
- *f* (bottom, at the foot of the vertical): the lower end of the straight position
- *c* (lower right): the blade of the oar

For let the straight line *a c* cut the straight line *e f* at *g*. The pair of triangles *a g z* and *b g d* are then equiangular, and therefore, as *a g* is to *b g*, so is *a z* to *b d*, by the fourth of the sixth book of Euclid. But *a g* is greater than *b g*; *a z* will therefore be greater than *b d*, and accordingly the handle of the oar traverses a greater distance than the ship — which was to be demonstrated.

If from the point *b* we now draw a straight line *h K* in both directions, of the length of the oar, making right angles with *b d* and cutting the straight line *a z* at *i*, we shall see plainly that the straight line *a z* is made up of *a i* and *i z*. The first of these answers to the curve *a h*, which is described by the handle's own motion; the second is equal to the straight line *b d*, which is run through by the motion of the ship.

Diagram. The same construction repeated with the line *h K* added. *a* is again at the left vertex, with the two straight lines opening out to *e* above and down past *b* and *g* to *f* and *c* below. The vertical through *e*, *z*, *d* and *f* is drawn as before, and a second vertical is now drawn through *b*, running up to *h* at the top and down to *K* at the bottom; the horizontal from *a* meets it at *i*.

- *h* (top): the upper end of the line drawn through *b*
- *e* (upper right): the handle at the end of the stroke
- *a* (left, at the vertex): the handle at the beginning of the stroke
- *i* (on *a z*, where the line *h K* cuts it) [?]
- *z* (on the vertical below *e*): foot of the perpendicular from *a*
- *b* (middle, on the line *h K*): the thole at the beginning of the stroke
- *d* (right of *b*): the thole at the end of the stroke
- *g* (below *b*): the crossing of the oar's two positions
- *K* (bottom left, on the line through *b*): the lower end of that line
- *f* (bottom, at the foot of the first vertical): the lower end of the straight position
- *c* (lower right): the blade of the oar

Proposition Two.

¶ *If the handle of the oar in its own motion and the ship pass over equal distances, it will be impossible for the blade to move; it will stay unmoved, like a centre.*

Let the oar again be *a c*, the handle *a*, and the thole *b*; and let the ship cover as much distance as *a* does in its own motion. I say that *c*, the blade of the oar, will stay unmoved. For suppose it were displaced from its place: let it pass backward over the distance *c d*, and, once that has been run through, let the oar *a c* have the straight position *f d*. The thole *b* will then have been carried over to *g*.

From the point *b* let the line *e b r* be raised in both directions at right angles to *b g*; from the point *a* let the straight line *a h* be raised perpendicular to *d f*; and likewise from the point *c* let the straight line *c e* be raised perpendicular to *e r*. Let the intersection of the straight lines *e r* and *a h* fall at *K*, and that of *c e* and *d f* at *z*.

Now *a K* is the distance which the handle of the oar passed over in its own motion — for it answers to the curved line *a r* — while *b g* is the distance the ship covered; the straight lines *a K* and *b g* will therefore be equal. But in the two equiangular triangles *e b c* and *b a K* you will conclude, either by proposition 26 of Euclid's first book or by the fourth of the sixth, that the straight lines *a K* and *e c* are equal. So *e c* will be equal to the straight line *b g*, by the common notion; and *e z* is equal to that same *b g* as well, in the parallelogram, by proposition 34 of that same first book. The straight line *e z* will therefore be equal to the straight line *e c* — a part equal to the whole, which is impossible. And so the blade *c* will stay unmoved, which is what we had to show.

You will prove the same thing another way, from what has already been shown [?] in the demonstration. Let the oar at the beginning of the motion have the position…

Diagram. The oar in its two positions with the perpendiculars of Proposition Two drawn in. *a* stands at the left vertex; from it one straight line runs up to the right toward *f* at the top, and another runs down to the right through *b* and on to *c* at the lower right. A curved arc labelled *r* joins the upper ends. A vertical rule runs down the right of the figure from *f* through *h*, *g* and *c*; a horizontal from *a* meets it, with *K* lettered on it, and further horizontals run through *b* and along the foot of the figure through *e* and *z* to *c*, with *d* below.

- *a* (left, at the vertex): the handle at the beginning of the stroke
- *r* (on the upper arc): the curve described by the handle's own motion

- *f* (top right): the upper end of the oar's final straight position
- *K* (middle, on the horizontal from *a*): the intersection of *e r* and *a h*
- *h* (right, on the vertical): the foot of the perpendicular from *a* on *d f*
- *b* (lower left): the thole at the beginning of the stroke
- *g* (lower right, on the vertical): the thole at the end of the stroke
- *e* (bottom left): on the perpendicular raised at *b*
- *z* (bottom, between *e* and *c*): the intersection of *c e* and *d f*
- *c* (bottom right): the blade of the oar
- *d* (below *c*): the point to which the blade would have moved back

a b c. From the point *c*, where the blade of the oar is, let the straight line *c g* be drawn, making right angles at the point *g* with the straight line along which the thole *b* moves under the ship's motion; and let that straight line *c g* then be produced as far as *e*, so that *g e* is equal to *a b*. Again, from the point *b* let the straight line *K b f* be raised at right angles to *b g*, and to it let the perpendiculars *a K* and *c f* be drawn from *a* and *c*. Since those same straight lines *a K* and *c f* are equal, by 26 of the first book of Euclid, and since the straight line *b g* is equal to *f c* in the parallelogram, *a K* will be equal to *b g*, by the common notion.

But *b* covers as much distance as the ship, and the ship as much as *a* does in its own motion, by hypothesis; and *a* covers the distance *a K*; *b* will therefore cover the distance *b g*. And since the angles at *g* are right, when the thole has reached *g* the oar *a c* will have the straight position *e c*, and the stroke will end in that place.

Diagram. The construction for the second proof, drawn as a rectangle with the oar's two positions across it. *a* stands at the left vertex; a line runs from *a* up to the right to *e* at the top right corner, and another runs down to the right through *b* to *c* at the lower right. A vertical rule closes the figure on the right, running from *e* down through *g* and *c*; a horizontal from *a* crosses the figure with *K* lettered on it; *b* sits at the left on a horizontal running across to *g*, and *f* stands at the lower left corner, with a diagonal drawn from *b* down to *c*.

- *a* (left, at the vertex): the handle at the beginning of the motion
- *e* (top right): the point to which *c g* is produced, so that *g e* equals *a b*
- *K* (middle, on the horizontal from *a*): the foot of the perpendicular from *a* on *K b f*
- *b* (lower left): the thole at the beginning of the motion
- *g* (lower right): the thole at the end of the motion
- *f* (bottom left): the lower end of the line raised at *b*

- *c* (bottom right): the blade of the oar
- (a further small letter below *c*, which will not resolve on the leaf) [?]

So the blade *c* was not displaced from its place, which was to be demonstrated. But note that the straight line *g c* is less than *b c*, half the oar; let their difference be *c t*. During the time, then, in which the thole *b* is carried over to *g*, the blade *c* runs out along that length *c t*; yet it will move neither backward nor forward, for this alone is what we set out to prove.

We do not doubt, however, that it can happen that the ship is sometimes driven with so uneven a thrust and so irregular a motion that the blade of the oar moves the contrary way for a little while — but it will return at once to its earlier place. And the thole will reach *g* neither sooner nor later than the blade brings itself to *c t*, just as if it had never left its place. Otherwise the ship and the handle of the oar would be seen to cover unequal distances, contrary to the hypothesis. And since, when this happens, the ship is carried faster at the end than at the beginning, it follows that some further force has come into play besides the thrust of the oars.

Converse of the proposition.

You will demonstrate the converse of this proposition: namely that if the blade of the oar is not displaced from its place, but stays there until the oar reaches the straight position, then the handle covers as much distance in its own motion as the ship does. For the straight line *e* [?] is equal to *a K* by 26 of the first book, and is equal to *b g* as well by 34 of that same first book; therefore *a K* and *b g* will be equal, by the common notion.

Proposition Three.

¶ *If the handle of the oar in its own motion covers twice the distance the ship does, the ship will be carried forward by that stroke by just as much as the blade has moved back.*

Let the oar at the beginning of the motion have the position *a c*, and at the end of it the straight position *f g*. The thole *b*, then, because of the ship's motion, will cover the distance *b d*. From the point *b* let the perpendicular *e z* be raised in both directions, and let there come to it from the point…

from the points *a* and *c*, at right angles to it, the straight lines *a e* and *c z*. Let the distance *a e*, run over by the handle in its own motion, be double the distance *b d*; and let the curve *c g*, which the blade of the oar describes, correspond to the straight line *c h*. I say that the two straight lines *b d* and *c h* are equal. For in the

two triangles *b a e* and *c b z*, the two straight lines *a e* and *c z* are equal. In the parallelogram *b h*, moreover, the two lines *b d* and *h z* are equal. But the straight line *a e* is double the straight line *b d*, by hypothesis; therefore *c z* is double the straight line *h z* as well. Consequently *c h* and *h z* will be equal, and so the two lines *c h* and *b d* are equal, by the common notion. And since the ship always runs over just as much distance as the thole [*scalmus*] does, it follows that if the handle of the oar has covered twice as much distance in its own motion as the ship has, the ship is carried forward by exactly as much as the blade [*palmula*] has moved back — which was to be demonstrated.

Diagram. The figure for Proposition Three, standing in the left-hand column. A long line descends steeply from *a*, at the upper left, across the whole figure to *c* at the lower right: this is the oar in its starting position, with the handle at *a* and the blade at *c*. A second long line descends from *f*, at the upper right, to *g* at the foot of the figure: this is the oar in its final straight position. A vertical line runs down the middle of the figure from *e*, above, through the thole *b*, to *z* at the bottom — the perpendicular raised at *b* in both directions. The horizontal *a e* closes the figure at the top, and the horizontal *c z* at the bottom, so that a four-sided figure is formed with *b* and *d* as its upper corners and *z* and *h* as its lower ones. A short curve joins *c* to *g*.

- *a* (upper left): the handle at the beginning of the motion
- *f* (upper right): the handle when the oar has come to the straight position
- *e* (top of the vertical line): the foot of the perpendicular dropped from *a*
- *b* (middle left, on the vertical line): the thole at the beginning of the motion
- *d* (middle right, level with *b*): the thole at the end of the motion
- *z* (foot of the vertical line): the foot of the perpendicular dropped from *c*
- *h* (on the bottom line, between *z* and *c*, directly under *d*): the fourth corner of the parallelogram
- *c* (right end of the bottom line): the blade at the beginning of the motion
- *g* (below the bottom line, between *h* and *c*): the blade when the oar has come to the straight position

Converse of the proposition.

If the ship has been carried forward by just as much as the blade of the oar has moved back, the handle will cover twice the distance in its own motion that the ship does. For if *b d* is put equal to *c h*, then, since *h z* is equal to that same *b d* in the parallelogram, *c h* and *h z* will be equal, by the common notion. *c z* will therefore be double *h z*, and so the same *c z* will be double the straight line *b d*.

Further, *c z* and *a e* are equal, by 26 of the first book; therefore *a e* will be double the straight line *b d*. Of these two lines, the

former is run over by the handle of the oar and the latter by the thole, and the ship is carried forward by just as much as the thole is. Therefore, if the ship has been carried forward by as much as the blade of the oar has moved back, the handle will cover twice the distance in its own motion that the ship does — which was to be shown.

Proposition Four.

¶ *If the ship runs over less distance than the handle of the oar, but more than half of it, it will be carried forward by more than the blade moves back; but if less than half, by less.*

In the figure already described, let *b d* be put smaller than *a e*, but greater than half of it. I say that *b d* is greater than *c h*. For *b d* and *h z* are equal; and besides this, the straight lines *a e* and *c z* are equal. *h z* will therefore be greater than half of *a e*, and so the remainder *c h* will be less than half of that same *a e*; and *c h* will therefore be less than *b d*. But *b d* is the distance the ship covers, while *c h* is the distance the blade of the oar runs in the contrary direction; and so the first part of the theorem is true. The second part will be shown in the same way. For if *b d* is less than half of *a e*, then *h z* too will be less than half of that same *a e*; and since *a e* and *c z* are equal, the remainder *c h* will be greater than half of that same *a e*, and consequently *b d* will be less than *c h*. The ship, therefore, runs over less distance forward than the blade of the oar does in the contrary direction — which is what we undertook to demonstrate.

Corollary.

From this proposition and the preceding one it follows that, if the handle of the oar in its own motion runs over a greater distance than the ship — whether that distance is double, or less than double, or greater than double — the distance which the ship meanwhile runs forward and the distance which the blade of the oar runs in the contrary direction, taken together, will be equal to the distance which the handle of the oar itself

covers in its own motion. For *b d* is always equal to *h z*, while the whole of *c z*, which is equal to *a e*, is made up of its parts *c h* and *h z*.

Converse of the proposition.

¶ *If the ship advances further than the blade of the oar moves back, it will cover more than half the distance which the handle of the oar runs in its own motion; if less, less than half.*

The demonstration of this can easily be gathered from what has been said above.

Proposition Five.

¶ *If the ship is carried faster than the handle of the oar, the blade will move forward and will never at any point move back; and it will run over the distance by which the ship's motion exceeds the motion of the handle.*

Let the oar have the position *a c* when the motion begins, and at its end the straight position *f g*. The thole *b*, then, carried over by the ship's motion, will be at *d*. Let the distance *b d* be greater than *a h*, which the handle of the oar has run over in its own motion; for this is what it means to say that the ship is carried faster than the handle. I say that the blade *c* will move forward. For when the thole *b* has been carried forward to *d*, the blade *c* itself will have been carried across to *g*, in the straight position, and will have covered the curved line *c g*, to which *c K* corresponds. The blade, therefore, will move forward.

Diagram. The figure for Proposition Five, standing in the left-hand column. From *a*, at the upper left, a short line runs to the right to *h*, and a long line descends from *a* through the thole *b* to *c* at the lower left, so that *a c* is the oar in its starting position. A horizontal runs from *b* to *d* on the right. At the right of the figure a long vertical descends from *f*, at the top, past *d*, to *K* at the bottom: this is the oar's final straight position *f g*. A horizontal line runs at the foot of the figure from the left side through *c* to *K*, and a curve sags from *c* down to *g*, which lies below and to the right of *K*.

- *a* (upper left): the handle at the beginning of the motion
- *h* (right of *a*, at the end of the short line): the point the handle reaches by its own motion
- *f* (upper right): the handle when the oar has come to the straight position
- *b* (below *a*, on the descending line): the thole at the beginning of the motion
- *d* (right, level with *b*): the thole after the ship's motion has carried it over
- *c* (lower left): the blade at the beginning of the motion
- *K* (right end of the bottom line): the end of the distance *c K*

- *g* (below and right of *K*): the blade when the oar has come to the straight position
- (a small letter standing just above the bottom line, close to *c*, apparently *i*) [?]

That it never at any point moves back is shown in this way. The point *a* toward *h*, and the point *c* toward *i*, are moved about the thole with the same speed. But by hypothesis the ship is carried faster than *a* [?] moves to *h*; therefore the ship itself is carried faster than *c* moves toward *i*. But that same *c* is moved toward *K* with the ship's own speed; therefore it is carried faster to *K* than to *i*. Consequently *c* will never at any point move back; on the contrary it will advance forward, and will run over the distance *c K*, which is what is left when *i c* is taken away from *i K*. For if the blade of the oar were moved with the ship's whole speed, it would pass beyond *K* by the time *b* reached *d*; but it is drawn back in the meantime because of the motion that takes place about *b*. And so the blade, retarded in the speed that comes to it from the ship's motion, will have *c K* for the distance it runs. It seems, however, that this cannot come about by the thrust of the oars alone, and that some further impelling [?] power is needed.

From these theorems it is clear how uncertainly Aristotle frames his question and how unskilfully he answers it. For it does not follow at once that, if the ship moves forward, the blade of the oar moves back; nor, even if the blade does move back, that it passes over less distance in the contrary direction than the ship advances. The second and third propositions demonstrate this. The handle of the oar, indeed, by its own motion about the thole together with the ship's motion, covers a greater distance than the ship does; but if by its own motion alone it happens to cover as much distance as the ship, it will be impossible for the blade to move. In vain, therefore, does he attempt to demonstrate universally that the handle of the oar runs over a greater distance than the blade does in the contrary direction. Besides, when the ship advances further than the blade of the oar goes back, the ship runs over less distance than the handle: therefore not an equal distance. And from this it is plain that neither the truth of what he proposes nor a demonstration of it is to be found in what he heaps together.

FINIS.

Some Annotations on the *Theoricae Planetarum* of Georg Peuerbach, by Pedro Nunes of Alcácer do Sal

Since these *Theoricae* of the planets were written by Georg Peuerbach according to the doctrine of Ptolemy and of Alfonso precisely so that the canons of the

tables might be more easily understood, we have chosen to annotate only those points which the commentators have explained either insufficiently or incorrectly — though we know that a great many of the things written in those same tables do not agree with the observations of certain other distinguished astronomers.

The theory of the Sun is set out by Georg Peuerbach roughly in the following way. The Sun's sphere consists of three orbs, divided from one another and yet contiguous. The middle orb is of uniform thickness, and it is to this that the solar body is attached; it has its convex surface in common with the concave surface of the highest orb, and its concave surface in common with the convex surface of the lowest, and the centre of these lies outside the centre of the world. But the concave surface of the lowest orb and the convex surface of the highest are concentric with the world; and so the whole sphere of the Sun is concentric with the world. The outermost orbs are partly eccentric and partly concentric, whereas the middle orb is eccentric throughout.

The two outermost orbs move about the centre of the world and about the axis of the zodiac, with exactly the same motion — according to the Alphonsine astronomers — by which the eighth sphere moves; and they are called the deferents of the Sun's *aux*. For since by their own motion they carry the centre of the orb that bears the Sun around the centre of the world, the Sun's *aux* must necessarily be moved with that same motion. Now the *aux* of the Sun, or apogee, is the point in the middle thickness of the deferent that is farthest from the centre of the world — that is, the end of the line drawn from the centre of the world itself through the centre of the deferent; while the point opposite the *aux*, or perigee, is the opposite point on that same orb that carries

the Sun. At the present time the Sun's *aux* is in the second degree of Cancer, though Ptolemy set it in a fixed seat in the sixth degree of Gemini.

Because of the motion of the middle orb, the Sun always advances beneath the ecliptic of the starry orb with an equal motion about its own centre — 59 minutes and 8 seconds very nearly each day, in the order of the signs. For that reason the apparent motion, which is referred to the centre of the world, is unequal: slower about the *aux*, and faster about the point opposite the *aux*.

The line of the Sun's true motion is the line drawn from the centre of the world through the centre of the solar body and extended to the zodiac. The Sun's true or apparent motion is reckoned in the zodiac from the beginning of Aries up to this line.

The line of the Sun's mean motion is the line drawn from the centre of the world as far as the zodiac, equidistant from the line understood to be drawn from the centre of the deferent to the centre of the solar body. The mean or equal motion is reckoned from the beginning of Aries up to the line of the mean motion. By the beginning of Aries we mean the vernal section of the ecliptic of the eighth sphere, not the beginning of the constellation; but according to Peuerbach it is the section of the ecliptic of the primum mobile and the equinoctial.

The argument of the Sun is the arc of the ecliptic between the line of the *aux* and the line of the Sun's mean motion; and it is similar to the arc of the eccentric between that same line of the *aux* and the centre of the Sun on the circumference described by the Sun's own centre in its annual revolution.

The equation, or the difference between the equal motion and the apparent motion, is the arc of the ecliptic between those two lines, of the equal motion and of the apparent. When there is no argument, or when there are six common signs, which comprise 180 degrees, there is no equation, because the lines of the true and of the equal motion then coincide.

When the Sun stands on the line drawn from the centre of the world perpendicular to the line of the *aux* — which Peuerbach calls the line of mean longitude [*mediae longitudinis*], and Ptolemy the mean passage — the equation, or difference, is at its greatest. In other positions, as the argument varies, the equations decrease toward the *aux* and toward the point opposite the *aux*.

When the argument is less than 6 signs, the line

Now let the straight line g h cut the straight line a c at the point k. Then f l will equal the right sine of the arc k p, and the straight line e l the right sine of the arc n g. So f l is found to be 3780 [?] of the parts of which the semidiameter of the eccentric circle contains 100,000, and e l is 99 of those same parts. Since the square on e f equals the two squares on f l and e l taken together, e f will be 3781 and three tenths of those same parts — that is, of the parts of which the semidiameter of the eccentric is 100,000. Therefore, in the parts of which that same semidiameter is 60, e f will be 2 parts, 16 minutes, 7 seconds, 41 thirds, or nearly so. And since, in the right-angled triangle e f l, e f is to f l as the whole sine [*sinus totus*] is to the right sine of the angle f e l, the right sine of that angle f e l will be about 99,966 parts. The arc of that same angle f e l will accordingly have 88 degrees 32 minutes. (I use the table of right sines of Petrus Appianus [Peter Apian].)

Consequently b m will be 1 degree 28 minutes of the sign Cancer. This does, however, differ somewhat from the place of the *aux* that Petrus Pitatus, mentioned above, set at the head of his table on the calculation of the motion of the eighth sphere. It follows that the Alphonsine hypotheses do not fit one another exactly.

Ptolemy, by contrast, assumed a different interval of time from the vernal equinox to the summer solstice and to the autumnal equinox, and a different quantity for the Sun's mean motion. So although he used this same method, he arrived at a different eccentricity — 2 parts with 29 minutes and about half of one minute — and put the place of the *aux* in the middle of the sixth degree of Gemini. And since exactly the same result had been reached many years earlier by Hipparchus, he concluded that the Sun's *aux* is immovable, and likewise the distance between the centres. Therefore what Georg Peuerbach writes about the motion of the two orbs that carry the Sun's *aux*, together with the corollary about the small circles described — on account of the motion of the eighth sphere — by the axis and poles of the orb that carries the Sun and by the centre of the eccentric circle, belongs to Alfonso's teaching, not Ptolemy's.

You will find that teaching altogether unreliable if you suppose, as Ptolemy himself testifies, that in his time the Sun's *aux* stood 24 degrees 30 minutes before the summer solstice. For in our own time — that is, in the year 1552 from the birth of Christ — it lies in the second degree of Cancer according to the Alphonsine calculation. The Sun's *aux* must therefore have advanced about 26 degrees from the time of Ptolemy's observation to our own, that is, in 1420 years. Yet the eighth sphere could not have run through that much, neither on Ptolemy's calculation, nor on Alfonso's, nor even on al-Battānī's.

Suppose instead you judge al-Battānī's observations the more trustworthy — for Alfonso must have relied on the observations of some highly skilled astronomer in order to attach the Sun's *aux* to the motion of the eighth sphere. You will fall into a difficulty of the same kind. The Sun's *aux* was found by al-Battānī to be 7 degrees 43 minutes before the summer tropic, whereas Alfonso placed it one degree and about 20 minutes before that same tropical point. About 377 years intervened between al-Battānī's determinations and Alfonso's. You will conclude this easily enough: from the 1420 years that have run from Ptolemy's time to ours, subtract the 743 years that fell between Ptolemy and al-Battānī; then, from what remains, subtract the 300 years that have run from the year 1252 after the birth of Christ to the present. Therefore, if al-Battānī's and Alfonso's determinations of the place of the Sun's *aux* are true, in the space of those 377

years the Sun's *aux* advanced 6 degrees 23 minutes. Yet you will find the motion of the eighth sphere slower over that period, whether you follow Alfonso's calculation or al-Battānī's. Indeed, if you compare the places of the fixed stars found by al-Battānī with the places of those same stars recorded in the Alphonsine tables, you will find a difference of only 5 degrees, with 38 minutes — not 6 degrees 23 minutes. They will therefore be unable to give any reason why, on the Alphonsine account, the motion of the Sun's *aux* should be the same as the motion assigned to the eighth sphere.

We should wonder no less that Alfonso placed the Sun's *aux*, at the time of Christ's incarnation, in the 12th degree of Gemini, while Ptolemy — who came about 132 years after Christ — placed it in the 6th degree of that same sign. There is a comparable disagreement between Arzachel [al-Zarqālī] and al-Battānī: al-Battānī had placed the Sun's *aux* in the 22nd degree of Gemini, and al-Zarqālī, later than he, placed it in the 17th degree of the same sign. Perhaps the cause of such wide variation among men so eminent was that the Sun's entry into the summer solstice is extremely hard to determine with instruments, because in that part of the zodiac the declination changes by an imperceptible difference — and the demonstration given above about the place of the *aux* depends precisely on knowing that entry. You will therefore establish the same thing by a far more reliable method through the Sun's entry into the equinoctial points, and into

the beginning of the next sign adjoining those equinoxes; or else through any three other places verified by observation, as Ioannes de Monteregio [Regiomontanus] taught, by a very subtle method, in the third book of the *Epitome* [*Epit.*] (*Epitome of the Almagest* III). Geber [Jābir ibn Aflaḥ], it is true, thought that the place of the *aux* and the eccentricity could not be found accurately enough in this way, because of the repeated multiplication and division of numbers and the extraction of square roots.

Once the place of the *aux* is known, the size of the argument is found; and from the argument itself the difference between the mean motion and the unequal or apparent motion becomes known for every position. For in the figure drawn above the two straight lines a c and k i are parallel; the angle i f [?] r at the centre of the eccentric is therefore equal to the angle a e m at the centre of the world, and the two arcs i r and a m are consequently proportional. Now arc a m contains 91 degrees 28 minutes, by what we have already demonstrated: that is what remains when the 88 degrees 32 minutes of arc m c are subtracted from the 180 degrees of the semicircle a m c. Arc i r of the Sun's eccentric therefore likewise contains 91 degrees 28 minutes. To these degrees and minutes we shall add the 2

degrees 10 minutes of arc l q — or k p — which has already been ascertained; the arc q r thus becomes known, namely 93 degrees 38 minutes.

And so, in the said year 1552 from the birth of Christ, when the Sun stood by apparent motion at the beginning of Aries [?], by mean motion it was found 2 degrees 10 minutes before the beginning of Aries itself; at that moment, then, it occupied the 27th degree, 50 minutes, of the sign Pisces, and its argument was 266 degrees 22 minutes — for that is what remains when the arc of the *aux*, a m, is subtracted from the 357 degrees 50 minutes of mean motion. From all this it will not be hard to establish the root of the Sun's mean motion for any era whatever. Suppose, for example, one wishes to establish the root of the Sun's mean motion at the beginning of the years of Christ. In the said year 1552 the vernal section — that is, the beginning of Aries — fell on the tenth day of March, about noon at the city of Venice, according to the calculation of Petrus Pitatus. There had therefore run up to that moment 1551 Roman years, two months, and about 10 days. In so much time the Sun's mean motion is 2 common signs [*signa communia*], 19 degrees, 33 minutes. Subtract these from 357 degrees 50 minutes — that is, from the 11 signs 27 degrees 50 minutes of mean motion reckoned from Aries — and we shall have the Sun's mean motion at the beginning of the years of Christ, at the meridian of the city of Venice: 9 signs, 8 degrees, 17 minutes.

Ptolemy, however, held that the Sun's *aux* has a fixed seat in 5 degrees 30 minutes of the sign Gemini. Taking the beginning of the mean motion from there, and setting the root at the beginning of the reign of Nabonassar, he constructed his tables. To be able to do this, he investigated the Sun's distance from the *aux* according to mean motion at the autumnal equinox, in the seventh year of the emperor Hadrian, and found it to be 116 degrees 40 minutes. You will reach the same figure far more easily than by that demonstration in the eighth chapter, using the figure drawn above, if you set a as the beginning of Libra, b of Cancer, c of Aries. For since arc e m at that time contained 65 degrees 30 minutes, arc a m, which remains from the semicircle, will be 114 degrees 30 minutes; and the arc i r of the eccentric proportional to it will comprise the same degrees and minutes. Further, the arc l q [?] — or k p — was shown by him to be 2 degrees 10 minutes; the whole arc q r will therefore be 116 degrees 40 minutes. And so, when the Sun stood at the point q of the eccentric and occupied the beginning of Libra, it was distant from the place of the *aux* by those 116 degrees 40 minutes.

Once the argument is known, and the ratio of the semidiameter of the eccentric to the eccentricity is known as well, it is easy to find the difference between the mean motion and the apparent. Let the Sun's eccentric be the circle a b c d about

the centre e; let f be the centre of the world; and let the line of the *aux*, passing through both centres, be b e f d. The line a c, making right angles with b d at the point f, is the line Ptolemy called the mean passage, and Peuerbach the line of mean longitude [*mediae longitudinis*]. When the Sun stands on it, the difference between the two motions, the mean and the apparent, is greatest — that is, the magnitude of the angle f a e, or f c e. That magnitude is made known from the ratio of the semidiameter

Diagram. A single circle — the Sun's eccentric — with a vertical diameter and a horizontal chord, set at the foot of the right-hand column. The vertical line runs from **b** at the top of the circle down through **e**, then **f**, to **d** at the bottom: this is the line of the *aux*, passing through the centre of the eccentric (e) and the centre of the world (f). The horizontal line runs from **a** at the left of the circle to **c** at the right, crossing the vertical line at **f** at right angles: this is the line of mean longitude. A further point **g** is marked on the circumference in the upper right quadrant — the Sun in another position — with straight lines drawn from g back to e and to f.

- b (top of circle): the *aux*
- e (upper of the two points on the vertical line, above f): centre of the eccentric
- f (lower of the two points on the vertical line, at the crossing): centre of the world
- d (bottom of circle): the point opposite the *aux*
- a (left of circle): end of the line of mean longitude
- c (right of circle): end of the line of mean longitude
- g (circumference, upper right): the Sun in another position

e c to e f. For if you understand c to be the centre of a circle equal to the eccentric, the straight line e f will be the right sine of the arc of the angle f c e. But of the parts of which e c contains 100,000, e f has been shown to be 3781 with three tenths; the arc of that angle f c e will therefore have 2 degrees, 10 minutes, 3 seconds, or nearly so. And so the angle b e c will embrace 92 degrees, 10 minutes, 3 seconds of the Sun's argument. Ptolemy, however, found a greater eccentricity, and therefore set the maximum difference between the mean and the apparent motion at 2 degrees with 23 minutes.

Let us now put the Sun in another position, at g, and let the angle b e g — its distance from the *aux* according to mean motion — be supposed known. Then the angle f e g of the triangle e f g will be known, as what is left over from two right

angles; and the two sides f e and e g containing that angle are known. The remaining angles of that same triangle will therefore be known by the 24th proposition of the first book of Geber; and so the angle f g e, the difference between the mean and the apparent motion, comes out known.

Annotatio secunda.

Although the Sun's mean motion is greater than the true motion in the second half of the eccentric — the half after the *aux* — and less in the first half, before the *aux*, when both are reckoned from Aries, we do not doubt that if the motions take their beginning from elsewhere it can happen that the mean motion and the true are sometimes equal. This will be made plain by the propositions [?] that follow.

Propositio prima.

If, for some period of time, the Sun's mean motion on the eccentric and its apparent motion, referred to the centre of the world, are equal, then the point of mean longitude — the mean passage — will fall within the limits of those motions.

Let the Sun's eccentric be the circle a b c, whose centre is d and whose line of the *aux* is a f; and on the eccentric let the arc e c be the arc traversed by the Sun during the time in which the mean motion and the apparent motion are equal. I say that the point of mean longitude will lie between c and e. For let g be the centre of the world, and let d c [?], d e, g c [?], and g e be joined. The angle c g e then subtends, in the zodiac, the arc of the ecliptic passed through by the Sun while with equal motion it runs through the arc c e of the eccentric — an arc similar and proportional to that arc of the ecliptic. Let c e be joined, and about the triangle c d e let the circle c d e be described. It will necessarily pass through g, because, on account of the equality and similarity of the motions, the two angles e d c and e g c are equal to one another and will therefore stand in the same segment; and so, if the circle did not pass through g, an impossibility would follow, contrary to the 16th proposition of the first book of Euclid. It does pass through g, then. Accordingly, in the quadrilateral d e c g the two opposite angles d g c and d e c are together equal to two right angles, by the 22nd of the third book. But d e c is acute, since the triangle c d e is isosceles; the angle d g c will therefore be obtuse. Moreover, the angle d c e at the base of that isosceles triangle is acute, and the angle d g e is equal to it, standing as it does in the same segment d g e [?]. The angle d g e will therefore itself be acute. So the straight line a g makes an acute angle with g e, and an obtuse angle with g c. Let the straight line g i therefore be erected from the point g upon a g at right angles.

Diagram. Two circles set in the lower half of the right-hand column. The larger circle is the Sun's eccentric a b c: **a** stands at the top, **b** at the left, **c** at the lower right; **f** is marked below the circle at the foot of the vertical line a f, the line of the *aux*. The smaller circle, drawn inside the larger and tangent to it in the lower right region, is the circle described about the triangle c d e: **d** lies on the vertical line above **g**, and **g** — the centre of the world — sits at the bottom of the vertical line where the two circles meet. **e** is marked on the circumference at the right. Straight lines are drawn from g and from d to the points on the right-hand arc, forming the triangle c d e and the quadrilateral d e c g. A short mark on the right-hand arc between e and c carries the label **i** [?], the point of mean longitude.

- a (top of the larger circle): end of the line of the *aux*
- f (below the circle, foot of the vertical line): the other end of the line of the *aux*
- b (left of the larger circle)
- c (lower right, on the larger circle)
- d (on the vertical line, above g): centre of the eccentric
- g (bottom of the vertical line, where the circles meet): centre of the world
- e (right, on the larger circle)
- i [?] (right-hand arc, between e and c): the point of mean longitude

That perpendicular will therefore fall between g e and g c; and so i will be the point of mean longitude. Accordingly, if for some period of time the Sun's mean motion and apparent motion are equal, the point of mean longitude will be included within their limits — which was to be demonstrated.

Diagram. A woodcut circle standing at the head of the left column, illustrating the case in which the centre of the world falls *outside* the circle described about the triangle. The large outer circle is the Sun's eccentric; a smaller circle is drawn inside it, passing through the points of the triangle, and several chords are ruled across the figure between the lettered points.

- b (left of the circumference): label *b*
- d (upper middle of the interior, on the eccentric's line of centres): label *d*
- e (right, on the circumference): label *e*
- g (far left of the interior, below the chord from *b*, standing outside the smaller circle): label *g*
- c (lower right, on the circumference): label *c*
- f (foot of the figure, on the circumference): label *f*

- o (at the centre, where the line drawn from *g* cuts the smaller circle): label *o*
- [?] (a further label stands at the top of the circle, buried under show-through from the reverse of the leaf and not resolvable)

the angles [?] and *d o e*, taken together, will be greater than the two angles *d g c* and *d g e* taken together. But these were shown to be equal; so this is impossible. And therefore the point *g* is in no way left outside the described circle.

By the same method we shall show that it cannot be left inside that circle either — the circle described about the triangle *c e d*. For let *g e* be produced until it meets the circumference of that same circle at the point *l*, as in the third figure, and let *d l* and *c l* be joined. Then the two angles *d g c* and *d g e*, taken together, will be shown equal to the two angles *d l c* and *d l e* taken together, exactly as before. But *d g c* is greater than *d l c*, by the 21st proposition of Euclid's first book, and *d g e* is likewise greater than *d l e*, by the 16th of that same first book. The two angles *d g c* and *d g e* taken together will therefore be greater than *d l c* and *d l e* taken together — so they are at once equal and greater, which is again impossible.

Diagram. The third figure: the Sun's eccentric drawn as a large circle with the line of the aux running vertically through it, and a smaller circle drawn about the triangle at the right of the interior, cutting the produced line at *l*. Chords join the lettered points across both circles.

- a (top of the circumference): label *a* — the aux
- b (left, on the circumference): label *b*
- d (centre of the eccentric): label *d*
- e (upper right, on the circumference): label *e*
- g (inside the smaller circle, below and right of *d*): label *g*
- L (immediately left of *g*, on the smaller circle): label *l*
- c (lower right, on the circumference): label *c*
- f (foot of the circumference): label *f* — the point opposite the aux

And therefore the circle described about the triangle *d e c* must pass through *g*, as in the first figure; and consequently the two angles *c g e* and *c d e* are equal. So when the Sun has traversed the arc *e c* of the eccentric with its mean motion, and has passed over that arc of the zodiac with its apparent motion — the zodiacal arc being bisected by the line of mean longitude — the mean motion for that period will be just as great as the apparent motion. This is what we undertook to demonstrate.

Propositio tertia.

¶ *For any given interval of time, to find the arc of the zodiac which the Sun traverses in that time by its apparent motion, and over which it makes its mean motion and its apparent motion equal in the same time.*

How great an arc of the zodiac is traversed by the line of the Sun's mean motion in a given time will be known from the tables. If it turns out to be 180 degrees, then the semicircle of the zodiac to be taken is the one reckoned from the aux to the point opposite the aux following the order of the signs, or else the one reckoned from the point opposite the aux to the aux. But if it is less, we shall subtract half of it from 90 degrees, and the remainder will be the distance of the beginning of that arc from the point of the aux. And since the place of the aux is known from the tables, it cannot be unknown how far the beginning of the given arc stands from the first point of Aries. You will therefore add the whole quantity of the arc to that beginning, and the arc of the zodiac will be established which the Sun traverses in the given time by its apparent motion, and over which it meanwhile travels an equal amount by its mean motion.

Remember, though, that this holds in common for two arcs of the zodiac lying at equal distances from the point of the aux. One of them recedes from the point of the aux following the order of the signs, the other in the contrary direction. Let the series of the signs in the figure below run from *a* to *f* by way of *i*. Then if the Sun's mean motion in the given time is 180 degrees, the Sun will pass through the semicircle *a i f*, or *f b a*, on the eccentric. But the diameter of the ecliptic passes through *a* and *f*; and therefore the apparent motion in the given time will likewise

be 180 degrees. But suppose the Sun's mean motion comprises fewer degrees than those [?]. We shall then construct, with the line *g i*, the angle *i g e*, which subtends in the zodiac half of those degrees and minutes, and we shall make the angle *i g c* equal to it. The whole angle *e g c* therefore subtends the arc of the zodiac of the apparent motion in the given time. And since it is bisected by the line *g i* of mean longitude, the mean motion for that time will be as great as the apparent, by the preceding proposition. Now let the straight lines *g e* and *g c* be extended until they meet the circumference of the circle at the points *b* and *h*. Since vertically opposite angles are equal to one another, when the Sun has passed through the arc *h b* of the eccentric, the mean motion for that time will be as great as the apparent.

Diagram. The Sun's eccentric drawn as a circle with the line of the aux running vertically from *a* at the top to *f* at the foot; *d* is the centre of the eccentric and *g*, below it, the centre of the world, from which straight lines are ruled out to the circumference on both sides.

- a (top of the circumference): label *a* — the aux
- b (upper left, on the circumference): label *b*
- e (upper right, on the circumference): label *e*
- d (interior, on the line of the aux): label *d*
- g (interior, below *d*, the point from which the radiating lines are drawn): label *g*
- h (lower left, on the circumference): label *h*
- c (lower right, on the circumference): label *c*
- f (foot of the circumference): label *f* — the point opposite the aux
- [?] (the label at the right of the figure, at mid-height, does not resolve on the scan; the text requires a point *i* on the line of mean longitude)

From the 90 degrees, then, that the right angle *a g i* contains, we shall subtract the degrees of the acute angle *e g i* — that is, half of the given motion — and what is left will be that arc of the zodiac, now known, which the angle *a g e* subtends. And since the place of the aux *a* is known from the tables, the place of the point *e* will therefore be known. If you add to it the whole arc of the zodiac of apparent motion subtended by the angle *e g c*, the beginning and the end of the arc sought will be evident. And since the arc of the zodiac subtended by the angle *b g h* is set opposite to it, both arcs of apparent motion will be known.

Nor will it be difficult to find how far the Sun, when it is at *e*, stands from the point of the aux according to its mean motion. For let the straight lines *d e* and *d c* be joined, and from the point *d* let *d r* be drawn at right angles upon *g e*. It will fall inside the triangle *d g e*, because the angle *e g d* has been shown to be acute, and *d e g* is acute as well, subtending as it does the smaller side. In the right-angled triangle *g d r*, then, the acute angle *d g r* is now known, and the line *d g* is supposed known in the parts of the semidiameter *d e*. And since as the whole sine is to the right sine of the angle *d g r*, so is *d g* to *d r*, the line *d r* will be known in those same parts. But *d r* is the right sine of the arc of the angle *d e r*; and therefore the arc of the angle *d e r* will be known. Now the angle *a d e*, which measures the Sun's distance from the aux according to mean motion, is equal to the two interior angles *d e r* and *d g e* in the triangle *e d g*. The angle *a d e* will therefore be known, and consequently it cannot be unknown how far the Sun, when at *e*, stands from *a* according to mean motion.

Astronomers call the angle *d e g* itself the angle of the equation; and it is certainly equal to the angle of the equation *d c g* belonging to the point *c*, since the two stand in one and the same segment of the circle described about the triangle *d c e*, by the demonstration of the preceding proposition.

But let us now suppose the mean motion answering to the given time is found to be greater than 180 degrees. We shall subtract it from 360, and work with the remaining arc in the manner already described. For once the arc of the zodiac of apparent motion which is less than a semicircle is known, the arc left over from the whole circle cannot be unknown. Thus, if the mean motion answering to the given time, subtracted from 360 degrees, leaves the arc *c e*, we shall make known by the method already described the arc of the zodiac of apparent motion subtended by the angle *e g c*. And it will be known at the point where it is known how far its endpoints stand from the point of the aux.

The Sun will therefore traverse the remaining part of the zodiac by its apparent motion in the given time, and its mean motion will equal its apparent motion over that same time.

For example: let the year be A.D. 1502, in which I was born, and the given time 60 days; and let it be required to find the arc of the zodiac traversed by apparent motion in those 60 days, to which the mean motion of that same interval is equal. From the resolved tables, then, I extract the mean motion of 60 days: 1 sign, 29 degrees, 8 minutes, 20 seconds. Half of this comes to 29 degrees, 34 minutes, 10 seconds, and such will be the angle *e g i*. Subtracting it from 90, there will remain 60 degrees — or 2 [?] signs — 25 minutes, 50 seconds, for the distance of the beginning of that

arc from the point of the aux, the arc which the angle *d g e* subtends in the zodiac. And since the same tables assign the Sun's aux for that year as 3 signs, 1 degree, 11 minutes, 55 seconds, we shall add these together and find that the beginning of the sought arc of apparent motion stands 5 signs, 1 degree, 37 minutes, 45 seconds from the first point of Aries. It will therefore be 1 degree, 37 minutes, 45 seconds of Virgo. To these 5 signs, 1 degree, 37 minutes, 45 seconds we shall add the arc of the mean motion — namely 1 sign, 29 degrees, 8 minutes, 20 seconds — and we shall arrive at last at 7 signs, 46 minutes, 5 seconds, which was the distance of the end of the sought arc from the first point of Aries. We shall therefore conclude that in that year, over a span of 60 days, the Sun passed by its apparent motion from 1 degree 37 minutes 45 seconds of Virgo to 46 minutes 5 seconds of the first degree of Scorpio, traversing an arc of the zodiac equal to its

mean motion; and that it behaved in the same way in the opposite arc of the zodiac.

And since in that same time the line of mean motion was running ahead, it will be worth finding the equation, so that we may understand how far the Sun stood from the first point of Aries by its mean motion. The method is this. The same tables assign the greatest equation of the solar motion as 2 degrees, 10 minutes, to which, in a table of right sines that assigns the circle's semidiameter 100000 equal parts, there answer 3780 parts. The ratio of the semidiameter of the eccentric to the eccentricity will therefore be that of 100000 to 3780. And since as the whole sine is to the right sine of the angle *d g r*, so is *d g* to *d r*, we shall multiply the 3780 parts contained by *d g* into 86976, which are in the sine of the arc of the angle *d g r*, now known — that is, 60 degrees, 25 minutes, 50 seconds. We shall then divide the product by the whole sine, simply by striking off the last five figures, and there will come out in the quotient about 3288, to which in that same table of right sines there answers 1 degree with 53 minutes, for the magnitude of the angle of the equation *d e g*.

Now the angle *a d e* is equal to the two interior and opposite angles *d g e* and *d e g*. So by adding together 60 degrees 25 minutes 50 seconds and 1 degree 53 minutes we shall obtain an arc of 62 degrees 18 minutes 50 seconds for the magnitude of the angle *a d e*; and consequently the arc *a e* of the eccentric subtended by it will comprise the same number of degrees, minutes and seconds. But the arc of the ecliptic between the point of the aux and the line of mean motion is proportional to *a e*. To those 62 degrees, 18 minutes, 50 seconds, then, we shall add the Sun's aux — namely 3 signs, 1 degree, 11 minutes, 55 seconds — and 5 signs, 3 degrees, 30 minutes, 45 seconds will result. So when the Sun was at *e*, the line of mean motion stood at 3 degrees, 30 minutes, 45 seconds of Virgo.

Or you will work more easily if you add the equation found — 1 degree, 53 minutes — to the Sun's true place in the zodiac when it is at *e*, the point of the eccentric; and you will add the same amount above its true place when it is at *c*.

Propositio quarta.

¶ *To find what the preceding proposition taught, but in another and far easier way, and without the help of the other propositions.*

Let the mean motion answering to the given interval of time be found from the tables. If it is found to be 180 degrees, you will declare that the Sun's motion in that time runs from the aux to the point opposite the aux, or from the point

opposite the aux to the aux. If it is less, let it be subtracted from those 180 degrees — for two right angles at the centre of the circle contain just that many — and let half of the remainder be taken; and you will have the Sun's distance from the point of the aux, traversed by apparent motion, at the moment when it begins to run through the arc sought. You will then add the arc of the mean motion drawn from the tables, and by that method you will have the beginning and the end of that arc of the zodiac which the Sun, traversing it by apparent motion, makes equal in the same time to the mean motion. And since the place of the aux is known from the tables, the arc sought will be entirely known.

For let *a b c* be the Sun's eccentric, and *d* its centre; let *c e* be the mean motion answering to the given time; and let the straight lines *d c*, *d e* and *c e* be joined. Then about the triangle *d c e* let the circle *d c e* be described, and in it let the straight line *d g*, equal to the eccentricity, be understood to be fitted, and let it be extended in both directions until it meets the circumference of that circle at the points *h* and *K*; and let *c g* be joined. We shall set *g* as the centre of the world; and *h K* will therefore be the line of the aux. Now in the isosceles triangle *d c e* the angle *c d e* is known, since it subtends the known arc *c e*, the mean motion in the given time. The two angles at the base *c e* will therefore be left known, if the angle *c d e* itself is subtracted from 180 degrees. And so half of that re-

Diagram. A woodcut circle at the head of the left column, belonging to *Propositio quarta*. The large outer circle is the Sun's eccentric; a smaller circle is drawn within it, on the right, passing through *d*, *e* and *c* — the circle described about the triangle *d c e*. The line of the aux runs down through the small circle from the label at its top, through *g*, to *K* at its foot; chords are ruled from *d*, *e* and *c* to *g*.

- a (upper left, on the eccentric's circumference): label *a*
- h[?] (top of the smaller circle, where the line of the aux meets it): the letter is faint on the scan; the text of the proposition names this point *h*
- b (left, on the eccentric's circumference): label *b*
- d (centre of the eccentric, at the upper left of the smaller circle): label *d*
- e (right, on both circumferences): label *e*
- g (below *d*, inside the smaller circle): label *g* — the centre of the world
- K (foot of the smaller circle): label *K*
- c (lower right, on both circumferences): label *c*

will therefore be known: namely the angle *d c e*. Now let the straight line *g e* be joined. The angle *d g e* will then be equal to the angle *d c e*, since the two stand in one and the same segment. So with the angle *d g e* known, if we place the Sun at *e*

— that is, at the beginning of the given time — its distance from the point of the aux by apparent motion will be known, and consequently its distance from the beginning of Aries as well. In the same way, when the Sun is at *c*, its distance from that same beginning of Aries will be found. That the mean motion and the apparent motion are equal to one another you will conclude from this: that the two angles *c d e* and *c g e* are equal to each other.

The angle of the equation *d e g* will be made known by a single inference, from the equation that occurs at mean longitude [?] and from the known angle *d g e*. For the eccentricity *d g* is equal to the right sine of the angle of the equation that occurs at mean longitude. So if the equation at mean longitude is taken as 2 degrees 10 minutes, as the *tabulae resolutae* set it down, the distance between the centres will be 3780 of the parts of which the semidiameter of the eccentric contains 100000. And in the rectilinear triangle *e d g*, as *d e* is to *d g*, so is the right sine of the angle *d g e* to the right sine of the angle *d e g*. Therefore, by the common rule of proportional numbers, from *d e* and *d g* and the known sine of the angle *d g e*, you will conclude the right sine of the angle of the equation *d e g*; and so the angle of the equation itself will be found from the table of right sines. This will give you the Sun's distance from the beginning of Aries by mean motion at either of the limits *e* and *c*, as before in the preceding proposition.

Annotatio tertia.

When the Sun is at mean longitude, the difference between its mean motion and its apparent motion is greatest. At places removed from that point by equal intervals of apparent motion the differences are equal; and they become greater the nearer the line of apparent motion lies to mean longitude, and smaller the further off it lies.

For in the eccentric *a b c* let *b g i* be the line of mean longitude. I say that at the points *b* and *i* the greatest difference between mean and apparent motion occurs. Let the Sun be placed at any point of the eccentric other than *b*, in the semicircle *a b f* — let it be *K* — and let *d K* and *g K* be joined, and also *d b*. We shall then show that the angle of the equation *d b g* is greater than the angle of the equation *d K g*.

At the point *g*, the centre of the world, we shall make with *b g* an angle equal to the angle *b g K*; let it be *b g l*. Let *K l* be joined, and let a circle be described about the triangle *d K l*; and let the straight line *g b*, produced, meet the circumference of the described circle at the point *m*, and let *d m* be joined. Now since the line of

mean longitude *g b* bisects the angle of apparent motion *K g l*, the circle *K l d* will pass through *g*: this was shown in the [?]. proposition of the second annotation.

Diagram. A large woodcut circle filling the foot of the page beneath the right column: the eccentric, with the vertical line of the aux running from *a* at the top through *d* and *g* to *f* at the foot, and the line of mean longitude running across from *b* on the left through *g* to *i* on the right. A second circle — the one described about the triangle *d K l* — is drawn through the interior on the left, and many chords are ruled between the lettered points. The same woodcut is printed again on the following page.

- a (top of the eccentric's circumference): label *a* — the aux
- o (upper left, on the circumference above *K*): label *o*
- K (upper left, on the circumference): label *K*
- z[?] (just inside the circumference to the right of *K*, on the described circle): the printed letter resembles a figure 2 on the scan; the text names this point *z*
- r (left, in the interior, on the described circle): label *r*
- n (interior, to the right of *r*): label *n*
- d (centre of the eccentric): label *d*
- i (right, on the circumference, at the end of the line of mean longitude): label *i*
- m (far left, beyond *b*): label *m*
- b (left, on the circumference): label *b*
- g (below *d*): label *g* — the centre of the world
- L (lower left, on the circumference): label *l*
- f (foot of the circumference): label *f* — the point opposite the aux

Therefore the angle *g m d* will be equal to the angle *g K d*, since the two stand in the same segment. But the angle *d b g* is greater than *g m d*, by the 16th proposition of Euclid's first book; so the angle *d b g* will be greater than the angle *g K d* — which was to be demonstrated.

We shall show the second part in the same figure. Let the Sun be placed at the positions *K* and *l*, at which it stands at equal intervals of apparent motion from the point *b*. I say that the two angles of the equation *d K g* and *d l g* will be equal to one another. For since at those positions the Sun stands equally distant from the point *b* by apparent motion, the two angles *K g b* and *b g l* will be equal to each other. So if a circle is described about the triangle *K l d*, it will pass through

g; and therefore the two angles of the equation *d K g* and *d l g*, standing in the same segment, will be equal to one another — which was to be shown.

The last part will be demonstrated in that same figure again. Let there be two points of the eccentric: *n*, the nearer to *b*, and *K*, the further off. I say that at the point *n* the difference between mean and apparent motion is greater. For from the point *g* let the straight line *g K* be drawn to the more distant point, and let the angle *b g l* be constructed equal to the angle *b g K*, and let a circle be described about the triangle *d K l*, as before. Then let the straight line *d n* be produced until it meets the circumference of the described circle at the point *r*, and let *g r* be joined. The angle *d r g* will therefore be equal to the angle *d K g*, since the two stand in the same segment. But the angle *d n g* is greater than *d r g*, by the 16th proposition of Euclid's first book; so that same angle *d n g* will be greater than the angle *d K g*. Consequently, when the Sun stands at *n*, a point nearer to mean longitude than *K* is, the angle of the equation — that is, the difference between mean and apparent motion — will be greater than at *K* itself.

And in that same figure and by that same method of demonstration we shall show that it is smaller at the point *o*, which is further off still, than at *K*. For let the straight line *g o* cut the circumference of the described circle at *z*, and let *d z* and *d o* be joined. The two angles *d z g* and *d K g*, standing in the same segment, will be equal to one another; but *d z g* is greater than the angle *d o g*, by the 16th proposition of Euclid's first book. Therefore *d K g* will be greater than *d o g*, by the common notion; and consequently the difference between mean and apparent motion will be greater at *K* than at *o*. So when the Sun is at mean longitude the difference between mean and apparent motion is greatest — together with the rest of what had to be demonstrated.

The difference at the point *i* will be just as great as at *b*. For since the straight line *d g* cuts the straight line *b i* at right angles, the two lines *b g* and *g i* will be equal; and therefore the two angles *d b g* and *d i g* will be equal, by the 4th proposition of the first book.

Diagram. A second impression of the same woodcut that stands at the foot of the preceding page, printed here in the right column so that the reader has the figure to hand on this side of the leaf as well: the eccentric with the vertical line of the aux *a … f*, the line of mean longitude *b g i*, the circle described about the triangle *d K l*, and the chords ruled between the lettered points. Its labels are those listed for the figure on the preceding page.

De Luna.

Annotatio prima.

The equation of the centre is the arc of the epicycle lying between the epicycle's true aux and its mean aux. Peuerbach writes, moreover, that this equation is greatest when the centre of the epicycle stands a little below the mean longitudes of the deferent. Those points are commonly called the mean longitudes which are determined by the straight line drawn from the centre of the world perpendicular to the line of the aux. Ioannes Baptista [Giovanni Battista], the old expositor of these *Theoricae*, and certain others, hold instead that the place on the eccentric at which the equation of the centre is greatest is the point at which a certain straight line terminates — the line which, at the point opposite the centre of the eccentric in the small circle, stands at right angles to the line of the aux.

For (says Baptista) let the Moon's eccentric be the circle *a b c d*, whose centre is *e*, and let the diameter of the aux be *a e d*; let the centre of the world be *f*, and let the point opposite to the centre *e* in the small circle be *g*. And from the point *f* let the line *f b*, and from the point *g* the line *g c*, be drawn perpendicular to the line of the aux as far as the circumference of the eccen-

Diagram. A woodcut circle at the head of the left column: the Moon's eccentric *a b c d*, with the vertical diameter of the aux running down the middle of the figure through *e*, *f* and *g* to the foot. The perpendiculars *f b* and *g c* run out to the left to the circumference, and a second circle — *f c g*, described about the right-angled triangle *c f g* — is drawn within the left half, cutting the eccentric at *c* and again at *i*. Chords are ruled from *f* and *g* out to *c*, *K*, *r* and *i*.

- [?] (top of the vertical diameter): the printed letter does not resolve on the scan and reads much like the *d* at the foot; the text names this end of the diameter of the aux *a*
- b (upper left, on the eccentric's circumference): label *b* — the mean longitude
- c (left, below *b*, on the circumference): label *c*
- K (lower left, on the circle *f c g*): label *K*
- r (to the right of *K*, on the eccentric's circumference): label *r*
- e (middle of the figure, on the vertical diameter): label *e* — the centre of the eccentric
- f (below *e*, on the vertical diameter): label *f* — the centre of the world
- g (below *f*, on the vertical diameter): label *g*

- i (lower middle, on the circumference beside the vertical): label *i*
- d (foot of the vertical diameter): label *d*

-tric. Then the line *b f* will be the line of mean longitude and the point *b* the mean longitude; while the point *c*, which lies a little below the mean longitude, will be (he says) the place at which the greatest equation of the centre occurs.

But he is mistaken, as we shall show at once from the very figure he himself drew. For let the straight lines *f c* and *e c* be joined. Next, about the right-angled triangle *c f g* let the circle *f c g* be described. The straight line *f c* will be that circle's diameter, by the converse of the first part of the 31st proposition of Euclid's third book; and therefore the centre of that circle will lie at the midpoint of that same diameter *f c*, and not on the straight line *e c*. Consequently the circle *f c g* does not touch the circle *a b c d* at all. For if it did touch it, then the point of contact — which would be *c* — and the centres of both circles would lie on one and the same straight line, by the 11th proposition of Euclid's third book. But the centre of the circle *a b c d* lies on the straight line *e c*, while the centre of the circle *f c g* lies on *f c*; and therefore they do not touch, but each cuts the other. And since, when a circle cuts a circle, it cuts it in two places only, by the 10th proposition of that same third book of Euclid, let one of their intersections be at *c* and the other at *i*, between *c* and *d*, and let the straight lines *f i* and *g i* be joined. I say, then, that at any point between *c* and *i* the equation of the centre is greater than at *c*, while at *c* itself and at *i* the equations are equal.

For let *r* be any point on the circumference of the eccentric between *c* and *i*, and let *f r* and *g r* be joined; and let *g r* itself, produced in a straight line, meet the circumference of the circle *f g c* at the point *K*, and let *f K* be joined. Then the two angles *f c g* and *f K g*, since they lie in the same segment *f c K g*, will be equal to each other, by the 21st proposition of Euclid's third book. But the angle *f r g*, being exterior in the triangle *f r K* to the interior and opposite angle *f K r* — that is, *f K g* — is greater, by the 16th proposition of Euclid's first book. The angle *f r g* will therefore be greater than the angle *f c g*, and consequently the equation of the centre at *r* is greater than at *c*. But the two angles *f c g* and *f i g* are equal to one another, since they stand in one and the same segment of the same circle *f g c*; and therefore the equations of the centre at *c* and at *i* will be equal — which is just what we undertook to demonstrate.

Lemma.

We assumed that the second intersection of the described circles lies at *i*, between *c* and *d*, and not between *c* and *a*. This we shall demonstrate by the following

device. For if the second intersection of the circles *a c d* and *f g c* were to fall between *a* and *c* — let it then be at *l*, and let *f l* be joined. And since the straight line *[?] c*…

Diagram. A woodcut circle filling the foot of the right column: the figure for the lemma. The eccentric is drawn with its vertical diameter running from the top of the circle through *e*, *f* and *g* to *d* at the foot; the point *l* is set on the circumference at the upper left, *c* on the circumference at the left, and straight lines are ruled from *f* out to *l* and to *c*. The second circle is drawn through the left half of the interior.

- [?] (top of the vertical diameter): the printed letter does not resolve on the scan and reads much like the *d* at the foot; the text names this end of the diameter of the aux *a*
- L (upper left, on the circumference): label *l*
- e (upper middle, on the vertical diameter): label *e* — the centre of the eccentric
- f (below *e*, on the vertical diameter): label *f* — the centre of the world
- c (left, on the circumference): label *c*
- g (below *f*, on the vertical diameter): label *g*
- d (foot of the vertical diameter): label *d*

is a diameter of the circle *f g c*, and therefore *f c* is greater than *f l*. But now, in the circle *a c d*, two straight lines *f c* and *f l* have been drawn from the point *f* — which is not the centre of that circle — as far as the circumference of that same circle *a c d*; and of these *f l* is nearer to the centre than *f c*. It follows that *f l* must be greater than *f c*, by Euclid III.7. Yet it was shown to be less, and this is impossible. Consequently the two circles described, *a c d* and *f g c*, cut one another at the point *c* and at some other point between *c* and *d*, and not between *a* and *c* — which is exactly what was assumed.

Annotatio secunda.

Although at every point between *c* and *i* the equation of the centre is greater than it is at *c* and *i* themselves, still at every position between *a* and *c*, and likewise at every position between *d* and *i*, the equation of the centre will be less than at *c* and *i*. For let the centre of the epicycle be placed at the point *m*, between *a* and *c*. I say that the equation of the centre will be greater at *c* than at *m*. Join the two straight lines *f m* and *g m*; and from the point *g* let the straight line *g n* be drawn to the point *n*, where the straight line *f m* cuts the circle *f g c*.

Diagram. A large circle — the eccentric — with a vertical diameter running from *a* at the top to *d* at the foot; on that diameter, in order downward from *a*, are marked *e*, *f* and *g*. On the left-hand circumference, again in order downward, stand *m*, then *c*, then (set a little inside the circumference) *o* and *p*, and lastly *i* near the bottom left. A second, smaller circle is drawn through the left-hand part of the figure, meeting the vertical diameter in the neighbourhood of *f* and *g* and crossing the eccentric on the left. Straight lines are drawn from *f* and from *g* out to *m*, to *n* (which lies just inside the circumference beside *m*, on the line *f m*) and to *c*.

- *a* (top of the vertical diameter): the apogee
- *e*, *f*, *g* (on the vertical diameter, in that order below *a*)
- *d* (foot of the vertical diameter)
- *m* (upper left, on the eccentric)
- *n* (just inside the circumference beside *m*)
- *c* (left, on the eccentric)
- *o* (below *c*)
- *p* (below *o*)
- *i* (lower left, on the eccentric)

Then the two angles *f n g* and *f c g* lie in one and the same segment, *f n c g* [?]. They are therefore equal to one another, by Euclid III.21. But the angle *f n g* is greater than the angle *f m g*, by Euclid I.16. Therefore the angle *f c g* is greater than *f m g*, and consequently the equation of the centre at *c* will be greater than at *m* — which is what had to be demonstrated. In the same way you will show that every equation between *d* and *i* is less than the one that occurs at the point *i* itself.

Annotatio tertia.

It should further be noticed that although the equations of the centre which occur between *c* and *i* are greater than any others that occur at points lying between *a* and *c* and between *d* and *i*, they cannot all be equal to one another. Take any point *o* between *c* and *i*, and describe a circle through the three points *f*, *g* and *o*. Either this circle cuts the eccentric at *o*, or it touches it without cutting. If it cuts, then it cuts again in some other place, by Euclid III.10. Let the second point of section in the figure drawn be *p*, between the points *c* and *i*. Then by the same method we used to show that the equations made at the points *c* and *i* are equal to one another — but less than those which occur at other points placed between those same *c* and *i*, and greater than the rest of the semicircle *a c d* — it can likewise be shown that the equations made at the points *o* and *p* are equal to one

another; less, however, than those which occur between those same *o* and *p*, and yet greater than the rest of that same semicircle. For the point of section which we posited as *p* cannot fall between *c* and *a*, nor between *d* and *i*, nor indeed at *c* and *i* themselves. Since the equations which occur at the points *o* and *p* are equal to one another, while the equation made at *o* is greater than the one made either at *c* or at *i*, or at any other points whatever of the circumferences *a c* and *d i* — as we have demonstrated — the second section, the one at *p*,

will therefore fall between *c* and *i*, so that no impossibility follows.

But let us now suppose that the circle described through *f* and *g* and the point *o* does not cut the eccentric but touches it, as appears in the figure below. Then the equation of the centre made at *o* will be greater than all the rest belonging to the semicircle *a c d*. For let *r* be any other point in the same semicircle, and let *f r* and *g r* be joined. From the point *t*, where the straight line *f r* cuts the circle *f g o*, let the straight line *g t* be drawn to the point *g*. The angle *f t g* will then be greater than the interior and opposite angle *g r t* of the triangle *g t r*, by Euclid I.16. But the two angles *f o g* and *f t g* are equal to one another, because they stand in one and the same segment of the circle *f g o*. Therefore the angle *f o g* is greater than the angle *g r t*, that is, than *g r f*. For this reason the equation of the centre at *o* will be the greatest of all those that can occur at the other points of the semicircle *a c d*; and consequently the equations which occur at the points of the circumference *c i* were not all equal to one another — which is what we had to demonstrate.

Diagram. The same eccentric circle, with the vertical diameter from *a* at the top through *e*, *f* and *g* down to *d* at the foot. On the left-hand circumference, in order downward, are *r*, then *c*, then *o*, and *i* near the bottom. Inside, a smaller circle is drawn through the neighbourhood of *f* and *g* and touches the eccentric on the left. Straight lines join *f* and *g* to *r*, to *t* (a point just inside the circumference beside *r*, where *f r* meets the smaller circle) and to *o*.

- *a* (top of the vertical diameter)
- *e*, *f*, *g* (on the vertical diameter, in that order below *a*)
- *d* (foot of the vertical diameter)
- *r* (upper left, on the eccentric)
- *t* (just inside the circumference beside *r*)
- *c* (left, on the eccentric)
- *o* (below *c*, at the point of contact)
- *i* (lower left) [?]

From all this you will conclude at the same time that if a circle described through *f* and *g* touches the eccentric, then at the point where it touches it the equation of the centre will be greatest. Conversely, at whatever point the equation of the centre is greatest, there the circle described through *f* and *g* must touch the eccentric. For let the greatest equation be at *o*, and let a circle be described about the triangle *f g o*. It either touches the eccentric at *o* or cuts it. Suppose it touches: then the greatest equation occurs at that point. Suppose it cuts: then it cuts in two places, and at these the equations will be equal, while at the points between them they will be greater — contrary to the hypothesis. Therefore it does not cut, but touches.

Annotatio quarta.

It is not yet clear from what we have demonstrated whether there is on the eccentric some point *m* at which the circle described through *f* and *g* touches it, nor by what method this is to be found out — that is, so that we may know for certain whether, among all the equations of the centre that occur in one semicircle, the semicircle running from the apogee to the point opposite the apogee, there is one that is the greatest of all. It will therefore be worth while to settle first the problem that follows.

Given a straight line *a b*, cut at random at the point *c*, to cut it again in such a way that the square of the greater segment exceeds the square of the lesser by the square of the straight line *a c*.

To do this, construct on *a b* the square *a b e d*; draw the diagonal *a e*; and construct on *a c* the square, which we shall call *a c f g*. Then on the given straight line *b e*, in the given angle *a b e*, construct the parallelogram *b e k i* equal to the rectilinear figure *c f e b*, by Euclid I.44 and I.45. I say that the given straight line *a b* has been cut at *i* into two unequal segments — *a i* the greater and *b i* the lesser — and that the square on *a i* exceeds the square on *b i* by the square on *a c*.

For from the point *l*, where the straight line *k i* cuts the diagonal *a e*, let the straight line *m n* be raised parallel to *a b*. The two parallelograms *m i* and *k n* will then be squares, by the corollary to Euclid II.4, and the two complements *b l* and *k m* will be equal, by Euclid I.43. And since the quadrilateral *c f e b* is by construction equal to the parallelogram *b k*, let the common rectilinear figure *b e l i* be taken away; then by the common notion there will be left, equal to one another,

the rectilinear figure *c f l i* and the triangle *k l e*. But you will show, by that same common notion, that the rectilinear figure *c f l i* is equal to the rectilinear figure *g f l m*; and the two triangles *k l e* and *l e n* are likewise equal to each other. Therefore the gnomon *g f c i l m* — which is what remains when the square *c g* is taken from the square *m i* — will be equal to the square *k n*, by the common notion. And so the two squares *k n* and *c g* taken together will be equal to the square *m i*. The square *m i* therefore exceeds the square *k n* by the square *c g*. But the square *m i* is constructed on the straight line *a i*, and the side of the square *k n*, which is *l n*, is equal to the straight line *b i* [?]. So in the given straight line *a b*, with the point *c* marked on it, we have cut the line a second time in such a way that the square on *a i*, the greater segment, exceeds the square of the lesser by the square on *a c*. This is what had to be done.

Diagram. A square standing on the base *a b*, lettered *a* at the bottom left, *b* at the bottom right, *e* at the top right and *d* at the top left, with the diagonal drawn from *a* up to *e*. On the base, between *a* and *b*, are marked *c* and, further to the right, *i*; from *i* a vertical rises to *K* on the top side, cutting the diagonal at *L*. Through *L* a horizontal *m n* is drawn, *m* on the left side and *n* on the right. Lower down on the left side stands *g*, and the small square built on *a c* has its far corner *f* on the diagonal.

- *a* (bottom left corner)
- *c* (on the base, right of *a*)
- *i* (on the base, right of *c*)
- *b* (bottom right corner)
- *d* (top left corner)
- *K* (on the top side)
- *e* (top right corner)
- *m* (on the left side)
- *L* (centre, on the diagonal)
- *n* (on the right side)
- *g* (on the left side, below *m*)
- *f* (interior, on the diagonal)

It will not be difficult to find these segments numerically, following the present demonstration. Let *a b* be a straight line of 60 equal parts, and let the square of the straight line *a c* be 600. Let the same *a b* be so cut at *i* that the square on *a i* exceeds the square on *b i* by those same 600, and let it be required to find how great *a i* and *b i* are. Since the square on *a b* is 3600, we shall subtract 600 from

that number, and 3000 will be left; the half of this is 1500, which we shall divide by 60, and the division will yield 25. Such will be *b i*, and the remaining segment *a i* will accordingly be of 35 parts. This agrees exactly with what was proposed: the square of 35 is 1225 and the square of 25 is 625; take 625 from 1225 and 600 is left, and it is by this that the square of the greater segment exceeds the square of the lesser segment.

Now that all this has been shown, we shall find the point on the eccentric at which the equation of the centre necessarily becomes greatest, and we shall indicate numerically how far that same point is distant from the apogee. Let the Moon's eccentric be the circle *a b c d*, its centre *e*, the diameter of the apogee *a c*, and the centre of the world *f*; and let *g* be the point opposite to the centre *e*, from which the line of the epicycle's mean apogee is drawn. I say that in the semicircle *a b c* there is one point at which the equation of the centre is greatest, and we shall find it by the following method. Describe a square on *e f*, and in the semidiameter *e c* set a straight line *e i* equal to the diagonal of that same square. The square on *e i* will then be equal to twice the square on *e f*, by Euclid I.47 and the common notion. Next we shall cut the given straight line *e c* in such a way that the square of the greater segment exceeds the square of the lesser by the square on *e i*, by the preceding problem. Let the greater segment be *e o*, and let the lesser be *c o*. And since, on the doctrine of Ptolemy assumed above — that the point *[?] g* lies between

Diagram. A large circle, the Moon's eccentric, lettered *a* at the top, *b* on the left, *d* on the right and *c* at the foot. A vertical diameter runs from *a* down to *c*; on it, in order below *a*, are marked *e*, then *f*, then *t*, then *g*, then *o*. A smaller circle is drawn within the lower left part of the figure, passing through the neighbourhood of *f* and *g*. On the left, *k* stands inside the circle and *r* on the circumference at the lower left; several straight lines are drawn from *k* and *r* to the points *f*, *t* and *g* on the diameter. Two further small letters stand inside the smaller circle, at the upper left, which do not resolve on the leaf [?].

- *a* (top of the vertical diameter): the apogee
- *b* (left of the circumference)
- *d* (right of the circumference)
- *c* (foot of the vertical diameter)
- *e* (on the diameter, below *a*): the centre of the eccentric
- *f* (on the diameter, below *e*): the centre of the world
- *t* (on the diameter, below *f*)
- *g* (on the diameter, below *t*)

- *o* (on the diameter, below *g*)
- *k* (interior, left)
- *r* (lower left, on the circumference)
- two interior letters at the upper left, unresolved [?]

between *f* and *c*, and that *e f* and *f g* are equal, it follows necessarily that of the three straight lines *e o*, *c o*, and *e f*, any two taken together are greater than the remaining one. For the moment we simply assume this; below, however, we shall prove it.

From two straight lines equal to *e o* and *c o*, then, together with the straight line *e f*, we shall construct the triangle *e f K*, by Euclid I.22. Let *e K* be equal to the straight line *e o*, and *f K* equal to the straight line *c o*; and let *e k* be produced straight on and continuously until it meets the eccentric at the point *r*.

I say that at *r* itself the equation of the centre is the greatest of all those that arise in the semicircle *a b c*. For since the square on *e o* is equal to the two squares on *c o* and on *e i*, the square on *e k* will be equal to the two squares on *f k* and *e i*. But the square on *e i* is equal to twice the square on *e f*. The square on *e K*, therefore, will be equal to two squares on *e f* together with the square on *f k*, and for that reason the angle *K f e* will be obtuse.

Now from the point *K* draw the straight line *K t* [?], making right angles at the point *t* [?] with *c f* produced in a straight line, by Euclid I.12. The square on the side *e K* will therefore be equal to the squares on the sides *e f* and *f k* together with twice the rectangle contained by *e f* and *f t*, by Euclid II.12. Consequently two squares on *e f*, together with the square on *f K*, will be equal to the squares on *e f* and *f k* together with twice the rectangle contained by *e f* and *f t* — by the common notion that things equal to one and the same thing are equal to one another. From these let us take away the common squares on *e f* and *f k*; there will then be left, equal to one another, the square on *e f* and twice the rectangle contained by that same *e f* and *f t*. But twice the rectangle contained by *e f* and half of *e f* is equal to that same square on *e f*, by Euclid II.2. Therefore the rectangle contained by *e f* and *f t*, and the rectangle contained by *e f* and half of *e f*, are equal to one another; and for that reason the straight line *f t* will be equal to half the straight line *e f*. Moreover *e f* and *f g* are equal by hypothesis. Therefore *f t* and *t g* will be equal to one another, by the common notion.

Next we shall join the straight line *K g*; and in the two right triangles *f t K* and *g k t* the bases *f K* and *K g* will be shown equal to one another, by Euclid I.4. But we set *e k* and *e o* equal; and when these are taken from the equals *e r* and *e c*, the

remainders *K r* and *c o* are left equal. Now *f K* was itself set equal to *c o*. Therefore *f K* and *K r* will be equal to one another by the common notion, and consequently the three straight lines *K f*, *K g*, and *K r* will be equal to one another. So let us describe a circle about *K* as centre with the distance *k r*; it will necessarily pass through the points *g* and *f*. And since the centres *K* and *e* of the circles *a b c d* and *f r g* lie on one and the same straight line *e r*, and the point *r* lies on both circles, the circle *f r g* will touch the circle *a b c d* at that same point *r*. It does not cut it: for by Euclid III.10 and I.20 an impossibility would follow, against the definition of a circle.

So let us join the straight lines *f r* and *g r*. The angle *f r g* will then be the greatest of those that can be constructed at the remaining points of the semicircle *a b c* by lines drawn from the points *f* and *g*, by what we demonstrated in Annotation 3. These same angles, moreover, are vertically opposite to the angles that at the centre of the epicycle subtend the equation of the centre. Consequently the greatest equation of the centre occurs at the point *r* — which is what we undertook to find.

Diagram. The Moon's eccentric drawn as a large circle, with *a* at the top, *c* at the foot, *b* on the left, and *d* on the right. The vertical diameter *a c* carries, in order below *a*: *e* (centre of the eccentric), *f* (centre of the world), *t*, *g*, and *o*. A smaller circle, centred at *K* on the left, passes through *r* on the lower-left circumference of the eccentric and through the neighbourhood of *f* and *g*; straight lines are drawn from *K* to *f*, *t*, *g*, *o*, and *r*, and from *r* to the points on the diameter.

- *a* (top of the eccentric): apogee end of the diameter
- *b* (left): point of the eccentric
- *c* (foot): perigee end of the diameter
- *d* (right): point of the eccentric
- *e* (on the diameter, below *a*): centre of the eccentric
- *f* (on the diameter, below *e*): centre of the world
- *i* [?] (on the diameter, between *f* and *t*): faint, not securely read on the scan
- *t* (on the diameter, below *f*): foot of the perpendicular from *K*
- *g* (on the diameter, below *t*): point from which the line of the epicycle's mean apogee is drawn
- *o* (on the diameter, below *g*): division point of *e c*
- *K* (left, interior): centre of the small circle
- *r* (lower left, on the eccentric): point of greatest equation of the centre

Lemma.

Our assumption — that of the three straight lines *e o*, *c o*, and *e f*, any two taken together are greater than the remaining one — will be easy to demonstrate. For *e o* and *c o* are greater than *e f*. Again, since *e o* is greater than *c o* [?], *e o* and *e f* are much greater than *c o* [?]. That *c o* and *e f* are greater than *e o* we shall show as follows.

f o is less than *c o*. For suppose it equal to *c o*. Then, since the square on *c o* together with twice the square on *e f* is equal to the square on *e o*, while the squares on *e f* and *f o*, together with twice the rectangle contained by *e f* and *f o*, are equal to that same square on *e o*, by Euclid II.4 — two squares on *e f* together with the square on *f o* will, by the common notion, be equal to the squares on *e f* and *f o* together with twice the rectangle contained by *e f* and *f o*. Take away from them one square on *e f* and the square on *f o*: there will be left, equal to one another, the square on *e f* and twice the rectangle contained by that same *e f* and *f o*. Consequently the straight line *e f* will be double the straight line *f o*, by the converse of Euclid I.36 and the common notion — and so equal to the whole of *f c*, contrary to the hypothesis. For following the doctrine of Ptolemy and of the author of the *Theoricae* we set *e f* and *f g* equal, and much less than *f c*.

By a similar chain of reasoning you will show that *f o* is not greater than *c o*. For since the two squares on *e f* together with the square on *c o* are equal to the square on *e o*, while the squares on *e f* and *f o*, together with twice the rectangle contained by *e f* and *f o*, are likewise equal to that same square on *e o* — the two squares on *e f* with the square on *c o* will, by the common notion, be equal to the squares on *e f* and *f o* together with twice the rectangle contained by *e f* and *f o*. From the two squares on *e f* and the square on *c o*, then, we shall take away one square on *e f* and the square on *c o*, and one square on *e f* only will be left. But from the squares on *e f* and *f o*, together with twice the rectangle contained by *e f* and *f o*, we shall take away the squares on *e f* and *f o* — which are the greater, if *f o* is greater than *c o* — and for that reason a greater remainder is left: the square on *e f* over twice the rectangle contained by *e f* and *f o*. And therefore the segment *f o* is less than half of *e f*, by the common notion.

The segment *c o* [?], therefore, is much less than half of that same *e f*; so that the straight line *e f* will be much greater than *f c*, again contrary to the hypothesis. Consequently *f o* is less than *c o*; and therefore *c o* and *e f* together are greater than *e o*, by the common notion — which was what we assumed.

Annotatio quinta.

The next step now is to show how far the point *r* itself — at which the greatest equation of the centre occurs — is distant from the apogee. This will be quite easy, provided only that we take the ratio of the semidiameter *e c* to the eccentricity *e f* as known from the doctrine of Ptolemy.

For since the straight line *e i* [?] was set as the diagonal of the square whose side is the straight line *e f*, the square on *e i* will be known: it is double the square of the straight line *e f*. And the straight line *e c* has been cut, by that same technique, into the segments *e o* and *c o* in such a way that the square on *e o* exceeds the square on *c o* by that same square on *e i*. Consequently the straight lines *e o* and *c o* themselves (as we taught above) will be made plain as known, in the same parts in which *e c* and *e f* are known. And since the straight line *f t* has been shown to be half of *e f*, or of *f g*, the whole *t e* will be known. Similarly, since *e K* was set equal to the straight line *e o*, it will be known; and *K f* likewise will emerge known, since it is equal to the straight line *c o*.

Diagram. The same construction repeated: the eccentric with *a* at the top, *c* at the foot, and *d* on the right; on the vertical diameter, in order below *a*, the points *e*, *f*, *t*, *g*, and *o*; the smaller circle centred at *K* on the left, cutting the diameter and passing through *r* on the lower-left circumference; straight lines drawn from *K* and from *r* to the points on the diameter.

- *a* (top): apogee end of the diameter
- *c* (foot): perigee end of the diameter
- *d* (right): point of the eccentric
- *e* (on the diameter, below *a*): centre of the eccentric
- *f* (on the diameter, below *e*): centre of the world
- *t* (on the diameter, below *f*)
- *g* (on the diameter, below *t*)
- *o* (on the diameter, below *g*)
- *K* (left, interior): centre of the small circle
- *s* (interior, below and right of *K*): lettered on the figure but not used in the text on this page
- *r* (lower left, on the eccentric)

Now, then, in the right triangle *K e t*: as the whole sine is to the right sine of the angle *t K e*, so is the side *e K* to the side *e t*. The first, third, and fourth quantities are known; therefore the second — the right sine of the acute angle *t k e* — will come out known, and from the table of right sines the angle *t k e* itself will be known. By a similar piece of reasoning, in the right triangle *K t f*, from the two

known sides *f k* and *f t* the angle *f K t* will be found; and if we subtract it from 90 degrees, the remaining acute angle *k f t* is left known. Further, we shall subtract the angle *f K t* itself from the angle *e K t*, and the angle *e K f* is left known. This last is an exterior angle in the isosceles triangle *f k r*, in which the two angles *K r f* and *k f r* are equal to one another; the angle *e k f* is therefore double the angle *K f r*, and for that reason the angle *k f r* itself will be known. We shall subtract it from the angle *K f t*, which is already known; and so the angle *r f t* [?] — the distance of the point *r* from the point opposite the apogee — will emerge known. The distance of that same point from the apogee cannot then remain unknown.

Now Ptolemy found the straight line *e f* to be 10 parts 19 minutes of the parts of which *e c* contains 49 parts 41 minutes; for the straight line *a f* contains 60 of those same parts. Consequently, if we put *e c* at 100000 equal parts, the straight line *e f* will contain 20765. If we double the square of this number, subtract it from the square of the straight line *e c*, and divide half of what is left — which is 456881475 [?] — into parts of 100000, that division will yield 45688. Just so great, therefore, will the straight line *c o* be; and consequently the remaining *e o* will be 54312 parts. And since *f t* is equal to half the straight line *e f*, the whole *t e* will be 31147 ½ [?] parts. If we multiply this by the 100000 parts of the whole sine, and divide the product by 54312 — that is, by the parts of the straight line *K e* — the quotient will give the right sine of the angle *t k e*, whose arc is found to be 34 degrees 59 minutes 40 seconds. Next we shall multiply the straight line *f t* — 10381 ½ [?] parts — by the whole sine, and divide the product by the 45688 parts contained in *f K*; and the quotient will give the right sine of the angle *f k t*, whose arc will be found to be 13 degrees 8 minutes 7 seconds.

The remaining angle *K f t* of the right triangle *K t f* will therefore be 76 degrees 51 minutes 53 seconds. Again, from the angle *t K e*, now known — that is, 34 degrees 59 minutes 40 seconds — subtract the 13 degrees 8 minutes 7 seconds of the angle *t k f*, and 21 degrees 51 minutes 33 seconds are left as the magnitude of the angle *e k f*. Half of that angle — namely the angle *K f t* [recte: *k f r*] — will be 10 degrees 55 minutes 46 seconds. These being subtracted from the 76 degrees 51 minutes 53 seconds of the angle *K f t*, there are left 65 degrees 56 minutes 7 seconds. Just so much, then, will the angle *r f t* contain, the distance of the point *r* from the point opposite the apogee; and so the distance of that same point from the apogee will be 114 degrees 3 minutes 53 seconds. Just so far, then, will the Moon's centre stand, when the epicycle has been set at that point of the eccentric at which the equation of the centre is greatest.

Annotatio sexta.

How great this greatest equation of the centre itself is, you will at once conclude from what we have just demonstrated. For the angle *t K e* was found to be 34 degrees 59 minutes 40 seconds; and the angle *f k t* contains 13 degrees 8 minutes 7 seconds, to which the angle *t K g* is equal, by Euclid I.4. The whole angle *g K e*, therefore, will be 48 degrees 7 minutes 47 seconds. And since in three…

Diagram. The same construction once more: the eccentric with *a* at the top, *c* at the foot, *b* on the left and *d* on the right; on the vertical diameter, in order below *a*, the points *e*, *f*, *t*, *g*, and *o*; the smaller circle centred at *K* on the left, touching the eccentric at *r* on the lower left; straight lines drawn from *K* and from *r* to the points on the diameter.

- *a* (top): apogee end of the diameter
- *b* (left): point of the eccentric
- *c* (foot): perigee end of the diameter
- *d* (right): point of the eccentric
- *e* (on the diameter, below *a*): centre of the eccentric
- *f* (on the diameter, below *e*): centre of the world
- *t* (on the diameter, below *f*)
- *g* (on the diameter, below *t*)
- *o* (on the diameter, below *g*)
- *K* (left, interior): centre of the small circle
- *s* (interior, below and right of *K*): lettered on the figure but not used in the text on this page
- *r* (lower left, on the eccentric)

the triangle k g r is isosceles, the exterior angle g K e[?] is double the interior angle opposite it, g r K, by Euclid I.5 and I.16[?]. The angle g r K, therefore, will be 24 degrees 3 minutes 53 seconds. From this we shall subtract 10 degrees 55 minutes 46 seconds — the angle k r f, which was shown to be equal to the angle K f r — and there will remain 13 degrees 8 minutes 7 seconds for the size of the angle f r g, the greatest equation of the centre. In these computations we use the table of right sines constructed by Petrus Appianus, which sets the radius of the circle at 100000 equal parts.

Annotatio septima.

If you want to know the distance of the epicycle from the centre of the world when the epicycle is at the point r — the place where it has the greatest equation of the centre — you can reach that easily by dropping the perpendicular K s from

the point K onto the line f r. For the two straight lines f s and r s will be equal to one another; and the angle K f s is already known, so the remaining angle f K s will be known as well, by Euclid I.32. Now as the whole sine is to the right sine of the angle f K s, so is the straight line f K to the straight line f s. Three of these quantities are known, so the last of them, f s, will come out by the common rule of proportional numbers. But f s is half of the straight line f r, and therefore the whole of f r will be known, and with it the distance of the centre of the epicycle from the centre of the world in that position, in parts of the radius e c. By this method we find the line f s to be 44859; the whole line f r will therefore be 89718 of the parts of which the radius of the eccentric contains 100000.

Annotatio octaua.

We also think it worth noting that, among the equations of the centre which occur on the arc a r — that is, between the apogee and the point r, where the greatest equation occurs — those made at points nearer to r will be the greater, while those made at points further off will be the smaller. Likewise, of the equations that occur on c r, the remaining segment of the semicircle a r c, those made at points nearer to r will be greater than those made at points further from r.

On the Moon's eccentric, then, let r be the point at which the greatest equation of the centre occurs, and on the arc a r let K be a point nearer to r than l is. I say that a greater equation of the centre will occur at k than at l. Let the straight lines f K and g K be joined, and about the triangle f g K let a circle be described [?]. We shall show that this circle does not touch the eccentric at all, but cuts it at K, and again at another point between c and r. For if it touched it at k itself, the equation at r would be smaller than at k, by what we demonstrated in the third annotation; for the point of contact is one only, by the thirteenth proposition of Euclid's thirteenth[?] book — whereas the greatest equation was posited at r, and this is impossible, since it contradicts the hypothesis. The circle f g K, therefore, cuts the eccentric at K; and since it must cut it in two places, we shall show that the other section lies between c and r. For it is not in [?]; and if it is at r[?] itself, then the two angles of the equations, f r g and f K g, will be equal to one another, but smaller than those made between the points k and r, by what we demonstrated in the first annotation. And so the greatest equation of the centre would not be at r, contrary to the hypothesis.

Diagram. A large circle — the Moon's eccentric — drawn at the foot of the page across the foot of both columns, with its vertical diameter running from the top of the circle down to a point at the foot, and with a smaller circle overlapping it on

the left, crossing its circumference and carrying straight lines drawn in to the point on the diameter below (the construction of the circle f g K of the eighth annotation). Show-through from the other side of the leaf overlies much of the figure, and only a few of the engraved letters are legible on the scan.

- upper left, on the small circle: l
- left, at the centre of the small circle: K
- lower left, on the small circle: m
- top of the vertical diameter: [?]
- remaining labels: [?]

Nor will that same circle f g K be able to cut the eccentric at any point other than K lying between a and r. For if it cuts the arc a r at some other point, the angles of the equations at those points of section will be greater than at r, by the demonstration of the second annotation — again contrary to the hypothesis. It does not, therefore, cut the arc a r a second time, and so it will cut between c and r. Let it cut at the point m. The angles of the equations at the points K and m will then be equal to one another, and greater than the one made at l, or at any other points whatever between a and K and between c and m, by the demonstration of the second annotation already cited. Accordingly, at the points of the arc a r nearer to the point of the greatest equation of the centre, the equations that occur will be greater than at the points further off. And we shall show the same of the equations made between c and r — which is what we undertook to demonstrate.

Annotatio nona.

When the Moon lies on the straight line drawn from the centre of the world so as to touch the epicycle, the equation of the argument is at its greatest for whatever position the epicycle occupies; a greater one, however, occurs when the epicycle is in a position nearer the centre of the world than when it is in a position further off. So if the centre of the epicycle is set at the point opposite the apogee of the eccentric, and the Moon lies on the straight line that comes from the centre of the world and touches the epicycle, the greatest equation of the argument of all will be obtained.

Let the Moon's eccentric be the circle a b c d, whose centre is e while the centre of the world is f; let the line of the apogee be a c, and let the epicycle be set in any position b. From the centre of the world f let the straight line f g be drawn in the plane of the eccentric, touching at the point g the greatest circle of the epicycle that lies in that same plane; and let the line f b be produced as far as K on the circumference of the epicycle, while the body of the Moon is placed at g. K, then,

will be the point of the true apogee. Let the Moon's motion on the eccentric be from the place b towards d, by way of a. The true argument will therefore be the arc k g, which is less than a semicircle; and the equation of that argument will be the arc of the zodiac which the angle b f g subtends. Let the point opposite the true apogee of the epicycle be l. It is plain, then, that from the point f no other straight line can be drawn to touch the semicircle k g l except f g; for on any other an impossibility would follow, contrary to the last common notion. All the rest, therefore, which fall on that semicircle, cut it; and for that reason the angle b f g, the angle of the equation of the argument, will be the greatest.

Now let the epicycle be placed in the position m, between b and the point opposite the apogee of the eccentric, and let f m be joined. The straight line f m will accordingly be smaller than f b, by Euclid III.7. Again from f, the centre of the world, let a straight line be drawn to touch the greatest circle of the epicycle that lies in the plane of the eccentric, and let the point of contact be n; and let f m be produced as far as r, the point of the true apogee. The greatest angle of the equation of the argument in the position m will therefore be m f n[?]; and I say that this is greater than the angle b f g, the greatest equation of the argument in the position b. For let the straight lines b g and m n be joined. The angles at the points g and n will then be right angles, by the converse of Euclid III.26[?]; and since b f was shown

Diagram. The Moon's eccentric drawn as a large circle at the foot of the page, with its vertical diameter running from a at the top down through e (the centre of the eccentric) and f (the centre of the world) to the foot of the circle, and d marked on the circumference at the right. Two positions of the epicycle are drawn as small circles overlapping the eccentric on the left: the upper one carries b on the eccentric, with K above it and L within it, and the lines from f run out to touch it at g, with o marked on the line between; the lower small circle carries m, with a second point marked at its lower left and the tangent lines again drawn back to f.

- top of the vertical diameter: a
- on the vertical diameter, middle: e (centre of the eccentric)
- on the vertical diameter, below e: f (centre of the world)
- right of the circumference: d
- upper small circle, above left: K
- upper small circle, on the eccentric: b
- upper small circle, within: L
- lower left of the upper small circle: g

- on the line below L: o
- lower small circle, within: m
- lower left of the lower small circle: m or n [?]
- outer left, on the tangent line: r [?]
- foot of the vertical diameter (expected c): [?]

greater than f m, f g will therefore be greater than f n — in the right-angled triangles b f g and m f n, by Euclid I.47 and the common notion. From f g, then, let the straight line g o be cut off equal to f n, and let the straight line b o be joined. The two angles b o g and m f n, of the triangles g b o and n m f, will therefore be equal to one another, by Euclid I.4. But the angle b o g is greater than the angle b f g, by the sixteenth proposition of that same first book — for it is the exterior angle of the triangle o b f and opposite to it. Therefore, by the common notion, the angle m f n is greater than the angle b f g. And so the greatest equation of the argument that occurs in a position nearer to the centre of the world exceeds the greatest equation of the argument that occurs in the position b, further from that same centre. For this reason, when the Moon's epicycle has been set at c, the point opposite the apogee of the eccentric — that is, in the position closest of all to the centre of the world — the greatest equation of all will occur, which was the last thing to be demonstrated.

You should remember, however, that in whatever position the greatest equation of the argument exceeds that of another position, there too the true argument will be greater than the other true argument. For since the angle m f n is greater than the angle b f g, the remaining angle f m n will be smaller than the remaining angle f b g; and therefore the argument n r will be greater than the argument K g.

Annotatio decima.

In the semicircle of the epicycle running from the true apogee to the point opposite the apogee, if the true arguments are equal but the positions of the epicycle are at unequal distances from the centre of the world, the greater equation will occur in the nearer position than in the further one. For with the epicycle set in the position b, the more distant from the centre of the world, let the Moon stand at l; and with it set at m, the nearer position, let it stand at p; and let the true argument K l be taken as equal to the true argument r p.

I say that a greater equation answers to the argument r p than to the argument k l. For let the straight lines f l and f p be joined, and from the greater, which is b f, let b q be cut off equal to f m, and let q l be joined. In the two triangles q l b and f p m, then, the two angles b q l and m f p will be equal to one another, by Euclid

I.4 — for the two angles l b f and p m f are equal. But the angle b q l is greater than b f l, by the sixteenth proposition of that same first book of Euclid, since it is the exterior angle of the triangle q l f and opposite to it. Therefore, by the common notion, the angle m f p will be greater than the angle b f l. And so in the nearer position an equal argument has the greater equation, which is what we undertook to demonstrate. From this you will infer that any argument whatever has a greater equation at the point opposite the apogee of the eccentric than in any other position.

Diagram. The Moon's eccentric drawn as a large circle in the lower half of the page, its vertical diameter running from a at the top through e down to c at the foot, and the centre of the world f marked on that diameter below e. On the left, two positions of the epicycle are drawn as small circles cutting the eccentric: the upper one carries b, with K and L marked at its left; the lower one carries m, with r at its left and p below; q is marked on the line running from b down to f, and straight lines are drawn from f out to l and p.

- top of the vertical diameter: a
- on the vertical diameter, below the centre: e
- on the vertical diameter, lower: f (centre of the world)
- foot of the vertical diameter: c
- upper small circle, on the eccentric: b
- left of the upper small circle: K
- below K: L
- on the line from b towards f: q
- lower small circle: m
- left of the lower small circle: r
- below r: p
- right of the circumference (expected d): [?]

Annotatio vndecima.

¶ *When, in one and the same position of the epicycle, equal equations answer to unequal arguments, the end of the smaller argument is further from the end of the argument of greatest equation for that position than the end of the greater argument is.*

In the circle a b c, then, from a point d taken outside it, let the straight line d e a be drawn through the circle's centre; let d f g likewise be drawn, but not through

the centre; and let the straight line d c h [?] be drawn so as to touch the circle at c. I say that the arc c g is greater than f c.

Join the straight lines f c and g c [?]. In the triangle c d g, then, the exterior angle g c h is equal to the two interior and opposite angles c g d and c d g. But the angle c f g is equal to that same angle g c h, by the 32nd proposition of the third book of Euclid, since it stands in the alternate segment. The angle c f g is therefore equal to those same two angles c g d and c d g, by the common notion, and so the same angle c f g is greater than c g d. A greater angle, moreover, has a greater arc answering to it, by the 33rd proposition of the sixth book of Euclid. The arc c g is therefore greater than the arc c f.

Let us now take the circle a b c to be the Moon's epicycle, d the centre of the world, a the point of the true apogee, a g the smaller argument, and a f the greater argument — arguments to which one and the same equation angle a d g answers. The point c will then be the point of contact, at which the equation of the argument is greatest in that position. So when the Moon stands at f and at g, the equations of the unequal arguments a g and a f will be equal; but the point g, the end of the smaller argument, will be further from c than f, the end of the greater. This is what had to be demonstrated.

Diagram. A circle stands at the foot of the left column, with the point d placed well below it outside. Three straight lines run up from d: one passes through e to a, that is, through the centre; a second passes through f to g; a third touches the circle at c and runs on to h. The chords drawn from c to f and from c to g close the small triangle at c.

- a (top of the circle): a
- b (right of the circle): b
- c (left of the circle): c
- g (upper left of the circle): g
- f (foot of the circle, left of e): f
- e (foot of the circle): e
- h (outside, upper left, on the tangent produced): h
- d (well below the circle, where the three lines meet): d

Annotatio duodecima.

It was shown in the tenth annotation that the equations of equal arguments increase from the apogee of the eccentric to the point opposite the apogee in proportion as the centre of the epicycle comes nearer to the centre of the world.

To find the Moon's true motion, therefore, one would have had to construct as many tables of the equations of the arguments as there are positions of the epicycle, laid out at least for every second or third degree. But since this was laborious, Ptolemy devised an easy procedure by which the equations of the arguments could be found for every position, even though it departs somewhat from an entirely exact computation.

To carry it out, he first computed the greatest equations of the argument for each position. And since these too increase continuously from the apogee of the eccentric to the point opposite the apogee, as we demonstrated above, he subtracted the greatest equation of the argument that occurs at the apogee from the greatest equation at the point opposite the apogee, and divided the difference into 60 equal parts, or sixtieths, which in the tables of equations are called the proportional minutes [minuta proportionalia].

In the same way he subtracted the greatest equation of the argument at the apogee from the greatest equation of the argument that occurs at every other position, and found by the rule of proportional numbers how many sixtieths, or proportional minutes, each difference amounted to.

For as that greatest difference of the greatest equations — the difference divided into 60 parts — is to the difference found for a given position of the centre of the epicycle, so is the number 60 to the number of sixtieths owed to that position.

In this proportion the first three terms are supposed known, so the fourth will become known as well. By this device the proportional minutes for any centre, that is, for any distance of the epicycle from the apogee of the eccentric, have been set down in the table of the Moon's equations.

He further assumed that, universally, as the differences of the greatest equations of the argument stand to one another, so do the differences of the equations of equal arguments, whatever those arguments may be, at those very same places of the eccentric — even though this strays somewhat from the true and exact proportion.

He therefore judged it enough to construct a single table only, of the equation of the individual

arguments for the position of the apogee, with the differences of those same equations from the equations that occur at the point opposite the apogee set out alongside them; and these differences are what they call the variation of the diameter of the small circle [diuersitas diametri circuli breuis].

So when it is worth finding how great the equation of a given argument is, one first finds the proportional minutes by means of the Moon's centre [centrum Lunae]; then one draws from the table itself the equation of the given argument for the position of the apogee, together with the variation of the diameter — that is, its difference from the equation which an equal argument would have at the point opposite the apogee. And since the number of proportional minutes is known, the rule of proportional numbers at once shows how much of that variation must be added to the equation already found for the given position.

For as 60 is to the number of proportional minutes found opposite the given centre, so is the variation of the diameter found opposite the given argument to the variation owed to the given position. Of these four quantities the first three are known; the fourth will therefore come to light. We shall add it to the equation already found, and so the equation of the given argument will at last emerge known.

You will gather this doctrine of the proportional minutes and of the equations of the arguments from Ptolemy, book 5, chapters 7 and 8 (*Almagest* V.7–8), and from Ioannes de Monteregio, proposition 11. From it, it is plain that the 60 proportional minutes are not the sixtieths of the excess of the longer line — the line drawn from the centre of the world to the apogee of the eccentric — over the shorter line, which goes from the same centre to the point opposite the apogee, however plainly Georg Peuerbach writes this. They are rather the sixtieths of the excess of the greatest equation of the argument that occurs at the point opposite the apogee over the greatest equation of the argument that occurs at the apogee.

Ioannes Baptista, having rehearsed both opinions about the proportional minutes, says this: "But whichever opinion be held, the first or the second, the operation is in no way at fault here, because wherever there happen to be thirty proportional minutes — that is, parts of the excess of the longer line over the shorter, outside the circumference — there too thirty parts of the sixtieths of the variation of the diameter must be added, and conversely." But the error is manifest, as we shall show at once.

For let the circle a b c, whose centre is d, be the Moon's eccentric, and let the centre of the world be e, at which the straight line b c makes right angles with the line of the apogee, which is a f. Let the interval between the two centres, which is d e, be bisected at g; from that midpoint let the straight line g K be erected at right angles to a f; and let d K and e K be joined. In the two right-angled triangles d g K and e g K, then, the two sides d K and e K will be equal to one another, by the fourth proposition of the first book of Euclid.

Diagram. A circle with the vertical line a f running from top to bottom through its centre. On that vertical, below a, stand d, then g, then e. The horizontal line b c crosses the vertical at e, meeting the circumference at b on the left and c on the right. From g a line runs to the right at right angles to a f, meeting the circumference at K; and the straight lines d K and e K are drawn.

- a (top of the circle): a
- f (foot of the circle): f
- b (left of the circle): b
- c (right of the circle): c
- d (on the vertical, below a — centre of the eccentric): d
- g (on the vertical, between d and e): g
- e (on the vertical, at the crossing of b c — centre of the world): e
- K (on the circumference, upper right): K

So when the centre of the Moon's epicycle is placed at K, it will be distant from the centre of the world by an interval equal to the radius of the eccentric; whereas the straight line a e exceeds the radius of the eccentric by the interval d e — that is, by thirty proportional minutes, on Peuerbach's account. With the centre of the epicycle placed at the point K, therefore, there will be 30 proportional minutes, and accordingly in that position k thirty sixtieths of the variation ought to be added, namely half of it.

But when the centre of the epicycle is at c, the Moon's centre — that is, the distance of the epicycle from the apogee of the eccentric — comprises ninety degrees, to which the table of the Moon's equations assigns 26 proportional minutes. At K, then, where the Moon's centre is less than 90 degrees, fewer proportional minutes than 26 are due; and so, with the centre of the epicycle placed at k, far less of the variation is to be added than 30 sixtieths. Ioannes Baptista is therefore in error here, which is what we undertook to demonstrate.

Georg Peuerbach, I suppose, meant to define the proportional minutes in this way so that less practised readers might understand that the equations of the arguments increase in proportion as the centre of the epicycle comes nearer to the centre of the world.

¶ De Marte, Ioue, atque Saturno. [On Mars, Jupiter, and Saturn]

Annotatio prima.

When Georg Peuerbach understood that the axes of the orbs carrying the epicycles of the [?] superior planets incline toward the axis of the zodiac, he supposed on that account — as I suspect — that the axes of the eccentrics themselves cut the axis of the zodiac. That is true, and indeed necessary, in the case of Mars alone; in the case of Jupiter and Saturn it is impossible.

For the surface of the eccentric of Mars always declines from the surface of the ecliptic, its greatest amount remaining unchanged, as was said of the Moon. We shall therefore understand the surface of that eccentric circle to be produced until it reaches the convex surface of the eighth sphere. The common section of this surface and the eighth sphere will then be a circle, by the first proposition of the first book of Theodosius, and a great circle, by the 6th proposition of that same first book.

Let such a circle in the figure below be a b c d, with centre e; let the circle of the ecliptic be a f c g, and let their common section be the diameter a e c; let the northern pole of the ecliptic be i, and let the pole of the circle a b c d that lies nearer to the pole i be K. Through those two poles i and k let a great circle d i f be described, by the 30th proposition of that first book of Theodosius. Let its common section with the plane of the ecliptic be the diameter f g, and its common section with the plane of the circle a b c d the diameter b d; and let the straight lines i e and K e be joined, in the plane of the circle d i f.

Now since the circle d i f passes through the poles i and K, it will pass through the remaining poles as well, by the corollary to the 13th proposition of that same first book of Theodosius. It will therefore cut those same circles a b c d and a f c g at right angles, by the 19th proposition.

And since the point i is the pole of the great circle a f c g, the circumference i f will be a quadrant, by the 24th proposition of the first book; and so the circumference i b will be less than a quadrant. The semicircle b i d is thus cut unequally at the point i. And since it has been shown that the semicircle b i d stands at right angles to the circle a b c d upon the diameter b d, the straight line drawn from the point i to b will be the smallest of all those that can be drawn from that point to the circumference of the circle a b c d, by the 25th proposition of the second book.

And accordingly the circumference i b is the smallest of all those that come from i to any points whatever of the semicircle a b c, by the 27th proposition of the third book of Euclid. For that reason i b is the complement of the greatest latitude of the circle a b c d, while the circumference b f is itself the greatest latitude, or declination, of that same circle a b c d from the ecliptic.

The apogee of Mars, moreover, is the point in the plane of the circle a b c d of the eccentric's greatest latitude from the ecliptic, as is had from Ptolemy and from Peuerbach himself; and the straight line drawn from the centre of the world to the point of the apogee passes through the centre of the eccentric. The point of the apogee and the centre of the eccentric therefore lie on that same straight line e b [?].

Let the point l be the centre of the eccentric, and from it, in the plane of the circle d i f, let the straight line l m be erected parallel to K e, by the 31st proposition of the first book of Euclid. And since the straight line K e comes from the point e, that is, from the centre of the circle a b c d, to K, the pole of that same circle, the line K e will be perpendicular to the plane of the circle a b c d, by the 40th proposition of the first book of Theodosius. And because we have drawn the straight line l m parallel to K e, the line l m too will be perpendicular to that same plane of the circle a b c d, by the 8th proposition of the eleventh book of Euclid. Therefore, if that same straight line l m, passing through the centre of the eccentric of Mars, is extended in both directions, it will pass through the poles of that eccentric, by the 9th proposition of the same first book, and it will become the axis of the orb carrying the epicycle of Mars.

But the straight line i e passes through the centre of the ecliptic and through its northern pole; so if it is produced straight on and continuously, it will terminate at the remaining pole as well, by the 13th proposition of the first book of Theodosius, and it will be the axis of the ecliptic.

We shall now show that these two axes, i e and l m, meet on the side of i and m. For since the straight line K e has been shown to be perpendicular to the plane of the circle a b c d, the angle K e l, in the plane of the circle d i f, will be a right angle, by the 2nd definition of the 11th book of Euclid. But in that same plane of the circle d i f three straight lines K e, i e, and e l are joined at the point e. The angle K e l is therefore greater than the angle i e l, by the 9th common notion, and so the angle i e l is itself less than a right angle. The angle m l e, however, is a right angle, by the 2nd definition of the 11th book, since the straight line l m has been shown to be perpendicular to the plane of the circle a b c d. The two straight lines i e and l m, then, together with the straight line e l, in the pla

Diagram. Two intersecting circles of nearly equal size drawn over the head of the leaf, filling the width of the left-hand column. A horizontal diameter runs from *c* on the left through the common centre *e* to *a* on the right; a second, oblique diameter crosses it at *e*. The circles are the eccentric *a b c d* and the ecliptic *a f c g*, which share the two points *a* and *c*.

- d (top of the circle): *d*
- g (inside, a little below *d*, on the second circle): *g*
- K (above and to the right of the centre): *K*
- i (just below and right of *K*): *i*
- c (left end of the horizontal diameter): *c*
- e (centre, where the diameters cross): *e*
- a (right end of the horizontal diameter): *a*
- l (short segment below and right of the centre): *l*
- m (below and right of *l*, at the end of the same short segment): *m*
- b (lower part of the figure, on the inner arc): *b* [?]
- f (foot of the figure): *f*

—ne of the circle *d i f*, make two angles, *i e l* and *m l e*, less than two right angles. They will therefore meet on the side of *i* and *m*, by the 5th postulate. It follows that the axis of the orb carrying Mars's epicycle does cut the axis of the zodiac, which was the first thing to be proved. From this it is also clear that the poles of the orb carrying the epicycle are unequally distant from the poles of the zodiac. For since the axes *i e*

Diagram. A second pair of intersecting circles at the foot of the left-hand column, drawn on the same construction but turned a quarter round: the point *a* now stands at the top and *c* at the bottom, so that the common section *a c* of the two circles is the vertical diameter, with the common centre *e* on it. Near the centre, just above *e*, a small four-sided figure *r o n p* is drawn, representing the erected plane *o p*.

- a (top of the figure): *a*
- K (upper left, inside the circle): *K*
- i (immediately to the right of *K*): *i*
- r (small quadrilateral, upper left corner): *r*
- o (small quadrilateral, upper right corner): *o*
- n (small quadrilateral, below *o*): *n*
- p (small quadrilateral, lower corner, on the vertical diameter): *P*
- b (upper right, on the circumference): *b*
- f (right end of the oblique diameter): *f*
- d (left end of the oblique diameter): *d*
- g (lower left, on the circumference): *g*
- e (centre, below *p*): *e*

- c (foot of the figure): *c*

and *l m* meet on the side of *i* and *m*, the further they are produced on the side of *e* and *l*, the further apart they stand.

That in Jupiter and Saturn the axis of the orb carrying the epicycle cannot cut the axis of the zodiac we shall show in the same figure. Now the point *b* is the point of the deferent's greatest latitude from the ecliptic; but in Saturn the point of the deferent carrying the epicycle that declines most from the ecliptic stands 50 degrees before the apogee — that is, against the succession of the signs — while in Jupiter it stands 20 degrees after the apogee.

Let us therefore set, in the plane of the circle *a b c d*, the point *n* as the centre of the eccentric, whether of Jupiter or of Saturn. From that point *n* let the perpendicular straight line *n o* be erected above the same plane, by the 12th proposition of the 11th book of Euclid. From the same point *n*, in the plane of the circle *a b c d* itself, let the straight line *n p* be drawn, by the 12th proposition of the 1st book, at right angles upon the straight line *a e*, the common section of the two circles *a b c d* and *a f c g*. Then from *n o*, along the straight line *n p*, let the plane *o p* be extended. That plane *o p* will therefore be at right angles to the plane of the circle *a b c d*, by the 18th proposition of the 11th book of Euclid. In that plane *o p*, then, given the straight line *p n* and the point *p* given on it, we shall erect the straight line *p r* at right angles, by the 11th proposition of the 1st book. The angle *n p r* in the plane *o p* will therefore be a right angle; but the angle *n p e*, which lies in the plane of the circle *a b c d*, is also a right angle; and the plane *o p* is at right angles to the plane of the circle *a b c d*. The angle *e p r*, therefore, will be a right angle, by the converse of the defi-

—nition, the 3rd of the 11th book. And for that reason the straight line *e p* will be perpendicular to the plane *o p* itself, by the 4th proposition of the 11th book.

We shall show that this same line *e p* is perpendicular to the plane of the circle *d i f* as well. For it was shown above that the circle *d i f* is at right angles to the circles *a b c d* and *a f c g*; the common section of those two circles — namely the straight line *a c* — will therefore be perpendicular to the plane of that same circle *d i f*, by the 19th proposition of the 11th book. And since the straight line *e p* stands at right angles both to the plane *o p* and to the plane of the circle *d i f*, those two planes, *o p* and the circle *d i f*, will be parallel, by the 14th proposition of the 11th book of Euclid.

Consequently the straight line *n o*, which lies in the plane *o p*, will not be able to meet the straight line *i e*, which lies in the plane of the circle *d i f*, even if the two

are produced without limit. For if they do meet, then the planes in which they lie — planes shown to be parallel — will meet, which is impossible. The straight line *n o*, therefore, does not meet *i e*.

Moreover, the straight line *n o*, passing through the centre of the eccentric, will, if produced in both directions, pass through the poles of that eccentric, by the 9th proposition of the 1st book of Theodosius, and will become the axis of the orb carrying the epicycle. The straight line *i e*, on the other hand, since it comes through the centre of the ecliptic and through the ecliptic's northern pole, will, if produced straight on and without break, end at the other pole, by the 13th proposition of that same 1st book of Theodosius, and will be the axis of the ecliptic. The axis, therefore, of the orb carrying the epicycle of Jupiter or of Saturn does not cut the axis of the zodiac at all — which is what we undertook to prove.

Nor, however, are these axes parallel. For suppose they are parallel. Since the straight line *K e* has been shown to be perpendicular to the plane of the circle *a b c d*, and *n o* is likewise perpendicular to that same plane, the two straight lines *K e* and *n o* will be parallel, by the 6th proposition of the 11th book of Euclid. And so if *i e* is parallel to that same straight line *n o*, then the two straight lines *k e* and *i e*, which meet at the centre *e*, will be parallel, by the 9th proposition of the same 11th book of Euclid — which is impossible.

The axes *n o* and *i e*, therefore, are neither parallel nor do they meet; from which you can conclude that they do not lie in one plane. For if they lie in one plane, then either they meet within that plane or they are equidistant. Since, then, they neither meet nor are parallel, they do not lie in one plane at all.

Annotatio secunda.

Although the axis of the orb carrying Mars's epicycle does cut the axis of the zodiac, that intersection nevertheless falls outside the orb itself, as far as possible from the orb's pole, once the axis has been produced still further in a straight line.

For let the diameter of the ecliptic *a b*, together with the diameter of Mars's eccentric *c d*, make the angles *b e d* and *a e c*, the angles of the greatest latitudes of that eccentric; let the centre of the eccentric be *f*, and the centre of the ecliptic *e*; and let the axis of the orb carrying the epicycle meet the axis of the ecliptic at *g*. Now since the greatest latitude of the deferent carrying Mars's epicycle is only one degree, according to Ptolemy's teaching, in the right-angled triangle *e f g* the acute angle *g e f*, which is the complement of the greatest northern latitude, will

be 89 degrees, and the remaining angle *f g e* will therefore be one degree, by the 32nd proposition of the 1st book of Euclid and by the common notion.

And since the right sine of the acute angle *e g f* is to the right sine of the acute angle *f e g* as the side *e f* is to the side *f g* — a proportion you will grasp at once if you imagine circles described about the centres *e* and *g* with the interval *e g* — while the side *e f* contains six of the parts of which, according to Ptolemy, the radius of the eccentric contains 60: we shall therefore multiply the right sine of 89 degrees, that is, 99984 parts, by 6, and divide the product by 1745, the number of parts in the right sine of one degree. The division will yield nearly 344 parts. Of the parts, then, of which the radius of the eccentric contains 60, the straight line *f g* contains 344. And yet the poles

Diagram. A figure of straight lines only, set in the right-hand column. A long horizontal line runs from *a* on the left through the centre *e* to *b* on the right: this is the diameter of the ecliptic. A second, oblique line crosses it at *e*, running from *c* at the lower left up to *d* at the upper right: this is the diameter of Mars's eccentric. From *e* a nearly vertical line rises to *g* at the top; from *g* a further line runs down to the right and meets the oblique diameter at *f*, so that *e f g* forms the right-angled triangle of the demonstration.

- g (top of the figure): *g*
- d (upper right, end of the oblique line): *d*
- f (on the oblique line between *e* and *d*, where the line from *g* meets it): *f*
- a (left end of the horizontal line): *a*
- b (right end of the horizontal line): *b*
- e (centre, where the two diameters cross): *e*
- c (lower left, end of the oblique line): *c*

of the orb carrying the epicycle lie on the surface of that orb, by the definition; and therefore the axis of the orb carrying Mars's epicycle meets the axis of the zodiac very far indeed from the northern pole of that same orb — which was to be proved.

Annotatio tertia.

Since the orbs that carry the apogees of Jupiter, Mars, and Saturn are moved by the motion of the eighth sphere upon the axis and poles of the zodiac, the points that are at present northern with respect to the ecliptic always have been northern and always will be; and likewise those that are southern of the ecliptic always will be southern, and always have been. Those, on the other hand, that

are at present in the section of the ecliptic's surface have always been in it, and will be so perpetually. Yet the declinations of those same points from the equinoctial circle will be now one thing and now another. But because the orb that carries the epicycle is moved upon its own axis in the succession of the signs, the surface of the eccentric will therefore cut the surface of the ecliptic successively at each one of its points.

Annotatio quarta.

The equation of the centre on the epicycle is proportional to the equation of the centre in the zodiac. For the angle of the equation of the centre on the epicycle is equal to the vertically opposite angle contained by the two straight lines that come from the centre of the equant and from the centre of the world to the centre of the epicycle. And equal to that same angle is the alternate angle contained by the line of the epicycle's true motion and the line of its mean motion. The two angles, therefore — that of the equation of the centre on the epicycle and that of the equation of the centre in the zodiac — are equal to one another.

The greatest equation of the centre, moreover, occurs when the centre of the epicycle is set at the mean longitude of the deferent, which is determined by the line drawn from the centre of the eccentric perpendicular to the line of the apogee; and it occurs there because at that place the greatest angle of equation is produced, as we shall show at once. Let the point *c* be the centre of the eccentric *a b f g*, let *d* be the centre of the world and *h* the centre of the equant, and let the line of the apogee be *b g*, upon which, at right angles over the centre of the eccentric, let the straight line *a c f* fall.

Diagram. A large circle occupying the head of the right-hand column: the eccentric *a b f g*, with *b* at the top and *g* at the foot, so that the apse line *b g* is the vertical diameter, and with *a* at the left and *f* at the right, so that *a f* is the horizontal diameter. Within it a smaller circle is drawn — the circle described about the triangle *d a h* — and straight lines are drawn from the points near the centre out to the circumference on the left. The three points *c*, *d*, and *h* stand close together near the middle of the figure: *c* the centre of the eccentric, *d* the centre of the world, *h* the centre of the equant. The point *e*, taken at large on the semicircle *b a g*, stands on the circumference above and to the left of *a*, and *r* lies on the line drawn from *d* to *e*, where that line cuts the smaller circle.

- b (top of the large circle): *b*
- e (on the circumference, upper left): *e* [?]
- r (inside, on the line *d e*): *r*

- K (inside, between *r* and *h*): *K* [?] — a label on the leaf for which the text of this annotation gives no point
- h (on the smaller circle, right of *K*): *h*
- a (left of the figure, at the end of the horizontal diameter): *a*
- c (near the middle of the figure): *c* [?]
- d (just below *c*): *d* [?]
- f (right of the figure, at the end of the horizontal diameter): *f*
- g (foot of the large circle): *g*

The point *a*, then, is the point of mean longitude, according to Peuerbach's definition. Now let any point whatever other than *a* be taken on the semicircle *b a g* — let it be *e* — and let the straight lines *a d*, *a h*, *e d*, and *e h* be joined. I say that the angle *d a h* is greater than the angle *d e h*.

For let a circle be described about the triangle *d a h*. Since the straight line *a c* cuts the straight line *d h* within that circle into equal parts and at right angles, the centre of the circle *d h a* will lie on that same straight line *a c*, by the corollary to the first proposition of the 3rd book of Euclid. And since the point *c* — that is, the centre of the circle *a b f g* — lies on that very same straight line *a c*, the circle *d h a* therefore touches the circle *a b f g* at *a*. For it does not cut it: by the 10th proposition of the 3rd book of Euclid and the 20th of the 1st book, an impossibility would follow, contrary to the definition of a circle.

We shall therefore draw a straight line from the point *h* to the point *r*, at which the straight line *d e* cuts the circle *d h a*. The angle *d r h* is then equal to the angle *d a h*, by the 19th theorem of the 3rd book of Euclid. But that same angle *d r h* is greater than the angle *d e h*, by the 16th proposition of the 1st book of Euclid. The angle *d a h*, therefore, will be greater than the angle *d e h*. And so the angle of equation *d a h* is the greatest of all those that occur at the remaining points of the semicircle *b a g* from the meeting of the straight lines coming from the points *d* and *h* — which was to be demonstrated.

Now when [the name begins the following leaf] undertook to demonstrate this,

Erasmus Reinhold assumed a false theorem. He drew the straight line from *a* to *e*; then, since in the two triangles *e d a* and *e h a*, which have the common base *a e*, the side *a d* is equal to the side *a h*, while the side *e d* is greater than the side *e h*, he concluded that the angle *e d a* is less than the angle *e h a*. His words are these: "Since, then, of the two triangles *e d a* and *e h a* two sides, *a d* and *a h*, are equal, while two are unequal — *e d* the greater and *e h* the less — it follows that the angle *e d a* is less than the angle *e h a*." And he supposes that this can easily be

shown by describing a circle about *a* as centre, with radius equal to *a h*. But we shall show by a very easy demonstration that no such thing follows.

In a circle whose centre is *o*, let two straight lines *a h* and *a d* be drawn, not through the centre and equal to each other; and out of the arc *d h a*, which is greater than a semicircle, let the straight line *d e* cut off the arc *d h e*, not greater than a semicircle; and let *e h* and *a e* be joined. In the two triangles *e d a* and *e h a*, then, the two sides *a d* and *a h* are equal, while two are unequal — *e d* the greater and *e h* the less — by the 14th theorem of the 3rd book of Euclid. Yet the angles *e d a* and *e h a* are equal to one another, by the 19th theorem of that same 3rd book, since *e*, *h*, *d*, *a* lie in the same segment.

Further, about *a* as centre and with radius *a h* (as he himself directs) let the circle *h d p* be described, and let the straight line *a e*, produced in both directions, meet the circumference of that described circle at the points *l* and *q*. In the adjoining arc *h l* let a point *z* be taken, and let the straight lines *a z* and *e z* be joined; and from the point *t*, at which *e z* cuts the arc *e h*, let a straight line be drawn as far as *a*. So in the two triangles *e d a* and *e z a* the two sides *a d* and *a z* are equal, while two are unequal — *e d* the greater and *e z* the less, by the 7th proposition of the 3rd book of Euclid. Yet the angle *e d a* is greater than the angle *e z a*. For the two angles *e d a* and *e t a* are equal to one another, since *e*, *t*, *d*, *a* lie in the same segment; and that angle *e t a* is greater than the interior and opposite angle *e z a* of the triangle *t z a*, by the 16th proposition of the first book of Euclid. Therefore the angle *e d a* will be greater than the angle *e z a*, by the common notion.

Diagram. An uncaptioned geometric figure occupying the outer half of the page. A large circle carries *l* at the top, *p* at the left, *q* at the bottom, and, on its right-hand side, *z* (upper right), *h*, and *d*. A vertical straight line runs from *l* down through the middle of the figure to *q*, where a small right-angle mark is set. A second, smaller circle lies within, its centre marked *o*, carrying *e* on the left and, at the lower left, the point about which the larger circle is described. From that lower-left point straight lines radiate to *e*, to *t*, to *z*, to *h*, and to *d*; further straight lines join *e* to *z*, to *t*, to *h*, and to *d*.

- *l* (top of the larger circle, on the vertical line)
- *z* (upper right, on the larger circle)
- *t* (just below and left of *z*)
- *h* (right, on the larger circle)
- *d* (right, below *h*)
- *e* (left of centre, on the smaller circle)

- *o* (centre of the smaller circle)
- *a* (lower left; centre of the described circle *h d p*)
- *p* (far left, on the larger circle)
- *q* (bottom, on the vertical line)

Moreover, in the figure described above, where the two straight lines *a h* and *e d* cut each other, let the point be marked *k*, and let two triangles *a d k* and *e h k* be considered, in which the two vertically opposite angles *a k d* and *e k h* are equal to one another. But the angle *d a k* has been shown to be greater than *k e h*; the angle *h d a*, therefore, is left less than the angle *k h e*, by the 32nd proposition of the 1st book of Euclid and the common notion. And since the two straight lines *a d* and *a h* are equal to one another, by the 4th proposition of the 1st book, while the straight line *e d* is greater than *e h*, by the 7th proposition of the 3rd book taken twice: in the two triangles *e d a* and *e h a* the two sides *a d* and *a h* are equal, while two are unequal — *e d* the greater and *e h* the less — and the angle *e d a* is less than the angle *e h a*. Consequently, if two triangles have a common base, and two of their sides are equal while two are unequal, it no more follows that the angle contained by the greater sides is the smaller than that it is the greater, or that the one is equal to the other.

The same slip was made by the old commentator, who from the same premisses tried to conclude, by the 21st proposition of the 1st book of Euclid, that the angle contained by the greater sides is the smaller. Yet it is clear that this cannot be concluded from that 21st proposition, which runs as follows: if from the ends of one side of a triangle two straight lines are set up inwards, meeting at a single point, those lines will be less than the two remaining sides of the triangle, and will contain a greater angle. The other man, Erasmus, went wrong in the very same way — and worse. For he not only misapplied the 21st proposition of the 1st book of Euclid; he also supposed that the two lines *e h* and *a h* (as in the earlier figure of the present annotation) are less than the two lines *e d* and *a d*, by the 7th proposition of the 3rd book of Euclid, on the ground that they are further from the centre *d*, about which is described the circle representing the zodiac, than the lines *e d* and *a d*, which are drawn from that same centre of the zodiac.

But that is not the reason why the two lines *e h* and *a h* are less than the two *e d* and *a d*. It is rather that *e h* is further from *c*, the centre of the eccentric circle, than *a h* is; *e h* is therefore less than *a h*, by the 7th proposition of the 3rd book. Now *a h* and *a d* are equal, by the 4th proposition of the 1st book, once the two right-angled triangles *a h e* [?] and *a d c* have been conceived; and therefore, by the common notion, *e h* will be less than *a d*. In the same way it will be shown

that *a h* is less than *e d*. For *e d* is nearer to that same centre *c* than *a d* is, so that *a d* is less than *e d*; and therefore *a h* too, being equal to *a d*, will be less than *e d*, by the common notion. Thus the two straight lines *e h* and *a h* will be less than the two *a d* and *e d* — which was to be demonstrated.

Annotatio quinta.

Ptolemy calls the mean distances of the centre of the epicycle from the earth the mean longitudes of the deferent; for distances of this kind exceed the shortest — those at the point opposite the apogee — by just as much as they are exceeded by the longest, which are at the apogee of the eccentric. This comes about when the centre of the epicycle is distant from the centre of the world by an interval equal to the radius of the deferent. It is for that position that the tables of the equations of the argument were constructed, and from it the proportional minutes take their start, so that, from the ratio of those minutes to 60, the rule of increase and decrease may be had for the other positions [?].

Now Georg Peuerbach defined the mean longitudes otherwise — namely as those points at which the equations of the centre become greatest, points determined by a certain straight line that makes right angles with the line of the apogee. Nevertheless he asserts that the equations of the argument themselves were computed for the position of the mean longitude. He confirms this openly further on, where he is speaking of Mercury: "the equations of the argument of Mercury," he says, "which are written in the tables, are those that occur while the centre of the epicycle is at the mean distance from the earth; whereas in the other planets it was made with the centre of the epicycle standing at the mean longitude of the deferent." But that in those three superior planets the equations of the argument were computed for the position of mean distance — that is, for the position in which the centre of the epicycle is distant from the centre of the world by an interval equal to the radius of the deferent — and not [?] for the mean longitudes as Peuerbach defined them, we shall show by a plain argument.

Let *a b c d* be the eccentric deferent of Mars, its centre *e*, the centre of the world *f*, and the centre of the equant *g*. Let the diameter *a c* be the line of the apogee, and let *b d* cut it at right angles at *e*, the centre of the deferent. The two points *b* and *d* will therefore be called mean longitudes, following Peuerbach's definition. Now let the straight lines *b f* and *b g* be joined, and let the centre of the epicycle be placed at *b*. The angle *f b g* will then be the angle of the greatest equation of the centre — which in the table of the equations of Mars is found to be 11° 24′.

Diagram. An uncaptioned circle, the eccentric deferent of Mars. The point *a* stands at the top, *b* at the right, *c* at the bottom, and *d* at the left; the vertical diameter *a c* is the line of the apogee and the horizontal diameter *d b* cuts it at right angles. Three points are marked close together on the vertical diameter, in order from the top: *g*, *e*, *f*. Straight lines are drawn from *b* to each of them.

- *a* (top): apogee
- *b* (right): mean longitude by Peuerbach's definition
- *c* (bottom): the point opposite the apogee
- *d* (left): the other mean longitude
- *g* (on the vertical diameter, above *e*): centre of the equant
- *e* (on the vertical diameter, at the middle): centre of the deferent
- *f* (on the vertical diameter, below *e*): centre of the world

So if from the 180 degrees of the two right angles, to which the three angles of the triangle *b g f* are equal, we subtract those 11° 24′, there will remain 168° 36′ for the two angles *b f g* and *f g b*. And since these are equal to one another, because the straight lines *f b* and *g b* are equal, the angle *b f g* [?] will therefore comprise half of those degrees and minutes, that is, 84° 18′. To this, in the table of the equations of Mars, four proportional minutes correspond. The points *b* and *d*, then, are not the places for which the table of the equations of the argument of Mars was drawn up.

You will find the same in Jupiter and Saturn. Accordingly these equations were not computed for the mean longitudes of the deferent as Peuerbach defined them — which is what we undertook to demonstrate.

But if you wish to know the position of the epicycle for which the tables of the equations just mentioned were drawn up, proceed thus. From the midpoint of the straight line *e f*, call it *K*, erect a straight line *K o* at right angles to the line of the apogee, extended as far as the circumference of the deferent. The point *o* will then be distant from the centre of the world by an interval equal to the radius of the deferent — as you will conclude by the 4th proposition of the first book of Euclid, once the straight lines *e o* and *f o* have been drawn. And the radius of the deferent is exceeded by the apogeal line *a f* by just as much as it exceeds the line *f c* drawn to the point opposite the apogee. The centre of the epicycle at the point *o*, therefore, will be said to be at the mean distance from the centre of the world.

So let the centre of the epicycle be placed at *o*, and from *f*, the centre of the world, let the straight line *f r* be drawn in the plane of the eccentric, touching the circle of the epicycle at *r*, by the 17th proposition of the 3rd book of Euclid; and let *o r*

be joined. The angle *o r f* will then be a right angle, by the 18th; and the angle *o f r* subtends the greatest equation of the argument in that position. And since, in the parts of which the radius of the deferent is 60, the radius of the epicycle was shown by Ptolemy to be 39 and a half, it follows that in the parts of which *f o* is 60000, *o r* will be 39500. Therefore, if a circle is understood to be described about the centre *f* with radius *f o*, the straight line *o r* becomes the right sine of the arc of the angle *o f r*. Now to those 39500 parts there correspond, in a table of right sines that sets the whole sine at 60000 equal parts, 41 parts of the circumference with 10 first minutes. The greatest equation of the argument of Mars, then, is 41° 10′ in the position in which the centre of the epicycle is distant from the centre of the world by an interval equal to the radius of the deferent — namely at *o*. And since the value set down in the Alphonsine and Ptolemaic tables is found to be of just so many degrees and minutes, it is clear that the table of the equation of the argument of Mars was drawn up for no other position than *o*.

You will find the same likewise in Jupiter and Saturn, if you care to test it by the method just described. How many degrees the true centre comprehends in that same position *o* will be easy to find. For the ratio of the radius of the deferent to the eccentricity *e f* was found by Ptolemy to be as 60 to 6. Therefore *f o* will stand to *f K* in the ratio of 60 to 3, or of 60000 to 3000. So we shall understand a circle described about *o* as centre, with radius *f o*; and *f K* will accordingly be the right sine of the angle *f o K*. To this, in a table of right sines that puts the radius at 60000 equal parts, there corresponds an arc of 2° 52′; and when the same is subtracted from 90 degrees, there is left the angle *K f o* of the right-angled triangle *f K o*, of 87° 8′. Therefore, with the centre of the epicycle standing at *o*, the true centre contains 87° 8′. To this, in the table of the equations of Mars, no proportional minutes correspond, precisely because the table was composed for that very position *o*. These proportional minutes, however, are not sixtieths of the excess of the distances at the three positions of the epicycle, but of the greatest equations — in accordance with what we said about the proportional minutes of the Moon.

Diagram. An uncaptioned circle, the eccentric deferent of Mars, with a second, smaller circle — the epicycle — drawn on its right-hand side and overlapping the circumference. The point *a* stands at the top, *d* at the left, *c* at the bottom, and *b* at the right; the vertical diameter runs from *a* to *c*. On that vertical, close together below the middle, stand *e* and, beneath it, *K*, with *f* below *K*, and *g* above *e*. A horizontal straight line runs from *d* across through *K* to the right-hand circumference, where *o* is marked just below *b*; the epicycle is drawn about *o*, with *r* on its lower right. Straight lines are drawn from *f* to *b* and from *f* to *r*.

- *a* (top): apogee
- *d* (left)
- *c* (bottom): the point opposite the apogee
- *b* (right): mean longitude by Peuerbach's definition
- *g* (on the vertical, above *e*) [?]: centre of the equant
- *e* (on the vertical): centre of the deferent
- *K* (on the vertical, below *e*): midpoint of *e f*
- *f* (on the vertical, below *K*): centre of the world
- *o* (right-hand circumference, just below *b*): centre of the epicycle at mean distance
- *r* (lower right, on the epicycle): point of tangency of *f r*

De Venere. [On Venus]

Annotatio prima.

The mean motion of Venus is always as great as the mean motion of the Sun. And since the apogee of the Sun's eccentric lies at the same place of the zodiac in longitude as the apogee of Venus's eccentric, the mean centre of Venus will be as great as the Sun's argument. Moreover, in the tables of Alfonso and of Ptolemy the equation of the centre is found to be just as great as the equation of the argument; those tables therefore assume that the centre of Venus's epicycle and the Sun are always at one and the same place of the zodiac in longitude. For since the mean motion of the Sun is as great as the mean motion of Venus's epicycle, then, once equal equations — of the argument and of the centre — have been added or subtracted, the true motion of the epicycle and the true motion of the Sun are left equal; and consequently the centre of the solar body and the centre of Venus's epicycle will always be at the same place of the zodiac in longitude.

Ptolemy showed in Book 10 that the distance of the centre of the world from the centre of Venus's equant, taken with respect to her own deferent, is as great as the solar eccentricity he finds — namely 2 parts 30 minutes, of the parts of which the radii of the deferents contain 60. It follows, then, that between the position of the apogee and that of the point opposite the apogee the centre of Venus's epicycle and the Sun are truly in the same place of the zodiac in longitude only once: namely when the distance of the centre of the epicycle from the centre of the equant is equal to the radius of the deferent. To show this the more easily, we shall set the eccentric of Venus, together with the eccentric of the Sun, in the plane of the ecliptic.

Let the eccentric of Venus be the circle *a b g*, and the line of the apogee *a b*, on which the centre of the world is *c*, the centre of the eccentric *d*, and the centre of the equant *e*. Now as *c e* is to the radius of the orb carrying the epicycle, so is the Sun's eccentricity to the radius of its own deferent; therefore, alternately, as the straight line *c e* is to the Sun's eccentricity, so is the radius *a d* to the radius of the orb carrying the Sun. But the radius *a d* is less than the radius of the orb carrying the Sun; the straight line *c e* is therefore less than the Sun's eccentricity. Let us accordingly place the centre of the Sun's eccentric at *f*, and let the centre of Venus's epicycle be at *g*, in which position it is distant from the centre of the equant by an interval equal to the radius of the orb carrying the epicycle. Let the straight lines *e g* and *c g* be joined, and from *f*, the centre of the Sun's eccentric, let the straight line *f K* be drawn so as to pass through the centre of the Sun;

Diagram. An uncaptioned geometric figure running the full width of the page beneath both columns. A large bold circle — the eccentric of Venus — is cut by a vertical straight line, the line of the apogee, which carries *b* at the top of the circle and *a* at its foot; on that line, between *b* and *a* and in order downward, stand *f*, *e*, *d*, and *c*. A smaller bold circle, whose centre is *g*, lies to the left within the large circle, and a point *i* is marked just to the right of *g*. To the far left, outside the large circle, stands *K*, from which straight lines run to *f* and to the interior of the figure. Below and to the left of the small circle stand *m* and, further out, *l*. A second, fainter circle is drawn to the left, overlapping the first. Straight lines cross the figure joining *K* to *f*, *K* through *g* to the line of the apogee, and *l* and *m* to points on that line.

- *b* (top of the large circle, on the vertical line): apogee
- *f* (on the vertical line, below *b*): centre of the Sun's eccentric
- *e* (on the vertical line, below *f*): centre of the equant
- *d* (on the vertical line, below *e*): centre of the eccentric deferent
- *c* (on the vertical line, below *d*): centre of the world
- *a* (foot of the large circle, on the vertical line): the point opposite the apogee
- *g* (centre of the small circle, left of the vertical line): centre of the epicycle of Venus
- *i* (just right of *g*): a second, assumed position of the centre of the epicycle
- *K* (far left, outside the large circle): centre of the Sun
- *m* (lower left): centre of the solar body in the assumed case
- *l* (lower left, outside the small circle): the meeting point of *c i* produced with *f l*

and let *e g* be produced in a straight line, which will meet *f K* — as indeed it must. For since we have set the plane of Venus's eccentric in the plane of the ecliptic, one and the same straight line drawn from the centre of the world will be the line of the Sun's mean motion and the line of the mean motion of Venus's epicycle. The straight lines *e g* and *f K* will therefore both be parallel to that same line of mean motion, by the definition of the line of mean motion; and so those same straight lines *e g* and *f K* will be parallel to one another, by the 30th proposition of the 1st book of Euclid. Consequently the two angles *c e g* and *c f K* — the exterior and the interior — which the straight line *c f* makes with those same lines *e g* and *f K*, will be equal to one another, by the 29th proposition of the 1st book of Euclid. But the two interior angles *c e g* and *g c e* of the triangle *e g c* are less than two right angles, by the 17th proposition of the 1st book of Euclid; the two angles *g c f* and *c f K*, therefore, will be less than two right angles, by the common notion. Accordingly the straight lines *c g* and *f K* will meet on the side of *g* and *K*; let them meet, then, at *K*. And since *e g* and *f K* have been shown to be parallel, the two triangles *c e g* and *c f K* are equiangular, and therefore as *c e* is to *e g*, so must *c f* be to *f K*, by the 4th proposition of the 6th book of Euclid.

Now *c e* is the distance of the centre of the world from the centre of the equant, while the straight line *e g* was set equal to the radius of the orb carrying the epicycle; and *c f* is the eccentricity of the orb carrying the Sun. It has been shown besides that the eccentricity of the equant circle has the same ratio to the radius of the orb carrying Venus's epicycle as the eccentricity of the orb carrying the Sun has to that orb's own radius. The straight line *f K*, therefore, is equal to the radius of the orb carrying the Sun; and since that same *f K* passes through the centre of the solar body, the point *K* is the centre of the Sun. And therefore, when the centre of Venus's epicycle is distant from the centre of the equant by an interval equal to the radius of her own deferent, the centre of the epicycle and the centre of the Sun lie on one and the same straight line coming from the centre of the world — which is what we undertook to demonstrate.

Now the centre of the epicycle will be distant from the centre of the equant by an interval equal to the radius of the deferent when it stands at the end of that line which is drawn at right angles to the line of the apogee from the midpoint between the centre of the eccentric and the centre of the equant; and this you will conclude at once from the 4th proposition of Euclid's 1st book. In that position the angle *c g e*, the angle of the equation of the centre, is equal to the angle *c K f*, the angle of the equation of the Sun's argument. That in no other position between *b* and *a* can a straight line drawn from the centre of the world to the centre of the epicycle, and extended straight, pass through the centre of the Sun

Diagram. A repetition, beneath both columns, of the uncaptioned figure printed on the preceding leaf: the large bold circle of Venus's eccentric cut by the vertical line of the apogee, with the same lettering. *b* stands at the top of the circle and *a* at its foot; on the vertical line, in order downward from *b*, stand *f*, *e*, *d*, and *c*. The small bold circle at the left has its centre at *g*, with *i* marked just to its right; *K* stands at the far left outside the large circle, with straight lines running from it to *f* and across the figure; *m* and *l* stand at the lower left. A fainter circle overlaps the figure on the left. Two further marks appear at the upper left and at the right of the leaf, too faint on the scan to identify [?].

- *b* (top of the large circle): apogee
- *f* (on the vertical line, below *b*): centre of the Sun's eccentric
- *e* (on the vertical line, below *f*): centre of the equant
- *d* (on the vertical line, below *e*): centre of the eccentric deferent
- *c* (on the vertical line, below *d*): centre of the world
- *a* (foot of the large circle): the point opposite the apogee
- *g* (centre of the small circle): centre of the epicycle of Venus
- *i* (just right of *g*): the assumed second position of the centre of the epicycle
- *K* (far left, outside the large circle): centre of the Sun
- *m* (lower left): centre of the solar body in the assumed case
- *l* (lower left): the meeting point of *c i* produced with *f l*

will not be difficult to demonstrate. For suppose it were possible. Let the centre of the epicycle then stand at the point *i*, between *g* and *a*, and let the straight line *c i*, drawn from the centre of the world to *i* and extended straight, meet the centre of the Sun at *l*; and let the straight lines *e i* and *f l* be joined, which you will show to be parallel by the same method we used to show that *f K* and *e g* are parallel. The two triangles *c e i* and *c f l* are therefore equiangular, by the 29th and the 32nd propositions of the 1st book of Euclid, and have their sides proportional, by the 4th of the 6th — that is, as *c e* is to *c f*, so is *e i* to *f l*. But in the two likewise equiangular triangles *c g e* and *c f K*, as *c e* is to *c f*, so is *e g* to *f K*; therefore as *e i* is to *f l*, so is *e g* to *f K*, by the 11th proposition of the 5th book of Euclid. But *e i* is greater than *e g*, by the 7th proposition of the 3rd book of Euclid; *f l*, therefore, will be greater than *f K*, by the 24th [?] proposition of the 5th book — which is impossible, being contrary to the definition of a circle, since *f* is the centre of the orb carrying the Sun. Consequently, when the epicycle stands at *i*, the straight line *c i* coming from the centre of the world does not pass through the centre of the Sun — which was to be demonstrated.

From this it appears that, with the epicycle set at *i*, the equation of the centre is less than the equation of the Sun's argument. For from the point *f* let the straight line *f l* be drawn through the centre of the Sun, meeting the straight line *c i* at the point *l*. It is established from what we have demonstrated that the two straight lines *f l* and *e i* are parallel, and that *f l* and *c i* do meet. And since *f l* has been shown to be greater than *f K*, and *f K* to be the radius of the orb carrying the Sun, let the centre of the solar body be the point *m*, and let *c m* be joined. In the triangle *c m l*, then, the interior angle *c l m* will be less than the exterior angle *c m f*, by the 16th proposition of the 1st book; and the angle *c i e* is equal to that same angle *c l m*, by the 29th proposition of that same 1st book. The angle *c i e* itself, therefore, will be less than *c m f*. But the angle of the equation of the centre is the alternate angle to that same *c i e*, while the angle of the equation of the Sun's argument is the alternate angle to *c m f*; the equation of the centre is therefore less than the equation of the argument, and the difference is the angle *l c m*, which in the tables is reckoned to be of imperceptible magnitude.

By exactly the same method you will demonstrate that, with the epicycle set between *b* and *g*, the equation of the centre is greater than the equation of the Sun's argument. For let the epicycle be placed at *n*, and let *e n* and *c n* be joined, and let the straight line *f p* be drawn through the centre of the Sun so as to meet *c n* at *o*. Then as *e g* is to *f K*, so is *e n* to *f o*; but *e g* is greater than *e n*, and therefore *f K* will be greater than *f o*. And since *f K* is the radius of the orb carrying the Sun, the centre of the Sun lies beyond the point *o*. Let it be at *p*, then, and let *c p* be joined. The angle *c p f* is the alternate angle to the angle of the equation of the argument, while the angle *c n e* is the alternate angle to the angle of the equation of the centre. But *c n e* is greater than *c p f*, because the angle *c o f*, which is equal to that same *c n e*, is greater than *c p f*. Accordingly the equation of the centre is greater than the equation of the argument; and the difference is as great as the angle *o c p*, which in the tables is neglected on account of its smallness.

Diagram. An uncaptioned figure set in the outer half of the leaf, alongside the text: the same construction as on the two preceding pages, but with the further points of this demonstration added. The large bold circle of Venus's eccentric is cut by the vertical line of the apogee, which carries *b* at the top and *a* at the foot; on that line, in order downward from *b*, stand *f*, *e*, *d*, and *c*. *K* stands at the left, outside the circle, with straight lines running from it across the figure, and *g* stands just to its lower right at the centre of the epicycle. Above and to the left, well outside the circle, stands *p*, with *o* below it and *n* below that, on lines running down into the circle.

- *p* (upper left, outside the circle): centre of the Sun in this position
- *o* (below *p*): the meeting of *f p* with *c n*
- *b* (top of the circle): apogee
- *n* (within the circle, between *b* and *g*): the assumed position of the centre of the epicycle
- *f* (on the vertical line, below *b*): centre of the Sun's eccentric
- *e* (on the vertical line, below *f*): centre of the equant
- *d* (on the vertical line, below *e*): centre of the eccentric deferent
- *c* (on the vertical line, below *d*): centre of the world
- *K* (left, outside the circle): centre of the Sun in the first position
- *g* (left of the vertical line): centre of the epicycle of Venus
- *a* (foot of the circle): the point opposite the apogee

De Mercurio. [On Mercury]

Annot. prima.

Of such parts as the radius *d g* of the eccentric is 60, Ptolemy takes each of the three lines *d c* [?], *c b*, *a b* to be [numeral: ?] such parts. The centre of the deferent lies at *d*, the apogee of the small circle; the centre of the epicycle lies at *g*, the apogee of the deferent, at that same place on the circle where *e*, the apogee of the equant, falls. And this is the greatest distance of the epicycle's centre from the centre of the world, namely 69 parts. In any other position it will stand less far from the centre of the world, and you can prove this geometrically as follows.

Let the centre of the deferent come to the point [?] of the western semicircle; and since the centre of the epicycle moves in the contrary direction, let it come to *t*. [Illegible: about four lines, badly abraded on the leaf; what can be made out is "… & centro æquantis … recta linea erit …".] The sides *a t* and *t r* of the triangle *a t r* are therefore greater than that side [?], by proposition 20 of the first book of Euclid. But *e g* and *a t* are equal [?], since [Illegible: about one line, on the radii of the circles]; and *a d* is greater than *a t*, by proposition 8 of the third book. Therefore *a g* is greater than the two sides *t r* and *r a*, and consequently *a g* is far greater than *a t*. Accordingly, when the centre of the epicycle stands at *g*, it will be at its greatest distance from the centre of the world.

That the centre of the epicycle must also be at the apogee of the equant whenever it is at the apogee of the deferent is concluded from the likeness of the motions. For suppose it were possible for the centre of the epicycle to be at the apogee of

the deferent when it is not at the apogee of the equant. Let it then be at *z*, and let the straight line *a z* be drawn; on which

Diagram. An uncaptioned geometric figure filling the inner column of the leaf. Two large bold circles are cut by a vertical straight line — the line of the apogee — running the full height of the figure. At the head of that line a small circle is drawn with *g* at its centre, labelled *aux deferētis* ("the apogee of the deferent"). Lower on the same line, where it meets the second arc, stands *e*, labelled *aux æquātis* ("the apogee of the equant"). At the middle of the figure a small circle is drawn on the same line: *d*, labelled *deferentis* [?] (the centre of the deferent), stands at its top; *c*, labelled *parui circuli* (the centre of the small circle), at its centre; and *b*, labelled *æquantis* (the centre of the equant), at its foot. Immediately below the small circle stands *a*, labelled *centrū mūdi* ("the centre of the world"). Lower down the same line stands *h*, labelled *oppm augis defe[rentis]* ("the point opposite the apogee of the deferent"), and at the very foot of the line *f*, labelled *oppm augis æquātis* ("the point opposite the apogee of the equant"). At the left edge of the figure is set *ORIĒS* [?] (East), and at the right edge *occidens* (West).

Diagram. A second, smaller uncaptioned figure standing below the first in the same column, its lettering much fainter on the leaf. Two long straight lines rise steeply and nearly parallel from the lower part of the figure; the left of the two is lettered *g* [?] at its head, the right *z* [?]. At the left a heavily inked point sends two straight lines down and to the right, and these converge on a small circle drawn at the foot of the figure, within which stands *d* [?]; a second heavily inked point is marked just below that circle. A single letter *e* [?] is set between the inked point at the left and the small circle. None of the lettering of this figure resolves securely under magnification.

the centre of the deferent must necessarily lie. Let the centre of the deferent then be at the point *r* of the small circle, since it moves on the circumference of that small circle itself. But in that case the centre of the epicycle and the centre of the deferent have moved toward the same side, contrary to the hypothesis. For the likeness of the motions is such that, by as much as the centre of the deferent moves westward from the apogee of the small circle about the centre of the small circle, by just so much does the centre of the epicycle withdraw eastward from the apogee of the equant about the centre of the equant itself. It is therefore impossible for the centre of the epicycle to stand at the apogee of the deferent without standing at the apogee of the equant as well. And since the centre of the deferent lies on the same straight line as the apogee of the eccentric, it will accordingly lie at the apogee of the small circle, as appears in this first figure.

Now let the centre of the deferent withdraw from the apogee of the small circle toward the west. When it has passed over the space *d i* of 4 signs — that is, 120 degrees — it will lie on the line *a i*, which touches that circle; and therefore, when the centre of the deferent is at *i*, it will be at its greatest western distance from the apogee of the small circle. For since *d i* contains 120 degrees, the circumference *b i* will take in a sixth part of the circle, that is 60 degrees; and hence the straight line *b i* will be equal to the radius of the circle, by the corollary to proposition 15 of the fourth book of Euclid. Consequently, if a circle is described about *b* with radius *b a* or *b c*, it will pass through the point *i*; and therefore the angle *a i c* will be shown to be a right angle by proposition 31 of the third book. The straight line *a i* will therefore touch the small circle at *i* itself, by the corollary to proposition 16.

So when the centre of the orb carrying the epicycle has been carried to *i*, the apogee which was at *g* will have been carried to *K*, and the point opposite the apogee, which was at *h*, will have been carried to *l*. Let a circle then be described about the point *i* itself with radius *i k* or *i l*, and let the straight line *b i*, produced straight on, meet it at *m*. The circle carrying the epicycle will then have the position *K l m*, and the centre of the epicycle will be at *m*, as appears in the second figure. For the equilateral triangle *b c i* is equiangular. The angle *c b i* will therefore be equal to the angle at the centre of the small circle, namely 60 degrees; and hence the exterior angle *d b m* will be 120 degrees at the centre of the circle.

By the likeness of the motions, then, the centre of the epicycle will be at *m*; for the centre of the epicycle, descending from the apogee of the deferent about *b*, must move by just as much as the centre of the deferent moves about *c*. It will therefore not be at *l*, the point opposite the apogee of the deferent, nor at the end of the line that touches the small circle.

Since *a c* and *c i* together are greater than *a i*, by proposition 20 of the first book, the straight line *a d* is greater than *a i*, and the whole *a g* will be greater than *a K*. Therefore the remainder *a h* will be less than the remainder *a l*. You will show the same in every other position of the deferent. But this same *a l* will be less than *a m*, by proposition 7 of the third book; for the point *a*, other than the centre *i*, lies on the diameter *K l*. The straight line *a g* has 69 parts; *a h*, therefore, will have 51. And since the square on *a c* is 36 and the square on *c i* is 9, the square on *a i* will be 27. The straight line *a K* therefore has 60 parts plus ℞ 27 [that is, 60 + √27], and the straight line *a l* 60 minus ℞ 27. As for the straight line *a m*, which is the shortest distance of the epicycle's centre from the centre of the world, it is found

by Ioannes de Monteregio's calculation to be 55 [?] parts 33 minutes; but when the centre of the epicycle stands on the tangent line, he finds its distance from the cen-

Diagram. An uncaptioned figure — the "second figure" of the text — set in the outer column beneath the type. It repeats the construction of the first figure with the points of this demonstration added. Two large bold circles are cut by the vertical line of the apogee. At the head of that line stands *g*; to the upper right, on the outer circle, stands *K*, joined by a straight line running down and to the left across the figure. On the vertical line below *g* stands *e*, and lower down *d*, at the top of the small circle; within the small circle stand *c* at its centre and *b* at its foot. The point *i* is marked on the small circle to the right of *b*, with straight lines running from *i* up to *K* and down across the figure. Below the small circle, on the vertical line, stands *a*, from which straight lines run to *i* and down to the lower left. In the lower left of the figure a small circle is drawn with *m* at its centre; below and to the right of it stands *L*, and to the right of *L*, on the vertical line, *h*. At the foot of the vertical line stands *f*.

tre of the world to be about 56 parts 22 minutes. In that case the centre of the eccentric will lie between *b* and *i*, while the point opposite the apogee of the deferent will lie between the tangent line and the point opposite the apogee of the equant, that is, between *l* and *f*.

Let the centre of the deferent then travel on [?], and, running through the circumference *i b*, let it arrive at *b*, the centre of the equant. In that position, therefore, one and the same circle that carries the epicycle will serve as the equant as well; and accordingly the point of the apogee will be the same as *e*, the space *k e* having been run through, while the point *l*, opposite the apogee, will in the same time return to *f*, the point opposite the apogee of the equant, the space *l f* having been run through; and at the same moment the centre of the epicycle will be at *f*. For since the two angles *b e i* [?] and *f b m* are equal to one another, and the motion of the centre of the deferent is like and proportional to the motion of the centre of the epicycle, and both begin to move together, they will therefore complete the angles *b c i* and *f b m* in the same time. So whenever *i* coincides with *b*, the centre of the epicycle will coincide with *f*, the point opposite the apogee of the equant.

From there, by the same law and with a like figure of motion, the centre of the deferent will go to *n*, the eastern point of contact; and at the same time the apogee of the deferent will run through the space *e p* from west to east, and the point opposite the apogee the space *f q*. The centre of the epicycle will therefore arrive

at *o*, the end of the line coming from the point *n* through the centre of the equant. In that place it will stand as far from the centre of the world as it stood before when it was at *m*; and you will be able to conclude this at once by proposition 4 of the first book of Euclid, from the equality of the angles at *b* and of the given sides *b m* and *b o*, which are what remain when the equal straight lines *b i* and *b n* are subtracted from the radii of the deferent. Finally, the point *n*, which has become the centre of the deferent, returns to *d*, the highest point from which it had begun to move; and in the same time the apogee of the deferent reaches *g*, from which it had departed, the space *p g* having been completed. At the same moment the point opposite the apogee comes to *h*; for when the centres of the equant and of the deferent lie on one straight line, the apogees and the point opposite the apogee must lie on that same line. The centre of the epicycle, moreover, will likewise return to *g*, from which it had set out at the beginning of the motion; and in this third figure you will conclude this from the likeness of the motions and from the equality of the angles *n c d* and *d b o*, just as you concluded it in the second.

That the centre of the epicycle is less distant from the centre of the world at the points *m* and *o* than it is when it stands at *f*, the point opposite the apogee of the equant, Ioannes de Monteregio proved in the 9th book of the *Epitome* [*Epito.*], proposition 21, in this way. The angle *a b o* contains a third part of two right angles; the two remaining angles of the triangle *b a o* therefore contain two thirds of two right angles, by proposition 32 of the first book of Euclid. But the angle *b a o* is greater than the angle *a o b*, by proposition 18 of that same first book; for *b o* is 57 parts, while *a b* is 3 of those same parts. The angle *b a o* therefore takes in more than a third part of two right angles, and hence that same angle *b a o* will be greater than the angle *a b o*; and consequently the side *b o* will be greater than the side *a o*, by proposition 19 of that same first book of Euclid. But *b o* and *a f* are equal — which you will conclude by the common notion, the two equal lines *b a* and *b n* being subtracted from the radii *b f* and *n o*. Therefore *a f* will be greater than *a o*, which was to be demonstrated.

Diagram. An uncaptioned figure — the "third figure" of the text — set in the outer column beneath the type, repeating the construction of the second with the points of this demonstration added. Two large bold circles are cut by the vertical line of the apogee. At the upper left, outside the circles, stands *p*, with straight lines running from it down and to the right across the figure; at the head of the vertical line, partly cut at the edge of the block, stands *g*. On the vertical line below stand *e* and then *d*, at the top of the small circle; within the small circle stand *c* at its centre and *b* at its foot. To the left of the small circle stands *n*, joined

by straight lines to *p* and to points across the figure. Below the small circle, on the vertical line, stands *a*. To the lower right a small circle is drawn with *o* at its centre, joined by straight lines running back to *a* and to the small circle at the middle. At the lower left stands *h*; below and to the right of it, on the inner arc, *q*; and at the foot of the vertical line *f*.

But this is not enough to let the expounders of the theorics conclude that the centre of the epicycle, when it is at *m* or at *o*, stands at its very least distance from the centre of the world. The same author demonstrated, in his *Disputations against the Cremonese*, that although the centre of the epicycle is carried with uniform motion about the centre of the equant, no other point whatever of the deferent can be moved with uniform motion about that same centre.

Annotatio secunda.

Only once in the year is the centre of the deferent the same as the centre of the equant; at every other time the centre of the deferent is farther from the centre of the world than the centre of the equant is. Peuerbach therefore infers rightly that the centre of Mercury's epicycle moves more swiftly around the apogee of the equant — that is, about the centre of the deferent — and more slowly around the point opposite the apogee.

In the western semicircle of the small circle take an arc *d i* less than a quadrant, and let *i l* be drawn [?] to the diameter [?]. To the rectilinear angle *d c i* set equal the angle *d b k* at *b*, the centre of the equant. About *i* as centre, with an interval equal to the radius of the deferent, let a circle be described, which the straight line *b k*, produced in a straight and continuous line, meets at *m*. Likewise let a circle be understood to be described about *l* as centre with an interval equal to the radius of the deferent, which the straight line *b m*, extended in the other direction, meets at *n*; and let the straight lines *i g*, *i m*, *l n* and *l f* be joined.

Now when the centre of the deferent, departing from the point *d*, has completed the space *d i*, the centre of the epicycle — because the two motions are alike — will be at *m*, having run through the space *g m* of the oval figure, the space which is subtended at the centre of the eccentric carrying the epicycle by the angle *g i m*. Similarly, when the centre of the deferent has departed from the centre of the equant and reached the point *l*, the centre of the epicycle — again because the two motions are alike — will be at *n*, having completed the space *f n* of the oval figure, the space which is subtended at the centre of the deferent by the angle *f l n*.

Diagram. A large circular figure occupying the foot of the leaf beneath both columns. Two large concentric-looking circles are cut by a vertical apse line running from the top of the figure to its foot; at the head of that line stands *g*, and at its foot *f*. Set within, above the centre, is a small circle carrying the points named in the demonstration. The word **Occidens.** ["West"] is printed to the left of the figure and **Oriens.** ["East"] to the right.

- *g* (top, on the apse line): the apogee
- *m* (upper left, on the outer circle): endpoint of the line drawn from *b* through *k*
- *e* (below *g*, on the inner arc): unlabelled in the text of this page
- *d* (top of the small circle): starting point of the deferent's centre
- *l* [?] (immediately right of *d*): the letter is cramped against *d* on the scan
- *k* (left side of the small circle)
- *c* (centre of the small circle)
- *o* (just left of *c*)
- *l* [?] (lower left of the small circle): the *i*/*l* letterforms are not separable here on the scan
- *b* (foot of the small circle): the centre of the equant
- *h* (below the small circle, on an inner arc)
- *n* (lower right): endpoint of *b m* extended the other way
- *f* (foot of the apse line): the point opposite the apogee

In the same time in which the centre of the epicycle, departing from the apogee *g*, runs through the space *g m*, it runs through *f n* as it departs from *f*, the point opposite the apogee — because the two vertically opposite angles *g b m* and *f b n* at the centre of the equant are equal to one another. We shall show, however, that the angle *g i m* is greater than the angle *f l n*, and that for this reason the centre of the epicycle is carried more swiftly about the centre of the deferent near the apogee of the equant than near the point opposite the apogee.

For in the two triangles *i m o* and *l n o* the two sides *i m* and *l n*, being radii of the deferent, are equal to one another, and the two angles *i o m* and *l o n*, which lie opposite those same sides, are equal. But the side *i o*, which faces the angle *o m i*, is greater than the side *l o*, which faces the angle *o n l*; and the angle *o n l* is itself acute, since by I.16 it is less than the acute angle *k b l*, which stands in the greater segment. The angle *o n l* will therefore be less than the angle *o m i*; and for that reason the angle *n l o* is left greater than the angle *m i o*, by proposition 32 of the first book of Euclid and by the common notion.

Again, in the other two triangles *c g i* and *c f l*: since the two vertically opposite angles at *c* are equal, and the two sides *c i*, *c g* are equal, each to each, to the two sides *c l*, *c f*, the remaining angles under which the equal sides are subtended will be equal, each to each, by proposition 4 of that same first book. The angle *g i c* will therefore be equal to the angle *f l c*. So from the angle *g i c* we shall take away the angle *m i o*, and the angle *g i m* is left; while from the angle *f l c* we shall take away the angle *n l o*, which was shown to be greater than the angle *m i o*, and the angle left is *f l n*. By the common notion, then, *f l n* will be less than *g i m*.

The same thing can be shown by the same method with the centre of the deferent brought to any position whatever between *d* and *i*. Therefore the centre of the epicycle is carried about the centre of the deferent more swiftly near the apogee of the equant than near the point opposite the apogee — which was to be demonstrated.

Lemma.

What we assumed in the two triangles *i m o* and *l n o* — that, since the two sides *i m* and *l n* are equal, and the two angles *i o m* and *l o n*, which lie opposite those equal sides, are equal, while the side *i o*, which faces the angle *o m i*, is greater than the side *l o*, which faces the angle *o n l*, and the angle *o n l* is itself acute, that same angle *o n l* is therefore less than the angle *o m i* — we shall demonstrate in this way.

About the triangle *i m o* let the circle *m i o* be described; and upon the straight line *i m*, which is equal to the straight line *l n*, let a triangle *p i m* be constructed having its sides equal to those of the triangle *l n o*, by proposition 22 of the first book of Euclid. That triangle will also be equiangular with it, by proposition 8 of that same first book. Let the angle *p m i* be equal to the angle *o n l*, and the angle *p i m* equal to *n l o*; then the remaining angle *m p i* is equal to the remaining angle *l o n*, and consequently equal to the angle *i o m*, by the common notion.

Now the angle *m p i* must stand in that segment *m o i* of the described circle in which the angle *i o m* stands. For if it either overshot the circumference of that circle or failed to reach it, then by proposition 16 of the first book and 27 of the third it would follow that the two angles *m p i* and *l o n* are unequal — which is absurd — the conclusion being drawn once a straight line has been carried from the point *i* to that point at which the straight line *m p* meets the circumference of the circle.

Diagram. A single circle set in the outer column, with a triangle and its attendant lines inscribed. Four labels are printed around the circumference.

- *o* (upper left, on the circumference)
- *p* (left, on the circumference below *o*)
- *i* (lower left, on the circumference)
- *m* (bottom right, on the circumference)

Straight lines join *o* to *i* and to *m*, *p* to *i* and to *m*, and *i* to *m*.

The angle *m p i* therefore stands in the segment *m o i*. And since the angle *p m i* is acute — for it was shown to be equal to the angle *o n l* — it stands in a segment greater than a semicircle, by the converse of proposition 31 of the third book; and for that reason the segment *i p* [?] left over from the circle will be less than a semicircle. But the straight line *i p* is equal to the straight line *l o*, and that same *l o* is less than the straight line *i o*; therefore the straight line *i p* will be less than *i o*. Consequently the point *p* does not lie outside the circumference *i o* at all, but between the points *i* and *o* themselves, lest an impossibility follow contrary to III.27 in Campanus's version. Therefore the angle *i m p* will be less than the angle *o m i*, being indeed a part of it. But the angle *o n l* is equal to that same angle *i m p*. The angle *o n l* is therefore less than the angle *o m i* — which was what the demonstration assumed.

Annotatio tertia.

The equations of the argument written in the tables — not only for the three upper planets and Venus, but for Mercury as well — are those that hold while the centre of the epicycle is distant from the centre of the world by an interval equal to the radius of the deferent. But there is this difference: in the other planets that interval is the mean longitude, the mean distance between the farthest and the nearest position of the centre of the epicycle from the centre of the world. For the greatest distance from the centre of the world, the distance at the apogee of the eccentric, exceeds the length of the radius of the deferent by just as much as that same radius exceeds the shortest distance of the centre of the epicycle, the distance at the point opposite the apogee. In Mercury it falls out otherwise. For while the centre of the epicycle is at the apogee of the deferent it is at its greatest distance from the centre of the world, namely 69 parts, whereas at the point opposite the apogee it will be at its least distance from that same centre of the world, namely 51 parts; and the radius of the deferent is the mean between these two distances.

But although it may happen that the centre of the epicycle is at the apogee of the deferent, and then at some other position whose distance from the centre of the world is equal to the radius of the deferent, the centre of the epicycle will

nevertheless never be distant from the centre of the world by an interval equal to that least distance of the point of the deferent opposite the apogee, but always by a greater one. For in every disposition or position of the deferent its nearest point is the one opposite the apogee, as is concluded by proposition 7 of the third book of Euclid; and it was shown in the first annotation, by proposition 20 of the first, that the distance of the deferent's point opposite the apogee, while the centre of the epicycle is at the apogee, is the shortest of all the other distances of the point opposite the apogee in every other position of the deferent. For that reason the centre of the epicycle will always stand off from the centre of the world by a greater interval than that shortest distance of the point opposite the apogee. And so, when the centre of the epicycle has come to stand away from the centre of the world by an interval equal to the radius of the deferent, that distance will not be called the mean distance of the centre of the epicycle from the centre of the world — unless one speaks very improperly, as Peuerbach does in the present passage.

Ioannes de Monteregio used his teacher's own words at the end of book 11 [?] of the *Epitome*, where he writes:

> In the position for which the equations of Mercury's argument have been computed, the centre of the epicycle is distant from the apogee of the equant by nearly 60°.

But this is the copyist's error. For in mean motion it is distant from the apogee of the equant by about 67° 8′, and in true motion by 64° 30′. The mean distance of the centre of the epicycle from the centre of the world is 62 parts with about 16 minutes — the mean, that is, between 69 and 55 with about 33 minutes. But the equations of the argument written in the tables are not computed for that position; they are computed for the position in which it is distant by 60 parts, that is, by an interval equal to the radius of the deferent.

For let *b* be the centre of the world on the line of the apogee of the equant *a b*, and *c* the centre of the equant, and let the centre of Mercury's epicycle be set at *d*, in which place it is distant from the apogee of the equant by about 67° 8′.

Diagram. A small line figure inset in the outer column: a narrow V-shaped construction of straight lines, with no circle drawn.

- *a* (upper left): the head of the line of the apogee
- *d* (upper right): the centre of Mercury's epicycle
- *c* (lower left): the centre of the equant
- *b* (foot of the figure): the centre of the world

Straight lines run from *a* down to *b*, from *d* down to *b*, from *d* to *c*, and from *c* to *b*.

The equation of the centre will therefore be drawn from the table as 2° with 38′; and when these are subtracted from the mean centre itself, 64° 30′ of the true centre are left. To these no proportional minutes at all correspond in the table. The table of the equations of the argument is therefore constructed for the point *d* of the deferent; and so Peuerbach wrote truly that the centre of the epicycle is distant from the apogee of the equant by two signs, 4° 30′, in the very place for which the table of equations was computed.

You will find as follows that the distance of the centre of the epicycle from the centre of the world in that same position is equal to the radius of the deferent. Since the angle *a c d*, the angle of the mean centre, is 67° 8′ of the circumference of the equant, the right sine of that arc will have 55284 parts of the sort of which there are 60000 in the radius of the circle; and the angle *b d c*, the equation of the centre, is of two degrees with 38′, so that its right sine will be 2756 parts. And since, in the triangle *b c d*, as the right sine of the interior angle *d c b* — or of the exterior angle *a c d* — is to the right sine of the angle *b d c*, so is the side *b d* to the side *b c*, the ratio of *b d* to *b c* is therefore the ratio which the number 55284 has to 2756. The ratio of those numbers is nearly as 20 to one, or 60 to 3. But it was shown by Ptolemy that the radius of the deferent has that same ratio to the distance of the centre of the world from the centre of the equant, namely nearly 20 to one, or 60 to 3. Equal, therefore, is the straight line

b d to the radius of the deferent in that position for which Mercury's table of the equations of the argument was computed.

You may show the same thing in another way as well. Let the centre of the epicycle *d* be distant from the centre of the world *b* by the interval *b d* when it stands in that position for which the equations of the argument in the table were written; from the point *b* let the straight line *b e* be drawn tangent to the epicycle at *e*; and let *d e* be joined by a straight line.

The angle *b e d* will therefore be right, and for that reason the angle *d b e* will subtend the greatest equation of the argument in that position. In the table this is found to be nearly 22 degrees 2 minutes, and the angle *d b e* at the centre of the circle will be just as large; its right sine will be 22500 parts. Accordingly, in a circle described about the centre *b* to the measure of the straight line *b d*, which we set at 60000 parts, the straight line *d e* — the radius of the epicycle, that is, the right sine of the angle *d b e* — will be 22500 of those same parts, and the ratio of *b d* to *d e* is therefore as 60 to 22 and a half.

Diagram. A small circle — the epicycle — drawn about a centre marked *d*, with the point *e* on its circumference at the right. A long, nearly straight line runs from *a*, above and to the left of the circle, down to *b* at the foot of the figure, passing through *c* just above *b*. From *b* two straight lines rise to the circle, one to its centre *d* and one to the point of contact *e*, and a short straight line joins *d* to *e*.

- a (upper left, above the circle): a
- d (centre of the small circle): d
- e (right of the circle, on the circumference): e
- c (on the long line, just above *b*): c
- b (foot of the figure, where the lines meet): b

And since Ptolemy showed that the radius of the deferent has this same ratio to the radius of the epicycle, the straight line *b d* is equal to the radius of the deferent. There is therefore no doubt that Mercury's equations of the argument, as they are set down in the table, were computed for that position in which the centre of the epicycle is distant from the centre of the world by an interval equal to the radius of the deferent — just as with the three superior planets and with Venus.

¶ On the affections of the planets. Annot. 1.

On the direct motion, the station, and the retrogradation of the five planets.

Let the centre of the world be *a* and the centre of the epicycle *b*. From the point *a*, then, let straight lines fall upon the circle of the planet's revolution on the epicycle: namely *a c*, which passes through the centre as far as *h* [?], and *a d* and *a e*, which touch the epicycle itself at the points *d* and *e*. Let *e* be the eastern point of contact and *d* the western; let *k* be the true apogee of the epicycle, and *c* the point opposite the apogee.

Ptolemy shows in Book 12, drawing on Apollonius of Perga (*Almagest* XII), that if the straight line *b e*, the radius of the epicycle, has a greater ratio to the straight line *a c* — the line left over from the distance between the centres — than the motion of the centre of that same epicycle has to the motion of the planet on that same epicycle, then the planet will be retrograde at *e*. And since, in every position of the epicycle of any one of the five planets, *b e* has a greater ratio to *c a* than the motion of the centre of the epicycle has to the motion of the planet on the epicycle, the five planets will therefore be retrograde at *c*, the point opposite the apogee of the epicycle.

Diagram. A large circle representing the epicycle, with its centre marked *b* and the true apogee *K* at the top. From the point *a*, far below the circle, a fan of long straight lines rises to the circle: two of them touch it at *e* on the left and *d* on the right, and the others cut across its lower part. A curved line sweeps from *e* across the lower portion of the circle to *d*, and the cutting lines meet it and the circumference at the lettered points below the centre.

- K (top of the circle): K
- g (upper left, on the circumference beside *K*): g
- h (left side of the circle): h
- b (centre of the circle): b
- e (left, at the point of contact): e
- d (right, at the point of contact): d
- i (within the lower left part of the circle): i
- l (within the lower right part of the circle): l
- f (lower left, on the cutting line): f
- c (foot of the circle, on the line to *a*): c
- a (apex far below the figure, where all the lines meet): a

Now let the straight line *a g* be drawn, cutting the face of the epicycle at *f* and *g*. *f g* will therefore be less than *c K*; but *a f* is greater than *a c*, by proposition 8 of Book 3 of Euclid. Half the straight line *f g* will therefore have a smaller ratio to *a f* than *b c* has to *a c*, by the eighth proposition of Book 5. Again, if a straight line *a h* is drawn between *a g* and *a e*, cutting the epicycle at *i* and *h*, half the straight line *h i* will have a still smaller ratio to *a i* than half of *f g* has to *a f*. For the ratio which half the inner line has to the outer decreases in proportion as the cutting lines lie nearer to the tangent *a e*.

Let half of *h i*, then, have to *a i* the same ratio which the motion of the centre of the epicycle in that position has to the motion of the planet on the epicycle. The planet at *i* will therefore appear neither to advance nor to go back, but to stand still. For when the planet has been carried from *K* to *e* in the order of the signs, it will not go back the moment it has passed *e*. Since the equation of the motion of the argument at *e* is extremely small (as we shall show below), the planet at *e* will seem rather to descend than to move in longitude, and for that reason its motion toward the preceding signs will in that place be markedly overcome by the motion of the centre of the epicycle toward the following signs.

The point of station will therefore not be *e*, but that point at which the line of the planet's true motion begins to move faster toward the preceding signs than the

line of the epicycle's true motion moves toward the following. Apollonius showed such a point to be *i*, and it is the first station. Corresponding to it on the other side, before *d*, at the end of the arc of retrogradation, lies the point of the second station — let this be *l* — at which the line of the epicycle's true motion begins to move faster toward the following signs than the line of the planet's true motion toward the preceding.

You will recognize this from the equation due to the motion of the argument in a single day, if it is compared with the motion of the centre of the epicycle on the same day. For from *e*, the eastern point of contact, the motion in longitude begins to diminish; and the nearer the motion of the argument is to the point opposite the true apogee, the greater the equation of that motion of the argument becomes. When, therefore, the equation of the motion of the argument is found greater than the motion of the centre of the epicycle in the same time, the planet will be retrograde.

For on the circumference *e c* let two equal arcs of the motion of the argument be taken: *g n*, nearer to the point *c*, and *o r*, farther off; and let the angles of their equations be subtended at the centre of the world, namely *g a n* and *o a r*. I say that the angle *g a n* is greater than the angle *o a r*.

For let the straight lines *a g* and *a o* be produced as far as *p* and *q* on the circumference of the epicycle itself, and let *n p* and *r q* be joined by straight lines. If the angle *g a n* is not greater than the angle *o a r*, it will either be equal to it or less than it. Suppose it equal. Since the two angles *g p n* and *o q r* are equal to one another by III.27 — for by hypothesis they stand on equal circumferences — then in the two triangles *a p n* and *a q r* the two remaining angles *a n p* and *a r q* will be equal, by proposition 32 of Book 1 and the common notion. The sides of those triangles which are subtended under equal sides [recte: *angulis*, angles] will therefore be proportional, by proposition 4 of Book 6: that is, as *a p* is to *a q*, so *a n* is to *a r*. But *a p* is greater than *a q*, by III.8; *a n* will therefore be greater than *a r*, which is impossible, since it runs against that same III.8. The angle is therefore not equal to the other.

Nor is the angle *g a n* less than the angle *o a r*. For if it is less, then at the point *a*, the end of the line *a o*, we shall construct the angle *o a t* equal to *g a n*, by proposition 23 of Book 1, drawing the straight line *a t* to cut the straight line *q r* at *t*.

By a similar chain of reasoning we shall then conclude, in the two triangles *a p n* and *a q t*, that as *a p* is to *a q*, so *a n* is to *a t*; and therefore *a n* will be greater than

a t, which is likewise impossible, since it runs against that same proposition 8 of Book 3.

So if the angle *g a n* is neither equal to *o a r* nor less than it, it will be greater; and therefore the equation of the arc *g n* — which is the arc of the zodiac corresponding to that arc — will be greater than the equation of the arc *o r*.

Diagram. A circle representing the epicycle, with the centre *b* and the apogee *K* at the top, *p* set just to the left of *K* and *q* higher on the left of the circumference. From the point *a*, far below, a fan of long straight lines rises through the circle, cutting its lower circumference at the lettered points and running up to *p*, *K* and *q*. A curved line crosses the lower part of the circle from *e* on the left to *d* on the right.

- K (top of the circle): K
- p (on the circumference just left of *K*): p
- q (upper left of the circumference): q
- b (centre of the circle): b
- e (left of the circle): e
- d (right of the circle): d
- t (within the circle, left of centre, on a cutting line): t
- l (lower right, on the curved line): l
- r (lower part of the circle, on a cutting line): r
- o (lower part of the circle, beside *r*): o
- n (lower part of the circle, beside *o*): n
- g (at the foot of the circle, left of *c*): g
- c (at the foot of the circle): c
- a (apex far below the figure, where all the lines meet): a

The equation of the arc nearer the point opposite the true apogee is therefore the greater, that of the more remote arc the less; which was to be shown.

That the two points of station *i* and *l* are themselves at equal distances from the point *c*, opposite the true apogee, we shall now show — provided it be granted that the motion of the centre of the epicycle bears the same ratio to the motion of the planet on the epicycle at those two points *i* and *l*, which you will necessarily concede so long as the position of the epicycle is not changed.

Diagram. A circle representing the epicycle, with the centre *b* and the apogee *K* at the top, *h* on the upper left of the circumference and *z* on the upper right. Straight

lines run from the centre *b* down to the lower part of the circle, and two long straight lines rise from the point *a*, far below the figure, through the circle to *h* and *z*. A curved line crosses the lower part of the circle from *i* on the left through *c* at the foot to *l* on the right.

- K (top of the circle): K
- h (upper left of the circumference): h
- z (upper right of the circumference): z
- b (centre of the circle): b
- i (lower left, on the curved line): i
- c (at the foot of the circle): c
- l (lower right, on the curved line): l
- a (apex far below the figure, where the lines meet): a

For let *a l*, produced in a straight line, cut the epicycle again at *z*. Since the ratio between those motions is the same at *i* and *l*, the points of station, then as half the straight line *h i* is to the straight line *a i*, so is half the straight line *l z* to the straight line *a l*, by proposition 11 of Book 5. Half of *h i* will therefore be equal to half of *l z*. For if it were greater, *a i* would be greater than *a l*, by proposition 14 of the fifth book; *h i* too would be greater than *l z*, by the common notion; and therefore the rectangle contained by the whole *a h* and by *a i* would be greater than the rectangle contained by *a z* and *a l* — which is impossible, since it runs against the corollary to proposition 35 of Book 3 in Campanus's version.

By the same device you will show that half of *h i* is not less than half of *l z*; and therefore the halves of *h i* and *l z* will be equal. And since, as half of *h i* is to half of *l z*, so is *a i* to *a l*, by alternation, the two straight lines *a i* and *a l* will therefore be equal.

Let *b i* and *b l* then be joined by straight lines. By proposition 8 of Book 1 you will show that the two angles *a b i* and *a b l*, belonging to the triangles *a i b* and *a l b*, are equal; and from this you will conclude, by proposition 26 of Book 3, that the two arcs *c i* and *c l* are equal. The two points of station must accordingly stand at equal distances from the point opposite the true apogee of the epicycle.

Capuanus [Capuano di Manfredonia], the expositor of the *Theoricae*, considers only the motion of the planet on the epicycle and neglects the motion of the deferent; he therefore places the points of station at *e* and *d* in the first figure of this annotation. That these lie at equal distances from the point opposite the true apogee of the epicycle you will establish by an ostensive demonstration.

For the angles at *e* and *d*, the points of contact, are right in the triangles *b a d* and *b a e*, by proposition 18 of Book 3; and therefore the squares on *a d* and on *a e* will be equal, by proposition 47 of Book 1 and the common notion; and so the sides *a d* and *a e* themselves will be equal. Now, by proposition 8 of that same first book, you will show that the two angles *d b a* and *e b a* are equal, and consequently, by proposition 26 of Book 3, that the two arcs *d c* and *e c* are equal.

But in Peuerbach *d* and *e* are not the points of station, since he says that the points of station come nearer to the point opposite the apogee of the epicycle because of the slowness of the motion of the argument. Yet it is clear that even if the motion of the planet on the epicycle has become slower, by no means…

by no means will the points of contact be changed on that account.

The arc of the first station is K h i; the arc of the second is K i l; the arc of direction is l K i; the arc of retrogradation is i c l. If, therefore, the arc K h i, that of the first station, is taken away from the whole circle, the arc left over will be i l z k [?], which is equal to the arc K h i l, that of the second station. And if from this last the arc K h i itself is taken away, what remains is i c l, the arc of retrogradation; and when this is subtracted from the whole circle, the arc of direction, l k i, is left.

Diagram. Two circles — the two positions of the epicycle — set one above the other, with a pair of straight lines drawn up to them from a single point *a* at the foot of the figure. The upper-left circle is the epicycle in its more remote position; the lower-right circle is the epicycle in its nearer position. Labels as they stand on the leaf:

- b (centre of the upper circle): the centre of the epicycle in the more remote position
- d (left of the upper circle, where the left-hand line from *a* touches it): the point of the first station in the more remote position
- c (within the upper circle, below the centre): the point opposite the apogee of the epicycle
- i (within the upper circle, beside *c*): the point in which the joining line b K cuts the circle
- K (on the left-hand line, below the upper circle): the point cut off on a d so that d K equals a f
- e (centre of the lower-right circle): the centre of the epicycle in the nearer position
- f (left side of the lower-right circle, where the right-hand line from *a* touches it): the point of the first station in the nearer position

- g (below the centre of the lower-right circle): the point opposite the apogee in the nearer position
- a (apex at the foot, where the two lines meet): the centre of the world

Annotation 2 [Annotatio secunda.]

Since Peuerbach says that the points of station are the nearer to the point opposite the true apogee of the epicycle the nearer the centre of the epicycle has come to the point opposite the apogee of the equant, and the larger the epicycle the planet has, some accordingly suppose that he has assigned the causes from which a smaller arc of retrogradation follows. This would indeed not be doubted in the least, if the points of contact were the stations. For while the centre of the epicycle of any planet (Mercury excepted) is nearer to the point opposite the apogee of the equant or of the eccentric, it comes closer to the earth.

Let us therefore suppose that the centre of the epicycle stands off from a, the centre of the world, by the interval a b; that in the more remote position the point of the first station is d and the point opposite the apogee of the epicycle is c; and that in the nearer position the centre of the same epicycle stands off from the centre of the world itself by the interval a e, the point of the first station being f and the point opposite the apogee g. I say that the arc f g — half the retrogradation in the nearer position — is smaller than the arc d c, which likewise contains half the retrogradation, but in the more remote position.

For since the straight lines a d and a f touch the circles of the epicycle by hypothesis, the angles a d b and a f c [?] will be right angles; and a b is supposed greater than a e, so the square on the line a b will be greater than the square on a e. You will therefore conclude, by proposition 47 of Book 1, that the two squares on a d and on b d are greater than the two squares on a f and on f e. But the square on b d is equal to the square on e f; therefore the square on a d will be greater than the square on a f, and consequently the line a d itself will also be greater than the line a f. So from the greater line a d let us cut off a straight line d K equal to the line a f, by proposition 3 [?] of Book 1, and let b K be joined, cutting the circle at i.

By proposition 4 of that same first book we shall then conclude that the angles of the triangle b d K are equal to the angles of the triangle e a f — those, that is, which are subtended by equal sides. The angle d b K will therefore be equal to the angle a e f, and consequently the arcs d i and f g will be equal. But d i is smaller than d c; therefore f g will be smaller than that same d c.

So in the nearer position the points of station are closer to the point opposite the true apogee than in the more remote position — on the supposition that the stations of the planets occur at the points of contact.

Now let the distances from the centre of the world be equal, but let the epicycles themselves be posited unequal. Then the points of station in the larger epicycle will be nearer to the point opposite the true apogee than in the smaller. For let the centre of each epicycle be understood as placed at b, so that its distance from a, the centre of the world, is the same. In the smaller epicycle let the point opposite the apogee be c, and let d be the other point of contact where the station is supposed to occur; in the larger, let the point opposite the apogee be e, and let f be the other point of station at which the contact falls. The straight line a f, touching the larger epicycle, cannot then fall between a b and a d [?], or an impossibility would follow, contrary to the last common notion — that two straight lines do not enclose a surface. Nor can it be extended in a straight line with a d itself, since the two angles formed at d and at f are right angles, arising from the meeting of the tangent lines with the radii of those epicycles. Therefore,

if the line a f were in one straight line with a d, the three interior angles of the triangle a b f would be less than the three interior angles of the triangle a b d — which again is impossible.

And therefore the line a f falls outside a d, and makes the angle b a f greater than the angle b a d. From this it follows that the remaining angle a b f turns out smaller than a b d. Next let the line b d be produced as far as the circumference of the larger epicycle, at the point g. The two arcs c d and e g then subtend, in unequal circles [?], one and the same acute angle e b g. But as the angle e b g is to a right angle, so are the arcs c d and e g to the quadrants of their own circles, by the last proposition of Book 6. The arcs c d and e g will therefore be similar, or proportional; and consequently the arc e f

Diagram. Two concentric circles — the smaller and the larger epicycle sharing one centre — with straight lines running down from the figure toward the centre of the world at the foot of the page. Labels as they stand on the leaf:

- b (centre of both circles): the common centre of the two epicycles
- d (upper right, on the smaller circle): the point of station in the smaller epicycle
- c (just below the centre, on the smaller circle): the point opposite the apogee in the smaller epicycle

- e (lower left, on the larger circle): the point opposite the apogee in the larger epicycle
- f (lower right, on the larger circle): the point of station in the larger epicycle
- g (right, on the larger circle): the point at which b d, produced, meets the larger circle

will be less than the arc which in its own circle is proportional to the arc c d of the smaller epicycle. And therefore the point of station of the larger epicycle is nearer to the point opposite the true apogee than the point of station of the smaller epicycle — which was to be shown.

Annotation 3 [Annotatio tertia.]

The third cause they assign for the greater nearness of the points of station — the slowness of the motion of the argument — can achieve nothing where the eccentric is understood to be at rest, since the points of contact will be the same whether the motion of the argument is fast or slow, provided everything else is posited equal.

And therefore, since the points of station are nearer to the point opposite the apogee of the epicycle than the points of contact themselves are, we must inquire whether what some have asserted about the threefold cause of the variation of the points of station is universally true. First we shall show that the points of station will not be nearer to the point opposite the true apogee of the epicycle merely because the centre of the epicycle is nearer to the centre of the world.

For let a be the centre of the epicycle, b the centre of the world, and a b the shortest distance of the epicycle's centre from the centre of the world; let c be the true apogee of the epicycle and d the point opposite the apogee. Let the line a e be perpendicular to c d; the point e will then lie at the middle of the semicircle, between c and d. From b, the centre of the world, let a straight line b g be drawn to g, a point falling between c and e, and let it cut the lower quadrant at f. Then b f will be greater than b d. But f g is less than c d, by proposition 8 of Book 3; and therefore b f will have a greater ratio to half of f g than b d has to d a, by proposition 8 of Book 5 and proposition 31, which was added by Campanus. By proposition 12 of Book 6 of Euclid let a straight line i d be found which has to d a the same ratio that b f has to half of f g. And since b f itself, to half of f

Diagram. A single circle — the epicycle — with its diameter drawn from the apogee at the top to the point opposite the apogee at the foot, and a bundle of

straight lines running down from the circle to the centre of the world well below it, and beyond that to two further points on the same line. Labels as they stand on the leaf:

- c (top of the circle): the true apogee of the epicycle
- a (centre of the circle): the centre of the epicycle
- d (foot of the circle): the point opposite the apogee
- e (right of the circle, on the horizontal radius from a): the middle point of the semicircle
- o (inside the circle, just left of e): the foot of the perpendicular dropped to a e
- r [?] (upper right of the circle, between c and e)
- s [?] (upper right of the circle, beside r)
- t (right of the circle, below s)
- m (lower right of the circle)
- n (lower right of the circle, left of m)
- f (lower right of the circle, left of n)
- b (well below the circle, where the lines converge): the centre of the world at the least distance
- K (below b, on the same straight line): the centre of the world at the greatest distance
- i (below K, on the same straight line)

g, has been shown to have a greater ratio than b d has to d a, i d will therefore have a greater ratio to d a than b d has to that same d a, by proposition 13 of Book 5; and consequently it will be greater than b d itself, by proposition 10 of that same fifth book.

So on the straight line c b, produced in a straight line, let K d be taken, less than i d but greater than b d. K d will then have a smaller ratio to d a than i d has to d a, and accordingly k d will have a smaller ratio to d a than b f has to half of f g. Now let k be posited as the centre of the world for the case when the centre of the epicycle is at its greatest distance from the earth, and let us assume that the motion of the planet on the epicycle has to the motion of the centre of the epicycle the ratio which b f has to half of f g. The planet at d will then be retrograde whether the centre of the epicycle is nearest to the centre of the world or farthest from it, provided the same proportion of the motions is preserved. But when it is nearest, the planet, on reaching f, will be at the point of station; for we posited the proportion of the motions to be that which b f has to half of f g.

We shall show, moreover, that when the centre of the epicycle stands off from the centre of the world by the interval a K, the point of station will be nearer to the point opposite the true apogee than f is. For the point of station will not be f itself. Suppose it is: then let the straight line k f be joined, and let it, produced in a straight line, meet the circumference of the epicycle at the point l, between c and g. It will then follow that as k f is to half of f l, so is the motion of the planet on the epicycle to the motion of the centre of the epicycle. But as b f is to half of f g, so likewise is the motion of the planet on the epicycle to the motion of the centre of the epicycle, by hypothesis. Therefore as k f is to half of f l, so is b f to half of f g, by proposition 11 of Book 5. And as half of f l is to the whole f l, so is half of f g to the whole f g; therefore as K f is to the whole f l, so is b f to the whole f g, by proposition 22 of Book 5.

Now let the line g l be joined. Since the vertically opposite angles b f K and g f l are equal, the two triangles f K b and f g l will be equiangular, and will have equal those angles which are subtended by sides of the same ratio, by proposition 6 of Book 6; and so the angle b K f will be equal to the angle g l f.

The line k f l, then, falling upon the two straight lines b k and g l, makes the alternate angles b K l and g l K equal; and therefore the lines b k and g l will be parallel, by proposition 27 of Book 1. Next, from the point g let the perpendicular straight line g o be dropped upon a e, by proposition 12 of Book 1, and let it, produced in a straight line, meet the lower quadrant d e at m. The straight line c d — that is, b K — will then be parallel to g m, by proposition 28 of Book 1. But g l and b K have been shown to be parallel; therefore the two lines g m and g l will be parallel, by proposition 30 of that same first book — which is impossible. For they meet at the point g, where they make the angle l g m; and the three points l, g and m lie on the circumference of a circle, not on one straight line.

So when the centre of the epicycle stands off from the centre of the world by the interval a k, the point of station will not be f. By the same method we shall show that the point of station does not lie between f and that point at which the straight line drawn from the point K touches the epicycle.

For suppose it does. Let n be the point of station, and let K n, produced, meet the point t on the circumference of that circle. Then, as the motion of the planet on the epicycle is to the motion of the centre of the epicycle, so is K n to half of n t; and therefore as k n is to half of n t, so is b f to half of f g; and consequently as k n is to the whole n t, so is b f to the whole f g. But K n has a greater ratio to n t than K f has to f l; therefore b f will have a greater ratio to f g than K f has to f l. So let a straight line f r be found to which k f has the ratio that b f has to f g. Then f r will

be less than f l, by proposition 10 of Book 5. Let the straight line g r be joined; the two triangles b f K and g f r will therefore be equiangular, by proposition 6 of Book 6. Hence we shall conclude, as before, that the two straight lines g m and g r are parallel — which is impossible.

And therefore, when the centre of the epicycle stands off from the centre of the world by the interval a K, the point of station will not lie between f and the point of contact. And since at the point d the planet is retrograde, while at the point of contact itself and above it it moves direct, the point of station will lie between d and f. It will therefore be nearer to the point opposite the true apogee when the centre of the epicycle is farther from the centre of the world — which was to be shown.

We shall show once more, in another figure, that the points of station are nearer to the point opposite the true apogee of the epicycle at the greater distances from the centre of the world. On the straight line c d, produced in a straight line, and from the point b touching [?] on it, let the line b e be drawn to e, the point at the middle of the semicircle, cutting the lower quadrant at f. Then b f will have a greater ratio to half of f e than b d has to d a. Now let a point k be taken somewhat below b, by the method stated above, such that it has a still smaller ratio

K d to d a than b f has to half of f e. And let b be the centre of the world, with a, the centre of the epicycle, at the point opposite the apogee — that is, at its shortest distance from the centre of the world. Let such a slowness be assumed for the motion of the epicycle's centre, and such a swiftness for the planet in the epicycle, that b f to half of f e, and the planet's motion in the epicycle to the motion of the epicycle's centre, have one and the same ratio. Therefore, when the centre of the epicycle stands off from the centre of the world by the interval a b, the planet will be retrograde at d and stationary at f.

Again, when the centre of the epicycle stands off from the centre of the world by the interval a K, the planet itself will be retrograde at d, but it will not be stationary at f, provided the same ratio of the motions has been kept. For

Diagram. A single circle — the epicycle — with a vertical diameter running through it and produced far downward to the centre of the world. A short horizontal radius runs from the centre to the right-hand side of the circle, and a separate straight line stands vertically outside the circle on the right, touching it. Long straight lines rise from the two points at the foot of the figure into the lower right quadrant of the circle. Labels as they stand on the leaf:

- c (top of the circle, on the vertical diameter): the upper end of the diameter
- a (centre of the circle): the centre of the epicycle
- d (bottom of the circle, on the vertical diameter): the point opposite the true apogee
- e (right of the circle, at the end of the horizontal radius): the point at the middle of the semicircle
- z (upper right, inside the circle above e): the point in which K f, produced, meets the upper quadrant
- t (immediately below z): a point on the same line
- x (left of z, inside the circle): the point taken on f z shorter than f z itself
- f (lower right, on the circle): the point in which b e cuts the lower quadrant
- r (immediately right of f): the point supposed to lie between f and e
- u (top of the outside vertical line): the upper end of the line touching the circle at e
- S (foot of that same outside line): its lower end
- b (on the produced diameter, below the circle): the centre of the world in the nearer position
- K (at the foot of the figure, on the same produced line): the centre of the world in the more remote position

if it is stationary at f, let the straight line K f be drawn through K and f, meeting the upper quadrant at z, and let e z be joined. Then as b f is to half of f e, so is K f to half of f z; and consequently, as b f is to the whole of f e, so is K f to the whole of f z. So the two triangles b f k and e f z will be equiangular by *Elements* VI.6, and the angle f z e will be equal to its alternate angle b K f. The straight lines a K and e z will accordingly be parallel.

Now let the straight line S u touch the circle of the epicycle itself at e. The angle a e u will then be a right angle; and since the angle e a K [?] is a right angle as well, a K and S u are parallel. For that reason the two straight lines e z and S u, which make an angle at e, will be parallel by proposition 30 of Book 1 — which is impossible. The planet will therefore not be stationary at f.

Nor will it be stationary at any point between f and e. Suppose it is: let it be at r, and let k r be joined and produced in a straight line as far as t on the circumference of the epicycle. Then as k r is to half of r t, so is the planet's motion in the epicycle to the motion of the epicycle's centre; and accordingly, as k r is to half of r t, so is b f to half of f e, and as K r is to the whole of r t, so is b f to the whole of f e. But k r has a greater ratio to r t than k f has to f z. Therefore b f will

have a greater ratio to f e than K f has to f z. So let K f have to f x — a line shorter than f z itself — the ratio that b f has to f e, and let e x be joined. The two triangles b f k and f e x will then be equiangular, and we shall conclude, as before, that the two straight lines a k and e x are parallel; and consequently that e x and S u are parallel, though at the point e they make the angle u e x — which is indeed impossible.

The planet will therefore not be stationary between f and e, but between d and f. And so, at the greater distances from the centre of the world, the points of station will be nearer to the point opposite the apogee of the epicycle — which was to be demonstrated.

Someone will perhaps suspect that the points of station have been shown nearer to the point opposite the apogee of the epicycle at the greater distances for the following reason: that the ratio of the planet's motion in the epicycle to the motion of the epicycle's centre exceeds the ratio of K d to d a by a smaller difference than that by which it exceeds the ratio of b d to d a. For K d has a greater ratio to d a than b d has to d a. And when straight lines are drawn from the centre of the world cutting the epicycle, the ratios of the outer lines to the halves of the inner portions increase continuously from the point d as far as the tangent lines. At the greater distances, then, the ratio of the outer line to half the inner will sooner become equal to that ratio which the planet's motion in the epicycle keeps to the motion of the epicycle's centre. And so, at a greater distance of the epicycle from the centre of the world than at a smaller one, the same planet will reach the point of sta-

tion sooner from the point d, the midpoint of the retrogradation; and consequently, at the greater distances, the points of station will be nearer to the point opposite the true apogee.

This reasoning, however, is deceptive. For the ratio of the motions — and likewise of the distances — may be such that the planet does not reach the point of station sooner at a greater distance of the epicycle's centre from the centre of the world than at a smaller one. Indeed, the point of station may be one and the same whether that centre is nearest to the earth or farthest from it.

For let b be the centre of the world for the case in which the centre of the epicycle is nearest to the earth, and let f be the point at which the straight line coming from that centre of the world touches the epicycle. From a point g lying between f and the point e, which is at the middle of the semicircle, let the straight line g h be drawn parallel to the straight line a b, cutting the upper quadrant at h; and let the

straight line b g be joined, cutting the lower quadrant of the epicycle's circle at i. Let the straight line h i also be joined, which when produced in a straight line meets the straight line a b at K. And let the ratio of the planet's motion in the epicycle to the motion of the epicycle's centre be assumed to be that which b i has to half of g i. The planet will then be retrograde at d, the point nearest the centre of the world, and stationary at i. Now let k be the centre of the world for the case in which the centre of the epicycle is farthest from the earth, and let the same ratio of the motions be kept.

I say that the planet will be retrograde at d and again stationary at i.

Diagram. A single circle — the epicycle — with a vertical diameter produced downward well past the circle, and a horizontal radius running to the right-hand side. Straight lines rise from the two points at the foot of the figure into the right-hand part of the circle. Labels as they stand on the leaf:

- c (top of the circle, on the vertical diameter): the upper end of the diameter
- a (centre of the circle): the centre of the epicycle
- d (bottom of the circle, on the vertical diameter): the point opposite the true apogee
- h (upper right, on the circle): the point in which g h cuts the upper quadrant
- e (right of the circle, at the end of the horizontal radius): the point at the middle of the semicircle
- g (just below e, on the circle): the point taken between f and e
- f (below g, on the circle): the point of contact of the tangent drawn from b
- i (lower right, on the circle): the point in which b g cuts the lower quadrant
- b (on the produced diameter, below the circle): the centre of the world in the nearer position
- K (at the foot of the figure, on the same produced line): the centre of the world in the more remote position

For since g h and b k are parallel, the two alternate angles g h i and b K i will be equal; and the angle g i h is equal to its vertically opposite angle b i K. The remaining angle K b i of the triangle b k i will therefore be equal to the remaining angle i g h of the triangle i h g, by *Elements* I.32 and the common notion. Accordingly these triangles will have their sides proportional, by VI.4: as b i is to i g, so is K i to i h. But as i g is to its own half, so is i h to its own half; therefore, as b i is to half of i g, so is K i to half of i h, by equality of ratio. And since we supposed the ratio of the motions to be that which b i has to half of i g, it follows that as the

planet's motion in the epicycle is to the motion of the epicycle's centre, so is k i to half of i h. The planet itself will therefore be stationary at that same point i when the centre of the epicycle is at its greatest distance from the centre of the world — which was also the case at the same point when the centre of that epicycle was nearest to the earth. It will likewise be retrograde at d, since k i has a greater ratio to half of i h than K d has to d a; and consequently the planet's motion in the epicycle must have a greater ratio to the motion of the epicycle's centre than K d has to d a. From this it is concluded that at the point d itself the planet is retrograde.

By a similar device it can be shown that the ratio of the motions may be such that in the nearer position the points of station are closer to the point opposite the true apogee than in the more remote position.

For let the point K be the centre of the world when it is at its greatest distance from a, the centre of the epicycle. From that point K let the straight line k e be drawn to the point e, which is at the middle of the semicircle, cutting the circle of the epicycle at f; and let the ratio of the motions be assumed to be that which K f has to half of f e. The planet will then be stationary at f and retrograde at d. Now let b be the centre of the world when the centre of the epicycle is nearest to the earth, and let the straight line b f be joined, which when produ-

ced reaches the circumference of the circle at g; and let e g be joined. Then the planet, the same ratio of the motions being kept, will be retrograde at d; but it will not be stationary at f. For suppose it is. Then, as K f is to half of f e, so will b f be to half of f g; and as K f is to the whole of f e, so will b f be to the whole of f g. Consequently we shall conclude, as before, that the two straight lines b K and g e are parallel — which is impossible. For the straight line b K is itself parallel to the line that touches the circle of the epicycle at the point e. And so the point of station will not be at f.

Nor will it lie beyond f. For suppose it does lie beyond f. Then the outer line drawn from the centre b to the point of station will have, to half of the inner portion lying inside the circle, the ratio that K f has to half of f e — for that is the ratio of the motions by hypothesis — and accordingly, as the outer line is to the whole inner portion, so is K f to f e. But the outer line that cuts the epicycle at a point beyond f has a greater ratio to the whole inner portion than b f has to f g. K f will therefore have a greater ratio to f e than b f has to f g. So let b f have to f o — a line shorter than f g itself — the ratio that K f keeps to f e, and let e o be joined. We shall then show, as before, that the two triangles b f k and e o f are equiangular by *Elements* VI.6, and the angles

Diagram. A single circle — the epicycle — with a vertical diameter produced downward past the circle, and a horizontal radius running to the right-hand side. Two straight lines rise from the points at the foot of the figure into the lower right of the circle. Labels as they stand on the leaf:

- c (top of the circle, on the vertical diameter): the upper end of the diameter
- a (centre of the circle): the centre of the epicycle
- e (right of the circle, at the end of the horizontal radius): the point at the middle of the semicircle
- o [?] (just below e, between e and f): the point taken on f g
- g [?] (beside o): the point in which b f, produced, meets the circumference
- d (bottom of the circle, on the vertical diameter): the point opposite the true apogee
- f (lower right, on the circle): the point in which k e cuts the circle of the epicycle
- b (on the produced diameter, below the circle): the centre of the world in the nearer position
- K (at the foot of the figure, on the same produced line): the centre of the world in the more remote position

b K f and f e o alternate and equal. The two straight lines b k and e o will therefore be parallel — which is impossible.

And since the planet is retrograde at d, the ratio of the motions being greater than that of b d to d a, while it cannot be stationary at f nor at any other point between f and the point at which the straight line drawn from the point b touches the epicycle, it will be stationary between d and f. Consequently the points of station will be nearer to the point opposite the true apogee in the nearer position than in the more remote one.

From all this it is plain that the greater nearness of the points of station does not arise from position alone — from a nearer or a more distant relation to the centre of the world. Nor is a greater size of epicycle the cause of the points of station being nearer to the point opposite the true apogee, if everything else is supposed equal.

For let two epicycles be understood about the centre a: let the diameter of the larger be b c and that of the smaller d e. Parallel to b c, in the plane of one and the same great circle, let h k be drawn, [?] cutting the smaller epicycle at the points f and g; and let the straight lines e f and c k be joined, which we shall produce in a

straight line until they meet, the point in which they meet being y — for they must necessarily meet on the side of e and c.

For if from the points f and K straight lines f l and K m are drawn at right angles to b c, then l e in the smaller circle will be greater than m c in the larger. For the rectangle contained by b m and m c is equal to the square on k m; and likewise the rectangle contained by d l and l e is equal to the square on f l, by *Elements* III.3 and III.35. But f l and K m are equal, since f m is a parallelogram. Therefore the rectangle contained by b m and m c will be equal to the rectangle contained by d l and l e. Now b m is greater than d l; m c will therefore be less than l e itself.

Next let l o be cut off equal to it, and let f o be joined. The angle l f o will then be equal to the angle m K c in the two right-angled triangles f l o and k m c, by proposition 4 of the first book of Euclid; and consequently the angle l f e will be greater than that same angle m K c, by the common notion. And since the two angles l f K and m K f of the parallelogram f m are right angles, the angle l f e being taken away from the angle l f K, and the angle m c K [recte: m K c] being added to m K f, there are left the two angles e f K and c K f — of which one is acute and the other obtuse — together less than two right angles. The straight lines f e and K c will therefore meet at

Diagram. A woodcut at the head of the inner column: two concentric circles drawn about a common centre *a*, representing the two epicycles of the preceding demonstration. A near-horizontal line runs across both, carrying (from left to right) *b* on the outer circle, *d* on the inner circle, the centre *a*, *e* on the inner circle, and *c* on the outer circle — the diameters *b c* and *d e*. A second straight line is drawn parallel to *b c* above it, from *h* on the left of the outer circle to *K* on the right, cutting the inner circle at *g* (left) and *f* (right); *q* is marked on the inner circle just above *g*. Between *e* and *c* on the diameter stand the feet of the two perpendiculars, *l* [?] and *m*, dropped from *f* and *K*. Below the centre, *p* is marked where the line from *a* crosses the inner circle and *n* where it crosses the outer circle; the two long straight lines drawn from *f* through *e* and from *K* through *c* run far down the page and meet at *y*, well below the circles.

- *a* (centre): common centre of the two circles
- *b*, *c* (left and right, outer circle): ends of the diameter of the larger epicycle
- *d*, *e* (left and right, inner circle): ends of the diameter of the smaller epicycle
- *h*, *K* (upper left, upper right, outer circle): ends of the line drawn parallel to *b c*

- *g*, *f* (upper left, upper right, inner circle): where that parallel cuts the smaller circle
- *q* (upper, inner circle): second intersection of the line *a y* produced with the smaller circle
- *l* [?], *m* (on *b c*, between *e* and *c*): feet of the perpendiculars
- *p* (below centre, inner circle), *n* (below centre, outer circle): intersections of the line *a y* with the two circles
- *y* (far below the figure): meeting-point of *f e* and *K c* produced

on the side of *e* and *c*.

Now let *a y* be joined and produced in a straight line as far as *t* on the circumference of the larger circle; let the point diametrically opposite *t* be *n*, and let the intersections of that same straight line with the smaller epicycle be *p* and *q*. Let the centre of the world be set beneath at *y*. Let us further suppose that the planets have altogether the same motions in the two epicycles, and that the ratio of the planet's motion in the epicycle to the motion of the epicycle's centre is greater than the ratio of the line *y p* to *a p*, and therefore very much greater than that of *y n* to *a n*.

Now let the planet of the smaller epicycle be stationary at *e*. I say that the planet of the larger epicycle will be stationary at *c*. For since the planet of the smaller epicycle is stationary at *e*, the motion of the planet in the epicycle will have to the motion of the epicycle's centre the same ratio that the line *y e* has to half the line *e f*. But as *y e* is to the whole of *e f*, so is *y c* to *c k*, because *e c* and *f k* are parallel. Therefore as *y e* is to half of *e f*, so is *y c* to half the line *c k*; and accordingly, as *y c* is to half of *c k*, so is the motion of the planet in the larger epicycle to the motion of the epicycle's centre.

And so the planet of the larger epicycle will be stationary at *c*. But the arcs *e p* and *c n* are proportional, since they are subtended by one and the same angle *n a c*; and therefore *e* and *c*, the points of station in the two epicycles, approach the points *p* and *n* to the same degree. It is therefore not the case that the larger the epicycle is (other things being supposed equal), the nearer the points of station will be to the point opposite the true apogee of the epicycle; and so the size of the epicycle is not the cause of the greater nearness of the points of station. That we have enclosed one epicycle within the other cannot in the least obstruct this demonstration of ours: let them be understood as separate, yet as moved by the same motions.

We shall also show, in another figure, that the points of station can lie nearer the perigee of the epicycle in the larger epicycle than in the smaller.

Let two circles *b c d* and *e f g*, belonging to unequal epicycles, be described about the centre *a*. Let a straight line *b h* be drawn not through the centre, cutting the smaller circle at *i* and *K*; and parallel to it let the line *c l* be drawn, farther from the centre and on the same side, cutting the smaller circle at the points *m* and *n*. Let the straight lines *m K* and *c h* be joined; if we produce them in a straight line, they must necessarily meet on the side of *h* and *K*. For suppose they were parallel: then the two angles *i K m* and *b h c*, the exterior and the interior, would be equal.

Diagram. A second woodcut at the foot of the outer column : two concentric circles about the centre *a*. On the outer circle, *b* stands at the left, *h* at the right, and *p* at the foot; *c* is marked at the right and *l* at the upper left, so that the straight line *b h* and the straight line *c l* cross the figure as two parallels, neither passing through *a*. On the inner circle *e* stands at the left, *f* and *n* above, *m* and *K* to the right, and *g* and *q* below. Two long straight lines are produced downward from the figure and meet at *o*, well below the circles.

- *a* (centre): common centre
- *b*, *h* (left, right, outer circle): ends of the first secant
- *c*, *l* (right, upper left, outer circle): ends of the parallel secant
- *i* [?], *K* (inner circle): where *b h* cuts the smaller circle
- *m*, *n* (right, upper, inner circle): where *c l* cuts the smaller circle
- *e*, *f*, *g*, *q* (inner circle)
- *p* (foot of the outer circle)
- *o* (far below the figure): meeting-point of *m K* and *c h* produced

They will be equal, the exterior and the interior. Now let the straight lines *a i*, *a b*, *a c* and *a m* be joined. The angle *i a m* will then be double the angle *i K m*, and the angle *b a c* likewise double the angle *b h c*, by the 20th proposition of Book 3 [?] of Euclid; and therefore the angle *i a m* will be equal to the angle *b a c* — a part equal to its whole, which is impossible.

Nor do they meet on the side of *c* and *m*. For if they met on that side, the exterior angle *i k m* of the triangle would be greater than the interior and opposite angle *K h c*, that is, *b h c*. The angle *i a m*, which is double the angle *i K m*, would therefore be greater than the angle *b a c*, which is double the angle *b h c*; so a part would be greater than its own whole, which again is impossible. By this same device you can also show, in the preceding figure, that the two straight lines *e f* and *c K* meet.

The straight lines *m K* and *c h* will therefore meet on the side of *h* and *K*. Let their meeting-point be *o*, and let the straight line *a o* be joined, cutting the nearest circumferences of the epicycles at *p* and *q*. Let us understand the epicycles themselves to move in such a way that in each of them the motion of the epicycle's centre always keeps to the motion of the planet in the epicycle the ratio which half the line *K m* has to the line *k o*. And since *a q* has a greater ratio to *q o* than half the line *k m* has to *K o*, the planet of the smaller epicycle will be retrograde at *q* and stationary at *K*. But as *m K* is to *K o*, so is *c h* to *h o*, by the 3rd [?] proposition of Book 6 of Euclid; therefore as half of *K m* is to *K o*, so is half of *c h* to *h o*.

Further, *a p* has a greater ratio to *p o* than half of *c h* has to *h o*; the planet of the larger epicycle will therefore be retrograde at *p*, but stationary at *h*. And since the point *K* lies between *b* and *h*, the ends of the straight line *b h*, which was drawn clear of the centre *a*, then if straight lines are drawn from *a* itself to *K* and to *h*, the line *a h* will lie nearer to the line *a o* than *a K* does — for *K* is left outside the triangle *a h o*. Consequently the arc *k q* is greater than that arc of the smaller circle which is similar to the arc *h p*; and so in the larger epicycle the point of station is nearer the perigee, which was to be demonstrated.

Finally, in yet another figure, we shall show that it can happen that in the smaller epicycle the points of station stand less far from the perigee of that epicycle, and in the larger epicycle farther off.

For let us conceive, as before, two circles of unequal epicycles about the centre *a*; let the diameter *b c* of the larger epicycle be drawn, and upon it, at right angles, let two straight lines *d e* and *f g* be drawn. Let *d e* be the farther from the centre *a* itself, and *f g* the nearer; and let their intersections with the smaller circle be *k h* and *i l*. The straight lines *d e* and *f g* will therefore be parallel. Now let *K i* and *e g* be joined; these must meet on the side of *g* and *i*, as we shall show at once.

Between the centre *a* and the line *d e* let the straight line *m n* be drawn parallel to *d e*, but standing off from *a* by the same interval as *f g* does — that is, with the intervals *a t* and *a z* equal. Let this line cut the smaller circle at *p*, between *K* and *i*, and the larger at *n*, between *e* and *g*; and let *e n* and *K p* be joined. The straight lines *p k* and *e n* will then meet on the side of *p* and *n*, just as was demonstrated in the preceding figure. And therefore *e k* will be shown greater than *n p*, by the 4th proposition of Book 6

Diagram. A woodcut at the foot of the outer column : two concentric circles about the centre *a*, crossed by three parallel straight lines. *c* is marked at the top of the

outer circle and *b* at its lower left, the ends of the diameter; *d* stands at the upper left and *e* at the right, the ends of the outermost of the parallels; *f* at the left and *g* at the lower right, the ends of the nearest parallel; *m* at the left and *n* at the far right, the ends of the intermediate parallel. On the inner circle *h* and *k* are marked above, and *l* and *i* below and to the right, where the parallels cut it; *P* stands to the right of the centre and *t* and *z* on the diameter above and below *a*, the feet of the parallels. Two long straight lines run down from the figure and meet at *o* far below. One further letter at the foot of the circle is not resolvable [?].

- *a* (centre): common centre of the two circles
- *b* (lower left), *c* (top): ends of the diameter of the larger epicycle
- *d* (upper left), *e* (right): ends of the line drawn farther from the centre
- *f* (left), *g* (lower right): ends of the line drawn nearer the centre
- *m* (left), *n* (far right): ends of the intermediate parallel
- *h*, *k* (above), *l*, *i* (below right): intersections of the parallels with the smaller circle
- *P* (right of centre): intersection of the intermediate parallel with the smaller circle
- *t*, *z* (on the diameter): the equal intervals *a t* and *a z*
- *o* (far below the figure): meeting-point of the two produced lines

of Euclid. But *n p* is equal to the line *g i*, as you will conclude by a common notion: for from the equals *t n* and *g z*, when the equals *t p* and *z i* have been taken away, the remainders are left. The line *e K* will therefore be greater than *g i*; but they are parallel, and so the straight lines *k i* and *e g* will meet on the side of *g* and *i*. For if they were parallel, the straight lines *e k* and *g i* would be equal, by the 34th proposition of Book 1; yet *e k* has been shown greater than *g i*. And they cannot meet on the side of *K* and *e*: for if they met on that side, *g i* would be greater than *e K*, by the 4th proposition of Book 6, whereas it has been shown to be smaller. The straight lines *K i* and *e g* therefore meet on the side of *g* and *i*.

Let their meeting be at the point *o*, and from *o* let a straight line *o a* be drawn to the centre *a*, cutting the nearest circumferences of the epicycles at *r* and *s*. We shall lay down that the epicycles are moved by the same motions, and in such a way that the motion of the epicycle's centre has to the motion of the planet in the epicycle the ratio which half of *k i* has to the line *i o*, and therefore the ratio which half of *e g* has to *g o*. For as *K i* is to *i o*, so is *e g* to *g o*, by the second proposition of Book 6 of Euclid. Accordingly the planet of the smaller epicycle will be retrograde at *s*, its perigee, but stationary at *i*; while the planet of the larger epicycle will be retrograde at *r* and stationary at *g*. And since, if a straight line *a g*

is drawn from the point *a* to the point *g*, it cannot cut the line *g z* before *g* — otherwise an impossibility would follow, against the last common notion, that two straight lines do not enclose a space — the arc *s i* will therefore be less than that arc which, in the same circle, is similar to the arc *g r*. And so in the smaller epicycle the points of station are nearer the perigee than in the larger, which is just what we undertook to demonstrate in the present figure. From all of this you will conclude that the greater size of the epicycle is not the cause (other things being supposed equal) of the greater nearness of the points of station — which was what we had to show.

The slowness of the motion of the argument — that is, a slower motion of the planet in the epicycle — is the true cause of the points of station drawing closer to one another. For let the centre of the epicycle be *a* and the centre of the world *b*; and let the motion of the planet in the epicycle have to the motion of the epicycle's centre a greater ratio than *b d* has to *d a*, but let it be as *b c* is to half of *c e*. The planet will then be retrograde at *d* and stationary at *c*. I say, then, that if the motion of the planet itself in the epicycle is set slower, yet in such a way that it still keeps a greater ratio to the motion of the centre of that same epicycle than *b d* has to *a d*, the planet will still be retrograde at *d*, but the point of station will lie between *c* and *d*, and in that way will come nearer the point opposite the true apogee of the same epicycle.

For it cannot make its station at the point *c* itself: if it did, the line *b c* would have to half of *c e* a greater ratio and a smaller one at the same time, which is impossible. For a slower motion of the planet in the epicycle has a smaller ratio to the same motion of the epicycle's centre than a faster one has. Nor, for the same reason, can the point of station be *f*, beyond *c*. For *b f* has a greater ratio to half of the inner line — let it be *f g* — than *b c* has to half of *c e*; and therefore the slower motion of the planet in the epicycle would have a greater ratio to the same motion of the epicycle's centre than the faster motion has, which again is impossible. And so, if the motion of the argument is set slower, the point of station will fall before *c*, that is, nearer the point opposite the true apogee of the epicycle.

You will reach the same conclusion if, keeping the same motion of the planet in the epicycle, you set the motion of the epicycle's centre somewhat faster than before. For however the proportion is diminished, provided that what is left is still greater than the ratio of the line *b d* to *d a*, the planet will likewise be retrograde at *d* and again stationary before *c*.

From all this it is plainly clear that Peuerbach, in the *Theoricae*, is not assigning the causes of the greater nearness of the points of station, but that he must be understood in this sense.

Diagram. A small woodcut set into the foot of the outer column, the text running down its right-hand side. A single circle is drawn with its centre marked *a* at the left of the interior. On the circumference, *e* stands at the upper right and *g* at the right; *c* is marked low in the interior with *f* immediately to its right, and *d* at the lower left of the circumference. Two nearly straight lines run from the top of the circle down past *d* and *c* and converge far below the figure at *b*.

- *a* (interior, left): the centre of the epicycle
- *b* (far below the figure): the centre of the world
- *c*, *d* (low in the figure): the point of station and the point of retrogradation
- *e*, *f*, *g*: the further points named in the argument

In Saturn, Jupiter and Mars, and in Venus too, it has been established by computing the stations themselves that the nearer the centre of the epicycle stands to the point opposite the apogee of the equant — that is, the nearer the centre of the epicycle stands to the centre of the world — the nearer the points of those same stations lie to the point opposite the true apogee

of the epicycle. This is not to say that, universally, a greater nearness of the centre of the epicycle makes the points of station less distant from one another. We have in fact shown that a greater nearness of the centre of the epicycle to the centre of the world sometimes yields a greater distance between the points of station, sometimes a smaller one, and sometimes an equal one.

In each of the three superior planets, however, and in Venus as well, the epicycle is of such a size, the radius of the orb carrying it likewise, and the eccentricity too, and the ratio of the planet's velocity on the epicycle to the motion of the centre of the epicycle is so far diminished in the positions nearer the centre of the world, that as the centre of the epicycle approaches the centre of the world, the points of station do become nearer to the point opposite the true apogee of the epicycle. This rule is exact, and confirmed by demonstration, for the position at the apogee of the equant, at mean longitude, and at the point opposite the apogee. For the other positions Ptolemy assumes, for the sake of an easier computation, that the arcs of station and the distances from the centre of the world are proportional; and Peuerbach appears to follow him when he says that the nearer the centre of the epicycle stands to the point opposite the apogee of the equant,

the nearer the points of station will lie to the point opposite the true apogee of the epicycle.

Mercury, it is agreed, was excepted, since with that planet it is not the case that the closer the centre of the epicycle comes to the point opposite the apogee of the equant, the less it is distant from the centre of the world — as was shown above in the theory of Mercury itself. There is a further reason: in Mercury the points of station follow the contrary law. The nearer the centre of Mercury's epicycle is to the centre of the world, the further those points stand from the point opposite the true apogee of the epicycle. For this planet's epicycle is of such a size, and such are its eccentricity and the radius of its eccentric, that a greater distance of the centre of the epicycle from the centre of the world yields a greater nearness of the points of station, as the computations show. Nor should this seem surprising, since it was shown above that a greater nearness to the centre of the world can sometimes produce a greater distance of the points of station from the point opposite the true apogee of the epicycle.

Ptolemy assumed that in the three superior planets and in Venus, as the centre of the epicycle comes nearer the centre of the world, so the points of station stand less far from the point opposite the true apogee of the epicycle, and that the differences of the stations are therefore proportional to the distances from the centre of the world. Against this assumption Geber maintains that in these same planets the arcs of station can be equal for unequal distances from the centre of the world, so that the distances of the points of station from the point opposite the true apogee of the epicycle come out equal. Ioannes de Monteregio [Regiomontanus] follows him, with the following argument borrowed from Geber himself.

Let a b g be the circle of the epicycle, its centre d, and let the centre of the world be e. Let the epicycle be placed at the mean longitude of the eccentric, and in that position let g be the point of station. Join the straight lines e d and e g, and produce them straight as far as the upper circumference of the epicycle — e d to a, the true apogee of the epicycle, and e g to b. Draw b z parallel to a e, and let the straight line h t, passing through the point g (wherever it may fall), cut b z at the point l; but so that d t is greater than the shortest distance of the centre of the epicycle from the centre of the world.

The two triangles b l g and e g t[?] will then be equiangular, and therefore b g is to g e as g l is to g t. But g h is greater than g l, so g h will have a greater ratio to g t than b g has to g e, and consequently half of g h will have a greater ratio to g t than half of g b has to e g.

Diagram. The epicycle is drawn as a large circle with its centre marked, and long straight lines descend from it to a point far below the circle representing the centre of the world; the whole construction is set in the outer column, with the text narrowed to run beside it.

- a (at the top of the circle, on the circumference): the true apogee of the epicycle; the vertical line a–d–e runs from it down through the centre
- d (at the centre of the circle): the centre of the epicycle
- b (upper left, on the circumference): the end of the line drawn from e through g and produced
- h (outer left, just outside the circumference): the upper end of the line h t
- l (inside the circle, upper left, printed as a capital ʟ): where h t cuts b z
- z (lower left, below the circle): the lower end of the line b z, drawn parallel to a e
- g[?] (lower left, on the circumference): the point of station
- t (well below the circle, on the descending line): the point defining the distance d t
- e (at the foot of the figure, where the descending lines converge): the centre of the world

Let us now suppose the same epicycle to move away from this position of mean longitude toward the point opposite the apogee of the eccentric. The motion of the centre of the epicycle will then be swifter in every position that is nearer the point opposite the apogee, and therefore the velocity of the centre of the epicycle will bear a greater ratio to the velocity of the planet on the epicycle in the nearer position than in the more remote. So when the centre of the epicycle stands from the centre of the world at an interval equal to the straight line d t, then in that position — whatever it may be — the velocity of the centre of the epicycle will bear a greater ratio to the velocity of the planet on the epicycle than it did when the epicycle was at the mean longitude of the deferent.

But it has also been shown that half of h g has a greater ratio to g t than half of b g has to g e. The ratios of the motions and the ratios of the lines therefore increase together. It is consequently possible that the ratio of half h g to g t exceeds the ratio of half b g to e g by exactly as much as, at the distance d t[?], the ratio of the velocity of the centre of the epicycle to the velocity of the planet on the epicycle exceeds the ratio which, at the distance e d, the velocity of the centre of the epicycle has to the velocity of the planet on the epicycle. And so at the distance d t the velocity of the centre of the epicycle will be to the velocity of the planet on the epicycle as half the straight line h g is to g t. In that case the point of station will

be g itself, just as it was before, when the epicycle stood at the mean longitude of the deferent.

This demonstration assumes, however, that not only e g but also t g, in every position between mean longitude and the point opposite the apogee of the deferent, meets the upper part of the epicycle when produced in a straight line. Otherwise the ratio of half h g to g t will not be greater in the nearer position than in the more remote; on the contrary, it will be smaller.

For in the figure below let a b be a quadrant of the circle, and let the line t g, produced straight, meet the circumference of the epicycle at the point h before b. From that point h raise a straight line h i parallel to the straight line a e, and let o be its intersection with e b. Then o g will be to g e as h g is to g t, because the angles are equal and the triangles g o h and g e t are similar. Therefore b g will have a greater ratio to g e than h g has to g t; and so half b g will likewise have a greater ratio to g e than half h g has to g t. In the nearer position, then, the ratio of half the interior line to the exterior line is not increased; on the contrary, it is diminished.

You will show the same if both e g and t g meet the circumference of the epicycle in the lower quadrant. And finally, suppose t g meets the lower quadrant, as in the figure drawn, but e g meets the upper quadrant before i: it will be shown in the same way that half b g has a greater ratio to g e than half h g has to g t.

Diagram. A circle divided into quadrants by two diameters drawn through the centre, with lines descending from the left and lower-left of the circumference to a point far beneath it; the outer column is narrowed above the figure to accommodate it.

- a (top of the circle, on the circumference): the true apogee of the epicycle
- d (at the centre of the circle): the centre of the epicycle
- i[?] (inside the circle, upper left): the far end of the line h i drawn parallel to a e
- b (left, on the circumference): the end of the quadrant a b
- o (below b, on the line e b): the intersection of h i with e b
- h (left, below o): where t g produced meets the circumference, before b
- g[?] (lower left, on the circumference): the point of station
- t (below the circle, on the descending line): the point defining the distance d t
- e (at the foot of the figure): the centre of the world

But even if we grant, as they assume, that the points b and h lie in the upper half of the epicycle, that argument still does not show that in these planets the station can occur at g both in the nearer position and in the more remote. For although half the straight line b g[?] has a greater ratio to g t than half b g has to g e, and although the nearer the epicycle stands to the point opposite the apogee of the eccentric the greater the ratio which half the interior line bears to the exterior — and although, moreover, the ratio of the motion of the centre of the epicycle to the motion of the planet on the epicycle always increases — they still do not prove that in one and the same position of the epicycle the ratio of the motions can add as much, in these planets, as the ratio of the lines does, unless they simply assert this to be possible, which is doubtful and uncertain.

Ptolemy's own rule is no less uncertain: that in the three superior planets and in Venus, the nearer the epicycle is to the centre of the world, the greater the arcs of station. Peuerbach nevertheless followed Ptolemy.

Since it is now accepted, then, that in the three superior planets and in Venus the nearer the epicycle is to the centre of the world the nearer the points of station lie to the perigee of the epicycle, while in Mercury the contrary holds — the nearer the epicycle is to the centre of the world, the further the points of station stand from the perigee of the epicycle — Erasmus Reinhold takes the cause of this difference to be the following. In the three superior planets and in Venus, the ratio which the radius of the epicycle bears to the outer line lying between the epicycle and the centre of the world exceeds the ratio which the motion of the centre of the epicycle keeps to the velocity of the planet on the epicycle by less in the nearer position than in the more remote; in Mercury, however, the opposite occurs.

For in the position more remote from the earth, the ratio of the radius of the epicycle to the outer line between the epicycle and the centre of the world surpasses the ratio which the motion of the centre of the epicycle has to the velocity of the planet on the epicycle by a smaller difference than when the same epicycle stands in the nearer position. For the greater (he says) the ratio left over when the ratio of the motions has been subtracted from the ratio of the lines, the further the points of station must stand from the perigee of the epicycle; and the smaller the remaining ratio, the nearer the points of station will be.

We shall show as follows, however, that this cause has not been rightly assigned.

Let the circle c g d be described about the centre a, and divided into quadrants by the two diameters c e and g d. On the line g d, produced straight, take two points:

b, the nearer to the centre, and k, the more remote; and join the straight line k e, cutting the circumference of the circle described at f. Let us now take this circle to be the epicycle of some planet whose greatest distance from the centre of the world is a K, while its least is supposed equal to the straight line a b.

Diagram. A circle divided into quadrants by a horizontal and a vertical diameter, with the vertical diameter produced downward well below the circle to two further points; a straight line rises from the lowest of these to the right-hand end of the horizontal diameter, cutting the circumference on the way. The figure sits at the foot of the inner column, with the text narrowed beside it.

- a (at the centre of the circle): the centre of the circle
- g (top of the circle, on the circumference): the upper end of the diameter g d
- c[?] (left end of the horizontal diameter): one end of the diameter c e
- e (right end of the horizontal diameter): the other end of the diameter c e
- d (bottom of the circle, on the circumference): the lower end of the diameter g d
- f (lower right, on the circumference): where the line k e cuts the circumference
- r (inside the circle, lower right, between d and h): the point through which b r is drawn
- b (below the circle, on g d produced): the point nearer the centre
- K (at the foot of the figure, on g d produced): the more remote point

Two further small letters stand beside f in the lower right of the figure but are not legible on the scan [?].

We shall further set the ratio of the motion of the centre of the epicycle to the motion of the planet on the epicycle equal to the ratio which half the straight line e f keeps to the straight line f K, and the same in every position. So when the epicycle stands at the apogee of the eccentric, the point of station will be f; but when it stands at the point opposite the apogee, the point of station will lie between d and f, for this we showed above.

Let h, then, be the point of station in the position opposite the apogee, and join the straight line b h. Produced straight, it will meet the circumference of the epicycle at the point i in the lower quadrant, for it cannot reach the point e, nor can it fall between e and g; otherwise half the interior line would have a greater ratio to h b than half e f has to f k — and that is the ratio of the motion of the centre of the epicycle to the motion of the planet on the epicycle.

Half e f is therefore to f K as half i h is to h b, since each of these is the ratio of the motion of the centre of the epicycle to the motion of the planet on the epicycle. But that ratio is smaller than the ratio of d a to d K and than the ratio of d a to d b. Moreover, d a has a greater ratio to the smaller line d b than the same d a has to the greater line d k. Consequently, if the ratio of the motion of the centre of the epicycle to the motion of the planet on the epicycle is subtracted from each of the two ratios — from d a to d b, and from d a to d K — the ratio left over will be greater when it has been subtracted from the ratio of d a to d b than when it has been subtracted from the ratio of d a to d k. On this reasoning, then, the point of station at the point opposite the apogee of the eccentric would stand further from the point d than f does, and so would not fall at h — which is impossible.

Suppose again — as does in fact happen in the superior planets and in Venus — that the centre of the epicycle moves somewhat more swiftly at the point opposite the apogee of the eccentric than at the apogee. We shall still show that where the greater ratio is left over, the points of station lie nearer the perigee of the epicycle. Let us suppose a straight line b r drawn from the point b to a point r lying between d and h, which, produced straight, cuts the epicycle again at a point o between e and i — yet in such a way that when the ratio which half the straight line o r has to b r is subtracted from the ratio of d a to d b, a greater ratio is still left over than the one left when the ratio of half the straight line h i to h b (that is, of half f e to f K) is subtracted from the ratio of the straight line d a to d K. We shall then suppose the centre of the epicycle to move at such a velocity at the point opposite the apogee of the eccentric that its motion keeps to the motion of the planet on the epicycle the same ratio which

the half of *o r* has to *r b*. So the point of station will be *r*, whereas at the apogee it was *f*. The station therefore falls nearer the perigee of the epicycle at the point opposite the apogee of the eccentric than at the apogee — even if the centre of the epicycle moves more swiftly at the point opposite the apogee, and a greater ratio is left over at that same point opposite the apogee. No planet, admittedly, actually has an epicycle of the kind we have imagined; even so, our argument is clear enough to show that a smaller ratio being left over at the point opposite the apogee than at the apogee is not the true cause from which the greater approach of the points of station arises.

We assumed that when equal ratios are taken away from two unequal ratios, more is left from the greater than from the less. This will be proved as follows. Let *a* have to *b* a greater ratio than *c* has to *d*. From the ratio of *a* to *b* let the ratio which *e* has to *f* be taken away; let *g* be to *h* as *e* is to *f*; and from the ratio which *c*

has to *d* let that same ratio of *g* to *h* be taken away. I say that a greater ratio will be left from the ratio of *a* to *b* than from the ratio of *c* to *d*. For let *i* be to *b*, and *k* to *d*, as *e* is to *f*, or as *g* is to *h*.

The ratio of *a* to *b*, then, will be compounded of the ratio of *a* to *i* and the ratio of *i* to *b*. Likewise the ratio of *c* to *d* will be compounded of the ratio of *c* to *k* and the ratio of *k* to *d*; this was shown by Eutocius of Ascalon on Book 2 of Archimedes' *On the Sphere and Cylinder* (*Sphere and Cylinder* II). And so, if the ratio of *i* to *b* is taken away from the ratio of *a* to *b*, the ratio of *a* to *i* is left; and when in the same way the ratio of *k* to *d* is subtracted from the ratio of *c* to *d*, the ratio of *c* to *k* is left. But *a* will have to *i* a greater ratio than *c* has to *k*.

For the first term *a* has to the second, *b*, a greater ratio than the third, *c*, has to the fourth, *d*, by hypothesis; while the second, *b*, has to the fifth, *i*, the same ratio that the fourth, *d*, has to the sixth, *k*, by the converse ratio. Therefore the first term *a* will have to the fifth, *i*, a greater ratio than the third, *c*, has to the sixth, *k*. This can indeed be demonstrated by the same device Campanus used to prove *Elements* V.31 (`31. quinti libri Euclidis`). And so, if equal ratios are taken away from unequal ratios, more is left from the greater than from the less — which is what we assumed.

¶ *Planets are said to be slow and of diminished course, etc.* [*Tardi dicuntur planetæ & minuti cursu. &c.*]

Annotation 4 [Annotatio quarta.]

The Sun supplies the example of the first part, when it moves from the apogee to the mean longitude. For at that place the mean motion exceeds the true motion by the greatest amount. But from the mean longitude itself as far as the point opposite the apogee the Sun will be called swift. Suppose that, in going from the apogee to the mean longitude, the line of true motion did not move more slowly than the line of mean motion over some interval of time: then the two would move either more swiftly or at equal velocity. In that case, at the end of the interval so conceived, the equation found would be either equal to the earlier one — the one found at the beginning of that same interval — or smaller than it; and both are impossible. For it has been shown that the equation is greatest at the point of mean longitude, and that from the apogee up to that place it increases continuously.

It will be shown in the same way that from the mean longitude as far as the point opposite the apogee the line of true motion moves more swiftly than the line of mean motion. From this you will conclude that in the true motion of the Sun the

passage is from less to greater, but not by equal steps. You have besides that from the mean longitude to the point opposite the apogee the Sun will be called swift in course but diminished in number. And note that, although the matter stands so, what we set down above about the mean and apparent motion of the Sun in connection with the Theory of the Sun is nonetheless true.

¶ *There are three reasons why the Moon, after conjunction, appears sometimes later and sometimes sooner.* [*Triplex est ratio, cur Luna post coniunctionem quandoq; tardius, quandoq; citius appareat.*]

Annotation 5 [Annotatio quinta.]

On the first cause [De prima causa.]

Let us suppose that the conjunction of the Sun and Moon took place in signs that descend slowly, and that the Moon herself has no latitude. I say that the Moon, once she has moved away from the Sun, will appear sooner than if that same conjunction had occurred in signs that descend swiftly.

For when the Sun has reached the horizon at setting and the signs occupying it are ones that descend directly, the Moon, having moved away from the Sun, is left above the horizon. The arc of the zodiac between the Moon and the Sun will therefore descend along with a greater arc of the equinoctial. And the arc of the Moon's parallel intercepted between the Moon and the horizon is similar, or proportional, to this arc of the equinoctial, by proposition 17 of Book 2 of Theodosius (*Spherics* II.17), or by what we demonstrated on the seventeenth proposition of Book 2 in *On Twilights*; they will therefore descend together, and in the same time. But if the conjunction has occurred in signs that descend obliquely, an arc of the zodiac between the Sun and the Moon equal to the former one will descend with a smaller arc of the equinoctial, and likewise with a smaller arc of the parallel of the Moon's place.

From this you will conclude that if the conjunction takes place in signs that descend directly, the Moon herself will reach her setting further into the night than if it had taken place in signs that descend obliquely. And since stars that reach their setting further into the night are seen better — for they are less darkened by the Sun's brightness, while those that go down immediately after sunset are scarcely observed at all — the Moon will be able to be seen sooner if the conjunction has taken place in signs that descend directly, and later in those signs whose descent is oblique.

This, I take it, is how the author means to conclude that the Moon, once separated from the Sun, appears sooner in northern climates if she occupies the signs running from the beginning of Capricorn to the end of Gemini.

But the point he takes for granted — that in northern climates the arcs of the ecliptic of the ascending semicircle descend directly — we shall establish with complete certainty in the following way. Let *a b c* be the descending semicircle of the ecliptic, *a* the beginning of Cancer, *b* of Libra, *c* of Capricorn; let *d b e* be the equinoctial; and let the arc *f b*, terminated at the point *b*, ascend together with the arc *g b*, in the oblique horizon *k g i* of a northern place in which the elevation of the equinoctial is 78 degrees 15 minutes or less — that is, in which the elevation of the pole is 11 degrees 45 minutes or more. I say that *g b* is greater than *b f* itself.

Diagram. An uncaptioned woodcut in the outer column. A straight horizontal line runs from *d* at the left to *e* at the right — the equinoctial. A curve descends from *a* at the upper left, crosses the horizontal line, and continues down to *c* at the lower right — the descending semicircle of the ecliptic. A straight oblique line runs from *i* at the lower left up to *K* at the upper right — the oblique horizon.

- a (upper left, on the curve): *a*, the beginning of Cancer
- c (lower right, on the curve): *c*, the beginning of Capricorn
- d (left end of the horizontal line): *d*
- e (right end of the horizontal line): *e*
- b (on the horizontal line, where the curve crosses it): *b*, the beginning of Libra — the autumnal section
- f (above the horizontal line, where the oblique line meets the curve): *f*
- g (just below the horizontal line, where the oblique line crosses it): *g*
- K (upper right end of the oblique line): *K*
- i (lower left end of the oblique line): *i*

For the three interior angles of the spherical triangle *b f g* are greater than two right angles, by proposition 49 of Book 3 of Regiomontanus' *On Triangles*. Hence, with the angle *f b g* assumed to be the angle of the greatest obliquity of the zodiac, nearly 23 degrees 30 minutes, the two angles *g f b* and *f g b* taken together will be greater than 156 degrees 30 minutes. But the angle *f g b* is supposed to be 78 degrees 15 minutes or less; the remaining angle *g f b* will therefore be greater than 78 degrees 15 minutes. And a greater side subtends a greater angle, by the seventh proposition of Menelaus' first book (*Spherics* I.7); so the arc *b g* will be greater than *b f*.

Accordingly this same *b f*, an arc of the quadrant *a b* terminated at the point *b*, ascends directly in every oblique horizon of a northern place in which the elevation of the north pole is 11 degrees 45 minutes or more — provided the altitude of the north pole is not so great that the proposed arc *b f* has neither rising nor setting in that horizon, but is always visible instead. For the altitude of the north pole above the horizon must be less than the complement of the declination of the point *f*, if that same *f* is to have both a rising and a setting in the same horizon within one revolution of the world.

Among the arcs of the quadrant *a b*, moreover, the one nearer to the point *a* — whether the arcs are contiguous or not — ascends with a greater arc of the equinoctial than the one further from that same point; and this you will conclude by III.6 and III.10 of Theodosius (*Spherics*). Every arc of the quadrant *a b*, therefore, ascends directly in the horizons of northern places just described, that is, with a greater arc of the equinoctial. But in the two quadrants of the ecliptic *a b* and *b c*, equal arcs terminating at the point *b*, the autumnal section, have equal arcs of ascension in one and the same horizon, by 14 of Book 3 of Theodosius (*Spherics* III.14). With the help of the common notion that when equals are taken from equals the remainders are equal, you will therefore conclude at once that any arcs of the two quadrants *a b* and *b c* which are equal and equally distant from *b*, the point of the autumnal section, have ascensions equal to one another. And consequently

every arc of the ecliptic in the descending semicircle ascends directly, that is, with a greater arc of the equinoctial. But in the time in which one arc of the descending semicircle rises, the arc opposite it in the ascending semicircle sets. Every arc of the ascending semicircle therefore descends in northern climates with a greater arc of the equinoctial, just as the author supposed — which was to be demonstrated.

As for what Capuano di Manfredonia, the old commentator, says in expounding this text on the appearance of the Moon — that Pisces and Aries have the greatest descensions in the oblique sphere — that is a delusion. Whether, however, every arc of the ecliptic in the descending semicircle sets obliquely, that is, with a smaller arc of the equinoctial, we shall examine next.

Let *a b c* be the ascending semicircle of the ecliptic and *d b e* the equinoctial, *a* the beginning of Capricorn, *b* of Aries, *c* of Cancer; and in the oblique horizon *K g i* of any northern place let the arc *b f* of the quadrant *b c* ascend together with the arc *b g* of the equinoctial. I say that *b g* is less than *b f* itself.

Diagram. The first of two uncaptioned woodcuts on this leaf, in the inner column. A straight horizontal line runs from *d* at the left to *e* at the right — the equinoctial. A curve rises from *a* at the lower left, crosses the horizontal line, and continues upward to the right — the ascending semicircle of the ecliptic. A straight oblique line runs from *i* at the lower left up to *K* at the upper right — the oblique horizon.

- d (left end of the horizontal line): *d*
- e (right end of the horizontal line): *e*
- b (on the horizontal line, where the curve crosses it): *b*, the beginning of Aries
- f (above the horizontal line, at the meeting of the curve and the oblique line): *f*
- g (below the horizontal line, where the oblique line crosses it): *g*
- a (lower left, on the curve): *a*, the beginning of Capricorn
- i (lower left end of the oblique line): *i*
- K (upper right end of the oblique line): *K*
- [?] (upper right of the figure): a further label, probably *c* for the beginning of Cancer, but not legible with certainty on the leaf

For the angle *b g i* is the angle of the elevation of the equinoctial, and so it will be acute, while the remaining angle *b g f* will be obtuse. But the two sides *b g* and *b f* of the triangle *f b g* are less than one semicircle; the angle *b g i*, exterior to the triangle *f b g*, will therefore be greater than the interior angle *b f g*. For that reason the angle *b f g* will itself be acute, and so the side *b g* that subtends it will be less than the side *b f*, which is subtended by the obtuse angle *b g f*.

And since equal arcs terminated at the point *b*, belonging to the quadrants of the ecliptic *a b* and *b c*, ascend with equal arcs of the equinoctial — as we demonstrated earlier for those that terminate at the autumnal section — and since in the time in which the arcs of the ascending semicircle of the ecliptic rise above the horizon the opposite arcs of the other semicircle descend, it follows that all the arcs of the descending semicircle of the ecliptic which terminate at the autumnal section descend obliquely, that is, with smaller arcs of the equinoctial circle. This was the first thing to be shown.

We further assumed that the two arcs *b g* and *b f* are less than one semicircle, and this we shall demonstrate at once. For since the angle *d g i*, the angle of the elevation of the equinoctial, is acute, the remaining angle *b g f* will be obtuse. Let there be erected from the point *g* an arc *g l* of a great circle, lying between the

equinoctial and the ecliptic and making right angles with the equinoctial at that same point *g* — which you will accomplish immediately if you draw a great circle through that same *g* and through the pole of the equinoctial, following Theodosius' rule in his first book.

Since, then, the angle at *b*, the angle of the greatest obliquity of the zodiac, is acute, the side *l g* of the right-angled triangle *b l g* will be less than a quadrant. But the side *b l*, which subtends the right angle, is less than a quadrant; therefore the remaining side *b g*, which contains the right angle, will also be less than a quadrant. And the arc *b f* itself is not greater than a quadrant. The two sides *b f* and *b g* of the triangle *b f g* are therefore less than one semicircle — which is what had been assumed.

Diagram. The second uncaptioned woodcut on this leaf, in the outer column, repeating the previous construction with the perpendicular arc *g L* added. A straight horizontal line runs from *d* at the left to *e* at the right — the equinoctial. A curve rises from *a* at the lower left across the horizontal line and continues to *c* at the upper right — the ascending semicircle of the ecliptic. A straight oblique line runs from *i* at the bottom up to *K* at the top right — the oblique horizon. A short segment rises from *g* to *L*, standing at right angles to the horizontal line.

- d (left end of the horizontal line): *d*
- e (right end of the horizontal line): *e*
- a (lower left, on the curve): *a*
- c (upper right, on the curve): *c*
- b (on the horizontal line, where the curve crosses it): *b*
- f (above the horizontal line, at the meeting of the curve and the oblique line): *f*
- g (on the horizontal line, where the oblique line crosses it): *g*
- L (just above *g*, the foot of the erected perpendicular arc): *L*
- i (lower end of the oblique line): *i*
- K (upper end of the oblique line): *K*

But let *e c* be the arc of the colure between *c*, the beginning of Cancer, and the equinoctial. *b e* will therefore be a quadrant, because that same colure, falling upon the ecliptic and the equinoctial and coming from their poles, makes right angles at *c* and at *e*. Now let an arc of a great circle be drawn from the poles of the equinoctial through *f*, a point of the ecliptic, cutting the equinoctial itself at *l*. In the right-angled triangle *b l f*, then, since the side *b l* is less than a quadrant, the angle *b f l* [?] will be acute. But the angle *f l b* is right; therefore the side *b f*,

subtended by the greater angle, will be greater than *b l*. For that reason the arc *f c*, which is left over from the quadrant *b c*, will be less than the arc *l e*, which is left over from the quadrant *b e*.

Let the arc *e y* be the difference between *f c* and *l e*, and through the two points *c* and *y* let an arc of a great circle *c y* be described. This arc represents the oblique horizon of that northern place in which the angle of the elevation of the equinoctial is equal to the acute angle *c y e*. The point *c* of the ecliptic will therefore rise together with the point *y* of the equinoctial in that same horizon. Now let the point *f* of the ecliptic come to that same horizon, which in that position is represented by the circle *f o*, with *o* the point of the equinoctial. The ecliptic arc *f c* will therefore ascend with the equinoctial arc *o y*. But *o y* is greater than *f c* itself, since *l y* is equal to that same *f c*; therefore *o y* is greater than *f c*, and accordingly the arc *f c* ascends directly [*rectè*] in that same horizon in which the elevation of the equinoctial is equal to the angle *c y e*.

Now let *a g* be an arc equal to the arc *f c*, ascending in that same oblique horizon with the equinoctial arc *m n*. And since *f c* and *a g* are equal arcs standing at equal intervals from the point *b*, the vernal section, they will have equal ascensions *o y* and *m n* — just as we demonstrated above for arcs of the descending semicircle. So if we set *f c* to be the sign of Gemini, *a g* will be the sign of Capricorn, and both will ascend directly in that same oblique horizon *c y*.

Diagram. An uncaptioned woodcut in the inner column. A straight horizontal line runs from *d* at the left to *e* at the right — the equinoctial; *m* and *n* are marked on it just to the right of *d*, and *o*, *L* and *y* on it between the crossing point and *e*. A curve rises from *a* at the lower left, crosses the horizontal line at *b*, and continues up to *c* at the upper right — the ascending semicircle of the ecliptic. At the left, two straight lines rise from *a* to the horizontal line, forming a narrow wedge, with *g* marked within it; at the right, two short straight lines drop from the curve to the horizontal line, one from *f* and one from *c*.

- d (left end of the horizontal line): *d*
- m (on the horizontal line, right of *d*): *m*
- n (on the horizontal line, right of *m*): *n*
- b (on the horizontal line, where the curve crosses it): *b*, the vernal section
- o (on the horizontal line, right of *b*): *o*
- L (on the horizontal line, right of *o*): *L*
- y (on the horizontal line, right of *L*): *y*
- e (right end of the horizontal line): *e*

- a (lower left, at the foot of the curve): *a*
- g (below the horizontal line, between *a* and *b*): *g*
- f (above the horizontal line, on the curve): *f*
- c (upper right end of the curve): *c*, the beginning of Cancer

But at the time when the sign of Gemini ascends, Sagittarius descends; and at the time when Capricorn ascends, Cancer descends. The two signs Cancer and Sagittarius therefore descend with the greater arcs in that same oblique horizon. You will show the same for any other arc terminated at the beginning of Cancer or of Capricorn. And you will likewise show the same for all those arcs which are parts of the arcs of the ecliptic that are exceeded by their own right ascensions by the greatest amount, even if they are not terminated at the beginnings of Cancer or Capricorn — as we have written at greater length in the book *On the Ascensions of the Signs* [*de Ascensionibus signorum*].

At an elevation of the north pole of 12 degrees, then, the sign of Gemini ascends with an arc of the equinoctial of 31 degrees and nearly 23 minutes, but the sign of Libra with 30 degrees 23 minutes. Sagittarius will therefore descend in that same horizon with 31 degrees 23 minutes, and yet the sign of Aries with 30 degrees 23 minutes. And up to a latitude of 15 degrees the sign of Gemini ascends with a greater arc of the equinoctial than Libra does, and consequently Sagittarius will descend more directly than Aries. But these latitudes are smaller than the latitude of the middle of the first clime, and the author's opinion is entirely certain for places further north. Since he holds that a slower descent is the cause of an earlier appearance, he must by no means deny that in places near the equinoctial circle the Moon appears later in Aries or Pisces than in Cancer or Sagittarius — a matter we shall discuss below.

¶ *On the second cause* [*De secunda causa,*]

Annotation 6 [Annotatio sexta.]

The Moon will also appear sooner after conjunction, says the author, if she has northern latitude; for she will descend later, and a later descent of the Moon after sunset is (on the author's view) the cause of an earlier appearance. You will find this perfectly certain in those northern places that lie between the tropic of Cancer and the arctic circle.

For in those places situated between that same tropic and the equinoctial circle the contrary can happen: the Moon may have northern latitude and yet descend sooner; sometimes she will descend together with the degree of the ecliptic in

which she stands, and sometimes later. The demonstration of the first part is as follows.

Let *e b g* be the ecliptic and *e c h* the equinoctial, and let their vernal section be *e*; let *a b c d* be the western part of the horizon of a northern place whose latitude is not less than the greatest obliquity of the zodiac. Let the point *b* of the ecliptic descend together with the point *c* of the equinoctial, and let the Moon's place be *b* — that is, without latitude, after her conjunction with the Sun.

Since, then, we have supposed the latitude of the place to be not less than the greatest obliquity of the zodiac, the complement of the altitude of the pole will not be greater than the complement of the greatest obliquity of the zodiac.

Moreover, in that same horizon *a b c d* the angle *e c b* is the angle of the complement of the pole's altitude, and the angle *b e c* subtends the greatest obliquity of the zodiac. The two angles *b e c* and *e c b* are therefore not greater than one right angle. And since the three interior angles of the spherical triangle *e b c* are greater than two right angles, the angle *c b e* will be greater than a right angle; and the vertically opposite angle *a b g*, being equal to it, will likewise be greater than a right angle. So let a quadrant of a great circle *k b* be drawn from the north pole of the ecliptic to *b*. That *K b* will fall between *a b* and *b g*, because the angle *a b g* has been shown to be obtuse, and the angle *K b g* is right by Theodosius, *Spherics* I.19.

Diagram. The first of two uncaptioned woodcuts on this leaf, in the inner column. From *e* at the left two lines run out to the right: the upper one is the ecliptic, passing through *b* and curving away to *g* at the right; the lower one is the equinoctial, running through *c* and down to *h* at the lower right. A straight line descends from *a* at the top through *b*, then past *c* to *d* at the bottom — the western part of the horizon. A second straight line descends from *K* at the top right through *f* to *b* and is continued below *b* to *i*.

- e (left): *e*, the vernal section
- b (upper middle, on the ecliptic): *b*
- g (right end of the upper curve): *g*
- c (below and left of *b*, on the equinoctial): *c*
- h (lower right end of the equinoctial): *h*
- a (top): *a*
- d (bottom): *d*
- K (top right): *K*, the north pole of the ecliptic
- f (between *K* and *b*): *f*

- i (just below *b*, on the continuation of the quadrant): *i*

The Moon at *b*, then, descends together with the point *c*; but if she is placed between *b* and *K*, as at *f* — that is, with northern latitude — she will come to the western horizon later than *b* or *c*, and much later than she would if she had southern latitude. You will conclude this at once if you extend the quadrant *b K* as far as the southern pole of the zodiac and place the Moon herself at the point *i*: for *b* descends later than *i* itself, and therefore *f* much later than that same *i*.

But since in the figure described we placed the Moon's position at *b*, away from the sections of the ecliptic and the equinoctial, let us now set *a b c* as the equinoctial and *d b e* as the ecliptic, and let *b* be the autumnal section, that is, the beginning of Libra; and let the western part of the horizon be *f b g*. We shall then show far more easily that the Moon placed at *b* without latitude descends sooner, but later if she stood in that same degree with northern latitude.

For since the angle *a b f* is that of the complement of the pole's altitude, it will be acute, and therefore the remaining angle *f b c* obtuse. The angle *f b e*, which arises from the meeting of the horizon with the ecliptic, will accordingly be more obtuse still. So let a quadrant *K b* of a great circle be drawn from the north pole of the zodiac; it will make right angles with the ecliptic itself at *b*. And that quadrant *k b* will fall between *f b* and *b e*.

And therefore, if the Moon is placed between *k* and *b* — that is, with northern latitude — she will descend later than at *b*, even if the latitude of the place is less than the greatest obliquity of the zodiac, as is concluded from this demonstration. For the angle *a b f* is acute in every oblique horizon, and the angle left over from two right angles is obtuse; and so the angle *f b e* will be more obtuse still, and therefore the quadrant *b K* will fall between *f b* and *b e*.

Diagram. The second uncaptioned woodcut on this leaf, in the outer column. Four lines meet at *b* near the centre. A straight line runs from *a* at the left through *b* to *c* at the right — the equinoctial. A curve runs from *d* at the upper left through *b* to *e* at the right, below *c* — the ecliptic. A second curve runs from *f* at the upper left through *b* down to *g* at the lower right — the western part of the horizon. A straight line runs from *K* at the upper right down to *b* — the quadrant drawn from the north pole of the zodiac.

- a (left): *a*
- b (centre, the meeting point): *b*, the autumnal section, the beginning of Libra
- c (right): *c*

- d (upper left): *d*
- e (right, below *c*): *e*
- f (upper left, above *d*): *f*
- g (lower right): *g*
- K (upper right): *K*, the north pole of the zodiac

Again, let us set *a b c* as the equinoctial, *l b i* as the ecliptic, *b* as the point of the vernal section, and *r b z* as the western part of the horizon. Let the altitude of the pole be greater than the greatest obliquity of the zodiac; the angle *a b r* will therefore be less than the angle of the complement of the greatest obliquity of the zodiac. For that reason the two angles *a b r* and *a b l* taken together are less than one right angle, and consequently the remaining angle *r b i* will be obtuse.

Diagram. The first of two uncaptioned woodcuts on this leaf, in the inner column. Four lines meet at *b* near the centre. A straight line runs from *a* at the left through *b* to *c* at the right — the equinoctial. A curve runs from *L* at the lower left up through *b* to *i* at the upper right — the ecliptic. A second curve runs from *r* at the top down through *b* to *z* at the lower right — the western part of the horizon. A straight line runs from *K* at the top down to *b* — the quadrant drawn from the pole.

- a (left): *a*
- b (centre, the meeting point): *b*, the vernal section
- c (right): *c*
- L (lower left): *L*
- i (upper right): *i*
- r (top left): *r*
- z (lower right): *z*
- K (top, right of *r*): *K*

So when the quadrant *b K* has been drawn to *b* itself, making right angles with *b i*, that quadrant will fall between *b r* and *b i*; and therefore, if the Moon is placed between *b* and *k*, she will descend later than at *b*. If, however, we set the latitude of the place equal to the greatest obliquity of the Sun, it will follow that the angle *l b r* is right, and therefore the circle of the horizon itself will pass through the poles of the ecliptic. And so, if you place the Moon at the beginning of Aries, she will descend together with *b*, whether she has northern latitude or southern. But when the Moon is in the southern signs, one must use the method of demonstration we used in the first figure, constructing the triangle at the autumnal section.

Further, to demonstrate the latter part of what was assumed, let us understand the great circle *a b c d* as the horizon of those places that lie between the equinoctial and the circle of Cancer; let the north pole of the world be *e*; and let the circle described by the north pole of the ecliptic through the motion of the first sphere be *d f b g*, with *d* and *b* the points where that arctic circle and the horizon intersect. Let the eastern semicircle of the horizon be *a b c*, and the western *a d c*. Then, with the pole of the zodiac set at the intersection *b*, let the ecliptic have in the western horizon the position *h i K*; at *d*, the position *l m n*; but at the point *f*, below the horizon, the position *o p q*; and finally at *g*, at any point above the horizon, let it have the position *r s t*.

I say that the Moon at *i* or at *m* will set neither sooner nor later than she would if she had latitude, whether southern or northern. For since the circle of the horizon, coming from the pole of the ecliptic, cuts the ecliptic at *i* and at *m*, that circle of the horizon will be a circle of latitude; and therefore the Moon herself at *i* or at *m* will set neither sooner nor later than she would if she had latitude, either southern or northern.

Diagram. The second uncaptioned woodcut on this leaf, in the outer column, and the largest figure of the three leaves. A large circle stands for the horizon *a b c d*, with *c* at the bottom and *d* at the upper left. A small circle at the upper right is the circle described by the north pole of the ecliptic, with *e*, the north pole of the world, at its centre; *a* is marked inside it. The small circle meets the horizon at *d* and at *b*, with *f* on its arc at the upper right and *g* on its arc at the left. Four further arcs cross the left half of the figure, each a position of the ecliptic: *h i K* uppermost, then *o p q*, then *l m n*, and *r s t* lowest.

- a (inside the small circle, above *e*): *a*
- b (right, where the small circle meets the horizon): *b*
- c (bottom of the large circle): *c*
- d (upper left, where the small circle meets the horizon): *d*
- e (centre of the small circle): *e*, the north pole of the world
- f (upper right, on the small circle): *f*
- g (left, on the small circle): *g*
- h (far left, the outer end of the uppermost arc): *h*
- i (on the uppermost arc, at the crossing): *i*
- K (right end of the uppermost arc): *K*
- o (left end of the second arc): *o*

- p [?] (on the second arc, at the crossing — the glyph is thickened with ink and reads as *p* or *f*): *p* [?]
- q (right end of the second arc): *q*
- L (left end of the third arc): *L*
- m (on the third arc, at the crossing): *m*
- n (right end of the third arc): *n*
- [?] (between *m* and *n*, below *q*): a further glyph resembling *z*, not resolved on the leaf
- r (left end of the lowest arc): *r*
- s (on the lowest arc, at the crossing): *s*
- t (right end of the lowest arc): *t*
- [?] (lower left, outside the large circle): a mark resembling *y* with a short lead-line, not resolved on the leaf

But let us place the Moon at *p*. I say that she will set later than she would if she had northern latitude, and sooner than if she had southern latitude. For with the north pole of the zodiac set at *f*, the circle of the ecliptic has the position *o p q*. So let a great circle be drawn from *f* itself to *p*, and let it be extended to *z*, toward the southern pole of the zodiac; the circle *f p z* will therefore be a circle of latitude. Now when the point *p* reaches the western horizon, every other point between *f* and *p* has already gone below the horizon, while the points between *p* and *z* have not yet set. The Moon placed at *p*, then, sets later than she would if she had northern latitude, and sooner than if she were allotted southern latitude.

Finally, let us place the Moon at *s*. I say that she will set sooner than she would if she had northern latitude, and later than if she had southern latitude. For with the north pole of the zodiac set at *g*, the circle of the ecliptic has the position *r s t*. So let a great circle be drawn from *g* itself to *s*, and let it be extended to *y*, toward the other pole of the zodiac; the circle *g s y* will therefore be a circle of latitude. Now when the point *s* of the ecliptic reaches the western semicircle of the horizon, every other point of the quadrant *g s* is still left above the horizon, while the points between *s* and *y* have already gone below it. The Moon placed at *s*, then, sets sooner than she would if she had northern latitude, and later if she were allotted southern latitude — which is what we undertook to demonstrate.

And so these two causes of the Moon's appearing sooner after her conjunction with the Sun, together with the third, which concerns her swifter motion, converge into a single cause: that she comes later to setting. This is also how the Arabs define the Moon's first appearance — by the time, that is, or by the degrees

of the equinoctial, that go below the horizon after sunset. We, however, see the matter differently. In our view the cause of the Moon's appearing sometimes sooner and sometimes later after her conjunction with the Sun is rather the Sun's concealment beneath the horizon, and not the greater or smaller descent of the arc of the ecliptic between the two luminaries.

For unless a slower descent calls for a greater concealment of the Sun, there will be no reason for the Moon to appear sooner, however far into the night the descent runs out. And it does happen that the equal descents of equal arcs of the ecliptic call for unequal concealments of the Sun. It also happens at times that equal arcs of the zodiac have unequal descents, and that the smaller concealment nevertheless corresponds to the greater descent.

A slower descent, moreover, shows that a greater interval of time has run within the night, whereas a greater concealment of the Sun shows a greater darkness about the horizon — and it is from that darkness that the stars near the horizon are better seen by us.

I admit that the Moon is sometimes seen in daylight; but at that time she is further from the Sun and fuller of light.

So let *a b c* be the ascending semicircle of the ecliptic, *d b e* the equinoctial, *b* the vernal section, *f* the Sun's place, and *b* the Moon's place after their conjunction. Let the western semicircle of any oblique horizon, at a place whose latitude is greater than the greatest declination of the Sun, be *h g f K*, cutting the equinoctial at *g* and the ecliptic at *f*. The equinoctial arc *b g* will then be the descension of the ecliptic arc *f b*; and when that arc has been depressed, let the oblique horizon itself have the position *l b m*. Now from the point *n*, the pole of the horizon, let a quadrant of a great circle be drawn to the horizon *l b m*: let it descend as far as *f*, the Sun's place below the horizon, and let it cut the horizon circle *l b m* at *o*. For it will not cut that circle at *b*, nor below *b*, since the pole of the horizon stands above *c* [?].

Diagram. An uncaptioned woodcut filling the foot of the outer column; the same block is printed again on the following leaf. A straight horizontal line runs from *d* at the left to *e* at the right — the equinoctial *d b e* — with *b* near the middle. A curve rises from *a* at the lower left, crosses the horizontal line at *b*, and continues to *c* at the upper right: the ascending semicircle of the ecliptic *a b c*. Four further curves run down across the figure from the upper edge to the lower right, each a position of the western semicircle of the oblique horizon. The first descends from *h*, with *g* marked where it meets the equinoctial and *f* where it meets the ecliptic,

and ends at *K*. The second descends from *L* through *b* and ends at *m*. The third descends from *q*, with *p* marked at its crossing of the ecliptic and *s* at its crossing of the equinoctial, and ends at *r*. The fourth is the short curve marked *x*, *z* and *y*, nearest the right. A long shallow curve enters from *n* at the upper right and sweeps down to the left, passing through *o* on the horizon *L* ... *m* and reaching *f*.

- d (left end of the horizontal line): *d*
- b (on the horizontal line, near the middle): *b*, the vernal section
- e (right end of the horizontal line): *e*
- a (lower left, foot of the ecliptic): *a*
- c (upper right, head of the ecliptic): *c*
- h (upper left): *h*
- g (below the horizontal line, left of *b*): *g*
- f (below the horizontal line, left of *g*): *f*
- K (lower middle): *K*
- L (upper left, right of *h*): *L*
- m (lower middle, right of *K*): *m*
- q (upper left, right of *L*): *q*
- p (on the horizontal line, right of *b*): *p*
- s (below the horizontal line, right of *p*): *s*
- r (lower right): *r*
- x (upper right): *x*
- z (upper right, below and right of *x*): *z*
- y (right, just above the horizontal line): *y*
- o (above the horizontal line, between *L* and *b*): *o*
- n (upper right corner): *n*, the pole of the horizon

The arc *o f*, then, will be the Sun's concealment below the horizon corresponding to the arc *f b*, taken beneath that same depressed horizon; and it will make right angles with the circle *l o m* at the point *o*, by Theodosius, *Spherics* I.13 [?].

Next let us consider another conjunction of the Sun and Moon, in which the Sun's place is *b* and the Moon's is *p*, the two arcs *f b* and *b p* being equal to each other. When the Moon has come to setting, let that same oblique horizon have the position *p q r*, cutting the equinoctial at *s*; and let the arc of the Sun's concealment at that same moment of time be *b t*. It makes right angles with the horizon at the point *t*, since it comes from the pole of that horizon.

The two arcs of the ecliptic *f b* and *b p* are therefore equal, and the arcs of their descensions — namely *b g* and *b s* — are equal, by Theodosius, *Spherics* III.14. But we shall show that the arcs of the Sun's concealment, *f o* and *b t*, are unequal, *b t* being the smaller.

For the two angles *b p t* and *b p s* are together equal to two right angles, while the three interior angles of the triangle *b s p* are greater than two right angles. So when the common angle *b p s* is taken away, the angle *b p t* will be left smaller than the two angles *p b s* and *p s b* taken together, by the common notion. Of these, one — *p b s* — is the angle of the greatest obliquity of the zodiac, while the other, *p s b*, is that of the complement of the pole's altitude in the oblique horizon under discussion.

But the angle *f b o* is equal to two angles taken together: the angle *f b g*, of the greatest obliquity of the zodiac, and the angle *g b o*, of the complement of the pole's altitude in that same horizon. The angle *b p t* is therefore smaller than the angle *f b o*.

Now the two triangles *f o b* and *b p t* have right angles at the points *t* and *o*. Therefore, as the whole sine is to the right sine of the angle *b p t*, so is the sine of the side *b p* to the sine of the side *b t*. Likewise, as the whole sine is to the right sine of the angle *o b f*, so is the sine of the side *f b* to the sine of the side *f o*. But the whole sine has a greater ratio to the sine of the angle *b p t* than to the sine of the angle *f b o*, since the angle *b p t* has been shown to be smaller than the angle *f b o*, both being acute. The sine of the side *b p* will therefore have a greater ratio to the sine of the side *b t* than the sine of the side *f b* has to the sine of the side *f o*. But by hypothesis the two sides *f b* and *b p* are equal, and so their sines will be equal. The sine of the side *b t* will therefore be smaller than the sine of the side *f o*, since it is the one to which the greater ratio is held, and the other the one to which the smaller.

Now the sides *b t* and *f o* are themselves less than quadrants; therefore the arc *b t* will be smaller than *f o*. So we have equal arcs of the ecliptic, and they have equal ascensions; but the concealments of the Sun are unequal — which is what we undertook to demonstrate. You will show that the sides *b t* and *f o* are less than quadrants because the angle *f b o* is less than a right angle, and likewise the angle *b p t*.

Furthermore, let us take an arc of the zodiac *p z* again equal to *f b* or to *b p*, and let us consider another conjunction of the Sun and Moon in which the Sun's place is *p*, below the horizon, and the Moon has moved away from the Sun by the arc *p*

z, to her setting; and let the point *y* of the equinoctial come to that same horizon, which then has the position *z y*. The arc *s y* will then be the descension of the arc *p z*, and greater than the descension of the arc *f b* or *b p*, because the arc *p z* itself lies further from the vernal section.

Now let an arc of a great circle *p x* be drawn from the point *p*; it will make right angles with the horizon at the point *x*. And by the same method of demonstration we used a little earlier, we shall show that the angle *p z x* is smaller than the angle *f b o*, and consequently that the arc of the Sun's concealment *p x* is smaller than the arc of concealment *f o*.

So *f b* and *p z* are equal arcs of the zodiac having unequal descents; and unequal concealments of the Sun correspond to them as well — that is, where the descent is greater, there the Sun's concealment is smaller. This is what we had to demonstrate.

We hold it entirely certain, moreover, that the Moon appears sooner after her conjunction with the Sun if the distance between them falls within the ascending semicircle of the ecliptic,

Diagram. The same woodcut as on the preceding leaf, printed again here across the foot of both columns, with the same lettering: the equinoctial *d b e*, the ascending semicircle of the ecliptic *a b c*, the four positions of the western semicircle of the oblique horizon (*h g f K*, *L b m*, *q p s r*, and the short curve *x z y*), and the long curve from *n* at the upper right passing through *o* to *f*.

- d (left end of the horizontal line): *d*
- b (on the horizontal line, near the middle): *b*
- e (right end of the horizontal line): *e*
- a (lower left): *a*
- c (upper right): *c*
- h (upper left): *h*
- L (upper left, right of *h*): *L*
- q (upper left, right of *L*): *q*
- o (above the horizontal line, between *L* and *b*): *o*
- t (above the horizontal line, right of *o*): *t*
- x (upper right): *x*
- z (upper right, below and right of *x*): *z*
- n (upper right corner): *n*
- p (on the horizontal line, right of *b*): *p*

- y (right, just above the horizontal line): *y*
- g (below the horizontal line, left of *b*): *g*
- f (below the horizontal line, left of *g*): *f*
- s (below the horizontal line, right of *p*): *s*
- r (lower right): *r*
- K (lower middle): *K*
- m (lower middle, right of *K*): *m*

and later if it falls within the descending semicircle — just as the author wrote. But this is not, as he asserted, because the descents are greater in the one semicircle than in the other. It is because the Sun, as it descends, will be more deeply hidden below the horizon when its distance from the Moon lies in the ascending semicircle, and less hidden if in the descending semicircle.

This greater or smaller concealment of the Sun arises from the angles formed at the meeting of the ecliptic and the oblique horizon: where an angle of this kind is smaller, there the Sun's concealment will be smaller, as is clear from what we demonstrated above. It is worth demonstrating, therefore, that every western and northern angle formed at the meeting of the ascending semicircle of the ecliptic with the western semicircle of the oblique horizon is greater than every angle formed at the meeting of that same semicircle of the horizon with the descending semicircle of the ecliptic.

We shall show this easily if it is first demonstrated that angles of this kind, formed at points of the ecliptic that lie at equal intervals from either section of the equator, are equal to one another.

Let *a b c* be the ascending semicircle of the ecliptic,

Diagram. An uncaptioned woodcut in the inner column. A straight horizontal line runs from *d* at the left to *e* at the right — the equinoctial *d b e* — with *m*, *b* and *i* marked on it in that order. A curve rises from *a* at the lower left through *g*, crosses the horizontal line at *b*, and continues through *f* to *c* at the upper right: the ascending semicircle of the ecliptic. Two further curves descend from the upper edge toward the lower right, each a position of the oblique horizon: one from *L* at the upper left, crossing the equinoctial at *m* and the ecliptic at *g* and ending at *n*; the other from *h*, crossing the ecliptic at *f* and the equinoctial at *i* and ending at *K*. A faint unlettered arc stands clear of the figure at the far left.

- d (left end of the horizontal line): *d*
- m (on the horizontal line, right of *d*): *m*

- b (on the horizontal line, at the middle): *b*, the vernal section
- i (on the horizontal line, right of *b*): *i*
- e (right end of the horizontal line): *e*
- a (lower left, foot of the ecliptic): *a*
- g (on the ecliptic, below the horizontal line, left of *b*): *g*
- f (on the ecliptic, above the horizontal line, right of *b*): *f*
- c (upper right, head of the ecliptic): *c*
- L (upper left, head of the first horizon): *L*
- n (lower middle, foot of the first horizon): *n*
- h (upper middle, head of the second horizon): *h*
- K (lower right, foot of the second horizon): *K*

d b e the equinoctial, and *b* the vernal section; and let *f* and *g* be two points of that semicircle of the ecliptic lying at equal intervals from the section *b* itself. Let the oblique horizon *h f i K* pass through *f*, making at the point *f* itself the western and northern angle *a f h*; and when the point *g* has come to its setting, let that same oblique horizon have the position *l m g n*, making the western and northern angle *a g l*. I say that the two angles *a g l* and *a f h* are equal to each other.

For in the spherical triangle *b f i*, as the right sine of the angle *b i f*, the complement of the pole's altitude, is to the right sine of the angle *i b f*, the greatest obliquity of the zodiac, so is the right sine of the side *b f* [?] to the right sine of the side *f i*. Again, in the spherical triangle *b g m*, as the right sine of the angle *b m g* is to the sine of the angle *g b m*, so is the sine of *b g* to the sine of *g m*.

But the two angles *b i f* and *b m g* are equal — the one being the angle of the complement of the pole's altitude, the other that of the altitude of the equator. Therefore, as the sine of *b f* is to the sine of *f i*, so is the sine of *b g* to the sine of *g m*; and since the two arcs *b f* and *b g* are equal by hypothesis, the right sines of the two arcs *f i* and *g m* will be equal, by the fifth book of Euclid. And since the arcs *f i* and *g m* are themselves less than quadrants — for they are the amplitudes of setting of the points *f* and *g*, that is, parts of the quadrants of the horizon lying between the sections of the meridian and the points *m* and *i* themselves — the two arcs *f i* and *g m* will accordingly be equal to each other.

Now let the two horizons, which are taken as one and the same, meet at the northern point *o*. For it makes no difference whether the sphere is moved while the horizon stays fixed, or whether you make the horizon movable while the sphere is at rest.

And since the two angles *d m o* and *d i o*, both of the complement of the north pole's altitude, are equal, the two sides *m o* and *i o* of the spherical triangle *m o i* taken together will be equal to one semicircle. To the side *m o* we shall add the arc *g m*, and from the side *i o* we shall subtract the arc *f i*; and the two arcs *g o* and *f o* will again be equal to one semicircle. For that reason, in the spherical triangle *g o f*, the angle

Diagram. A spherical figure at the head of the left column, serving the demonstration just concluded. A straight horizontal line runs across the figure from left to right, carrying the labels **d** (left end), **m**, **b**, **i**, and **e** (right end). Two great-circle arcs sweep down from the upper left across this line: the first, labelled **l** on its upper stretch, descends from a heavy point at the upper left and crosses the horizontal line at **m**; the second, labelled **h** at its crest, runs from the upper left through **f** to **c** at the upper right. A further arc crosses the horizontal line at **i** and runs down to **K** at the lower right. Below the horizontal line, the label **g** stands just under **m**, and a second arc descends to **n** at the foot of the figure, ending near a second heavy point at the lower left. The two heavy points (upper left and lower left) carry no letter that resolves on the scan.

- d (left end of the horizontal line)
- m (on the horizontal line, left of centre)
- b (on the horizontal line, at centre)
- i (on the horizontal line, right of centre)
- e (right end of the horizontal line)
- l (on the arc rising to the upper left)
- h (at the crest of the upper arc)
- f (on the upper arc, above the horizontal line)
- c (upper right, end of the upper arc)
- g (below the horizontal line, under m)
- K (lower right, end of the descending arc)
- n (foot of the figure)

the angle a g o will be equal to the angle a f o — that is, the angle a g l is equal to the angle a f h.

If you prefer, you can also show the two arcs f i and g m equal to one another without appealing to sines. For the two arcs b i and b m are equal by Theodosius, *Spherics* III.14; therefore f i and g m will be equal by Menelaus, *Spherics* I.4.

You will prove the same thing in the same way, and by exactly the same technique, for the angles formed in the descending semicircle. But for those formed at the beginning of Capricorn and the end of Gemini, no side of the triangle can be a semicircle; so we shall construct a different demonstration, along these lines.

Let the oblique horizon be a b c d; let the north pole of the world, the visible one, be e, and the hidden pole f. Let the western semicircle of the horizon be b a d c, and let the remaining one be the eastern. When the beginning of Cancer sets at a, let the zodiac have the position g a h; and when the beginning of Capricorn sets at d, let the zodiac itself have the position K d i. I say that the exterior angle b a g — the western and northern angle formed at the point a — is equal to the angle a d k, which stands at d and is likewise western and northern.

Diagram. In the right column, a large circle representing the oblique horizon, with **b** at the top, **a** on the circumference at the upper left, **d** on the circumference at the lower left, and **c** on the circumference at the lower right. Outside the circle at the upper left stands **g**, and outside at the left stands **k**; from each of these an arc runs inward — the arc through a is continued to **h** inside the upper part of the circle, and the arc through d is continued to **i** [?] within. A further arc descends from **b** across the circle and out to **f** below it. A heavy point inside the circle, above and right of centre, marks the visible pole; its letter does not resolve on the scan.

- b (top of the circle)
- a (on the circle, upper left)
- d (on the circle, lower left)
- c (on the circle, lower right)
- g (outside the circle, upper left)
- k (outside the circle, at the left)
- h (inside the circle, upper part)
- i [?] (inside the circle, at centre)
- f (below the circle)
- (heavy point inside the circle, upper right — letter not legible)

For let a great circle be drawn through a and e, and another through f and d, and let the meridian b e c f be drawn. Now take the two spherical triangles a b e and d c f. Since the meridian passes through the poles of the horizon, it makes the angles e b a and f c d right angles. The two sides b e and c f are equal, for b e is the elevation of the visible pole and c f the depression of the hidden pole. The two

sides a e and f d are equal as well, since they are the complements of the greatest obliquities of the zodiac. The remaining sides, together with the remaining angles, will therefore be equal to one another, by the last proposition of the third book of Regiomontanus (*On Triangles* III); so the angle b a e is equal to the angle c d f.

You can also reach this conclusion from the proportion of the sides and the angles, in the following way. The two sides a e and b e are equal, each to each, to the two sides d f and f c, and the sines of the sides and of the angles keep the same proportion. Therefore, as the whole sine is to the sine of the angle b a e, so the whole sine is to the sine of the angle c d f. The angles b a e and c d f are moreover acute, because the sides opposite them are less than quadrants; so these angles will be equal.

But the two angles g a e and f d i are right angles, since the arcs a e and f d, when produced, pass through the poles of the ecliptic. So when the equal angles b a e and c d f are taken away, the remaining angles b a g and c d i will be equal to each other, by the common notion. And the angle c d i is equal to its vertically opposite angle a d k,

so the two angles b a g and a d k — the western and northern angles formed at the very beginnings of Cancer and of Capricorn by the meeting of the ecliptic with the horizon — are equal; which was what remained to be proved.

It will now be easy to show that every western and northern angle formed by the meeting of the ascending semicircle of the ecliptic with the western semicircle of the oblique horizon is greater than every western and northern angle formed by the meeting of that same semicircle of the horizon with the descending semicircle of the ecliptic.

Let the circle a b c d be the ecliptic, its northern semicircle a b c and its southern semicircle c d a; let l a e be the equinoctial; and let a be the beginning of Aries, c of Libra, b of Cancer, d of Capricorn. The ascending semicircle will therefore be d a b, and the descending semicircle b c d. I say that every western and northern angle formed by the meeting of the oblique horizon with the circle of the ecliptic at points of the ascending semicircle d a b is greater than every angle, likewise western and northern, formed at points of the descending semicircle b c d.

Diagram. In the left column, a lens-shaped figure formed by two great-circle arcs meeting at the left and right. A long, nearly straight line — the equinoctial — runs across the figure from **L** at the far left, through **a** where it meets the left cusp, past **h** at the centre and **i** to the right of centre, out through **c** and on to **e** at the far

right. The upper arc of the lens carries **b** at its crest and **K** to the right of b. From **f**, outside the figure at the upper left, an arc descends across the upper half, passing the label **g** and meeting the equinoctial near h; a second arc from f crosses further right, meeting the equinoctial near i. The lower arc of the lens carries **m** at the lower left, **d** at the bottom, and **r** at the lower right.

- L (far left, on the equinoctial)
- a (left cusp of the lens, on the equinoctial)
- h (centre of the figure, on the equinoctial)
- i (right of centre, on the equinoctial)
- c (right cusp of the lens)
- e (far right, on the equinoctial)
- b (crest of the upper arc)
- K (upper arc, right of b)
- f (outside the figure, upper left)
- g (inside the upper half, left)
- m (lower arc, lower left)
- d (bottom of the lower arc)
- r (lower arc, lower right)

For let the oblique horizon f g h make, on its western side, the western and northern angle f g a with the ecliptic at the point g of the ascending semicircle — that is, of the first quadrant; and at the point k of the second quadrant let it make the angle f K g, likewise western and northern, the horizon now having the position f k i. Let h and i be the two points at which the horizon cuts the equinoctial.

Now take the triangle f h i. Its two angles a h f (the exterior one) and h i f (the interior one) are equal, since both are the complement of the altitude of the pole in the same horizon; therefore the two sides f h and f i taken together are equal to one semicircle. And for that reason the two sides f g and f K of the triangle f g K are together less than a semicircle. From this you will conclude that the exterior angle f g a is greater than the interior angle f K g.

By this same technique you will prove that angles of this kind steadily decrease from a to b and from b to c, and that the angles of the first quadrant are greater than those of the second; but that from the point c to d, and from d [?] to a — that is, in the southern semicircle — such angles steadily increase.

Next take any point you please in the quadrant c d; call it r. I say that the angle formed at g, any point whatever of the quadrant a b, is greater than the angle formed at r. For let K and r stand at equal distances from the point c, the beginning of Libra. Then the two western and northern angles at the points K and r will be equal, by what we demonstrated above. But the angle at g is greater than the angle at K; therefore the angle at g is greater than the angle at r. Consequently the angle formed at any point of the first quadrant will be greater than the angle formed at any point of the descending semicircle.

Next take any point you please in the quadrant d a; let it be m. I say that the angle formed at m is greater than every angle formed in the descending semicircle. For let m and g stand at equal distances from a, the beginning of Aries; then the angles at m and at g will be equal. But the angle at g is greater than every angle formed in the descending semicircle. Therefore every western and northern angle made in the ascending semicircle will be greater than every western and northern angle of the descending semicircle.

And just as the angles from a to c by way of b steadily decrease, so those at the points running from c back to a by way of d steadily increase. The angle formed at the beginning of Aries will therefore be the greatest of all, and the one at the beginning of Libra the least of all. The angle at the beginning of Aries contains the greatest obliquity of the zodiac together with the complement of the altitude of the pole; the angle at the beginning of Libra is what remains when the angle of the zodiac's obliquity is subtracted from the angle of the complement of the pole's altitude. And therefore, when the Moon occupies

the beginning of Aries and stands in the western part of the horizon, the Sun's concealment below that horizon after conjunction will be greatest; least, by contrast, at the beginning of Libra.

The inclinations of the horizon and the ecliptic are equal to each other on both sides — which becomes obvious at once if you understand the intersections themselves to be the poles of a certain great circle passing through the ends of the quadrants. The western and eastern angles of each semicircle of the ecliptic, ascending and descending alike, which are formed with the oblique horizon, will therefore be interchanged by this rule: the eastern angles of the one are equal to the western angles of the other. So the eastern angle is least at the beginning of Aries, but greatest at the beginning of Libra. And therefore, when the Moon is at the beginning of Aries and in the eastern part of the horizon before its conjunction with the Sun, the Sun's concealment below the horizon will be least; greatest, by contrast, at the beginning of Libra.

Just as the new Moon after conjunction, in the evening after sunset, appears sooner when it is in Aries, so the waning Moon before conjunction, in the morning before sunrise, will for the same reason be hidden sooner — that is, much before the conjunction itself. In Libra the opposite holds.

Suppose, though, that you wish to refer the early or late appearances of the new Moon, and the concealments of the old, not to the concealment of the Sun (as our own view has it) but rather to the swift or sluggish ascensions and descensions of the arcs of the ecliptic lying between the two luminaries, as Georg Peuerbach and the Arabs do. Then you will say that after conjunction the new Moon appears soonest in Capricorn and Gemini, and latest in Virgo and Libra; and that before conjunction the old Moon is hidden soonest in Pisces and Aries, but latest in Cancer and Sagittarius.

Furthermore, just as the Moon's swifter motion lengthens its distance from the Sun after conjunction and makes the new Moon appear sooner, so before conjunction it contracts that distance, and the old Moon is on that account hidden later. The Moon's northern latitude is likewise a reason, in these northern climates, why the new Moon appears sooner and the old and waning Moon is hidden later — that is, not long before conjunction.

The author's three causes will therefore concur, so that on the very day on which the Moon is old, the new Moon is seen in the evening. If only two concur, it will appear on the second day; if only one, on the third day — but not so that [?] on one and the same day, the day on which the old Moon is seen in the morning before sunrise, the new Moon appears in the evening. For the conjunction's occurring in the ascending semicircle of the ecliptic is equally a cause that the new Moon appears sooner, and that the old is hidden sooner and lies concealed for a longer time before conjunction.

But the old Moon ought not to be concealed at all — let alone later — if it is to be seen in the morning of the same day before conjunction and the new Moon to appear in the evening after that conjunction. And if the author thought this could happen, as his own words seem to state, and as some assert they have witnessed, he was nevertheless wrong to claim that it arises from those three causes acting together.

Al-Battānī, however, the most careful of astronomers, in setting out one by one the causes of the Moon's early appearance after conjunction, added three others. Account must be taken (he says) of parallax, so that the arc of the zodiac contained between the Moon's apparent place and the setting Sun may be known;

the Moon's own distance from the Earth must be measured as well; and likewise the true interval between the Moon and the Sun. For when the Moon is 180 degrees distant from the Sun it shows the fullness of its light; and since 12 times 15 makes 180, if the interval between the two luminaries is divided by 15, the quotient yields the digits of light — that is, the twelfths. Finally he concludes that the Moon cannot be seen within the space of one natural day after conjunction; so he will be far less willing to grant that the old Moon and the new are observed within a single artificial day.

¶ On parallax [De diuersitate aspectus.]

Annotation 7 [Annotatio septima.]

Let the centre of the Earth be a, the planet seen in the upper hemisphere b, and the place on the Earth's surface from which it is observed c. Then, with the triangle a b c set up, let its plane be extended as far as the firmament. In that plane let d be the point above the vertex — the end, that is, of the line a c produced in a straight line — and let the straight line e f in the same plane be the line of the horizon. Let a b and c b be produced as far as g and h in the firmament; the planet, then, will be seen at g, but its true place will be at h. So the apparent distance from the zenith will be d g, and the true distance d h; and therefore the parallax in the circle of altitude…

will be the arc g h.

Now from the point a let a straight line a K be drawn out as far as the firmament, parallel to the straight line c g. And since the straight line a c, the radius of the Earth, is of imperceptible magnitude in comparison with a K, the arc g K will accordingly be reckoned of imperceptible magnitude in the circle d f e; and for that reason the arc h K will be taken as equal to the arc g h, the arc of parallax.

Diagram. In the left column, a figure of three nested circles. The outermost circle is the firmament, carrying **d** at the top, **e** at the far left, and **K** on its right-hand side with **f** just below and outside it; **h** stands on the outer circle above and left of **g**, and **g** itself lies on the outer circle at the upper right. Within it a second, middle circle carries **L**, **b**, and **m** in a close group above and right of the centre, with **i** just below and right of m. At the very centre a small circle represents the Earth: **a** is marked at its centre and **c** on its upper-left circumference. A horizontal line runs across the whole figure from e at the left, and a vertical line descends from d through the centre; from a and from c a fan of straight lines runs out toward b, m, g, K, and i.

- d (top of the outer circle)
- e (far left, on the horizontal line)
- f (right, just outside the outer circle, below K)
- K (on the outer circle, at the right)
- g (on the outer circle, upper right)
- h (on the outer circle, above and left of g)
- L (on the middle circle, above centre)
- b (on the middle circle, right of L)
- m (on the middle circle, right of b)
- i (below and right of m)
- c (on the small central circle, upper left)
- a (at the centre of the small circle)

But the angle c b a is equal to its alternate angle b a K, which subtends that same arc h K. The same angle c b a will therefore define the parallax in the circle of altitude d f e itself, provided it has been set at the centre of that circle. This angle of parallax will be greatest at the horizon, and the further from the horizon it stands, the smaller it will be.

For let the planet be placed at i, a point of the horizon, and let the straight line a i, the line of the true place, be drawn to it. I say that the angle a i c, the parallax at the horizon, will be greater than the angle a b c, the parallax of that same planet above the horizon. And let the same planet be placed again in another, higher position, say at l, and let the straight lines a l and c l be drawn: the angle a b c will then be greater than the angle a l c.

You will demonstrate these points by exactly the technique we used earlier in the *Theory of the Sun*, to show that the difference between mean and apparent motion — that is, between mean and true motion — is greatest at the point of mean longitude, and that the nearer the Sun comes to the point opposite the apogee, the smaller that difference is. You will conclude all this easily if you take the point a to be the centre of the Sun's eccentric, c the centre of the world, and the line c i accordingly to be the line of mean longitude.

But that the further a star is from the centre of the world the smaller the parallax it has, you will grasp at once if from a, the centre of the world, you draw a straight line to the point m, set between b and g. Since in the triangle a b m the exterior angle c b a is greater than the interior and opposite angle a m b, a planet seen at m will have the smaller parallax. And so the parallax of a nearer planet will be greater than that of a more distant one; which was what had to be shown.

At the ninetieth degree of the ecliptic from the ascendant no parallax arises in longitude, because that ninetieth degree of the ecliptic always lies on the circle that runs through the zenith and the poles of the ecliptic.

For let a b c be the circle of the horizon, with e as its pole; let a d be the segment of the ecliptic set above the horizon; and from f, the pole of the zodiac, let a great circle come through e, cutting the ecliptic at g and the horizon at h. I say that g is the ninetieth degree from the ascendant.

Now since the circles of the horizon and of the ecliptic bisect one another, by Theodosius, *Spherics* I.15, the circle f g h — coming through the poles of both — will therefore bisect the semicircles a b d and a g d, by Theodosius, *Spherics* II.12. Accordingly a g will be a quadrant, and the point g will be the 90th degree from the ascendant, at which indeed no parallax in longitude will occur, since that same great circle f g h, beneath which the star is seen, comes from the poles of the ecliptic.

Diagram. In the right column, a single circle representing the horizon, with **b** at the top, **d** on the circumference at the upper right, **h** on the circumference at the left, **c** on the circumference at the lower right, and **a** on the circumference at the lower left. Two arcs cross inside the circle: one runs from a up to d, the other from h down past **e** (marked right of centre) and through **g** (marked at the centre, with a heavy point) toward the lower right, ending at a short mark labelled **f** on the circumference between d and c.

- b (top of the circle)
- d (upper right, on the circle)
- h (left, on the circle)
- c (lower right, on the circle)
- a (lower left, on the circle)
- e (inside, right of centre)
- g (at the centre, heavy point)
- f (on the circle, between d and c)

Parallax will nevertheless be present then in latitude; and that will be no other arc than the one we defined above as parallax simply so called, or as parallax in the circle of altitude.

There is a further point to be noted: the distance…

between the ninetieth degree of the ecliptic from the ascendant and the meridian, measured along the divisions of the horizon, is as great as the amplitude of rising [*amplitudo ortus*] of the ascendant.

For let the circle a b f be the meridian, d e f the semicircle of the equinoctial, and b e c the semicircle of the horizon; let g k be the segment of the ecliptic between the meridian and the horizon; let the point a be the pole of the horizon; and let the great circle a o r, coming through the poles of the ecliptic, cut the ecliptic at o and the horizon at r. Then o K and K r, the segments of the ecliptic and of the horizon, will be quadrants, by Theodosius, *Spherics* I.15 and II.12; and the point o will therefore be the ninetieth degree from the ascendant.

Again, since the meridian a b f passes through the poles both of the equinoctial and of the horizon, the segments d e of the equinoctial and b e of the horizon will be quadrants, by the same propositions of Theodosius, namely I.15 and II.12. So from the two quadrants K r and b e let us take away the common arc e r, and the two arcs e k and b r will be left equal to one another.

Diagram. In the left column, a single large circle — the meridian — with **c** at the top, **f** at the right, **b** at the lower left, and **d** on the circumference at the left. **a** stands inside the circle at the upper left with **o** immediately to its right, and **g** stands just outside the circle to the left of a. **K** is marked at the top, right of centre, and **e** at the centre; **r** stands below and left of e, toward the foot of the figure. Three arcs cross the disc: one runs from d through e to f, a second from b through e up toward c and on to the right, and a third descends from a through o and r.

- c (top of the circle)
- f (right, on the circle)
- b (lower left, on the circle)
- d (left, on the circle)
- a (inside, upper left)
- o (inside, right of a)
- g (outside the circle, left of a)
- K (inside, upper part, right of centre)
- e (at the centre)
- r (below and left of e)

Now e K is the amplitude of rising of the ascendant, while b r is the distance between the ninetieth degree from the ascendant and the meridian, measured along the divisions of the horizon. The distance between the 90th degree from the

ascendant and the meridian, taken along the horizon, is therefore as great as the amplitude of rising of the ascendant. And accordingly, whenever a star's distance from the meridian along the horizon is found equal to the amplitude of rising of the ascendant, that star will in that position have no parallax in longitude. Regiomontanus rightly gave this warning in his book on the comet, problem 5.

¶ On latitude and declination [De Latitudine & declinatione.]

Annotation 8 [Annotatio octaua.]

The more recent astronomers assign the true places of the stars on the concave surface of the ninth sphere, or of the primum mobile. They define the declination of a star as the arc of a great circle passing through the poles of the world — that is, of the equinoctial — between the true place of the star and the equinoctial; and they determine the declination of any star from its meridian altitude and from the latitude, once found, of the place where the observation is made. The latitude of a star, by contrast, will be the arc of a great circle passing through the poles of the ecliptic of the eighth sphere, between the true place of that same star and the ecliptic of the eighth sphere itself.

For they take the latitudes of the fixed stars which Hipparchus and Ptolemy observed many centuries earlier to be unaltered, even though the eighth sphere itself is admitted to be stirred by the motion of trepidation. The latitudes of the fixed stars must therefore be referred to the ecliptic of the eighth sphere, not to the ecliptic of the ninth sphere or of the primum mobile. Peuerbach holds the same view when he says that the Sun has no latitude.

They investigate the true place of the observed fixed star by means of a spherical triangle, one side of which is equal to the greatest declination of the Sun, a second the complement of the latitude of that star, and the third, finally, the complement of the declination — thereby making known the angle formed at the pole of the zodiac of the eighth sphere. It is plain, then, that the distance found by that technique must be referred to the point of the summer tropic at which the greatest declination of the Sun falls, not to the beginning of Cancer of the primum mobile; and accordingly that the starting point for computing the motion of the fixed stars is taken from the vernal section, not from the beginning of Aries of the primum mobile.

And since the wandering stars and the fixed alike ought to have one and the same starting point in the tables, from which their motions may be computed, the vernal section will on the view of the more recent astronomers be that starting point — a movable and wandering one, which we by no means approve. We have

written more about this in the previous book, chapter 4, on the declination of the Sun.

¶ The three superior planets have a further latitude arising from the plane surface of the epicycle, and the rest. [Tres planetæ superiores latitudinem aliam habent ex parte superficiei planæ epic. & reliqua.]

Annotation 9 [Annotatio nona.]

Since the centre of the epicycle lies in the plane of the deferent, while Peuerbach writes that the surface of the epicycle sometimes declines from the surface of the deferent, someone may perhaps suspect that the planes of the epicycle and of the deferent sometimes cut one another, the one declining from the other, and sometimes come together as one — whereas in truth the planes of the epicycle and of the deferent always cut one another, and never come together.

The remaining features of this motion, moreover, have not been sufficiently set out by the author himself. We shall therefore expound this theory of the latitude arising from the epicycle more clearly, in the following way.

When the centre of the epicycle is placed at the node of the head [*nodus capitis*], the plane surface of the epicycle will lie in the plane surface of the ecliptic; only the diameter of the true apogee will be in the surface of the deferent, namely in the common section of the plane of the ecliptic and the plane of the deferent.

Then, as the epicycle moves away from the node, the diameter of the true apogee begins to decline from the surface of the deferent: for the true apogee will recede toward the surface of the ecliptic, and the point opposite the apogee in the opposite direction. The diameter of the mean longitudes, which is the axis of this motion, will also cut through the surface of the deferent: the eastern half-axis will be left between the two surfaces, that of the ecliptic and that of the deferent, while the western half-axis will lie outside both surfaces, yet equidistant from the surface of the ecliptic.

The surface of the epicycle will therefore be inclined to the surface of the deferent, and likewise to the surface of the ecliptic on the side of the point opposite the apogee. And so, as the centre of the epicycle advances, the true apogee and the point opposite it will recede more and more from the surface of the deferent; the axis of this motion, however, will approach that surface continually, while remaining nonetheless equidistant from the ecliptic, until the centre of the epicycle reaches that point of the deferent which declines most from the ecliptic. For then the diameter of the true apogee will decline as much as it

possibly can from the surface of the deferent, while the diameter of the mean longitudes will be set in the surface of the deferent itself.

From this place on to the node of the tail [*nodus caudæ*], the diameter of the true apogee will continually approach the surface of the deferent, while the diameter of the mean longitudes will cut that surface again — but with the halves interchanged. For its western half-diameter will be left between the surfaces of the ecliptic and of the deferent, and the eastern outside both surfaces, equidistant (as before) from the surface of the ecliptic. And so, when the centre of the epicycle reaches the node of the tail, the surface of the epicycle will again be set in the surface of the ecliptic, and the diameter of the true apogee in the surface of the deferent.

But once the centre has moved through the remaining, southern semicircle, the point opposite the true apogee will decline from the surface of the deferent toward the southern side, and the remaining consequences will follow as before. And since, when the centre of the epicycle stands at the nodes, the surface of the epicycle coincides with the surface of the ecliptic, but outside those nodes is inclined to it, while it always cuts the surface of the deferent — the axis about which the epicycle moves in longitude will therefore be equidistant from the axis of the ecliptic only twice in a single revolution of the centre of the epicycle, namely when the centre of the epicycle is at the nodes. From the axis of the eccentric, however, it will never be equidistant.

Annotation 10 [Annotatio decima.]

Since the centre of the epicycle lies in the surface of the deferent, and the plane surface of the epicycle always cuts the surface of the deferent, the straight line that is the common section of those surfaces will therefore be a diameter of the epicycle.

And so, when the centre of the epicycle stands outside the nodes, that half of the surface of the epicycle which contains the point of the apogee — the upper half, that is — is enclosed between the two surfaces, that of the deferent and that of the ecliptic; while the lower half, in which the point opposite the apogee lies, is left outside both. So long, then, as the planet is in the lower half of the epicycle, it is removed further from the ecliptic than the deferent is from that same circle [?]. The planet will accordingly not always be found between the deferent and the ecliptic, as Gerardus Cremonensis [Gerard of Cremona] supposed.

But when the centre of the epicycle stands at the point of the deferent of greatest latitude, that common section will be the diameter of the mean longitudes; in

other positions it will be a different diameter. And accordingly the half of the surface of the epicycle — whether the upper, which is contained between the surfaces of the deferent and of the ecliptic, or the lower, which is left outside both — will not be one and the same in every position of the epicycle.

¶ The eighth sphere has a threefold motion. [Octauæ sphæræ triplex inest motus.]

Annotation 1 [Annotatio prima.]

We have written more about the motion of the eighth sphere according to Alfonso in the previous book, in the chapter on the Declination of the Sun. And since the theory of that motion has been most lucidly set out by Georg Peuerbach, we shall note only a few small points here.

You will easily show that the ecliptic of the eighth sphere cuts the small circles into alternately equal portions. For those small circles are equal, by Theodosius, *Spherics* I.33; they are also equidistant, by [proposition] 2 of the second book; and therefore alternate portions of them will be equal, by [proposition] 22 of that same second book.

Now, so that you may understand when the equation of the motion of access and recess is to be added to the motion of the ninth sphere and when it is to be subtracted, let us set a as one pole of the small circle; let b be the western intersection of that small circle with the ecliptic of the ninth sphere and c the eastern; and from the point d, the northern pole of that ninth ecliptic, let a great circle come through a, cutting the small circle at e and f. That great circle will therefore make right angles with the small circle, by Theodosius, *Spherics* I.20, and for that reason the small circle will be cut into the quadrants b e, e c, c f, and f b.

From the same pole d let great circles also come through two points of the small circle, g and h, whose distances from e are equal, and let them cut that same small circle on the other side at K and i, and the ecliptic of the ninth sphere at l and m. The ninth's ecliptic and the circle d g K will then cut one another at right angles, by Theodosius, *Spherics* I.19. The ninth's ecliptic therefore passes through the poles of the circle d g K, by [proposition] 17; it passes also through the poles of the small circle. Accordingly it will bisect the arc g c K at the point c, and likewise the arc g l K at the point l, by Theodosius II.12. So, if from the quadrants e c and c f the equal circumferences g c and c K are taken away, the two arcs e g and f k are left equal, by the common notion.

By the same technique you will conclude that the two arcs e h and f l [recte: f i] are equal; and therefore the four arcs e g, f k, e h, and f i will be equal to one another, by the common notion.

Now from the point a let the arcs of great circles a g and a h be drawn to g and h. In the two spherical triangles a g d and a h d the two angles d a g and d a h will be equal to one another, because the two arcs g e and e h are equal; the side a g is equal to the side a h; and the side a d is common to both triangles. The two angles a d g and a d h, contained by equal sides, will therefore be equal to one another, by Menelaus I.4. And accordingly the two arcs of the ninth's ecliptic, a l and a m, subtended by those angles, will be equal to one another.

Now a l is the equation of the motion of access and recess when the head of the eighth sphere is at g or at K, and a m is the equation of that same motion when the head is at h or at i. Whenever, therefore, the motion of access and recess is the arc e g, or e c k, or e c i, or e c h, the same equation will be found in the table.

The equation is added to the motion of the ninth sphere when the head is at K, even though in that position the head is moving backwards by its own proper motion. For when it was at c, the whole arc a c was being added; at K, then, the arc of regression will be c l, and when this is subtracted from a c there is left the arc a l, still to be added to the motion in longitude.

Diagram. In the right column, a single circle — the small circle of access and recess — with the point **d** set well above it, outside the circumference. A horizontal line runs across the circle from **b** at the far left to **c** at the far right; on it, **a** stands at the centre, with **m** between b and a and **l** between a and c. On the upper circumference stand **h** (left), **e** (at the top, on the vertical through a) and **g** (right); on the lower circumference, directly beneath them, **i** (left), **f** (at the bottom) and **K** (right). Three curved lines descend from d and cross the circle from top to bottom: one through h to i, one through e and a to f, and one through g to K.

- d (above the circle, outside it: the northern pole of the ninth sphere's ecliptic)
- b (left end of the horizontal line)
- m (on the horizontal line, between b and a)
- a (centre of the circle, on the horizontal line)
- l (on the horizontal line, between a and c)
- c (right end of the horizontal line)
- h (upper circumference, left)

- e (top of the circle)
- g (upper circumference, right)
- i (lower circumference, left)
- f (bottom of the circle)
- K (lower circumference, right)

But for the same reason the equation is subtracted when the head is at h, even though in that position it advances in the order of the signs: for when it was at b, the whole arc a b was being taken away from the motion of the ninth sphere. At h, then, the arc of advance in the order of the signs will be b m; and when this is subtracted from a b, the arc a m is left, still to be taken away from the motion in longitude.

Further, while the head of Aries of the eighth sphere moves through the first quadrant of the small circle — that is, from e to c — the equations do indeed increase, but by unequal increments: for the differences of those equations become continually smaller.

For let the arcs e g, g k, and k m be set equal, and let the head of Aries of the eighth sphere stand at g, at K, and at m; and let great circles be drawn through the poles of the ninth's ecliptic. The one coming through the point g cuts the small circle again at h, and the ecliptic of the ninth at i; the one through k cuts the small circle at q, and the ninth's ecliptic at o; and the one through m cuts that same small circle again at n, and the ninth's ecliptic at r. The arc a i will therefore be the equation of the motion e g, the arc a o the equation of the motion e K, and the arc a r the equation of the motion e m. I say, then, that the arc o r — the diffe-

—rence, that is, of the equations a r and a o — is less than the arc i o, which is the difference of the equations a o and a i; and finally that the arc i o itself is less than a i.

For the straight line b c is the common section of the small circle and the plane of the ninth's ecliptic. Let the great circle e a f, coming through the point p, which is the centre of the sphere, cut the plane of the ninth's ecliptic along the straight line p a, and the small circle along the straight line e f; the intersection of those two straight lines is l, the centre of the small circle. Let the great circle g i h likewise cut the plane of the ninth's ecliptic along the straight line p i, and the small circle along the straight line g h; let the intersection of those straight lines be the point z. Further, let the great circle K o q cut the plane of the ecliptic along the straight line p o, and the small circle along the straight line K q; let the intersection of those straight lines be the point t. And finally let the great circle m r n cut the

plane of the ecliptic along the straight line p r, and the small circle along the straight line m n; let the intersection of those straight lines be the point s.

Now since the straight line p l joins the centre of the sphere to the centre of the small circle, it is therefore perpendicular to the plane of the small circle, by Theodosius, *Spherics* I.7; and for that reason the rectilinear angle p l c will be a right angle, by *Elements* XI, definition 2.

We shall therefore conceive the right-angled triangle p l t in the plane of the ninth's ecliptic. Its side p t, which subtends the right angle, will be greater than the side p l, which subtends an acute angle, by *Elements* I.19. The straight line p z cuts the remaining, acute angle l p t unequally. For if the line p z did cut the angle l p t into two equal angles, l p z and z p t, then as p t is to p l, so t z would be to z l, by *Elements* VI.3. But p t has been shown greater than p l; t z would therefore be greater than z l, which is impossible. For since t z is further from the centre, it will be less than z l. Accordingly the straight line p z does not bisect the angle l p t but cuts it unequally, and the angle l p z, which looks upon the greater segment of the base, will be greater than z p t, which looks upon the smaller.

For if the angle l p z were less than the angle z p t, we should then bisect the whole angle l p t, by *Elements* I.9; and the straight line dividing the angle l p t would accordingly fall between z and t. By that same *Elements* VI.3 we should then conclude something impossible a second time — namely that a part of the segment t z is much greater than the segment z l, which again is impossible.

Diagram. A large figure occupying the middle of the leaf and running across both columns, so that the left column is narrowed alongside it. A circle represents the small circle, crossed from left to right by a straight line running from **b**, at the left of the circumference, to **c**, at the right — the common section of the small circle with the plane of the ninth's ecliptic. Near the centre of that line stands **l**; to the right of centre, just below the line, stand **z**, **t** and **s** in order, and just above the line stand three further marks, read as **a**, **i** and **o** [?] (with **r** [?] to their right). On the upper circumference stand **g**, **K** and **m**, and on the lower circumference **f**, **h**, **q** and **n**; near-vertical chords descend from the upper points to the lower ones, crossing the horizontal line at z, t and s. From a point far outside the circle at the lower left — the centre of the sphere, called p in the text, though the woodcut letter itself is not legibly cut [?] — a fan of long straight lines runs up to the points on the horizontal line and to the circumference. A further letter, read as **d** [?], stands outside the circle at the lower right.

- b (left, on the circumference; left end of the horizontal line)

- c (right, on the circumference; right end of the horizontal line)
- l (on the horizontal line, near the centre — the centre of the small circle)
- z, t, s (on the horizontal line, right of l, in that order)
- a, i, o [?] (marks set just above the horizontal line, above z, t and s respectively; r [?] further right)
- g, K, m (upper circumference, left to right)
- f, h, q, n (lower circumference, left to right)
- [?] (the vertex of the pencil of straight lines, far outside the circle at the lower left; p in the text)
- d [?] (outside the circle, at the lower right)

For this reason the straight line p z will cut the angle l p t unequally, and l p z will be greater than z p t; and accordingly the ecliptic's arc a i will be greater than the arc i o, by the last proposition of *Elements* VI.

It will be demonstrated in the same way that the side p s is greater than the side p z, since in the triangle s p z the angle p z s is obtuse — being the exterior angle opposite the right angle p l z — while the angle z s p is acute. But the straight line t s, being further from the centre, is less than z t; the straight line p t will therefore cut the angle z p s unequally, and the angle z p t will be greater than the angle t p s. And for that reason the arc i o will be greater than the arc o r.

The equations of the motion of access and recess for the quadrant e c therefore increase by unequal increments: for the differences of those equations are drawn in more and more from a to c — which is what we undertook to demonstrate.

Lemma [Lemma.]

That l z, which we assumed to be greater than the straight line z t, and that z t in turn is greater than the straight line t s, we shall easily conclude in the following way. Let the straight lines f g and h K be joined. Since the two arcs e g and l h have been shown equal, by Theodosius II.12, the angle e f g will therefore be equal to its alternate angle f g h, by *Elements* III.27; and for that reason the straight lines e f and g h will be parallel, by [*Elements*] I.27. You will demonstrate in the same way that the two straight lines g h and K q are parallel. And finally, by that same [proposition] of the third book, 27, you will conclude that the two angles e f g and g h K are equal to one another.

From the points g and K let the perpendicular straight lines g x and K y be drawn to the straight lines e f and g h, by *Elements* I.12. The two right-angled triangles f g x and h K y will then be equiangular, by *Elements* I.32 and the common notion,

and accordingly they will have their sides proportional, by *Elements* VI.4: as f g is to h K, so g x is to K y. Now let the straight lines l g and l K be joined. The angle f l g will be greater than the angle h l K, and for that reason f g will be greater than h K, by *Elements* I.24; and therefore the straight line g x will be greater than K y.

But the straight lines l z and g x are parallel, since the angles at l and at x are right angles; and K y and z t are likewise parallel, since the angles at z and at y are also right angles. In the parallelograms l g and z k, then, the side g x is equal to the side l z, and further the sides K y and z t will be equal, by *Elements* I.34. But the straight line g x has been shown greater than K y. And for that reason l z will be greater than z t; and by the same technique you will conclude that z t is greater than t s — which was assumed in the demonstration.

Diagram. In the left column, a circle crossed from left to right by a straight line running from **b**, at the left of the circumference, to **c**, at the right. On the upper circumference stand **e** (left) and **g** (right of e), with **K** below and right of g, inside the circle, and **n** [?] on the circumference further to the right. Beneath e, inside the circle, stands **x**, and **y** stands below and right of it. On the horizontal line stand **l** (left of centre), then **t** and **s** toward the right, with **z** just below the line beneath l. On the lower part of the figure stand **f**, **h** and **q** in order from left to right. Near-vertical chords descend from the upper points to the lower ones, and short straight lines join l to g and to K.

- b (left, on the circumference; left end of the horizontal line)
- c (right, on the circumference; right end of the horizontal line)
- e (upper circumference, left)
- g (upper circumference, right of e)
- K (inside the circle, below and right of g)
- n [?] (right of K, toward the circumference)
- x (inside the circle, below e)
- y (below and right of x)
- l (on the horizontal line, left of centre)
- z (just below the horizontal line, beneath l)
- t, s (on the horizontal line, toward the right)
- f, h, q (lower part of the figure, left to right)

And so, just as the equations increase with ever smaller differences while the head of Aries of the eighth sphere moves through the quadrant e c, so through the quadrant c f those equations decrease with ever greater differences. And since the head of Aries of the eighth sphere moves through the quadrant f b, the

equations increase as they did through e c; and through b e they decrease again with greater differences, as they did through c f.

The motion of the fixed stars that arises from the motion of the ninth sphere together with the trepidation of the eighth is therefore as follows. While the head of Aries of the eighth sphere moves from the point b as far as e, that motion is swift and increasing in speed. From there to c it is swift but diminishing in speed. From there, through the quadrant c f, it is slow and increasing in slowness, as far as the point h, which is 21 degrees distant from the point f; for when the head of the eighth sphere stands at that 21st degree before f, the fixed stars will be stationary.

For the difference of the equations at h itself, as the table set out degree by degree shows, contains 8 minutes and 49 seconds, which is the amount of the motion of the ninth sphere in 20 years. But the mean motion of access and recess in those same 20 years is a little more than one degree. From the 21st degree before f, then, as far as the 20th before that same f, the head of Aries of the eighth sphere goes backwards by its own proper motion, but advances by just as much through the motion of the ninth sphere; and for that reason the fixed stars will appear stationary during that time. From there on to f they will be retrograde, increasing their regression; and from f as far as the 20th degree beyond that same f they will still be retrograde, but diminishing their regression. From the 20th to the 21st they will again be stationary. From there as far as b they will now move in the order of the signs, though their motion will be slow, diminishing in slowness.

¶ On the motion of the eighth sphere, according to Thebit [De motu octauæ sphæræ, secundùm Thebith.]

Annotation 2 [Annotatio secunda.]

Let the circle a b c be the fixed ecliptic, and the point b the head of Aries of that ecliptic — that is, the pole of the small circle d e f, on which the head of Aries of the movable ecliptic turns. Through that same point b let the arc of a great circle e b f come at right angles upon the fixed ecliptic a b c itself, cutting the small circle at e and f. Let a b and b c be quadrants, the point a the beginning of Capricorn and c the beginning of Cancer. And a and c will therefore be the poles of the circle e b f, by the first book of Theodosius. Through a and e let the great circle a e also come, which must pass through the point c, by proposition 15 of that same first book of Theodosius. Since, then, a and c are the poles of the circle e b f, the two segments a e and

— and *e c* will be quadrants, and the angles which that circle *a e c* [?] makes with *e b f* will be right angles. The poles of that same circle *a e c*, therefore, will lie on the circle *e b f*, by [proposition] 17. And so the circle *e b f* cuts the circle *a e c* and passes through its poles; it also cuts the small circle *d e f* and passes through its poles. For that reason the two circles *a e c* and *d e f* touch one another at that same point *e*, by proposition 4 of the second book of Theodosius.

In the same way let a great circle *a f c* [?] be drawn through *a* and *f*, which will touch that same small circle at *f* itself. Next, let the equinoctial cut the circle *a e c* at *g*, in the western part; let it cut the circle *a f c* at *h*, in the eastern part; and let it cut the small circle at *d* and *k*.

We shall take *a e c* to be the movable ecliptic while the head of Aries of that ecliptic is at the point *e*, at the northern contact. Since *a e* and *e c* are quadrants, the beginning of Cancer of that movable ecliptic will be at *c*, and the beginning of Capricorn at *a*. The same will happen when that same head of Aries is at the southern contact *f*; and so the movable heads of Cancer and Capricorn will coincide with the heads of the fixed ones.

In the position *a e c* the movable ecliptic is separated from the equinoctial by the arc *b g*, which is the greatest distance of the movable section from the fixed section *b*; in the position *a f c*, by the arc *b h*. Now these arcs *b g* and *b h* are equal. You will readily conclude this from the two right triangles *b f h* and *b e g*. The vertically opposite angles *f b h* and *e b g* are equal, and the two sides *b f* and *b e* are equal; therefore the remaining angles and the remaining sides will be equal to one another. The side *b g*, then, will be equal to the side *b h*, by the first book of Menelaus — a result you can also reach by the ratios of the right sines.

For the angles at *b* are acute; and since the sides *b e* and *b f* are less than quadrants, the angles *b g e* and *b h f* will be acute. Now in the triangle *b f h*, as the whole sine is to the sine of the complement of the arc *b f*, so is the sine of the angle *f b h* to the sine of the complement of the angle *f h b*. Likewise in the triangle *b e g*, as the whole sine is to the sine of the complement of the side *b e*, so is the sine of the angle *e b g* to the sine of the complement of the angle *b g e*. You will conclude, therefore, that the right sines of the angles *f h b* and *b g e* are equal.

And since, as the whole sine is to the sine of the angle *f h b*, so is the sine of the side *b h* to the sine of the side *b f*; and likewise, as the whole sine is to the sine of the angle *b g e*, so is the sine of the side *b g* to the sine of the side *b e* — while the right sines of the sides *b f* and *b e* are equal — the sines of the sides *b h* and *b g*

will therefore be equal. And because the whole arc *g h* is less than a semicircle, the two arcs *b g* and *b h* will be equal to one another.

Each of them contains ten degrees and 45 minutes; the whole *g h*, then, will be of 21 degrees 30 minutes. An equal arc corresponds to it in the eastern part, arising from the alternate contact of the small circle described about the head of Libra of the fixed ecliptic; and this is why the author writes that the quantity of each is about 21 degrees and 30 minutes.

Diagram. A lens- or eye-shaped construction spanning the full width of the page, formed by two great arcs meeting at the far left and far right points; a small circle is drawn about the centre, with further great-circle arcs crossing it. The outer left and right vertices are the poles; the top and bottom vertices are the two contacts of the small circle; the centre is the fixed vernal section.

- *a* (far left vertex): pole of the circle *e b f*; beginning of Capricorn
- *c* (far right vertex): pole of the circle *e b f*; beginning of Cancer
- *e* (top vertex): northern contact of the small circle
- *f* (bottom vertex): southern contact of the small circle
- *b* (centre): head of Aries of the fixed ecliptic
- *d* (left of centre, on the small circle): intersection of the equinoctial with the small circle
- *g* (upper left, inside the figure): intersection of the equinoctial with the circle *a e c*
- *l* (upper left, immediately right of *g*)
- *i* (upper centre-left, marked by a small point) [?]
- *k* (right of *b*)
- *k* (lower left, on the descending arc) [?]
- *m* (right of centre, on the upper arc)
- *h* (right, where the equinoctial meets the circle *a f c*)

Now let a great circle come through *b* at right angles upon the equinoctial, cutting the small circle at *i* and *k*. The arc *d i* will then be a quadrant of that small circle; and with the head of Aries of the movable ecliptic placed at *i*, let that movable ecliptic cut the equinoctial at *l*. The arc *i l* will accordingly be the greatest equation of the eighth sphere — that is, the greatest distance of the head of Aries of the movable ecliptic from the section of that ecliptic with the equator — which he sets equal to the arc *b g*, namely 10 degrees 45 minutes.

For the arcs *b g* and *i l* are in fact unequal. They are nevertheless reckoned equal, because the two angles *e g b* and *i l b*, the greatest declinations of the movable ecliptic at the positions *e* and *i*, differ imperceptibly.

Of the two right triangles *b e g* and *i b l*, then, the two sides *e b* and *i b* are equal, and the two angles *g e b* and *l b i* are right angles; while the two angles at *g* and *l*, the greatest declinations, are assumed equal on account of their imperceptible difference. For that reason the sides *b g* and *i l*, which subtend the right angles of those same triangles, will be equal, by the first book of Menelaus.

You can also show this by the ratios of the right sines. In the right triangle *b e g*, as the whole sine is to the sine of the angle *b g e*, so is the sine of the side *b g* to the sine of the side *b e*. In the right triangle *i b l*, moreover, as the whole sine is to the sine of the angle *i l b*, so is the sine of the side *i l* to the sine of the side *b i*. And since the two angles *b g e* and *i l b* are assumed

equal, it follows that as the sine of *b g* is to the sine of *b e*, so will the sine of *i l* be to the sine of *b i*; and, by alternation, as the sine of *b g* is to the sine of *i l*, so is the sine of *b e* to the sine of *b i*. But *b e* and *b i* are equal; therefore the sines of *b g* and *i l* will be equal. And because each of the arcs *b g* and *i l* is less than a quadrant, the arcs themselves will be equal — which was to be shown.

While the head of Aries is at the contact *e*, the greatest declination of the movable ecliptic is greater than the greatest declination of the fixed one. For the two arcs *b c* and *e c* have been shown to be quadrants; the arc *g c*, therefore, will be greater than a quadrant, and so the two arcs *b c* and *g c* taken together are greater than a semicircle. For that reason the exterior angle *c b h* will be less than the interior and opposite angle *c g b* of the triangle *b g c*. But that same angle *c b h* is the angle of the greatest declination of the fixed ecliptic, while the angle *c g h* is the angle of the greatest declination of the movable ecliptic while the head of Aries is at the contact *e*. In the position *e*, then, the greatest declination of the movable ecliptic is greater than that of the fixed one — which was to be shown.

But now let the head of Aries of the movable ecliptic be placed at the point *d*, the intersection of the equator and the small circle. I say that the greatest declination of the movable ecliptic will be less than that of the fixed one. For in that position let the movable ecliptic cut the fixed ecliptic at *m*. The arc *d m* will then be a quadrant, and the point *m* will accordingly be the point of greatest declination — that is, the beginning of Cancer in that position. The arc *b m*, however, is less than a quadrant, being a part of the quadrant *b c*. In the triangle *b m d*, then, since the two sides *b m* and *d m* taken together are less than a semicircle, the exterior angle

m b h will be greater than the interior and opposite angle *b d m*. But that angle *m b h* is the angle of the greatest declination of the fixed ecliptic, while the angle *b d m* is the angle of the greatest declination of the movable one. With the head of Aries of the movable ecliptic standing at the intersection of the equator and the small circle, therefore, the greatest declination of the movable ecliptic is less than that of the fixed one — which was to be shown.

The End of the Annotations on the *Theoricae planetarum* of Georg Peuerbach

AT COIMBRA.

From the Press of António de Mariz.

Year 1573

ON THE ERRORS OF ORONCE FINE, ROYAL PROFESSOR OF MATHEMATICS AT PARIS [Lutetia]

Who supposed that he had found, between two given lines, two mean proportionals in continued proportion; that he had squared the circle; that he had doubled the cube; that he had delivered the art of describing any rectilinear polygon whatever in a circle; and that he had made the differences of longitude of places evident by a means other than lunar eclipses, and indeed at any given time.

By Pedro Nunes of Alcácer do Sal [Petri Nonii Salaciensis]

One Book.

SECOND EDITION.

AT COIMBRA

Printed by António de Mariz.

Year 1573.

What, besides the refutations of Oronce's arguments, is contained in this book

¶ Plato's discovery concerning the finding of two mean proportionals and the duplication of the cube.

Archimedes' altogether lucid demonstration of the ratio of the circumference to the diameter, with the true numbers. For those given in the recently printed book of Archimedes himself are corrupt.

By what method the difference of longitude between places is to be derived from the motion of the Moon.

An explanation of the definitions of the fifth book of Euclid's *Elements*.

The theory and construction of horizontal and vertical sundials.

A demonstration, and the use, of the principal tables of directions of Ioannes de Regiomonte [Regiomontanus].

Petrus Nonius Salaciensis [Pedro Nunes] to the Reader

I should like you, kind reader, to be warned of this above all: that I resolved to publish this little work not out of any wish to attack anyone, but for the sake of bringing the truth to light. For what better suits a mathematician than the defence of truth itself and of the discipline he professes? Since it is the duty of a good man not to conceal the art he possesses but rather to contribute everything to the common good, he is bound to do so above all when he sees studious men entangled in errors under someone else's guidance. That has perhaps happened to many who, swayed by the authority of Oronce Fine, persuade themselves of a great deal which our own careful work can easily show to be false.

Oronce's errors are few, but so conspicuous that they cannot be passed over in silence. He goes wrong only when he ventures to construct mathematical demonstrations — and he ventures that rarely, unless perhaps you choose to call demonstrations the ones he quite openly borrowed from Theon [Theon of Alexandria] and Campanus, without ever mentioning them. In these he could not have gone wrong unless Theon or Campanus had gone wrong first. But Theon never slips, while Campanus, in expounding the definitions in Book V, was badly deluded — and so, then, was Oronce.

Thirteen years ago I had already decided to warn him by letter to test his discoveries more prudently and let them ripen before sending them out. But I

changed my mind, since I thought this was rather the business of men who live in the same city where Oronce teaches mathematics publicly and who are trained in the same arts and disciplines. Yet since I still do not see him recalled from the errors he had committed, whether by others' warning or of his own accord — see him rather piling new faults on the old — I have concluded that the matter should be passed over no longer. My purpose, then, is to remedy this trouble, and to set out all those errors briefly.

I should like Oronce to receive this in the same spirit in which I shall receive it whenever anyone points out my own errors. It belongs to human weakness to slip often, and I judge that this can happen to me too. I hold it the office of a good man not to gloss over the faults of others, but rather, if it could be done, to deliver all men out of the darkness of ignorance into the light of truth.

Farewell.

On the errors of Oronce Fine of the Dauphiné, who supposed that he had found two mean proportionals in continued proportion between two given lines, had squared the circle, had doubled the cube, had delivered the art of describing any rectilinear polygon whatever in a circle, and had made the differences of longitude between places plain by a method other than lunar eclipses, and that at any given time. A single book by Petri Nonii Salaciensis [Pedro Nunes].

There has just been brought to me a new book by the mathematician Oronce Fine, entitled *On the Quadrature of the Circle*. In it he boasts of having solved those five extremely difficult problems which, though sought through every age and over the longest stretch of time by the most learned men, with great industry and unremitting labour and reflection, have nevertheless not yet been found out.

The first is: in what way a cubic body is to be doubled without its shape being altered. Over this the Greek philosophers sweated a great deal. But since no method of finding it presented itself to them, Hippocrates of Chios was the first to see that the duplication of the cube could be found only once two mean proportionals in continued proportion had been found between two straight lines of which the greater was double the lesser. And so they slid into another problem no less difficult, and not a few turned to investigating it: Eratosthenes, Plato, Archytas, Hieron [Hero of Alexandria], Philo of Byzantium, Apollonius of Perga, Diocles, Pappus, and others. Their demonstrations are not accepted, on the ground that by them the mean proportionals themselves cannot be found without the assistance of some mechanical instrument.

The third problem is the most difficult of all: by what art the circle is to be reduced to the form of a square. On this the great Archimedes expended great anxiety — to say nothing of Hippocrates, Bryson, and the remaining mathematicians before Aristotle. Archimedes did as much as he could, but he could not find the exact quadrature of the circle. For he demonstrated only this: that a circle is equal to a right-angled triangle one of whose sides containing the right angle is the radius of that same circle, while the other is equal to the circumference. He did not find a straight line equal to the circumference; but he did establish with the greatest certainty that the circumference, set against the diameter, has a ratio less than triple-and-a-seventh [*tripla sesquiseptima*] but greater than triple-and-ten-seventy-firsts [*tripla super decupartiente septuagesimas primas*]. So that if the diameter of a circle is assumed to be 497 equal parts, the circumference will be greater than 1561 yet less than 1562. From that ratio of the circumference to the diameter, therefore, no exact quadrature of the circle will be able to be gathered — only a very close one, which by the same method Archimedes used could come nearer the mark.

In the fourth problem the difficulty is no less. For in it the question proposed for investigation is how any regular rectilinear figure whatever is to be described in a circle. Euclid handed down only the art of describing in a circle an equilateral triangle, a square, a pentagon, a hexagon, and next a fifteen-sided figure; besides these an octagon, a decagon, and a dodecagon, and the remaining figures in which the number of sides is always increased by doubling. But if by some device an isosceles triangle were constructed having each of the angles at its base triple the remaining angle, we could certainly describe in a circle a rectilinear figure of seven equal sides; for its smaller angle would subtend a seventh of the circumfer[ence]

ence of the circumference. And if an isosceles triangle could be constructed in which each of the angles at the base was four times the remaining angle, we could beyond doubt describe in a circle a rectilinear figure of nine equal sides, since its smaller angle would subtend a ninth part of the circumference. That figure too would be easy to describe if we could divide into three further parts the third of the circle's circumference which the side of an equilateral triangle subtends. But none of this has yet been discovered, and so we do not know how to describe the heptagon, the nonagon, or the further regular rectilinear figures that follow from them, in which the number of sides is always increased by doubling.

The last problem is as arduous and difficult as its investigation is useful. What he proposes to investigate is this: by what means, other than lunar eclipses, and at any given time, the differences of longitude of places — that is, the intervals between their meridians — may be found. For the latitudes of places, that is, their distances from the equinoctial circle, can be determined not only at the time of the meridian but at any other time as well, in the way we have devised. But no one so far has been able to find differences of longitude except by lunar eclipses, or by the measurement of journeys, which is uncertain. And since eclipses happen rarely, and since the globular shape of the earth makes it impossible for the same eclipse to be observed in every place on the globe, the positions of places — which are settled and defined chiefly by these two elements — are for the most part unknown.

Oronce Fine, however, thinks that he has discovered all of this and has demonstrated it with the greatest clarity. It has come about by divine providence, he says, that things which are outstanding and difficult are deferred to their own times and are reserved for those discoverers whom God alone knows to have been chosen for them. For my part I think the man has lost his senses. Otherwise he would have recognized the first errors he committed twelve years ago, and would then have feared these new and enormous ones, which I shall set out in this little book with the utmost clarity.

The method of Oronce Fine for finding, between two given straight lines, two others continuously proportional in the same ratio.

CAPVT. I.

Oronce Fine supposes that not only could the cube be doubled, but the circle could also easily be squared, if between two given straight lines two mean proportionals in continued proportion were found. Accordingly, in the book [*liber*

[?]] in which he sets himself to square the circle, he attempts to find two mean proportionals in the following way between a b, the side of the square described about a given circle, and b c, the side of the square described in the same circle.

Diagram. A square, with a circle inscribed in it touching all four sides, and a second square inscribed within that circle.

- b (lower left vertex of the inner square): b
- c (lower right vertex of the inner square): c
- a (below and to the left, outside the outer square): a
- (one further label at the lower right of the figure, unresolved): [?]

Let a b and b c be set, he says, as the sides at the right angle contained by a b c; and with b as centre and b a as radius let the circle a d e f be described. Then let both a b and b c be produced in a continuous straight line until they reach the points d, e, f on the circumference of that circle.

Diagram. A circle with two perpendicular diameters, a e horizontal and d f vertical, meeting at the centre b; a network of chords and joining lines is drawn across it.

- d (top of the circle, on the circumference): d
- s (immediately to the right of d, on the circumference): s [?]
- r (within the circle, just below d, on the vertical): r
- a (left end of the horizontal diameter): a
- o (on the horizontal diameter, between a and b): o
- b (centre): b
- n (on the horizontal diameter, between b and e): n
- t (on the circumference, upper right, above e): t [?]
- e (right end of the horizontal diameter): e
- v (just below n, toward e): v
- c (on the vertical below the centre b): c
- m (lower left, on the circumference): m
- k (at the foot of the vertical, right of m): k
- l (at the bottom of the circle): l
- f (bottom of the vertical, right of l): f

a e and d f will therefore be diameters of the same circle, cutting each other at right angles at its centre b. Now let the straight line c f, the remaining part of the radius b f, be divided in extreme and mean ratio, so that c k is the greater

segment and k f the lesser. Thus, as f c is to c k, so is c k to k f. The line b k, says Fine, will therefore be the second proportional line. Now let e k be joined, and, produced in a straight line, let it meet the circumference of the circle at the point l. Then, with a l joined, let the straight line m n be drawn through the point c parallel to e l, cutting the line

a l at the point m, and the radius b e at the point n. He then calls the line b n the third proportional: that is, as a b, the first line, is to b k, the second, so the same b k is to b n, the third, and in the same ratio again b n is to b c, the fourth line.

To demonstrate this he cuts off from the radius b d a line b r equal to b k; he joins m r, which cuts the diameter a e at the point o; and he joins the straight line a r, which, produced in a straight line, meets the circumference of the circle at the point s. Finally, from the points r and s he draws the straight lines r n and s e to n and e, and the rest as the figure itself contains.

With the construction so made, he argues in this way. Since b a [?] is equal to b e, and b r to b k, and the angles standing about b are equal to one another, the two triangles a b r and e b k have two sides equal to two sides, the one to the other, and the angles contained by the equal sides equal. Therefore the base a r is equal to the base e k, the remaining angle b a r to the remaining b e k, and the remaining angle a r b to the remaining b k e, by *Elements* I.4.

The straight line a e, then, falling on the straight lines a s and l e, makes the alternate angles equal. Therefore a s is parallel to l e, by proposition 27 of the first book of the *Elements*, and m n is likewise parallel to it, by proposition 30 of the same first book. And a l, falling on the parallels a s and l e, makes the interior angles on the same side, a l e and l a r, equal to two

Diagram. The same figure as on the preceding page, repeated: a circle with the perpendicular diameters a e and d f meeting at the centre b, with the joining lines drawn.

- d (top of the circle): d
- s (on the circumference, immediately right of d): s
- r (within the circle, below d): r
- a (left end of the horizontal diameter): a
- o (on the horizontal diameter, between a and b): o
- b (centre): b
- n (on the horizontal diameter, between b and e): n
- t (on the circumference, upper right, above e): t

- e (right end of the horizontal diameter): e
- v (below n, toward e): v
- c (on the vertical below b): c
- m (lower left, on the circumference): m
- k (at the foot of the vertical, right of m): k
- l (at the bottom of the circle): l
- f (bottom of the vertical, right of l): f

right angles, by proposition 29 of the first book of the *Elements*. But the angle a l e is right, by proposition 31 of the third book of the *Elements*; therefore the angle l a r is right as well. In the same way the angle l e s will be shown to be right, because the angle a s e is also right by the same proposition 31 of the third book of the *Elements*.

And therefore a l is parallel to e s, by *Elements* I.28. The quadrilateral a l e s is accordingly a rectangle and a parallelogram. Again, since the straight line a n falls on the parallels a r and m n, the angle a n m is equal to the alternate angle n a r. Similarly, since the straight line m r falls on the same parallels, the angle a r m will be equal to the alternate angle r m n, by *Elements* I.29. Moreover the vertical angles about o, contained by a o r and m o n, are equal by proposition 15 of the same first book. The triangles a o r and m o n are therefore equiangular; the sides about the equal angles are proportional to one another; and the sides subtended by the equal angles are of the same ratio, by *Elements* VI.4. As a o is to o r, then, so is n o to o m. The sides a o and o n, and likewise r o and o m, are therefore of the same ratio.

Furthermore, since a o is to o r as n o is to o m, and the angles contained by a o m and n o r are equal to one another by *Elements* I.15, the triangles a o m and n o r have one angle equal to one angle, and the sides about the equal angles reciprocally proportional. The triangle a o m is therefore equal to the triangle n o r, by proposition 15 of the sixth book of the *Elements*. And since the bases a m and n r subtend equal angles in equal triangles, they are of necessity, he says, of the same ratio. But a o and o n, and likewise r o and o m, are of the same ratio too, for as a o is to o r, so is n o to o m. The sides of those same triangles a o m and n o r are therefore proportional. Hence the triangles are equiangular to one another, and the angles subtended by the sides of the same ratio are equal, by *Elements* VI.5. For just as in equiangular triangles the sides subtended by equal angles are of the same ratio, by *Elements* VI.4, so in triangles whose sides are proportional the sides of the same ratio subtend equal angles conversely.

The angle a m o is therefore equal to o r n, and the remaining angle m a o to the remaining o n r. So the straight lines a n and m r, falling on the straight lines a m and n r, make the alternate angles equal to one another. Therefore n r is parallel to a m, and likewise parallel to e s, by *Elements* I.27 and I.30. The quadrilateral a m n r is accordingly a parallelogram. He says that it is a rectangle as well: for the angles contained at the points a and m are right, and therefore the opposite angles a r n

and m n r are right, by *Elements* I.34. Both a l e s and a m n r, and consequently the quadrilateral e t n v as well, are therefore parallelograms and rectangles. Hence the triangles a r n and r n c are right-angled, and the angles standing at the points r and n are right — which was the first thing Fine had undertaken to demonstrate.

Having shown this beforehand, he concludes that the straight lines b r and b n are the two lines in the middle place, continuously proportional in the same ratio, between a b and b c, the sides of the squares mentioned above: as a b is to b r, so the same b r is to b n, and b n to b c.

For since the triangle a r n is right-angled, and the perpendicular b r is dropped from the right angle at r to the base a n, b r is the mean proportional between a b and b n, the segments of that base, by the corollary to *Elements* VI.8. As a b is to b r, then, so is that same b r to b n. Again, since the triangle r n c is likewise right-angled, and the perpendicular b n is dropped from the right angle at n to the base r c, that same b n is the mean proportional between r b and b c, the segments of that base, by the same corollary to *Elements* VI.8. As b r is to b n, then, so is that same b n to b c. But it has been shown above that as a b is to b r, so is that same b r to b n. Therefore, by *Elements* V.11, as a b is to b r, so is b n to b c. Given, then, the two sides of the squares, a b and b c, of which one is described in a given circle and the other about it, Fine finds by this device the two mean straight lines continuously proportional in the same ratio, namely b r and b n — which is what he had undertaken to do.

Fine's corollary [Orontij corollarium.]

> If you wish (he says) to find these two straight lines, continuously proportional between the sides of the squares just mentioned, by a mechanical and very ready device, proceed as follows [*sic pendêter facito* [?]]. First have a gnomon made of some hard and choice material, similar to t e m. Then, with the sides of the said squares set at a right angle in the manner written above — such as a b and b c — and produced in a straight line on either side toward the parts in which they meet at the

right angle, as b d and b e are, let the diagonal line e n of the rectangular parallelogram e t n v be laid exactly in a straight line with b e [?], that is, with the longer line produced; and let the inner side of the gnomon be brought to the point c, the limit of the shorter side, the diagonal e n never being moved from the straight line b e. For the remaining, inner side of the gnomon will cut off for you the second proportional line from the shorter line produced, while the inner angle of the gnomon at n will at the same time bound the third of those same four continuously proportional lines.

Fine on the remaining straight lines [Orontius de cæteris lineis rectis.]

Although the discovery of proportional lines set out above seems to serve in a special way the sides of the proposed squares — of which one is described about the circle and the other within it — it can nevertheless be applied without distinction to any given unequal straight lines between which it may be worth finding two mean straight lines continuously proportional in the same ratio, the beginning of the construction alone being changed a little. For the excess of the greater line over the smaller will not always be divided proportionally, as c f was in the description of the preceding figure, but only so long as the smaller of the given lines exceeds half the greater. For whenever the smaller line is half of that greater one, or less than that half, the second proportional line — such as b k, or b r, was in the same preceding figure — must be sought out by another method; and the formula of the investigation must be varied as often as that same smaller line makes a different aliquot part of the greater, denominated by an evenly even number, or falls bounded between the divisions of those aliquot parts. Once that is done, the figure will have to be completed as described above, for everything else, together with the demonstration itself, remains the same in every part. But these first stages of the constructions, he says, we have judged it superfluous and useless to set out here one by one, since the side of the square described in a circle is always greater than half the side of the square that is circumscribed about the same circle.

This is Oronce Fine's discovery and demonstration concerning the two mean proportionals, in the same words and the same order in which we [took it] from the book

on the quadrature of the circle. We have transcribed it faithfully, since we resolved to alter nothing but rather to examine it in what follows.

Plato's method for finding two mean proportionals in continued proportion between two given lines — a method Oronce Fine partly imitates and partly tries to perfect.

CAP. II.

When Plato set himself to find two mean proportionals in continued proportion between two given lines, and other philosophers were trying various methods, he saw that two mean proportionals could easily be assigned between two given lines a b, b c, provided that a b and b c were set at the right angle a b c and produced in a straight line beyond b; then from their end points a and c certain straight lines were to be drawn to both of the produced lines, lines which, together with a third line joining them, would make right angles.

For let a b and b c be produced to d and e; and suppose that straight lines are drawn from a and c to a b and b c produced, in such a way that at certain points on them — let these be d and e — they make the right angles a d e and c e d with the third line d e, which joins a d and c e.

Diagram. A plane figure in the left column: two straight lines cross at an interior point b. The point d stands at the apex; a lies to the right, c to the left, and e below, near c. Lines are drawn from d to a and from c through b to e, with d joined to e, so that the figure shows the two triangles a d e and c e d of the text sharing the line d e.

- d (apex): d
- a (right): a
- c (left): c
- b (interior crossing point): b
- e (below b, toward the left): e

I say that b d and b e are the mean proportionals, in continued proportion, between the given lines a b and b c. For in the right triangle a d e the angle contained by a d e is right, while the line d b cuts the base a e at right angles in the point b; therefore, by the corollary to *Elements* VI.8, b d is the mean proportional between a b and b e. In the same way the triangle d e c has a right angle contained by d e c, and the line b e cuts the base d c at right angles; hence the perpendicular b e is the mean proportional between d b and b c. Therefore, as a b is to b d, so is b d to b e; and as b d is to b e, so is b e to b c. And so, by Euclid V.11, the four lines a b, b d, b e, b c are continuously proportional in the same

ratio as b d to b e; therefore b d and b e are the mean proportionals between the two lines a b and b c, which was to be demonstrated.

Plato, however, could not reach by a sure and undoubted demonstration a way of drawing the straight lines from the points a and c to the lines b d and b e, and to those points at which right angles would be made with the third line joining them; so he achieved the same result at least by a mechanical instrument and contrivance. The following device occurred to him.

Diagram. A perspective woodcut of the mechanical instrument in the right column: a heavy gnomon (a rectangular frame open at the left) with an upright at the left lettered f above and g below, an upright at the right lettered m above and h below, and a sliding cross-piece lettered k at its left end and L at its right end, seated in grooves cut in the two uprights.

- f (top of the left upright): f
- g (foot of the left upright): g
- m (top of the right upright): m
- h (foot of the right upright): h
- k (left end of the sliding ruler): k
- L (right end of the sliding ruler): L

Let a gnomon f g h be built out of any hard material, and let one of its legs, f g, be hollowed out, with a groove of square section cut in it, into which a ruler k l is fitted, so that it can be moved now toward f, now toward g, yet always parallel to the leg g h. In that way it will stay at right angles to f g. This will work still more conveniently if a second ruler h m is fitted and fastened at right angles to g h, itself carrying a groove into which the other end of the ruler k l is set. That way the ruler k l will not wobble at all, but will move straight and evenly.

Once the instrument is built, if one wishes to find two mean proportionals between two given lines a b and b c, then — the two straight lines having been joined at a right angle — the instrument is fitted so that the edge of the leg g h touches the point c and the ruler k l reaches the point a, with the angle g lying on b e and the angle k on b d. The ruler g h will then take the position c e, the ruler k l the position d a, and f g the position e d; and so the angles at d and e [?] will be right angles. Consequently, as a b is to b d, so is b d to b e, and b e to b c.

Eutocius of Ascalon reported this method of Plato's, and the methods of the other philosophers as well, in his commentary on the second book of Archimedes' *On the Sphere and Cylinder*; so did Giorgio Valla in that great work of his, *On Things*

to Be Sought and Things to Be Shunned. Most recently the learned Ioannes Vernerus [Johannes Werner] of Nuremberg has expounded all those methods far more lucidly.

Oronce Fine, by contrast, undertakes to draw straight lines from the points a and c to the lines b e and b d, and to those very points d and e at which right angles are made with the straight line d e that joins them, without any mechanical contrivance at all — something Plato could not do. Besides that, in imitation of Plato he builds a gnomon by which he can produce those same right angles at the same points, and so find the two mean proportionals; and he adds the rest, which it will be worth examining in what follows.

That Oronce Fine went astray over the finding of the two mean proportionals, through ignorance of the geometrical elements of Euclid's sixth book.

CAP. III.

First reproof.

Now that Oronce Fine's demonstration and figure have been set out again, we shall answer him word for word, so as to refute him the more plainly.

We grant, then, that the quadrilateral a l c s [?] is a parallelogram and a rectangle. We further admit that the two triangles a o r and m o n are equiangular, and hence that the sides about the equal angles are proportional and that the sides subtending equal angles are in the same ratio, as *Elements* VI.4 shows. And therefore, as a o is to o r, so is n o to o m; so the sides a o and o n, and likewise r o and o m, of those triangles a o r and m o n, are in the same ratio. Nor do we doubt that the angles contained by a o m and n o r are equal. We likewise concede that the triangles a o m and n o r — which have one angle equal to one angle at the vertex, and the sides about those equal angles reciprocally proportional — are equal, as the second part of Euclid VI.15 demonstrates.

But when he infers that, because the bases a m and n r in the equal triangles a o m and n o r subtend equal angles, they must therefore be in the same ratio, we deny that this follows. And when he adds that a o and o n, and r o and o m as well, are in the same ratio, since as a o is to o r, so is n o to o m — we do indeed grant that those sides are proportional and in the same ratio in the similar triangles a o r and m o n, for that had already been shown. But from this he wrongly gathers that the sides of the triangles a o m and n o r are proportional.

Diagram. Oronce Fine's figure, repeated here from the passage under attack: a large circle with a nearly horizontal chord running from a on the left, through the interior points o and b, to the right side of the circle, and with several straight lines drawn from a point r at the top of the circumference down across the interior to points on the lower and right-hand arcs. Further lettered points lie on the circumference at the lower left and at the foot of the circle, and two letters stand above the top of the circle. The lettering is small and partly broken on the leaf; the readings below are given with flags where the letter did not resolve.

- r (top of the circumference): r
- a (left end of the horizontal chord): a
- o (on the chord, left of centre): o
- b (on the chord, right of o): b
- n (on the chord, right of centre): n
- t [?] (on the chord near the right-hand arc): t [?]
- [?] (right-hand end of the chord, on the circumference): [?]
- v [?] (below the right-hand end of the chord): v [?]
- e (interior, below centre): e
- m (lower-left arc): m
- k [?] (at the foot, right of m): k [?]
- L [?] (below the foot of the circle, left): L [?]
- f [?] (below the foot of the circle, right): f [?]
- d [?] (above the circle, left): d [?]
- s [?] (above the circle, right): s [?]

Having taken that as true, he was then free to infer, by *Elements* VI.5, that the triangles a o m and n o r are equiangular, and finally to conclude that the quadrilateral a m n r is a parallelogram and a rectangle. Here Oronce Fine is at fault in many ways. For if he had only just shown that the triangles a o m and n o r have their sides a o, o m and r o, o n reciprocally proportional, on the ground that as a o is to o r, so reciprocally is n o to o m — how can he at once, in those very same triangles and resting on that very same foundation (namely that as a o is to o r, so is n o to o m), gather that those same sides are proportional and in the same ratio, and consequently that the triangles are equiangular and similar, when a moment before he had demonstrated them to be reciprocal?

But if he was caught by so slight a sophism: the two similar triangles a o r and m o n have proportional sides a o, o r and n o, o m about the equal angles contained by a o r and m o n. Yet those same sides, in the two triangles a o m and n o r,

contain the equal angles a o m and n o r; therefore the triangles a o m and n o r have proportional sides about the equal angles a o m and n o r, and on that account they are not merely reciprocal but similar as well. He ought to have noticed that the triangles a o m and n o r cannot be reciprocal and similar at once unless their sides are equal to one another — something he had neither assumed nor proved. Had he examined the argument he framed in this way and dissolved it, he would have been freed from his error.

He should also have pressed the point further. If the two triangles a o m and n o r are to be judged similar and equiangular just because two other triangles, a o r and m o n, stand similar about them — those, that is, with which they share common sides, and which contain equal angles at the vertex — then, since the same thing must hold of all triangles whose two angles are equal and whose containing sides are reciprocally proportional, it would now follow that all triangles having one angle equal to one angle and the sides about those equal angles reciprocally proportional would be not only equal, as *Elements* VI.15 demonstrates, but similar and equiangular besides. We shall show that this is false.

Let any line whatever, say a n, cut the line r m at the point o (to use the figure already drawn), and let proportionals be laid down in whatever ratio one pleases, so that as a o is to o r, so is n o to o m; and let those four lines be unequal. Join a m and n r. The triangles a o m and n o r will then be equal, having reciprocally proportional sides, by *Elements* VI.15. Join a r and m n; the two triangles a o r and m o n will accordingly be equiangular, by VI.6 of the same book, with the sides about the equal angles proportional, and with the sides subtending equal angles in the same ratio, by VI.4.

Now let us reason as Oronce Fine does, and use his own words. As a o is to o r, so is n o to o m; therefore the sides a o and o n, and likewise r o and o m, are in the same ratio. Moreover, since as a o is to o r, so is n o to o m, and since the angles contained by a o m and n o r are equal by *Elements* I.15, the triangles a o m and n o r have one angle equal to one angle and the sides about those equal angles reciprocally proportional; so triangle a o m is equal to triangle n o r, by *Elements* VI.15. And since the bases a m and n r subtend equal angles in equal triangles, they must therefore be in the same ratio. But a o and o n, and r o and o m as well, are likewise in the same ratio, for as a o is to o r, so is n o to o m. So the sides of those same triangles a o m and n o r are proportional, and consequently the triangles are equiangular to one another and have equal the angles under which sides of the same ratio are subtended, by *Elements* VI.5. The angle a m o is

therefore equal to o r n, and the remaining angle m a o to the remaining angle o n r.

So far we have reasoned as Oronce Fine does; but now let us go on for ourselves. Since in those two equiangular triangles a o m and n o r the angle a m o is equal to o r n, and the remaining angle m a o to the remaining angle o n r, therefore by VI.4, as a o is to o n, so is m o to o r. But in fact, by the hypothesis and by alternation of the proportion, as a o is to o n, so is r o to o m; and therefore, by *Elements* V.11, as m o is to o r, so is r o to o m. Yet by hypothesis m o and o r are unequal; therefore m o is at once greater and less than o r, which is impossible. Oronce Fine's syllogism is accordingly fallacious.

Add to this that the two triangles a o m and n o r, being reciprocal, will be equal; but if they are similar and the sides of the one are unequal to the sides of the other, those same triangles will be unequal, for their ratio is the duplicate of the ratio of the corresponding sides, by VI.19. Yet by that syllogism of his Oronce Fine is trying to prove that the same triangles are similar and reciprocal together no less when the sides are unequal than when they are equal; for if it established the one, it would establish the other too. But the triangles are not equal and unequal at once; therefore they are not similar, but only reciprocal.

He should therefore have examined the disposition of the proportional sides that cut one another, namely a n and r m. For if in the two triangles a o r and m o n it were true that, as a o is to o r, so m o is to o n, then the two triangles a o m and n o r would indeed be held equiangular and similar, but not reciprocal; for the two sides a o and o m would be proportional to the two sides o r and o n. But Oronce Fine neither assumed this nor had any right to assume it, because in those two triangles a o r and m o n the angle a r m is equal to the alternate angle r m n, and the remaining angle n a r equal to the remaining angle a n m. So the sides subtending equal angles are in the same ratio — as a o is to o r, so is n o to o m — and therefore it is not the case that as a o is to o r, so is m o to o n. For that reason the triangles a o m and n o r are shown to be reciprocal, not similar.

We shall set out this divergent nature and rule of similar and of reciprocal triangles more plainly by an example in which the sides of the triangles are rational lines. Let a right triangle a b c be drawn, having its right angle at b; let the side a b be 4 feet, and b c 3; then a c will be 5 feet, by *Elements* I.47. Let a b be produced as far as d, and let a d be

let *a d* be eight feet; and from the point *d* let a straight line *d e* be erected making a right angle with *a d* itself; and let *c a* be produced until it meets *d e* at the point *e*.

Diagram. A geometric construction set in the head of the left column. A vertical line runs down the middle of the figure; at its top stands the point *d*, and below *d*, in order, the points *f*, *a*, and *b*. From *d* a horizontal segment runs to the left to the apex *e* [?], and the segment *e d* is marked **6**. The upper part of the vertical, from *d* down to *f*, is marked **6**. From *b*, at the foot of the vertical, a horizontal segment runs to the right to *c*, and *b c* is marked **3**. Straight lines are drawn from the apex *e* to *b*, to *f*, and through *a* to *c*; the long line from *e* down to *a* is marked **10**, the segment *a c* is marked **5**, and the segment *a b* is marked **4**. The right angles at *d* and at *b* are drawn square.

- *e* [?] (apex, upper left): vertex of the outer triangle
- *d* (upper right): foot of the perpendicular *d e*
- *f* (on the vertical, below *d*): the point cutting off *a f*
- *a* (on the vertical, below *f*): common vertex of the two triangles
- *b* (foot of the vertical): right angle of triangle *a b c*
- *c* (lower right): third vertex of triangle *a b c*
- **6** (top edge): length of *e d*
- **6** (upper vertical): length of *d f*
- **10** (long diagonal): length of *e a*
- **5** (short diagonal): length of *a c*
- **4** (lower vertical): length of *a b*
- **3** (bottom edge): length of *b c*

Since the angles at *b* and at *d* are right angles, and the angles at the vertex are equal, the triangles *a b c* and *a d e* are therefore equiangular, by *Elements* I.32. They are consequently similar and have their sides proportional, by *Elements* VI.4. So *d e* will be six feet and *a e* ten.

Now from the line *a d* let a part *a f* be cut off, two feet long, and let *c f* and *e f* be joined. In the two triangles *a b c* and *a f e*, then, the sides about the equal angles *b a c* and *e a f* are reciprocal: for as *a b*, 4, is to *a f*, 2, so is *a e*, 10, to *a c*, 5. The triangles *a b c* and *a f e* are therefore equal, by *Elements* VI.15 — and yet they do not have their sides proportional, nor are they equiangular. For the angle *e f a* is greater than the interior angle *e d f*, by *Elements* I.16, and therefore greater than the angle *c b a*; while the remaining angle *a e f* is less than the angle *a e d*, and therefore less than the angle *a c b*, by the common notion. The triangles *a b c* and

a f e are accordingly not equiangular, and they do not have their sides proportional. For if they did have their sides proportional they would be equiangular, by *Elements* VI.5; but equiangular they are not, and so neither do they have their sides proportional.

Now let *e b* and *c f* be joined. Two triangles will thus be set up, *a b e* and *a c f*, and these are similar. For as *a b* is to *a f*, so is *a e* to *a c*; and by alternation, as *a b* is to *a e*, so is *a f* to *a c*. The sides *a b* and *a e* are therefore proportional to *a f* and *a c*; and the angles *b a e* and *c a f*, contained by those proportional sides, are equal, by *Elements* I.15. So the triangles *a b e* and *a c f* are equiangular, by *Elements* VI.6; the sides about the equal angles are proportional; and the sides which subtend the equal angles are of like ratio.

Only in this way, then, are the two sides *a b*, *a e* of the triangle *a b e* shown to be proportional to the two sides *a f*, *a c* of the triangle *a c f* — and not *reciprocally* proportional. The two sides *a f*, *a e* of the triangle *a f e*, by contrast, are reciprocally proportional to the two sides *a b*, *b c* [?] of the triangle *a b c*, and yet they are not called proportional without qualification.

For in Euclid the sides of figures are said to be proportional when the ratio between the sides of the one figure is the same as the ratio between the sides of the other; and for that reason, if the proportion is alternated, both antecedent terms lie in the one figure and both consequent terms in the other. In the two triangles *a b e* and *a c f*, for instance, the two sides *a b* and *a e* are said to be proportional to *a f* and *a c*, since in the triangle *a b e*, as *a b* is to *a e*, so, in the triangle *a c f*, is *a f* to *a c*. Consequently, if we alternate, both antecedents will fall in the one triangle and both consequents in the other: for as *a b* is to *a f*, so is *a e* to *a c*. Thus *a b* and *a e* become the antecedents, and *a f* and *a c* the consequents.

Sides of figures are called *reciprocally* proportional, by contrast, when in each of the two figures one side is an antecedent and the other a consequent, and the antecedent of the one figure is referred reciprocally to the consequent of the other. Where two sides are equal to two sides, therefore, they will be called not merely proportional but reciprocally proportional as well; but if they are unequal, there is no way at all for them to be proportional and reciprocally proportional at once, as we demonstrated a little earlier.

Oronce Fine, however, deluded as he is, confounds all of this, and announces false theorems as true and doubtful ones as certain. Of that kind is this pronouncement of his, laid down without any proof at all:

> And since the bases *a m* and *n r* [?] subtend equal angles in equal triangles, they are therefore forced to be of like ratio.

He seems, then, to hold that if in two equal triangles one angle is equal to one angle, the sides which subtend those equal angles and the remaining sides will be in the same ratio. There is no other way to understand how the bases of two triangles can be said to be of like ratio, unless at least two of the remaining sides are in the same ratio as those bases. But if he wants this to hold universally, he is plainly mistaken. The bases will indeed be proportional to the other sides of those same triangles when they are equal to one another — indeed the remaining sides too will then be equal to the remaining sides. But if the bases are supposed unequal, it does not follow that the sides must stand to the sides in the same ratio.

For let there be two equal triangles *a b e* [?] and *d e f*, in which two angles, *b a c* and *e d f*, are supposed equal, and let the bases *b c* and *e f*, which subtend the equal angles, be given equal as well. I say that the two remaining sides of the triangle *a b c* are equal to the two remaining sides of the triangle *d e f*, and that consequently the sides of the one triangle stand to the sides of the other in the same ratio as the bases — the ratio, that is, of equality.

Diagram. A circle drawn in the left column with a horizontal chord *b c*, *b* on the left and *c* on the right. Inscribed on this chord are the triangle *a b c*, with *a* set high on the left of the circumference, and the triangle *b g c*, with *g* set high on the right; the two triangles cross, and *a g* is drawn joining their apexes. A further straight line runs from *b* out to the point *K*, which lies outside the circumference above and to the right, with *K c* completing the triangle *b K c*; the side *b K* cuts the circumference at *g*. The point *f* lies inside the circle where the lines cross.

- *a* (upper left, on the circumference): apex of triangle *a b c*
- *g* (upper right, on the circumference): apex of triangle *b g c*
- *K* (outside the circle, upper right): apex of the triangle falling beyond the circumference
- *f* (interior, centre): the intersection point
- *b* (left end of the chord)
- *c* (right end of the chord)

Diagram. A separate scalene triangle drawn below and to the right of the circle: *e* at the lower left, *f* at the lower right, and *d* at the upper right, with *e d*, *d f*, and *e f* drawn.

- *d* (upper right)

- *e* (lower left)
- *f* (lower right)

For about the triangle *a b c* let a circle be described, by *Elements* IV.5; and out of three straight lines equal to the sides of the triangle *d e f* let a triangle be constructed on the line *b c* — which is equal to one of them — on the side of *b a c*, by *Elements* I.22 of the first book. The constructed triangle must then be described within that same circle. For if it is not — since two of its angles are at the points *b* and *c* — the remaining angle either fails to reach the circumference of that circle or passes beyond it.

Suppose first that it fails to reach the circumference: let such a triangle be *b f c*. Then the two angles *e d f* and *c f b* will be equal, by *Elements* I.8. But the angle *b a c* is equal to that angle *e d f* by hypothesis; therefore the two angles *b a c* and *c f b* are equal to one another, by the common notion. Let the straight line *b f* be produced until it meets the circumference of the described circle at the point *g*, and let *c g* be joined. Then the two angles *b a c* and *b g c*, standing in the same segment *b a g c*, are equal to one another, by *Elements* III.21. For that reason the two angles *c f b* and *b g c* are equal, by the common notion. But the angle *c f b*, being exterior, is greater than the interior and opposite angle *b g c* of the triangle *c f g* — which is impossible.

Now suppose it is said that the triangle constructed out of the three straight lines equal to the sides of the triangle *d e f* passes beyond the circumference of the circle. Let such a triangle be *b K c*, and let the side *b K* cut the circumference of the circle at *g*. Then, joining *g c*, we shall show in the same way that the angle *b a c* is equal to the angle *k*, and that the angle *c g b* is equal to that same angle *k*, by the common notion. And since the angle *c g b* is exterior, while *K* is the opposite and interior angle in the triangle *g c k*, it will therefore be at once greater than that angle and equal to it — which is impossible.

The constructed triangle must therefore be inscribed in that same described circle. So let the constructed triangle inscribed in the circle be *b g c*, and let *a g* be joined. The triangle *b g c* is then equal to the triangle *d e f*, by *Elements* I.8 and I.4. But the triangle *a b c* is equal to that same triangle *d e f* by hypothesis; therefore the two triangles *a b c* and *b g c* are equal, by the common notion. And since they stand on the same base *b c*, the straight lines *a g* and *b c* are parallel, by *Elements* I.39 of the same first book. The angle *a g b* is therefore equal to the alternate angle *g b c*, by *Elements* I.29 of that same first book of the elements. But that same angle *a g b* is equal to the angle *a c b*, by *Elements* III.21, for they stand in the same circumference *a g c b*; therefore the two angles *g b c* and *a c b* are equal, by

the common notion. The two circumferences *a b* and *g c* are consequently equal, by *Elements* III.29; and the straight line *a b* is equal to the straight line *g c*, by the same 29. If, then, the circumference *a g* is added to the two equal circumferences *a b* and *g c*, the two circumferences *b a g* and *a g c* will be equal, by the common notion; and consequently the straight lines *b g* and *a c* will be equal, by that same *Elements* III.29.

So the triangle *a b c* has its sides equal to those of the triangle *b g c*; the triangle *d e f* likewise has its sides equal to those of the same triangle *b g c*; therefore the triangle *a b c* has its sides equal to those of the triangle *d e f*. The sides of the one are consequently proportional to the sides of the other in the ratio of equality, exactly as the bases are — which was the first thing to be demonstrated.

But there is no necessity for this to hold universally, for any equal triangles whatever that have one angle equal to one angle. In the figure set out earlier, the two triangles *a b c* and *a f e* are equal; yet the bases *b c* and *e f*, which subtend the equal angles at *a*, are not in the same ratio as the sides.

Diagram. The rational-sided construction repeated at the foot of the right column. *e* stands at the upper left and *d* at the upper right, the top segment *e d* marked **6**. From *d* the vertical descends, its upper part, down to *f*, marked **6**; below *f* on the same vertical lies *a*, and below *a* the point *b*, the segment *a b* marked **4**. From *b* the base runs right to *c*, marked **3**. Lines are drawn from *e* to *b*, to *f*, and through *a* to *c*; the long line *e a* is marked **10** and the segment *a c* is marked **5**.

- *e* (upper left) · *d* (upper right) · *f*, *a* (on the vertical, in that order below *d*) · *b* (foot of the vertical) · *c* (lower right)
- **6** (*e d*) · **6** (*d f*) · **10** (*e a*) · **4** (*a b*) · **5** (*a c*) · **3** (*b c*)

For as *d e* is to *b c*, so is *d a* to *a b*, because of the similarity of the triangles *a d e* and *a b c*. But *d a* has to *a b* a greater ratio than *a f* has to *a b*, by *Elements* V.8; therefore *d e* has to *b c* a greater ratio than *a f* has to *a b*, by *Elements* V.13 of that same fifth book. Moreover, since by *Elements* I.19 *e f* is greater than *d e*, *e f* will have a greater ratio to *b c* than *d e* has to that same *b c*, by *Elements* V.8. But *d e* was shown to have a greater ratio to *b c* than *a f* has to *a b*; therefore *e f* will have a far greater

ratio to *b c* than *a f* has to *a b*, by the twelfth rule of the art of demonstration. The bases *e f* and *b c* of the equal triangles *a f e* and *a b c* are therefore not in the same ratio as their sides *a f* and *a b*.

It will be shown in the same way that those same bases are not in the same ratio as the two sides *e a* and *a c*. For *e f* has a greater ratio to *b c* than *d e* has to *b c*; and as *d e* is to *b c*, so is *e a* [to *a c*] [?], because of the similarity of the triangles *a d e* and *a b c*; therefore *e f* will have a greater ratio to *b c* [?] than *e a* has to *a c* — which was to be demonstrated.

The same points can be demonstrated far more easily by a demonstration that leads to an absurdity. For if *e f* is to *b c* as *a f* is to *a b*, then by alternation, as *e f* is to *a f*, so is *b c* to *a b*. But each of the two angles *a e f* and *b c a* is less than a right angle, and the angles at *a* are equal; therefore the triangles *a b c* and *a f e* are equiangular, by *Elements* VI.7. Yet it has been demonstrated that they are not equiangular — so this is impossible. Likewise, if *b c* is to *e f* as *a c* is to *a e*, then by alternation, as *b c* is to *a c*, so is *e f* to *a e*; and since neither of the two angles *c b a* [?] and *e f a* is less than a right angle, the triangles *a b c* and *a f e* will be equiangular by that same *Elements* VI.7 — which is absurd. The bases of equal triangles, therefore, will not automatically be of like ratio simply because they subtend equal angles.

Oronce Fine, however, supposed that they were forced to be of like ratio, and on that basis concludes, by *Elements* VI.5, that the triangles *a o m* and *n o r* are equiangular and have equal angles subtended by sides of the same ratio — that is to say, that the angle *a m o* is equal to *o r n*, and the remaining angle *m a o* to the remaining angle *o n r* [?].

From all this, then, we can conclude that Oronce Fine's syllogism is not a demonstration but sheer delusion, and that he accordingly went wrong over the finding of two mean proportionals through ignorance of the geometrical elements of Euclid's sixth book — which is what we had undertaken to show.

Orontium Finæum errasse circa inuentionem duarum mediarum proportionaliū, ob imperitiam artis demonstrādi. **CAP.IIII.**

Reprehensio secunda.

[*That Oronce Fine went wrong over the finding of two mean proportionals, through inexperience in the art of demonstration.* **Chapter IIII.** — **Second rebuke.**]

But Oronce Fine deserves the sharper rebuke on this further count: even granting that he had not learned the art of demonstration from Aristotle's books, he could still have picked up that same art from daily practice, since he was borrowing his demonstrations out of Euclid's books from Theon and Campanus. For in those

books no principles are assumed that are not aimed at the conclusion, and nothing is constructed that does not serve the demonstration.

Oronce Fine proceeds very differently. Setting out to find two mean proportionals between two given lines, he first divided the excess of the greater over the lesser in extreme and mean ratio; he then added the greater segment to the lesser of the two given lines, and set up the compounded line as the second proportional — and yet in the whole of his demonstration he was never to make any further mention of a division of that kind. What need was there, then, of that division, if he was not going to bring it up again in the proof? Or how does he elicit the mean proportionals from it, if neither the conclusion of the demonstration nor any intermediate proposition used to infer it is tied to the excess having been divided in that particular way?

Therefore, if the conclusion is not proved by that division, and is not referred back to it, and has no answer to give to it either, then cutting the greater and the lesser into any parts you please, equal or unequal, would have contributed no less toward finding the mean proportionals than division in extreme and mean ratio did. For it would still be open to us to draw some straight line through the mark of the division and the end of the quadrant [?], and another parallel to it through the end of the lesser line, and to construct the rest just as Oronce Fine did. The demonstration would be no more broken or weakened on that account than if it were carried through in his own way; on the contrary, the same procedure and the same method would remain — if that false, but scarcely deceptive, argumentation deserves to be called a method at all. Countless different and unequal pairs of mean proportionals would then be placed between the two given lines — which is absurd.

It is therefore plain from a manifest argument that Oronce Fine went wrong over the finding of mean proportionals through ignorance of the art of demonstration, which is what we had undertaken to show.

The absurdity is easy to set out. Between the two straight lines *b f* and *b c*, on Oronce Fine's teaching, the two mean proportionals are *b K* and *b n*. But let us divide the difference *c f*, not in extreme and mean

ratio, but into the equal parts *c z* and *z f*; and then let a figure be constructed in imitation of his. Draw a straight line through the point *z* and the end of the quadrant at *e*; and parallel to it, through *c*, let another straight line be drawn, which will necessarily cut the radius *b e*. Let the equal piece so cut off be set down as *b i*, and let everything else be constructed in order. And suppose that

Oronce Fine's reasoning holds good for showing that *b K* and *b n* are the mean proportionals between *b f* and *b c*. Then it will hold equally well for our demonstrating that *b z* and *b i* are likewise mean proportionals between the same two lines; for we alter nothing that could change the demonstration.

For whether the difference *f c* is divided in extreme and mean ratio at the point *k*, or into two equal parts at *z*, it will be shown all the same that the quadrilateral inscribed in the circle is a parallelogram and right-angled. And then, from the similar triangles, by exactly the same technique Oronce Fine used, the two straight lines *b z* and *b i* will be demonstrated to be mean proportionals between *b f* and *b c*.

That this is absurd we shall show as follows. *b f* has a greater ratio to *b z* than the same *b f* has to *b K*, by *Elements* V.8. But as *b f* is to *b z*, so is *b z* to *b i*; and as *b z* is to *b i*, so is *b i* to *b c*. As *b f* is to *b k*, on the other hand, so is *b k* to *b n*; and as *b K* is to *b n*, so is *b n* to *b c*. Therefore *b i* has a greater ratio to *b c* than *b n* has to *b c*, by *Elements* V.12; and for that reason *b i* will be greater than *b n*, by *Elements* V.10 — the part greater than its own whole, which is impossible.

By an evident and necessary argument it is concluded [?] that the two straight lines which Oronce Fine set up as mean proportionals are not the true ones: rather, the one exceeds the just magnitude and the other fails to fill it out.

CAP.V. Reprehensio tertia.

[Chapter V. — Third rebuke.]

Let *a b*, or *b f* equal to it, be the side of the square described about the circle set before us, and *b c* the side of the square described within that same circle. And let *c f*, the difference of these two lines, be divided in extreme and mean ratio, and let the greater segment be *c K*. Oronce Fine asserts that if we set *b f* as the first of four continually proportional lines and *b c* as the fourth, then the line *b K* is the second proportional — the first, that is, of the two means.

We, however, shall show by an evident argument, using rational quantities, that *b k* is less than the second proportional line. For the line *a b*, the side of the square described about the given circle, is equal to the diameter of that circle, and so equal to the diagonal of the inscribed square. Therefore, as the diagonal of the inscribed square is to the side of that same square, so is *a b*, or *b f*, to *b c*; and that ratio is half the double ratio [*dimidium rationis duplæ*].

Let us therefore set *c f*, the difference between the diagonal and the side of that square, at 4. Then *b f* will be 8 + √32 [printed `8 p̄ R̄ 32`, where p̄ stands for *plus* and R̄ for *radix*, the root; m̄ below stands for *minus*], and *b k* will be 2 + √20 + √32.

Now 5 21/32 is less than the root of 32, since it is the root of 31 1017/1024; therefore 13 21/32 is less than *b f*, the first line. But 4 9/19 is the root of 20 5/361, so that 4 9/19 is a little greater than the root of 20. Moreover, 5 23/35 is the root of 32 4/1225, and so 5 23/35 is greater than the root of 32. Let these numbers now be heaped together — 2, and 5 23/35, and 4 9/19 — and their sum will be 12 87/665, which is therefore greater than *b K*. Consequently *b f* has to *b k* a greater ratio than 13 21/32 has to 12 87/665.

Let 13 21/32 and 12 87/665 be reduced to one and the same denomination: then, as 13 21/32 is to 12 87/665, so is 290605 to 258144. And accordingly *b f* has to *b k* a greater ratio than 290605 has to 258144. The cubes of these numbers are 24541960163195125 and 17202283700649984, and their ratio is greater than the ratio of 17 to 12.

But 17 has to 12 a greater ratio than half the double ratio, since their squares, 289 [?] and 144, are in a greater ratio than the double one. The ratio of cube to cube is therefore far greater than half the double ratio; and consequently cube has to cube a greater ratio than *b f* has to *b c*. Now cube has to cube triple the ratio that side has to side; and the line *b f* likewise has to the line *b c* triple the ratio that the same *b f* has to the second proportional. The sides of the cubes are therefore in a greater ratio than the line

b f and the second proportional stand in. But it was demonstrated that *b f* has to *b K* a greater ratio than the side of the greater cube has to the side of the lesser; therefore *b f* will have a greater ratio to *b K* than the same *b f* has to the second proportional. Consequently the second proportional will be greater than *b k* itself.

Diagram. A circle occupying the middle of the left column. A straight line crosses it horizontally from *a*, on the left of the circumference, through *o*, then *b* at the centre, then *n*, and out to *e* on the right of the circumference. From a point at the top of the figure, marked *d* with *s* beside it, several straight lines descend into the circle and cut one another, forming a star of chords that meet the circumference at the left, the right and the foot. The remaining letters are set at the points where these chords meet or cross.

- *a* (left, on the circumference)
- *o* (on the horizontal line, left of centre)
- *b* (centre)

- *n* (on the horizontal line, right of centre)
- *e* (right, on the circumference)
- *d* (top, above the circle)
- *s* (top, beside *d*)
- *y* [?] (inside the circle, just below the top)
- *t* (upper right, on the circumference)
- *v* [?] (inside, below *t*)
- *c* (inside, below the centre)
- *m* (lower left, on the circumference)
- *k* (inside, lower right of the centre)
- *L* [?] (at the foot of the circle)
- *f* (at the foot of the circle, to the right of *L* [?])

Again, since of four proportional magnitudes, as the first is to the second, so is the third to the fourth, therefore by alternation, as the first is to the third, so is the second to the fourth. But *b c* [?] is the fourth proportional line, and *b e* is equal to the first; therefore, as the second proportional line is to *b c*, the fourth, so is *b e* to the third. And the second proportional was shown to be greater than *b K*, so that the second proportional will have a greater ratio to *b c*, the fourth, than *b k* has to that same *b c*; and consequently *b e* will have a greater ratio to the third than *b k* has to *b c*. But as *b K* is to *b c*, so is *b e* to *b n*, because of the similarity of the triangles *k b e* and *b c n*. Therefore *b e* will have a greater ratio to the third than the same *b e* has to *b n*, and consequently the third proportional line will be less than *b n* itself. So, just as *b k* fails to fill out the true magnitude of the second, so *b n* exceeds the third — which is what we undertook to demonstrate.

There follows the method we used to show that *b f* is to *b K* as 8 + √32 is to 2 + √20 + √32. The rest is easy, and ready to hand for anyone versed in the elements.

Thus *b f* is the diagonal of the square and *b c* its side; and let the excess *c f* be divided at the point *k* in extreme and mean ratio, *c K* being the greater segment and *f K* the lesser. I say that as *b f* is to *b K*, so is 8 + √32 to 2 + √20 + √32.

Positio prima.

[First position.]

¶ Let *c f* first be set at 4 equal parts; its half is therefore 2. The two squares taken together — that of the whole *c f* and that of its half — will be 20. Accordingly *c K*, the greater segment, will be √20 − 2; and consequently *f K*, the lesser segment, is left as 6 − √20.

c f, 4; its half, 2. Their squares, 16 and 4.

Therefore *c k* will be √20 − 2. Subtract √20 − 2 from 4. There will be left *f k*, 6 − √20.

Positio secunda.

[Second position.]

¶ But now let the whole *b f* be set at 2. Then *b c*, the side of that same square, will be √2, since it is the mean proportional between 2 and 1. Subtract √2 from 2, and the excess *c f* is left as 2 − √2.

Let *b f* be 2. Then *b c* will be √2. And *c f*, 2 − √2.

¶ Now, since when *f c* is 2 − √2, *b f* is 2 in those same parts, therefore when the same *f c* is 4, *b f* will be found to be 8 + √32, by the common rule of four proportional quantities. For multiplying 4, the third term of the proportion, by 2, the second term, gives 8; then, dividing 8 by 2 − √2, the first term, the division yields 8 + √32, the fourth term of the proportion.

2 − √2 | 2 | 4 8.

2 − √2 × 2 + √2, the common multiplier. 4 − 2. (For the other two cross-multiplications cancel each other out.) Therefore the divisor is 2.

2 + √2. 16 + √128, the dividend. Therefore 8 + √32, the fourth term [?].

¶ So, when *c f* is 4 parts, *b f* is 8 + √32; and since, when *c f* is 4, *f K* is 6 − √20, we shall accordingly subtract 6 − √20 from 8 + √32, and *b K* will be left as 2 + √20 + √32. Therefore, as *b f* is to *b k*, so is 8 + √32 to 2 + √20 + √32 — which was to be shown.

b f, 8 + √32. Subtract *f k*, 6 − √20. There will be left *b k*, 2 + √20 + √32.

¶ That the greater cube has to the lesser a greater ratio than 17 has to 12 you will not doubt, if you multiply that greater cube by 12 and divide the product by 17. The division will yield 17323736585784794 2/17. Accordingly, as 17 is to 12, so is the greater cube to this number that comes out of the division. But that same number exceeds the lesser cube; therefore the greater cube will have to the lesser a greater ratio than 17 to 12.

That Oronce Fine's instrument does not point out the true mean proportionals.

CAP. VI. Reprehensio quarta.

[Chapter VI. — Fourth rebuke.]

From what has been said it will be easy to establish that the gnomonic instrument of Oronce Fine cannot supply the true mean proportionals. Let the gnomon itself, *r e m*, be constructed; and let two straight lines *g b* and *b z* be set out, joined at the right angle *g b z*, of which the greater, *g b*, is the side of a square described about some circle, while the lesser, *b z*, is the side of the square described within that same circle. Suppose it is required to find, by this gnomonic instrument, two continually proportional means between *g b* and *b z*.

Diagram. Oronce Fine's gnomonic instrument, drawn in the left column as two long parallel-sided rules — the two arms of the gnomon — laid across the figure on the diagonal, running from the lower left up to the upper right. A straight line comes in horizontally from *a* at the far left, passes *g*, and reaches *b* near the middle of the figure; from *b* a line descends to *z*. The upper end of the outer arm is at *d*, and *r* is marked on the arms near that end. To the right of *b* the two arms cross a further set of straight lines, and the remaining letters are set where those lines meet the arms or one another; the lower ends of the two arms carry *m* and a second *m* [?] at the lower left of the figure.

- *a* (far left, at the end of the horizontal line)
- *g* (on the horizontal line, right of *a*)
- *b* (on the horizontal line, at the centre of the figure)
- *q* (above the horizontal line, left of centre)
- *z* (below *b*)
- *p* (below and right of *b*)
- *t* (right of *p*)
- *e* (right of *t*)
- *n* [?] (right of *e*, on the upper arm)
- *v* [?] (below *e*)
- *d* (top of the figure)
- *r* (upper left, on the arms)
- *r* [?] (a second *r*, beside the first)
- *m* (lower left, at the foot of one arm)
- *m* [?] (a second *m*, at the foot of the other arm)
- *k* [?] (on the lower arm, right of centre)

Then, as Oronce Fine teaches, let the instrument be fitted in such a way that the diagonal line *e n* is laid exactly along the straight line of *b e*, that is, of the greater

line produced; and let the inner arm of the gnomon, *n m*, be forced to come to the point *z*, the boundary of the lesser line *b z*, while *e n* is kept always unmoved from the straight line of *b e*. For then, on Oronce Fine's teaching, the remaining and inner arm of the gnomon, *r n*, will take up the position *r p*; out of the lesser line produced it will cut off the line *b q*, the second proportional; and the inner angle of the gnomon will mark out *b p*, the third proportional.

But we shall demonstrate that this is false, in the following way. Let *b r* and *b n* be the lines found between *a b* and *b c* in that figure from which the constructed gnomon was derived. Then the straight line *b e*, falling on the straight lines *p q* and *n r*, makes the interior angle *q p e* equal to the exterior angle *r n e*. Therefore *p q* is parallel to *r n*, and consequently the triangles *b p q* and *b n r* are equiangular and similar. In the same way the two triangles *b p z* and *b n c* are equiangular and similar. Consequently, as *b r* is to *b n*, so is *b q* to *b p*; and as *b n* is to *b c*, so is *b p* to *b z*.

But by hypothesis the line *b r* is equal to the line *b K* in the figure described above; and *b k* does not reach the true magnitude of the second proportional, while *b n* exceeds the third. Therefore *b r* has a far smaller ratio to *b n* than the true second proportional has to the true third. And for that reason *b n* will have a greater ratio to *b c* than the true third has to *b c*, the fourth; and likewise *a b*, the first, will have a greater ratio to *b r* than the same *a b* has to the true second.

Now *a b* is to *b c* as *g b* is to *b z*, since each of these is half the double ratio — the ratio, that is, of the diagonal to the side of the same square. Therefore *b q* and *b p* are not the mean proportionals between *g b* and *b z*. On the contrary, *b q* has to *b p* a smaller ratio than the true second has to the true third; *g b* has to *b q* a greater ratio than the first has to the true second; and *b p* has to *b z* a greater ratio, likewise, than the true third has to *b z*, the fourth. Oronce Fine's instrument, therefore, cannot supply two mean proportionals between the sides of two squares

— which is what we undertook to demonstrate.

It is important to notice, however, how great a difference there is between Plato's instrument and Oronce Fine's gnomon. By Plato's instrument two mean proportionals are found universally, between any two straight lines whatever, even though no discovery of mean proportionals between two other lines of the same ratio has gone before it. But if you wish to find two mean proportionals between two given lines by Oronce Fine's gnomon, you must first have in hand an entirely certain discovery of two proportionals between other lines of the

same ratio. Only then will you be able to find two mean proportionals between any two lines you please of a like ratio; otherwise not. For that gnomonic instrument cannot be correctly constructed unless two continually proportional means have already been found between some pair of lines of the same ratio — and this Oronce Fine did not achieve. Even if he had achieved it, it would still be better to satisfy the question by *Elements* VI.12 [?], or to use Plato's general instrument, than to use any particular instrument whatever.

From this you may recognize that the gnomon by itself is not sufficient for taking two mean proportionals universally between two given lines, even if it were correctly built. For suppose you set before yourself two lines *a b* and *b c*, between which two means *b r* and *b n* have been found; suppose you then vary *a b*, and now set up *g b* as the first of the two lines proposed, and look for two mean proportionals between *g b* and *b c*. The gnomon will point out no other means afresh than those very lines *b r* and *b n* which had been found between *a b* and *b c* — which is absurd.

That Oronce Fine erred universally concerning the finding of two mean proportionals between two given lines, the lesser of which exceeds half the greater.

CAP. VII. Reprehensio quinta.

[Chapter VII. — Fifth rebuke.]

Now I do not see how Oronce Fine is to be excused. Either he thinks that all lines whose lesser exceeds half the greater are incommensurable, or else he does not know which magnitudes are called commensurable and which incommensurable. For he says of the remaining straight lines that, although they are not the sides of squares of which the one is described about a circle and the other within that same circle, still, if the lesser of them exceeds half the greater, two mean proportionals under a continued proportion can nonetheless be found between them by the same technique he has used. In this too we shall show that he has gone badly wrong.

Let two straight lines *b f* and *b c* be set out, between which two mean proportionals under a continued proportion are to be found. Let the greater of them, *b f*, be 125 feet, and the lesser 64; the lesser will then exceed half the greater. Let the excess *c f* be divided in extreme and mean ratio, the greater segment being *c K* and the lesser *k f*. Then, by Oronce Fine's rule for the remaining straight lines, the straight line *b K* will be the second proportional — which we shall show to be false by the most evident principles.

For the two magnitudes *b f* and *b c* have to one another the ratio a number has to a number; therefore the magnitudes *b f* and *b c* are commensurable, by *Elements* X.6. Hence *b c* will also be commensurable with *c f*, by *Elements* X.15. This is perfectly clear as well if 64 is taken from 125, for *c f* will be left as 61. But *c f* and *c K* do not have to one another the ratio a number has to a number, as Campanus showed on *Elements* IX.16, and as is clear also from *Elements* XIII.6; therefore *c f* and *c K* are incommensurable, by *Elements* X.8. Now *b c* and *c f* were commensurable; and for that reason *b c* and *c k* are incommensurable, by *Elements* X.13, or by the lemma to X.12. The whole *b k*, then, will be incommensurable with *b c*, by *Elements* X.16; and *b f* likewise incommensurable with that same *b k*, by *Elements* X.13. Consequently, if *b f* the first is incommensurable with *b k* the second, then *b K* the second will be incommensurable with the third, and the third also incommensurable with *b c* the fourth. But *b f* is 125, the second proportional 100, the third 80, and the fourth *b c* 64; therefore they are commensurable by *Elements* X.6, and not incommensurable. Oronce Fine's rule for finding two continually proportional means between two given lines of which the lesser

exceeds half the greater is therefore false.

Now when the sides of cubes stand in the sesquiquartan ratio [5 : 4], the ratio of the cubes is less than double; and the sesquiquintan ratio [6 : 5] is less than the sesquiquartan, and the sesquisextan [7 : 6] less than the sesquiquintan, and the remaining superparticular ratios are successively less. Therefore the ratio of the remaining cubes whose sides stand in a superparticular ratio less than the sesquiquartan will be far less than double. So the lesser of those cube numbers whose sides have a superparticular ratio less than the sesquiquartan will exceed half the greater; and between those cube numbers there will fall two continually proportional mean numbers, by *Elements* VIII.12.

For that reason, if we suppose two lines *b f* and *b c* to have the ratio of any two cube numbers whose sides stand in a superparticular ratio less than the sesquiquartan, two mean proportionals will fall between *b f* and *b c*, and those four lines will be commensurable. Yet Oronce Fine will be forced to concede that they are incommensurable, for our demonstration is universal. And it will be possible to observe this not only where the sides of the cube numbers have the sesquiquartan ratio or some smaller superparticular one, but also where they have a superpartient ratio less than the sesquiquartan. For in that case too the ratio of the cubes will be less than double, and accordingly the lesser of them will exceed half the greater.

Moreover, as number is to number, so is straight line to straight line — a truth rightly assumed in the corollary to *Elements* X.6. Consequently, if one straight line is set to another in the ratio which those cube numbers have, the mean proportionals must necessarily be commensurable. Oronce Fine has therefore blundered shamefully in a matter so clear and so manifest; and he appears, in consequence, not to have known which magnitudes are commensurable and which incommensurable.

It is evidently demonstrated that Oronce Fine erred also concerning the finding of mean proportionals between two straight lines of which the greater is double the lesser, and that he consequently did not double the cube at all.

CAP.VIII.

Reprehēsio sexta.

[Chapter VIII. — Sixth rebuke.]

Setting out to double the cube whose side is *b c*, Oronce Fine joined to that side the double line *a b* at the right angle at *b*; and about the centre *b*, with radius *a b*, he described the circle *a d e f*. He produced the lines *a b* and *b c* in a straight line as far as the circumference of the circle so described, and through *a* and *c* he drew a straight line which meets the circumference of that circle at the point *g*.

Diagram. A circle occupying the upper half of the right column. Its centre is *b*. A horizontal line runs across the figure from *a*, on the circumference at the left, through *b*, past *n*, and out to *e* on the circumference at the right. A vertical line runs from *d*, at the top of the circumference, down through *b* and *c* to *f* at the foot. From *a* two straight lines run off to the right, and a further set of chords crosses the lower half of the circle, meeting the circumference at the left, at the foot and at the lower right; the remaining letters stand where those chords cut one another or the two diameters.

- *d* (top, on the circumference)
- *r* [?] (inside the circle, on the vertical line a little below *d*)
- *a* (left, on the circumference)
- *b* (centre)
- *n* (on the horizontal line, between *b* and *e*)
- *e* (right, on the circumference)
- *c* (on the vertical line, below *b*)
- *h* (inside, below and to the right of *c*)

- *k* (inside, below *c* and to its left)
- *l* (at the foot of the circle, left of *f*)
- *f* (at the foot, on the vertical line)
- *g* (lower right, on the circumference)

He then looks for two mean proportionals between *a b* and *b c* in this way. He divides the straight line *c g* at the point *h* in extreme and mean ratio, so that *g h* is the greater segment and *c h* the lesser. Next he draws the straight line *e h* through *e* and *h*, which meets the circumference of the same circle at the point *l* and cuts the radius *b f* at *k*. He further draws the line *c n* parallel to *e k*, which cuts the radius *b e* at *n*. Finally he cuts off from *b d* the line *b r* equal to *b k*, and constructs the rest just as in the first problem.

He therefore asserts that *b K*, or *b r*, is the second proportional and *b n* the third: that is, as *a b* is to *b r*, so *b r* is to *b n*, and *b n* to *b c* itself; and he asserts that this can be demonstrated just as in that first problem. He accordingly maintains that the cube described on the line *b n*, the third proportional, is double the cube set before us whose side is *b c*. But since this procedure of Oronce Fine's is proved by no other argument, it will be dis-

proved in the same way as we refuted the demonstration of the first problem in the third and fourth chapters. Furthermore, since Oronce Fine perhaps supposed that even if his technique were false the falsehood would still be impossible to disentangle, we shall for that reason show at once, by the most certain and evident principles, that the line *b k* is greater than the second proportional and the line *b n* less than the third proportional — and hence that the cube described on *b n* has to the cube on *b c* a ratio less than double.

For let *b f* be double *b c*, and let *c g* be divided at the point *h* in extreme and mean ratio, *g h* being the greater segment; and let the straight line *e h*, produced to *l*, cut *b f* at *k*. I say that *b K* is greater than the second proportional.

Positio prima.

[First position.]

¶ Let *e g* first be 8. Now the right-angled triangles *a b c* and *a e g* are equiangular, and therefore have proportional the sides about their equal angles. But *a b* is double *b c*; therefore *a g* is also double the straight line *e g*, and so *a g* itself is 16. Hence the diameter *a e* is √320, and *b e*, the radius, √80, while the straight line *b c*, half the radius, is √20. Consequently the straight line *a c*, subtending the right

angle at *b*, will be 10, and *c g* is left as 6; *g h*, the greater segment, is √45 − 3, while the remaining segment *c h* is 9 − √45, and *a h* is 19 − √45.

e g, 8. *a g*, 16. *a e*, √320. *a b*, √80. *b c*, √20. *a c*, 10. *c g*, 6. *g h*, √45 − 3. *c h*, 9 − √45. *a h*, 19 − √45.

Positio secunda.

[Second position.]

¶ But now let *a b* be put at 100 and *b c* at 50, and through *h* let *h t* be drawn parallel to *b f*. The two triangles *a b c* and *a t h* will then be equiangular and similar. Therefore, as *a c* is to *a h*, so is *b c* to *t h*, and *a b* to *a t*. So, by the common rule of proportional quantities, in such parts as *a b* is 100 and *b c* is 50, *t h* is found to be 95 − √1125, and *a t*, double *t h*, 190 − √4500. Let *a t* be taken away from the diameter *a e*, and *t e* will be left as 10 + √4500.

10 | 19 − √45 | 50. 950 − √112500. *t h*, 95 − √1125. *a t*, 190 − √4500. *a b*, 100. *b c*, 50. *t e*, 10 + √4500.

And since the two triangles *b K e* and *t h e* are equiangular and similar, it follows that as *t e* is to *b e*, so is *t h* to *b K*. But *t e*, *b e* and *t h* are known; therefore, by that same common rule of four proportional quantities, the straight line *b k* will become known as √(20977 157[?]/484) + √(58 53/484) − √(2614 449/484) − 21 13/22, in such parts as those of which *b e*, or *b f*, is 100, and *b c*

10 + √4500 | 100 | 95 − √1125.

9500 − √11250000[?], the dividend.

10 + √4500, the divisor.

The fourth term of the proportion, which is *b K*: √(20977 257/484) + √(57 53/484) − √(2614 449/484) − 21 13/22.

Diagram. The same circle repeated at the head of the right column, now with the parallel *h t* drawn in. Its centre is *b*. A horizontal line runs from *a*, on the circumference at the left, through *b* and *t* to *e* on the circumference at the right. A vertical line descends from *d*, at the top of the circumference, through *b* and *c* to *f* at the foot; a second, shorter vertical drops from *t* to *h*. Chords run from *a* down to the lower right of the circle and from *e* down across to the foot at the left, meeting one another at the letters set in the lower half of the figure.

- *d* (top, on the circumference)

- *a* (left, on the circumference)
- *b* (centre)
- *t* (on the horizontal line, immediately right of *b*)
- *e* (right, on the circumference)
- *c* (on the vertical line, below *b*)
- *k* (inside, below *c* and to its left)
- *h* (inside, below and to the right of *c*)
- *g* (lower right, on the circumference)
- *l* (at the foot of the circle, left of *f*)
- *f* (at the foot, on the vertical line)

b c, 50.

Now √(20977 257/484) is greater than 144 5/6, since the square of the latter is only 20976 25/36; and likewise √(58 53/484) is greater than 7 13/22, since the square of the latter is only 57 201/484 [?]. Therefore, if from the sum of these squares 21 13/22 and √(2614 449/484) are taken away, what is left will fall short of the line *b K* itself.

But √(2614 449/484) is less than 51 1/7, for the square of the latter is 2615 39/49 [?]. So if 51 1/7 and 21 13/22 are taken away from that same sum, what is left will fall much further short of that same line *b K*. And what is left is 79 29/42. The line *b K* is therefore greater than 79 29/42.

Now the cube of the line *b f* is 1,000,000, and so the cube of the second proportional is half of it, namely 500,000; but the cube of that same 79 29/42 is much greater, for it exceeds the number 506,000. The second proportional is therefore less than that same 79 29/42, and much less, then, than *b K* — which is what we undertook to show first of all.

From this, moreover, you will easily demonstrate that the third proportional is greater than the line *b n*. For since *c n* is parallel to the straight line *K e*, the two triangles *K b e* and *c b n* are equiangular and therefore similar; and so, as *b e* [?] is to *b K*, so is *b n* to *b c*. But *b K* is greater than the second proportional, and therefore *b e* will have a smaller ratio to *b K* than that same *b e* has to the second proportional. Consequently *b n* will also have a smaller ratio to *b c* than *b e* has to the second proportional. But as *b e* is to the second proportional, so is the third to *b c*, the fourth; therefore *b n* will have a smaller ratio to *b c* than the third proportional has to that same *b c*. For that reason *b n* is less than the third

proportional, and accordingly the ratio of the cube described on the line *b n* to the cube described on *b c* is less than double — which was to be demonstrated.

We deliberately chose this method, one familiar to trained mathematicians, for investigating the length of the line *b K*, rather than proceed by the measurement of angles; for in a matter of this kind it was not open to us to use tables of arc and chord, which cannot be exact, however useful they are for other purposes.

Modus Orontij Finæi ad quadrandum circulum.

CAP. IX.

[*Oronce Fine's method for squaring the circle.* — **Chapter IX.**]

So that we may report Oronce Fine's method as faithfully as possible, we shall transfer to this place the very technique he used, set out in his own words.

> Let a circle *a h* be given (he says), about which one square equal to it must be drawn, and another of equal perimeter found. Accordingly, about that same circle *a h* let a square *a e* be described, by *Elements* IV.7; and within that same circle *a h* let another square *d h* be described, by IV.6 of the same book. Then, between the sides of these two squares — namely *a* and *d* — let two straight lines in continued proportion under the same ratio be found, by the method of the preceding problem; let these be *b* and *c*, so that as the side *a* is to the line *b*, so is that same *b* to *c*, and *c* to the side *d*. Next, on those straight lines *b* and *c* let squares *b f* and *c g* be described, by I.46 of the same *Elements*,
>
> and let the sides of those squares *b f* and *c g* be equidistant, or parallel, both to one another and to the sides of the squares *a e* and *d h* already named. Finally, in the squares *b f* and *c g* let single circles *b l* and *c m* be described, by IV.8 of the said *Elements*. These circles, by the very hypothesis about the sides, will have the same centre as the circle *a h*, namely *n*, and together with the squares themselves will be set about the same diameter. With the construction made in this way, Oronce Fine says that the square *b f* is equal, first of all, to the given circle *a h*; that the square *c g* is likewise equal to the circle *b l*, and the square *d h* correspondingly to the circle *c m*; and further, that the square *c g* has the same perimeter as that same circle *a h*.

Diagram. A large woodcut filling the foot of the left column. Four squares are nested one within another, all on the same centre and with their sides parallel; circles are drawn about and within them, so that the arcs cut across the corners

of the squares. A single straight diagonal runs from the upper right of the figure down to the lower left corner. The lettering falls in two groups, one descending along the upper right, the other rising from the lower left corner.

- *a* (upper right, on the outermost square)
- *b* (upper right, below *a*)
- *c* (upper right, below *b*)
- *d* (below *c*, set larger, toward the middle of the figure)
- *n* [?] (near the centre)
- *m* (lower left, on the diagonal)
- *l* [?] (lower left, below *m*)
- *g* (lower left, below *l*)
- *f* (lower left, below *g*)
- *e* (at the lower left corner of the outermost square)

So Oronce Fine, word for word, in the second problem of his book on the quadrature of the circle. He then set out his proofs in the fifth problem, in which he attempts to show by three arguments that the square *b f* is equal to the circle *a h*. The first proceeds from the equality of proportional numbers, using numbers very close to the truth taken over from Archimedes' rule for the ratio of the circumference to the diameter — which is roughly the triple sesquiseptimate ratio [*tripla sesquiseptima*, that is 3 1/7, or 22 : 7] — and it runs as follows.

From what Archimedes demonstrated it is settled that in parts of which the square *a e* is 14, the circle *a h* is 11. Again, in parts of which that same square *a e* is 14, the square *d h*, being its half, is 7. In the same parts, therefore, the square *a e* is 14, the circle *a h* 11, and the square *d h* 7. But once two mean proportionals between 14 and 7 have been found, the first of them must be the cube root of the number 1372, which is very close to the truth at 11. Now the square *b f* is the first mean proportional between the square *a e*, 14, and the square *d h*, 7; the square *b f* will therefore be 11 parts, in which the square *a e* is 14, the square *d h* 7, and the circle *a h* 11. So both alike, the square *b f* and the circle *a h*, are 11, and accordingly the square *b f* is equal to the given circle *a h*. In the same way he proves that the square *c g* would be equal to the circle *b l*, and likewise the square *d h* equal to the circle *c m*.

The second argument is taken from the equality of the sides, on the same hypotheses drawn from what Archimedes demonstrated. Let the side of the square *a e*, or the diameter of the circle *a h*, be set (he says) at 14 equal parts. The square *a e* itself will then be 196 square parts, and the square *d h*, being its half,

98 of the same parts. Again, in parts of which the diameter of the circle *a h* is 14, the circumference is 44, by Archimedes' rule for the measurement of the circle; half the circumference is therefore 22, and the radius 7. But half the circumference multiplied into the radius yields the area of the circle; the area of the circle *a h* will therefore be 154 parts, of which the square *a e* is 196 — and the ratio of these numbers is as 14 to 11. Now the square root of the number 154, close to the truth, is 12 with nearly 5/12. That, then, is the side of the square which is equal to that circle *a h*. But the side of the square *b f* is found to be just as great: for in parts of which the side of the square *a e* is 14, the side of the square *d h* is nearly 9 17/18, that is, the square root of the number 98. Now let two mean proportionals in continued proportion be found between 14 and 9 17/18. The first of them, which is the side of the square *b f*, will be the cube root of the number 1949 1/9 — that is, the number 12 together with 12/30 [?], which corresponds nearly to that 5/12. And on this account he affirms that the side of the square *b f* is exactly as great as the side of the square which is equal to that circle *a h*.

The third argument is from the impossible. For if the square *d h* were given as in any way greater or less than the circle *c m*, and consequently the square *b f* as greater or less than the circle *a h*, we should fall into an absurdity. For then those four squares would no longer stand in one continued proportion, nor would the circles described in them. Rather, on account of however small an inequality among the numbers, the very continued proportion by which (as he says) they are bound together would be utterly dissolved — as, for instance, if we were to grant the circle *c m* to be 7 1/10 parts, or 6 9/10, which he affirms can be gathered from the differences of the numbers themselves by the rule of proportional numbers. Therefore (he concludes) the circle *c m* is neither greater nor less than the square *d h*, nor the circle *a h* than the square *b f* itself; but in every way the square *b f* is equal to the circle *a h*, the square *c g* to the circle *b l*, and the square *d h*

to the circle *c m* — which is what he had undertaken to demonstrate.

From this he infers, against Archimedes, that the ratio of the circumference to the diameter is greater than 3 1/7, and that the square has a smaller ratio to the inscribed circle than 14 to 11. He proves this as follows. Since the four squares *a e*, *b f*, *c g*, *d h* are continuously proportional, and the first is double the last, the ratio of the first to the second taken three times makes up the double ratio. And therefore the first and the second, multiplied cubically, must produce the double ratio. Accordingly, since the first square is 14, the second will be 11 and about 1/9. For if 14 is cubically multiplied into itself, it comes to 2744; and 11 with 1/9,

likewise multiplied cubically, produce very nearly 1372, which is half the number 2744. The square *a e* therefore has to the square *b f* very nearly the ratio of 14 to 11 1/9.

But he says he has shown the circle *a h* to be equal to that same square *b f*; and so he concludes that the square *a e* has to the circle *a h* very nearly the ratio of 14 to 11 and 1/9. And since, as the square is to the inscribed circle, so is four times the diameter of the circle to the circumference of that same circle — for so he either meant to say or ought to have said, and not to half the circumference — it follows that in parts of which the diameter is seven, and four times the diameter 28, the circumference will be 22 and 2/9 [?]. And from this he concludes that the circumference has to the diameter a greater ratio than 3 1/7.

He confirms this also by inspection of the eye and by experiment. For if, with a very fine pair of compasses, the seventh part of the diameter is fitted to the circumference, it will subtend (he says) the twenty-second part of that same circumference, and twenty-two such sevenths will exactly complete the whole circumference. And since the arc is greater than the chord that subtends it, the whole circumference will be greater than 22 sevenths of the same diameter; and therefore the ratio of the circumference to the diameter will be greater than 3 1/7. This, he adds, appears to be corroborated by the reckoning of the numbers. For in parts of which the circumference is 360, the twenty-second part is 16 and about 22 minutes, while the chord subtending it is 17 parts and about 5 mi-

nutes, in parts of which the diameter is 120 [?]. The seventh part of the diameter, again, is in those same parts 17 and 8 minutes, and differs from the chord of the twenty-second part of the circumference by three minutes only. This, he says, has plainly fallen out from the calculation of the chords itself, since in dividing numbers (he says) and in extracting roots over and over again something is always lost, and on that account the numbers themselves are at last forced to fall short of the full complement of units owed to them. From this, he says, arises the shortfall in the ratio of the circumference to the diameter which is demonstrated, by the numbers of the right sines in imitation of Archimedes, to be less than 3 1/7 — whereas the truth of the matter (he says) stands thus: that the circumference has to the diameter very nearly the ratio of 22 and 2/9 to 7, and the square to the inscribed circle the ratio of 14 to 11 and 1/9. And accordingly, in parts of which the square *a e* is 14, the circle *a h* — and the square *b f* equal to it — is 11 and about 1/9; while the square *c g*, and the circle *b l* equal to it, is 8 parts together with nearly 19/23 [?]. And from these, again, with the help of the numbers, he

gathers that the perimeter of the third square *c g* is equal to the circumference of the given circle *a h*, just as he had undertaken to demonstrate.

Neque Orontium circulum quadrasse, neque rectam lineam æqualem circunferentiæ inuenisse.

CAP.X.

Reprehensio. VII.

[*That Oronce Fine neither squared the circle nor found a straight line equal to the circumference.* — **Chapter X.** — **Seventh rebuke.**]

Such, then [?], is Oronce Fine's discovery concerning the quadrature of the circle, which we shall show to be false in many ways. He assumes, first of all, that two mean proportionals between the side of the square circumscribed about a given circle and the side of the square described within that same circle have been found by him. But we have demonstrated above that the mean proportionals he set up are not the true ones — indeed, that one of them falls short of its proper magni-

—tude, while the other exceeds it.

Furthermore, Oronce Fine's quadrature of the circle is false, because he assumes on Archimedes' authority that the square has to the circle inscribed in it the ratio of 14 to 11, when in fact that ratio is not exact. He then goes on to prove that the square b f is equal to the circle a h, on the ground that the cube root of the number 1372 — which likewise comes close to the truth — is 11; but it is in fact a little greater. Now if for that reason he takes the square b f itself to be equal to the same circle a h because it agrees with Archimedes' quadrature (for it is on this foundation, and indeed on this alone, that he chiefly relies), then it is established from Archimedes himself that the circle a h does not fill out 11 parts. But the cube root of the number 1372 exceeds those same 11 parts, since 11 multiplied cubically into itself makes only 1331. The square b f will therefore not be equal to the circle a h.

In his second argument, taken from the equality of the sides, he maintains the same thing and on the same principles. He sets the side of the square a e — that is, the diameter of the circle a h — at 14 equal parts, and, assuming the ratio of the circumference to the diameter demonstrated by Archimedes, namely 22 to 7 or 44 to 14, he finds that the side of the square equal to the circle a h is 12 parts together with about 5/12. But he finds the side of the square b f to be 12 parts together with 12/30, which answers roughly to that 5/12; and on that account he

concludes that the side of the square b f is as great as the side of the square equal to that same circle a h. Here Oronce Fine's computations are to be laughed at rather than seriously refuted or overturned. For if he is now content, in proving the square b f equal to the circle a h, with the argument that it agrees with Archimedes' numbers — he had nonetheless demonstrated a little earlier, in his first argument, that if the square a e is 14 the circle a h will be 11, while the square b f is a little greater, namely the cube root of the number 1372. How, then, does he now conclude that the circle a h is a little greater than the square b f? For 5/12 is greater than 12/30.

Indeed, if the side of the square a e — the diameter of the circle a h — is set at 14 equal parts, and the ratio of the circumference to the diameter is taken as 22 to 7, or 44 to 14 (although Archimedes found it a little smaller), then the side of the square equal to that same circle a h will beyond doubt be the square root of the number 154; and accordingly the root will be the square root of the cube root of the number 3652264 [?].

But if the square b f is set as the first mean proportional between the squares a e and d h, the root will be the square root of the cube root of the number 3764768 — for that is all that is found by the rule Oronce Fine lays down for finding mean proportionals between two given numbers, a thoroughly commonplace rule. The square b f is therefore greater than the circle a h. And yet, from what Archimedes demonstrated, the circle a h itself does not reach the number 154: the square b f is therefore much greater than that same circle a h. So I can conclude at one stroke that Oronce Fine has neither found the quadrature of the circle nor proved it.

His third argument, taken from the impossible, proves nothing whatever. Suppose that, the square a e being 14 such parts, the circle a h is (as he assumes) 11 of them, and that the sides of the four squares are in continued proportion. Then the square b f will be the cube root of the number 1372, the circle b l the cube root of the number 665 with 1/2, the square c g the cube root of 686, and the circle c m the cube root of 332 with 3/4; while the square d h will be 7, that is, the cube root of 343. We grant, then, that the square d h is greater than the circle c m, since the cube root of 343 is greater than the cube root of the number 332 with 3/4 — and no absurdity follows from that.

In the corollary, however, he utterly overturns whatever he had built up before; and here it is truly worth the effort to observe the man's folly. He had assumed from Archimedes that the ratio of the circumference to the diameter is triple-and-a-seventh [*tripla sesquiseptima*], and on that account that the square has to the

inscribed circle the ratio of 14 to 11. Then, propped up by these supports, he proved as best he could that the square b f is equal to the circle a h, on the ground that it would be the cube root of the number 1372, which comes extremely close to the truth of 11, and so would agree with Archimedes' numbers. But now, in passing from the argument to the corollary, he finds that it [?] has already grown to 11 and 1/9. On that account he asserts that the square has to the inscribed circle roughly the ratio of 14 to 11 and 1/9, and that the circumference has to the diameter a ratio greater than triple-and-a-seventh, against Archimedes. But let us see how he convicts him. Given the square a e of 14 equal parts, Archimedes would concede that the square b f is very nearly 11 and 1/9 of those same parts; the circle a h, however, eleven

eleven parts. On that account, then, the square a e has to the inscribed circle a greater ratio than 14 to 11, but to the square b f a smaller one. Yet Oronce Fine has nowhere shown that the square b f is equal to the circle a h; he has only assumed, from what Archimedes demonstrated, that the circle a h is 11, while he demonstrated that the square b f is the cube root of the number 1372, which is a little greater than 11 and 1/9. He therefore infers wrongly that the ratio of the circumference to the diameter is greater than triple-and-a-seventh, and that the square has to the inscribed circle a ratio smaller than 14 to 11.

Moreover, the numerical computation drawn from the table of arc and chord is not opposed to Archimedes. I shall bring forward his demonstration on the measurement of the circle in the following chapter, so that it may stand out clearly that Archimedes did not find the ratio of the circumference to the diameter smaller than triple-and-a-seventh merely because something is always lost in dividing numbers and extracting roots, but because it truly is smaller than triple-and-a-seventh. Oronce Fine, by contrast, assumes that the ratio of the circumference to the diameter is greater than triple-and-a-seventh in order to show that the square c g is isoperimetric with the circle a h; and since he takes false and improbable hypotheses, he will be able to conclude nothing. Nor could he demonstrate his proposition even if they were granted, since he reasons from certain examples and from uncertain numbers. For the sides of the four squares just named are incommensurable, and yet he supposes them to be numbers in order to draw his conclusion. How, then, will he demonstrate what is true and necessary by means of what is false and uncertain?

Some other method, more certain and more expeditious, should therefore have been chosen — in this way, namely. The three straight lines a, b, c, the sides of the three squares a e, b f, c g, are in continued proportion, by hypothesis. Therefore

the rectangle contained by the two lines a and c is equal to the square b g, the square on the mean. But that square b g is (as Oronce Fine supposes) equal to the circle a h. Therefore the circle a h is, by a common notion, equal to the rectangle contained by a and c. Now the straight line a is equal to the diameter of the circle a h, and the straight line c is besides the side of the square c g. Therefore the circle a h is equal to the rectangle contained by the diameter of that same circle and the side of the square c g. But that same circle a h is also equal to the rectangle contained by the diameter and the fourth part of the circumference. Two rectangles will therefore be equal to one another by a common notion: the one contained by the diameter of the circle a h and the side of the square c g, the other by the same diameter and the quadrant of the circumference. And accordingly the quadrant of the circumference will be equal to the side of the square c g, and the whole circumference equal to all the sides of that same square together. And so the circle a h is isoperimetric with the third square c g — which is what Oronce Fine had undertaken to demonstrate, but by no means demonstrated.

¶ You will demonstrate the same thing in another way, by reduction to the impossible. For if they are not isoperimetric, let the quadrant of the circumference of the circle a h be greater than the side c. But the rectangle contained by a and the quadrant of the circumference is equal to the circle a h; and the circle in turn is by hypothesis equal to the square b f. The square b f will therefore be greater than the rectangle contained by a and c, and on that account a, b, c will not be in continued proportion — contrary to the hypothesis. The same absurdity will follow if the quadrant of the circumference of the circle a h is granted smaller than the side c. Therefore they are isoperimetric. If you happen to doubt that the rectangle contained by the diameter and the quadrant of the circumference is equal to the circle, you will conclude it from Archimedes with the greatest ease. For as the diameter is to the radius of the same circle, so is half the circumference to the quad-

Diagram. The woodcut occupies the lower two-thirds of the left column. It shows a large square in outline with three further squares nested within it, all sharing the lower-left corner region; the upper side of each successive square is drawn as a separate horizontal line, and these four lines carry labels at the top of the figure. Circular arcs — the inscribed circles of the successive squares — sweep from the upper edge down toward the lower left, so that the arcs cross the nested sides. A long straight diagonal runs from the upper right down to the lower-left corner.

- a (uppermost horizontal line): the side of the first (largest) square
- b (second horizontal line): the side of the second square
- c (third horizontal line): the side of the third square
- d (fourth, innermost horizontal line): the side of the fourth square
- n [?] (mid-figure, on the long diagonal): a single letter, crowded by the crossing arcs
- e (outermost point at the lower-left corner)
- f (next point inward at the lower left)
- g (next point inward)
- h, k [?] (a closely set pair of letters just inward of g)
- m (above and to the right of the h/k pair, against an arc)

—rant. Therefore the rectangle contained by the diameter and the quadrant of the circumference is equal to the rectangle contained by the radius and half the circumference. But the rectangle comprehended by the radius and half the circumference is equal to the circle, by what Archimedes demonstrated. Therefore the rectangle contained by the diameter and the quadrant of the circumference will be equal to that same circle.

It should be noted, however, that in this quadrature of Oronce Fine's we mark only his more conspicuous errors, passing over the smaller ones. Of that kind is the very construction of the figure of the second problem. For although the sides b and c are equidistant both from one another and from a and d, it does not follow that the inscribed circles must have the same centre. It ought therefore to have been constructed in another way — but these are very slight matters.

> *That Archimedes truly demonstrated that the circumference of a circle contains the diameter three times, and besides a part a little smaller than a seventh of that same diameter but greater than ten seventy-firsts: so that it may plainly appear how rashly, how falsely, and how ignorantly Oronce Fine asserts, against Archimedes, that the ratio of the circumference to the diameter is greater than triple-and-a-seventh.*

CAP.XI. Reprehēsio. VIII.

Let a c be the diameter of a circle whose centre is e. I say that the circumference of that circle is three times the diameter, and besides has a part smaller than a seventh of the same diameter, but greater than ten seventy-firsts. This was demonstrated by Archimedes in his book on the measurement of the circle (*On*

the Measurement of the Circle), and has been sufficiently explained by Eutocius of Ascalon in roughly the following way.

Let the diameter a c be cut at right angles at the centre e by the straight line b d, which is likewise a diameter of the circle. The whole circumference will thus be divided by these two diameters into quadrants. Let the semicircumference b c d be further divided into three equal parts, b z, z r, and r d, by the 15th proposition of the fourth book of Euclid's *Elements* (*Elements* IV.15). Since b z is a sixth part of the circumference, and b c a quarter of it, c z will accordingly be a twelfth, and c r likewise a twelfth. At the point c let the straight line c f be raised at right angles to a c, touching the circle; and from the centre e let straight lines be drawn through z and r, to meet the straight line c f at f and t [?]. The angle f e c will therefore be a twelfth part of four right angles, and accordingly a third part of one right angle; and the angle r e c is equal to it, a third part of one right angle. Moreover the two right angles at c are equal, and the side e c, which lies between equal angles, is common to the two triangles c f e and c t e. The remaining sides of those triangles are therefore equal, and the remaining angles equal, by the 26th proposition of the first book (*Elements* I.26): that is, the side t e is equal to the side e f, and t c to c f; and the two angles at f and t will besides be equal to one another. And since the whole angle t e f is a third part of two right angles, each of the two angles at f and t will likewise be a third part of two right angles, by the 32nd proposition of the first book (*Elements* I.32) and a common notion. The triangle t f e is therefore equilateral, by the sixth proposition of that same first book (*Elements* I.6); and c f is half of e f.

Diagram. The woodcut fills the lower part of the left column. A circle is drawn with its centre marked at e; a horizontal diameter runs across it and a vertical diameter crosses it at right angles at the centre. A straight line is drawn tangent to the circle at its topmost point and extended well beyond the circle on both sides; a fan of straight lines runs from the centre e out through points on the upper arc and on to meet that tangent, several of them very close together near the point of tangency.

- c (top of the circle, on the tangent line): the point of tangency and upper end of the vertical diameter
- a (bottom of the circle): the lower end of the vertical diameter
- d (left end of the horizontal diameter)
- b (right end of the horizontal diameter)
- e (centre of the circle)

- t (on the tangent, well to the left of c): the point where the line from e through r meets the tangent
- f (on the tangent, well to the right of c): the point where the line from e through z meets the tangent
- r (on the upper-left arc)
- z [?] (on the upper-right arc)
- m, l, k, h, g [?] (a crowded cluster of labels on the tangent line immediately on either side of c, marking the feet of the closely spaced lines from e; the individual assignment of these five letters is not securely resolvable)

Of the parts, then, of which e f is 306, c f [?] is 153; and the square built on e f will be 93636 such square parts, while the square on c f will be 23409. Now the square on e f equals the two squares on c f and c e together, by *Elements* I.47. We therefore subtract 23409 from 93636, and the square on e c is left as 70227. Its square root is a little more than 265, since the square of that number is only 7022[?]. Add 306 and 265: their sum is 571. Therefore 571 is less than e f and e c taken together.

Divide the angle f e c into two equal parts, then, by drawing the straight line e g, by *Elements* I.9. As e f is to e c, so is f g to g c, by *Elements* VI.3; and by composition, as e f and e c together are to e c, so is f c to g c. Alternando, therefore, as e f and e c together are to f c, so is e c to c g. But we set f c at 153, and we showed that e f and e c together are greater than 571; therefore e f, e c will have to f c a greater ratio than 571 to 153, by *Elements* V.8. Consequently e c will likewise have to c g a greater ratio than 571 to 153, by *Elements* V.13. Let c g, then, be 153 of the same equal parts: e c will therefore be greater than 571, by *Elements* V.10. And so the square on e g — which by *Elements* I.47 equals the two squares on the straight lines e c and c g — will be greater than the squares of 153 and 571. Now the square of the number 153 is the number 23409, and the square of 571 is 326041. The sum of these squares, namely 349450, will therefore be less than the square on e g, and e g itself will be greater than the square root of the number 349450. But the square root of 349450 is a little more than 591⅛: for if 591⅛ is multiplied into itself, it makes only 349428 47/64 [?]. Therefore e g is greater than 591⅛, and e c was likewise shown to be greater than 571. Accordingly e g and e c together are greater than 1162⅛, the number that 591⅛ and 571 come to when added. But c g was set at 153; and therefore e g, e c joined have to c g a greater ratio than 1162⅛ to 153, by *Elements* V.8.

Again, divide the angle g e c into two equal parts by drawing the straight line e h. Then by *Elements* VI.3, as g e is to e c, so is g h to h c; and by composition, as g e

and e c together are to e c, so is g c to h c. Alternando, therefore, as e g and e c are to g c, so is e c to c h. But it was shown that e g and e c joined have to c g a greater ratio than 1162⅛ to 153; therefore e c also has to c h a greater ratio than 1162⅛ to 153, by *Elements* V.13. Let c h be 153: e c will then be greater than 1162⅛, by *Elements* V.10 of the same book. And since the square on c h is 23409, and the square of 1162⅛ is 1350534 33/64, the two squares together come to 1373943 33/64. The two squares on e c and c h together must therefore be greater than 1373943 33/64. But the square on the straight line e h, which subtends the right angle, equals the two squares on e c and c h; therefore the square on e h is greater than 1373943 33/64, and so e h itself will be greater than the square root of 1373943 33/64. Now the square root of that number is greater than 1172⅛: for if 1172⅛ is multiplied into itself, it makes only 1373877 1/64 [?]. Therefore e h will be greater than 1172⅛; and e c was shown to be greater than 1162⅛. Accordingly e h and e c together are greater than 2334¼, which is what the two numbers 1172⅛ and 1162⅛ come to when added. But we set c h at 153; therefore e h and e c joined have to c h a greater ratio than 2334¼ to 153, by *Elements* V.8.

Again, divide the angle h e c into two equal parts by drawing e k. Then as e h is to e c, so is h k to k c, by *Elements* VI.3; and by composition, as e h and e c together are to e c, so is h c to k c; and alternando, as e h and e c joined are to c h, so is e c to c k. But it was shown that e h and e c joined have a greater ratio to c h than 2334¼ to 153. Therefore e c will also have to c k a greater ratio than 2334¼ to 153, by *Elements* V.13. So let us set c k at 153: e c will be greater than 2334¼, by *Elements* V.10. The square on c k will therefore be 23409,

and the square of 2334¼ is 5448723 1/16; the two together therefore make 5472132 1/16, and the two squares on e c and c K joined must be greater than that. But the square on the straight line e K, which subtends the right angle, equals the squares on e c and c K; therefore the square on e k will be greater than 5472132 1/16, and consequently e K itself will be greater than the square root of the number 5472132 and 1/16. But the square root of that number is greater than 2339¼: for if you multiply 2339¼ into itself, it makes only 5472090 9/15 [?]. Therefore e K is greater than 2339¼; and e c was shown to be greater than 2334¼. So e K and e c together are greater than 4673½, the number that grows out of them; and accordingly e K, e c joined have to c K a greater ratio than 4673½ to 153, by *Elements* V.8.

¶ Again, divide the angle K e c into two equal parts by drawing e l. Then as e K is to e c, so is K l to l c; and by composition, as e K and e c joined are to e c, so is K c to l c; and alternando, as e K and e c joined are to K c, so is e c to c l. But it was

shown that e K and e c joined have to K c a greater ratio than 4673½ to 153; therefore e c also has to c l a greater ratio than 4673½ to 153.

Now the angle f e c was shown to be a twelfth part of four right angles. Its half, g e c, will therefore be a twenty-fourth part; the half of that, h e c, will be a forty-eighth; the half of that again, K e c, will be a ninety-sixth part; and the half of that in turn, l e c, a one hundred and ninety-second. Now construct the angle c e m equal to l e c: the angle l e m will therefore be a ninety-sixth part of four right angles. Consequently the straight line l m will be the side of an equilateral polygon of 96 sides described about the circle, by the teaching of *Elements* IV.12. And c m is equal to c l, by *Elements* I.26 of Euclid; therefore l m is double c l, just as a c is double the radius e c. But parts have the same ratio as their equimultiples when taken correspondingly, by *Elements* V.15; therefore as e c is to c l, so is a c to l m. But e c has to c l a greater ratio than 4673½ to 153; therefore a c has to l m a greater ratio than 4673½ to 153, by *Elements* V.13 of the same book. So let us set a c at 4673½: by *Elements* V.10 of the same book, l m will be less than 153. Multiply 96 by 153: the result is 14688. Accordingly the perimeter of the polygon with 96 sides will be less than 14688.

Now the diameter a c is 4673½; three times the diameter will therefore be 14020½. Subtract this from 14688, and only 667½ is left — a number less than a seventh part of the diameter. For if you multiply it by seven, the result is 4672½, which the diameter exceeds by one. Therefore the number 14688 has to 4673½ a ratio less than triple-and-a-seventh [*tripla sesquiseptima*], and accordingly the perimeter of the polygon will have to the diameter a ratio less than triple-and-a-seventh, by *Elements* V.8. But the circumference of the circle is less than the perimeter of the polygon, by the first proposition of *On the Sphere and Cylinder*; therefore the circumference of the circle has to the diameter a ratio less than triple-and-a-seventh — which was the first thing to be shown.

As for the demonstration Oronce Fine brings forward to establish this in the second proposition of his book, worked out from numbers drawn from the table of sines: it is agreed that it is not Archimedes' own, as he himself admits, though he claims it is better than the one Archimedes made. Fine ought therefore to be asked whether that demonstration of his really does establish that the ratio of the circumference to the diameter is less than triple-and-a-seventh, or not. If it does establish it, why did he assert against Archimedes that the ratio is greater than triple-and-a-seventh? And if he thinks it does not establish it, why did he bring it forward at all? It would have been better to go through the author's own demonstration and point out its fault.

¶ But let us now demonstrate the second part: namely, that the circumference contains the diameter three times, and besides that a part greater than ten seventy-firsts. In the circle whose centre is e and whose diameter is a c,

let b c be the side of an equilateral hexagon inscribed in that same circle. Then b c will be equal to the radius, by the corollary to *Elements* IV.15 of Euclid, and therefore a c will be double b c. Join a b: by *Elements* III.31 the angle a b c will be a right angle, and it is triple the angle b a c, by the last proposition of *Elements* VI. Consequently the angle b a c will be a third part of one right angle. Let a c be set at 1560: b c will therefore be 780, and the square on a c will be 2433600, while the square on b c will be 608400. Now the square on a c equals the squares on a b and b c, by *Elements* I.47. We therefore subtract 608400 from 2433600, and the square on a b is left as 1825200. Its square root is a little less than 1351: for if you multiply 1351 into itself it makes 1825201. Therefore a b itself will be a little less than 1351.

Bisect the angle b a c by drawing a g, which cuts the straight line b c in f, and join c g. Then as a b is to a c, so is b f to f c, by *Elements* VI.3; and consequently, by composition, as a b and a c joined are to a c, so is b c to f c. Alternando, therefore, as a b and a c joined are to b c, so is a c to f c. Now a c is 1560, and a b was shown to be a little less than 1351; therefore a c and a b taken together are a little less than 2911. But b c is 780; therefore a b, a c joined have to b c a smaller ratio than 2911 to 780, by *Elements* V.8, and accordingly a c also has to f c a smaller ratio than 2911 to 780, by *Elements* V.13 of the same book.

Diagram. A woodcut occupying the upper right of the page. The straight line a c runs down the left side of the figure, a at the foot and the convergence point of the chords at the head; from the head of that line an arc sweeps to the right and downward, carrying the lettered points of the successive bisections. A fan of straight lines is drawn from a to each of the points on the arc, and chords run from the point at the head of the line down to the same points; two of the crossings are lettered.

- a (foot of the figure, lower left): the point from which the fan of straight lines radiates.
- [?] (head of the line at the upper left, where the chords converge): the letter is a solid blot of ink on the leaf and cannot be resolved; the text makes this point c.
- b (on the arc, lowest and furthest right of the lettered points on it).
- f (on the line joining b to the head of the figure): the crossing of a g with b c.

- g (on the arc, above b).
- o (on the line joining g to the head of the figure): the crossing of a h with g c.
- h (on the arc, above g).
- k (on the arc, above h).
- [?] (a further mark between k and h): a small heavily inked label that does not resolve.

Now the two triangles a g c and c f g are equiangular. For the angle g a c is equal to the angle b a g by construction, and the angle g c f is equal to that same angle b a g, by *Elements* III.27; therefore the two angles g a c and g c f are equal, by a common notion. The right angle c g a is common to both triangles; therefore the remaining angle a c g [?] will be equal to the remaining angle g f c, by *Elements* I.32 and a common notion. For that reason the sides of the triangles a g c [?] and c f g that subtend equal angles stand in the same ratio, by *Elements* VI.4. Therefore as a c is to f c, so is a g to g c. But a c was shown to have to f c a smaller ratio than 2911 to 780; therefore a g has to g c a smaller ratio than 2911 to 780, by *Elements* V.13. Let g c be set at 780: a g will therefore be less than 2911, by *Elements* V.10 of the same book. And since the square on a c equals the two squares on a g and g c, the square on a c will be less than 9082321 — the number that arises from adding 8473921, the square of the number 2911, and 608400, the square on g c. Accordingly a c itself will be less than the square root of the number 9082321. But that square root is a little less than 3013¾, since the square of 3013¾ is 9082689 1/16; therefore a c is less than 3013¾.

¶ Again, bisect the angle g a c by drawing the straight line a h, which cuts the straight line g c in o, and join c h. Then as a g is to a c, so is g o to o c; and consequently, by composition and then by alternation, as a g and a c joined are to g c, so is a c to c o. But a g was shown to be less than 2911, and a c is less than 3013¾. Therefore a g and a c joined are less than 5924¾, and accordingly a g and a c joined will have a smaller ratio to

to *g c* than 5924¾ has to 728 [?], by *Elements* V.8. And therefore *a c* will likewise have a smaller ratio to *c o* than 5924¾ has to 728, by V.13 of the same book. Now the two triangles *a h c* and *h c o* are equiangular, and the sides subtended by equal angles stand in the same ratio; therefore as *a c* is to *c o*, so is *a h* to *h c*. But *a c* has to *c o* a smaller ratio than 5924¾ to 728; therefore *a h* too will have to *h c* a smaller ratio than 5924¾ to 728 — that is, a smaller one than the number 1823 has to 240. For 5924¾ has to 1823 the triple-and-a-quarter ratio [*tripla sesquiquarta*], and 728 likewise has to 240 the triple-and-a-quarter ratio; and

therefore, by alternation, as 5924¾ is to 728, so is 1823 to 240. Accordingly *a h* has to *h c* a smaller ratio than 1823 to 240.

Let *h c* be set at 240; then *a h* will be less than 1823. But the square on *a c* is equal to the two squares on the lines *a h* and *h c*, by *Elements* I.47; therefore the square on *a c* is less than 3380929, a number that arises from 3323329, the square of the number 1823, together with 57600, the square of the number 240. Consequently *a c* itself is less than the square root of that number 3380929. But that same square root is less than 1838 9/11, since the square of 1838 9/11 is very nearly 3381252. Therefore *a c* is less than 1838 9/11.

¶ Again, let the angle *h a c* be bisected by drawing *a k*, which cuts the straight line *h c* in *r*, and join *c k*. Then as *h a* is to *a c*, so is *h r* to *r c*; and next, by composition and then by alternation of the ratio, as *h a* and *a c* joined are to *c h*, so is *a c* to *c r*. Now the two triangles *a c k* and *c r k* are equiangular; therefore as *a c* is to *c r*, so is *a k* to *c k*; and for that reason, as *h a* and *a c* joined are to *c h*, so is *a k* to *c k*. And since *c h* has been set at 240, while *a h* was shown to be less than 1823 and *a c* less than 1838 9/11, therefore *a h* and *a c* joined are less than 3661 9/11, and accordingly they have to *c h* a smaller ratio than 3661 9/11 to 240. Therefore *a k* will likewise have to *c k* a smaller ratio than 3661 9/11 to 240, by proposition 13 of the fifth book. Let 3661 9/11 be resolved into elevenths, and the number 40280 will be produced; let 240 likewise be resolved into elevenths, and the number 2640 will result. The ratio of these two, reduced to least terms, is as 1007 to 66. Therefore *a k* will have to *c k* a smaller ratio than 1007 to 66.

Let *c k* now be set at 66; then *a k* will be less than 1007. And since the square on *a c* is equal to the two squares on the lines *a k* and *c k*, the square on *a c* will therefore be less than 1018405 — for this number grows out of 4356, the square arising from *c k*, and 1014049, the square of the number 1007, collected into one. But the square root of that 1018405 is less than 1009 1/6, since the square of 1009 1/6 is 1018417 13/36 [?]; therefore *a c* is less than 1009 1/6.

¶ Again, let the angle *k a c* be bisected by drawing *a l*, which cuts the straight line *k c* in *t*, and join *c l*. In the same way it will hold that as *a k* is to *a c*, so is *k t* to *t c*; and next, by composition and then by alternation, as *a k* and *a c* joined are to *c k*, so is *a c* to *c t*. Now the two triangles *a l c* and *c t l* are equiangular; therefore as *a c* is to *c l*, so is *a l* to *l c*; and for that reason, as *a k* and *a c* joined are to *c k*, so is *a l* to *l c*. And since *c k* has been set at 66, and *a k* has been shown to be less than 1007, while *a c* is less than 1009 1/6, therefore *a k* and *a c* joined are less than 2016 1/6, and accordingly they have to *c k* a smaller ratio than 2016 1/6 to 66. Therefore *a l* will have to *l c* a smaller ratio than 2016 1/6 to 66.

Let *l c* now be set at 66; then *a l* will be less than 2106 1/6 [?]. Now the square on *a c* is equal to the two squares on *a l* and *l c*; the square on *a c* will therefore be less than 4069284 1/36 — for this number grows out of 4064928 1/36, the square of 1016 1/6 [?], and 4356, the square arising from *l c*, collected into one. But the square root of the number 4069284 and 1/36 is less than 2017¼, since the square of 2017¼ is 4069297 9/16; therefore *a c* is less than 2017¼.

Now the arc *b c* is a sixth part of the whole circumference, and *g c* a twelfth, and *h c* a twenty-fourth, and *k c* a forty-eighth; the remaining arc *l c* will therefore be a ninety-sixth. And *l c*, which has been set at 66, will be the side of a polygon of 96 equal sides inscribed in the circle. Let 66 then be multiplied by 96, the number of the polygon's sides, and the perimeter of that same polygon will come to 6336. And therefore it will have a greater ratio

— that same perimeter of the polygon, I mean — to the diameter *a c* than 6336 has to 2017¼, by proposition 8 of the fifth book. Now 6336 contains three times 2017¼, which is 6051¾, and there remain over 284¼, which is more than ten seventy-first parts; for ten seventy-first parts come to 284 17/42 [?]. And therefore much more will the perimeter of the polygon have to the diameter a ratio greater than triple-and-ten-seventy-firsts [*tripla super decupartiente septuagesimas primas*]. But the circumference of the circle is greater still than the perimeter of the polygon; therefore much more still does the circumference have to the diameter a ratio greater than triple-and-ten-seventy-firsts — which was to be shown.

But since one eighth is less than ten seventy-firsts, Archimedes infers from this that the circumference has to the diameter a ratio less than triple-and-a-seventh [*tripla sesquiseptima*], but greater than triple-and-an-eighth [*tripla sesquioctava*].

Oronce Fine, however, when he had described in a circle a polygon of 384 equal sides, concluded against Archimedes, by numbers drawn from the table of right sines, that the ratio of the circumference to the diameter is *less* than triple-and-ten-seventy-firsts. On this point this mathematician of the Paris academy would have to be questioned all over again. Let him consider whether his conclusion is true or not. If he concluded what is true, why then did he assert that the ratio of the circumference to the diameter is greater than triple-and-a-seventh? For ten seventy-first parts are less than a seventh part. But if what he concluded is false, what need was there of that false argument — especially since it is not Archimedes'? Or how will it be superior to Archimedes' demonstration, as he claims it is? Moreover, even granting that the perimeter of that polygon of 384 equal sides contained the diameter three times over, plus a part less than ten

seventy-firsts, it was not on that account to be inferred that the circumference of the circle contains the diameter three times over and less than ten seventy-firsts; for the circumference of the circle is greater than the perimeter of the polygon — not equal to it, and not less. And of unequal magnitudes, the greater has to one and the same magnitude a greater ratio than the lesser has, by *Elements* V.8. And therefore he concludes, like a man untaught, that the ratio of the circumference to the diameter is less than triple-and-ten-seventy-firsts.

That Oronce Fine, in the Protomathesis, *did not correctly hand down Archimedes' finding concerning the ratio of the circumference to the diameter.*

CAP. XII. Reprehensio. IX.

For when, in that work of his which he called the *Protomathesis*, he had undertaken to demonstrate, in Archimedes' commonly received manner, that the ratio of the circumference to the diameter is less than triple-and-a-seventh, he went wrong in his reasoning, because he supposed it made no difference if, in place of the true and exact roots, slightly greater ones were taken. Thus at the first bisection of the angle he took 42 19/42 for the square root of the number 1802, although the exact root is a little less than that 42 19/42. Now he had shown that the square on the line which bisects the angle at the centre and subtends the right angle has to the square on the tangent line a greater ratio than 1802 to 121; and from this he concluded that the ratio of side to side is greater than 42 19/42 to 11 — but with little skill. For the ratio of those sides will indeed be greater than the ratio which the exact root of the number 1802 has to 11, and consequently greater than the ratio which any number smaller than that same exact root has to 11. But it does not follow from these that it is also greater than the ratio which 42 19/42 has to 11, nor is that established on any other ground. And this is precisely why Archimedes, in gathering that the ratio of the circumference to the diameter is less than triple-and-a-seventh, always takes numbers smaller than the exact roots — just as, to show that this same ratio is greater than triple-and-ten-seventy-firsts, he always takes numbers greater than the exact roots.

Oronce Fine committed the same error at the third bisection of the angle, since he took 169 for the square root of the number 28552, although the exact root of that same number is a little less than 169. His other mistakes, so far as they bear on this demonstration, are slighter — yet they are the mistakes of a man who seems not to know the definitions set down at the beginning of Euclid's books. For he thinks that the ratio two lines have in length is the same as the ratio they have in power, a point he hammers home again and again. And it is roughly of this same

kind, what he wove into the second part of the demonstration in order to conclude, along with Archimedes, that the ratio

of the circumference to the diameter is greater than triple-and-ten-seventy-firsts:

> *b m* has to *m d* a smaller ratio (he says) than 458½ to 15. Therefore, taken together — by proposition 18 of the fifth book — *b m* and *m d* will accordingly observe to *d m* itself a smaller ratio than 458½ and 15 together observe to that same number 15. And again the squares on *b m* and *m d* will correspondingly have to the square on *d m* a smaller ratio than 458½ has to 15; for the ratio of the squares is the same as the ratio of the sides themselves. But the square produced from *b d* is equal to the two squares on *b m* and *m d*, by proposition 47 of the first book. Therefore the square on *b d* will likewise obtain to the square on *d m* a smaller ratio than 458½ to 15; and consequently the straight line *b d* will in length keep to *d m* a smaller ratio than that same number 458½ keeps to the said number 15. And conversely, in the end, *m d* will have to *b d* a greater ratio than 15 to 458½.

So Oronce Fine. But it is worth seeing how little he demonstrates, and what falsehoods he presses upon us. The right-angled triangle *b m d* in Oronce Fine's figure is like *a l c* in ours: for *b d* is the diameter of the circle, the straight line *d m* subtends the ninety-sixth part of the circumference, and the angle at *m* is a right angle. Let us suppose, then, that it is as he inferred from what had gone before, namely that *b m* has to *m d* a smaller ratio than 458½ to 15. It is then rightly inferred, by composition of the ratio, that *b m* and *m d* together have to *d m* a smaller ratio than 458½ and 15 together have to that same number 15. But from these he wrongly gathers that the squares on *b m* and *m d* in turn have to the square on *d m* a smaller ratio than 458½ to 15 — or even (if he prefers) than 458½ together with 15 has to 15, which would seem the more consistent claim. For two and one have to one a smaller ratio than nine and three have to three; and yet their squares, those of 2 and of 1, that is 5, have a greater ratio to 1 than 9 and 3 have to 3, and a much greater one still than 9 has to 3.

There are countless numerical examples by which reasoning of this sort will be overthrown. As for what he brings forward in proof — that the ratio of the squares will be the same as the ratio of the sides themselves — it is manifest from Euclid's sixth book that this is false. For it is not the same, but double that of the sides [*dupla*, i.e. the duplicate ratio]. Therefore, although (as he goes on to say) the square on *b d* is equal to the two squares on *b m* and *m d*, it does not follow that the square on *b d* has to the square on *d m* a smaller ratio than 458½ to 15; nor (in

the way he concludes) that the straight line *b d* has to the straight line *d m* a smaller ratio likewise than 488½ [recte: 458½] to 15. And this holds no better than if, *b m* being set at 4 and *m d* at 3, you should wish — because *b m* has to *m d* a smaller ratio than 15 to 10 — to prove by a like syllogism that *b d* has to that same *m d* a smaller ratio than 15 to 10, which is plainly false, since *b d* is 5.

But now, if we reason more carefully, keeping the earlier hypothesis (for perhaps his book is corrupt, and so that we may not always appear to be attacking Oronce Fine but sometimes to be helping him), even so it will not follow that the ratio of the polygon's perimeter to the diameter is greater than triple-and-ten-seventy-firsts. Suppose that, following his own procedure, we argue thus: *b m* has to *m d* a smaller ratio than 458½ to 15; therefore the square made from *b m* will have to the square made from *m d* a smaller ratio than the square of 458½ has to the square of the number 15. Accordingly, taken together, the squares made from *b m* and *m d* will have to the square on *m d* a smaller ratio than the squares of the numbers 458½ and 15 have to the square of that same number 15. Now the squares on *b m* and *m d* are equal to the square on *b d*, by proposition 47 of the first book; and the squares of 458½ and of 15 in turn — namely 210222¼ and 225 — make up 210447¼. Therefore the square on *b d* has to the square on *m d* a smaller ratio than 210447¼ to 225. And so the straight line *b d* will have to the straight line *d m* a smaller ratio than the square root of that 210447¼ has to 5 [?]; and conversely *d m* will keep to *b d* a greater ratio than 15 keeps to the square root of the same

of that same 210447¼.

And since the straight line *d m* is the side of a 96-sided equilateral polygon inscribed in that same circle, and 15 multiplied by 96 gives 1440, the perimeter of the polygon therefore has to the diameter *b d* a greater ratio than the number 1440 has to the square root of that 210447¼. But the ratio of these two is less than triple-and-ten-seventy-firsts; so even reasoning in this way, Oronce Fine could not conclude that the perimeter of the polygon has to the diameter a ratio greater than triple-and-ten-seventy-firsts.

We shall now show, in this way, that the number 1440 has to the square root of 210447¼ a ratio less than triple-and-ten-seventy-firsts. The squares of the two numbers 223 and 71 are 49729 and 5041; the square of the number 1440 is 2073600. Now 49729 has to 5041 approximately the ratio that 2073600 has to 210200. But this fourth proportional is less than 210447¼, and therefore 2073600 will have a greater ratio to 210200 than the same number 2073600 has to 210447¼; and consequently 49729 will have a greater ratio to 5041 than 2073600

has to 210447¼. The square root of the number 49729 — namely 223 — must therefore have a greater ratio to 71, the square root of the number 5041, than 1440, the square root of the number 2073600, has to the square root of that 210447¼. But 223 has to 71 the ratio triple-and-ten-seventy-firsts. Therefore 1440 will have to the square root of that same 210447¼ a ratio less than triple-and-ten-seventy-firsts, which was to be shown. And so it is clear that Oronce Fine did not report correctly, in the *Protomathesis*, Archimedes' finding about the ratio of the circumference to the diameter.

¶ *That another quadrature of the circle, devised by Oronce Fine and set out by him in the* Protomathesis*, is false.*

CAP. XIII.

Reprehensio. X.

Oronce Fine devised another method of squaring the circle, which he set out in the *Protomathesis*, and which I shall reproduce here word for word.

> Let a circle *b c d* be described (he says) about the centre *a*, its diameter being *b d*. Within it let a square *e f* be described, by *Elements* IV.6 and IV.7; and about the same circle *b c d* let a square *b g d* be circumscribed. Then, from the angle *e* of the inscribed square to the angle *g* of the circumscribed square, let a straight line be drawn by the first postulate, cutting the diameter *b d* at the point *h*, and the circle *b c d* at the point *i*. Next, from a given straight line double *a h*, let a further square *h l m* be described through the given point *h*, by *Elements* I.46, parallel both to the inscribed square *e f* and to the circumscribed square *b g d*. The square *h l m* will then be the mean proportional between the squares *e f* and *b g d*; for it is taken between the two squares by the intersection of the diameter with the line equidistant from the sides of each square — just as, in the common planisphere, following Ptolemy's own demonstration, we habitually describe a mean proportional between two given circles by similar intersections of the diametral and meridian lines. For given two magnitudes it is possible to assign a third proportional,
>
> by *Elements* VI.13. Consequently, from the point *i* let a straight line *i l* be drawn to the point *l*, by the same first postulate, cutting the same diameter *b d* at the point *n*. And with centre *a* and radius *a n* let a circle *n o* be described, by the third postulate. The circle *n o* will then be the third magnitude proportional in order after the square *b g d* and the inscribed circle *b c d*; for it is derived from the square *b g d*, from the circle *b c d*,

and from *e f*, the square which is the mean proportional between the squares *e f* and *b g d*, by the intersection of the diameter *b d*. For given two magnitudes it is possible to find a third proportional, by *Elements* VI.11. The circle *b c d*, therefore, is the mean proportional between the square *b g d* and the circle *n o*. Finally, about this circle *n o* let a square *n o p* be circumscribed, by *Elements* IV.7. Since, then, by *Elements* XII.2, circles stand to one another as the squares on their diameters, the square *b g d* is to the square *n o p* as the circle *b c d* is to the circle *n o*. And so, alternately, as the square *b g d* is to the circle *b c d*, so is the square *n o p* to the circle *n o*, by *Elements* V.18. The circle *b c d* and the square *n o p*, therefore, standing between the same square *b g d* and circle *n o*, are proportional, and for that reason equal to one another. The same conclusion (he adds) may also be reached in another way: the circle *a b c* and the square *n o p* have the same ratio to the same circle *n o* — namely the ratio that the square *b g d* has to the circle *b c d* — and things which have the same ratio to the same thing are equal to one another, by *Elements* V.9; therefore the circle *b c d* and the square *n o p* are equal to one another. Given the circle *b c d*, then, the square *n o p* equal to it is given.

He further adds that if anyone should say that some rectilinear figure, rather than the circle *n o*, is the third proportional after the square *b g d* and the circle *b c d*, the proposition will follow all the same. For a given figure can be reduced to a square, by the last proposition of *Elements* II; let that square be *R S*. Since, then, the square *d b g* is the greater extreme, it will be greater than the square *R S*, and consequently its side greater than the other's side. Let *g t* and *v x* be cut off equal to the sides of that square *R S*, and let *t x* be joined, by the first postulate. The rectangle *g x* will then be the mean proportional between the square *b g d* and the square *R S*, since it is formed from the sides of those squares. But the circle *b c d* is the mean proportional between the square *b g d* and the said square *R S*. Therefore the circle *b c d* and the rectangle *g x* are equal to one another. Given the rectangle *g x*, then, let a square be constructed equal to it, by the last proposition of *Elements* II; and let it again be *n o p*. So, the circle *b c d* being proposed, a square equal to it is described — which was what had to be done.

Again, if some captious person, or one so far unlearned, should deny that the square *h l m* — from which the square *n o p* is proportionally derived — is the mean proportional between the two squares, one of which is described within the circle *b c d*, as *e f*, while the other is circumscribed

> about the same circle: I shall give him a rectilinear figure, namely an octagon described within that same circle *b c d*, which I shall prove to be the mean proportional between those squares; then I shall turn that octagon into a square, by the last proposition of *Elements* II, and complete the rest as in the preceding demonstration.

So far Oronce Fine.

This, then, is how he supposed he had found and demonstrated the quadrature of the circle. And yet a captious and unlearned person will rightly deny that the square *h l m* is the mean proportional between the two squares *e f* and *b g d*. For if those three squares *b g d*, *h l m*, and *f e* are proportional, their sides will be proportional too, since the ratio of the squares is double that of the sides, by *Elements* VI.20; and therefore the halves of those sides will likewise be proportional. Now let the straight line *a b* cut the side of the square *e f* at *s*. Then, as *b g* is to *h l*, so is *h l* to *s e*. But the lines drawn from the centre *a* are equal to these, namely *a b*, *a h*, and *a s*; therefore as *a b* is to *a h*, so is *a h* to *a s*. And therefore, separando, by *Elements* V.17, as *b h* is to *h a*, so is *h s* to *s a*. But as *h s* is to *s a*, so is *h s* to *s e*, by *Elements* V.7; therefore as *b h* is to *h a*, so is *h s* to *s e*, by *Elements* V.11. Now the two right-angled triangles *h e s* and *h g b* are equiangular, by *Elements* I.32, because the vertically opposite angles at *h* are equal; therefore as *h s* is to *s e*, so is *b h* to *b g*, by *Elements* VI.4. And therefore as *b h* is to *h a*, so is *b h* to *b g*, by the same *Elements* V.11. Therefore *h a* and *b g* are equal to one another, by *Elements* V.9. But *a b* is equal to *b g*; and so the straight line *h a* will be equal to the straight line *a b* by the common notion — the part equal to the whole, which is impossible.

The impossibility will also be shown in another way, by *Elements* V.19, if those three squares are granted to be proportional. For as the whole *b a* is to the whole *h a*, so will *h a*, the part taken away, be to *s a*, the part taken away; and therefore as the whole *b a* is to the whole *h a*, so is the remainder *b h* to the remainder *h s*. But as *b h*

is to *h s*, so is *b g* to *s e*, because the triangles *b g h* and *s e h* are similar. Therefore as *b a* is to *h a*, so is *g b* to *s e*, by *Elements* V.11. But *g b* is equal to *b a*, and *s e* to *s a*; therefore as *b a* is to *h a*, so is *b a* to *s a*, and consequently the straight line *s a* will be equal to *h a*, by *Elements* V.9 — the part equal to the whole, which is impossible. And so the square *h l m* is not the mean proportional between the squares *e f* and *b g d*, as Oronce Fine asserted it to be. He therefore either failed to understand Ptolemy's demonstration or applied it wrongly.

Now suppose some other square is found that is the mean proportional between the same squares *e f* and *b g d* — such as the square described on the straight line that is the mean proportional between the sides of those squares — and suppose it is set parallel to them: then one of its sides will cut the straight line *a b* either before *h* or after *h*; and since two straight lines do not enclose a surface, the line drawn from *e* through the point of section will not run straight to *g*. Oronce Fine will therefore not derive the circle *n o* as the third magnitude proportional after the square *b g d* and the inscribed circle *b c d*, out of the two squares *b g d* and *e f* and the circle *b c d*, as his plan required.

And finally, however that might be managed — even if it were conceded that the square *h l m* is the mean proportional between the extreme squares *b g d* and *e f* — he still does not prove that the circle *b c d* is the mean proportional between the square *b g d* and the circle *n o*. He cites *Elements* VI.11, but beside the point: in that proposition Euclid teaches only how a third proportional is to be found to two given straight lines. Nor does he resolve the objection he himself raised — that some rectilinear figure, rather than the circle *n o*, might be said to be the third proportional after the square *b g d* and the circle *b c d*, on the ground that a given figure could be reduced to a square. We have no doubt about how any rectilinear figure whatever is to be reduced to a square; what we lack is the art of finding a rectilinear figure that is the third proportional after the square and the circle. His solution, then, would prove — if it proved anything at all — only that the quadrature of the circle is possible. But finding a square equal to a given circle, as he had undertaken to do, is another matter altogether. And so the quadrature of the circle set out by Oronce Fine in the *Protomathesis* is false, which was what we had to show.

¶ *That Oronce Fine's finding concerning the description of rectilinear figures is sheer delusion.*

CAP. XIIII.

Reprehensio. XI.

Oronce Fine composed an entire book on the complete description of all rectilinear and many-angled figures — those called regular — both within and outside a given circle, and upon any straight line offered. But since he leaned on false foundations, and laid down empty and deceptive ones, whatever he built on them must collapse. I shall set down word for word the second problem of that book, on which all the rest depend, so that I may not seem to have altered anything by reporting his writings in some other way.

Problem 2, of the book On the Description of Many-angled Figures.

> Given an isosceles triangle each of whose base angles is double the remaining angle, to construct the other isosceles triangles in which each of the base angles keeps to the remaining angle the ratio that a given number keeps to unity; and at the same time to make known the side of the many-angled figure that will be described by means of the isosceles triangle offered.
>
> ¶ Let there be given an isosceles triangle *a b c*, each of whose angles at the base *b c* is double the remaining angle at *a*, by *Elements* IV.10 of the geometrical elements; and let the base *b c* of that triangle *a b c* be moreover the side of a pentagon described in the circle circumscribed about that same triangle, by *Elements* IV.11. From this isosceles triangle *a b c*, then, as from a root and primary figure, we shall deduce and generate the remaining isosceles triangles, in each of which the base angles will keep to the remaining angle the other multiple ratios — triple, quadruple, quintuple, sextuple, and so on in order. Their bases, moreover, will fix in due order the sides of the other many-angled and regular figures described in the circles that are circumscribed about those same triangles. That no one up to now has either done this or

Diagram. The figure of Fine's quadrature, occupying the foot of the left column. A large circle stands inside a circumscribed square, with a second, slightly smaller square set concentrically inside the first, and a further square inscribed in the circle; a second, smaller circle is drawn concentric with the first, low in the figure. Straight lines run from the upper-left corner of the outer square down across the figure, and a vertical diameter runs from the top of the circle to its foot, crossed by horizontal lines at several of the square sides. The interior of the figure is disturbed by show-through from the reverse of the leaf, which obscures part of the lettering.

Labels read on the leaf, with position:

- *g* (upper-left corner of the outer square): the angle of the circumscribed square
- *b* (top, on the vertical diameter): the upper end of the diameter
- *h* (on the vertical diameter, below *b*): the point where the drawn line cuts the diameter
- *s* (on the vertical diameter, below *h*): the section on the side of the inscribed square

- *c* (left, on the circle): the left point of the circle *b c d*
- *a* (centre of the figure): the centre
- *d* (foot of the figure, on the vertical diameter): the lower end of the diameter
- *f* (lower left, inside the figure): a corner of the inscribed square
- [?] (upper right, on the circle): a single letter, not resolved
- [?] (right edge, mid-height): a single letter, not resolved
- [?] (upper middle, right of the diameter): a single letter, not resolved
- [?] (lower left, below *f*): a single letter, not resolved
- [?] (bottom centre, above *d*): two adjacent letters, not resolved

Diagram. The figure for the problem, at the foot of the right column: a circle with a triangle inscribed in it, its apex above and its base a chord low in the circle, with two further straight lines drawn from the apex down to the base, so that the shaded principal triangle is flanked by narrower ones.

Labels read on the leaf, with position:

- *g* (top, just inside the circle)
- *d* (interior, above the apex of the triangle)
- *a* (apex of the inscribed triangle)
- *b* (lower left, where the base meets the circle)
- *c* (lower right, where the base meets the circle)
- [?] (on the base, a little to the right of *b*): a single letter, not resolved

> …has either done or even thought out, while very many have both proposed it and often longed for it — this we have found to be the case.
>
> ¶ First of all, then (to come to the matter itself), let the base *b c* of that isosceles triangle *a b c* be divided into seven mutually equal parts, by the preceding first problem. Then, leaving one seventh part at each of the two ends of *b c*, let the five remaining intermediate parts be substituted as the base of a second isosceles triangle whose two sides are equal to the sides *a b* and *a c*; and let a triangle of this kind be *d e f*, its base being *e f*, of the five parts just described. I say, then, first, that the angle *e d f*, contained by the equal sides of the triangle *d e f*, subtends the side of the equilateral and equiangular heptagon described within the circle circumscribed about that same triangle *d e f*; and further, that each of the angles standing at the base *e f* will be triple the remaining angle contained by those same mutually equal sides.

For when two triangles have two sides equal to two sides, each to each, and the angles contained by the equal sides equal to one another, they also have base equal to base, by *Elements* I.4. Again, when two triangles have two sides equal to two sides, each to each, and base equal to base, they conversely have the angles contained by the equal sides equal to one another, by *Elements* I.8. Moreover, whenever two triangles have two sides equal to two sides, each to each, but the one angle contained by the equal sides greater than the other, the base of the first is correspondingly greater than the base of the second, by *Elements* I.24. Likewise, if those same two triangles have two sides alternately equal to two sides, but the one base greater than the other, then conversely they will have the angle contained by the equal sides greater, by *Elements* I.25.

From all this it is not hard to gather that in any two triangles having two sides equal to two sides, each to each, the magnitude of the angles contained by the equal sides follows proportionally upon the given magnitude of the bases — that is, the angles copy the proportion of the bases, and conversely. Since, then, the isosceles triangles *a b c* and *d e f* have two sides equal to two sides in every respect, while their bases *b c* and *e f* are unequal to one another, if the angle of the one triangle contained by the equal sides takes its denomination from the base of the other, the remaining angle must correspondingly take its denomination from the remaining base. For the one of the two unequal bases increases the angle it subtends by exactly as much as the other diminishes it, since the sides are by hypothesis kept equal to one another.

Now the angle *b a c* subtends the base *b c* of 7 parts, which is the side of the equilateral and equiangular pentagon — a figure named from the quinary number of the parts of the base *e f* — described in that circle which is circumscribed about the triangle *a b c* itself, by *Elements* IV.11. The angle *e d f*, then, subtends conversely the side of the equilateral and equiangular heptagon, which is named from the septenary number of the parts of the base [?], and which is described in the circle circumscribed about that same triangle *d e f* — since the base *e f* is of 5 such parts as *b c* is 7. For, of such parts as the circumference of the circle is 35, the side of the inscribed regular pentagon subtends 7, while the side of the heptagon subtends 5; for five times 7, or seven times 5, make 35. The base *e f*, then, of the isosceles triangle *d e f*, is the side of the equilateral and equiangular heptagon which is described in the circle circumscribed about that same triangle *d e f*.

> That each of the angles standing at the base *e f* of the isosceles triangle *d e f* is triple the remaining angle contained under *e d f* becomes clear of itself. For since the angle *e d f* subtends the base *e f*, which is the side of the equilateral and equiangular heptagon described in the circle circumscribed about the triangle *d e f*, it therefore subtends a seventh part of the circumference of that circumscribed circle. The remaining two angles *d e f* and *d f e*, which stand at the base *e f*, will accordingly claim for themselves the remaining six sevenths; and since these are equal to one another, by *Elements* I.5, each of those equal angles will subtend three sevenths of that circumference. Each of them, therefore, will be triple the remaining angle contained by the equal sides — as may be seen from the preceding figure.
>
> ¶ Likewise, if the said base *b c* of the same isosceles triangle *a b c* is divided by the preceding problem into nine mutually equal parts. And so on.

Oronce supposes, then, that in those two isosceles triangles *a b c* and *d e f*, because the two equal sides *a b* and *a c* are equal to the two equal sides *d e* and *d f*, angle *b a c* will for that reason have

the same ratio to angle *e d f* as base *b c* has to base *e f*. And therefore, if base *b c* is 7 such parts, of which its part *e f* is 5, and if the angle *b a c* is a fifth of two right angles, it follows that angle *e d f* is a seventh of two right angles, and that each of the angles *d f e* and *f e d* is triple that same angle *e d f*. All of this I shall show to be utterly false, as briefly and as plainly as possible.

Let a circle *b c g* be described about the centre *a* with the radius *a b* or *a c*. In it let *c K* be fitted equal to the straight line *e f*, which is less than the diameter, by *Elements* IV.1; and let *a K* be joined.

Diagram. A circle carrying the construction of the refutation, set in the left-hand column. Labels legible on the leaf:

- g (upper part of the circumference)
- d (inside the circle, above the centre)
- a (the centre)
- b (on the circumference, lower left)
- c (on the circumference, lower right)
- K (on the circumference, right-hand side)

From the centre two straight lines run down to *b* and to *c*, and *b* and *c* are joined by a chord; a further straight line runs from the centre across to *K*. A second, narrower pair of lines descends from the same point toward the base, but the letters marking its ends are lost to show-through from the reverse of the leaf [?].

The triangle *a c K* is therefore isosceles, and its two sides *a c* and *a K* are equal to the two sides *d e* and *d f* of the triangle *d e f*. The angle *e d f* of triangle *d e f* is consequently equal to the angle *c a K* of triangle *a c K*, by *Elements* I.8. Angle *b a c*, then, has the same ratio to each of the two angles *e d f* and *c a K*.

But as angle *b a c* is to angle *c a K*, so is the circumference *b c* to the circumference *c K*, by the last proposition of *Elements* VI. Therefore, as angle *b a c* is to angle *e d f*, so is the circumference *b c* to the circumference *c K*.

Yet by what Ptolemy demonstrated in the first book of the *Great Construction* [*Almagest*], chapter nine, the circumference *b c* has a greater ratio to the circumference *c K* than the straight line *b c* has to the straight line *c K*. Angle *b a c*, therefore, will have a greater ratio to angle *e d f* than the straight line *b c* has to the straight line *c K*, by *Elements* V.13 [?]. But as *b c* is to *c K*, so is that same *b c* to *e f*, by *Elements* V.7 — for *c K* and *e f* are equal. Angle *b a c*, then, will have a greater ratio to angle *e d f* than the straight line *b c* has to the straight line *e f*. Oronce is therefore mistaken, and the descriptions of many-angled figures that he has brought forward are false.

For the angle *e d f* does not in the least subtend the side of the equilateral and equiangular heptagon; nor will the angle *d f e* or *f e d* be triple the angle *e d f*. And this demonstration of ours proves, quite generally, that everything else that follows — about the description of the nonagon and of the other figures, and about the division of angles, right to the end of his book — is false.

Oronce was taken in by a very flimsy argument. For although, when two sides of one triangle are equal to two sides of another, it is true that if the one angle is equal to the other the base is equal to the base, and that if the one angle is greater than the other the base is greater than the base, and that if the one angle is less than the other the base is less than the base, it still does not follow that the angles and the bases are proportional.

It is just as with two squares: if side is equal to side, square is equal to square; but if one side is greater than the other, the square must be greater than the square, and if one side is less than the other, the square too is less than the square. Yet the squares and the sides are not proportional; rather, the ratio of the squares is always the duplicate of the ratio of the sides.

Again, if the denomination of one of two ratios is equal to the denomination of the other, the ratios themselves will be equal; if one denomination is greater than the other, the one ratio will also be greater than the other; but if one denomination is less than the other, the one ratio too will be less than the other. It is nevertheless not necessary that the ratios and the denominations be proportional. For the denomination of a sextuple ratio is 6, and that of a triple ratio 3, so that the one denomination is double the other; but a sextuple ratio is not double a triple ratio.

By these and many other examples, then, he might have been torn away from that error, since no demonstration came to his aid. And a certain master of Alcalá [Complutensis] fell into the same error, who wished to appear ingenious in his work on the geometry of Thomas Brauardinus [Thomas Bradwardine].

For if in those two isosceles triangles the ratio of the angles contained by the equal sides were one and the same as the ratio of the bases, then that table of arc and chord which Ptolemy put together in the *Great Construction* with so many syllogisms and so much labour could certainly be constructed with the least possible trouble. Suppose, for example, that it were worth knowing how many parts that straight line is which subtends a fifth part of the circumference, that is 72 degrees, in parts of which the diameter is 120. Since the radius of the circle subtends a sixth part, and the sixth has to the fifth the ratio of 5 to 6, and the ratio of the circumferences is the ratio of the angles, the straight line subtending a fifth [?] part of the whole circumference would then be of 72 parts. Accordingly the side of the decagon, since it subtends half that angle, would be 36; the straight line subtending a quadrant of the circumference would be 90; the one subtending a single degree would be 1; and so on through the very same numbers as the parts of the circumference divided into 360. And so none of those lines which Ptolemy computed would come out irrational. But that is not how things stand.

That Oronce erred grievously in investigating the longitude of places, through ignorance of the first rudiments of astronomy

CAP. XV.

Reprehensio. XII.

There is no need to set out at length Oronce's method for finding the longitudes of places, for it is nearly the one that Ioannes Vernerus of Nuremberg had already handed down in his notes on Ptolemy's *Geography*, and after him Petrus

Appianus — namely, by an observed position of the moon, though only in outline. Ioannes Vernerus, however, handled the matter more simply.

Oronce teaches first how the numbers of the lunar motion are to be drawn from the ordinary almanac [*diarium*], and how a table is to be constructed by which one may easily learn, for each single day, at what hour and minute the moon will reach the meridian of the reference place [*locus radicalis*], and what its true motion is at that moment.

He teaches besides how the instrument of Ptolemy's rulers [*regulae Ptolemaei*] is to be constructed — an instrument that must be kept ready to hand, together with some clock of moving wheels and an ordinary sphere, either solid or built up of rings. Then, when the moon has reached the meridian of a place whose longitude is unknown, one is to set beneath the meridian of the globe, by the time elapsed since noon (which the clock will show), the degree of the ecliptic that comes to the meridian together with the moon. At that moment the moon's altitude above the horizon is to be found by Ptolemy's rulers themselves and reckoned off on the globe's meridian; and through its end [?] a semicircle is to be drawn from the poles of the ecliptic, which will show the moon's place in the ecliptic. With these preparations made, so that the difference of longitude between the given place and the reference place may be found, he adds a certain rule and precept, which I shall set down word for word.

> ¶ You are to observe (he says) that the moon reaches the meridian of an eastern place sooner, with respect to the reference place, and therefore under a larger reckoning of time, than it reaches the meridian of the reference place itself; but that it reaches the meridian of a western place later, and under a smaller number of time, that is, of hours and minutes. For in eastern places the stars
>
> rise above the horizon sooner than in western ones. With the moon's true motion, however, which runs from west through mid-heaven toward the east, the case is otherwise: in eastern places that motion will always be less than in western ones. For in the interval during which the moon is carried by the motion of the universe from the eastern to the western meridian, it travels some distance of the zodiac's longitude in the contrary direction by its own proper movement, and by that amount its true motion is increased.
>
> The place, then, to whose meridian the moon is found to have come under a larger number of hours and minutes and with a smaller motion than to the reference meridian, will be further east than the reference

> place itself. But if this happens under a smaller reckoning of those same hours and minutes, yet with a larger motion of the moon, that same place will be further west than the reference place.
>
> By what difference the given place is further east or further west than the reference place you will find in this way. If the given place proves to be further east than the reference place, the time of the moon's arrival at the meridian of the reference place is to be subtracted from the time of the same moon's arrival at the meridian of the given place; while the moon's true motion found under the meridian of the given place at the time of that arrival is to be taken from the true motion of the same moon which you determined while the moon was being brought to the reference meridian. What is left will be the difference of time, and the difference of the moon's own true motion caught between the two observations. Next you are to convert that difference of time into degrees of the equator in the usual way; and for the difference of the true lunar motion you will compute the right ascension, which you will subtract from that difference of time. What is left at last will be the proposed difference of longitude — that, namely, by which the given place lies further east than the reference place.
>
> But if the given place is further west than that same reference place, you will observe throughout the contrary procedure. You will subtract the time of the moon's arrival at the meridian of the given place from the time at which the moon was brought to the reference meridian, and the moon's true motion occurring under the reference meridian from the motion found at the time of that same moon's arrival at the meridian of the given place. And by means of these differences you will collect the difference of longitude itself.

So Oronce has laid out his art of finding the differences of longitude of places, and he explains it by two examples. In them he sets the meridian of Paris as the reference meridian, with which two other meridians are to be compared — one further east than the reference meridian, the other further west than it — so that finally the differ-

so that finally the difference may be determined.

In his first example, then, Oronce puts the Moon as having reached the meridian of some given place at the 14th hour and 17 minutes after noon on 15 November, and as having been found at that moment — by the instrument of rules and by the sphere, in the manner set out above — in the 20th degree of Cancer together

with [numeral: ?] minutes. In the second example he supposes the Moon to have reached the meridian of some given place at the 13th hour, 17 minutes after noon of the same 15 November, and to have occupied at that moment the 20th degree of Cancer and 55 minutes of it. In both examples alike he lays down that the Moon reached the reference meridian, that of Paris, at the 13th hour and about the 47th minute after noon of that same fifteenth day of November, and that it then held the 20th degree of the same Cancer with 40 minutes. And so, once the computation is made, he concludes that the first meridian lies 7 degrees 14 minutes east of the meridian of Paris, and that the second lies 6 degrees 59 minutes west of that same meridian of Paris.

Someone might perhaps suspect that lunar motions found with Ptolemy's rules and with the sphere, in the way he has taught, are not true but only apparent, because of parallax. To remove that ambiguity he speaks as follows:

> Finally it must be noted that, while the Moon is placed under the meridian itself, its visible place — that is, the place observed by the visual ray through *a d* [?], the rule — marks out at the same time the true place of that same Moon in the sky, because there is then no parallax, that is, no difference between the true place and the visible place along the longitude of the zodiac itself.

I have set out Oronce's procedure for finding the differences of meridians at somewhat greater length than I had intended, but I judged that nothing shorter would do, so that the refutations and rebuttals of it should be plainly grasped by anyone at all, even those with little practice in astronomy.

Oronce is in error — or rather out of his mind — because he thinks the Moon reaches the meridian of eastern places under a larger number of hours and minutes than that of western ones. Let us suppose, so that beginners may take this in more easily, that the Sun occupies the first point of Capricorn and is below the earth on the meridian of some place — at midnight, that is, the twelfth hour after noon — while the Moon holds the opposite point, the first point of Cancer, and is on the meridian above the earth. And let us understand the Moon then to be carried westward by the diurnal motion along with the other spheres. Because at the same time it moves in the contrary direction, eastward in the order of the signs, it will therefore reach the meridian of a second, more westerly place later than the first point of Cancer itself does. This lag will be that small portion of time, or arc of the equator, with which the part of the zodiac the Moon has meanwhile traversed rises in the right sphere — just as we are used to understanding the matter in the case of the inequality of the natural days.

So as soon as the first point of Cancer reaches the meridian of the second place, it will beyond doubt be midnight at that second place, which lies west of the first — that is, the 12th hour after noon — since the Sun will then be at the opposite point of that same meridian, below the earth. But for the Moon to reach that same meridian of the second place, the small portion of time just described has to be added. This example makes it plain, then, that the Moon reaches the meridian of a *western* place not under a smaller but under a larger number of hours and minutes than that of an eastern one; and conversely, that it reaches the meridian of a more easterly place under a smaller measure of time.

The fixed stars, by contrast, move so very slowly that at whatever reckoning of time they reach one meridian on a given day, at that same reckoning they will reach all the rest on that same day. For example, if in this present year 1546, on 12 or 13 January, Cor Leonis reaches the meridian of the people of Coimbra at the 13th hour after noon, it will also reach the meridian of the Parisians, who lie further east, on that same day and under the same measure of time, 13 hours after noon; and likewise the meridian of the Fortunate Isles, which lie further west, and all the remaining meridians of the whole world — though it does reach the meridians of eastern places sooner than those of western ones. Nor need we press here a scrupulous accounting of the minutes traversed meanwhile by the Sun's own proper motion, which might seem to take something away from a calculation of this kind: we reckon that as imperceptible.

If the stars did come to different meridians under a varying measure or naming of the time, nothing whatever would be easier than to measure the differences of longitude of places on any day one pleased. The difference of those unequal times would likewise be the difference of longitude of the places. But that is not how it stands: for while the fixed stars are moving, the Sun will not remain in its motion [?]. On what reasoning, then, Oronce flatly asserts the contrary of this I do not understand. I only see that he is caught up in a great error and delusion — one from which even people who have merely tasted the first introductions to astronomy for three or four days could extricate themselves.

Again, he is in another great error, because he

thinks that the Moon's true place in the zodiac differs in no way from the seen or apparent place whenever the Moon stands on the meridian, and that it then has no parallax in ecliptic longitude. Here again he shows himself utterly ignorant of astronomy. For the Moon to be free of parallax in longitude, it must be located on a great circle passing through the poles of the ecliptic and of the horizon. But the meridian never passes through the poles of the ecliptic except when the first

points of Cancer and Capricorn are on that meridian. Therefore, for the Moon standing on the meridian to be free of parallax in zodiacal longitude, the first points of Cancer and Capricorn must be on that same meridian.

Consequently, whenever the Moon is on the meridian together with points of the ecliptic other than the first points of Cancer and Capricorn, its true place in zodiacal longitude will be different from the one sight has shown. But the Moon holds the first points of Cancer and Capricorn only twice a month. Only twice a month, then, will the Moon standing on the meridian have a true place in zodiacal longitude that does not differ in the least from the seen place. And it is impossible for this to happen in one month at more than two meridians of the terrestrial globe. So Oronce is in error.

Add that the Moon's seen place on the ecliptic cannot be determined exactly by that technique either. What good is it to have found its distance from the zenith by Ptolemy's rules, down to minutes and seconds, if that distance then has to be counted off and laid down on that foot-and-a-half globe, whose parts cannot be divided into so many small subdivisions? And if you deviate from the correct value by even the smallest amount in so delicate a matter, you will not hit the Moon's seen place in the zodiac, but a different one, greater or smaller by a perceptible difference. For it matters a great deal to which point of the meridian the semicircle drawn through the poles of the ecliptic is to be fitted.

As for the clock of moving wheels, there is much to suspect there, and not without reason. Besides, since Oronce assumes the Sun's [?] to be known — although it must be unknown on a meridian not yet determined — it would be better on that account to find the hour by the altitude of some star whose place is known, as in my book *On Twilights*. The clock would then be superfluous.

And in any case, what need was there of that foot-and-a-half globe for finding the Moon's seen place in the zodiac, which he thinks differs not at all from the true place? For once the altitude of the pole has been determined, and the Moon's distance from the zenith while it stands on the meridian, and once the right ascension is also known of the degree of the ecliptic that is on that meridian together with the Moon, then the Moon's place in the zodiac can be found by problem 55 of the *tabula primi mobilis*, or by some easy geometry of spherical triangles. For when two sides of a triangle are known, together with the angle they contain, the remaining side and the remaining angles become known.

Now how far the visible pole of the world is from the Moon standing on the meridian has been learned from the observation. The distance of that same pole

from the nearer pole of the ecliptic is known besides — about 23½ degrees. And the angle enclosed at the pole of the world itself by these two distances, or arcs of great circles, is known, since it measures the right ascension of an arc of the ecliptic less than a semicircle, intercepted between the first point of Capricorn and the point that reached the meridian together with the Moon. So the base of such a triangle, which is the complement of the Moon's apparent latitude, and the angle at the pole of the ecliptic subtending the Moon's apparent distance from the first point of Cancer measured along zodiacal longitude, will both become known.

He could moreover have found the right ascension of the degree of the ecliptic standing on the meridian together with the Moon without any help from a globe, even with the Sun's place and the time elapsed since noon unknown, by the altitude of some known star — working out numerically the magnitude of the angle at the pole of the world which subtends that star's distance from the zenith. The Moon's seen place would then have been known in the way just described. Again, the seen place could also be found from the Moon's distance from two known stars, as in book VII of the *Epitome* of Ioannes de Monteregio.

Oronce also computed with little skill at what hour the Moon would reach the meridian of the reference place, neglecting the equation of days [*æquatio dierum*], which counts for a great deal in the Moon's case. He was therefore wrong to go on and use the common hours read off that clock of moving wheels.

It is clear, then, that Oronce erred in many ways, and disgracefully, in his investigation of the difference of longitude of places. Many years earlier the learned Ioannes Vernerus also attempted to find the same thing, likewise by the Moon's motion but in a different way, as he left in writing in the annotations he composed on Ptolemy's *Geography*. He directs that at a place of unknown longitude, at a known moment, the Moon's distance be taken with the astronomical staff [*baculus astronomicus*] from some fixed star that departs little or not at all from the ecliptic.

That distance is then to be divided by the Moon's hourly motion, and the time of the Moon's conjunction with that same fixed star will come out. Next, the time of that same conjunction is to be drawn from the tables of the Moon's motion for the meridian of known longitude. Finally, by comparing the two times with each other, the difference of longitude of those places will become known.

The parallax in longitude he calls slight, and so judges that it may be disregarded, or else determined from book V of Ptolemy's *Almagest*: for at once, he says, the true elongation of the Moon and the fixed star is found from that seen distance

between them. But I find this method too not a little deceptive. For if the place where the observation is made has an unknown longitude, the Sun's motion for that place's meridian must be unknown as well. The time of the observation will therefore be unknown, unless it is weighed by clocks of moving wheels or by other devices of the kind.

It is deceptive on a further count: it will sometimes happen that the Moon's distance from the star is not wholly a matter of longitude but of latitude, and the observer's eye cannot always tell the difference — especially when the Moon is near its rising or setting. Nor, on a meridian of unknown longitude, will it be possible to resolve that ambiguity by tables constructed for a meridian of known longitude: for over the intervening time, if the meridians are different, the latitude must vary, and that variation cannot be discerned in any way. And even if it were already established that at the very time of the observation the Moon had latitude, it would still not be permissible to take the Moon's distance from a fixed star lying on the ecliptic, or having latitude of the opposite denomination, as an arc of zodiacal longitude.

The parallax in longitude, which he rates so cheaply, I value more highly than the other objections I have raised. For the Moon's motion in one hour is agreed to be about half a degree. Since, then, the parallax in longitude can amount to a whole degree, if we choose to treat it as negligible we shall sometimes fall into an error of two hours — that is, of 30 degrees — in the very difference of meridians we are seeking.

As for his claim that the true elongation of the Moon and the fixed star is found at once from their seen distance by book V of Ptolemy, we do not deny it, provided that the Moon's distance from the centre of the earth in that position has been ascertained and that the other things Ptolemy assumes for the demonstration are granted. But these are unknown on that meridian of unknown longitude where such an observation is made. And so the Moon's true elongation must be unknown too.

Still, I think the learned Vernerus was not unaware of all this; he merely passed it over and left it to the observer's judgment. The observer has to see how far the fixed star and the Moon stand from the ecliptic, and toward which side. And there will be no difficulty if he weighs this by tables computed for a meridian of known longitude, since the Moon's latitude varies little over a twelve-hour span, which embraces the whole range of longitude. So one should take the Moon's distance from some fixed star having nearly the same latitude and on the same side: the seen interval will then differ from the arc of the seen elongation on the

ecliptic by an imperceptible excess. The time elapsed since noon will be shown by the altitude of that same fixed star, or of some other known one, together with the Sun's place taken from the same tables — and this by calculation, as in *On Twilights*. For the greatest error that can occur in the Sun's place drawn from tables for a meridian of unknown longitude is half a degree; and in the reckoning of the times themselves this does not exceed two minutes of an hour.

The case is different with Oronce's procedure. There, by the time elapsed since noon and the right ascension owed to the Sun's place, that degree of the ecliptic is set under the globe's meridian which reaches the meridian together with the Moon. So if a slip of half a degree occurs in computing the Sun's motion, the error in the right ascension will be about as great, and likewise in the very degree of the ecliptic placed under the meridian, and finally, no less, in the Moon's seen place, if the Moon is seen on the ecliptic. But the Moon traverses half a degree in about one hour; the error in the difference of longitude of places will therefore be about one hour, or 15 degrees.

But let us return to Ioannes Vernerus's procedure. I see nothing left ambiguous in it except the parallax, and that we shall examine by the following technique. We shall suppose the Moon's seen place in zodiacal longitude to be the true one, although it is not. Then, having taken the Moon's altitude with the instrument of rules [*instrumentum regularum*], we shall compute the parallax for the given time of observation from the Moon's own true motion, from its seen distance from the pole of the horizon along the circle of altitude, and from the known altitude of the pole, by book V of Ptolemy['s *Almagest*]

[Ptole]my. We shall then subtract the parallax itself from the Moon's seen place, which we have supposed to be the true one, if the parallax is found to fall between the ascendant degree and the ninetieth; but we shall add it if it falls between the setting degree and that same ninetieth. The Moon's true motion for that same moment of observation will then emerge.

I should not want anyone to be troubled that I have looked for the parallax while treating the Moon's seen place as though it were the true one. For the difference between the true place and the seen place, in zodiacal longitude, cannot be great enough for the Moon's distance from the centre of the world to vary perceptibly: the Moon will hold practically the same position, whether on the eccentric or on the epicycle. And so, if we look for the parallax with respect to the seen place as though to the true one, the same value must necessarily be found.

By this technique, then, handed down by Ioannes Vernerus, the difference of longitude between places can easily be found. Alternatively, let the Moon's place be investigated — either by its seen distance from two known stars, as I mentioned above, or with the armillary instrument [*instrumentum armillarum*] — and, once the parallax has been added or subtracted, its true place will result. Then let there be drawn from the tables, for the meridian of known longitude, the most exact time at which the Moon occupies that same place of the zodiac; for the difference between those two times will also be the interval between the meridians.

One must note, however, that because instruments are unreliable, some error always occurs. An error of a quarter of one degree in the Moon's motion will produce an error of nearly half an hour in the difference of longitude of the places — that is, of 7½ degrees.

For that reason, to measure the difference of longitude of meridians whose interval is not large, some other way would have to be sought. But the most reliable method is Ptolemy's, and modelled on his: by day he caught the Moon's seen place with the armillary instrument by means of the Sun's place, while by night he found the places of the stars by means of the Moon's place. What Oronce Fine devised, by contrast, is false, outrageous, and contrary to the art.

¶ *That Oronce Fine also erred grievously in investigating the longitude and latitude of a place whose travelled distance from the reference place is known together with the angle of position.*

CAP. XVI.

Reprehensio XIII.

Oronce Fine thinks it no small matter that with the common astrolabe he can investigate the intervals between two places, together with the angle of position, whenever their latitudes and their difference of longitude are known; and conversely, that from the known latitude of one place, its travelled distance from the other, and the angle of position, he can find the difference of longitude and of latitude — and this on the astrolabe plate drawn up for one of the two given places. It is just as, from the hours elapsed since midday, or generally from a star's distance from the meridian together with its declination, the star's distance from the zenith is usually elicited; and conversely, once that interval and the declination are known, how far the same star stands from the meridian becomes known.

As if this were not thoroughly common knowledge — not only to mathematicians, but even to those craftsmen who make representations of the world in plano, draw sea charts, and measure differences of longitude, on globes or on astrolabes, by latitudes and angles of position or by travelled distances. For among them it is a commonplace and unquestioned maxim that the same must be done in laying down the places of the world, or in finding distances, as in fixing the positions of the fixed stars.

If Oronce Fine were to handle this matter somewhat more deeply, it would have been worth his while to draw up a general plate, capable of representing the parallels of altitude — the *Almicantarath* — of every horizon, so that the relations of any places whatever could be seen on it without a confusion of lines. I have set out the complete description of that plate, its use and its demonstration, in my book on the astrolabe — which, along with most of my other small works, I would already be putting out, if I had a man skilled at engraving and printing. In France and Germany there are today a great many such men, and highly gifted ones at that.

But even in these thoroughly commonplace matters that Oronce Fine brings forward, he errs grievously. For he says, at the end of the second problem, that if the angle of position is 90 degrees, the given place will lie under the same parallel as the reference place itself, differing from it in longitude alone; and then, in the third:

> You must not forget (he says) that whenever the same angle of position is right, the traveller's arc then expresses the longitudinal difference of those same places, and the places themselves
>
> have the same latitude from the equator.

But we shall show most clearly that this is false and mistaken.

For the angle of position of one place with respect to another is formed at the zenith by the meeting of the meridian with the great circle drawn through the zenith of the other place. This is how Ptolemy took the angle of position of one place with respect to another in book I of the *Geography* (*Geography* I). Vertical circles, therefore, descending from the zenith itself — that is, from the pole of the horizon — of the place we set as the reference place, make innumerable angles of position with the meridian of that same place at the same point of the zenith. Geographers, however, admit no angle of position except a right or an acute one. For if a great circle is inclined to a great circle, it makes two angles, one acute and the other obtuse; they take the acute angle, as being the smaller, and leave the

remaining one, the obtuse and larger. The greatest angle of position is therefore right, and it is produced by the vertical circle drawn through the two points of equinoctial rising and setting; indeed, when they speak strictly, they call this circle alone the vertical circle. These angles of position are represented by those vertical circles which are drawn on the common astrolabe or on Ptolemy's planisphere. And others like them are laid out on the surface of the terrestrial globe under the feet of a person standing on it — that is, at the point where the straight line produced through the centre of the world and the zenith cuts that convex surface.

The measures of those same spherical angles are certain plane rectilinear angles, formed in the plane of the horizon of the reference place and at its centre, or in any other plane parallel to it, out of the common section of those great circles; and these too may be called angles of position. For you will affirm that the spherical angle of position, formed at the zenith by the coincidence of the meridian with any other great circle whatever, contains just as many degrees as you find the rectilinear angle to contain which the two common sections make in the plane of the horizon and at its centre — one of them being the section of the plane of the horizon with the meridian, the other the section of that same plane with the remaining great circle — as you can easily gather from book I of Theodosius (Theodosius, *Spherics* I) and from book XI of Euclid (*Elements* XI).

By the same principles you will also demonstrate that the meridian line, and another line perpendicular to it in the plane of the horizon itself, are the common sections of the meridian circle and of the vertical circle with that same plane of the horizon, but not of a lesser circle. And therefore, when the Sun is in the vertical circle, it will cast the shadows of gnomons stretched out into that same straight perpendicular line. This is the line from which equinoctial rising and setting is discerned — that is, the sixth hour, on those sundials whose centres are directed to the cardinal points of the world.

Lesser circles, by contrast, cannot cut the horizon into equal parts, nor pass through its centre. Indeed, the parallel circle standing 45 degrees from the equinoctial touches at a single point the horizon of that same latitude; while the remaining parallels, inclining toward the visible pole, can neither touch nor cut the horizons of their own latitudes, because they stand at a smaller interval from that same pole than those horizons do. But those parallels which are left toward the equinoctial do cut their own horizons, though unequally.

For this reason the right rectilinear angle formed at the centre of the horizon with the meridian line shows the situation of those places which lie in the plane

of the vertical circle, not of those which lie in the parallel of the reference place. Whenever, therefore, the angle of position is found to be right — whether it is rectilinear or spherical — we shall say that the zenith of the second place is situated in the vertical circle of the first, the reference place, and not in the parallel of the reference place, as Oronce Fine supposes. For that same vertical circle touches the parallel at a single point; and therefore the latitude of the reference place will be greater than the latitude of the second place, and the difference of longitude of those same places will take in more degrees than the traveller's arc does, as we shall demonstrate in the figure set below.

¶ Let

Diagram. A sphere drawn in perspective within a bounding circle, carrying the horizon, meridian, equinoctial, vertical circle, and the parallel of the reference place, together with a quadrant let fall from the visible pole. The labels are small and partly broken by the woodcut's line-work; several are given below with `[?]`.

- a (top of the figure, on the outer circle): terminus of the meridian a e c and of the horizon a b c d
- c (foot of the figure, on the outer circle): opposite terminus of the meridian
- e (centre): the zenith of the reference place
- f (between a and e, on the meridian): the visible pole
- g (lower middle, on the equinoctial): point of the equinoctial b g d
- d (right): point where the equinoctial and the horizon meet
- b [?] (left, on the dashed arc): the other point where the equinoctial and the horizon meet
- l (left) and m (upper right): the extremities of the parallel l e m of the reference place
- i [?] (right of centre): the zenith of the second place, on the vertical circle
- h [?] (on the parallel l e m): where the quadrant f i K cuts the parallel
- K [?] (lower right, on the equinoctial): foot of the quadrant f i K
- n [?] (right edge, on the horizon): the point where the great circle e h n cuts the horizon

For let a b c d be the horizon of the reference place — that is, of the place to which the situations of the others are referred; let e be the zenith and f the visible pole. Let a e c be the meridian, b g d the equinoctial, b e d the vertical circle, and l e m the parallel of that same place. Let there be understood one place whose zenith is set at i, a point of the vertical circle; and let there come from the pole f, through i itself, the quadrant f i K, cutting the parallel l e m at h.

Since the meridian a e c passes through the poles of the equinoctial circle, it will therefore cut that circle at right angles, by proposition 19 of book I of Theodosius (Theodosius, *Spherics* I.19); and the poles of the meridian will lie on the equinoctial itself, by proposition 21 of that same book I (*Spherics* I.21). In the same way you will conclude that the poles of the meridian also lie on the horizon. The points b and d, therefore, at which the equinoctial and the horizon meet, will be poles of the same meridian. The vertical circle b e d will accordingly be at right angles to the meridian, by that same proposition 19 of book I of Theodosius, and the angle of position of the point i at the zenith e will be found to be right.

For that reason, by what Geber demonstrated: as the right sine of the arc f K is to the right sine of the arc f i, so is the right sine of the arc g K to the sine of the arc e i. But the sine of the arc f K is greater than the sine of the arc f i; therefore the sine of the arc g K will also be greater than the sine of the arc e i. And since each of those arcs is less than a quadrant, the arc g K will on that account be greater than the arc e i. Now the arc g K is the difference of longitude of those places whose zeniths are at e and at i, while the traveller's arc is e i. The difference of longitude is therefore greater than the traveller's arc — which was to be demonstrated.

Further, you will demonstrate that the latitude of the place having its zenith at i is smaller than the latitude of the reference place, which has its zenith at e; for the parallel l e m touches the vertical circle b e d at the point e, by proposition 4 of book II of Theodosius (*Spherics* II.4). The arc K i therefore becomes a part of the arc h K, and accordingly the latitude of the place whose zenith is at i will be less than the latitude of the reference place.

But if the given place and the reference place were set under the equinoctial circle — as are those places which have their zeniths under g and K — then the angle of position of the one to the other would indeed be right, and g K would be the traveller's arc and at the same time the difference of longitude. If, however, they are set under some other parallel — say at e and h, points of the parallel l e m — then, once a great circle has been drawn through e and h cutting the horizon at n, the angle under a e n becomes acute, subtending the arc a n of the horizon; and similar to it there will be formed, at the centre of the horizon a b c d, a certain rectilinear angle which will rightly also be called an angle of position, arising from the common sections of the plane of that same horizon with the planes of the meridian and of the great circle e h n — for it subtends the same arc a n of the horizon. The difference of longitude will be g K, and the traveller's arc e h, a part of the quadrant e n.

The same may be inspected on the astrolabe, and in

Diagram. Oronce Fine's own figure, redrawn: an outer circle with a smaller circle inside it sharing the centre a, crossed by a horizontal diameter c–e and a vertical diameter b–d, with two oblique arcs drawn from the upper region toward the lower left. The lettering is small; the doubtful readings are marked.

- a (centre): the pole of the world
- b (top, on the outer circle) and d (foot, on the outer circle): termini of the vertical diameter; b is one terminus of the meridian a b
- c (left, on the outer circle) and e (right, on the outer circle): termini of the horizontal diameter; c g e is the traveller's and vertical circle
- g (directly above the centre a): the zenith of the reference place
- n (upper left of g, on the oblique arc): the zenith of the other place
- h (upper left, on the outer circle): terminus of the meridian a h
- m (on the outer circle, left of and below h): terminus of the traveller's circle g n m [?]
- r [?] (left of g, on the parallel): the point Oronce Fine would place the zenith at
- z [?] (below g): point of the parallel g z r
- p (lower left, on the outer circle): terminus of the traveller's circle g r p
- t [?] (left, mid-height): unresolved label on the oblique arc

Oronce Fine's own figure, in which the point g represents the zenith of the reference place, and n the zenith of the other place, whose distance from g itself is the arc g n. The traveller's circle g n m [?] makes an acute angle of position at g; the rule a n h, produced in place of the meridian through a, the pole of the world, as far as the equinoctial circle, indicates the latitude h n, and likewise shows the difference of longitude b h between the two meridians a b and a h.

But let us suppose the angle of position has been found to be right. The zenith, then, of the place which has such a position will be situated on the traveller's circle, which is also the vertical circle, c g e. If not, let it instead be placed at the point r, on the parallel g z r — for this is what Oronce Fine will maintain — and let the traveller's circle g r p be drawn through g and r. That traveller's circle g r p will therefore show the angle of position contained under the two great circles g p and g d to be acute — a part, that is, of the right angle under g c and g d — and will show the traveller's arc to be g r, a segment of that same great circle g p, and not of the parallel g z r, under which a longer interval is measured.

For while the magnitude of the angle of position was being investigated, the observer's eye and the point r — or the place lying beneath it — together with g, the zenith of the reference place, formed a triangle by three straight lines drawn. But every triangle lies in one plane, by proposition 2 of book XI of Euclid's *Elements* (*Elements* XI.2). That plane, therefore, by proposition 1 of book I of Theodosius (*Spherics* I.1), will cut the sphere along the circumference of a circle, and that circle, by proposition 15 of the same book I (*Spherics* I.15), will be a great circle. The traveller's arc, therefore, is not a segment of a parallel, nor will the angle of position be right, if the zenith of the observed place is put at r.

Therefore the point must be located on the traveller's circle and on the vertical circle *c g e*. Let it accordingly be placed at its point *t*. In that case the angle of position at *t* will be right, since it is contained by that same vertical circle and by the meridian, and so takes in ninety degrees; and the traveller's arc will be *g t*. But the rule drawn from the pole *a* through *t* will show that the latitude is less than the one *b g* represents, and consequently that the difference of longitude is something other than the traveller's arc — which is what we undertook to demonstrate.

As for the claim that on the plane charts of the world which sailors use, straight lines cutting the meridians at right angles represent the parallels; and that so long as the helmsman, holding the tiller during the voyage, has aimed and fixed the ship's prow at the equinoctial east or west, we stay under the same parallel, so that either no advance at all is made along the vertical circle, or an imperceptible one — that is a separate question, and one we have discussed at length in the commentaries on the art of navigation that we wrote in Spanish.

¶ That Oronce Fine erred concerning the composition of ratios and the definitions of proportional magnitudes.

CAP. XVII.

Reprehensio. XIIII.

A ratio is said to be composed of two ratios, or of several (this is the fifth definition of the sixth book of the *Elements*), when the quantities of the ratios, multiplied together, produce some quantity. For the quantity of a ratio is the denomination of that ratio; and the quantity called the denomination is the one which, multiplied into the consequent term of the ratio, produces the antecedent. For example, the quantity or denomination of the sesquialterate ratio [*ratio sesquialtera*] is one and a half, whereas the quantity or denomination of the

subsesquialterate [*subsesquialtera*] is two thirds. For suppose one magnitude, or one number, is proposed as the consequent of a sesquialterate ratio — say 6. If that number 6 is multiplied by one and a half, the number produced will be nine, which has to six a sesquialterate ratio. But now let that same 6 be the consequent term of a subsesquialterate ratio, and let it be multiplied into two thirds: the result will be 4, which is the antecedent of that subsesquialterate ratio.

The denomination of a ratio is therefore rightly called the quantity of the ratio, since it expresses the relation of the antecedent term to the consequent — that is, how great the antecedent term is when compared with the consequent. So Jordanus [Jordanus de Nemore] in his *Arithmetica*, and before him Eutocius of Ascalon, the celebrated commentator on Archimedes, on the second book of *On the Sphere and Cylinder*, at the fourth theorem. In that passage Eutocius demonstrates plainly that, when a single middle term is set between the two terms of any ratio — whether that middle term is less than the greater, or greater than the less, or less than both, or greater than both — the denominating quantities of the two ratios, multiplied by one another, produce the denomination of the ratio found between the first and last terms. He therefore concludes that the ratio of the extremes is compounded of the ratios of the intermediates, a thing Theon had established only by induction. Eutocius's demonstration is extremely easy.

The examples are these. Let the middle term 4 fall between 12 and 2, greater than the less and less than the greater. Then the ratio of 12 to 2 will be compounded of the ratio of 12 to 4, which is the triple, and of the ratio of 4 to 2, which is the double. For multiply 3, the denomination of the triple, by 2, the denomination of the double: the result is 6, and that number is the denominating quantity of the sextuple ratio, which 12 has to 2.

Now let 12 be set as the middle term between 9 and 6, greater than both of them. Then the sesquialterate ratio of 9 to 6 is compounded of the subsesquitertian [*subsesquitertia*] ratio which 9 has to 12, and of the double ratio which the number 12 has to 6. For the quantity of the subsesquitertian is three quarters, and the quantity of the double is 2; so multiply 2 into three quarters, and one and a half results — that is, the quantity of the sesquialterate ratio.

Again, let 4 be taken as the middle term between 9 and 6, less than both of them. Then the sesquialterate ratio will be compounded of the double-and-a-quarter ratio [*dupla sesquiquarta*] and of the subsesquialterate. For if 2¼, the quantity of the double-and-a-quarter ratio, is multiplied into the quantity of the

subsesquialterate, which is two thirds, one and a half will come out, the quantity of the sesquialterate ratio; and the same holds in the other cases.

But since Oronce Fine was in error about the quantity, or denomination, of ratios, he could not help erring about their composition as well; and for that reason he understood and expounded that fifth definition of the sixth book perversely.

He supposes that both ratios — that of the greater term to the less and that of the less to the greater — are allotted the same quantity: that is, that the quantity of the subdouble ratio is two, just as the quantity of the double is; that the quantity of the triple and of the subtriple alike is 3; and that of the sesquialterate and the subsesquialterate alike 1½. And so, when 12 is put as the middle term between 9 and 6, he sees that the quantity of the subsesquitertian — which he takes to be 1⅓ — multiplied by 2 produces 2⅔, the quantity of the double-superbipartient-thirds ratio [*dupla superbipartiens tertias*], and not one and a half. He is therefore driven to assert that Euclid's definition is true only where the ratios are either wholly of greater inequality or wholly of lesser inequality.

> For if one of the proposed ratios (he says) were of greater inequality and the other of lesser, then the quantity of the greater would have to be divided by the quantity of the lesser; for the resulting quantity will show the ratio produced from them.

But Oronce Fine is mistaken, for the greater ratio will have the greater quantity. Now the triple ratio is greater than the subtriple — this is plain from *Elements* V.8, if 9 and 1 are compared with 3. It follows necessarily that the quantity of the triple is greater than the quantity of the subtriple, and the same must be laid down in the other cases.

Nor did Oronce Fine understand the definitions of the fifth book. He followed Campanus throughout: whatever he set down in his expositions of them he borrowed from Campanus entirely. But whenever he comes upon a definition that is not found in Campanus's version, then, with no one to guide him, he errs the more violently. Campanus's fault is nevertheless the lighter, since he did not persist in the error; and so our quarrel will be with Oronce Fine.

In the fifth definition Euclid says that magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another. The clear sense of this is as follows. He had defined a ratio to be a certain relation between two magnitudes of the same kind. But although a line, a surface, and a body are of the same kind — for they are all placed under the continuous — they do not have a ratio to one another; nor does a finite line have any ratio to an infinite

one, nor a rectilineal angle to the angle of contact. A rectilineal angle can, however, have to a curvilineal angle a ratio of equality, and of greater inequality, and of lesser inequality.

That plane rectilineal and curvilineal figures do have a ratio to one another is established, since Hippocrates of Chios squared the lune exactly, and Archimedes the parabola. So, to make it more clearly understood which magnitudes he was calling of the same kind — those, that is, which are to be compared with one another in respect of ratio — Euclid adds that this can be recognized from multiplication. For if any one of them, when multiplied, exceeds the other, those same magnitudes will be said to have a ratio to one another; otherwise not. On that account, in the demonstrations of the theorems that follow the definitions, he repeatedly orders one of the proposed magnitudes between which there is some ratio to be multiplied until it exceeds the other. He does the same in *Elements* X.1 and in a good many other places.

But Oronce Fine expounds it quite otherwise, in this way: that if magnitude *a* is compared with magnitude *b*, and equimultiples of both are taken — *c* of *a* and *d* of *b* — then whatever ratio the multiple *c* has to the multiple *d*, the magnitude *a* will keep the same ratio to the magnitude *b*. He fails to notice, however, that what he states is not a definition but the fifteenth theorem, in which Euclid demonstrates that parts have the same ratio as the equimultiples of them taken correspondingly; whereas the fifth definition does not say this, but rather that those magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another.

He errs in the same way about the sixth definition, which runs as follows. Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when the equimultiples of the first and third and the equimultiples of the second and fourth, according to any multiplication whatever, either alike exceed, or are alike equal to, or alike fall short of one another when taken correspondingly.

For Euclid had defined, in the first and second definitions, a part and a magnitude that is a multiple of a part; in the third, ratio; in the fourth, proportion; and then in the fifth, two magnitudes having a ratio to one another. In the sixth, therefore, he defines what it is for four magnitudes to be in the same ratio — as the first is to the second, so the third to the fourth. This he could not explain in general terms by anything more evident than arithmetical excesses or defects, or the differences of the multiples of the first and third from the multiples of the second and fourth. For it has already been made known from the second definition what

it is for one magnitude to be a multiple of another. Arithmetical proportion, moreover, is simpler and

plainer, and far clearer, than geometrical proportion, since it looks only to the differences of numbers or of magnitudes, and not, as geometrical proportion does, to other and different relations besides. And within arithmetic itself, again, nothing is more primary, nothing simpler or better known, than absolute excess or defect, where there is no comparison of differences at all. For this knowledge comes before the knowledge by which differences are compared with one another, so as to make out whether they are equal or unequal. From this arose that threefold kind of ratio — the ratio of equality, of greater inequality, and of lesser inequality.

So then (he says) of four magnitudes the first will be said to have the same ratio to the second as the third has to the fourth, when, according to any multiplication whatever, the equimultiples taken of the first and third stand to the equimultiples of the second and fourth, taken according to any multiplication whatever, in such a way that if the multiple of the first exceeds the multiple of the second, the multiple of the third also exceeds the multiple of the fourth; and if the multiple of the first is equal to the multiple of the second, the multiple of the third too is equal to the multiple of the fourth; and if, finally, the multiple of the first falls short of the multiple of the second, the multiple of the third likewise falls short of the multiple of the fourth. And this holds whether the excesses or defects are equal or unequal, provided only that each multiple magnitude either alike exceeds each multiple, or is alike equal to it, or alike falls short of it — that is, provided only that each multiple magnitude has to each multiple either alike a ratio of equality, or alike one of greater inequality, or alike one of lesser inequality.

For example: let four magnitudes be proposed — A the first, B the second, C the third, and D the fourth. Let E and F be taken as equimultiples of the first and third, according to multiplication by the number 3; and further let G and K be equimultiples of the second and fourth, according to multiplication by the number 2. Let E exceed G, and let F alike exceed K. Then let L and M be taken as equimultiples of the first and third according to multiplication by the number 4, and let N and O be taken as equimultiples of the second and fourth according to multiplication by the number 7. Let L fall short of N, and let M likewise fall short of O. Again, let P and Q be understood as equimultiples of the first and third according to multiplication by the number 6, and R and T as equimultiples of the second and fourth according to multiplication by the number 9. Let P be equal to

R, and let Q also be equal to T. On that account A will be said to be to B, and C to D, in the same ratio, if not only according to the multiplications just named but according to any others whatever, and in like manner, the equimultiples of the first and third and the equimultiples of the second and fourth alike exceed, or are alike equal to, or alike fall short of one another. And the magnitudes themselves, keeping the same ratio, will be called proportional, by the seventh definition.

P.18	Q.36
L.12	M.24
E.9	F.18
A.3	C.6
B.2	D.4
G.4	K.8
N.14	O.28
R.18	T.36

But when, of equimultiples (this is the eighth definition), the multiple of the first has exceeded the multiple of the second while the multiple of the third has not exceeded the multiple of the fourth, then the first will be said to have a greater ratio to the second than the third has to the fourth. And you are not to understand that this must hold according to every multiplication, as was said of four proportional magnitudes. For it will happen that equimultiples of the first and third, taken according to certain multiplications, and equimultiples of the second and fourth, both alike exceed or both alike fall short of one another; and yet none the less the first will be said to have a greater ratio to the second than the third has to the fourth, because according to some other multiplication, the multiples being equally taken, the multiple of the former exceeds the multiple of the second while the multiple of the third does not exceed the multiple of the fourth. So, for four magnitudes to be called proportional, the equimultiples taken according to any multiplications whatever must either alike exceed, or be alike equal, or alike fall short, in the way stated above. But for the first to be said to have a greater ratio to the second than the third has to the fourth, it is enough if, according to some one multiplication, the multiple of the first exceeds the multiple of the second while the multiple of the third does not exceed the multiple of the fourth, as appears in the example set out below.

E.9 [?]	F.12 [?]	E.9	F.12	K.15	L.20
A.3	C.4	A.3	C.4	A.3	C.4
B.2	D.3	B.2	D.3	B.2	D.3
G.4	H.6	M.8	N.12	O.14	P.21

But Oronce Fine goes wrong, together with Campanus, expounding it thus:

> Four magnitudes are in the same ratio, the first to the second and the third to the fourth, when, of the first and third, equimultiples being taken …
>
> …multiples, and likewise of the second and the fourth, equimultiples being taken according to any multiplication whatever, the multiple of the first preserves the same ratio to the multiple of the second that the multiple of the third preserves to the multiple of the fourth — whether the ratio be one of greater or of lesser inequality; for these things (he says) are to be understood of a proportional excess or defect. But if the multiple of the first has a greater ratio to the multiple of the second than the multiple of the third has to the multiple of the fourth, then the first magnitude will preserve a greater ratio to the second than the third does to the fourth.

Since this interpretation of Fine's so plainly defines the same by the same, it is laughable enough to need no other refutation. He ought, though, to have noticed that on his own exposition the converse of the sixth definition turns out to be the fourth theorem of that same fifth book (*Elements* V.4) — and that theorem is in fact demonstrated by Campanus and Theon by means of that very converse of the sixth definition. Since Fine borrows their demonstration, he is quite openly trying to prove the very thing that he takes, in that same place, as his definition; and no madness can be greater than that. He blunders in exactly the same way in all those propositions of the fifth book for whose demonstration he assumes these same definitions.

¶ That Oronce Fine plainly erred in his description of the nocturnal dial.

CAP. XVIII.

Reprehensio. XV.

In the first book of the *Horologia*, at proposition 18, Fine fell into a great error in describing the nocturnal dial. He supposed that the star of the Lesser Bear — of second magnitude, the southern star of the last side, which has a northern latitude of 72°50′ and in our own day lies in the sign of Leo — comes to mid-heaven with nearly the last degree of Libra, and that he had found this by the method of the second, fourth, and sixth problems of the *Tables of Directions* of Regiomontanus. I shall show at once, by those very problems that Fine used, that this same star comes to mid-heaven with the end of the fifteenth degree of Scorpio, and that consequently anyone who measures elapsed time by the dial he constructed will be led into an error of one hour.

Now according to Fine the star's place in the year 1530 was the seventh degree of Leo with 27 minutes, and in this, he says, he followed Werner. So, entering the general table of declinations with 7°27′ of Leo, following the method of the second problem, we shall find an arc of 19° and nearly 3′, and the multiplicand 97017. And since the arc found and the star's latitude have the same denomination — northern, that is — we shall join the one to the other, and there will result an arc of 91°53′ of the circle of latitude, contained between the equator and the star's true place. We shall multiply the right sine of this arc, 59967, by 97017, the multiplicand kept above; and when five figures are cast out of the product, 58178 remains — the right sine, namely, of 75°51′. That arc is the star's northern declination for the given time.

Next, entering the general table of mid-heavens [*tabula cœli mediationum*] with 7°27′ of Leo, following the method of the fourth problem, we shall obtain the radix of ascensions [radix ascensionum] as 125°5′, and the multiplicand 14995. Then we shall enter the fruitful table [*tabula fœcunda*] with 75°51′, the star's declination, and multiply the number found there, 396907, by 14995; and when five figures are cast out of the product, 59516 remains — the right sine, that is, of two arcs, of which the one, less than a quadrant, has 82°43′, while the other, greater than a quadrant and left over from the semicircle, contains 97°17′. But since the arc of the circle of latitude contained between the equator and the star's true place was found greater than a quadrant, and the star's place lies in the

descending semicircle of the ecliptic, we shall add those 97°17′ of the greater arc to the radix of ascensions, which contains 125°5′; and 222°22′ will result as the right ascension of the star in question. Now, then, by the sixth problem we shall look in the table of right ascensions reckoned from Aries for that number, 222°22′; and in the side column of the same table we shall find the fifteenth degree of Scorpio, with which the proposed star must come to the meridian.

Since Fine appears not only to be ignorant of the reasons and foundations of the tables, but not even to know how to use them, and so that it may also be clear that I have proceeded correctly, I have thought it worth

while to set out the construction of those general tables themselves — of declinations and of mid-heavens, and of the fruitful table as well; and to do so on the very example we have just been handling, that of finding the degree of the ecliptic with which the said star of the Lesser Bear culminates. For in the remaining cases the procedure is the same.

Let the circle a b c d, then, be the colure that marks off the greatest declinations. Let b e c be the semicircle of the ecliptic descending through Libra, b the beginning of Cancer and c of Capricorn; on the same side let g e d be the semicircle of the equinoctial, and the point e the beginning of Libra; let a be the northern pole of the world, and f the pole of the ecliptic. Let the star z be conceived at the seventh degree with 27 minutes of Leo, and through that star let great circles be drawn from the poles a and f to the equinoctial and the ecliptic — namely f z k, cutting the ecliptic at n, and a z m, cutting the equinoctial at l.

Diagram. A single heavy circle (the colure a b c d) with two further great-circle arcs drawn across it. One arc runs from the upper left to the lower right of the circle, crossing the second; the other runs from the left toward the right, and short arcs from the top of the circle descend through the interior. The labels, with their positions:

- a (top of the circle): the northern pole of the world
- [?] (a further letter set just to the right of a, at the top of the circle; not resolved on the leaf)
- b (upper left, on the circle): the beginning of Cancer
- z (interior, upper middle): the star
- n (interior, left of centre): intersection with the ecliptic
- g (left, on the circle): end of the equinoctial semicircle
- K (interior, lower left)
- e (centre): the beginning of Libra

- m (interior, lower right of centre): point of the ecliptic
- l (interior, right of centre): intersection with the equinoctial
- d (right, on the circle): end of the equinoctial semicircle
- c (lower right, on the circle): the beginning of Capricorn

So f z k will be the circle of latitude of the star z, and the arc n z its latitude; f z the complement of the latitude; and the arc n k the segment of that same circle lying between the ecliptic and the equinoctial. The arc z l, again, will be the declination of the star z, and a z the complement of the declination.

Let the arc n z, then, be given — namely 72°50′ — and suppose we have to find, by the *Tables of Directions*, the point m of the ecliptic with which the star z comes to mid-heaven. We shall first investigate the arc of declination z l, by the second problem, in this way. We shall enter the general table of declinations with the degree and minute of the ecliptic that the point n marks, and under the headings "arc" and "multiplicand" we shall find the arc k n and the multiplicand — which is in fact the right sine of the angle e k n.

Numbers of this kind were devised by the following technique, so that they might be set in the table itself. Since the spherical triangle e n k has a right angle at n, and the angle n e k is known, being the angle of the greatest declination, the side e n is known too; for it is what is left of the semicircle when the arc of the star's longitude is taken away from the northern semicircle of the ecliptic. The remaining angle e k n, therefore, and the remaining sides k n and k e, will be known. For as the whole sine is to the sine of the complement of the arc e n, so is the sine of the angle n e k to the sine of the complement of the angle e k n. So by the rule of proportional numbers and the table of right sines the right sine of the angle e k n will become known; and it is set in that general table of declinations as the multiplicand, because it is by multiplying this number into a certain other number — as I shall add in a moment — that the right sine of the arc z l has to be found.

In exactly the same way the multiplicands are found for the remaining points of the quadrant b e, and these will serve for the other three quadrants as well, because of the equality of the angles which the circles of latitude make with the equinoctial when they pass through points of the ecliptic that are equally distant on either side from the tropic point.

Next, since, as the whole sine is to the right sine of the complement of the arc k n, so is the right sine of the angle e k n (which has just been found) to the right sine of the complement of the angle n e k, the right sine of the complement of the arc k

n will therefore become known by that same rule of proportional numbers; and the arc of that right sine, taken away from a quadrant, leaves the arc K n itself, which is set in the same general table of declinations.

The arc e k, moreover, can be found in several ways: either because, as the whole sine is to the right sine of the complement of the arc e n, so is the sine of the complement of the arc k n to the sine of the complement of the arc e K; or because, as the whole sine is to the sine of the angle e K [?], so is the sine of the arc e k to the sine of the arc k n. For in either way, three terms being known, the remaining term of the proportion will become known. When e K is then subtracted from the semicircle, the arc that remains is the one entered in the general table of mid-heavens as the radix of ascensions.

We join K n to the arc of latitude n z, as the author directs, so that the arc K z may be made up, the arc contained between the equinoctial and z, the star's place. And since the spherical triangle k z l has a right angle at l — because of the circle a l, which comes through the poles of the equinoctial — it will therefore be: as the whole sine is to the

sine of the angle z K l, so is the sine of the arc K z to the sine of the arc z l. Now the sine of the angle z k l is known — it is the multiplicand kept earlier — and the sine of the arc K z is known from the table of right sines. We shall therefore multiply the number of the right sine of that angle l k z by the right sine of the arc k z, and divide the product by the whole sine, casting out five figures; for the right sine of the arc z l will then become known, and the arc z l itself, the star's declination, will be known from the table of right sines.

Diagram. The same construction as on the preceding leaf, redrawn with the two great circles produced beyond the bounding circle so that they meet at a point outside it on the right. The labels, with their positions:

- a (top, on the circle): the northern pole of the world
- f (top right, on the circle): the pole of the ecliptic
- z (interior, below a; marked by a heavy dot): the star
- b (upper left, on the circle)
- n (interior, left of centre)
- g (left, on the circle)
- K (interior, lower left)
- e (centre)
- l (interior, right of centre)

- m (interior, lower right of centre)
- d (right, on the circle)
- o (outside the circle, right): the meeting-point of K z and k l
- c [?] (lower right, on the circle; the letter is not clearly resolved on the leaf)

I should not want anyone to be troubled by the author's dividing the product by a whole sine of 100000 equal parts, and yet drawing the sine of the arc K z, and the arc z l, from a table that supposes that same whole sine to be of 60000 parts. For when he was investigating the multiplicand — which is the right sine of the angle l K z — he used a table of right sines that supposes the radius to be of 100000 equal parts. The ratio of those 100000 to the multiplicand, therefore, is the same as the ratio the sine of the arc k z has to the sine of the arc z l; and since the sine of the arc K z is drawn from the table that supposes the radius to be of 60000 equal parts, the arc z l must be drawn from that same table.

He adds a unit to the quotient, furthermore, whenever the rejected figures denote a number greater than 50000; for if the number left undivided exceeds half the divisor, a unit may then be added to the quotient without appreciable error.

Once the arc of declination z l is known, then, the author could have settled the business with a single division. For in this example, if the right sine of the difference between the arc k z and the quadrant is multiplied by the whole sine — by the addition of five ciphers — and the product divided by the right sine of the complement of the star's declination, the division will yield the right sine of the difference between the quadrant and that arc which is intercepted between the circle of latitude and the circle of declination; and so the intercepted arc itself will be known. The difference between the arc K z and the quadrant is 1°53′; we shall multiply its right sine by 100000, and divide the product by 24446, the right sine of the complement of the declination, and the division will give 13442, the right sine of 7°44′. The arc K l will therefore be 9[?]°44′. We shall accordingly add it to the radix of ascensions, and the right ascension that was sought will come out at 222°49′ — somewhat greater, indeed, than the one found by the table, because the number drawn from the fruitful table according to the proportion of the minutes to 60 falls appreciably short of the true number.

The ground of this operation is evident. For in the right-angled triangle k z l the arc k z was found greater than a quadrant, and z l less than a quadrant; therefore the arc k l will be greater than a quadrant. Let K z and k l meet at the point o. Then l o will be less than a quadrant, and z o likewise less than a quadrant. And therefore, as the whole sine is to the sine of the arc a z, so is the sine of the complement of the arc l o to the sine of the complement of the arc z o. But the

complements of those arcs l o and z o are the excesses of the arcs k l and K z over quadrants. Therefore, as the whole sine is to the sine of the arc a z, so is the sine of the difference between the quadrant and k l to the sine of the difference between the quadrant and K z. Of these proportional terms the first, the second, and the fourth are known; the third, then, will become known in the way already described. The right sine of the arc z a is to be taken from the table that supposes the radius to be of 100000 equal parts; and the method is universal. For whenever k z is found less than a quadrant, the arc of declination being less than a quadrant, the remaining side containing the right angle — the side intercepted between the circle of latitude and the circle of declination — will likewise be less than a quadrant; and therefore, as the whole sine is to the sine of the complement of the declination, so is the sine of the complement of the intercepted arc to the sine of the complement of the star's distance from the equinoctial along the circle of latitude.

Yet either this method did not occur to the author of the tables, or he gave it up of his own accord and chose another, more difficult one. For observe that from the ends of the two arcs f g and l g, two arcs a l and f k, drawn across, cut one another at the point z; and therefore the ratio of the right sine of the arc l K to the right sine of the arc K g will be compounded of the ratio of the sine of l z to the sine of z a, and the ratio of the sine of f a to the sine of f g. So if the ratio of l z to z a is taken away from the ratio of l k to K g, the ratio of f a to f g will be left. Let r be made by multiplying l K into z a, and t by multiplying l z into K g; the ratio of r to t, therefore, [leaves]

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l k	l z	
k g	z a	
r		
r [?]		
l a [?]	l K	a z
f g.	k g.	z l

will be left, once the ratio of *l z* to *z a* has been taken away from the ratio of *l K* to *K g*. Therefore, as *f a* is to *f g*, so is *r* to *t*.

Now let us take *l K* and *z a* as the antecedents of the two ratios *l K* to *K g* and *z a* to *l z*. Since *r* arises from *l k* multiplied into *z a*, and *t* from *k g* into *l z*, the ratio of *r*

to *t* is for that reason compounded of those same two ratios — that is, of *l k* to *k g* and of *z a* to *l z*. This is clear from the technique of subtracting and adding ratios.

The ratio of *f a* to *f g*, then (we are always speaking of sines), is compounded of these two ratios, *l K* to *k g* and *a z* to *z l*. And therefore the product of *f a* into *k g* will have to the product of *f g* into *l k* the same ratio that *a z* has to *z l*. So if each of these two products is divided equally by *f g*, the same ratio will still be preserved between the quotients as holds between *a z* and *z l*; and therefore the product of *f a* into *k g*, divided by *f g*, is to the product of *f g* into *l k*, divided by *f g*, as *a z* is to *z l*. But the product of *f g* into *l k*, divided by *f g*, is simply *l K*; and therefore the product of *f a* into *k g*, divided afterwards by *f g*, has to *l K* the ratio that *a z* has to *z l*.

Now the arc *f a* is the distance of the poles, equal to the greatest declination of the ecliptic. The arc *K g* is the arc of the equator lying between the solstitial colure and the terminus of the root of the ascensions, which had already been made known through the star's true place in longitude. But the arc *f g* contains a quadrant together with the arc of the greatest declination, and so its right sine is the sine of the complement of the greatest declination.

Therefore, as the right sine of the complement of the star's declination is to the right sine of that same declination, so is the product of the right sine of the greatest declination into the right sine of the arc of the equator intercepted between the circle of latitude and the solstitial colure — divided afterwards by the right sine of the complement of the greatest declination — to the right sine of the arc of the equator intercepted between the circle of latitude and the circle of declination.

For that reason, when he set out to compose a general table from which the third of these four proportional terms could be drawn, he used this demonstration-figure in the following way. He supposed the arc *b u* to be the first degree of Cancer. He therefore found the arc *g K*, the excess of the root of the ascensions above the quadrant, to be 55 minutes by the method described above. Its right sine, 959, taken from a table that assumes a radius of 60000 equal parts, he multiplied by 23924, the right sine of the greatest declination of the ecliptic; and the product, 22943116, he divided by 55023, the right sine of the complement of the greatest declination. The division yielded the number 417, which he made the third term of the proportion just mentioned, and which he entered in the general table of mid-heavens opposite the first degree of Cancer. He also called it the *number to be multiplied*, because it is then to be multiplied by the second term —

that is, by the right sine of the star's declination — according to the rule of the fourth problem and of the present demonstration.

And since a circle of latitude passing through 29 degrees of Gemini cuts off from the equator, together with the colure itself, an arc equal to *K g*, and the same arrangement is preserved, and the numbers by which the multiplication and the division are performed likewise remain the same, he therefore placed that same number 417 a second time in the same general table of mid-heavens, opposite 29 degrees of Gemini. For the same reason he also set it opposite 29 degrees of Sagittarius and the first degree of Capricorn.

By exactly the same art, having supposed the arc *b n* to be ten degrees and having found its root of the ascensions to be 99 degrees 11 minutes, he multiplied 9575, the right sine of 9 degrees 11 minutes, by 23924; the product, 229072300, he divided by 55023, and found the number to be multiplied to be 4163 — which also answers to the 20th degree of Gemini and of Sagittarius, and to the 10th of Capricorn. Working on in this way, he completed the table for the whole circle.

But since the right sine of the greatest declination is always the multiplier and the right sine of its complement always the divisor, we shall set that divisor at 100000; the right sine of the greatest declination in those same parts will then be 43480, and we shall do the work with half the labour by multiplying by 43480 and throwing away five figures from the product.

So whenever the star proposed has a known place and a known declination, three of the four proportional terms named above will be known: the first, which is the right sine of the complement of the declination; the second, which is the right sine of that same declination; and the third, the number to be multiplied, which we have just explained. And therefore, if that same number to be multiplied is multiplied by the sine of the declination, and the product divided by the sine of the complement, the division will yield the right sine of the arc of the equator intercepted between the circle of latitude and the circle of declination. In the example assumed, that arc is greater than a quadrant, for the reason given above, and it must be added to the root of the ascensions so that the right ascension of the star *z* may come out known.

But the author saw that the work of dividing is laborious, and for that reason composed a certain table

which he called the *fœcunda* [fruitful] table, contrived in such a way that if the first of the four terms named above — the right sine of the complement of the star's declination — is supposed to be 100000 equal parts, there may be drawn

from that table a number to which those 100000 have the same ratio as the right sine of the complement of that star's declination has to the right sine of the same declination.

For he took from the table of right sines the right sine of one degree, namely 1047; he multiplied this number by 100000 merely by adding five ciphers, divided the product by 59990, the right sine of 89 degrees, and found the quotient 1745. He accordingly placed it in the *fœcunda* table opposite one degree — that is, to signify that of such parts as the right sine of 89 degrees is 100000, the right sine of one degree is 1745, as the evident rule of proportional numbers concludes.

By the same art he multiplied 2093, the right sine of two degrees, by 100000, divided the product by 59963, the right sine of 88 degrees, and found the quotient to be the number 3490: of such parts, then, as the arc of 88 degrees is 100000, the arc of two degrees is 3490. He therefore set the number 3490 opposite two degrees, and kept the same method for the rest.

Since, then, as the right sine of the complement of the star's declination is to the right sine of that same declination, so is the number to be multiplied to the right sine of the arc of the equator intercepted between the circle of latitude and the circle of declination, we shall for that reason enter the *fœcunda* table with the arc of the star's declination and multiply the number to be multiplied by the number found there. From the product we shall throw away five figures, for the number left is the right sine of the arc intercepted between the circle of latitude and the circle of declination; and so, by a table of sines assuming a radius of 60000 equal parts, the arc itself will be known. From all this the remaining matters contained in the second and fourth problems — namely, when one arc is to be joined to another, and when one is to be subtracted from the other — will easily become clear.

This (as we conjecture) was the author's device in these problems: ingenious, certainly, but full of labour, both in the construction of the tables and in their use, and liable to mislead the operator in taking the proportional parts from the *fœcunda* table when minutes are attached to the degrees. This can be seen in the example assumed.

For the star's declination was found to be 75 degrees 51 minutes. Its right sine therefore has 58179 parts, and if five ciphers are added to it we get 5817900000. We shall divide this number by 14667, the right sine of 14 degrees 9 minutes, the complement of the declination, and the division will yield 396666 as the true number that would answer to 75 degrees 51 minutes in the *fœcunda* table, if that

table were extended not only by whole degrees but by minutes. But since we were following proportional parts according to the author's own precept, the number we drew from that same *fœcunda* table was 396907; and the difference between these two numbers produces a perceptible error. For if we multiply 396666 by 14995, the result is 5948006670; and when five figures are thrown away from this number, 59480 is left — that is, the right sine of 82 degrees 27 minutes. Let these be taken from 180, and 97 degrees 33 minutes remain as the magnitude of the arc of the equator intercepted between the circle of latitude and the star's circle of declination. In this way the right ascension will be 222 degrees 38 minutes, whereas earlier it had been found to be 222 degrees 22 minutes. There is therefore an excess of 16 minutes, which are not to be neglected in the investigation of right ascension.

For that reason we think the result is found more exactly by the rule of the sixth proposition of our book *On Twilights*, in the following way. Let the number 39874, which is the right sine of the 23 degrees 30 minutes of the greatest declination, be multiplied by 29515, the right sine of the complement of the star's latitude: the result is 1176881110. We shall then multiply this number by 20612, the versed sine of the 37 degrees 27 minutes by which the star proposed is distant, in longitude along the zodiac, from the beginning of Cancer; and from the product we shall throw away ten figures. The number 2456 is left, which we shall subtract from 99389, the right sine of 83 degrees 40 minutes — the degrees contained in the complement of the difference found between the greatest declination and the complement of the star's latitude. There will therefore remain 96933, the right sine of 75 degrees 46 minutes; and so great, accordingly, is the declination of the star proposed.

Then, to find the right ascension by the rule of that same sixth proposition, we shall subtract from the arc of the greatest declination of the ecliptic the 14 degrees 14 minutes of the complement of the star's declination. The difference of these arcs will be 9 degrees 16 minutes, and its complement therefore contains 80 degrees 44 minutes. From the right sine of this arc, 98694, we shall subtract 95545, the right sine of the star's latitude, and the number 3149 will be left, the fourth term of the proportion. This we shall multiply by the square

of the whole sine, the first term, merely by adding ten ciphers. We shall divide the product by 980382038, which arises from the multiplication of the right sine of the greatest declination into the right sine of the star's declination, and the division will yield 32120, the versed sine of 47 degrees 15 minutes.

So great, then, is the right ascension of that arc of the ecliptic which is contained between two termini, one of which is the point of the ecliptic with which the star named mid-heavens, the other the end of Sagittarius. If the reckoning is made from the beginning of Aries, therefore, the right ascension of that same star will be what is left over from three quadrants — namely 222 degrees 45 minutes. However we compute it, then, Fine was in error.

¶ That Oronce Fine handed down false descriptions of horizontal and vertical dials.

CAP. XIX. Reprehensio. XVI.

We also find false the descriptions of dials, both horizontal and vertical, which Fine handed down in the same book. This will be plainly established once the rationale of constructing dials is known.

Let there be, then, in the plane of a given horizon whose centre is *c*, the meridian line — that is, the common section of that same horizon and the meridian — the straight line *a b*; and let *a* face toward the parts of the visible pole, and *b* toward those of the hidden pole. Let the radius of the horizontal dial to be made be *c d*. And let us conceive one plane parallel to the equator, such as *e f*, cutting the plane of the horizon along the straight line *f d g*; and let the common section of the meridian and a plane of this kind, *e f*, be the straight line *d K*.

Since, then, the meridian itself passes through the poles of the horizon, it will stand at right angles to that horizon, by Theodosius, *Spherics* I.19. Moreover, since the common section of the plane *e f* and the sphere is the circumference of a circle, by the first proposition of that same first book, the equator and the circle whose plane is *e f* will have the same poles, by the first proposition of the second book; and therefore the meridian will also be at right angles to the plane *e f*, by the same I.19. So the straight line *f g*, the common section of the horizon and the plane *e f*, will be at right angles to that same meridian, by *Elements* XI.19 of Euclid; and therefore the angles *f d K* and *f d b*

Diagram. A perspective construction for the horizontal dial, filling the upper half of the right column. A large quadrilateral drawn in oblique view represents the plane *e f*, parallel to the equator, cutting the plane of the horizon; its right-hand corners are lettered *e* (upper) and *g* (lower), and *f* stands at its left-hand edge, so that the line *f … g* running across the figure is the common section of the two planes. A straight vertical line descends through the whole figure from *a* at the top to *b* at the foot: this is the meridian line *a b*. Two circles are drawn tangent to

the line *f g* at the single point *d* — an upper circle about the centre *l*, and a lower, larger circle about the centre *c* — with a cluster of small lettered points around *d* where the two circles and the lines meet.

- a (top of the vertical line): toward the visible pole
- b (foot of the vertical line): toward the hidden pole
- e (upper right corner of the tilted plane)
- f (left-hand edge of the tilted plane, on the line of section)
- g (lower right corner of the tilted plane, on the line of section)
- K (upper left, on the upper circle): end of the section *d K*
- h (left of the upper circle)
- l (centre of the upper circle)
- z (right of the upper circle) [?]
- d (point of contact on the line *f g*, between the two circles)
- m (left of *d*), o (further left of *d*), p (right of *d*), t (below *d*)
- n [?] and r [?] (further small points in the cluster about *d*)
- q (left of the lower circle)
- c (centre of the lower circle)
- v (right of the lower circle)
- s (foot of the lower circle)

will be right angles, by the second definition of that same eleventh book; and consequently the angle *b d K* will be the inclination of the plane *e f* to the plane of the horizon.

In the same way you will demonstrate that the inclination of the plane of the equator to the plane of the horizon is the rectilineal angle at *c*, the point formed by the meeting of *b c* with the common section of the planes of the equator and the meridian — an angle which, at the centre itself, subtends the arc of the altitude of the equator above the horizon. And you will conclude that the angles of inclination of the plane *e f* and of the equator to the plane of the horizon are equal, by the sixteenth proposition of the eleventh book and the twenty-ninth of the first. Therefore the rectilineal angle *c d K* will be equal to the angle of the altitude of the equator.

Now let there be constructed, by the twenty-third proposition of the first book, on the given straight line *c d* and at the given point *c* in it, in the plane of the meridian, a rectilineal angle *d c l* equal to the angle of the complement of the altitude of the equator in the given horizon. The straight lines *d l* and *c l* must

then necessarily meet, since the two angles at *c* and at *d* taken together are equal to only one right angle. And since it has been shown that the angles *f d l* and *f d c* are right angles, then, if the circles *d h K z* and *d q s v* are described about the centres *l* and *c* with the distances *d l* and *d c*, the line *f g* will touch each of them at *d* itself, by the corollary to III.15. Since, therefore, the point *z* [meridian…]

looks toward those parts of the meridian line that lie toward the visible pole, and the angle *d c l*, set up in the plane of the meridian, is equal to the angle complementary to the altitude of the equator — that is, to the angle of the altitude of the pole — the straight line *c l*, produced in both directions, must therefore pass through the poles of the equator. Consequently *c l* will be perpendicular to the plane *e f*, and the point *l* will be the centre of that parallel circle whose plane is *e f* itself, by the twelfth proposition of Theodosius, *Spherics* I. The angle *d l c*, then, will be a right angle, either by the second definition of *Elements* XI, or by *Elements* I.32 together with the common notion. And the straight line *c l* itself will be that part of the axis of the equatorial circle which is contained between the centre of the world and the centre of the parallel we have conceived.

Let us further imagine two great circles passing through the poles of the equator, the ones that mark the first hour before noon and the first hour after noon. It is plain that such circles, taken together with the meridian, cut off equal arcs from the circumference of the equator — arcs of fifteen degrees, that is, on the usual division of the day into twenty-four equal hours — and they must pass through the centres *c* and *l*, as Theodosius demonstrates in *Spherics* I.19.

Let the common sections of those circles with the plane *e f* be the lines *l o* and *l p*, cutting the circumference of the circle *d h K z* at *m* and at *n*. Let the straight lines *c o* and *c p* also be joined, cutting the circumference of the circle *d q s u* at *r* and at *r* [?]. Then *c o* and *c p* will be the common sections of the plane of the horizon with those same great circles that separate the hour-intervals. Each of the two arcs *d m* and *d n* will contain fifteen degrees of the circumference of the circle *d h K z*, by the fourteenth proposition of Theodosius, *Spherics* II; and the arcs *d r* and *d t* will be proportional to the arcs that the said great circles — the ones that mark off the hours — cut from the circumference of the horizon on the side of the visible pole.

Let us suppose, then, that the arc *d r* belongs to the first hour before noon and *d t* to the first hour after noon, and let us imagine that the axis of the equatorial circle can cast a shadow. It follows that the centre of the Sun's body, the axis of the equator, and the shadow thrown out in the opposite direction must always lie in the plane of a single great circle — though the shadow does of course have to

be caught on some body set to receive it. Therefore, whenever the Sun, carried by the daily motion, reaches the circumference of the circle of the first hour before noon, the shadow of the axis *c l* will lie exactly along the straight line *c o* [?] in the plane of the horizon; when the Sun stands on the meridian it will throw the shadow onto *d s*; but when it is on the circle of the first hour after noon it will throw the shadow of the axis onto *c p*.

Now the distance between the centres *c* and *l*, compared with that immense distance at which the Sun stands from the centre of the world, is reckoned imperceptible; the plane *e f* may therefore be taken, without inconvenience, for the plane of the equator. So throughout the whole time in which the Sun traverses the southern signs, the part of the axis *c l* will by day indicate the hours in like manner on the circumference of the circle *d h K z*, so that *l o* is the line of the first hour before noon and *l p* the line of the first hour after noon in the equatorial circle, just as *c r* and *c r* [?] are in the plane of the horizon. But when the Sun has ranged through the northern signs, the remaining part of the axis beyond *l* will show the hours on the opposite face; for we are here assuming, to make the matter easier to grasp, that the plane of the equator is slightly thick, and that each of its two faces stands for that same equatorial plane. The same account holds for the other hour-spaces before and after noon, as far as the fifth.

The line of the sixth hour, however — both in the plane of the equator, or of any parallel, and in the horizon — cuts the meridian line at right angles, because the circle of the sixth hour, together with the meridian, divides both circles into quadrants. For this reason the straight line *q v*, which divides the circle of the horizon into quadrants, will indicate the sixth hour before noon and after: the straight line *q c* the sixth before noon, and *c v* the sixth after noon. Likewise, in the horary equator *d h K z*, the straight line *h z*, which divides the circle *d h K z* into quadrants, will show the sixth hour.

The spaces *d r* and *d t*, which belong to equal times and are equally distant from the point of noon, are equal to one another. For in the two triangles *l d o* and *l d p* the two angles at *d* are equal, being right angles; the two angles at *l* are likewise equal to one another, because the arcs *d m* and *d n* are equal; and the side *d l* is common. The two sides *d o* and *d p* will therefore be equal, by *Elements* I.26. And so in the two right-angled triangles *d c o* and *d c p* as well, the two angles at *c* will be equal by I.4 of the same book, and consequently the arcs *d r* and *d t* equal by *Elements* III.26. In the same way, with the help of the common notion, the arcs of the second hour after noon and before noon will be shown to be equal, and so

with any arcs that answer to equal intervals of time equally distant from the meridian.

It should be noted, moreover, that if the semicircle *q d v* is divided into twelve hour-spaces, and straight lines are then drawn from the points of division through the centre, those straight lines will mark the remaining twelve hours

in the other semicircle.

Let the horizon be the circle *a b c* described about the centre *e*, and the meridian *a d f*; let the common section of the meridian and the horizon be *a e f*.

Diagram. A large circle (the horizon) with an inner concentric circle (the circle of the dial), crossed by several straight lines drawn through the centre and by an arc representing the meridian; two arcs sweep across the upper right of the figure. Readable labels, by position:

- *a* (upper left, on the outer circle): the angle of midnight
- *g* (upper centre, on the outer circle, just right of *a*)
- *r* (upper right, on the outer circle)
- *h* (inside, upper left of the inner circle)
- *K* (inside, upper centre of the inner circle, right of *h*)
- *d* (right of centre): the visible pole
- *b* (left edge of the outer circle)
- *o* (inside, left of the centre)
- *e* (centre): the centre of the horizon
- *c* (right edge of the outer circle)
- *t* (lower centre, left of *f*)
- *f* (foot of the figure, on the outer circle)

The demonstration also names the points *i*, *l* and *z*; these are not resolvable on the scan and are not supplied here.

Let *g d i* be the semicircle that shows the first hour after noon, and let the common section of it and the horizon be *g e i*. Let *d* be the visible pole, the arc *a d* the elevation of the pole above the horizon, the straight line *d e* the semi-axis, and the point *a* the angle of midnight. Let the circle of the dial, concentric with the horizon, be *h l o* [?], in which the arc *a l K* [?] is proportional to the arc *a g*.

Since the Sun throws its shadow in the opposite direction, then whenever it is in the arc *d f* it will throw the shadow of the semi-axis *d e* onto *e a*; and therefore, although that straight line inclines toward the angle of midnight, it will indicate

the time of noon. But when the Sun has reached the arc *d i* — that is, the first hour after noon — it will throw the shadow of the semi-axis along the length of the straight line *e g*; and so, although *e g* is inclined toward the region near the angle of midnight, it will nevertheless, for the reason just given, show us the end of the first hour after noon.

Again, at that elevation of the pole at which the Sun for some part of the year stands by day in the arc *d g*, it will throw the shadow of the semi-axis onto the straight line *e i* [?]. And since the arc *d g* is distant from the point of midnight by the interval of a single hour, the straight line *e i* will therefore mark the end of the first hour after midnight, or the beginning of the eleventh hour before noon, even though that straight line faces toward the south. But at that elevation of the pole at which the horizon touches the circle of Cancer, on the very day of the solstice the Sun — rising, as it were, and setting at *a* — will throw an infinite shadow onto *e f*; and then the straight line *e f* will represent neither noon nor midnight. In other regions, however, where the natural day is divided into daylight and night, it will be called the line of midnight.

The arc *i f*, then, will represent the first hour after midnight; and since it subtends at the centre the angle *f e i*, which is equal to the vertically opposite angle *a e g*, it will therefore be equal to the arc *a g* of the first hour after noon. In the same way you will show that the remaining arcs after midnight are equal to the corresponding arcs of the same name after noon, and likewise that the arcs before midnight are equal to those before noon.

The arc of the second hour, however, is greater than the arc of the first hour, and the arc of the third greater than that of the second, and so on to the end of the sixth: to a time further from noon there answers a greater arc in the horizon, and likewise in the horizontal dial. For let *g r* be the arc of the second hour in the circumference of the horizon, marked off by the hour-circle *z d r*. Then the three circles *z d r*, *i d g* and *f d a* cut off equal arcs from the circumference of the equator. But if the arc *g r* were granted equal to *a g*, or less than it, it would follow, by Theodosius, *Spherics* III.4, that the equatorial arc of the first hour is greater than the equatorial arc of the second — which is absurd and contrary to the hypothesis. Therefore *g r* is greater than *g a*; and the same will be demonstrated of the rest, as far as the sixth.

In vertical dials whose planes face south, only twelve hours are marked out. For since the face of the dial lies in the plane of a vertical circle, through the summer it is lit by the Sun after the sixth hour of the morning; at the equinox it is lit from sunrise to sunset, and so again not before the sixth hour; and in the rest of the

year it is agreed that the Sun rises after the sixth hour of the morning and sets before the sixth of the evening. The centre of the dial is taken to be the centre of the horizon, as it is in the horizontal dial. The inclination of the axis above the plane of that vertical dial contains the angle complementary to the altitude of the pole in the given horizon. The hour-spaces are separated by the same hour-circles that pass through the poles of the world and divide the entire circumference of the equator into twenty-four equal parts.

Vertical and horizontal dials are interchanged by this rule: if the latitudes of two places together make up a quadrant, the horizontal dial of the one place becomes the vertical dial of the other, and conversely the vertical becomes the horizontal. For let *a b c* be the northern semicircumference of the horizon, and let *a f c* be the semicircumference of the vertical circle that runs through the intersections of the horizon and the equator; let *b e f* be the quadrant of the meridian, and the point

Diagram. An obliquely drawn circle (a sphere seen edgewise, so that the circle appears as an ellipse), with a centre from which a fan of straight lines is drawn to points spaced along the upper left quadrant of the circumference. Readable labels, by position:

- *a* (left end of the horizontal diameter)
- *b* (top of the figure)
- *e* (centre)
- *c* (right end of the horizontal diameter)
- *f* (foot of the figure)

the zenith *f*. Let the visible north pole be *e*, whose altitude above the horizon is *b e*; the semicircle *a e c* will show the sixth hour before noon and after noon. Let the remaining five circles that separate the hours be drawn. By them the quadrant *a b* will be divided into arcs proportional to the arcs of the circle of the horizontal dial described about the same centre; and by those same circles the quadrant *f c* will be divided into arcs proportional to the arcs of the circle of the vertical dial described about the same centre. For the planes of those circles, passing through the centre of the horizon, cut the circumferences of the horizontal and the vertical dials exactly as they cut the quadrants *a b* and *f c*.

Now let us conceive another place on the globe under the same meridian, whose zenith is at *b*: let the northern semicircle of its horizon be *a f c*, and its vertical semicircumference *a b c*. The altitude of the pole will then be the arc *e f*, which before was the complement of the altitude; and the quadrant *f c* will now be divided by the same spaces for the horizontal dial by which it was formerly

distributed for the vertical. For a like disposition of the circles is preserved, but the altitude of the pole is exchanged for the complement of the altitude, and the two latitudes added together make 90 degrees. The horizontal dial, then, of the place that has *b e* for the altitude of its pole becomes the vertical dial for the place whose altitude is *e f*; and conversely the horizontal dial of the latter place becomes the vertical dial of the former — which was to be demonstrated.

So much for the theory of horizontal and vertical dials. Constructing them on a single plane will be easy if a right-angled triangle is first set up in that plane, one of whose acute angles subtends as many degrees of the circumference of a circle as the altitude of the pole amounts to in the given horizon. For then the side opposite that angle will be the radius of the equatorial dial, from which the distributions of the hours in the horizontal dial are derived; the side that subtends the right angle will be the radius of the horizontal dial itself; and the side that subtends the remaining angle — the altitude of the equator — will be the portion of the axis lying between the centre of the horizon and the centre of the equatorial dial.

But so far as the construction of the horizontal dial is concerned, it makes no difference at all whether the plane of the equatorial dial cuts the plane of the horizon, making with it the inclination equal to the altitude of the equator, as we conceived and imagined it in our mind, or whether both circles are described in one and the same plane. For the intersections of the lines coming from the centre of the equatorial dial with the tangent line fall at the same points, and consequently the divisions of the hour-intervals on the circumference of the dial are the same.

Whenever, therefore, you set out to construct a horizontal dial, let a circle *b c d e* be described on some plane surface, about a centre *a* and with any interval you please, say *a b*; this circle will serve the office of the equatorial dial. We shall then divide it into quadrants by drawing dia-

Diagram. Two circles, one above the other, forming a single construction. The upper circle has its centre *a*, with a vertical and a horizontal diameter drawn through it; from the centre a close fan of straight lines runs down toward the lower right, and one long straight line runs far out to the right to a point *f*, where the lines converge. The lower circle has its centre *K*, likewise crossed by a vertical and a horizontal diameter. Readable labels, by position:

- *d* (top of the upper circle)
- *c* (left end of the upper circle's horizontal diameter)

- *a* (centre of the upper circle)
- *e* (right end of the upper circle's horizontal diameter)
- *b* (lower left, on the upper circle)
- *h* (below the upper circle, right of *b*)
- *g* (below the upper circle, right of *h*)
- *f* (far right, where the fan of lines converges)
- *n* (left end of the lower circle's horizontal diameter)
- *K* (centre of the lower circle)
- *l* (right end of the lower circle's horizontal diameter)
- *m* (foot of the lower circle)

Two further small letters stand between *b* and *h* at the foot of the upper circle; they are not resolvable on the scan [?].

meters *c e* and *b d*, which cut one another at right angles at the centre *a*; and from the point *b* we shall draw *b z* perpendicular to *a b*. We shall then divide the quadrant *b e* into six equal parts, each of which will take in fifteen degrees, and through each of the division marks we shall draw straight lines from the centre; these cut the straight line *b z* at the points *f*, *g*, *h*, *r*, *o*. Next, on that same quadrant *b e*, counting from *e* toward *b*, we shall reckon off the number of degrees of the altitude of the pole in the given horizon; and through *i*, the end of that count, we shall draw from the centre a straight line cutting *b z* at the point *t* [?].

A right-angled triangle *a b t* will thus have been constructed, in which the angle *a t b* is equal to the angle *e a t* [?], the altitude of the pole, being alternate to it. The straight line *a b* answers to the radius of the equatorial dial; and therefore the straight line *a t*, which subtends the right angle at *b*, will be the radius of the horizontal dial in the given latitude of the region.

Let the straight line *d b*, then, be produced; and about the centre *K*, with the interval *b K* equal to the straight line *a t*, let the circle of the horizontal dial *b l m n* be described, and let it be divided into quadrants by the two diameters *b m* and *l n*, which cut one another at right angles at that centre. Next let straight lines be drawn from the centre *k* to the points *f*, *g*, *h*, *r*, *o* of that perpendicular — or tangent — line; for these, taken together with the radii *b k* and *k l*, will cut the quadrant *b l* into six unequal arcs answering to as many equal hours. The line *b k* will stand for noon, *k l* for the end of the sixth hour after noon, and the remaining lines, in their order, for the ends of the other five hours; the proper numbers are to be written against them.

Finally, with the compasses, let spaces equal to those of the quadrant *b l* be laid off in the quadrant *b n*, and we shall have the remaining six hours before noon. Then from each of the points of division let straight lines be drawn through the centre *K* to the opposite points of the circumference; and the circle of the dial will at last be divided into the spaces of the twenty-four hours. But only those spaces should be marked on the dial itself which will indicate the number of hours of the longest day in the given region; the rest are superfluous.

It is also open to you to describe the circle of the horizontal dial first, to whatever size you please, and then to derive the radius of the equatorial from it, in the following way. About the centre *k*, with any interval you please, say *k b*, let the circle of the horizontal dial be described and divided into quadrants by drawing the diameters *b m* and *l n*. Then from one of the diameters let the right sine of the arc of the pole's altitude in the given horizon be taken off; for that right sine will be the radius of the equatorial dial, from which the hour-intervals are to be derived. We shall therefore produce the straight line *b m* to *d*, and from it take the straight line *b a* equal to that same sine of the pole's altitude. Then about *a* as centre, with the interval *a b*, we shall describe the circle *b c d e*, which will serve the office of the equatorial dial, from which the divisions of the hours for the horizontal dial are to be drawn out; and let the rest be completed as before.

Last, let an iron style be fixed at the centre *k*, raised above the line *k b* just so far that it makes with it, at the point *k* itself, an angle equal to the angle *a t b* — or *e a t* — the altitude of the pole; and let its tip stand at an equal distance from the points *l* and *n*. So placed, it will remain in the plane of the meridian and will represent the axis of the world. Or, if you prefer, let a triangle be made out of any hard, thin material, with sides equal to the sides of the triangle *a b t*, and set upright at right angles on the flat surface of the dial, in such a way that the line *a t* lies exactly along *b k*, with *a* falling on *b* and *t* on *K*. Then, with the dial set parallel to the horizon, the line *b m* laid along the meridian line, and the point *b* facing the angle of midnight, the shadow of the style will show the hour of the day.

The vertical dial of a given place is to be built in the same way as the horizontal dial of the place whose altitude of the pole is equal to the altitude of the equator of that given place.

From all this it will be clear how gravely Oronce Fine goes wrong in his account of dials. In the first proposition of the first book he lays out a general prototype for constructing dials, in this manner. About the centre *a* he describes the circle *b c d e*, and divides it into four quadrants by the two diameters *b d* and *c e*, which

cut one another at right angles at that same centre *a*. Of these quadrants he divides the right-hand upper one, *b c*, into 90 equal parts, and counts off from the point *b* toward *c* the altitude of the pole above the horizon of the place for which one wishes to build the dial. The end of the count he marks with the small letter *f*, and from the centre *a* to the given point *f* he draws the straight line *a f*. Then from the same point *f* he lets fall the perpendicular *f g* on the straight line *b d*; it meets the circumference of the circle he has described at the point *g*, and cuts *b d* at *h*. He then sets *a i*, on the radius *a c*, equal to *f h*, and from the point *h* to the point *i* he draws

Diagram. A circle with two diameters through the centre *a*, one vertical and one horizontal, together with the construction of Fine's "general prototype": the radius *a f* drawn to a point on the upper right arc, a line dropped from *f* to the horizontal diameter at *h*, a point *i* taken on the upper radius, the line *h i* drawn across and cutting *a f* at *K*, and a short line from *h* meeting *a f* at *r*. Readable labels, by position:

- *c* (top of the circle)
- *i* (on the upper radius, just below *c*)
- *f* (upper right, on the circumference)
- *K* (inside, on *a f*, where *h i* crosses it)
- *r* (inside, on *a f*, below *K*)
- *a* (centre)
- *h* (on the horizontal radius, between *a* and *b*)
- *b* (right end of the horizontal diameter)
- *d* (left end of the horizontal diameter)

The label at the foot of the circle does not resolve [?], and the point *g* named in the text — where the line from *f* is said to meet the circumference — is not legible on the woodcut [?].

the straight line *h i*, which cuts the straight line *a f* at the point *K*. *f h* will therefore be the right sine of the altitude of the pole, and *a h* the right sine of the complement of that same polar elevation. This construction he calls the general prototype, for building both horizontal and vertical dials.

Afterwards, in the second proposition, setting out to construct a horizontal dial for the latitude given by the arc *b f*, he makes a line equal to the straight line *a h* of this general prototype of his the radius of the dial, and takes a straight line equal to *a K*, or *f K*, for the radius of the horary equator. Again, in the third proposition he sets down a line equal to the straight line *f h* for the radius of the

vertical dial of the same latitude *b f*, and for the radius of the horary equator he puts a line equal to *a k*, or *K f*. In all this he is plainly mistaken.

For by *Elements* I.4 the two straight lines *a f* and *h i* are equal to one another, and the two angles *h a f* and *a h i* are equal; likewise the two angles at *f* and at *i* are equal. The two angles *K f h* and *K h f* will therefore be equal, by I.29 of that same first book together with the common notion; and so *a K* is equal to *h K*, and *f K* equal to that same *h K*, by I.6 of the same book. *a K* is therefore half of *a f*, and *h K* half of *h i*.

And so, whenever the latitude of the place — that is, the arc *b f* — is greater than half a quadrant, as it is in the latitude of Paris and in a great many other places, each of the equal angles *K a h* and *K h a* will be greater than half a right angle, and the remaining angle *a K h* will consequently be less than a right angle, by *Elements* I.32 and the common notion. Hence, if we make the straight line *a h* the radius of the horizontal dial for the latitude *b f*, *a K* — or the equal line *K h* — will not be the radius of the horary equator from which the divisions of the hour-intervals are drawn. Instead we shall let fall from the point *h* a perpendicular *h r* upon the straight line *a f*; and since the angle *a K h* is acute, that perpendicular will fall between *a* and *K*. *h r*, then, will be the radius of the horary equator, and *a r* the portion of the axis.

It is clear, moreover, that *h r*, subtended by a smaller angle, is less than the straight line *h K* or *a K*; and that the angle *a h r*, which is what is left over from the right angle, represents the altitude of the equator, that is, the complement of the latitude — and not the angle *h a K* or *a h K*, as follows from what Oronce Fine says.

But if the latitude *b f* is supposed less than half a quadrant, the angle *a K h* will be greater than a right angle, and the perpendicular dropped from *h* will therefore fall between *K* and *f*. That perpendicular will then be the radius of the horary equator, and it too will be less than the same *K h*, or *a k*. Only where the latitude of the place is equal to half a quadrant will the perpendicular fall on *K*; there each of the angles *K a h* and *a h K* will be half a right angle, the straight line *a h* will be the radius of the horizontal dial, and *a K* the radius of the horary equator.

From this it is plain that he has gone wrong in his account of the vertical dial as well. For if the latitude of the place is *b f*, the angle *a f h* will be the complement of that same place's latitude; and therefore, if *f h* is made the radius of the dial, the perpendicular *h r* will be the radius of the horary equator, and *f r* the portion of the axis, exactly as we demonstrated above. *a K* will be the radius of the horary

equator only where the latitude of the place is equal to half a quadrant; for there one and the same dial is both horizontal and vertical.

No less false is what he asserts in the seventh proposition of the same first book on dials. He describes the quadrant of the meridian *a b c*, divides its circumference *b c* into 90 equal parts, and draws from the centre *a* straight lines to the ends of the arcs of the several climates. He then cuts off from *a c* the part *a d*, for the size the dial is to have, and at the point *d* erects on *a c* the perpendicular *d e*, parallel to *a b*, which cuts the lines drawn from the centre at the points *f*, *g*, *h*, *i*, *K*, *l*, *m*, *n*. He says, accordingly, that the straight line *a d* will be the radius of the horizontal dials; that the perpendicular *d f* is the radius of the vertical dial of the first climate, *d g* of the second, *d h* of the third, and so on for the rest; and that the subtending line *a f* is the diameter [*dimetiens*] of the horary equator of the first climate, *a g* of the second, *a h* of the third, *a i* of the fourth, and so on for the others.

But all this is quite plainly false. For *a f* is the radius of the horizontal dial of the first climate, and the perpen-

Diagram. A quadrant: a vertical radius rising at the left from the corner point *a*, a horizontal radius running right along the foot, and an arc joining their ends. From *a* a fan of straight lines is drawn out to points marked along the arc. A vertical line is erected at *d*, on the horizontal radius, parallel to the vertical radius, and the fan of lines crosses it at a series of lettered points. Readable labels, by position:

- *b* (top left, at the head of the vertical radius)
- *a* (bottom left corner, the centre)
- *d* (on the horizontal radius, at the foot of the erected perpendicular)
- along the erected perpendicular, from the foot upward: *f*, *g* [?], *h*, *i*, *k*, *l*, *m*, *n* [?]

The label at the top of the erected perpendicular, where it meets the arc, does not resolve [?]; neither does the label at the right-hand end of the horizontal radius [?]. Small characters — apparently numbers — are set outside the arc against the ends of the fan lines; they are not legible on the scan [?]. A short line with a further character stands at the right edge of the woodcut, likewise unresolved [?].

dicular *d f* the radius of the horary equator, while *a d* is the portion of the axis. Again, *a f* is the radius of the vertical dial of that same first climate, *a d* the radius of the horary equator, and the remaining line *d f* the portion of the axis. But if we take *a d* for the radius of the horizontal dial, then a perpendicular *d r* must be

dropped from *d* upon *a f*; this will become the radius of the horary equator, and *a r* the portion of the axis. And if we make the straight line *d f* the radius of the vertical dial, *d r* will still be the radius of the horary equator. All this is evident from what has been shown above.

Any other dials, then, that are put together on foundations of this kind of Oronce Fine's are false as well. As for the remaining inclined and hanging sundials that he sets out, there is no leisure to examine them.

FINIS.

PEDRO NUNES OF ALCÁCER DO SAL [PETRI NONII SALACIENSIS]: ON TWILIGHTS, ONE BOOK

Likewise, one book of Allacen [Alhazen], the most ancient of the Arabs, on the causes of twilight, long ago rendered into Latin by Gerardo Cremonensi [Gerard of Cremona], and now revised afresh by the same PEDRO NUNES.

SECOND EDITION.

COIMBRA.

Printed by Antonius à Marijs [António de Mariz].

In the year 1573.

Preface to the work *On Twilight* by Pedro Nunes, Geographer, addressed to the most exalted and most powerful King of Portugal, John III — of Africa, Ethiopia, Arabia, Persia and India

Talk turned lately to twilights, most unconquered King, in the presence of that most upright prince, your most illustrious brother the Infante Henrique — a man distinguished for holiness of life and for learning, and the keenest defender of our faith. He never lets a moment pass without either looking to the salvation of souls, or working through the best authors, or listening to the conversation of learned men; and he takes a remarkable delight in the theorems of astronomy. Not, of course, in that shifting and by now almost discredited kind that deals with judgments touching life and fortune, but in the astronomy that discusses the

course of the stars and the whole order of the heavens. It was you, most gracious King, who ten years ago put him in my charge to be trained in the mathematical sciences. He learned, with great diligence and in a short time, arithmetic and the *Elements* of Euclid's geometry, the treatise on the Sphere, the *Theorics of the Planets*, part of Ptolemy's *Great Composition of the Stars* [*Almagest*], Aristotle's *Mechanics*, the whole of cosmography, the use of certain instruments of the ancients, and of several besides that I had devised myself for the art of navigation. Had he been able to spend longer on these subjects, he would certainly have come out a finished mathematician. But he had to be initiated and consecrated in holy orders, and to apply himself to the noble study of theology.

Even so, he asks me every day about some problem or other, hard, difficult and ingenious. And since his time does not allow him to work at geometrical demonstrations, he lays the burden of demonstrating on me. A few days ago, then, he asked about the length of the twilights in different climates. There was no lack of men who tried to settle the matter on the spot, and who even asserted that they had found the answer — for we have plenty of Gorgiases of Leontini among us. But when I saw that nothing was being put forward except something trite and commonplace, never yet (so far as I know) demonstrated by anyone, I decided to set the matter out more clearly by way of the surest and most evident principles of the mathematical art. And so, by reflection and investigation, I found results that I had read nowhere, and which — had they not become known to me by demonstration — would have been plainly beyond belief:

namely, that when the Sun has entered the first degree of Capricorn the days begin to lengthen, but the twilights to shorten; that before it has completed the whole winter quarter of the zodiac it produces the shortest twilight — on the horizon of Lisbon, on the twenty-fifth day of February (as the most exact calculation has shown) in our own age; and that from that point they increase again, as far as the summer tropic. But for those living under the equator — a region that lies open very widely under your empire — the twilights are shortest when the Sun is carried above the zenith, at the time of the equinox; all the rest grow daily greater toward either tropic. So different is the rule governing the increase of the twilights and that of the days. I have demonstrated a good many other things besides, most worth knowing and most delightful.

Furthermore, this method of demonstration of mine is, I admit, in places different from the one used by those older authorities, the most learned Menelaus, Ptolemy and Geber [Jābir ibn Aflaḥ]; but it is in no way foreign to Euclid and Theodosius. Whether it is easier, or readier for practical work, all the learned will weigh for

themselves. Slight as these pages are, and far from making up a proper volume, I judged nonetheless that they should be published for the common good — for I work constantly at studies of this kind so as to be of some use to students of these liberal arts. I have appended a short work of the very ancient Arab Alhazen, translated into Latin long ago by Gerard of Cremona, in which the causes of twilights are examined with exactness. But I had found it so debased and so corrupted with errors that I sweated more over correcting another man's book than over hammering out my own from the beginning.

These things I wished to dedicate to you, wisest of Kings, as the patron and cultivator of the sciences, since you protect, foster and advance letters and all men of letters. Not that I reckoned such trifles worthy of your majesty; rather, I wanted to find some occasion to excuse myself for having delayed my translation of Vitruvius so long. Ill health is the reason I have not finished a work that was begun and more than half done; and partly, too, the fact that I expound the books of Aristotle in daily lectures to that great-souled prince, your brother the Infante Luís, a most devoted student of letters. For he did not think it enough to have crossed the sea with the Emperor Charles [Charles V] to storm Tunis, the most strongly fortified city of Africa, and to have shown himself most vigorous in every campaign and in the charge of battle: not unless he also called back the studies he had interrupted, and gained a remarkable mastery of arithmetic, geometry, music and astrology. Indeed, even now he does not cease to cultivate his mind with the adornment of the remaining sciences — unlike most philosophers of our age, who treat ignorance of mathematics as a saving of effort. Yet I ought, I confess, to have devoted myself all the same, with my whole mind, to the duty assigned me: I have no just excuse before my king. But you will pardon me, most Chris-

tian and most merciful King, especially since I hope to finish the promised work shortly. Farewell, and may your renowned majesty live among us as long as possible.

Lisbon, in the year 1541 from the redemption of the world, the fifteenth day before the Kalends of November.

A poem by Antonius Pinarius in praise of the work

> Where Cynthia [printed *Cimbia* [?]] brings on the twilights of night with her swift team, or where the Sun goads his glowing horses; within what fixed bounds the middle air holds its place, and where the stars stand fixed in the ethereal pole: all of this, tracking it out skilfully [?] and in

order, Pedro Nunes gives you, reader, by a labour worthy of Hercules. Put off low thoughts, and strip your breast of the cares of earth [?]: a book that is not great holds great things.

THE FIRST PART OF THE BOOK *ON TWILIGHTS* BY PEDRO NUNES OF SALACIA BEGINS

Johannes de Sacrobosco [Sacrobosco], the author of the standard *Sphere*, Stöffler in his elucidation of the astrolabe, and all the other astronomers I have read who treat of twilight, define twilight as the doubtful light that lies midway between day and night. Every day, therefore, must have two twilights: one the morning twilight, which falls about dawn, the other the evening twilight, which falls about dusk. They further maintain that the morning twilight begins, and the evening twilight ends, when the Sun — before its rising, or after its setting — stands 18 degrees away from the horizon, measured on that great circle of the world-sphere which passes through the zenith of the region and through the Sun.

So whenever we wish to measure the interval of time that twilight takes up, we must observe how much time the degree of the zodiac opposite the Sun takes to be lifted 18 degrees above the horizon on the eastern side; for that is precisely the time owed to the evening twilight. We must likewise learn how long that same degree opposite the Sun takes to reach its setting, once it has been raised through an equal arc on the western side of the horizon; the time that has elapsed in the interval fixes the length of the morning twilight.

Times of this kind could perfectly well be gathered by arithmetical computation, in accordance with the geometrical demonstrations of spherical arcs and angles. Nevertheless, because astronomers can reach the proposed result easily by this route, they look the times up on the tympans of the astrolabe [*tympana astrolabij*], drawn for various altitudes of the pole. Now once that first foundation is granted — that the Sun, depressed 18 degrees below the horizon, begins to light the upper hemisphere before its rising, with the morning twilight, and after its setting brings the evening twilight to an end — the method we use for measuring the intervals of the twilights is entirely certain.

For it is plain, from what has been demonstrated both by me and by others elsewhere, that points of the ecliptic diametrically opposite have equal alternations of days and nights, and that equal spaces of time correspond, at an equal altitude, to a descending point and to its opposite ascending point. Within

one and the same interval of time, therefore, the degree of the ecliptic that the Sun itself occupies is depressed 18 degrees below the horizon, while the opposite degree is raised. So the lengths of the twilights are quite properly drawn out from the rising or the setting of the opposite degrees — which is what the more recent astronomers observe.

Appendix. I.

Since equal altitudes before noon and after noon have equal intervals of time from rising and from setting, it follows that the two twilights of one and the same day, the morning and the evening, are equal to each other.

Appendix. II.

It is also clear from this that when the Sun stands in degrees of the ecliptic that lie at an equal distance on either side from one or the other of the tropic points, the twilights are equal. For on those same days the semidiurnal and seminocturnal arcs are equal, the one to the other; and again, equal altitudes above the horizon correspond to equal spaces of time. So the twilights too must be equal.

Lemma, or assumption.

To demonstrate that points of the ecliptic diametrically opposite have nights equal to days by turns, and the remaining things we have assumed.

Let the centre of the sphere be a, the axis e a d, and the poles accordingly e and d. Let the points of the ecliptic diametrically opposite be b and c, and let the meridian pass through b: it will therefore pass through c as well, since the meridian and the ecliptic cut each other only into equal parts, by Theodosius, *Spherics* I.15. Let the common sections of the equator, and of the circles parallel to it which are described by the points b and c in the diurnal motion, with the meridian e b d c, be b h, f g, i c. These circles will therefore have the same poles e and d, by the first proposition of the second book of Theodosius. And this same meridian e b c d will cut those parallel circles into equal parts and at right angles, by the 19th proposition of the first book. Moreover the axis e a d will be perpendicular

Diagram. A large orthographic projection of the sphere on the plane of the meridian, drawn as a circle with the horizon and a set of parallel circles projected onto it. A horizontal chord runs across the upper part of the figure from b on the left to h on the right, passing through the lettered points near the centre; a second, lower chord runs from i on the left to c on the right; and a third from f on

the left to g on the right passes through the centre a. Several oblique straight lines cross the figure — one running from the upper left down to the lower right, another from the lower left up to the upper right — representing the oblique horizon and the sections of the circles parallel to the horizon. Two arcs are drawn outside the main circle at the top, carrying m and R. The lower half of the figure is crowded with short crossing lines and small letters where the perpendiculars are dropped to the horizon line.

- m (top of the figure, on the outermost arc): m
- R (just to the right and below m): R
- e (upper middle, on an inner arc): e — the visible pole
- o (upper left, on an inner arc): o
- G (right side, upper): G
- C (right side, beside q): C
- q (right side, beside C): q
- E (left side): E
- K (left of centre, on the upper chord): K
- A (right of centre, on the upper chord region): A [?]
- t (right side, below the upper chord): t
- b (left end of the upper chord): b
- h (right end of the upper chord): h
- s (centre, on the upper chord; printed as S): s
- z (right of centre, below the upper chord): z
- a (centre of the figure): a
- f (left side, on the middle chord): f
- g (right side, on the middle chord): g
- i (left side, on the lower chord): i
- c (right side, on the lower chord): c
- x (left of centre, lower half): x
- r (left of centre, lower half, beside x): r
- l (centre, lower half): l
- N (lower middle-right): N
- V (right side, lower half): V
- y (lower left): y
- P (lower left, below y): P
- Q (lower left, below P): Q
- p (bottom left, at the rim): p

- d (bottom of the figure, at the rim): d — the hidden pole
- n (bottom of the figure, below the rim): n
- [Further small letters in the crowded lower left of the figure are not resolvable: [?]]

to their planes and will pass through their centres, by 12 of the first book. Consequently the straight line f a g, coming through the centre, will be the diameter of the equator, while b h and i c will be the diameters of the two parallel circles just named, and the points K and l, lying on the common sections with the axis, will be their centres.

Now in the triangles a b K and a c l the two angles at a are equal, by the 15th proposition of the first book of Euclid, and the angles at the centres K and l are right; furthermore the sides a b and a c that subtend them are equal. So by *Elements* I.26 the remaining sides of the one triangle must be equal to the remaining sides of the other. The radii b K and l c are therefore equal, and so too the parallel circles described from the points b and c are equal, by the definition. Let these circles be b m N and i n o.

Further, let any oblique horizon cut the meridian just drawn along the straight line p q, the circle b m N along the straight line m s N, and the remaining circle i n o along the straight line n r o. Then m b N will be the diurnal arc and the remaining arc m h N the nocturnal, for those who have the pole e visible. In the same way n [?] o will be the diurnal arc, and the remaining arc n c o the nocturnal.

Moreover, let us consider the two triangles a K s and a l r. Their angles at K and l are right, and their angles at a are equal, by *Elements* I.15; and the sides a K and a l have been shown to be equal. Therefore, by *Elements* I.26, the straight lines K s and l r are equal to each other.

But since both the horizon and the circle b m N cut the meridian at right angles, by the 19th proposition of the first book of Theodosius, their common section m s N will cut the meridian at right angles, by *Elements* XI.19. Now the straight line b s h is the diameter of the circle b m N and lies in the plane of the meridian; therefore the angles that b s h and m s N make at the point s are right. By a similar argument it will be proved that the angles which the straight lines n r o and i r c make at the point r are right.

For this reason, in the two right-angled triangles K m s and l n r, the two straight lines m s and n r will be equal to each other, by *Elements* I.47 and the common

notion. Accordingly the angles m k s and n l r are equal, by *Elements* I.8, and the arcs m h and n i are equal

by the 25th proposition of the third book. And since the semicircumferences b m h and i n c are equal, by the common notion the remaining arcs b m and n c will be equal. Now the arc b m is the semidiurnal arc of the ecliptic point b, and m h its seminocturnal arc; while of the others, i n is the semidiurnal arc and n c the seminocturnal. Therefore the semidiurnal arc of one point is equal to the seminocturnal arc of the opposite point, and conversely the seminocturnal to the semidiurnal — which was to be demonstrated.

This could also have been demonstrated by a simpler argument. It was enough to have shown that the angles at s and r are right and that the straight lines k s and l r are equal; for in that way the straight lines b s and c r come out equal, being the versed sines of the arcs b m and n c in circles that are themselves equal, and the remainders s h and r i are equal, being the versed sines of the arcs m h and n i. That the seminocturnal arcs in the same circle are equal to each other, and the semidiurnal arcs likewise equal one to the other, becomes plainly evident once k N is joined. For by *Elements* I.47 and I.8, in the two triangles k m s and k N s the angles at the point k come out equal; and therefore the seminocturnal arcs will be equal, by *Elements* III.26 and by the common notion, and the semidiurnal arcs too will be equal one to the other. This can also be shown from proposition 12 of the second book of Theodosius alone, and the earlier part from the 22nd.

Let us further conceive that the ecliptic point b has descended from the horizon and has traversed the arc m R of its own parallel, while the point c has ascended and has completed the arc n P of its own parallel. Let the circle parallel to the horizon which passes through R cut, in the lower hemisphere, the plane of the meridian along the straight line Q z t, and the circle b m N along the straight line R z v. Then the arc q t, or p Q, will be equal to the arc of concealment of the point b on the vertical circle, when it is at R. Again, let another circle parallel to the horizon, which passes through P in the upper hemisphere, cut the plane of the meridian along the straight line y x G, and the circle i n o along the straight line P x E. Then in like manner the arc q G, or p y, will be equal to the arc of ascension of the point c on the vertical circle, when it is at P.

I say that if the arcs of time m R and P n are supposed equal, then the arc of concealment q t must be equal to the arc of elevation above the horizon p y; and conversely, if the arcs of concealment and of elevation are given as equal to each other, the arcs of time m R and n P must be equal to each other.

From the points t, z, y, x let the perpendiculars t C, z A, y F, x D be drawn down to the straight line p q; and let it first be granted that the arcs m R and n P are equal to each other. Then, since the two arcs m h and n i have been shown to be equal, the two remaining arcs R h and P i will be equal, by the common notion. Accordingly the angle R k z of the triangle z K R will be equal to the angle P l x of the triangle x l P, by *Elements* III.27. The angles at z and x are equal, being right; and the radii K R and L P are equal. Therefore K z and l x

will be equal to each other, by *Elements* I.26. So when the equal segments K s and l r are taken away from them, the two straight lines s z and r x are left equal, by the common notion. Now in the triangles A z s and D x r the angles at s and r are equal — as is easily proved by *Elements* I.15, I.28 and I.29 — the angles at A and D are right, and the sides s z and r x are equal, as we have just demonstrated. The side A z will therefore be equal to the side D x, by *Elements* I.26. But t C is parallel to A z, and y F is parallel to D x, by *Elements* I.28; and the two straight lines y G and Q t are parallel to p q, by *Elements* XI.16. Therefore, by *Elements* I.34 and the common notion, the two straight lines y F and t C will be equal to each other. But these are the right sines of the arcs t q and p y; so those arcs t q and p y will be equal. One of them is the arc of concealment of the point b below the horizon, when b is at R; the other is the arc of elevation of the point c in the upper hemisphere, when c is at the point P of its own parallel.

Now let the arcs t q and p y be supposed equal. I say that the two arcs m R and n P — the arcs by which the times of concealment and of equal elevation are measured — will be equal to each other. To demonstrate this we shall use the very same figure already drawn, in which the perpendiculars t C and y F, the right sines of equal arcs, are proved equal to each other. The perpendiculars z A and x D will therefore be equal to each other, by *Elements* I.34 and the common notion. The angles at the points s and r in the triangles A z s and D x r have been shown to be equal, and the two angles at A and D are right; so the two sides s z and r x will be equal to each other, by *Elements* I.26. But we have demonstrated that the two straight lines K s and l r are equal; therefore, by the common notion, K z and l x will be equal to each other. Accordingly, in the two right-angled triangles K R z and l P x, the side z R will be equal to the side x P, by *Elements* I.47 and the common notion. Therefore, in those same right-angled triangles, the angles at the points K and l will be equal, by *Elements* I.8, and so the arcs R h and P i are equal, by *Elements* III.26. Finally we shall take these away from the equal arcs m h and n i, and the two arcs m R and n P will be left equal — the arcs by which we measure the times of concealment and of equal elevation. That is what had to be demonstrated.

To demonstrate the same another way.

For any two points of the sphere diametrically opposite, one of them must be raised above the horizon by exactly as much as the other is hidden below the horizon. Let a great circle be drawn through the zenith and through one of those points; it will necessarily pass through the other as well, since otherwise they would not be diametrically opposite, and such a circle will reach the point opposite the zenith. Let us now consider two semicircles of this circle: one lying wholly above the horizon; the other, between the two opposite points,

made up of the arc of concealment, which is less than a quadrant, and a further arc above the horizon, which is greater than a quadrant. We shall then subtract this arc greater than a quadrant from those two semicircles, and by the common notion the two arcs — the arc of concealment and the arc of elevation of those opposite points — will be left equal.

That the times are equal is demonstrated as follows. Let the sphere be moved until one of the two points touches the horizon. The other must then be on the horizon as well; otherwise they would not be diametrically opposite. It is clear, then, that in one and the same interval of time the one is depressed and the other raised, right up to the two contacts with the horizon. That is what had to be demonstrated.

We shall now easily demonstrate that under equal elevations on the eastern side and on the western, within one and the same day, equal times slip by, and conversely that equal spaces of time run out only under equal elevations. Let us conceive some one of the circles parallel to the horizon cutting the circumference of the circle i n o — the circle which the point c describes in the diurnal motion — on the rising side at P, and on the setting side at E. The points P and E must therefore be raised above the horizon by equal arcs. I say that the eastern arc n P is equal to the western arc E o.

For let that circle parallel to the horizon cut the plane of the meridian along the straight line y x G; it will then cut the circle i n o as well, along the straight line P x E. The horizon, moreover, cut that same circle along the straight line n r o. So the two straight lines P x E and n r o will be parallel, by *Elements* XI.16. Hence, if the points o and P are joined, the two alternate angles at o and P become equal, by *Elements* I.29; and therefore the arcs n P and E o will be equal to each other, by *Elements* III.26.

But now let the arcs of time n P and E o be equal. I say that the points P and E will be raised equally above the horizon. Let P E be joined, and through the point x,

the common section of the straight lines P E and i c, let the straight line y x G be drawn in the plane of the meridian parallel to p q, the diameter of the horizon, by *Elements* I.31. So if the points P and o are joined by a straight line, the alternate angles at P and o, standing on equal circumferences, will be equal by *Elements* III.27; and therefore the straight lines n o and P E are parallel, by *Elements* I.27. And since the straight lines y x G and P x E cut each other, they will lie in one plane, by *Elements* XI.2. A plane of this kind must cut the sphere along the circumference of a circle, by Theodosius, *Spherics* I.1. But those two straight lines y x G and P x E are parallel to the two straight lines p r q and n r o; therefore the planes drawn from them will be parallel, by *Elements* XI.15.

And so the circle drawn from the intersecting straight lines y x G and P x E is parallel to the horizon; therefore the arcs by which such a circle stands away from the circuit of the horizon, measured along the vertical circles, are equal to each other. So the points P and E of the circle i n o will have equal altitudes above the horizon, equal also to the arcs y p and G q. That is what had to be demonstrated.

Note that the arcs lying between parallel circles, on those great circles which pass through the poles of the parallels, are equal to each other, as Theodosius proves in *Spherics* II.14. For the descending arcs of the great circles are equal, by *Elements* III.17 [?], since the straight lines that subtend them are equal by the definition of a pole; and therefore, by the common notion, the arcs between the parallels are equal.

It must further be understood that every straight line perpendicular to the diameter of a circle is the right sine of the intervening [?] arc of the circumference. For the perpendicular so drawn is half of the whole subtending straight line, by *Elements* III.3; and so, by *Elements* I.4 and III.26 or III.28, it will be half of the straight line subtended by the double arc. That in one circle, or in two equal circles, equal arcs have equal sines is proved by *Elements* III.27 and I.26; and conversely it will be demonstrated that equal sines correspond to equal arcs.

To demonstrate the same another way.

Let a circle be described on the sphere parallel to the horizon, at an interval equal to the complement of the elevation of the given point. And since this circle and the parallel to the equinoctial described by the motion of the sphere intersect each other at those two points of equal elevation, the meridian will therefore cut each of the two portions between those two points into equal parts, by

Theodosius, *Spherics* II.12. We shall then subtract those parts from the equal semidiurnal arcs, and equal arcs will be left, by the common notion.

We shall demonstrate the converse in this way. If the arcs of time are given as equal, the points will be equally distant from the point of noon. Let a circle parallel to the horizon be described through one of those points. I say that it will pass through the other. If it does not, it follows, by Theodosius, *Spherics* II.12, that a part is equal to the whole — which is impossible. So if the times are equal, the altitudes will be equal, which was to be shown.

For the rest, to make it known that days and nights become equal to each other when the Sun occupies those points of the ecliptic that lie at an equal distance on either side from one or the other of the tropic points, it will only be necessary to demonstrate that points of this kind, carried round by the diurnal motion, describe one and the same circle. Let us therefore conceive, in this small figure drawn here,

Diagram. A circle carrying two diameters that cut one another at right angles at the centre, together with a chord drawn parallel to one of them and a further line drawn parallel to the common section of the two planes. Readable labels:

- b (top of the circle, on the vertical diameter)
- d (bottom of the circle, on the same diameter)
- e (centre)
- f (at the left, outside the circumference, at the end of a drawn line)
- o (on the vertical diameter above the centre, where the chord crosses it)
- i (to the right of the centre, on the line drawn from o)
- h (lower right, within the circle)
- [?] (a further label at the right-hand edge of the circumference, not resolvable — the text calls this point g)
- [?] · [?] (the two ends of the horizontal diameter, called a and c in the text, are not resolvable on the scan)

in this small figure drawn here, that the circumference a b c d — divided into quadrants by the two diameters a c and b d, which cut one another at right angles at the centre e — is the ecliptic; that a c is the common section of the plane of this circle with the plane of the colure that marks off the equinoxes, and with the equinoctial as well; and that b d is the common section of that same plane with the colure that marks the solstices. Then a and c will be the equinoctial points, and b and d the tropic points.

Now take two points f and g standing at an equal distance on either side from the point b (or from the point d). I say that f and g describe one and the same circle by the diurnal motion.

Join the straight line f g, cutting the diameter b d at the point o. Since the plane of the colure that passes through the tropic points cuts the plane of the equator, let e h be the straight line lying in their common section; and from the point o, which lies in the plane of that same colure, draw the straight line o i parallel to e h, by *Elements* I.31. The two straight lines f g and o i therefore cut one another and so lie in a single plane, by *Elements* XI.2. But o g and e c are parallel, since the arcs a f and c g are equal and subtend equal alternate angles at the circumference; and o i and e h are parallel too. The planes drawn through f g and o i, and through a c and e h, must therefore be parallel to one another. Now the common section of a plane and a sphere is the circumference of a circle, by Theodosius, *Spherics* I.1. So a circle parallel to the equator passes through the points f and g — and this is precisely the circle described by the diurnal motion.

To demonstrate the same otherwise, by reduction to the impossible. About the pole of the world let a circle be described parallel to the equinoctial, passing through one of the two points. I say that it will pass through the other as well. If it does not, it follows that a part is equal to the whole, by Theodosius, *Spherics* II.12. And in this way you will demonstrate that points standing at an equal distance from a tropic point have equal declinations.

Corrolarium.

And since, as the first part of the lemma makes clear, the circles described from opposite points of the ecliptic are equal, it is evident from this that the parallels described by points standing at an equal distance on either side from one or other of the equinoctial points are equal as well.

Appendix.III.

It follows besides that unequal twilights belong to points standing equally distant on either side from one or other of the equinoctial points: longer twilights to the northern points in the northern region, but shorter ones to the southern points; while in the southern region the reverse holds.

For let the meridian be the circle a b c d; let the straight line b d be the section of the equator; and let the straight lines e f and g h be the sections of any two parallel circles that the Sun describes by the diurnal motion when it occupies those degrees of the ecliptic which stand at an equal distance on either side from

one or other of the equinoctial points. Let a be the north pole, and let it be taken as the visible pole [*manifestus polus*]. Let the diameter l y be the section of the horizon, and let it cut the straight lines e f and g h at the points k and o.

Conceive further, beneath the horizon, a circle parallel to it, at which the Sun begins the morning twilight. Let the straight line p r be the common section of this circle and the meridian, and let the points at which it cuts the straight lines e f and g h be t and s.

Diagram. The meridian circle, carrying the section of the equator, the sections of two parallel circles, the section of the horizon, and the section of the circle parallel to the horizon beneath it. Readable labels:

- y (top of the circle)
- f (upper right)
- e (at the left, outside the circumference)
- b (left, on the horizontal diameter)
- d (right, on the horizontal diameter)
- k (within the circle, upper left)
- t (within the circle, to the right of k)
- o (within the circle, below the centre-left)
- s (within the circle, to the right of o)
- h (at the right, within the circle)
- l (lower left)
- p (at the foot of the circle)
- [?] (a label above the circle at the upper left, called a in the text)
- [?] (a label at the foot, to the right of p, called c in the text)

Now by *Elements* XI.16 the straight lines e f and g h — the common sections of parallel circles — are parallel; and by the same proposition l y and p r are parallel as well. The two straight lines o s and k t are therefore equal to one another, by *Elements* I.34.

But the meridian circle passes through the poles of the equator and of the circles parallel to it, by Theodosius, *Spherics* II.1, and likewise through the poles of the horizon and of the circles parallel to it. By Theodosius, *Spherics* I.19, then, it will cut all those circles at right angles. The common sections of the horizon and of the circles parallel to the equator accordingly stand at right angles to the plane of the meridian as drawn, at the points k and o, by *Elements* XI.19. Moreover the common sections of the circle parallel to the horizon and of the circles parallel to

the equator likewise stand at right angles to the meridian itself, at the points s and t.

The lines that stand over k and t take in that arc of the northern parallel which defines the length of the morning twilight, while those that stand over o and s enclose the arc of the southern parallel which marks the interval of the morning twilight in the same way.

Since the points of the ecliptic we have conceived stand at an equal distance on either side from one or other of the equinoctial points, the parallels they describe will be equal, by the corollary to the preceding lemma. The meridian a b c d cuts these into two equal parts, by Theodosius, *Spherics* I.19; so the common sections e f and g h are diameters of those parallels, and are equal to one another.

And because the portions e k and o h are greater than the semidiameters — as the first part of the lemma demonstrated — the straight line k t lies farther from the centre of its own parallel, from which it certainly recedes, than o s lies from the centre of its own parallel, which it in fact approaches. But it has been shown that the straight lines k t and o s are equal to one another. The straight lines that stand at right angles to the meridian over the points k and t therefore take in a greater arc of the parallel's circumference than those over o and s.

The twilight accordingly lasts longer when the Sun occupies the northern point of the ecliptic than when it occupies the southern point that is equally distant from the equinoctial point. This holds in the northern region; in the southern the reverse, as is plainly clear from this same figure once the names of the parallels are exchanged.

Lemma.

To demonstrate that the straight lines perpendicular to the plane of the meridian over the points k and t cut off a greater arc of the parallel circle than those that stand at right angles to the meridian over the points o and s, let us conceive the parallel circles themselves. One of them has e f for its diameter — the northern one; let it be e f m, described about the centre q. The other is the one that has

Diagram. A circle carrying a set of chords and perpendiculars, with a second point marked as centre below the middle. Readable labels:

- S (top of the circle)
- O (below S, within the circle)
- D (upper left, within the circle)

- C (upper right, within the circle)
- B (at the left, outside the circumference)
- P (within the circle, to the right of B)
- s (small, near the middle)
- E (to the right of the middle)
- A (at the right, on the circumference)
- R (below the middle, at the centre marked in the text)
- h (at the foot of the circle)
- [?] (two or three further small letters between the middle and the foot are not resolvable on the scan)

g h for its diameter; let it be A g h, described about the centre R.

Further, let the perpendicular lines drawn on either side be these: over k and t, m n and v x; over o and s, A B and C D. Since these are at right angles to the plane of the meridian, in which the diameters e f and g h of the parallel circles lie, the angles at the points k and t in the plane of the circle e f m will be right angles, by *Elements* XI, def. 2; and likewise the angles at the points o and s, in the plane of the circle A g h, will be right.

From the points v and D draw v z and D P at right angles to m n and A B, and join q n and A R. Then in the two right-angled triangles n q k and A R s, since the semidiameters q n and A R are equal, the two squares on q k and k n are equal to the two squares on R s and A s, by *Elements* I.47 and the common notion. But the square on R s is less than the square on q k, since R s has been shown to be less than q k, because the point k is farther from the centre of its own circle. The square on A s will therefore be greater than the square on k n, and so the straight line A s is greater than k n. In the same way it will be demonstrated that the straight line O D is greater than the straight line t v.

Now the two quadrilaterals O D P s and t v z k are parallelograms, by *Elements* I.28; so by *Elements* I.34, O D is equal to P s, and t v to k z. The straight line P s will therefore be greater than the straight line k z, by the common notion. Hence the whole A P is also greater than the whole n z. From A P, the greater, cut off the straight line E P equal to the lesser, and join E D, A D and v n.

Since k t is equal to v z, and O s is equal to the straight line D P — also by *Elements* I.34 — while k t and O s have been shown to be equal, the straight lines D P and v z will be equal to one another. In the two right-angled triangles E D P and n v z, therefore, the angle D E P will be equal to the angle v n z, by *Elements* I.4. But the angle D E P is itself greater than the angle D A P, by *Elements* I.16; so the angle v n

z will also be greater than the angle D A P. By the last proposition of *Elements* VI, then, the arc v m will also be greater than the arc B D.

These arcs are cut off from the circumferences of the parallel circles by straight lines standing at right angles to the plane of the meridian at the points k, t, O and s. The perpendicular straight lines over k and t therefore cut off a greater arc than those over O and s — which is what had to be demonstrated.

Appendix.IIII.

It likewise follows that, while the Sun occupies the northern signs, longer twilights correspond, in the northern region, to the points nearer the summer tropic.

Let a b c d be the meridian circle, described about the centre e and divided into quadrants by the diameters b d and a c, which cut one another at right angles. Let a c be the common section of the meridian and the equinoctial, and b d the common section of the right horizon of those who live under a. Let the two straight lines h y and K l be the diameters of two circles parallel to the equator, so placed that the one whose diameter is K l lies nearer the summer tropic than the one whose diameter is h y. And let the straight line f g be the common section of the oblique horizon of a northern place. I say that the twilights are longer when the Sun describes the parallel whose diameter is K l than when it describes the one whose diameter is h y.

For let the straight line m n be the common section of a certain circle parallel to the horizon — the circle at which, as day begins to break, the Sun opens the morning twilight. Let m n cut the straight lines k l and h y at the points r and p, and let the straight line f g cut the same lines at the points o and q. Then, by what we demonstrated in the preceding appendix, the two straight lines o p and q r will be equal to one another.

Next, from the points o and p, which lie in the plane of the meridian, let perpendiculars be raised, cutting the diameter K l at the points s and t. Then o s and p t will be parallel, by *Elements* I.28. But K l and h y are the common sections of the meridian and of the parallel circles, and are themselves parallel; therefore s t and o p will be equal, and will stand at equal distances from the centres — since, as was shown above in the first lemma, the straight lines K l and h y are the diameters of circles parallel to the equator, and the centres of those circles lie at the common intersections of the straight line b d with those same diameters.

Accordingly, let the more northerly circle, the one whose diameter is k l, be k i l A, described about the centre v; and in its plane, about the same centre, let a circle z

R B be described with a radius equal to half the diameter h y. From the points s and t, in that same plane, let straight lines be raised on either side at right angles to K l, cutting on one side the interior circle at the points i and G, and the exterior circle at the points z and x, and on the other side the interior circle at A and Q, and the exterior at R and B. Let i Q and z B be joined, and let their intersection be the point D.

Then, in the triangle i D z, the exterior angle A i D is greater than the interior angle i z D, by *Elements* I.16. A right angle therefore has a greater ratio to the angle i z D than to the angle A i D, by *Elements* V.8. But in equal circles the angles have the same ratio as the circumferences on which they stand, by the last proposition of *Elements* VI. Therefore the quadrant of the exterior circle also has a greater ratio to the arc R B than the quadrant of the interior circle has to the arc A Q, by *Elements* V.13.

Moreover, the straight lines raised at right angles to the diameter k l from the points q and r — which are farther from the centre v — cut off a greater arc of the interior circle than A Q, as the lemma of the preceding appendix demonstrated. The quadrant of the exterior circle will therefore have a greater ratio to the arc R B than the quadrant of the interior circle has to the arc contained by the two straight lines drawn at right angles from q and r — the arc, that is, which is contained between the horizon and the circle parallel to it.

Further, the exterior circle is equal to the parallel whose diameter is h y and which lies farther from the summer tropic; and the arc R B is proved equal to the arc contained, in that same parallel, between the horizon and the circle parallel to it, by what we demonstrated in the first lemma. Therefore the quadrant of the parallel farther from the summer tropic will also have a greater ratio to the arc lying between the horizon and the circle parallel to it than the quadrant of the nearer parallel has to the arc lying between the horizon and that same circle which is parallel to it.

Diagram. A large woodcut circle — the meridian — occupying the foot of the left column. Across it run a heavy horizontal line terminating at k on the left and l on the right, a second horizontal line from h to y below it, and a third from a to c lower still; a line descends to d at the bottom of the circle, and an oblique line crosses the whole figure from the lower left. Two smaller concentric circles are drawn within the upper right portion of the figure, about a centre on the heavy horizontal line.

- z (top of the figure, on the outer arc): z

- x (top, to the right of z): x
- i (below z, inside): i
- G (below i, inside): G
- b (top of the figure, on the circumference of the meridian, directly above e, at the head of the diameter b d): b
- k (left end of the heavy horizontal line): k
- l (right end of the same line): l
- n (immediately to the left of l): n
- s, t, q (on the heavy horizontal line, near its middle and right): s, t, q
- D (above the heavy horizontal line, near the centre): D
- h (left end of the second horizontal line): h
- y (right end of the second horizontal line): y
- o, P (on that same second horizontal line h y, standing directly above s and t on the heavy horizontal line): o, P
- a (left end of the third horizontal line): a
- e (on the third horizontal line, at the centre of the figure): e
- c (right side, below y): c
- A, Q (inside, right of centre, below the heavy horizontal line): A, Q
- R, B (inside, below A and Q): R, B
- f (lower left, on the circumference): f
- d (bottom of the figure): d
- v (on the heavy horizontal line k l, at the centre of the figure, to the left of s — the counter is closed by ink, so the letter prints as a solid wedge): v
- m (lower left, on the circumference below f): m

And since the intervals of time correspond in equal proportion to the parts of the equator and of the circles parallel to it, the space of six hours will therefore have a greater ratio to the length of the twilight of the parallel farther from the summer tropic than to the length of the twilight of the nearer parallel. Therefore, by *Elements* V.10, the twilight of the parallel nearer the summer tropic must be the longer — which is what had to be demonstrated.

Appendix. V.

For those who live under the equator, when the Sun occupies points of the ecliptic that are equally distant on either side from one or the other of the equinoctial points, the twilights are equal; but when the points are unequally distant, the twilights are unequal. The longer twilights correspond to the more distant points,

the shorter to the nearer. And as the right sine of the complement of the declination of the nearer point is to the sine of the complement of the more distant point, so is the right sine of the arc of the length of the twilight of the more distant point to the sine of the arc of the length of the twilight of the nearer point.

Let the circle a b c d be the meridian, as in the preceding figure; let the diameter a c be the section of the equator and the meridian, and the diameter b d the section of the right horizon of those who live under a, a point of the equator. Let b be the north pole and d the south. Let the two straight lines f g and h y be the diameters of the two parallels which the Sun describes when it stands at an equal distance on either side from one or the other of the equinoctial points; and let the common sections of these with the diameter b d be the points o and n — the remaining features of the parallels so conceived being as was shown in the first lemma.

Next, let a certain circle be understood beneath the right horizon and parallel to it, marking the beginning of the morning twilight and the end of the evening twilight; and let the common section of this circle and the meridian be the straight line K l, which cuts the straight lines f g and h y at the marks m and p. It is clear from what was shown in the third appendix that the straight lines o m and n p are equal to one another, and that each of them is equal to the right sine of the arc which, in its own parallel, defines the length of the twilight. And since the parallels themselves are equal, as is plain from the corollary of the first lemma, the intercepted arcs will accordingly be equal to one another; and so the twilights themselves are equal to one another — which was the first thing to be demonstrated.

Further, let the straight line q s be the diameter of one of the parallel circles, more northerly than the one whose diameter is set as f g; let its centre be r, and let the straight line K l cut it at the point t. Again it is clear, from what we demonstrated in connection with the third appendix, that the straight line r t is equal to the right sine of the arc which, in its own parallel, defines the length of the twilight. We shall therefore conceive two meridians passing through the ends of this arc, which will cut off from the circumference of the equator an arc proportional to it, by Theodosius, *Spherics* II.14, and which will thereby exhibit the times of the length of the twilight. Of these two meridians, one will be the right horizon itself, the other the one described beneath the earth.

Now let there be taken, on the radius e c, a certain straight line e z equal to the right sine of that arc of the equator. Let the same be understood in the parallel whose diameter is f g: let the straight line be e v — which we shall show presently to be less than e z — equal to the right sine of that arc of the equator which is

proportional to the arc that the two conceived meridians cut off from the parallel whose diameter is f g.

And since, when arcs of circles are similar, both their right sines and the radii of the circles themselves are proportional, it follows that as a e, the radius of the equator, is to f o, the radius of the nearer parallel, so is e v to o m; and further, as q r, the radius of the more distant parallel, is to a e, the radius of the equator, so is r t — or its equal o m — to e z. Therefore, by *Elements* V.23, as q r is to f o, so is e v to e z.

But q r is the right sine of the complement of the declination of the point q, the more distant and more northerly, and f o is the right sine of the complement of the declination of the point f, which is nearer the equator; while e v is equal to the right sine of the arc of the equator that measures the length of the twilight when the Sun occupies the nearer point of the ecliptic, and the straight line e z has been laid down equal to the right sine of the arc of the equator that shows the length of the twilight when the Sun stands at the more northerly point.

Diagram. A woodcut circle at the foot of the left column, headed by an ornamental initial E. Four horizontal chords cross the circle, and a near-vertical line runs from K at the top to l at the bottom, cutting them.

- b (top of the circle): b
- K (top, to the right of b): K
- q (left end of the uppermost chord): q
- r (on that chord, toward its middle): r
- t (on that chord, where the near-vertical line crosses it): t
- [?] (right end of the uppermost chord, the one whose left end is q — only a small hooked fragment of the glyph took ink, and it prints markedly smaller than the c and y below it; the letter is not identifiable)
- f (left end of the second chord): f
- o (on the second chord): o
- m (on the second chord, at the near-vertical line): m
- [?] (right end of the second chord, the one whose left end is f — a bowl-and-stem glyph with the bowl at the foot; b, g and a numeral 6 are not separable at this size, and the same figure's h at the left end of the lowest chord prints as a b, so the b/h/g confusion is demonstrable here)
- a (left end of the third chord): a
- e (on the third chord, at the centre): e
- v, z (on the third chord, right of e): v, z

- c (right end of the third chord): c
- h (left end of the lowest chord): h
- n (on the lowest chord): n
- P (on the lowest chord, at the near-vertical line): P
- y (right end of the lowest chord): y
- d (bottom of the circle): d
- l (bottom, to the right of d): l

But q r is less than f o; therefore e v is less than e z, and the one arc likewise less than the other — which could also have been demonstrated as in the fourth appendix. It is clear, then, that for those who live under the equator, when the Sun occupies points unequally distant from it, the twilights are unequal: the longer answer to the more distant points, the shorter to the nearer. And as the right sine of the complement of the declination of the nearer point is to the right sine of the complement of the more distant point, so is the right sine of the arc of the twilight which answers on the equator to the more distant point, to the sine of the arc of the twilight which belongs on the equator to the nearer point — which was the second thing to be demonstrated.

Lemma.

The right sines and likewise the versed sines of similar arcs have the same ratio as the radii of their circles.

Let a b c be a circle whose centre is d and whose diameter is b c, and f g h a circle whose centre is i and whose diameter is g h; and in them let a b and f g be similar arcs, and let a e and f K be the right sines of those similar arcs, and b e and g k the versed sines. I say that the ratio of a e to f K, and of b e to g K, is as the ratio of the radius b d to the radius g i.

For let a d and f i be joined. The circle a b c is either equal to the circle f g h or unequal. Let it first be equal: then the radii a d and f i will be equal. Now the straight lines a e and f K are perpendicular to the diameters b c and g h, by the definition of the right sine and by *Elements* III.3; therefore the two triangles e a d and k f i will have right angles at e and k. And since the arcs a b and f g are given as similar, the angle a d e will be equal to the angle f i K, by the last definition of *Elements* III. Therefore, by *Elements* I.32 and the common notion, those two triangles will be equiangular, and so they will have proportional the sides which subtend equal angles, by *Elements* VI.4. As a d is to f i, then, so is a e, the right sine of the arc a b, to f K, the right sine of the arc f g, and so is e d to K i. But a d is

equal to f i; therefore a e is equal to f k, and e d to K i — which the twenty-sixth proposition of *Elements* I alone could also have concluded.

Now let the equal straight lines e d and k i be taken away from the equal radii: then, by the common notion, b e, the versed sine of the arc a b, will be left equal to g k, the versed sine of the arc f g. The ratio of all these straight lines will therefore be the same, namely a ratio of equality.

But now let the circle a b c be supposed less than the circle f g h. About the centre i, with a radius equal to a d, let the circle r s o be described, cutting the straight line f i at o and the straight line g i at r; and let the right sine of the arc o r be o t. It will be demonstrated as above that the angles at K and t are right, and that the straight lines i t, t r and o t are equal to d e, e b and a e; and that the triangles K f i and t o i,

Diagram. A two-part woodcut set into the left column, with the text of the *Lemma* running around its right side. At the left stand two concentric circles crossed by a vertical diameter; an oblique line runs from the common centre out to the upper right, cutting the inner circle at o on its way to the outer circumference, and a short perpendicular drops from that meeting-point back onto the diameter. At the right stands a smaller single circle, likewise crossed by a vertical diameter, with an oblique line drawn from a point on the diameter out to the circumference at the right and a short line closing the figure.

Left figure (the two concentric circles):

- g (top of the vertical diameter, on the outer circle): g
- K (immediately below g, on the diameter): K
- t (below K, on the diameter): t
- o (upper right, where the oblique line meets the inner circle): o
- i (on the diameter, at the common centre): i
- s (on the diameter below the centre, at the inner circle): s
- [?] (foot of the vertical diameter, on the outer circle — a bold glyph reading as b or h; the two are not separable at this size)
- [?] (a short bold stroke at the top right of the outer circle, where the oblique line meets the circumference; the glyph does not resolve into a letter)

Right figure (the smaller single circle):

- b (top of the vertical diameter): b

- [?] (on the diameter just below b — a small round glyph; c and e are not separable at this size)
- a (right, on the circumference, at the outer end of the oblique line): a
- d (on the diameter, below the preceding mark): d
- [?] (foot of the vertical diameter — a small round glyph; c and e are not separable at this size)

by *Elements* I.32 and the common notion, are equiangular. Their sides that subtend equal angles will therefore be proportional, by *Elements* VI.4. So as the straight line *f i* is to *o i*, so is *f K* to *o t*, and *K i* to *t i*. But *f i* is equal to the straight line *g i*, and *o i* to *r i*; therefore, by *Elements* V.7, as *g i* is to *r i*, so is *k i* to *t i*. Consequently, as *g i* is to *r i*, so is the remainder *g K* to the remainder *r t*, by *Elements* V.19. And so, by repeating *Elements* V.7 as often as the case requires, the proposition will be concluded.

The same result can also be demonstrated more simply, and in a fully general way, without constructing the circle *r s o*. For the angles at *i* and *d*, the centres [?], are equal, by the last definition of *Elements* III, while the angles at *K* and *e* are right, by the definition of the right sine and by *Elements* III.3. The remaining angle at *f* will therefore be equal to the remaining angle at *a*, by *Elements* I.32 and the common notion. So the two triangles *k f i* and *e a d* will have proportional sides subtending equal angles. Therefore, as *f i*, the greater radius, is to *a d*, the smaller radius, so is *f k*, the right sine of the arc *f g*, to *a e*, the right sine of the arc *a b*; and likewise *K i*, the complement of the versed sine *g K*, is to *e d*, the complement of the versed sine *b e*. And so, by *Elements* V.7, as *g i* is to *b d*, so is *K i* to *e d*. Therefore, by *Elements* V.19, as the radius *g i* is to the radius *b d*, so is *g K*, the versed sine of the arc *g f*, to *b e*, the versed sine of the arc *a b*. And since — as we have just demonstrated, and by *Elements* V.7 — the radius *h i* is to the radius *c d* as *K i* is to *e d*, it follows by *Elements* V.12 that as radius is to radius, so is the whole *h K*, the versed sine of the arc *f h* left over from the semicircle, to the whole *c e*, the versed sine of the arc *a c*. Therefore the right sines, and the versed sines too, of similar arcs have the same ratio as the radii of their circles — which is what had to be demonstrated.

Appendix.VI.

In more northerly places the twilights are longer, whether the Sun occupies the northern signs, or the southern, or even the equinoctial points. Moreover, as the right sine of the complement of the lesser altitude of the pole is to the sine of the complement of the greater, so is the difference of the versed sines of the true and the

apparent semi-nocturnal arcs of the more northerly place to the difference of the versed sines of the true and the apparent semi-nocturnal arcs of the other place.

Let *a b c d* be the meridian circle about the centre *n*; let the straight line *a c* be the common section of the equinoctial; and let *i k* be the diameter of whichever northern parallel the Sun describes as it moves through the northern signs. Let the straight line *b d* be the axis of the sphere, the point *b* the north pole and *d* the south. Let *e f* be the section of that horizon above which the north pole itself is raised by the arc *b f*,

but let *o p* be the section of the horizon of those places for which the same pole is raised higher.

I say first that on the very day on which the Sun by its diurnal motion describes the parallel we have assumed, the twilight is longer in the more northerly place. For let the straight line *q r* be the common section of the meridian and of the circle parallel to the horizon of the more northerly place; and let the straight line *g h* be the common section of the meridian and of the circle parallel to the horizon of the other place, which lies nearer the equinoctial. Let the arc *f h* be put equal to the arc *p r*, and each of them equal to the Sun's distance from the horizon at the moment when it begins to light the upper hemisphere before sunrise, or ceases to light it after sunset. Moreover, from the points *l* and *s*, where the diameters *e f* and *o p* meet the straight line *i K*, let the perpendiculars *l y* and *s x* be dropped to the straight lines *g h* and *q r*.

Diagram. A large woodcut circle — the meridian — standing at the foot of the left column. A vertical diameter runs from *b* at the top, through the centre *n*, to *d* at the foot. Two horizontal chords cross it: an upper one whose right-hand end is lettered *K*, and a lower one through the centre, lettered *a* at the left and *c* at the right. Four further straight lines run obliquely across the whole figure from the upper right down to the lower left; two of them pass through the centre *n*, and their upper ends are lettered *f* and *h* at the top of the circle and *p* on the right side, with one further letter beneath *p*. Where these oblique lines cut the upper chord, a little to the right of the vertical diameter, stands a crowded cluster of very small interior letters.

- *b* (top of the circle, at the head of the vertical diameter): the north pole
- *d* (foot of the vertical diameter): the south pole
- *n* (centre of the circle, where the oblique lines converge): the centre of the meridian

- *a* (left end of the middle horizontal, on the circumference): the left end of the equinoctial section *a c*
- *c* (right end of the middle horizontal, on the circumference): the right end of the equinoctial section
- *K* (right end of the upper horizontal chord): the right end of *i K*, the diameter of the Sun's parallel
- *f* (upper right, on the circumference): the upper end of the horizon section *e f*
- *h* (immediately to the right of *f*): the upper end of *g h*, the section of the circle parallel to that horizon
- *p* (on the circumference at the right, below *f* and *h*; inked as a capital *P*): the upper end of *o p*, the horizon section of the more northerly place
- *m* (just above the upper chord, right of the vertical diameter): the vertex of the triangle *y l m*, on *g h*
- *s* (just above the upper chord, to the right of *m*): where *o p* meets *i K*
- *y* (just below the upper chord, beneath the preceding pair): the foot of the perpendicular *l y*
- *x* (below the chord, to the right of *y*): the foot of the perpendicular *s x*
- *t* (below the chord, to the right of *x*): where *o p* meets *q r*
- *z* (at the right of the same cluster, near the end of the upper chord): the point taken on *t x* produced
- *l* [?] (immediately to the left of *m*, above the chord): the glyph resolves only as far as *l* or *f*; the point where *e f* meets *i K*
- [?] (upper left, on the circumference at the left end of the upper chord): not resolved
- [?] (three small letters crowded together against the lower-left arc, at the left-hand ends of the two oblique diameters and the two chords parallel to them): not resolved
- [?] (immediately below *p*, on the circumference at the right): the glyph reads as either *t* or *r*; not resolved

Diagram. A small right-angled triangle at the lower left of the woodcut block, standing clear of the meridian circle below and to the left of it. A horizontal base runs from left to right; a perpendicular rises from a point on that base to the apex, and the hypotenuse falls from the apex to the acute vertex at the right-hand end of the base. An arc is struck about that right-hand vertex, sweeping from the apex down to the left-hand end of the base.

- *a* (left end of the base, where the arc meets it): the end of the arc *a l*
- *y* (on the base, at the foot of the perpendicular): the right angle of the triangle
- *m* (the acute vertex at the right end of the base): the vertex of the angle *l m y*, and the centre about which the arc is struck
- [?] (the apex, directly above *y*): the letter does not resolve; the construction names it *l*

Diagram. A second small right-angled triangle at the lower right of the woodcut block. Its base runs from an acute vertex at the left to the right angle at the right, where a vertical side rises to the apex; the hypotenuse closes the figure from the left-hand vertex up to the apex. An arc is struck about the left-hand vertex, cutting the hypotenuse and the base, and from the point where it cuts the hypotenuse a short perpendicular falls to the base.

- *t* (the acute vertex at the left end of the base): the vertex of the angle *s t x*
- *s* (the apex, at the top of the vertical side): the apex of the triangle
- *b* (on the hypotenuse, where the arc cuts it): one end of the arc *b c*
- *d* (on the base, below *b*): the foot of the perpendicular *b d*
- *c* (on the base, to the right of *d*): the other end of the arc *b c*
- [?] (at the right end of the base, at the foot of the vertical side): the letter does not resolve; the construction names it *x*

We shall then consider the two right triangles *y l m* and *x s t*. In them the side *l y* is equal to the side *s x*, for each of them is equal to the right sine of the arc *f h* or of the arc *p r*, as is clear from what we demonstrated earlier. The angle *l m y* is equal to the angle *a n e*, the altitude of the equinoctial, by the second part of *Elements* I.29 taken twice; and the angle *s t x* will be shown in the same way to be equal to the angle *a n o*, the altitude of the equinoctial for the other place. But the angle *a n e* is greater than the angle *a n o*; therefore the angle *l m y* is greater than the angle *s t x*.

Further, the side *t x* cannot be equal to the side *m y*. For since the angles at *y* and *x* are right and the sides *l y* and *l x* are themselves equal, the two angles *l m y* and *s t x* would be equal to one another by *Elements* I.4. Nor can *t x* be less; for suppose the straight line *t x* were produced beyond *t* until it became equal to *m y* — say at the point *z* — and *s z* were joined. Then in the two triangles *y l m* and *x s z* the two angles *l m y* and *s z x* would be equal to one another, by *Elements* I.4; but the angle *s t x* is greater than the angle *s z x*, by *Elements* I.16, and so it would be greater than the angle *l m y*, by the common notion. Since we have just

demonstrated the opposite, it follows that the straight line *t x* is greater than the straight line *m y*. The two squares on *s x* and *t x* will therefore be greater than the squares on *l y* and *m y*; and so, in those two right triangles *y l m* and *x s t*, the square on *s t* will be greater than the square on *l m*, by *Elements* I.47 and the common notion. The straight line *s t* itself will therefore be greater than *l m*, being nearer the centre.

For this reason the perpendiculars standing on the plane of the meridian at the points *l* and *m* will cut off a smaller arc of the circumference of the parallel than those standing at the points *s* and *t*, by what we demonstrated in the lemma of Appendix III. And since these latter sections are the common sections of the assumed parallel with the horizon of the more northerly place and with the circle parallel to that horizon below the earth, while the former are the common sections of the same parallel with the horizon of the other place and with the circle parallel to that horizon, it follows that when the Sun occupies the northern signs the twilights are longer in more northerly places — which was the first thing to be demonstrated.

We shall demonstrate the second thing we proposed in this way. It is clear from what was shown in the first lemma and in Appendix III that the common sections made by the Sun's assumed parallel, both with the horizons and with the circles parallel to them, are perpendicular to the plane of the meridian as drawn. Therefore the common section of that same parallel and of the horizon of the more northerly place will make right angles with the diameter *i k* at the point *s*; and that common section, terminated at the point *s*, will be the right sine of the true semi-nocturnal arc — the arc that is usually reckoned from sunset to the angle of midnight [*angulus mediae noctis*], or from the angle of midnight to sunrise. And so the straight line *s K* is the versed sine of that same semi-nocturnal arc. Moreover, the common section of the assumed parallel and of the circle parallel to the horizon of the more northerly place, which makes right angles with the straight line *i k* at the point *t*, is the right sine of the apparent semi-nocturnal arc; and therefore the straight line *t K* will be the versed sine of that arc. We call that the apparent semi-nocturnal arc which is left of the true semi-nocturnal

arc once the interval of twilight has been subtracted. The straight line *s t* will therefore be the difference of the versed sines of the true semi-nocturnal arc and of the apparent semi-nocturnal arc.

In the same way it will be demonstrated that the straight line *l m* is the difference of two versed sines, one answering to the true semi-nocturnal arc and the other

to the apparent semi-nocturnal arc of the other place — the place that lies toward the equinoctial, whose altitude of the pole is the arc *b f*. Now the complement of this arc is the arc *c f*, which subtends at the centre of the circle the angle *f n c*; and that angle is equal to the angle *l m y*, lying opposite it in the parallelogram, as *Elements* I.34 proves. Likewise the arc *c p* is the complement of the arc *b p*, the altitude of the pole of the more northerly place, and it subtends the angle *p n c*, which is equal to the angle *s t x*, lying opposite it in the parallelogram.

With all this now established, we shall conclude our demonstration as follows. In the triangle *y l m*, as the right sine of the angle *l m y* is to the whole sine, so is the straight line *l y* to the straight line *l m*. Again, in the triangle *x s t*, as the whole sine is to the right sine of the angle *s t x*, so is the straight line *s t* to the straight line *s x*. And since the straight lines *l y* and *l x* are equal to one another, it follows by *Elements* V.7 that as the whole sine is to the sine of the angle *s t x*, so is *s t* to the straight line *l y*. Therefore, by *Elements* V.23, as the sine of the angle *l m y* is to the sine of the angle *s t x*, so is the straight line *s t* to the straight line *l m*. But the sine of the angle *l m y* is equal to the sine of the complement of the arc *b f*, and the sine of the angle *s t x* is equal to the sine of the complement of the arc *b p*; and the arc *b f* is the altitude of the first place, the lesser of the two, while the arc *b p* is the altitude of the second place, the greater. Therefore, as the right sine of the complement of the lesser altitude of the pole is to the right sine of the complement of the greater altitude, so is the difference of the versed sines of the true and the apparent semi-nocturnal arcs of the more northerly place to the difference of the versed sines of the true and the apparent semi-nocturnal arcs of the place of lesser altitude — which is what we proposed to demonstrate. And this once again establishes the first part.

Although we have framed the present demonstration for the Sun's northern parallel, we shall nonetheless be able to apply the very same demonstration, without any variation at all, to the southern parallels, and likewise to the circle of the equinoctial. On the equinoctial, indeed, those same straight lines that we called the differences of the versed sines of the true and the apparent semi-nocturnal arcs are also equal to the right sines of the magnitudes of the twilights, since each of them terminates at the centre of the equinoctial.

Lemma.

In the demonstration it was assumed that the right sine of the angle *l m y* is to the whole sine as the straight line *l y* is to the straight line *l m*; and further, that in the triangle *x s t*, as that same whole sine is to the right sine of the angle *s t x*, so is the straight line *s t* to the straight line *s x*. To show this, let the straight line *m y* be

produced in a straight line, and about the point *m*, with radius *l m*, let *a l*, the arc of the angle *l m y*, be described. Then about the point *t*, to the measure of the same radius *l m*, let *b c*, the arc of the angle *s t x*, be described, and from the point *b* let the perpendicular *b d* be dropped to the straight line *t x*.

The first part of the lemma is then perfectly clear: the same straight line *l y* is the right sine of the angle *l m y*, and the straight line *l m* is the whole sine, that is, the radius of the circle. The second part is evident as well, for the two triangles *x s t* and *d b t* are equiangular, by *Elements* I.32 and the common notion. Therefore, by *Elements* VI.4, as *b t*, the whole sine, equal to the former one, is to *b d*, the right sine of the angle *s t x*, so is the straight line *s t* to the straight line *s x*.

But so that the demonstration of our appendix may establish what we proposed to prove in the second place, it is worth referring all these ratios to that whole sine which is equal to the radius of the meridian as drawn. For this reason we shall produce the straight lines *l m* and *m y* in a straight line, to a length equal to the radius of the meridian as drawn:

—dian; and likewise *s t*, *t x*; and about the centres *m*, *t* let us describe circumferences on which the angles *l m y*, *s t x* are subtended. From these let us draw their right sines — that is, perpendiculars onto the straight lines *m y*, *t x*, which we produced to a length equal to the radius of the meridian. Therefore, just as we demonstrated in the case of the two triangles *x s t*, *d b t*, we shall show in these figures as well that the right sine of the angle *l m y* is to the whole sine — that is, to the radius of the circle, which is equal to the radius of the meridian as drawn — as *l y* is to *l m*. Again, the whole sine, equal to the radius of that same meridian, is to the right sine of the angle *s t x* as the straight line *s t* is to the straight line *s x*, because the angles are equal and the triangles similar.

From this, then, what the appendix set out to prove follows correctly. For proposition 23 of the fifth book (*Elements* V.23) proves that as *s t* is to *l m*, so is the right sine of the arc of the angle *l m y* to the right sine of the arc of the angle *s t x* — and this in equal circles, with the meridian drawn as described. Now the sine of the angle *l m y* is equal to the sine of the complement of the arc *b f*, and the sine of the angle *s t x* is equal to the sine of the complement of the arc *b p*, since in equal circles equal angles are subtended by equal arcs, by proposition 26 of the third book (*Elements* III.26); and equal arcs have equal sines, as in the first lemma. Accordingly, by the seventh proposition of the fifth book (*Elements* V.7) it follows that the straight line *s t* bears the same ratio to the straight line *l m* as the sine of the complement of the arc *b f* bears to the sine of the complement of the arc *b p*.

PARS SECVNDA.

PROPOSITIO PRIMA.

To demonstrate that the arc of the Sun's distance from the horizon, at the beginning of morning twilight or at the end of evening twilight, cannot be fixed, but must necessarily vary with the changing seasons.

But surely that first assumption must be judged false. For no distance of the Sun from the horizon in the lower hemisphere — the distance that produces twilight — can be fixed and settled. Morning twilight begins at the moment when the air in our hemisphere starts to grow bright, and it starts to do so when the Sun's light can first be reflected at the surface of the horizon. It can be so reflected when the air it meets is not entirely pure, but, through the admixture of vapours, thicker and denser than air that stands too far off from the earth. So if the vapours happen to be far distant from the earth, the Sun's light will at that time be reflected from a larger arc below the horizon; but if they are only slightly distant, from a smaller one. Yet it is plain that the highest elevation of the vapours is subject to variation, and that one elevation is loftier than another as the seasons change. Therefore the vertical arc itself [?], by which the Sun stands off from the horizon, will not remain fixed and settled either.

But so that this may stand out more clearly, it is worth setting out the causes of twilight. Let *a b c d* be a great circle of the earth, lying within the surface of a certain great vertical circle which before daybreak passes through the Sun and through the zenith of the region. Let *a* be the centre, and let the straight line *e a f* touch the circle as drawn at the point *a*. Let the cone of the earth's shadow at this same hour of night be understood as *d f g b*. The air lying within a shadow-cone of this kind will not be lit by the Sun. And although the solar ray does reach the point *f* and other points outside the shadow, the air there will nevertheless not appear illuminated, because on account of its great distance from the earth it is purer and thinner than can produce a reflection. Let us therefore conceive the Sun to move up to the beginning of morning twilight — that is, to the moment when the air begins to grow bright.

Diagram. A circle (the earth) set below a horizontal straight line that touches it at its uppermost point. Two long straight lines descend from the upper left and upper right, crossing one another just above the horizontal line and running down past the circle on either side, so as to bound the shadow-cone. Labels read on the leaf:

- *e* (horizontal line, left end): left extremity of the tangent line
- *a* (horizontal line, at the point of contact with the circle): the point of tangency and centre of vision
- *i* (horizontal line, right of *a*): the point where the two oblique lines cross
- *f* [?] (horizontal line, right end): right extremity of the tangent line
- *h* (just below the horizontal line, near *i*): vertex of the shadow-cone within the figure
- *b* (upper left of the circle)
- *d* (right of the circle)
- *c* (foot of the circle)
- *K* (upper left, at the head of the left-hand oblique line)
- *l* [?] (upper right, at the head of the right-hand oblique line)

And let the circle *a b c d* again represent the greatest circle of the earth, beneath that vertical circle which at that very moment of time passes through the Sun; and let *a* be the point at the centre of vision, at which indeed the straight line *e a i* is tangent [?]; and the cone of the earth's shadow will be *h i K l*. Therefore, since the Sun's light is first reflected to the sight from the point *i* itself, and the air at *i* is the first to appear illuminated, the highest elevation of the vapours — the elevation that makes the air thicker and thereby visible — will be at *i*. But since the greatest height of vapours of this kind is variable, being sometimes less distant from the earth than *i* and sometimes more, according to the varying action of the Sun upon the matter out of which it raises the vapours, it is certain — as anyone will readily be able to demonstrate — that when that height stands off from the earth less than *i* does, the Sun itself will also be less distant from the horizon; but if more, then likewise it will stand off from the horizon by a greater arc. The Sun's distance from the horizon at the beginning of morning twilight, or at the end of evening twilight, cannot therefore remain fixed and settled — which was to be demonstrated.

[Margin: Strabo.]

Strabo, however, at the close of the second book of the *Geography*, sets this distance at 17 and a half degrees. For around the Borysthenes, he says, along the Bear-ward parts of the horizon, on almost all summer nights the sky is lit by the Sun, the light circling round from setting to rising. For the summer tropic stands off from the horizon by half and a twelfth part of one sign [*animalculum*, rendering Greek ζῴδιον (zōdion, "zodiacal sign")] — as far as the Sun stands off from the horizon in the dead of night. From this it is drawn out that the north

pole in those places is raised above the horizon by 48 and a half degrees. For he laid down that the distance between the equinoctial and the tropic amounts to four sixtieth-parts of a great circle, namely 24 degrees. The distance of the summer tropic from the north pole must therefore at that time have been 66 degrees; and when 17½ are subtracted from these — the amount contained in the half and the twelfth part of one sign — there remain 48½ degrees for the altitude of the pole above the horizon. The same follows also from the number of stades by which he supposed those places to be distant from the equinoctial.

[Margin: Hipparchus.]

The same author also reports that Hipparchus said that at the Borysthenes and in Gaul, through whole summer nights, the Sun's brightness shines as it passes round from setting to rising. But all of this seems doubtful to me, especially because these statements appear to be at odds with one another.

[Margin: Allacen.]

Alhazen, for his part, asserts that this distance of the Sun from the horizon amounts to 19 degrees, without support from any reasoning.

[Margin: Vitelo,]

Then Witelo, who is later than Alhazen and from whom he borrowed nearly the whole of that vast work of his on the theory of vision, likewise writes that it comprises 19 degrees, and that he had established this with the instrument of armillae, or by tables, through astronomical observation. Finally, all the more recent writers maintain that the said distance amounts to 18 degrees. Now if each of these authors truly did determine the Sun's distance from the horizon at the beginning of morning twilight to be as great as he claimed, we can no longer deny that it is variable; and if he did not, then no authority of theirs need move us away from believing that a distance of this kind is subject to variation. But while we are labouring to hand down an art

by which a distance of this kind may be correctly determined, we shall meanwhile suppose it to be as great as the more recent astronomers make it — namely, 18 degrees.

Propositio. II.

To find the declination of a given point of the ecliptic. For the ratio of the whole sine to the right sine of the greatest declination is as the ratio of the right sine of the arc

of distance from the vernal or autumnal section to the right sine of the declination of that same point.

Let the circle *a b c d* be the colure that distinguishes the solstices; *b*, *d* the poles of the ecliptic; *f*, *p* the poles of the equator. Let the common section of this circle and the ecliptic be the straight line *a c*, and let the straight line *g h* be the common section of the same colure and the equator. Since, then, these common sections are diameters of great circles — this by Theodosius — they will intersect one another at *e*, the centre of the world.

Diagram. A circle — the solstitial colure — crossed by a set of straight chords. One chord runs from the left of the circle across to the right (the common section of colure and ecliptic); a second, inclined to it, is the common section of colure and equator; two further parallel chords, drawn in the lower left quadrant, are the common sections of the parallel circles; and short lines are let fall perpendicular between them. Labels read on the leaf:

- *b* (top of the circle)
- *f* (upper left, outside the circle)
- *h* (upper right, within the circle)
- *K* (upper right, on the inclined chord)
- *r* (right, inside the circle)
- *c* (right extremity of the circle)
- *s* (centre-right, an intersection of chords)
- *e* (centre of the figure)
- *l* [?] (left of centre, an intersection of chords)
- *a* (left extremity of the circle)
- *o* (below and left of centre)
- *y* (lower left, within the circle)
- *g* (lower left, within the circle)
- *q* (lower left, below *g*)
- *d* (foot of the circle)
- *P* (bottom right, outside the circle)

Further, let the common section of the colure as drawn with a circle parallel to the equator that passes through the given point of the ecliptic be either the straight line *y K* or the straight line *q r*. Let the intersections of these with the straight line *a c* be the points *l*, *s*; and from these let two straight lines *l o*, *s t* be drawn perpendicular to the straight line *g h*. Now since the poles of the equator

and of the ecliptic lie on the colure itself, the colure cuts each circle at right angles, by proposition 19 of the first book of Theodosius (*Spherics* I.19). Their common section will therefore be at right angles to the plane of that same colure, and its extreme points must terminate at the beginnings of Aries and Libra. By like reasoning it will also be shown that the common sections of the parallel circles and of the ecliptic are at right angles to that same plane of the colure at the points *l*, *s*.

Therefore, if we set *a* as the beginning of Cancer and *c* as the beginning of Capricorn, then the part of the common section standing at *l*, terminated at *l* itself, will be the right sine of the distance of the given northern point from the beginning of Cancer; and the straight line *e l* [?] will be equal to the right sine of the complement of the quadrant — that is, of the distance of the given point from the beginning of Aries or Libra — by propositions 28 and 34 of the first book of Euclid (*Elements* I.28 and I.34). Likewise the part of the common section standing at *s*, terminated at the point *s* itself, will be the right sine of the distance of the given southern point from the beginning of Capricorn; and the straight line *e s* will be equal to the right sine of the distance from the beginning of Aries or Libra. But since the straight lines *g h*, *y K* are parallel, by proposition 16 of the eleventh book of Euclid (*Elements* XI.16), while the straight line *l o* is parallel to the right sine of the arc *y g*, by proposition 28 of the first book (*Elements* I.28), it follows by proposition 34 of that same book (*Elements* I.34) that the straight line *l o* will be equal to the right sine of the arc *y g*. But, as we demonstrated in the first lemma, the arcs lying between parallel circles and belonging to those great circles which pass through the poles of the parallels themselves are equal; and equal arcs have equal right lines. Therefore, by the common notion, the straight line *l o* will be equal to the right sine of the declination of the given northern point, and the straight line *s t* equal to the right sine of the declination of the given southern point.

Further, in the right-angled triangle *e l o*, the whole sine is to the right sine of the arc *a g* — the arc of the angle *o e l*, the arc of greatest declination — as the straight line *e l* is to *l o*, by the lemma of the sixth appendix. Therefore, by the seventh proposition of the fifth book (*Elements* V.7), it will hold that as the whole sine is to the right sine of the arc of greatest declination, so is the right sine of the distance of the given northern point from the nearest vernal or autumnal section to the right sine of the declination of that same point. The same will be proved in the right-angled triangle *e s t*. For as the whole sine is to the right sine of the arc *c h*, the greatest declination of the ecliptic, which subtends the angle *t e s*, so is *e s* — equal to the right sine of the distance of the given southern point from the

nearest section — to *s t*, which is equal to the right sine of the declination of that same point.

Accordingly, we shall multiply the right sine of the arc of the ecliptic by which the given point stands off from the nearest section into the right sine of the greatest declination; we shall divide the product by the whole sine, casting away the last five figures; and there will come out of a division of this kind the right sine of the declination of the given point of the ecliptic. The declination itself will then be known from the table of right sines: northern, if the given point has its place in the northern signs; southern, if in the southern. But if the declination were the datum and the arc of distance unknown, one would have to carry the first of those four proportional terms into the fourth, and divide the product by the second

divide, for from a partition of this kind the third term would come out known — namely the right sine of the distance sought. These rules of proportional numbers are drawn from the sixteenth proposition of the sixth book, or the nineteenth of the seventh, of Euclid (*Elements* VI.16; *Elements* VII.19).

From this method of demonstration it is also clear that points of the ecliptic which stand off by an equal interval from either section, the vernal or the autumnal, have equal declinations. For those two triangles, e l o and e s t, are equiangular; and so, if the arcs of distance are put equal, we shall conclude — by *Elements* VI.4 or *Elements* I.26 — that the straight line o l is equal to the straight line s t. The right sines of the declinations are therefore equal, and so too are the arcs themselves, a result usually demonstrated by other chains of reasoning.

This further makes it clear that points of the ecliptic which stand off by an equal interval from either of the tropic points have equal declinations, since they are contained under one and the same circle parallel to the equator. For the straight line that is the common section of the ecliptic and of the circle parallel to the equator — the line which cuts the colure at right angles at l — makes right angles at that same point l with the diameter a c, by the second definition of Book XI (*Elements* XI, def. 2). Therefore, by what we demonstrated in the first lemma, this common section is divided at the point l into two equal right sines of equal arcs terminating at the point a. The last part of that same first lemma demonstrated this separately as well.

Propositio.III.

To construct a certain instrument, very well suited to observations of the stars, by which their altitudes may be determined exactly.

Let an astrolabe be constructed as exactly as it can be made, and let it have a dioptra — that is, a rule turning upon the centre — as straight as possible. To this let sight-vanes [*tabellæ*] be set upright, as is usually done, whose apertures should be no wider than allows the brighter fixed stars to be seen distinctly through them. For example, let the single flat and circular surface of an astrolabe of this kind be a b c d, divided into quadrants by the diameters a c and b d, and let its centre be the point e. Upon this surface, within that same circumference, let quadrants of circles be described one inside another, at any interval you please (it makes no difference whether even or odd), forty-four in number. Let the outer quadrant, a b, be divided into ninety equal parts; let the one of the inner quadrants nearest to it be divided into 89 equal parts; the next after that into 88; the one that follows immediately upon this into 8[?]; and let the process go forward in this order until the last and smallest of the inner quadrants is reached, which will be cut into 46 equal parts.

Diagram. A circle divided into four quadrants by two perpendicular diameters. Within the upper-left quadrant a close-set nest of concentric quadrant arcs is drawn, running from the circumference inward toward the centre. A straight rule is laid across the figure through the centre, cutting the nested arcs obliquely, and a second line runs from the centre up into the graduated quadrant.

- b (top of the circle, upper end of the vertical diameter)
- d (foot of the circle, lower end of the vertical diameter)
- a (left end of the horizontal diameter)
- c (right end of the horizontal diameter)
- e (centre of the circle)
- [?] (two further small marks — one inside the upper-right quadrant, one at the left of the figure — do not resolve)

In each quadrant let every tenth part be marked with certain very fine little lines running a little past the circumference. For unless the astrolabe were of enormous size, great confusion would arise from the excessive narrowness of the intervals if every fifth or tenth part were distinguished by numerals. But the number of parts that each quadrant has should be written near one of its ends, alongside the radius. Thus, if the reckoning runs from a toward b, let 90 be written over the point b itself in the algorists' figures [*notis algoristicis*], and beneath it, alongside the diameter e b, the remaining numbers will be set in their several proper places.

By this device, then, the number ninety of degrees — which each quadrant, the inner ones too, is understood to have, even though it is set out as divided into fewer parts — actually possesses every aliquot part named from any number less than ninety: namely the half of the whole, the third, the fourth, the fifth, the sixth, the seventh, the eighth, the ninth, the tenth, the eleventh, the twelfth, and the rest one by one as far as the ninetieth, which the outer quadrant actually has. For no one will deny that, moving up from the smaller parts to the greater as far as the forty-sixth, it has aliquot parts — the ninetieth, the eighty-ninth, the eighty-eighth, and the rest. And that it has all the others as well, those named from the numbers lying between unity and 46, can easily be established from this: whoever divides some number into a number divides also its half, its quarter, and the other submultiples that the dividing number has. Thus one who divides into ninety divides also into

forty-five; and one who divides into 88 divides also into 44; and so on with the rest. And indeed the individual numbers from 23 to 45 are the halves of those which are set out in the series of numbers from 46 to 90, one being always skipped; and these too are multiples of other, smaller numbers; and so with the rest, each standing to the others in the same way, right down to unity. Therefore the very number ninety of degrees which we understand to be contained in each quadrant has, through the divisions described above, every aliquot part, from the half up to the ninetieth.

So much for the structure of the instrument; its use will be very easy. Suppose one wishes, at night, to determine exactly the altitude of any star above the horizon. We shall raise an astrolabe of this kind aloft above the eye, so that it hangs free from the suspension ring fixed at the point b, and we shall direct its side a b toward the star itself; then we shall turn the dioptra gently up and down until we sight the star under observation through both apertures.

But since the dioptra is scarcely ever laid across the quadrants described without cutting one of them at some mark of division, we shall consider the number of whole parts contained in the portion cut off, and besides that the number into which the whole quadrant has been divided; and by the common rule of proportional numbers we shall convert these parts into ninetieths of the quadrant — the parts we usually call degrees — in the following way. We shall multiply their number by ninety, divide the product by the number of parts of the whole quadrant, and there will come out of that division the number of degrees which those parts have. But if some number is left over from the division (as very often happens), we shall multiply it by sixty, divide the product by that

same number of parts of the whole quadrant, the common divisor, and first minutes will come out. The number left over from this division we shall again multiply by sixty, divide the product by the common divisor, and second minutes will come out; and so it will go on until either nothing is left from the division, or the small quantities that come out of the division ought to be neglected on account of their smallness.

An example. The altitude of some star having been observed, let the outermost line of the dioptra, the one passing through the centre — which astronomers call the fiducial line [*fiduciæ linea*] — hold the position of the diameter f g; let it cut the quadrant i r, which is of 87 equal parts, at the point o; and let the arc of altitude o i comprise thirty parts. We shall therefore multiply 30 by 90, and 2700 will result. This number we shall divide by 87, and 31 degrees will come out of the partition, but 3 will be left over. This number we shall multiply by 60, and 180 will result. Finally we shall divide 180 by 87, the common divisor, and two first minutes will come out of the partition, and the number left over will be 6. This we shall then multiply by 60 to collect the second minutes, and 360 will result. We shall divide this by 87, and four third [?] minutes will come out of the partition, but the number left over will be 12. This, then, multiplied by 60, and the product divided by the common divisor, will give eight fourth [?] minutes, and 24 will be left over from the partition. And we shall go forward by exactly the same method as long as we please.

For the rest, so that an instrument of this kind may serve more conveniently for observations of the Sun, let two further, very narrow apertures be made in the upright sight-vanes; for the Sun's ray entering through these by day will show its altitude above the horizon more surely.

Propositio.IIII.

From the meridian altitude of the Sun, to find the elevation of the pole above the horizon of the place at which the observation is made — that is, the latitude of the region.

Through the known place of the Sun let its declination be obtained; and let this be added to the quadrant if it is southern, but subtracted if it is northern. For the number that comes out of an addition or subtraction of this kind will be the distance of the Sun from the arctic pole of the world. Next, whether the pole of the horizon lies between the Sun and the arctic pole, or on the contrary the Sun is set between the pole of the horizon and the arctic pole of the world, we shall elicit from the meridian shadow cast on the surface of the horizon. For if that shadow

points toward the north, it is clear that the pole of the horizon is situated between the Sun and that northern pole; but if toward the south, the Sun must have its position between the arctic pole of the world and the pole of the horizon. With these things known beforehand, then, we shall observe with the astrolabe — whose construction we taught in the preceding proposition — the greatest altitude of the Sun; and the Sun must have this at the time of the meridian. We shall then reckon on the astrolabe the complement of this greatest altitude of the Sun, namely the distance between the pole of the horizon and the Sun; and this we shall subtract from the arc by which the Sun stands off from the arctic pole of the world,

if the pole of the horizon has been found to lie between the two; but we shall add the same if, on the contrary, the Sun has had its place between the pole of the horizon and the arctic pole of the world. For the arc that is either left by a subtraction of this kind or made up by an addition will be the distance of the pole of the horizon from the arctic pole of the world. And now the latitude of the place we inhabit cannot fail to be known. For if that arc is equal to a quadrant, the pole of the horizon is of course set under the equator. But if it is unequal, its difference from the quadrant will be called the latitude of the place — northern if the arc found is less than a quadrant, southern if greater.

Where, however, the meridian altitude of the Sun is equal to a quadrant, the latitude of the place at which this is observed and the declination of the Sun will be equal to one another. Further, that the latitude of a place is equal to the altitude of the pole of the world above the horizon is demonstrated by the common notion alone.

One must also remember that there are certain places for which, at a certain season, the Sun neither rises nor sets, but is seen perpetually above the horizon: above the horizons of such places it has two meridian altitudes within twenty-four hours, one greatest and one least. In these cases too we shall use the greatest altitude, and the operation will not vary at all.

Latitudes of places can moreover be found by day, apart from noon. For, as may be seen in the commentary on the art of navigation which we composed in our mother tongue, we have devised a method by which, at any time of day, with the hour and the position of the meridian unknown, the elevation of the pole of the world above the horizon may be found, and together with it the hour and the position of the meridian itself; and this even if, wandering in mid-ocean or living in the wastes, we were ignorant not only of the hour and the position of the

meridian, but also of the place of the Sun and its declination, and finally of the year and the day on which an observation of this kind is made.

Propositio.V.

From the given latitude of a place, or the altitude of the pole above the horizon, to determine the declination of a star occupying the meridian.

By the third proposition let the altitude of the star in question be observed exactly when it has taken possession of the meridian. Then, if it has receded from the pole of the horizon toward the parts of the visible pole [*manifestus polus*] — the one seen elevated — we shall join the complement of the altitude of that same star to the arc of the latitude of the place at which the observation is made. For the number made up of these two, if it does not exceed a quadrant, will be the declination of the star itself. But if it is found greater than a quadrant, we shall subtract it from a semicircle, and the declination of the star in question will be left, of the same denomination as the latitude of the place.

But if the star has receded from the pole of the horizon toward the parts of the hidden pole, then, a comparison having been made between the latitude of the place and the complement of the altitude of the star: if the two are found equal, the star in question will have no declination. But if they are unequal, let the lesser number be subtracted from the greater, and the declination of that star will be left, of the same denomination as that which the latitude of the place has, if the latitude itself has been found the greater — but of the opposite denomination if greater [?]. If in truth no distance at all is found between the star and the pole of the horizon, the declination of the star will be equal to the latitude of the place, and toward the same side. The demonstrations of this and of the preceding proposition, being very easy, we have deliberately passed over.

Propositio.VI.

From the given longitude and latitude of a star, to find its declination; and conversely, from its latitude and declination, to find its longitude and its right ascension. For as the square of the whole sine is to the rectangle contained by the right sines of the greatest declination of the ecliptic and of the complement of the star's latitude, so is the versed sine of its longitude — taken as beginning from either of the tropic points — to a certain straight line, which we shall not without reason call the argument of the declination [argumentum declinationis]. For if that line is equal to the right sine of the complement of the difference of the two arcs named above, no declination at all will be had. But if it is unequal, the difference of

those straight lines will of course be the right sine of the declination sought: of the same denomination indeed as

— as the latitude, if it is smaller; but of the opposite denomination, if greater. Moreover, if the latitude is northern, the star's longitude must be reckoned from the first point of Cancer in the order of the signs, provided the star lies in the descending half of the ecliptic; the reverse holds if it lies in the ascending half. But if the latitude is southern, the reckoning begins from the first point of Capricorn, in the contrary order.

Another way. If the star under consideration lies between the arctic pole of the ecliptic and that southern parallel which everywhere recedes from the ecliptic by an arc equal to the greatest declination, then: as the square of the whole sine is to the rectangle contained by the right sines of the greatest declination of the ecliptic and of the complement of the star's latitude, so is the versed sine of its longitude reckoned from the first point of Cancer to a certain straight line. If that line is found equal to the right sine of the complement of the difference of two arcs — one of them the greatest declination itself, the other the distance of the proposed star from the northern pole of the ecliptic — no declination at all will be had. But if it is unequal to that right sine, the difference of those straight lines will of course be the right sine of the declination sought: northern if the line is smaller, southern if greater. If, however, the proposed star lies on the parallel circle just conceived, the fourth term of the proportion will itself be the right sine of its southern declination. And if the star lies outside that circle, the fourth term of the proportion, taken together with the right sine of the arc by which the star's latitude exceeds the greatest declination, will make up the right sine of the southern declination.

But if the proposed star has no latitude at all, then as the whole sine is to the right sine of the greatest declination of the ecliptic, so is the right sine of the arc by which the star is distant from the nearer section — vernal or autumnal — to the right sine of the declination it has.

Although we could find in many ways what the present problem sets out for investigation, I preferred to use our own [?] art of demonstrating, and the same figures we have used from the beginning of this little work — and not without reason. For besides the fact that on this method the business is settled with very few multiplications and divisions, diagrams of this kind have a certain beauty about them, which I have been accustomed, so far as I could, to work into my demonstrations elsewhere. They render the surfaces of the celestial circles so faithfully in the plane of a single meridian that in them, as in a kind of

instrument, we can find out what is proposed for investigation without any working of numbers.

Let the circle a b c d, whose centre is e, be the colure that passes through the first points of Cancer and Capricorn; let the point f be the arctic pole of the world, and b the nearer pole of the ecliptic; let the straight line a c be the section of the ecliptic, and g h the section of the equator. Let us suppose further that the star whose declination we wish to measure has a northern latitude equal to the complement of the greatest declination of the ecliptic, as in the first figure. Let us conceive a circle parallel to the ecliptic passing through the centre of the star, and let its section be f i. Then let a circle parallel to the equator be conceived — the circle the star itself describes by the daily motion — and let its section be k l.

Diagram. A woodcut circle at the foot of the right column, captioned *1ª figuratio* inside the lower half of the circle. A horizontal diameter runs from a at the left to c at the right; a second diameter rises steeply across it from g at the lower left to h at the upper right, the two crossing at the letter set in the middle of the figure. From f, on the circumference at the upper left, a chord descends to the right; a second, shallower chord crosses it, and the two intersections in the upper part of the figure are marked m and, below and to the left of it, n. Two further letters are engraved along the top of the circle to the right of b; neither resolves. A further inked letter stands on the left-hand circumference above a — a position distinct from g, which sits below a — and it does not resolve either. The lettering here is smaller and more broken up by the heavy woodcut rules than in the four companion figures on the following leaves.

- f (upper left, on the circumference): f
- b (top of the circle): b
- [?] (top of the circle, immediately right of b — a single bare vertical stroke, i or l)
- [?] (top of the circle, right of the preceding — a faint round mark)
- m (inside, in the upper part of the figure, right of the vertical axis): m
- n (inside, below and left of m): n
- [?] (left, on the circumference above a — an inked letter that does not resolve; distinct from g, which stands below a)
- a (left, at the left end of the horizontal diameter): a
- e [?] (at the middle of the figure, where the diameters cross — the glyph is of the c/e shape and does not resolve between them)
- g (lower left, on the circumference): g

- h (right, on the circumference, above c): h
- c (right, at the right end of the horizontal diameter): c
- [?] (at the bottom of the circle — a heavily inked mark that does not resolve as a letter)

Therefore the straight line that is the common section of these two circles, terminated at the proposed star, will stand at right angles to the plane of the drawn colure at the point m, the intersection of the straight lines f i and k l, by Theodosius, *Spherics* I.19 and *Elements* XI.19. And so the straight line f m will be the versed sine of the distance of the proposed star from the beginning of Cancer, taken on the parallel to the ecliptic just conceived. That distance we must reckon in the order of the signs if the star's place falls in the descending semicircle of the ecliptic, but the other way about if it falls in the ascending semicircle, so that the distance comes out less than a semicircle. Now let f e be joined, cutting the straight line k l at the point n.

Since the straight lines g h and k l are parallel, and a c and f i are likewise parallel, by *Elements* XI.16, the angle f m n of the triangle n f m will be equal to the angle a e g, the angle of the greatest declination of the ecliptic, by the first part of *Elements* I.29, taken twice. But the angle f e h is right, because f g and f h are quadrants — which Theodosius, *Spherics* I.10 also demonstrates. Therefore the angle f n m will likewise be right, by the second part of that same twenty-ninth proposition of the first book.

Consequently, in the triangle f n m, as the whole sine is to the right sine of the arc of the angle f m n, so is the straight line f m — the versed sine of the star's longitude reckoned from the beginning of Cancer on the conceived parallel of latitude — to the straight line f n, by the lemma to the sixth of the appendix. But again, as that same whole sine, namely the radius a e, is to the right sine of the complement of the star's latitude (that is, the radius of the conceived parallel of latitude), so is the versed sine of the same star's longitude reckoned in degrees of the ecliptic to the straight line f m, by the lemma to the fifth of the appendix. Therefore the ratio of the versed sine of the star's longitude in degrees of the ecliptic to the straight line f n is compounded of these same ratios: those which the whole sine has to the right sine of the arc of the angle f m n — an angle equal to that of the greatest declination — and to the right sine of the complement of the star's own latitude.

Moreover, by *Elements* VI.23, with V.7 assisting: as the square of the whole sine is to the rectangle contained by the right sines of the greatest declination of the

ecliptic and of the complement of the star's latitude, so is the versed sine of the longitude the star has to the straight line f n.

Now the complement of the star's latitude is posited equal to the arc b f, the interval between the poles, and b f is itself equal to the arc a g of the greatest declination of the ecliptic, by the common notion. It follows that no difference at all is left between the greatest declination and the complement of the star's latitude. The whole quadrant g f can therefore in a manner be called the complement of the difference of those equal arcs, and the whole sine e f is the right sine of such a complement.

Here the straight line e f [?] exceeds f n, the fourth term of the proportion; and e n is equal to the right sine of the arc g k, by *Elements* I.28 and I.34. But the arc g k and the arc of the star's declination are equal, and they have equal right sines, by what we demonstrated in the first lemma. Therefore, by the common notion, when the straight line f n — the fourth term of the proportion — is subtracted from the straight line e f, the whole sine, which we have called the right sine of the complement of the difference of the two arcs named above, the right sine of the declination sought will be left. And by the common rule of proportional numbers the fourth term of the proportion itself becomes known; therefore the right sine of the declination sought is left known as well, and so the declination itself is known from the table of right sines.

But now let it be supposed, as in the second and third figures, that the star's latitude is again northern, but that its complement is unequal to the greatest declination. Let the section of the circle that passes through the centre of the star's body and is parallel to the ecliptic be z r, and the section of the circle parallel to the equator be s o. Then the arc b z will be equal to the complement of the star's latitude, while b f is equal to the greatest declination of the ecliptic; and so the arc z f is the difference between the greatest declination of the ecliptic and the complement of the star's latitude. Now from the point z draw the straight line z p perpendicular

Diagram. The upper of two woodcut circles in the right column, captioned *2ª figuratio* inside the lower half of the circle. A horizontal diameter runs from a at the left to c at the right, with a letter set at the middle where the lines cross. A steep diameter rises from g at the lower left to h at the upper right, and a second line, roughly parallel to it, crosses the figure a little higher. Two chords descend from the top of the circle toward the right, their far ends marked at the top by o and a further letter beside it; inside the upper part of the figure stand t and, lower and to the left, q, with p below the middle. Three inked letters stand on the left-

hand circumference — two together at the upper left, and a third lower down, immediately above a — and none of the three resolves.

- [?] (upper left, on the circumference — the leftmost of two letters standing there)
- [?] (upper left, on the circumference, right of the preceding)
- b (top of the circle): b
- o (top of the circle, right of b — a heavy round mark): o
- [?] (top right, right of o — reads r or z)
- t (inside, below and right of b): t
- q (inside, left of the middle, above the horizontal diameter): q
- h (right, on the circumference, above c): h
- [?] (left, on the circumference immediately above a — an inked letter that does not resolve)
- a (left, at the left end of the horizontal diameter): a
- e [?] (at the middle of the figure, where the diameters cross — the glyph is of the c/e shape and does not resolve between them)
- c (right, at the right end of the horizontal diameter): c
- p (inside, below the middle and to the left): p
- g (lower left, on the circumference): g
- d (bottom of the circle): d

Diagram. The lower of the two woodcut circles in the right column, captioned *3ª* — the caption is set without the word *figuratio* — inside the lower right of the circle. This is the most cleanly cut of the five figures in this stretch, and every point-letter resolves. Five letters run along the top of the circle: f, z, b, o, r, reading left to right. A horizontal diameter runs from a at the left to c at the right, with e at the middle. A steep diameter rises from g at the lower left to h at the upper right, and a second line runs roughly parallel to it above. Two chords descend from the top toward the right. Inside the figure, t stands below b, q below and left of t, and p between q and e. The bottom of the circle is marked d, and s stands on the circumference at the left, above a.

- f (top of the circle, leftmost of the five letters along the top): f
- z (top of the circle, right of f): z
- b (top of the circle, at the summit): b
- o (top of the circle, right of b): o
- r (top of the circle, rightmost of the five): r

- t (inside, below b): t
- q (inside, below and left of t): q
- p (inside, between q and the middle of the figure): p
- e (at the middle of the figure, where the diameters cross): e
- s (left, on the circumference, above a): s
- a (left, at the left end of the horizontal diameter): a
- g (lower left, on the circumference): g
- h (right, on the circumference, above c): h
- c (right, at the right end of the horizontal diameter): c
- d (bottom of the circle): d

to g h. Since f g, or f h, is a quadrant, the straight line z p will be the right sine of the complement of the arc z f. Further, let the intersection of the straight lines z p and s o fall at the point q; then the straight line p q will be equal to the right sine of the northern arc g s, and the arc g s is itself equal to the declination of the proposed star. The straight line z q, therefore, will be the difference of two right sines: one of them the right sine of the star's declination, the other the right sine of the complement of the difference of the two arcs named above, namely the greatest declination of the ecliptic and the complement of the same star's latitude.

Now the angles of the triangle q z t are equal to the angles of the triangle n f m in the first figure: the angle at t is equal to the angle of the greatest declination, and besides, the angle at q is right. We shall therefore conclude, just as in the first figure, that four terms are proportional on the same footing. The first is the square of the whole sine; the second, the rectangle contained by the right sines of the greatest declination of the ecliptic and of the complement of the star's latitude; the third, the versed sine of the same star's longitude reckoned from the beginning of Cancer; and the fourth, finally, the straight line z q. The first three terms are supposed known, so the fourth too will become known by the common rule of proportional numbers. We shall then take z q away from the straight line z p, and there will be left p q known, equal to the right sine of the declination sought. The arc of the declination will therefore come out known from the table of right sines.

Again, with the latitude northern, let the common section of a circle drawn through the star under consideration and having the same poles as the world be the straight line g h; and let the common section of the circle that passes through the same star but is parallel to the ecliptic be z k, as in the fourth figure. Then the arc z f will be equal to the difference between the greatest declination of the

ecliptic and the complement of the star's latitude. From the point z draw z x perpendicular to the straight line g h; that straight line z x will then be the fourth term of the proportion just mentioned.

Further, let it be supposed that the right sine of the complement of the difference of the two arcs named above — one of them the greatest declination of the ecliptic, the other the complement of the star's latitude — is equal to that perpendicular z x. I say that the straight line g h is the section of the equator, and that the proposed star has no declination. For if the straight line g h is not the section of the equator, the section will be some other line, either above it or below it, yet parallel to it, as it must be by *Elements* XI.16. Let such a line be the straight line e n [?], cutting the straight line z x at the point i. Then, since the angles at x are right, the angles at i will be right too, by *Elements* I.29. But the arc e f, between the pole of the world and the equator, is a quadrant; so the arc e z will be the complement of the arc z f, and the straight line i z its right sine. Yet by hypothesis the straight line z x was equal to the right sine of the complement of the arc z f. Therefore i z and z x are equal to one another, by the common notion — the part and the whole — which is impossible. The equator therefore cannot cut the colure above g, nor below it either, on account of the same absurdity. It will therefore cut it on the straight line g h itself. Hence the proposed star must have no declination.

Diagram. The left-hand of the two woodcut circles set side by side in the lower half of the leaf, captioned *4ª figuratio* inside the lower left of the circle. A horizontal diameter runs from a at the left to c at the right. Three further lines cross the figure between the left-hand and right-hand circumferences. Their left-hand ends are lettered, reading downward, z, then a letter that does not resolve, then g; their right-hand ends, reading downward, o, then h, then K, the K engraved as a capital. The three lines run close together through the middle of the figure and their crossings do not resolve on the leaf, so which left-hand end belongs to which right-hand end cannot be settled from the woodcut itself. A short line drops from z to x, inside the figure at the lower left, and a faint letter stands inside a little to the right of the middle. The bottom of the circle is marked d, and f stands on the circumference at the upper left, with b at the summit.

- f (upper left, on the circumference): f
- b (top of the circle): b
- o (upper right, on the circumference): o
- h (right, on the circumference): h
- K (right, on the circumference, just below h — engraved as a capital): K

- c (right, at the right end of the horizontal diameter): c
- z (left, on the circumference, above a): z
- a (left, at the left end of the horizontal diameter): a
- [?] (left, on the circumference between a and g — a letter of the c/e shape that does not resolve)
- g (lower left, on the circumference): g
- x (inside, at the lower left): x
- e [?] (inside, a little right of the middle of the figure — a faint mark of the c/e shape; ink is present in this position but the glyph does not resolve)
- d (bottom of the circle): d

Moreover, let the latitude be northern as in the other cases, and the difference z f as stated. Let the circle drawn through the star under consideration and parallel to the equator cut the plane of the colure, as in the fifth figure, along the straight line r t, their common section; and let the right sine of the complement of the arc z f be the line z y, which we shall produce in a straight line until it cuts the straight line r t at the point v. The fourth term of the proportion will then be the straight line z v, which will exceed the straight line z y by the difference y v, equal to the right sine of the arc g r. And the arc g r is itself equal to the star's declination; equal arcs have equal right sines, by what we demonstrated in the first lemma. Therefore, when the straight line z y is taken away from the straight line z v, the fourth term — as often as z y is found to be the smaller — the straight line y v is equal to the right sine of the decli-

Diagram. The right-hand of the two woodcut circles set side by side in the lower half of the leaf, with the caption *5ª figuratio* set inside the upper part of the circle, below b. A horizontal diameter runs from a at the left to c at the right, with e at the middle. Two lines cross the figure from lower left to upper right, roughly parallel: the upper from g to h, the lower from r — just below g on the circumference — to t on the right. A short line descends from z, on the circumference above a, through y and on to v, where it meets the lower of the two oblique lines. f stands on the circumference at the upper left and d at the bottom of the circle.

- f (upper left, on the circumference): f
- b (top of the circle): b
- z (left, on the circumference, below f and above a): z
- a (left, at the left end of the horizontal diameter): a
- g (lower left, on the circumference): g

- r (lower left, on the circumference, just below g): r
- d (bottom of the circle): d
- e (at the middle of the figure, where the diameters cross): e
- y (inside, at the lower left): y
- v (inside, below and right of y): v
- h (right, on the circumference, above c): h
- t [?] (right, on the circumference between h and c — the glyph is small and reads t or e)
- c (right, at the right end of the horizontal diameter): c

clination of the star, will be left known; and the declination itself will finally become known from the table of right sines. This declination must necessarily be southern. For the straight line *z v*, the fourth term of the proportion, could not exceed the straight line *z y* — the right sine of the arc *z g* — unless the circle drawn through the star parallel to the equator cut the colure below *g h*. Moreover, when the latitude is southern we shall reach the same result in these same figures; but we shall take the point *b* to be the southern pole of the ecliptic, and *s* the antarctic pole of the world. The star's longitude will then be reckoned from the beginning of Capricorn, either in the order of the signs or against it, and the declination found will carry the denomination opposite to the earlier one.

There is also another way, corresponding to the way we set out the declinations of the stars, in which their longitudes and right ascensions can be found; and it differs only slightly from the first. For if the latitude is northern, the first three terms of the proportion will not vary at all, and the fourth will likewise be preserved intact, because one and the same arc is called both the complement of the star's latitude and its distance from the northern pole of the ecliptic. So whenever this is the case, one and the same demonstration will serve. But if the latitude is southern and smaller than the arc of the greatest declination, the proposition will be demonstrated in these same figures — provided that in them the circle parallel to the ecliptic passing through the star under consideration is understood to cut the colure between the points *a* and *g*.

Now let the star proposed have a southern latitude equal to the arc of the greatest declination. Again we shall take the point *b* to be the northern pole of the ecliptic, *a c* the section of the ecliptic, and *q h* the section of the equator; *a* is the beginning of Cancer and *c* the end of Sagittarius. Let the section of the circle parallel to the ecliptic passing through the star be *q k*, and let the straight line *o g* be the section of the circle parallel to the equator which is likewise drawn through it. Let this last be extended toward *o*, and from the point *q* let the perpendicular *q z* be

dropped onto it, as in the sixth figure. Then *q z* will be the fourth term of the proportion, and equal to the right sine of the arc *q o* — that is, of the southern declination of the star proposed. This can easily be shown by *Elements* I.34, once a perpendicular has first been dropped from the point *o* onto *q h*.

Diagram. A circle representing the solstitial colure, with a horizontal diameter and a vertical radius drawn from the top of the circle to the centre; several chords and a shorter transversal cross the lower part of the figure, and a dashed segment runs across the bottom toward the right. The engraved caption *6ª figuratio* is set inside the upper left of the circle, divided by the vertical radius.

- b (top of the circle, above the vertical radius): *b*
- a (left end of the horizontal diameter): *a*
- c (right end of the horizontal diameter): *c*
- h (on the circumference, upper right): *h*
- g (on the circumference, right side, below *c*): *g*
- K (on the circumference, right side, below *g*): *K*
- s (inside the circle at the left, below *a*): *s*
- P (right end of the dashed segment near the bottom of the circle): *P*
- [?] (a cluster of small letters at the lower left and along the bottom of the figure — four or five glyphs, none of which resolves on the scan)

Finally, let us suppose that the star under consideration has a southern latitude greater than the arc of the greatest declination. Let the section of the circle parallel to the ecliptic passing through it be *r p*, and let the section [?] of the circle parallel to the equator that also passes through it be *o g*. From the point *r* let the perpendicular *r t* be dropped onto that section; then let the perpendicular, once drawn, be extended until it meets the straight line *q h* at the point *s*. Then *r t* will be the fourth term of the proportion, while the arc *q r* will be the excess by which the star's latitude exceeds the arc *a q* of the greatest declination. The straight line [?] *s* is its right sine, and the whole line *s r*, made up of these two straight lines, is equal to the right sine of the arc *o q*, the star's declination — as can again be demonstrated by *Elements* I.34.

Therefore, once the fourth term of the proportion is known, we shall add it to the right sine of the difference between the star's latitude and the greatest declination of the ecliptic; the number so formed will be the right sine of the declination sought, and the declination itself will then be found from the table of right sines.

If, however, the star proposed has no latitude, we shall compute its declination far more easily. For the second proposition has demonstrated that as the whole sine is to the right sine of the greatest declination of the ecliptic, so the right sine of the arc by which a star without latitude is distant from the vernal or the autumnal section is to the right sine of its declination. So by the common rule of proportional numbers we shall draw the declination from the star's known distance and, conversely, from the declination the distance by which it stands off from one section or the other.

By the star's latitude and declination together we shall also easily compute its longitude from the beginning of Cancer, either in the order of the signs or against it. For when the latitude is northern and its complement is set equal to the arc of the greatest declination of the ecliptic, as in the first figure, we shall subtract the right sine of the declination from the whole sine, and the fourth term of the proportion will be left known. But if it is unequal, and the declination is northern, as in the second and third figures, we shall subtract from the right sine of the complement of the difference of those unequal arcs the right sine of the northern declination proposed, and the fourth term will be left known. If, again, it is unequal but the declination proposed is southern, as in the fifth figure, we shall add to the right sine of the complement of that difference the right sine of the southern declination proposed, and the fourth term will be formed and known. Finally, if it is unequal and the star has no de-

clination, as in the fourth figure, the fourth term itself will be the right sine of the complement of that difference.

So when the fourth term has become known by any of these ways, we shall multiply it into the first, divide the product by the second, and the third term will come out known: namely the versed sine of the star's longitude from the beginning of Cancer, reckoned in the order of the signs or against it. Its arc will therefore be found from the table of sines. In exactly the same way, if the latitude is southern, the fourth term of the proportion will be investigated; and by carrying it into the first and dividing the product by the second, the third term will result — namely the versed sine of the star's longitude reckoned from the beginning of Capricorn, in the succession of the signs or against it. The arc of the longitude itself will then become known from the table of sines.

Correlarium.

And since the altitude of the pole above the horizon can be determined every day from the Sun's meridian ray, and since by knowing it the declinations of the stars

can be traced out at night, as the fifth proposition taught, while their latitudes do not vary at all — it is clear from this in what way the true longitudes of the stars are to be found on any night whatever.

Again, from the star's known declination and latitude we shall investigate its right ascension in these same figures, merely exchanging the names. By the point *b* we shall understand the arctic pole of the world; by *s*, the nearer pole of the ecliptic; by *a c*, the section of the equator; by *g h*, the section of the ecliptic; by the straight line *f i* [?] of the first figure, the section of the circle parallel to the equator drawn through the centre of the star's body; and by *k l*, the section of the circle parallel to the ecliptic, which also passes through the star's centre — and in like manner in the remaining figures.

So whenever the declination is northern and its complement is set equal to the greatest declination of the ecliptic, as in the first figure, we shall subtract from the whole sine the right sine of the star's latitude, that is, the straight line *e n*, and the fourth term will be left known. But if it is unequal, and not only the declination is northern but the latitude as well, as in the second and third figures, we shall subtract from the right sine of the complement of the difference between the greatest declination and the complement of the star's declination the right sine of the latitude, that is, the straight line *p q*, and the fourth term will be left known. If, again, it is unequal and the latitude proposed is southern, as in the fifth figure, we shall add to the right sine of the complement of that difference the straight line *y v*, the right sine of the latitude proposed, and the fourth term of the proportion will be gathered and known. Finally, if it is unequal and the star proposed has no latitude, as in the fourth figure, the right sine of the complement of that difference will itself be the fourth term.

So when the fourth term of the proportion has become known by any of these ways, we shall multiply it into the square of the whole sine, the first term of the proportion, and divide the product by the number produced from multiplying together the right sines of the greatest declination of the ecliptic and of the complement of the star's declination — which is established as the second term of the proportion. From that division the third term of the proportion will come out known: namely the versed sine of the right ascension begun from that point of the equator which rises together with the beginning of Capricorn. The arc of the ascension will therefore be found from the table of sines.

This arc will be reckoned in the direction of the diurnal motion if the star lies in those signs which descend from the beginning of Cancer to the end of Sagittarius, but the other way if it lies in the ascending signs. By a similar procedure we shall

investigate the right ascension if the star's declination is proposed as southern; but it will take its beginning from the point of the equator that rises with the beginning of Cancer, and it will be reckoned in the direction of the diurnal motion if the star is found in the signs of the ascending semicircle, and the other way if it is in the remaining signs of the descending semicircle.

If, however, the star proposed has no declination, as in the seventh figure, we shall investigate its right ascension by means of the latitude — just as, by the second proposition, we drew from the declination alone the arc of the ecliptic between a star without latitude and one or other of the sections, vernal or autumnal — only with the names interchanged. For *b* and *d* will be the poles of the equator, *s* and *p* the poles of the ecliptic; the straight line *a c* the section of the equator, and the other, *g h*, that of the ecliptic; while *y k* [?], or *q r*, will be the section of the circle parallel to the ecliptic drawn through the star under consideration. Thus *l o*, or *s t*, will be equal to the right sine of the star's latitude, and *e l*, or *e s*, equal to the right sine of the arc of the right ascension. And it is clear from the second proposition that as the whole sine is to the right sine of the greatest declination, so is *e l* to *l o*. Therefore by the

common rule of proportional numbers, once the first, second and fourth terms are known, the third will become known, and the arc of the right ascension itself will likewise be found from the table of sines. This arc will be reckoned from the nearest section of the equator and the ecliptic.

Diagram. An uncaptioned circle standing for the equinoctial, crossed by a horizontal diameter and by two pairs of parallel chords that run obliquely from the lower left to the upper right; short transversals join the oblique lines within the circle, and the labelled points of the construction lie on these transversals.

- f (on the circumference, upper left): *f*
- b (on the circumference, top): *b*
- K (on the circumference, upper right, at the end of an oblique line): *K*
- h (on the circumference, right, below *K*): *h*
- r (on the circumference, right, below *h*): *r*
- c (right end of the horizontal diameter): *c*
- a (left end of the horizontal diameter): *a*
- l (inside the circle, on the horizontal diameter left of centre): *l*
- e [?] (inside the circle, between *l* and *t* on the upper oblique line)
- t (inside the circle, on the upper oblique line, right of centre): *t*
- o (inside the circle, below *l*): *o*

- s (inside the circle, below *t*): *s*
- g (on the circumference, lower left): *g*
- q (on the circumference, lower left, below *g*): *q*
- d (on the circumference, bottom): *d*
- P (on the circumference, bottom, right of *d*): *P*
- [?] (one further small glyph at the left, between *a* and *g*, which does not resolve on the scan)

Moreover, this seventh figure can be adapted to investigating the right ascension of a star that has no latitude — a computation we taught by means of the preceding fourth figure, under larger numbers. For let *a b c d* be the equinoctial circle; let the diameter *g h* be the section of the colure passing through the tropic points; let the straight line *y K* be the section of a certain circle parallel to that colure, drawn through the star; and let the straight line *a c* be the section of the circle of declination of the star under consideration. Then, by what we demonstrated in the first lemma, the arc *y g* will be equal to the arc of the ecliptic between the star and the tropic point, while the perpendicular *l o* will be equal to the right sine of that same arc. Next, the straight line *e l* will be equal to the right sine of the complement of the declination of the star proposed, since the circle of declination cuts both the colure and the circle parallel to it. Common sections of this kind are straight lines perpendicular to the plane of the equator at the points *e* and *l*, one of which terminates at the pole of the world and the other at the star under consideration. Further, the arc *a g* is the right ascension of the star itself, reckoned from that point of the equator which is nearest the neighbouring tropic point, either in the direction of the world's motion or against it.

Now in the triangle *e l o*, as the whole sine is to the right sine of the arc of the angle *l e o*, so is the straight line *e l* to the straight line *l o*, by the lemma of the sixth appendix; and the angle *l e o* is itself subtended by the arc *a g*. Therefore the ratio of the whole sine to the sine of the star's right ascension will be as the ratio of the right sine of the complement of that same star's declination to the right sine of the arc of the ecliptic by which the star stands off from the tropic point. We shall accordingly multiply the fourth term into the first simply by appending five ciphers; we shall divide the product by the third, namely the right sine of the complement of the declination; and the number that results will be the right sine of the arc *a g*, that is, of the star's ascension. So the ascension itself will become known from the table of sines.

The remaining part of this sixth proposition, which always refers longitudes to the head of Cancer, can likewise be adapted to this inquiry into right ascension, again with the names interchanged.

But to all these results, established by demonstration, we shall add a few examples. Let us suppose that Spica Virginis has its place in the eighteenth first-minute of the seventeenth degree of Libra, and that it also has two degrees of southern latitude; and let it be required to find its declination. The right sine of the 88 degrees which the complement of the latitude contains holds 99939 parts, of the kind of which the whole sine has 100000. The right sine of the greatest declination of the ecliptic has 39874 parts. The product of these two numbers is 3984967686 — that is, the second term of the proportion just mentioned. The arc of longitude reckoned from the beginning of Capricorn against the succession of the signs contains 73 degrees and 42 first-minutes; its versed sine, 71934, is the third term. So we shall multiply this third term into the second, and from that multiplication there will arise 286654665524724. We shall divide this number by the square of the whole sine, the first term, simply by striking off the last ten figures, and from that division there will come out 28665 and nearly half of one part — that is, the fourth term of the proportion.

Now since the difference between the greatest declination of the ecliptic and the complement of the said star's latitude is 64 and ½ degrees, the complement of this difference will contain 25 and ½ degrees, whose right sine is 43051. And because the fourth term of the proportion, 28665, is less than 43051, the right sine of the complement of that difference, and because the latitude of the star proposed is southern, its declination too will be southern. So we shall subtract the one number from the other, and 14386 will be left; and this number, finally, being the right sine of the

must be the right sine of the declination sought, according to the demonstration of the second figure. In the table of right sines there correspond to it 8 degrees and 16 minutes, which the southern declination of the Spike of Virgo will have.

Next, let us suppose that the brighter star of the northern pan of Libra has 8 degrees and 30 minutes of northern latitude, and likewise 7 degrees and 18 minutes of southern declination, and that its true place has to be traced out. The complement of the latitude, then, contains 81 degrees and 30 minutes, and its right sine amounts to 98901 parts of the radius. The right sine of the greatest declination of the ecliptic is 39874. The product of these two numbers is 3942578474, the second term of the proportion. The difference between the greatest declination and the complement of the latitude is 58 degrees; the

complement of that difference is 32 degrees, and its right sine will be 52991 parts. Now since the latitude is northern while the declination is southern, we shall — as in the fifth figure — add to that 52991 the right sine of the proposed star's declination, namely 12706 parts; and the fourth term of the proportion will be established as 65697. This we shall multiply into the square of the whole sine, the first term of the proportion, and 656970000000000 will result. Finally we divide this number by the second term, and the division yields the third term, namely 166660: the versed sine of the star's longitude reckoned from the beginning of Cancer. To this number there corresponds in the table of sines an arc of 131 degrees and 48 minutes, with some seconds added. On the hypotheses laid down, therefore, the proposed star will be placed within the forty-ninth minute of the twelfth degree of the sign Scorpio.

Its right ascension we shall compute as follows. The complement of the declination is 82 degrees and 42 minutes, and the right sine of that arc is 99189 parts. We shall multiply this number into 39874, the right sine of the greatest declination of the ecliptic, and 3955062186 results, that is, the second term of the proportion. Next, from the 82 degrees and 42 minutes of the complement of the proposed star's declination we shall subtract the 23 degrees and 30 minutes of the greatest declination, and 59 degrees and 12 minutes of difference will be left. The complement of this difference will contain 30 degrees and 48 minutes, whose right sine is 51204 parts. To this number we shall add 14780, the right sine of the star's latitude, following the demonstration of the fifth figure, because the declination is northern and the latitude southern; and from these there will be established a number of 65984 parts, namely the fourth term of the proportion. Finally we shall multiply this into the square of the whole sine, the first term, and the number 659840000000000 parts results. We divide it, then, by the second term, and from this division 166834 comes out, that is, the versed sine of the right ascension of the proposed star reckoned from the point adjoining the beginning of Cancer. Its arc we draw from the table of sines: 131 degrees of the equator and 56 minutes. If to these we add 90 degrees, the right ascension of the first quadrant of the ecliptic, there will finally be produced a right ascension of 221 degrees and 56 minutes reckoned from the beginning of Aries, which is what the bright star of the northern pan of Libra itself has.

Declinations too can be computed, by no longer a chain of reasoning, through the remaining part of the proportion, the one that always refers longitudes to the head of Cancer — in this way. Let the Greater Dog have 39 degrees and 10 minutes of southern latitude, and 7 degrees and 18 minutes of longitude from the beginning of Cancer, and let its declination have to be measured. The

complement of the latitude, then, will have 50 degrees and 50 minutes, and the right sine of this arc is 75531 parts. We shall multiply this number into 39874, the right sine of the greatest declination, and it becomes 3011723094, namely the second term of the proportion. Besides this we shall carry it into 811, the versed sine of the 7 degrees and 18 minutes which the proposed star's distance from the head of Cancer has — that is, the third term of the proportion — and the number produced is 2442507429234. We shall divide this by the square of the whole sine, the first term, by striking off the last ten figures, and from such a division there will come 244 parts of the radius. But since this star's latitude falls to the south and is greater than the arc of the greatest declination of the ecliptic by 15 degrees and 40 minutes, we shall gather 27004, the right sine of those degrees and minutes, together with the fourth term 244 into one sum, and 27248 results. To this number there correspond in the table of the right sine 15 degrees and 49 minutes, which, on the hypotheses laid down, the declination of the Greater Dog must have.

The places of the stars we have used in these examples we have taken from the common ephemeris, just as if they were the true ones — and even if we do not doubt that the fixed stars themselves have advanced further than the Alphonsine table shows. On that very question we hold, as we have found from many observations, that the opinion of al-Battānī comes closest to the

truth. But of these matters elsewhere.

Propositio. VII.

To compute, by a very short calculation, the lengths of days and nights and of twilights on any oblique horizon.

The lengths of days and nights and of twilights can be investigated in many ways; but the one we have thought out in these figures pleases us best, because it is easier than the rest and reproduces the true image of the thing itself.

We shall multiply the right sine of the Sun's meridian altitude — which we shall derive from the Sun's declination and the altitude of the pole — into the whole sine. The product we shall divide by the right sine of the complement of the altitude of the pole, and a certain number will come out of that division. This number we shall carry into the whole sine, and the product we shall divide by the right sine of the complement of the declination of the given point of the ecliptic; and out of that division will come the versed sine of the semidiurnal arc. The semidiurnal arc itself, then, will be known from the table of sines. We shall

subtract it from the space of twelve hours, and the magnitude of the semi-nocturnal arc will be left known.

The length of the twilight, further, we shall reckon in this way. We shall carry the right sine of 18 degrees into the whole sine, and divide the product by the right sine of the complement of the altitude of the pole of that same oblique horizon; and the number which comes out of this division we shall add to the number that came out of the first division. What is compounded of them we shall carry into the whole sine, and this product we shall then divide by the right sine of the complement of the declination of the proposed point of the ecliptic; and out of that division will come the versed sine of a certain arc which embraces the length of the semidiurnal arc together with the twilight. The whole of such an arc, therefore, will be known. So we shall subtract from this magnitude the semidiurnal arc, so that the length of the twilight may be left known.

And in order that we may demonstrate this operation by the truest and clearest principles of the mathematical art, let the figure of the sixth appendix be taken up again. In it the circle a b c d, described about the centre n, is the meridian; the straight line a c is the common section of the equinoctial and of the meridian itself; i K is the common section of the parallel described by the Sun for the given day; o p is the section of the oblique horizon; q r is the section of the circle parallel to it, under which the Sun begins to illuminate the upper hemisphere at the beginning of the morning twilight; the points s and t [?] are the intersections of the straight line i K with o p and with q r. From the point i let the straight line i R be drawn at right angles to the straight line o p, the section of the horizon; and further, from the point s, the straight line s x upon q r.

Diagram. A woodcut circle set into the upper part of the right column: the meridian. A vertical diameter runs from b at the top of the circumference to d at the bottom, and a horizontal diameter runs from a at the left of the circumference to a letter at the right; the two cross at the centre n. Above the horizontal diameter a chord runs across the figure from the circumference at the left to K at the right. Two parallel oblique lines cross the whole figure from the lower left to the upper right, their upper ends carrying P and a second letter just outside the circumference, their lower ends o and q inside it. A short line drops from the left end of the chord to the upper oblique, and further short lines mark the points s, t and x where the chord and the obliques meet.

- b (top of the circle, at the head of the vertical diameter): b
- [?] (left end of the upper chord, at the circumference — a tall stroke with a hooked top; the glyph will not discriminate between `i` and `f` at label size)

- a (left end of the horizontal diameter, at the circumference): a
- n (centre of the circle, at the crossing of the two diameters): n
- s (on the upper chord, at the crossing of the first oblique): s
- t (below and to the right of s, at the crossing of the second oblique): t
- x (below s and to the left of t, on the second oblique): x
- K (right end of the upper chord, just outside the circumference): K
- [?] (right end of the horizontal diameter, just outside the circumference — a small round letter, c or e; the two are not separable in this fount at label size)
- P (upper right, at the upper end of the first oblique, just outside the circumference): P
- [?] (immediately below P, at the upper end of the second oblique — reads as r, but c and e are not excluded)
- [?] (lower left quadrant, where the short line from the left end of the chord meets the upper oblique — **the letter is engraved in mirror image**: the bowl stands to the left of a stem set on the right, so the glyph cannot be read as any upright letter. Left unresolved rather than turned round to fit the demonstration.)
- o (lower left, at the lower end of the first oblique, inside the circumference): o
- q (below and slightly left of o, at the lower end of the second oblique): q
- d (bottom of the circle, at the foot of the vertical diameter): d

Then we shall consider the right-angled triangle i R s, in which, as the right sine of the angle i s R is to the whole sine, so is the straight line i R to the straight line i s. Now the angle i s R is known, being equal to the angle of the altitude of the equator; and the straight line i R, the right sine of the meridian altitude, is known too. By the common rule of proportional numbers, therefore, the straight line i s, the versed sine of the semidiurnal arc, will become known in parts of the radius of the greatest circle of the sphere. And since, in those same parts, the half of the diameter i k — namely the right sine of the complement of the declination of the given point — is known, the ratio of the radius of the parallel of the given point to the versed sine of the semidiurnal arc will be known. We shall therefore suppose that such a radius has 100000 equal parts; and by the common rule of proportional numbers the straight line i s, the versed sine of the semidiurnal arc, will be known in those same parts. Hence the semidiurnal arc itself is known as well. We shall subtract it from the semicircle, and the magnitude of the semi-nocturnal arc will be left known.

Again, the triangle s x t has the angle s x t right, and the angle x t s equal to the angle of the altitude of the equator, and known. Further, the side s x, the right sine of the 18 degrees by which the Sun is distant from the horizon, is known. By arguments exactly like the earlier ones, therefore, the straight line s t will become known; and so the whole i t will be made known, which is certainly the versed sine of the arc compounded of the semidiurnal arc and the arc of the length of the twilight. For this reason we shall subtract from that arc the semidiurnal arc, known by the earlier chains of reasoning, and the length of the twilight will be left known — which was what had to be demonstrated.

Propositio. VIII.

¶ *As the right sine of the complement of the declination of the given point is to the right sine of the meridian altitude, so is the square of the whole sine to the rectangle contained by the right sine of the altitude of the equator and the versed sine of the semidiurnal arc.*

¶ *Further: as the right sine of the complement of the declination of the given point is to the right sine of the meridian altitude taken together with the right sine of that arc by which the Sun is distant from the horizon at the beginning of the morning twilight, so is the square of the whole sine to the rectangle contained by the right sine of the altitude of the equator and the versed sine of the arc compounded of the semidiurnal arc and the arc of the length of the twilight.*

¶ *Otherwise. As the square of the whole sine is to the rectangle contained by the right sine of the altitude of the equator and the right sine of the complement of the declination of the given point, so is the versed sine of the semidiurnal arc to the right sine of the meridian altitude.*

¶ *Likewise. As the square of the whole sine is to the rectangle contained by the right sine of the altitude of the equator and the right sine also of the complement of the declination of the given point, so is the versed sine of the arc compounded of the semidiurnal arc and the arc of the twilight to a certain straight line compounded of two right sines, of which one is that of the meridian altitude, and the other that of the arc by which the Sun is distant from the horizon at the beginning of the morning twilight.*

Let the figure of the preceding proposition be repeated. Since the straight line i s is the versed sine of the semidiurnal arc in the given parallel, let a z, in the straight line a c, the diameter of the equator, be the versed sine proportional to it. Accordingly, as we demonstrated above, as i s is to i R, so is the whole sine to the

right sine of the altitude of the equator; and as the half of the diameter i k is to the straight line i s, so is the radius of the equator — that is, the whole sine itself — to the straight line a z, the versed sine of the semidiurnal arc, by the lemma of the fifth appendix. So the ratio which the half of the diameter i K has to i R will be compounded of two ratios, of which one is the same as the ratio which the whole sine has to the right sine of the altitude of the equator, and the other the same as that which the whole sine itself has to the versed sine of the semidiurnal arc. But out of those same two ratios there is put together the ratio of the square of the whole sine to the rectangle contained by the right sine of the altitude of the equator and the versed sine of the semidiurnal arc, by *Elements* VI.23. Therefore, as the half of the diameter i K — namely the right sine of the complement of the declination of the point conceived on the ecliptic — is to the straight line i R, the right sine of the meridian altitude, so is the square of the whole sine to that rectangle which is contained by the right sine of the altitude of the equator and the versed sine of the semidiurnal arc: which was what had to be demonstrated.

Diagram. The same woodcut circle repeated in the upper part of the right column, immediately below the ornamental initial R that opens the demonstration. Again a vertical diameter runs from b at the top of the circumference to d at the bottom, and a horizontal diameter from a at the left to a letter at the right, crossing at the centre n; a chord above the horizontal diameter runs from the circumference at the left to K at the right; and two parallel oblique lines cross from the lower left, where they end at o and q, to the upper right, where they end at P and a second letter. Two further letters stand inside the circle, below the horizontal diameter and to the right of the centre, which the preceding page's cut does not carry.

- b (top of the circle, at the head of the vertical diameter): b
- [?] (left end of the upper chord, at the circumference — again a tall hooked stroke; `i` or `f`, not discriminable)
- a (left end of the horizontal diameter, at the circumference): a
- n (centre of the circle, at the crossing of the two diameters): n
- s (on the upper chord, at the crossing of the first oblique): s
- t (below and to the right of s, at the crossing of the second oblique): t
- x (below s and to the left of t, on the second oblique): x
- z (inside the circle, below the horizontal diameter and well to the right of the centre): z
- [?] (immediately to the right of z, on the same level — a bowl with a descender; reads as `g`, but `q` is not excluded at this size)

- K (right end of the upper chord, just outside the circumference): K
- [?] (right end of the horizontal diameter, just outside the circumference — c or e, not separable)
- P (upper right, at the upper end of the first oblique, just outside the circumference): P
- [?] (immediately below P, at the upper end of the second oblique — reads as r, with c and e not excluded)
- [?] (lower left quadrant, where the short line from the left end of the chord meets the upper oblique — **the letter is engraved in mirror image** here as well, the bowl standing to the left of a stem set on the right. Left unresolved.)
- o (lower left, at the lower end of the first oblique, inside the circumference): o
- q (below and slightly left of o, at the lower end of the second oblique): q
- d (bottom of the circle, at the foot of the vertical diameter): d

And so, whenever it is desired to measure the length of the semidiurnal arc according to the present rule, then — by a table that sets the whole sine at 100000 equal parts — we shall add ten ciphers to the right sine of the meridian altitude of the point conceived, and divide the number so produced by the right sine of the complement of the declination of that same point. Next, the number which comes out of that division we shall divide by the right sine of the altitude of the equator; for the number which results from this second division

— will be the versed sine of the semidiurnal arc of the point conceived, for the given elevation of the pole. Finally, by the table of sines, the semidiurnal arc itself will be known.

To demonstrate the second point, let the straight line i R be produced as far as the point v on the straight line q r. Now the straight line i t is the versed sine of the arc compounded of the semidiurnal arc and the arc of twilight in the given parallel; so let a g, on the diameter of the equator, be the versed sine of the arc proportional to it in the equator itself. Then, as i t is to i v, so will the whole sine be to the right sine of the arc of the angle of the altitude of the equator. Moreover, as the half-diameter i k — that is, the right sine of the complement of the declination of the given point — is to the straight line i r, so is a n, the whole sine, to the straight line a g, the versed sine of the arc compounded of the semidiurnal arc and the arc of twilight.

The ratio, therefore, of the right sine of the complement of the declination of the given point to the straight line i v — which is made up of i R, the right sine of the meridian altitude, together with R v, the right sine of the arc of the Sun's distance from the horizon at the beginning of morning twilight — will be understood to be compounded of two ratios. One of these is the same as the ratio the whole sine bears to the right sine of the altitude of the equator; the other is the same as the ratio the whole sine itself bears to the versed sine of the arc compounded of the semidiurnal arc and the arc of twilight. But these two ratios compound into the ratio of the square of the whole sine to the rectangle contained by the right sine of the altitude of the equator and the versed sine of the arc compounded of the semidiurnal arc and the arc of twilight, by *Elements* VI.23. Therefore, as the right sine of the complement of the declination of the given point is to the straight line compounded of the right sines of the meridian altitude and of the arc of the Sun's distance from the horizon at the beginning of morning twilight, so is the square of the whole sine to the rectangle contained by the right sine of the altitude of the equator and the versed sine of the arc compounded of the semidiurnal arc and the arc of twilight — which was the second thing that had to be demonstrated.

Accordingly, to measure the length of twilight we shall gather into one the right sines of the meridian altitude and of the arc of the Sun's distance from the horizon. To the number so composed we shall append ten ciphers, and we shall divide the number thus produced by the right sine of the complement of the declination of the given point; then the number that comes out of this division we shall divide by the right sine of the altitude of the equator. For the number that results from this second division will be the versed sine of the arc compounded of the semidiurnal arc and the arc of twilight. That whole arc, however, embraces the length of time from the beginning of morning twilight up to noon; and so we shall subtract from it the span of the semidiurnal time, and the interval of twilight will be left known.

The demonstrations of the two remaining rules will run in this way. As the whole sine is to the right sine of the altitude of the equator, so is the straight line i s to the straight line i R, the right sine of the meridian altitude. Moreover, as the whole sine is to the right sine of the complement of the declination of the given point, so is the versed sine of the semidiurnal arc to the straight line i s. The ratio, then, of the versed sine of the semidiurnal arc to the right sine of the meridian altitude will be understood as compounded of the same ratios that the whole sine has both to the right sine of the altitude of the equator and to the right sine of the complement of the declination of the proposed point. But these make up the ratio which the square of the whole sine has to the rectangle contained by the right

sines of the altitude of the equator and of the complement of the declination. Therefore, as the versed sine of the semidiurnal arc is to the right sine of the meridian altitude, so is the square of the whole sine to the rectangle contained by the right sines of the altitude of the equator and of the complement of the declination of the proposed point — which had to be demonstrated.

Following this demonstration, then, we shall find the length of the semidiurnal arc of the proposed point in any horizon whatever in this way. To the right sine of the meridian altitude we shall append ten ciphers; the number so produced we shall divide by the number that arises from multiplying the right sine of the altitude of the equator into the right sine of the complement of the declination of the given point. The number that comes out of that division will be the versed sine of the semidiurnal arc of the point conceived in the given horizon.

Furthermore, it is clear from the demonstrations above that, as the whole sine is to the right sine of the altitude of the equator, so is the straight line i t to the straight line i v, which is made up of the right sines of the meridian altitude of the given point and of the arc of the Sun's distance from the horizon at the beginning of morning twilight. Likewise, as the whole sine is to the half-diameter i k — that is, to the right sine of the complement of the declination of the proposed point — so is a g, the versed sine of the arc compounded of the semidiurnal and the twilight arc, to the straight line i r. Therefore the ratio of the square of the whole sine to the rectangle contained by the right sines of the altitude of the equator and of the comple—

Diagram. A circle standing at the head of the left column, cut by two horizontal chords and two straight lines drawn across it. The horizontal diameter runs from a on the left to c on the right, passing through the centre; a second, higher horizontal chord runs from h on the left to i on the right. Two further lines cross the figure vertically, one of them running from b at the top to d[?] at the foot and the other from f at the top to g at the foot; their intersections with the upper chord fall at K and l. Three small letters stand close together at the centre and just below it; only their positions are certain on the leaf.

- b (top of the circle, left of the pair): terminus of one of the two lines drawn down across the figure
- f (top of the circle, right of the pair): terminus of the second line drawn down across the figure
- h (left end of the upper chord): end of the diameter of the parallel described by the Sun
- i (right end of the upper chord): the other end of that diameter

- K (on the upper chord, left of centre): intersection of the upper chord with the line from b
- l (on the upper chord, right of K): intersection of the upper chord with the line from f
- a (left end of the horizontal diameter): end of the section of the equator
- c (right end of the horizontal diameter): the other end of that section
- d (foot of the circle, left of the pair)[?]
- g (foot of the circle, right of the pair)
- [?] (at the centre)
- [?] (just right of the centre)
- [?] (right of the preceding, below the horizontal diameter)

— the right horizon, which marks the beginning of morning twilight or the end of evening twilight. Let h, i be the section of the parallel described by the Sun when it stands outside the equinoctial. Let the intersections of this straight line with the straight lines b d and f g be the points k and l, and let the intersection of the straight lines a c and f g be called o.

It is clear that when the Sun has entered the equinoctial points and by its daily motion has described the circle of the equator, its distance from the right horizon measured on the vertical circle, and the arc of the equator itself that marks the length of twilight, will be one and the same arc; and the straight line e o will be equal to the right sine of that arc. But now let the Sun stand outside the equator and describe a parallel whose diameter is h i. Then the straight line K l will be equal to the right sine of the arc b f, and equal also to the right sine of the arc of the length of twilight in the parallel described, by *Elements* I.34.

Let the straight line e p, then, be the right sine of the proportional arc in the equinoctial circle. As h K, the radius of the parallel conceived, is to the right sine of the twilight arc in that same parallel, so is a e, the radius of the equator, to e p, the right sine of the arc of the length of twilight proportional to it. And so, as h K is to the right sine of the arc b f, so is a e to e p, by *Elements* V.7. But by what was shown in the first lemma, b f and the Sun's distance from the right horizon at the beginning of twilight are equal arcs and have equal sines. It follows that, as a e, the whole sine, is to e p, the right sine of the magnitude of twilight, so is h K, the radius of the parallel conceived — that is, the right sine of the complement of the declination of the Sun's place — to the right sine of the arc by which the Sun itself stands away from the right horizon at the beginning of morning twilight or at the end of evening twilight, by that same *Elements* V.7.

So we shall multiply the right sine of the arc of the Sun's distance from the horizon into the whole sine, and divide the product by the right sine of the complement of the declination of the point conceived. For the number that comes out of this division will be the right sine of the arc of the length of twilight; and the arc that answers to it will then become known through the table of right sines.

Propositio. XI.

To find the semidiurnal arc of a star and its distance from the meridian. For the right sine of the meridian altitude of any star has the same ratio to the right sine of that star's altitude at the time of observation as the versed sine of the semidiurnal arc of the same star has to the excess by which that versed sine exceeds the versed sine of the distance from the meridian.

Otherwise. As the versed sine of the star's semidiurnal arc is to the versed sine of the arc of the same star's distance from the meridian, so is the right sine of the meridian altitude to the excess by which it exceeds the right sine of the altitude the star itself has at the time of observation.

Let the circle a b c d, described about the centre e, be the meridian. Let the diameter a c be the section of the equator; the straight line h y the section of the parallel described by the star conceived; the straight line f g the section of the oblique horizon in which that star has its rising and setting; and finally the straight line k l the section of a certain circle parallel to the horizon, understood to be described through the same star at the time of observation. Let the points at which the straight line h y cuts the straight lines f g and K l be m and n. It is clear from Theodosius, *Spherics* I.19, and from *Elements* XI, that the straight lines which are the common sections along which the plane of the parallel conceived cuts the horizon and the circle parallel to it stand at right angles to the plane of the meridian described, at the points m and o[?]. Therefore, by *Elements* XI, def. 2, they will make right angles with the straight line h y, which by the

Diagram. A circle in the right column, with b at the top and d at the foot. A horizontal chord runs from h at the left to y at the right, and is met at two points, n and m, by two further straight lines drawn obliquely across the figure from the lower left to the upper right; g stands at the upper right of the circle and f at the lower left. The horizontal diameter ends at c on the right. A letter stands at the centre and one or two more just below and left of it; these are too small to resolve on the leaf, and the letters k and l named in the text could not be found on the cut.

- b (top of the circle)
- d (foot of the circle)
- h (left end of the horizontal chord): end of the section of the parallel described by the star
- y (right end of that chord): the other end of that section
- n (on the chord, left of centre): intersection with one of the oblique lines
- m (on the chord, right of n): intersection with the other oblique line
- g (upper right of the circle): terminus of an oblique line
- f (lower left of the circle): terminus of an oblique line
- c (right end of the horizontal diameter): end of the section of the equator
- [?] (at the centre)
- [?] (below and left of the centre)
- [?] (a small letter within the lower left quadrant)

19th proposition already cited is established to be the diameter of the parallel described. Therefore the common section that stands over m will end at the point of rising, and on the other side at the point of setting. And so, by what we demonstrated in the first lemma, each of its two portions will be the right sine both of the star's semidiurnal arc and of its seminocturnal arc; while the eastern portion of the other section, lying between the point n and the observed star, will be the right sine of the arc of the distance of each from the meridian, in the parallel described.

Next we shall suppose that b is the pole of the world that is always visible. Then the straight line h m will be the versed sine of the semidiurnal arc, and m y, the remaining part of the diameter, the versed sine of the seminocturnal arc of the star conceived; the straight line h n will be the versed sine of the arc of distance from h, the point of midday; and n m the difference between the versed sines of the semidiurnal arc and of the arc of the same star's distance from the point of midday.

From the point h itself let the straight line h o be drawn at right angles to the straight line f g, cutting K l at the point r. Then h o will be the right sine of the arc h f, the meridian altitude; and the straight line b r[?] will be equal to the right sine of the arc f k, which is equal to the altitude of the star above the horizon at the time of observation.

Next we shall consider the triangle h o m, whose sides h o and h m are cut at the points n and r by the straight line n r, drawn parallel to the base. Therefore, by *Elements* VI.2 and by composition of the proportion, as h o, the right sine of the

meridian altitude, is to o r, the right sine of the star's altitude at the time of observation, so is h m, the versed sine of the semidiurnal arc, to n m, the difference between h m itself and h n, the versed sine of the arc of the star's distance from the point of midday.

Of these four terms, three are known. The right sine of the meridian altitude is known, for its arc becomes clear at once from knowing the elevation of the equator above the horizon and the declination of the star. The right sine of the star's altitude at the time of observation is assumed known. And the versed sine of the star's semidiurnal arc is found in the way we used in the seventh or the eighth proposition. Therefore, by the common rule of proportional numbers, the difference n m itself will be made known. This we shall subtract from the versed sine of the semidiurnal arc, and h n, the versed sine of the arc of the star's distance from the meridian point, will be left known; the arc itself will at last become known through the table of sines — which is what we set out to find.

Again, by a like proof, or by *Elements* VI.4, it will become clear that, as h m, the versed sine of the semidiurnal arc, is to h n, the versed sine of the arc of the star's distance from the meridian, so is h o, the right sine of the meridian altitude, to h s, the excess by which h o itself exceeds the straight line o r, the right sine of the star's altitude at the time of observation.

It makes no difference whether o s is taken in the demonstration as being the very right sine of the star's altitude or merely as equal to it, for by *Elements* V.7 the same proportion follows. Nor will the demonstration vary at all if the proposed star has no declination: for then the versed sine of the semidiurnal arc will be the whole sine itself, and the excess by which it exceeds the versed sine of the arc of distance from the meridian will be equal to the right sine of the arc of the hours from the star's rising. Therefore, as the right sine of the meridian altitude is to the right sine of the star's altitude at the time of observation, so is the whole sine to the right sine of the arc of the hours from the rising of that same star.

The proposition is true in the right sphere as well. There one and the same straight line will be the right sine of the meridian altitude and the versed sine of the semidiurnal arc in the parallel described; and the remainder, which is the difference between the versed sines of the semidiurnal arc and of the arc of the star's distance from the meridian in the same parallel, is equal to the right sine of the altitude the star has at the time of observation. Therefore, as the right sine of the meridian altitude is to the right sine of the star's altitude at the time of observation, so is the versed sine of the semidiurnal arc to the difference between

it and the versed sine of the arc of distance from the meridian. But this versed sine of the semidiurnal arc, in the parallel described by the star, is the whole sine, while the difference just named is equal to the right sine of the arc of the hours from the star's rising in the same parallel. In the right sphere, then, matters always stand as they do in the oblique sphere when the proposed star has no declination: namely, as the right sine of the meridian altitude is to the right sine of the star's altitude at the time of observation, so is the whole sine to the right sine of the arc of the hours from the rising of that same star.

Propositio. XII.

The ratio of the difference between the right sines of the meridian altitude of the Sun or of a star and of the altitude it has at the time of observation, to the versed sine of the arc of distance from the meridian, is as the ratio of the rectangle contained by the right sines of the complement of the declination of that same star and of the complement of the altitude of the pole, to the square of the whole sine.

Let the preceding figure be repeated, and let us consider the triangle r h n. In it the angle h r n is right, by *Elements* I.29; and by the same proposition the angle h n r is equal to the angle g e c, which subtends the complement of the altitude of the pole. Therefore, by the lemma of the sixth appendix, as h r is to h n, so is the right sine of the complement of the altitude of the pole to the whole sine.

But the straight line h n is itself the versed sine of the star's distance from the meridian on the parallel described, and it stands to the versed sine of the proportional arc of that same distance on the equator as the radius of the described parallel — that is, the right sine of the complement of the declination of that same star — stands to the radius of the equator, that is, to the whole sine, by the lemma of the fifth. Therefore, by *Elements* VI.23, as h r is to the versed sine of the arc of the equator proportional to the arc of the parallel of the star's distance from the meridian, so is the rectangle contained by the right sines of the complement of the altitude of the pole and of the complement of the declination of that same star to the square of the whole sine.

Diagram. A woodcut circle set into the left column about a third of the way down, interrupting the sentence above. Two nearly straight lines cross it from left to right: an upper chord running from h on the left circumference through n and m and out to the right-hand side, and a second line a little lower, beginning at a on the left circumference and running across to the right. A long oblique rises from the lower left, past k and f, crosses both of these lines, and reaches the circumference at the upper right; a shorter line descends from the upper chord

toward the lower left, the two crossing at r near the middle of the figure. The letter b stands at the top of the circle, d at the lower right.

- b (top, on the circumference)
- l [?] (upper right, at the circumference — a single upright stroke; l is the likeliest reading)
- g [?] (upper right, on the circumference just below the preceding stroke — a bold looped form, read as g but not certain)
- h (left, on the circumference — left end of the upper chord)
- n (on the upper chord, left of centre)
- m (on the upper chord, right of centre)
- a (left, on the circumference — left end of the second line)
- r [?] (interior, below the upper chord, where the two crossing lines meet — faint, read as r)
- c [?] (interior, above and right of r — a round form; c and e cannot be separated at this size)
- o [?] (interior, below and right of the preceding letter — a round form, read as o)
- k (lower left, inside the circumference)
- f (foot of the circle)
- d (lower right, on the circumference)

Now the straight line h r is itself the difference between the right sines of the meridian altitude and of the altitude which the same star has at the time of observation. Therefore, as the difference between the right sines of the two altitudes just named is to the versed sine of the star's distance from the meridian, so is the rectangle contained by the right sines of the complement of the altitude of the pole and of the complement of the declination of that same star to the square of the whole sine.

The present proposition is universally true, whether the star proposed has a rising and a setting in the given horizon, or completes a whole revolution above it. For a star whose declination is greater than the complement of the altitude of the pole above the horizon is seen turning about the pole all night long, provided the region and the star itself incline toward the same side — that is, both to the north or both to the south. Even so, such a star must have two meridian altitudes on each single day, one greatest and the other least. It will be greatest when the star is set at the pole of the horizon, or has receded least from it; least, by contrast, when it has receded most. So the right sine of the altitude which the star

has at the time of observation must be subtracted from the right sine of the greatest meridian altitude, in order that the first term of the proportion may be left. Moreover, the distance from the meridian is to be reckoned toward that side on which midday falls, as is clear from the demonstration itself.

Let it be, then, as in the preceding case: let the circle a b c d, about the centre e, be the meridian; the straight line a c the section of the equator; h y the section of the parallel described by a star conceived about b, the pole of the world; the straight line f g the section of the oblique horizon; the point h its pole. Next let the straight line k l be the section of a circle parallel to the horizon, understood to be drawn through the star at the time of observation; and let the point at which this straight line cuts the straight line h y be n.

Diagram. A woodcut circle at the head of the right column, with the text running round it. The horizontal diameter runs from a on the left, through the centre e, to c on the right. A chord cuts across the upper part of the circle, running from h on the upper-left circumference toward the right-hand side; two long lines cross one another near the centre, one descending from the upper left toward the lower right, the other rising from the lower left, past f, toward the upper right, where the two — carried beyond the circumference — meet at o outside the circle. Further letters stand above the top arc and along the upper chord, and one more at the far left, outside the circle.

- [?] (above the top arc, left — a bold letter, not resolved)
- [?] (above the top arc, right — a single upright stroke, not resolved)
- h (upper left, on the circumference)
- a [?] (interior, below the top arc and left of centre — read as a, not certain)
- y (interior, below the top arc, right of the preceding letter)
- [?] (interior, on the line running down from h — a small letter, not resolved)
- x [?] (far left, outside the circle, at about the level of the upper chord — read as x)
- a (left, on the circumference — left end of the horizontal diameter)
- e (centre)
- c (right, on the circumference — right end of the horizontal diameter)
- [?] (right, on the circumference where the rising oblique meets it — a bold letter, not resolved)
- o (outside the circle at the upper right, where the two lines carried beyond the circumference meet)

- f (lower left, on the circumference)
- d (foot of the circle)

Since, then, y c, the arc of the star's declination, is posited greater than g c, the complement of the altitude of the pole, the star proposed will describe a whole circle above the horizon. And since it is posited to pass through the pole of the horizon, it will have a quadrant of a circle as its greatest altitude when it reaches h, but the arc g y as its least when it reaches the point y, where the circumference of the described parallel will again cut the meridian. This follows from the fact that the straight line h y is the diameter of that parallel, by Theodosius, *Spherics* I.19, and is therefore greater than any other straight line in it that does not pass through the centre — which will be demonstrated by *Elements* I.20 together with the common notion. Straight lines of this kind are the common sections of the vertical circles and of the parallel just named, and they subtend the arcs lying between h, the pole of the horizon, and the star proposed. Therefore, by *Elements* III.27 in Campanus's transmission, the arc h y will be greater than any other arc lying between the pole of the horizon and the star. So by the common notion we shall conclude that the least altitude of the star proposed lies under the arc g y.

Now let the radius e h be drawn, and let k l cut it at the point r; and let h y and f g be extended until they meet at o. Since, then,

a c and h o are parallel, the straight line e o falling upon them will make the alternate angles h o e and o e c equal, by *Elements* I.29. But the angle h n r is equal to the interior angle h o e opposite to it, by the same proposition. Therefore, by the common notion, the angle h n r is equal to the angle o e c, which subtends the complement of the altitude of the pole. And the angles which e h makes with f g are right, by Theodosius, *Spherics* I.10, and by the second definition of *Elements* XI. Therefore the angles at r will be right, by that same *Elements* I.29; and so the straight line e r will be equal to the right sine of the arc k f, and therefore also equal to the right sine of the star's altitude at the time of observation, by what we demonstrated in the first lemma.

Accordingly h r will be left as the difference of the whole sine — that is, of the greatest altitude of the star proposed — and of the altitude the star has at the time of observation; while the straight line h n will be the versed sine of the arc of the star's distance from the meridian. Now therefore, in the triangle h n r, we shall conclude the proportion proposed just as we did in the first figure.

Diagram. The upper of two woodcut circles in the left column, separated from the lower one by four lines of text. A chord crosses the upper part of the circle, a

second chord lies below it, and two oblique lines cross one another inside the circle and run out to the circumference; a stretch of the left and lower-left arc is cut as a broken line rather than a solid one. The figure carries no point-letters anywhere — not one letter of the usual a b c d / f g / h y set appears on it. Its whole lettering is a single word standing just outside the circumference at the upper left, set sideways to the page: it runs vertically up the leaf, a quarter-turn counter-clockwise from the body type, so that the leaf must be turned clockwise to read it.

- zenit [?] (outside the circumference at the upper left, set sideways, reading upward): *zenit* — "zenith". This copy of the word is materially fainter than the one on the cut below and its middle letters could not be separated on their own; the reading is secured chiefly by the clearer copy.

Consequently, whether the pole of the horizon is placed between the equator and the star's parallel, or between that parallel and the visible pole, the demonstration is general, as one may see in the figures set below. In these it is plainly established, by Theodosius, *Spherics* II.28 and II.29, that one meridian altitude is the greatest and the other the least.

Diagram. The lower of the two woodcut circles in the left column, cut to much the same pattern as the one above it, though its pair of obliques is set at somewhat different angles: a circle with an upper chord, a second chord below it, and two oblique lines crossing within it and reaching the circumference, part of the left and lower-left arc again broken rather than solid. As in the figure above, no point-letter appears anywhere on the cut. The only lettering is again a single word outside the circumference at the upper left, set sideways and running vertically up the leaf.

- zenit (outside the circumference at the upper left, set sideways, reading upward): *zenit* — "zenith"

Note that if the star proposed has no declination, there is no need of a composition of ratios. For once a straight line has been drawn in the first figure from the point a perpendicular to f g, it will be clear from the lemma of the sixth appendix, together with the hypotheses and constructions already laid down, that the difference between the right sines of the meridian altitude and of the altitude which the star itself has at the time of observation has the same ratio to the versed sine of the distance from the meridian as the right sine of the complement of the altitude of the pole has to the whole sine.

Propositio. XIII.

In the right sphere, as the right sine of the meridian altitude — that is, of the complement of the declination — of the Sun or of a star is to the right sine of the altitude it has at the time of observation, so is the whole sine to the right sine of the arc of the complement of the distance from the meridian.

Although what is here proposed has already been shown at the end of the eleventh proposition, it will not be useless to demonstrate this same thing by a figure of its own. Let it be, as usual: let the circle a b c d, about the centre e, be the meridian described; the straight line a c the section of the equator; b d the section of the right horizon; f g the section of the parallel described by the Sun or the star; h y the section of a circle parallel to that right horizon, passing through the Sun or the star observed at the very time of observation. And let the straight line f g cut the straight lines h y and b d at the points l and K.

Diagram. A circle inscribed in the right column, with two horizontal and two vertical chords. The vertical diameter runs from b at the top to d at the bottom; a second vertical chord, set to the left of it, runs from h at the top to y at the bottom. The horizontal diameter runs from a on the left to c on the right through the centre e; a second horizontal chord, set above it, runs from f on the left to g on the right. The chord f g crosses h y at l and b d at K.

- b (top, on the circumference): upper end of the vertical diameter b d — the section of the right horizon
- d (bottom, on the circumference): lower end of the same diameter
- h (upper left, on the circumference): upper end of the chord h y — the section of the circle parallel to the right horizon
- y (lower left, on the circumference): lower end of the same chord
- a (left, on the circumference): left end of the horizontal diameter a c — the section of the equator
- c (right, on the circumference): right end of the same diameter
- e (centre): centre of the meridian circle
- f (left, on the circumference): left end of the chord f g — the section of the parallel described by the Sun or star
- g (right, on the circumference): right end of the same chord — the section of the parallel described by the Sun or star
- l (interior, left): intersection of f g with h y
- K (interior, right of l): intersection of f g with b d

Therefore, by what we demonstrated above, the straight line f K will be the right sine of the complement of the declination, or of the meridian altitude, of that same star — namely of the arc f b. And since the straight line l K is equal to the right sine of the arc b h, it will for that reason also be equal to the right sine of the star's altitude at the time of observation, by the common notion. But that same f K is the radius of the described parallel, and so its whole sine; and l K is equal to the right sine of the arc of the star's distance

from its rising or from its setting — that is, of the complement of its distance from the meridian — on this very same parallel. Therefore, by *Elements* V.7, as the right sine of the meridian altitude, or of the complement of the declination, of the Sun or of a star is to the right sine of the altitude which it has at the time of observation, so is the whole sine to the right sine of the complement of the distance from the meridian. Which was to be demonstrated.

Propositio. XIIII.

From the known altitude of the Sun or of a star above the horizon, to elicit the equal hour of the day; and conversely, from the known hour, to find out the altitude of the Sun or of a star.

By day we shall reckon the Sun's distance from the meridian from its altitude, by the preceding propositions. By night, from the known altitude and declination of a star, we shall likewise seek out its distance from the meridian. But we shall always compute that distance from the meridian to the star in the direction contrary to the one in which the heavens move, whether the star is inclining toward the east or toward the west; for in this way we shall be freed from a multiplication of rules. Thus if the star is 40 degrees distant toward the west, we shall subtract these from 360; 320 degrees will remain, and the observed star will be said to be that far from the meridian, on the side toward which the zodiac is carried by its proper motion, but contrary to the first motion.

Next we shall reckon the right ascensions of the Sun and of the star by the sixth proposition, and from these we shall gather by how great an arc of the equator the Sun is distant from the observed star, following the order of the signs. For when the right ascension of the Sun is found greater than the right ascension of the star, their difference is the quantity sought; but if it is less, the difference of the ascensions taken from the whole circle leaves the sought distance known.

Now we shall gather both these distances into one sum — namely the star's distance from the meridian and the Sun's distance from the star itself. For the

composite arc, if it is less than a whole circle (or else what is left of it when a circle has been taken away, if it is greater), will be the Sun's distance from the meridian, reckoned from midday to the Sun in the order of the signs; and so the hour cannot be unknown.

But whenever the right ascension of the star is found equal to the right ascension of the Sun, the Sun and the star will be equally distant from the meridian, and toward the same quarters of the world, either eastward or westward. And whenever the difference of their right ascensions is a semicircle, the star's distance from the meridian point becomes equal to the Sun's distance from the midnight point, and conversely, but toward opposite quarters. From this is elicited the method of constructing a universal nocturnal timepiece by means of the distance from the meridian of some one of the ever-visible stars.

But when one wishes, from a known number of equal hours, to investigate the altitude of the Sun or of a star proposed, we shall first weigh — from our knowledge of the semidiurnal arc of the Sun's place and of the arc of the given time — whether the given time is a daytime or a night-time one. If daytime, we shall find the Sun's altitude by the twelfth proposition. For the square of the whole sine has the same ratio to the rectangle contained by the right sines of the complement of the Sun's declination and of the complement of the altitude of the pole as the versed sine of the arc of the Sun's distance from the meridian has to the difference between the right sines of the meridian altitude and of the altitude which the Sun itself has at the time of observation. Therefore, by the common rule of proportional numbers, the fourth term becomes known from the first three, which are known. So when that fourth term has been subtracted from the right sine of the meridian altitude — which we know each day from the altitude of the pole and the Sun's declination — the right sine of the Sun's altitude answering to the given time is left known; and therefore its arc will be known from the table of sines.

But if the given time is found to be a night-time one, we shall convert the number of hours into degrees, and from their number we shall elicit the Sun's distance from the meridian, taken in the order of the signs. Besides this we shall elicit, in the way explained above and from the right ascensions, the ascensional distance of the Sun from the star proposed to us; and we shall subtract the lesser distance from the greater. For the remaining arc will be the star's distance from the meridian, and it is the one that will be reckoned from midday in the succession of the signs if the Sun's distance from the meridian is found the greater — the contrary if it is less.

Now therefore, from our knowledge of the star's semidiurnal arc and of its distance from the meridian, we shall easily recognize whether the star itself is below the horizon or above it. And if it is found above the horizon and set at the midnight point, we shall pronounce that it has its least altitude, as was shown in the twelfth proposition.

as was shown in the twelfth proposition. That least altitude must be what is left when the complement of the star's declination is taken away from the altitude of the pole; but this can happen only to those stars which are always visible. Again, if the star proposed is found to be no distance at all from the meridian, we shall look for its meridian — that is, its greatest — altitude by converting the fourth or the fifth proposition. In any other position, provided only that the star is above the earth, we shall learn its altitude from the distance already found, by the twelfth proposition. I think no one is so ignorant of astronomy that he cannot understand the reasoning of all these computations by himself, without a teacher.

An example will now make all this easier. Let the Sun have 20° 12′ of northern declination, and let it stand 36° above the horizon of Lisbon; and let it be required, following the teaching of the present proposition, to compute the time before or after noon. Since the elevation of the arctic pole on that horizon is 38° 40′, the altitude of the equator will be 51° 20′. To this we shall add the Sun's declination of 20° 12′, and there is made up an arc of 71° 32′ for the meridian altitude. The right sine of that arc contains 94850 parts. From this number we shall subtract 58778, the right sine of the Sun's altitude of 36°, and 36072 will be left as their difference. This number, as is clear from the twelfth proposition, will be the first term of the proportion. We shall therefore multiply it by the square of the whole sine, and 360720000000000 results. Next we shall multiply 93849, the right sine of the complement of the Sun's declination, by 78079, the right sine of the complement of the altitude of the pole, and 7327636071 is produced — the third term, that is. Finally we shall divide the product of the first term into the fourth by this, and 49227 comes out, the versed sine of the arc of the Sun's distance from the meridian, namely the second term; and to this there answer in the table 59° 29′. Therefore, reckoning 15° to the hour, we shall declare that at that very moment of time the Sun is three hours and nearly 58 minutes of an hour from the meridian.

Let us further suppose that the Sun occupies the beginning of Taurus and is four equal hours from the meridian, and that its altitude has to be found. The Sun's declination, computed by the second proposition, is 11° 30′; its complement is therefore 78° 30′, and the right sine of that complement is 97992. This number we

shall multiply by 78079, the right sine of the complement of the altitude of the pole, and 765217468 [?] is produced, the third term of the proportion. Then we shall multiply this by 50000, the versed sine of the Sun's proposed distance from the meridian, which is the second term of the proportion, and 382560873400000 results. Finally we shall divide this number by the square of the whole sine, the fourth term, by striking off the last ten figures; and from that division there will come 38256, that is, the first term of the proportion named. Now since a number of this kind is the difference between the right sine of the meridian altitude and the right sine of the altitude the Sun itself has when at the given time it stands away from the meridian, we shall subtract from 88968 — the right sine of the meridian altitude of 62° 50′ — the 38256 parts which the said difference contains; and 50712 will be left as the right sine of the Sun's altitude. To this number there answer in the table 30° 28′. Therefore, when the Sun has occupied the beginning of Taurus and has withdrawn four equal hours from the meridian of Lisbon, it will be seen raised above the horizon by those 30° 28′.

Again, let us suppose that at that moment of time at which the Sun holds 15° 13′ of Gemini, the Bright Star of the Northern Crown [*Lucida coronæ septentrionalis*] is setting toward the west, and stands 41° above the horizon; that its declination is northern, of 28° 51′; and further that it has a right ascension of 227° 44′, reckoned from the vernal section. It is required to elicit from these data what hour it is. The meridian altitude of the observed star will therefore be 80° 11′, and the right sine of that is 98535. From this we shall subtract 65605, the right sine of 41°, and 32930 will be left. This difference — the first term of the proportion — we shall multiply by the square of the whole sine, the fourth term, and 329300000000000 results. Next we shall multiply 87588, the right sine of the 61° 9′ which the complement of the observed star's declination contains, by 78079, the right sine of the complement of the altitude of the pole, and 6838783452 results, namely the third term of the proportion named. Finally we shall divide by this the number which had come from multiplying the first term into the fourth; and from that division 48152 comes out, the versed sine of the same star's distance from the meridian toward the west, to which there answer in the table 58° 46′. This distance, then, we shall subtract from the whole circle, and 301° 14′ will be left, by which the Sun will likewise be distant from the meridian — but the computation will run in the contrary direction.

Now the right ascension of the Sun is 73° 57′, and that of the star 227° 44′; the ascensional distance of the Sun from the star itself will therefore be 206° 13′, taken in that same order. Moreover, out of these two distances there is made up a number of 507° 27′, from which we shall then take away the 360° that make up

the whole circle, and 147° 27′ are at last left, by which the Sun will at that time be distant from the meridian — that is, by 9 hours 49 minutes 48 seconds before noon.

Let us further inquire, at that same time, about any star whose declination and right ascension are known from what precedes: whether it is below the earth or above, and how great an elevation it has above the horizon. Take, for example, that star which in Latin is called *vociferans* [the Clamourer] and in Arabic *Alramech*, whose northern declination is supposed to be 21° 45′ and its right ascension 207° 17 [?]′. Since the right ascension of the Sun is 73° 57 [?]′, the ascensional distance will accordingly be 226° 40′. From this we shall subtract the Sun's distance from the meridian, 147° 27′, and 79° 13′ will be left, by which the star conceived will be distant from the noon line toward the west. Now, so that the size of that star's own semidiurnal arc may be known, we shall multiply 92880, the right sine of the complement of its declination, by 78079, the right sine of the complement of the altitude of the pole, and 7251977520 results. By this, then, we shall divide 956720000000000, which is made by drawing the square of the whole sine into 95672, the right sine of the 73° 5′ which the meridian altitude of the star proposed contains; and from the division there will come 131925, the versed sine of the semidiurnal arc of that same star, as the eighth proposition demonstrates. Further, to that number of parts there answer in the table 108° 37′ for the size of the semidiurnal arc; and so the star conceived will be seen raised above the horizon. This can also be elicited from the declination alone, without computing the semidiurnal arc. For since that declination is supposed to be northern, by what we demonstrated above the semidiurnal arc of such a star must exceed a quadrant; but its distance from the meridian is 79° 13′, and therefore it will be seen raised above the horizon. Whenever, however, the star's distance from the meridian is greater than a quadrant, its semidiurnal arc will have to be computed, so that we may judge whether the star is below the horizon or above it. Now therefore, just as in the example taken up, from the star's known distance from the meridian…

let us determine its altitude. Following the plan of the present proposition, we shall multiply 81291, the versed sine of the 79° 13′ by which the star is distant from the meridian, by 7251977520, the product of multiplying the right sine of the complement of the altitude of the pole into the right sine of the complement of that star's own declination; and 589520504578320 results. Finally we shall divide this number by the square of the whole sine, and from that division there will come 58952, namely the difference between the right sines of the meridian altitude and of the altitude the star has at the time of observation. We shall

therefore subtract 58952 from 95672, the right sine of the meridian altitude of that same star, and 36720 will be left, the right sine of the 21° 33′ of its elevation above the horizon.

Propositio. XV.

¶ *To investigate the length of the twilight.*

At the beginning of the morning twilight or the end of the evening twilight, let there be observed, with the astrolabe whose construction we taught in the third proposition, the altitude of any star whose declination and right ascension are known by the sixth proposition; and by the preceding proposition let the arc of equal hours before or after noon be computed. Let there also be computed, by the seventh or the eighth, the length of the semidiurnal arc of the Sun's place; for the difference of the two arcs will be the interval, or magnitude, of the twilight.

An example. At Lisbon, in the passing year of salvation 1541, on the first day of the month of October, in the evening, the sky being clear, from the highest citadel of the city, when there was no longer any brightness at all in the western quarter, I observed the star of the Heart of the Scorpion [*cor Scorpij*] setting, and found it raised five degrees above the horizon. And since its place is the end of the fourth degree of Sagittarius — which agrees both with al-Battānī's opinion and with our own other observations — its declination will accordingly be 24° 56′ and its right ascension 241° 10′. Accordingly we shall subtract 8715, the right sine of 5°, from 44463, the right sine of 26° 24′, the meridian altitude of that same star, and there will be left as the difference of the right sines 35748. This difference, then, we shall multiply by the square of the whole sine; and the product we shall divide by the number which is made by drawing 90679 — the right sine, that is, of…

the complement of the declination of the star named — into 78079, the right sine of the complement of the altitude of the pole; and from the division there will come 50492, the versed sine of 60° 19′, the distance of that star from the meridian. And since the Sun at that time occupied the end of the 18th degree of Libra, whose right ascension is 196° 35′, the difference of those right ascensions is therefore 44° 35′. So the Sun's distance from the noon line, according to the diurnal motion, was 104° 54′. From this we shall take away the Sun's semidiurnal arc, 84° 18′, and 20° 36′ will be left for the magnitude of the twilight — that is, one hour, 22 minutes, 24 seconds. If, however, we wish to abide by an exact reckoning, this total is a little larger than the length of the twilight. For the evening twilight does not begin before the Sun's centre is hidden 14 minutes below the horizon. It will therefore be necessary, by the eighth proposition, to

compute the time from noon to the Sun's centre when it is set those 14 minutes below the horizon; this we shall then subtract from the distance found, and the true length of the twilight will be left.

Propositio. XVI.

¶ *From the given length of the twilight, to elicit the Sun's distance from the horizon.*

It was demonstrated above, in the eighth proposition, that as the square of the whole sine is to the rectangle contained by the right sines of the complement of the altitude of the pole and of the complement of the declination of the Sun's place, so is the versed sine of the arc compounded of the semidiurnal arc and the arc of the twilight to a certain straight line compounded of two right sines — of which one is the right sine of the meridian altitude, and the other that of the arc by which the Sun is distant from the horizon at the beginning of the morning twilight or the end of the evening twilight. We shall therefore compute, by the seventh or the eighth proposition, the magnitude of the semidiurnal arc of the Sun's place; to it we shall add the arc of the length of the twilight; the versed sine of the compounded arc we shall multiply by the number which is made by drawing the right sine of the complement of the altitude of the pole into the right sine of the complement of the declination of the Sun's place; the product we shall divide by the square of the whole sine; and out of the division there will come a certain number of parts of the diameter, from which…

we shall subtract the right sine of the Sun's meridian altitude; and there will be left the right sine of the arc of the vertical circle by which the Sun's centre stands away from the horizon at the beginning of the morning twilight or the end of the evening twilight. That arc itself, then, will be known from the table.

An example. On the same day the Sun's declination is 7° 5′, and its complement 82° 55′. The right sine of that complement, 99236, we shall multiply by 78079, the right sine of the complement of the altitude of the pole; and the number produced by that multiplication we shall multiply by 125713, the versed sine of the arc compounded of the semidiurnal arc and the twilight arc, which was found to be 104° 54′. The product we shall divide by the square of the whole sine, striking off the last ten figures, and 97405 will come. From these we shall take away 69779, the right sine of the 44° 15′ which the Sun's meridian altitude contains, and 27626 will be left for the right sine of the arc of the Sun's concealment at the end of the twilight. To these there answer in the table 16 degrees, 2 minutes, of the circumference of the circle. The magnitude of the arc of

the Sun's concealment at the end of the twilight is therefore known — which is what we set out to investigate.

Propositio. XVII.

¶ *To disclose the rule of the increase and decrease of the twilights.*

We find that the twilights keep a rule in their increase and diminution far different from the progression of the days and nights. For the day always increases from the beginning of Capricorn as far as Cancer, and at the very beginning of Aries it is equal to the night. The twilights, by contrast, begin to diminish from the beginning of Capricorn, and grow smaller day by day, always by a perceptible difference, as far as that point of the ecliptic at which the right sine of the altitude of the pole is to the whole sine as the right sine of the arc of the Sun's concealment is to twice the right sine of the declination of that same point. Yet before that diminution reaches the equator, we shall come upon a point of the ecliptic whose twilight arc will be equal to the twilight of the equator. The twilights therefore go on decreasing after that, though by an almost imperceptible quantity, as far as a certain point of the ecliptic before the beginning of Aries, at which the twilight is…

the shortest that can possibly be. From that point they always increase, as far as the beginning of Cancer. All of this we shall demonstrate in order. We shall also investigate the point of the ecliptic in the winter quadrant at which the twilight is equal to the twilight of the equator; and beyond that we shall look for the further point, before the beginning of Aries, at which the Sun produces the shortest twilight.

Diagram. A woodcut circle — the meridian — standing in the upper part of the left column, cutting into the sentence that introduces it. A heavy horizontal diameter runs from b at the left to d at the right, and a vertical diameter from c at the top to Q at the foot; the centre is marked a. Four further horizontal chords cross the circle below the diameter b d, their left ends lettered down the left arc and their right ends down the right arc, and two oblique lines run up from the lower left to the upper right, cutting all of them. The point-letters are engraved at roughly half the size of the text type; those that resolve are:

- c (top of the circle): upper end of the vertical diameter c Q, the section of the right horizon
- b (left end of the heavy horizontal diameter): left end of b d, the section of the equator

- d (right end of the same diameter)
- Q (foot of the circle): lower end of c Q
- a (centre of the circle)
- D (on the left arc, first below b): left end of the chord D G
- A (on the left arc, below D): left end of the chord A B
- l (on the left arc, below A): left end of the chord l m
- v (on the left arc, below l): left end of the chord v R
- e (on the lower left arc, below v): left end of the oblique line e f, the section of the oblique horizon
- g (on the lower left arc, below e): left end of the second oblique line g h
- f (on the upper right arc): right end of e f
- h (on the right arc, below f): right end of g h
- G (on the right arc, first below d): right end of the chord D G
- [?] (on the right arc between G and m, where the chord A B meets it — the glyph is overrun by the heavy arc and does not resolve)
- m (on the right arc, below the preceding): right end of the chord l m
- R (on the right arc, lowest of the run): right end of the chord v R
- z (interior, on the chord D G, left of the centre)
- s (interior, on the chord D G, between z and the centre)
- x (interior, below z, where the oblique line meets the chord next below D G)
- o (interior, immediately right of x)
- r (interior, below x)
- q (interior, lower left of the centre)
- p (interior, right of q, on the same chord)
- [?] (two further small marks below and to the right of the centre do not resolve)

To begin, then, let us draw the meridian b c d Q about the centre a. In it let e f be the section of the oblique horizon; g h the section of a circle parallel to it; e g, or f h, the arc of the Sun's concealment at the beginning of the morning twilight or the end of the evening twilight; c Q the section of the right horizon of those who live under b; and b d itself the section of the equator. Next let us conceive a circle parallel to the equator, passing through the common section of the right horizon and of the circle parallel to the horizon; and let the common section of that circle and the meridian be the straight line l m. Let a t be divided into two equal parts at the point s, and let the straight line D G be drawn through s parallel to b d. From

what we demonstrated in the first part it is clear that the straight line a s is equal to the right sine of the arc of declination from the equator of that parallel whose diameter is D G — that is, to the right sine of the arc b D, or d G. From the point t [?] let the perpendicular r o [?] be raised onto e f; and let two further parallels be drawn, one of which cuts the meridian along the straight line v R, below l m, the other along the straight line A B, between D G and l m. Let e f cut the diameters of those parallels at the points x, r, q, and let g h cut the same diameters at the points n, y, ı p [?]. Therefore, by what we demonstrated in the fourth appendix, the perpendiculars standing on p q cut off more degrees of the parallel's circumference than those standing on r t [?]; those on r r [?] more than those on x ı [?]; and finally those standing on these enclose more degrees than those standing on z n.

There are two grounds for the perceptible diminution from the beginning of Capricorn as far as that parallel whose diameter is D G: first, that the more southerly circles are the smaller; second, that in those same circles the straight lines upon whose ends the perpendicular common sections stand are further from the centres. These, in fact, were the two middle terms of the demonstration of the fourth appendix. But by the lemma of the sixth appendix, as the right sine of the angle t a o [?] — the angle of the altitude of the pole — is to the whole sine, so is o t, equal to the right sine of the arc e g of the Sun's concealment, to the straight line a t, which is double a s; and a s is equal to the right sine of the declination of the point D, or G. Therefore, three terms being known, a t will be found, and its half a s will become known; and so from the table of right sines the arc b D, the declination of the point of the ecliptic conceived, will be got.

For example, on the horizon of Lisbon the right sine of the altitude of the pole contains 62478 parts. We shall therefore divide by this number the product of the whole sine into 27626, the right sine of the arc of the Sun's concealment, and from the division there will come 44217. Half of this number contains 22108½ parts, the right sine of 12° 46′, the declination of the point D, or G. From the known declination, the point of the ecliptic to which it answers will be found by the second proposition — namely 3° 40′ of Scorpio and 26° 20′ of Aquarius. The twilights, therefore, decrease with a perceptible difference throughout, from the winter solstice up to the fifth day of February in our own age; while, as the day decreases, they begin to grow by a perceptible increment from the 17th day of October.

From this same figure the amplitude of rising of the point of the ecliptic conceived can also easily be got. For it is clear, from what we demonstrated in the

first part, that the common section of the horizon and of the parallel whose diameter is D G is perpendicular to that same diameter D G in the plane of that parallel, and becomes the right sine of the semidiurnal and seminocturnal arc. And since that line is perpendicular to the plane of the meridian, it will also be the right sine, in the plane of the horizon, of the complement of the amplitude of rising — that is, of the arc of the horizon contained between the point e and the intersection of the horizon's circumference with the parallel conceived. Accordingly the straight line e z will be the versed sine of the complement of the amplitude of rising, and the remainder a z will be equal to the right sine of the amplitude of rising of the point of the ecliptic conceived, which describes the parallel whose diameter is D G. We shall therefore investigate the magnitude of the straight line a z in this way.

Since the angle z a s is equal to the angle of the altitude of the pole, the remaining angle a z s will be equal to the angle of the altitude of the equator. But by the lemma of the sixth appendix, as the right sine of the angle a z s is to the whole sine, so is a s — the right sine of the declination of the point of the ecliptic conceived — to the straight line a z, the right sine of the amplitude of rising of that same point. Of these four quantities the first three are given; and so, by the common rule of proportional numbers, the fourth will become known. From the table of right sines, then, the arc of the amplitude of rising will come out known.

These things being shown, we shall next demonstrate that the diminution of the twilights is not continuous all the way to the equator. Rather, before the Sun enters the beginning of Aries, at a certain point of the winter quadrant of the ecliptic — which we shall indicate at once — the twilight becomes equal to the one the Sun produces when it stands on the equinoctial circle; while at the intermediate points of the ecliptic the twilights are always smaller than these. The end of the decrease of the twilights must therefore fall at one of those intermediate points, at which the twilight will be the shortest of all. From that point the twilights always increase; and when the Sun reaches the beginning of Aries the twilight will be equal to the earlier one, and they will go on growing in an unbroken series as far as the beginning of Cancer.

Diagram. A hemispherical figure. A great arc rises from the lower left to an apex at C and descends steeply to P at the lower right; a second arc crosses it, meeting the base region at the left. Four nearly parallel curves are drawn across the interior, running from the left edge toward the right, where they meet a short steeply inclined arc carrying a run of small letters. The labels are:

- C (apex): the northern intersection of the two horizons

- K (upper left, on the outer arc): end of the equinoctial circle B D K
- D (below and right of K): on the equinoctial circle
- E (interior, right of centre, on the crossing arc): the far end of the twilight arc B E, on the second horizon P E C K
- N (left edge, second of the stacked labels): left end of the parallel r y N
- M (left edge, third of the stacked labels): left end of the parallel f o M
- Q (left edge, lowest of the stacked labels): left end of the parallel A P Q
- y (interior, below and left of E): the point on the parallel r y N
- o (interior, below y): the point on the parallel f o M
- V (right edge, uppermost): foot of the perpendicular arc E V
- B (right edge, below V): the equinoctial rising, on the horizon A B C D
- z (right edge, first of the run of four small letters below B): foot of the perpendicular arc y z
- [?] (second of that run — the glyph does not resolve)
- [?] (third of that run — the glyph does not resolve)
- f (fourth of that run, immediately above g): where the parallel f o M meets the horizon A B C D
- g (right edge, low): foot of the perpendicular arc P g
- A (right edge, lowest): where the parallel A P Q meets the horizon A B C D
- P (foot of the figure, at the right): the point on the second horizon P E C K

The equinoctial circle is B D K; the oblique horizon A B C D; and B itself is the equinoctial rising. Let B E further be the arc of the length of the twilight which the Sun makes when it occupies the beginning of Aries. Through the point E let there pass a horizon P E C K, similar to the first — that is, of equal elevation of the pole — cutting it at C on the northern side. Now the angles C B E and C E D, being angles of the altitude of the equator, are equal to each other; the two arcs B C and C E taken together will therefore equal a semicircle, by the tenth proposition of the first book of Menelaus. But the angle B E C, being obtuse, is greater than the acute angle E B C, and so the arc B C is greater than the arc E C, by the seventh. B C is therefore greater than a quadrant, and E C less than a quadrant.

Let an arc o C be taken equal to B C, so that the two arcs E C and o C together make up a semicircle, and let the parallel f o M be drawn. Likewise, through the points y and P — one of which inclines toward the equator, the other toward the winter tropic — let the parallels r y N and A P Q be drawn; and from those points E, y, o, P let the arcs P g, o t, y z, E V be dropped at right angles onto the horizon A B C D. This is easily done: find the further pole of the horizon A B C D by the 31.

proposition of the first book of Theodosius, and from it draw great circles through the points E, y, o, P; these will cut the horizon A B C D at right angles, by the 19th proposition. Therefore, as the right sine of the arc P C is to the right sine of the arc E C, so is the right sine of the arc P g to the right sine of the arc E V, by the 12th proposition of the first book of Geber — a result that can also be shown by the demonstrations given above.

For by that same method of demonstration we have just used to show that the right sines of the declination of the point of the ecliptic conceived and of its amplitude of rising have the same ratio as the right sine of the altitude of the equator and the whole sine — or by the reasoning we followed in the inquiry into the declinations of the points of the ecliptic and into the length of the equinoctial twilight — it plainly follows that in a right-angled spherical triangle the right sines of the sides and of the angles they subtend are proportional in the same order; and by the 23rd proposition of the fifth book of Euclid we shall conclude the same of every other triangle. Accordingly, by the 11th proposition of the fifth book, as the right sine of the arc P C is to the right sine of the arc P g, so is the right sine of the arc E C to the right sine of the arc E V; and therefore, by alternation, as the right sine of the arc P C is to the right sine of the arc E C, so is the right sine of A g [recte: P g] to the right sine of the arc E V. I would not have anyone troubled that we have occupied ourselves only with sides less than a quadrant, since one and the same straight line subtends both an arc less than a quadrant and what that arc wants of a semicircle. However that theorem be demonstrated, our procedure will not on that account vary in the least. And so, as the right sine of the arc…

P C is to the right sine of the arc E C, so is the right sine of the arc P g to the right sine of the arc E V. But the right sine of the arc P C is less than the right sine of the arc E C, because the right sine of the arc P C is less than the right sine of the arc o C, and the arcs o C and E C have one and the same right sine. The right sine of the arc P g is therefore also less than the right sine of the arc E V. Now the arc E V is the arc of concealment at the beginning of the morning twilight, when the Sun occupies the equator. P g is therefore less than the arc of concealment at the moment when the Sun, describing the parallel A P Q, begins the morning twilight. So the twilight of that day had already begun before the Sun, by the motion of the first heaven, reached the point P. But all the arcs of the parallels contained between two horizons of equal elevation of the pole are proportional to the arc B E, the twilight of the equator; the twilight of the parallel A P Q, which inclines toward the winter tropic, is therefore longer than the equinoctial twilight.

We shall now demonstrate that the twilight of the parallel f o M and the equinoctial twilight are equal. For as the right sine of the arc o C is to the right sine of the arc E C, so is the right sine of the arc o t to the right sine of the arc E V; and one and the same straight line is the right sine of the arcs o C and E C. The right sines of the two arcs o t and E V are therefore equal to each other, and so the arcs o t and E V are themselves equal. Accordingly the arc o t will be the Sun's concealment at the beginning of the morning twilight when the Sun describes the parallel f o M; and so o f is the length of the twilight. But the arcs f o and B E are proportional. Therefore the twilight the Sun makes when it describes the parallel f o M, and the equinoctial twilight, are equal — which was what had to be demonstrated.

As for the twilight of the parallel r y N, and any other twilights belonging to the parallels that lie between f o M and the equinoctial circle, these must be less than the equinoctial twilight itself. For it is clear, on the same principles, that as the right sine of the arc y C is to the right sine of the arc E C, so is the right sine of the arc y z to the right sine of the arc E V. But the right sine of the arc y C is greater than the right sine of the arc B C, because y C lies between E C and o C, the arcs that make up a semicircle. The right sine of the arc y z is therefore greater than the right sine of the arc E V, and so the arc y z will be greater than E V. But E V is the arc of the Sun's concealment at the beginning of the morning twilight; therefore y z is greater than the arc of concealment itself, and the morning twilight will not yet have begun when the Sun, by the motion of the first heaven, has reached y. Now the arcs y r and E B, the measure of the equinoctial twilight, are proportional; therefore a shorter twilight is produced when the Sun describes the parallel r y N than when it occupies the equator or the parallel f o M — which is likewise what we set out to demonstrate. By this same method of demonstration it will also be proved that, when the Sun is in the northern signs, longer twilights belong to the more northerly points — which was shown in the first part by other middle terms.

But before we go on with the rest, let us demonstrate what we assumed: that the arcs of the parallel circles contained between similar horizons are proportional.

Diagram. A quadrant-shaped figure. A long arc sweeps from O at the left across the top of the figure and down to A at the lower right; a second arc runs from the upper region down to P at the foot; two shorter arcs cross the right-hand part of the figure and meet near B. The labels are:

- R (upper left of the crown of the figure)
- G (upper right of the crown, right of R)

- O (left extremity)
- n (right of centre, immediately left of B, on the arc of the equator): where the meridian through P cuts the equator
- B (right of centre, where the crossing arcs meet)
- m (right of centre, immediately right of B, on the same arc): where the meridian through A cuts the equator
- E (below and left of B, above P)
- P (foot of the figure, at the left of the pair)
- A (right extremity, below and right of E)

Let meridians be drawn through A and P, cutting the equator at m and n; the angles at m and n are therefore right, by the 19th proposition of the first book of Theodosius. The arcs A m and P n of declination are equal, by the common notion; the angles at B and E are equal; of the remaining arcs, A B and P E taken together are less than a semicircumference; and further, m B and n E taken together are also less than a semicircumference. Therefore A B and P E — the amplitudes of rising of the same point of the ecliptic on similar horizons — are equal. Likewise m B and n E — the ascensional differences, that is, the differences between the quadrant and the semidiurnal arc — are equal to each other, by the 16th proposition of the first book of Menelaus. If, then, the common arc B n is added to these, the two arcs m n and B E will become equal, by the common notion. But the arc A P is proportional to the arc m n, by what we demonstrated in the first lemma, or by the 14th proposition of the second book of Theodosius. Therefore the arc A P is itself proportional to the arc B E; and the demonstration will run in the same way for the rest.

But if you, industrious reader, do not have Menelaus's *Spherics* to hand, you can gather the same result another way from the demonstrations already given. For in the right-angled triangle P n E, as the whole sine is to the right sine of the angle E, so the right sine of arc P E is to the right sine of arc P n. Likewise in triangle A m B, as the whole sine is to the right sine of the angle B, so the right sine of arc A B is to the right sine of arc A m. But the whole sine has the same ratio to the right sine of the angle E as it has to the right sine of the angle B, by *Elements* V.7. Therefore, as the right sine of arc P E is to the right sine of arc P n, so the right sine of arc A B is to the right sine of arc A m, by *Elements* V.11. Moreover the arcs P n and A m are themselves equal. So by V.7 and V.9 of that same fifth book we shall conclude that the right sines of arcs P E and A B are equal — and since each of these arcs is less than a quadrant, the arcs themselves must be equal too.

Next we shall extend the arcs P n and P E to the measure of quadrants, as far as G and R; and about P as pole let the great circle G R O be described; and let the arc n E be produced until it meets the circle so described at O. The pole of the circle P n G therefore lies on the circle n E O, by Theodosius, *Spherics* I.17. And since the angle at G is right, by I.19, the pole of that same circle P n G will also lie on the circle G R O, by the same I.17. O is therefore that pole, and n O and G O are quadrants, by proposition 24 of the same book. Therefore, by what we demonstrated above, in the spherical triangle E R O, as the whole sine is to the right sine of the arc of the angle O, so the right sine of arc E O is to the right sine of arc E R. But n G, the arc of the angle O, is the complement of arc P n; E O is the complement of arc n E; and finally arc E R is the complement of arc P E. Therefore, as the whole sine is to the right sine of the complement of arc P n, so the right sine of the complement of arc n E is to the right sine of the complement of arc P E. The same will be demonstrated in any other right-angled triangle whose sides are less than quadrants.

Now, then, we shall conclude what we assumed, in this way. As the whole sine is to the right sine of the complement of arc A m, so the right sine of the complement of arc n E is to the right sine of the complement of arc P E, by *Elements* V.7. And as the whole sine is to the right sine of the complement of arc A m, so the right sine of the complement of arc m B is to the right sine of the complement of arc A B. Therefore, by *Elements* V.11, as the right sine of the complement of arc n E is to the right sine of the complement of arc P E, so the right sine of the complement of arc m B is to the right sine of the complement of arc A B. So by alternation, as the right sine of the complement of arc n E is to the right sine of the complement of arc m B, so the right sine of the complement of arc P E is to the right sine of arc A B. But the complements of the arcs P E and A B are equal. Therefore the complements of the arcs n E and m B are equal to one another as well, and so are the arcs n E and m B — which the theorems of Menelaus demonstrate more concisely.

But let us return to our undertaking and look for that point of the ecliptic at which the Sun, when it stands there, will produce a twilight equal to the equinoctial twilight. In the figure already drawn, the arc B E was the length of the equinoctial twilight — which belongs also to the point of the ecliptic that describes the parallel f o M — and E V was the arc of the Sun's concealment at the beginning of the twilight. And since arc E C is less than a quadrant, while arc o C is the remaining part of the semicircle, we shall describe about the point C as pole the arc G Z H of a great circle, cutting the equator at the point Z and the two horizons at G and H. The angles at G and H are therefore right, by Theodosius,

Spherics I.19, and the arcs C G and C H are quadrants, by proposition 24 of the same book. Now in the two triangles B Z H and G Z E the angles at Z are vertically opposite and equal — the common notion alone suffices to prove this. The angles at E and B are equal as well, since they belong to equal altitudes of the equator in similar horizons; and the remaining angles, at G and H, are right. Therefore, by Menelaus, *Spherics* I.17, the sides subtended by equal angles are equal; and so the arcs G Z and Z H are equal.

Diagram. A spherical figure in the right-hand column. A long, nearly straight arc runs across the whole width, lettered K at its left end and M at its lower left, and reaching f at its right end. Two further great-circle arcs cross it: one rises from the left through D and C to the upper right, the other passes through E toward B on the right. A short transverse arc drops from the upper-right region through Z, and G lies below E toward the right; o sits at the lower right, below the crossing near f. V and H stand close together at the upper right, engraved sideways along the arc.

- K (left end of the long arc): K
- M (lower left, at the foot of the same arc): M
- D (on the upper arc, left of centre): D
- C (top of the figure, on the upper arc): C
- E (centre, where two arcs cross): E
- V (upper right, set sideways): V
- H (upper right, beside V, set sideways): H
- B (right-hand margin of the figure): B
- Z (below V and H): Z
- G (below and right of E): G
- f (right end of the long arc): f [?]
- o (lower right, below the crossing): o

We shall reach the same conclusion if, as a little earlier, we pursue the matter through the propositions themselves. For from the arcs B C and C E, which together equal a semicircle,

and from the two arcs H C and C G, likewise equal to a semicircle, when the common arcs H C and C E are taken away, the two arcs B H and G E will be left equal. But the angles at Z are themselves equal. Therefore, proceeding by the route of the proportions, we shall demonstrate that the arcs B Z and Z E are equal, and again that G Z and Z H are equal.

Now in triangle B E V, as the tenth proposition demonstrated, the right sine of the angle B — the altitude of the equator — is to the whole sine as the right sine of arc E V, the Sun's concealment, is to the right sine of arc B E. So by the common rule of proportional numbers, once three terms are known the fourth will be found: namely the right sine of arc B E. The arc B E itself, the length of the equinoctial twilight, is therefore known, and its half B Z is known as well. Further, in triangle B Z H, as the whole sine is to the right sine of the angle B, the altitude of the equator, so the right sine of arc B Z is to the right sine of arc Z H. So from three known terms the fourth, Z H, will become known, and the whole arc G H with it. And since, as the right sine of arc G H is to the right sine of arc E V, so the right sine of arc G C is to the right sine of arc E C, from three known terms the fourth will become known — namely the right sine of arc E C — and the arc B C [?] is known as well. So when we subtract it from the semicircle, the arc o C is left known; and hence the difference o E is known, which is equal to the amplitude of rising of the point of the ecliptic we are looking for. Now, then, from the amplitude of rising known in the given horizon, the declination of the point that describes the parallel f o M will be found, and by the second proposition the point of the ecliptic to which that declination belongs.

Now let us pass to investigating the size of the shortest twilight that can occur in a given horizon, and the point of the ecliptic at which it occurs. Let the equinoctial circle be B A G and the oblique horizon Q B C. Let the Sun make the shortest twilight when it describes the parallel D F, and let the measure of that twilight be the arc A B of the equator. Let a horizon P A C, similar to the first, pass through A; and let the arc of the shortest twilight itself on the parallel D F be E D. I say first that the arc E C is a quadrant. Let any other parallels whatever be drawn, such as o t, r y, P Q; and onto the horizon Q B C let the perpendicular arcs A z, o m, t n, E K, P h be dropped. The arc E k is therefore the Sun's concealment at the beginning of the shortest twilight. Now all the arcs of the parallels intercepted between the horizons drawn are proportional, and the shortest twilight is E D. Therefore, when the Sun describes the parallel r y, the morning twilight begins before the motion of the first heaven carries the Sun to the point r; and so the arc r n of the vertical circle, through which it is still hidden below the horizon Q B C, is less than E K, the twilight concealment of the Sun. But as the right sine of arc E K is to the right sine of arc r n, so the right sine of arc E C is to the right sine of arc r C; and since the first term is greater than the second, the third is likewise greater than the fourth. It will be demonstrated in the same way that the right sine of arc E C is greater than the right sine of arc o C, and than that of any other arc which the Sun's parallels mark off, whether on the side of C or on the opposite side.

Therefore, if the right sine of arc E C is greater than the right sines of the arcs bounded by the neighbouring points on either side, that arc must be a quadrant.

Diagram. A quadrant-shaped spherical figure at the head of the right-hand column. Two long arcs run from the upper left (C) and the left (G) toward the lower right; a third springs from F on the left. Within them a fan of short nested parallels crosses from the interior toward the right-hand limb, meeting a curve that descends past B to D and h. A and o and s lie in the interior, E stands at the foot of the fan, and the rectangle-like figure at the bottom is closed by P on the left and Q on the right. Several of the smallest letters are engraved sideways along the nested arcs at the upper right and beside B, and will not resolve on the leaf.

- C (upper left, at the head of the outer arc): C
- G (left margin of the figure): G
- F (left margin, below G): F
- A (interior, left of the fan): A
- o (interior, below A): o
- s (interior, right of o): s
- E (foot of the fan, on the lower arc): E
- P (bottom left): P
- Q (bottom right): Q
- B (right limb, set sideways): B
- D (right margin, below B): D
- h (right margin, below D): h
- (four small letters set sideways along the nested arcs, upper right): [?]
- (two small letters set sideways beside B): [?]

Now, then, we shall easily find the quantity of the shortest twilight. Let the arc E k cut the arc A B of the equator at the point s. It is plain, from what we demonstrated a little earlier, that the arcs A s and B s are equal, and again that E s and k s are equal to one another. And since, in the right-angled triangle s B K, the right sine of the angle B, the altitude of the equator, is to the whole sine as the right sine of arc k s — half the twilight concealment — is to the right sine of arc B s, half the length of the shortest twilight, therefore from three known terms the fourth will become known: namely the right sine of arc B s. By the table, then, the arc B s belonging to that right sine will be known, and the whole arc A B with it; and so the length of the shortest twilight itself is known. Again, in that same right-angled triangle s B k, as the whole sine is to the right sine of the complement of arc B K, so the right sine of the complement of arc k s is to

the right sine of the complement of arc B s. But the first, third and fourth terms are known; the second will therefore become known by the common rule of proportional numbers. So we shall subtract the known complement of arc B K from the quadrant, and the arc B K itself will be left, or A E, which is equal to it. Moreover B D and A E have been shown to be equal; therefore B D, the amplitude of rising of the point of the ecliptic at which the shortest twilight occurs, will be known. And from the amplitude of rising known in the given horizon, the declination of the point of the ecliptic conceived will be found, and by the second proposition the point of the ecliptic itself to which that declination belongs.

Now that we have investigated by geometrical demonstrations everything we set out to do, it remains to work it all through in numbers. First, then, let us suppose the Sun to occupy the equator, and let us compute the length of the twilight in a given horizon — for example, where the arctic pole is elevated 38° 40′ — and let us also look for that point of the ecliptic at which a twilight equal to it occurs a second time. We shall therefore multiply 27626, the right sine of the Sun's concealment, into the whole sine; we shall divide the product by 78079, the right sine of the altitude of the equator; and there will result 35382, the right sine of the arc of the length of the twilight. To this there correspond in the table 20° 43′ 20″. The half of this has 10° 21′ 40″, whose right sine is 17985 parts. We shall multiply this number into 78079 and divide the product by the whole sine, and there will come 14042, the right sine of 8° 4′ 20″. The doubled arc is therefore 16° 8′ 40″, whose right sine is 27806. By this we shall divide the number produced by multiplying the whole sine into 27616 [?], the right sine of the arc of concealment, and there will come 99553 [?], to which there correspond very nearly 83° 29′. These we take out of the semicircle, and 96° 31′ will be left; and from these we shall again subtract 83° 29′, and there will be left 13° 2′ of the amplitude of rising. Its right sine, 22551, we shall multiply into 78079 and divide the product by the whole sine, and there will come from the division 17607 and a half: the right sine of 10° 8′ 30″ of declination. Finally we shall multiply 17607 and a half into the whole sine and divide the product by 39874, the right sine of the greatest declination of the ecliptic; and there will come 44158, the right sine of 26° 12′ of the sign Libra, or of 3° 48′ of the sign Pisces. Therefore on the tenth day of the month of October and the twelfth of February, in a common year, the twilights in our own age are equal to those the Sun makes again when it has entered the first degree of Aries or of Libra — this, however, in the horizon of Lisbon.

Further, in order to show the length of the shortest twilight and the point of the ecliptic at which it occurs, we shall multiply the whole sine into 13946, the right sine of 8° 1′, half the arc of concealment; and we shall divide the product by

78079, the right sine of the altitude of the equator; and there will come from the division 17861, the right sine of 10° 17′ 20″, which is what half the length of the shortest twilight has. The shortest twilight is therefore 20° 34′ 40″. But in order to find the point of the ecliptic at which it occurs, we shall multiply 98391, the right sine of the complement of half the length of the twilight, into the whole sine; we shall divide the product by 99022, the right sine of the complement of half the arc of concealment; and there will come from the division 99363, the right sine of 83° 32′, which the complement of the amplitude of rising of the sought point of the ecliptic must have. Taking these from the quadrant, then, there will be left the arc of the amplitude of rising, 6° 28′. Its right sine, 11262, we shall multiply into 78079 and divide the product by the whole sine, and there will come 8793, the right sine of 5° 2′ 40″ of southern declination. Then we shall multiply 8793 into the whole sine and divide the product by 39874, the right sine of the greatest declination; and there will come from the division 22052, the right sine of 12° 44′ of the sign Libra, or of 17° 16′ of the sign Pisces. Therefore in our own age the shortest twilights fall on the 26th day of September and the 25th of February in that same horizon of Lisbon.

It should be noticed, however, that it is not impossible for two shortest twilights to occur in some region on two consecutive days: as if, for example, on some day of the year the arc E C were 90° 15′, and on the next day r C were 89° 45′. But that the two days on which we assert the shortest twilights can occur should *not* be consecutive is altogether impossible; for it would follow that on the day in between a twilight would occur shorter than the shortest. Nor indeed is it necessary that the arc E C be a quadrant, even if there is only one shortest twilight, E D, in the winter quadrant and again one in the autumn quadrant. Rather, the arc E C will either be a quadrant, or greater or less than a quadrant by the smallest difference. For even so its right sine will be greater than the right sine of any other arc of the circle P A C that terminates at the point C. So although we always treat it as a quadrant, no departure from exact reckoning will arise on that account.

Table of the arcs of the twilights at the beginnings of the signs, for the various elevations of the north pole. [*Tabula arcuum crepusculorum ad initia signorum pro varia poli arctici sublimitate.*]

Elevation of the pole	Capri. g.	Capri. m.	Sagit. / Aqua. g.	Sagit. / Aqua. m.	Scorp. / Pisces. g.	Scorp. / Pisces. m.	Libra. / Ari. g.	Libra. / Ari. m.	Virgo. / Taur. g.	Virgo. / Taur. m.	Leo. / Gemi. g.	Leo. / Gemi. m.	Cancer g.	Cancer m.
30	20	5	19	35	18	46	18	36	19	33	21	16	22	15
33	20	48	20	15	19	22	19	14	20	19	22	20	23	31
36	21	37	20	42	20	5	19	58	21	15	23	38	25	5
39	22	39	21	58	20	54	20	49	22	21	25	15	27	6
42	23	51	23	4	21	52	21	49	23	41	27	19	29	47
45	25	16	24	21	23	00	23	00	25	18	30	5	33	39
48	27	1	25	55	24	81	24	23	27	16	33	56	40	3
5[numeral: ?]	29	8	27	48	25	54	26	2	29	49	40	13	*nox tota* (the whole night)	—

Propositio. XVIII.

To measure the greatest elevation of the vapours.

Before I take up the investigation the present problem proposes, I shall prove in order several things that must necessarily be premised.

First: if a luminous sphere illuminates another spherical body, the outermost luminous rays must touch both spheres; and if the rays that fall on it touch both bodies, those rays must be the outermost and the longest.

Let the sphere whose centre is A illuminate the sphere whose centre is b. Join the straight line A b, and through it pass a plane cutting both spheres. By Theodosius, *Spherics* I.1 it is clear that the common sections of the plane and the spheres are circles, and great circles at that. Let these circles be c d e and f g h.

Now light spreads out in every direction along straight lines. So let the straight line c g — or e h — be the outermost ray on the surface of the luminous cone or cylinder along which the illumination takes place, drawn out from the base to the apex. I say that c g, or e h, touches both circles of the spheres.

For if the straight line c g does not touch the circle f g h, then it cuts it. In that case, if from the point c a straight line is drawn touching the circle f g h — by *Elements* III.17 — it will fall either between the line c g and the line joining the point c to b, the centre of the circle, or else outside c g. On the first alternative, two straight lines, namely c g and the tangent, will enclose a surface, which is impossible. If the second is granted, then, since light is spread along the tangent line and along the other lines falling between it and c g, c g will not be the outermost ray, nor the longest either; for any of the others is more remote, and longer, by *Elements* III.8. Therefore the straight line c g, the outermost ray, must touch the circle at the point g.

It will be shown in the same way that it touches the circle c d e at the point c, since on the opposite side the same absurdities follow. Therefore the outermost ray that goes out from a luminous spherical body into the spherical body it illuminates touches both bodies.

Suppose now, conversely, that the straight line c g touches the two circles at the points c and g. I say that the ray c g will be the outermost and the longest. If it were not, it would run out to g from some other point, or from c to some other point, or in some other way whatever; and then two straight lines would enclose a surface, which is impossible. So if a straight line touches both spheres, it will be the outermost ray, and the longest — which was to be proved.

Diagram. A large geometrical figure occupying the lower half of the left column. Two circles lie one above the other on a common vertical axis: a small circle above, with its centre marked b, and a large circle below. From a point at the top of the figure, marked l, two long straight lines run down on either side, touching the small circle and meeting the large circle at the two ends of a horizontal chord; the whole forms the triangular section of the cone described in the text. The vertical axis runs from l down through b and through the large circle to a point d at its lowest part. An outermost great circle encloses the whole figure, and carries

three labels on its circumference: Q at the left, Z above it at the upper left, and R at the upper right. Three long straight chords are drawn across the figure: one from Z and one from Q, both running down toward the right and cutting both tangent lines, and a third from R running down toward the left. Readable labels:

- l (apex, top centre): the point at which the two tangent lines and the axis meet; printed as a capital ɪ in the woodcut.
- R (upper right): on the enclosing great circle; the third chord runs down to the left from this point.
- Z (upper left): on the enclosing great circle; one chord runs down to the right from this point.
- Q (left, below Z): on the enclosing great circle, at the left edge of the figure; the second chord runs down to the right from this point.
- b (centre of the small circle).
- h (on the small circle, left): the tangent point of the left-hand line.
- [?] (on the small circle, right): the tangent point of the right-hand line, which the text calls g; at this point the leaf carries a crowded cluster of small glyphs crossed by the Z- and Q-chords and by the line from R, and no ɡ can be made out among them.
- m, f, n (on the lower arc of the small circle, left to right): f on the axis, m and n on either side of it.
- x (on the axis, inside the large circle, where the horizontal chord crosses it).
- c (left end of the horizontal chord, on the large circle).
- K (on the horizontal chord, between x and its right-hand end).
- c (right end of the horizontal chord, on the large circle; the woodcut cuts the same c-form at both ends of this chord).
- d (lowest point of the large circle, on the axis).
- [?] (on the axis between the chord and d): the centre of the large circle, which the text calls A; the glyph does not resolve on the scan.
- Two or three further small letters stand close against the right and left sides of the small circle (one appears to be ſ, another ʏ); they do not resolve and are not used in the text of these pages.

Second. A luminous sphere illuminates more than half of a smaller sphere, and the two are contained under one and the same cone, whose vertex lies toward the smaller sphere.

Let the greater sphere, whose centre is A, illuminate the smaller sphere, whose centre is b. Join A b, and through it pass a plane cutting both spheres, so that the

common sections are the circles c d e and f g h. Let the outermost rays that enclose the illuminated part be c g and e h, and let the illuminated portion be the arc g f h. I say that this arc is greater than a semicircle.

Let f be the point common to the straight line A b and the arc g f h, and join A c, A e, b g, b h. Then from the greater lines A c and A e cut off e i and c k equal to the lesser lines b h and b g, by *Elements* I.3, and join b i and b k; let these cut the arc g f h at the points m and n.

Since the straight line e h touches both circles, by what was proved first, the angles A e h and b h e will be right angles, by *Elements* III.18. Therefore A e and b h are parallel, by *Elements* I.28. But e i and b h were made equal; so the angles A e h and i b h will be equal, by *Elements* I.34. The angle i b h is therefore a right angle, and the arc h m is a quadrant. It will be shown in the same way that the arc g n is a quadrant. So the whole arc g f h is greater than a semicircle, and the remaining arc g h less than a semicircle.

The straight line b f, moreover, bisects the illuminated part of the circle. For in the two triangles A i b and A k b the two angles at the points i and K are right angles, and the straight lines A i and A K are equal; so by *Elements* I.47 and the common notion the remaining sides will be equal. By *Elements* I.8, then, the angles A b i and A b K will be equal; therefore the arcs f [?] n and f m are equal, by *Elements* III.26; and so, by the common notion, the arc f g is equal to the arc f h.

That the straight lines A b, c g, and e h, produced in a straight line, meet at a single point can be shown very briefly. The angles at i and k are right angles, so the angles at A are acute; therefore, by the fifth postulate of *Elements* I, the two straight lines A b and e h, produced in a straight line, will meet on the side of b h. Let them meet, then, at the point l. It will be shown in the same way that A b and c g meet on that same side. That this meeting takes place at the point l itself is clear from the fact that the two angles l b h and l b g are equal, since they are equal

to the two angles made at A, by *Elements* I.28 and I.29; while the angles which the straight lines b g and b h make at the points g and h are equal, being right angles. Therefore, when the straight line c g is produced, two equal triangles will be formed on the equal bases b g and b h, by *Elements* I.26. So the meeting must take place at the point l itself; for if it took place anywhere else, a part would be equal to the whole, which is impossible.

Now, then, we shall carry over to solids what we have proved in a single plane about the arcs of circles and about straight lines meeting at one point in the shape of a cone [*in effigie metae*]. Let the straight line e c be joined, cutting the straight line A l at the point x, and consider the two triangles A c x and A c x [recte: A e x], which must be equal, by *Elements* I.4. By the same proposition, then, the two triangles e l x and c l x will be equal and will have equal angles at x; and so the angles at x are right angles, by definition 10 of *Elements* I.

Now conceive the straight line A l produced as far as d, so that it divides the greater circle into semicircles. Conceive further that, with the straight line d l remaining as the common axis, the right-angled triangle e l x, together with the semicircles belonging to e h, is carried round until it returns to the point from which it began to move. The semicircles will generate spheres, and the triangle a cone containing both spheres, whose base is a circle having e c for its diameter.

Corollary. From this it is clear that, as the part of the greater sphere which illuminates the smaller sphere is to the shaded part of the smaller, so is the part that does not illuminate to the part illuminated. For the angles g b h and c A e are equal; therefore the arcs e c and g h are similar, and the remaining arcs are proportional as well.

Otherwise, as Aristarchus of Samos has it in his book *On the Sizes and Distances of the Sun and Moon*: if a sphere takes its light from a sphere greater than itself, more than half of it is bathed in light, and both spheres are contained by the same cone.

Let the sphere whose centre is b be bathed in light by a sphere greater than itself, whose centre is A. I say that the part of the sphere whose centre is b that is bathed in light is greater than a hemisphere, and that both spheres are contained by the same cone.

Join A b, and through A b pass a plane cutting both spheres. Let the common sections be the circles c d e and f g h, which are great circles, by the first book of Theodosius. Then, A b being produced in a straight line, let a straight line b l be found, by *Elements* VI.12, such that A b has to it the same ratio that the difference of the radii of the said circles has to f b, the radius of the smaller circle. Therefore, by the composition of ratios shown in *Elements* V.18, the radius of the greater circle is to the radius of the smaller as A l is to b l.

Next, from the point l draw a straight line l h touching the circle f g h at the point h, by *Elements* III.17. Extend l h in a straight line, join b h, and through the point A draw the straight line A e parallel to b h, by *Elements* I.31. The two triangles A l

e and b l h will therefore be equiangular, by I.29 of the same first book, and so will have their sides proportional, by *Elements* VI.4. Hence, as A l is to b l, so is A e to b h. But f b and b h are equal, being radii of the same circle; therefore, by *Elements* V.7, as A l is to b l, so is A e to f b. But it has been shown that as A l is to b l, so is the radius of the greater circle to f b. Therefore the straight line A e will be the radius of the circle c d e, by *Elements* V.9.

Now the angle l h b is a right angle, by *Elements* III.18; therefore the angle l e A, being equal to it, is a right angle too. So, by the corollary to *Elements* III.16, the straight line e l touches the circle c d e at the point e. From the point e draw the straight line e x perpendicular to A l, and produce A l as far as d. Then if, with d l remaining fixed, the right-angled triangle e x l, together with the semicircles belonging to e h, is revolved back to the point from which they began to move, the semicircles will generate the spheres themselves, whose centres are A and b, and the triangle a cone touching them — since in every position of the revolution the straight line e l touches the semicircles.

Accordingly the part of the sphere bathed in light is greater than a hemisphere. For since the angle at h is a right angle, the angle l b h must be acute. In the same way, if from the point l the straight line l g c is drawn touching both circles, and b g and A c are joined, the angle l b g will be found to be acute. The two angles h b f and g b f are therefore obtuse, and the arc g f h greater than a semicircle. So the part of the smaller sphere contained under that arc is greater than a hemisphere — which was to be proved.

Third thing to be premised. Given the known distance between the centres of the said spheres and the ratio of their radii, to express in numbers the arc of the great circle of the smaller sphere under which its illuminated part is contained.

Let us use the very same figure. Let the ratio of A e to b h be supposed known, and A b, the distance of the centres, known in the same units as the radii.

and let it be proposed to make clear the arc *g f h*, under which the part of the smaller sphere lit by the rays is contained.

Since, then, the ratio of the line *A e* to *b h* is supposed known, the line *A i*, their difference, will be known in the same units as the diameters of the greater and the smaller sphere. But the distance *A b* is also given in those same units, so the ratio of *A b* to *A i* will be known as well; and the angle at *i* is a right angle. Therefore

Diagram. A large woodcut circle spanning both columns of the printed page — it fills the width of the left column and runs on into the right. Inside it, near the centre, sits a small circle; two long straight lines rise from the lower part of the figure, touch the small circle on either side, and meet at a point at the top of the large circle. Two further long straight lines cross the whole figure obliquely from the upper left to the right, ending in the margin of the right-hand column. At the foot of the large circle a shallow lens-shaped arc is drawn, cut by short chords. Several labels on the cut do not resolve on the scan; only the letters listed below were read from the leaf.

- *z* (upper left, on the great circle)
- *Q* (left, on the great circle)
- *R* (upper right, on the great circle)
- *P* (right, on the great circle, at the far end of the chord *P Q*, hard against the right-hand text column)
- *v*[?] (right, on the great circle, at the far end of the chord *z v*, about one line-height above *P*)
- *l* (top centre, at the point where the two rising lines meet)
- *h* (immediately left of the small central circle)
- *s* (immediately right of the small central circle)
- *y* (small letter above and left of the small central circle)
- *o*, *r* (crowded on the upper right of the small central circle, at about two o'clock, where three lines cross; at least one further letter in that cluster does not resolve — [?])
- *m*, [?], *n* (a row of three small letters below the centre of the small circle; the middle glyph is a tall crossbarred letter that does not separate on the scan)
- [?] (a small bold glyph on the vertical axis just above the lens-shaped arc at the foot)
- *c* (at the left end of the lens-shaped arc)
- [?] (at the right end of the lens-shaped arc, where it meets the great circle — a single small rounded glyph mirroring the *c* at the left end; on the scan it does not separate into *c* or *e*)
- *A* (large capital, bottom centre, within the lens-shaped arc)
- *d* (below the foot of the figure, outside the great circle)
- Remaining labels — the letter at the centre of the small circle, and the further crowded letters to its right — [?]

in the right-angled triangle *A i b*, as the line *A b* is to the line *A i*, so the whole sine is to the right sine of the angle *A b i*, by the lemma of the sixth appendix. Of these four proportional terms the first three are known, and so the fourth is known too. We shall multiply the difference of the radii by the whole sine and divide the product by the distance between the centres; the quotient gives the number of units in the right sine subtending the arc *f m*, which is the measure of the angle *A b i*. The arc *f m* itself will then be found from the table of right sines. And since it has been shown that the arc *h m* is a quadrant, the whole arc *f h*, made up of the two together, will be known.

But we demonstrated above that the arcs *f g* and *f h* are equal. The whole arc *g f h*, therefore — the arc according to which the smaller sphere is illuminated by the greater — is determined; which was to be demonstrated.

From this we can gather how large a portion of the earth is lit by the sun, once we assume, from Book V of Ptolemy's *Great Composition* [*Almagest*], that the distance between the centres is 1210 units, of which the earth's radius is one and the sun's radius five and a half.

Now that these foundations have been laid, we shall easily demonstrate at what distance from the earth the highest vapours stand — the vapours that condense and thicken the air. Let us take up again the very figure we used a moment ago. Let the sphere whose centre is *A* be the body of the sun, and the sphere whose centre is *b* the globe of the earth. Next, let a great circle *A P R Q* be conceived, described about *b*, the centre of the world, with radius *A b*, and drawn through the pole of the horizon and through the sun's centre at the beginning of morning twilight. Let the common section of the plane of this conceived circle with the sun be the circle *c d e*, and its common section with the earth the circle *f g h*. From the arc *e c* let the solar rays *e l* and *c l* fall, touching the earth at the points *g* and *h*. Under the arc *g f h*, then, is contained the part of the terrestrial globe lit by the solar rays, while under the remaining arc *g h* lies the part darkened by shadow. Let the point *R* be, further, the pole of the horizon, and let *b R* be joined, cutting the circle *f g h*[?] at the point *t*, where the centre of vision [*centrum visus*] is placed. Then let the straight line *P Q*, passing through the centre of the world, be the common section of the horizon and of the circle *A P R Q* just described; and let the straight line *z v* be the common section of that same circle and of another circle which we call the sensible horizon. Because the terrestrial globe is so small, this second circle is parallel to the conceived horizon — the horizon perceived by reason alone — and stands off from it by an imperceptible difference. And although the distance between these two horizons is exceedingly small in

comparison with the interval that separates the sun from the earth, the difference between their diameters is nevertheless considerable.

For the horizon grasped by reason cuts the whole world in two and reaches to the sphere of the fixed stars, whereas the one taken by the senses has, on Proclus's view, a diameter of two thousand stadia. Macrobius, by contrast, thinks it only three hundred and sixty: "the gaze of one looking straight ahead," he says, "does not exceed a hundred and eighty stadia; when sight has travelled that far it falls short as it goes on, and is bent back, curving along the roundness." Albertus Magnus sets it at a thousand stadia, but calls it the sensible horizon on other grounds. Yet whether the diameter of the sensible horizon is as long as Proclus supposed, or shorter, as Macrobius holds, our demonstration will not vary on that account — for we do see the sun rise and set, and the stars as well. In what sense these authors are to be understood when they fix the limits of vision at the distances just named, whether greater or smaller, belongs to another discipline to settle.

Let us return to the matter in hand. The two lines *P Q* and *z v* are equidistant, by *Elements* XI.16. The angle *R b P* is a right angle, since *R P* is a quadrant; therefore the angle *b t v* is a right angle also — a conclusion that could equally be drawn from Book I of Theodosius. The line *z v*, then, touches the circle at the point *t*, by the corollary of *Elements* III.16.

Now since pure, thin air produces no reflection of light, let us conceive in the mind a sphere of the vapours that rise from land and sea and that thicken and condense the air to the point where it can reflect the sun's light. For whatever lies beyond this sphere toward the heavens, although the sun lights it during the night, is invisible for want of reflection. Let *y r s* be an arc of a great circle of such a sphere described about the centre *b*, and let the straight line *z v* cut it at the point *r*. Therefore, although before morning twilight the sun's light was reflected from every point of the arc *r s*, no ray could reach *t*, the centre of vision — because below the straight line *t v* no straight line can be drawn that does not cut the circle, as is demonstrated in *Elements* III.16. The roundness of the earth was thus an obstacle preventing anything placed below the straight line *t v* from being seen. Again, whatever is contained within the cone of the earth's shadow [*turbinata terræ vmbra*] *g l h* cannot be looked upon. The first point, then, that appears illuminated at the beginning of morning twilight, when day breaks, is *r*: for it lies neither in that thinnest and clearest air which returns no sunlight to us, nor within the earth's shadow, nor below the plane of the sensible horizon.

Accordingly, let *b r* be joined — a straight line cutting the circle of the earth at the point *o*. Then *o r* will be the full height of the vapours raised aloft from the earth, and we shall work out its length in this way. The angle *P b r*[?] is a right angle, and the angle *A b P*, the depression of the sun below the horizon, is known by the preceding proposition. The whole angle *A b t* is therefore known. From it we shall subtract the angle *A b g*, likewise known, which subtends half the arc of the earth lit by the sun, and the angle *g b t* will be left known. Further, the angle which *b g* makes with the line *g l* touching the circle at *g* is a right angle, by *Elements* III.18, and the angle at *t* is right as well. The two triangles *b r g* and *b r t* therefore have equal sides, by *Elements* I.47 together with the common notion; and so those triangles are equiangular, by *Elements* I.8. The angle *t b r* is half the angle *t b g*; but the angle *t b g* has just been found, and so *t b r* will be found too. The remaining angle *t r b* of the triangle *b r t* will then be known.

Now as the right sine of the angle *t r b* is to the whole sine, so is the line *b t* to the line *b r*, by the lemma of the sixth appendix. Of these four quantities the first two are known; and the third — namely the line *b t*, and how many stadia it contains — is found once the number of stadia in the whole circle *f g h* is assumed from Ptolemy or Eratosthenes, and the ratio of that circle to its diameter is assumed from Archimedes. By the common rule of proportional numbers, then, the number of stadia in the line *b r* will be known. From it we shall take away the number of stadia in the radius, and there will be left known the line *o r*, that is, the distance at which the highest vapours stand off from the earth — which is what we set out to investigate.

But so that this same computation may be made more easily, one must observe the following. If the sun did not begin to light the upper hemisphere before it stood below the horizon by an arc equal to the difference between the quadrant and half the illuminated arc, no morning twilight would occur: its topmost ray would merely graze the outermost[?] horizon. But morning twilight does occur. The sun, therefore, begins to light the upper hemisphere before it is hidden by an arc equal to that difference between the quadrant and half the illuminated arc. And so the arc of the sun's concealment below the horizon, at the beginning of morning twilight or at the end of evening twilight, is always greater than the difference between the quadrant and half the illuminated arc. That difference, subtracted from the arc of concealment, will therefore leave an arc equal to the one lying between the point where the sun's ray touches the terrestrial globe and the centre of the sensible horizon,

into which all lines of sight run together — as may be observed in the figure itself. For the two angles *n b g* and *P b t* are right angles; and when the common angle *P b g* is taken away from them, the two angles *n b P* and *g b t* are left equal. Moreover, that same angle *n b P* is what remains when the angle *A b n* — the difference between the quadrant and half the illuminated arc — is subtracted from the angle *A b P*, the sun's concealment at the beginning of morning twilight. For the same judgment holds of angles as of arcs, since arcs are the measure of angles.

Whenever, then, we wish to measure the total height of the vapours, we shall multiply the difference between the radii of the sun and the earth by the whole sine, adding five ciphers; we shall divide the product by the distance between the centres, and there will result the right sine of the difference between the quadrant and half the illuminated arc. We shall subtract the arc of that sine from the arc of the sun's depression, and there will be left the arc between the centre of the sensible horizon and the point at which the sun's ray touches the terrestrial globe. Next we shall take the complement of half this arc; and by the right sine of that complement we shall divide the number produced by multiplying the whole sine into the number of stadia in the earth's radius. This division will indeed yield the distance of the highest vapours from the centre of the earth; and once the measure of the radius has been removed, the greatest height to which the vapours are raised will be left known.

[Margin: Ptolemy's and Marinus's view on the measurement of the earth]

One must further note, concerning the measure of the earth's radius, that on the view of Ptolemy and Marinus five hundred stadia on the earth's surface answer to one degree in the heavens. On this reckoning the whole circuit of the earth along its greatest circle will comprise a hundred and eighty thousand stadia.

[Margin: Pliny. Strabo.]

Pliny and Strabo, however, count seven hundred stadia to each degree, so that the entire circumference amounts to two hundred and fifty-two thousand stadia; and they say that Eratosthenes found it so.

[Margin: Eratosthenes' observation, from Cleomedes.]

Cleomedes, though, records an observation and computation of Eratosthenes from which only two hundred and fifty thousand stadia are drawn. The substance of that observation and demonstration is this. Let it be supposed that Syene and Alexandria lie under the same meridian, and that the interval between

the two cities is five thousand stadia. Suppose further that Syene lies under the summer tropic; and likewise that the sun's rays are parallel at the earth, as many have demonstrated. They do in fact converge, but because of their immense length they appear equidistant — which is why trees, too, cast shadows separated, so far as the senses can tell, by equal intervals. Here Pliny went wrong.

[Margin: Pliny's error.]

For the cause of the shadows being parallel is not the sun's size but its immense distance. Indeed, if the sun were exceedingly small and yet placed at the same remove, then provided its rays could reach the earth it would none the less cast the shadows of trees separated by equal intervals. Let this much suffice as a passing remark; let us now return to Eratosthenes' observation.

With a gnomon standing upright to the plane of the horizon at Alexandria, and the sun at the first point of Cancer, the acute angle made at noon by the sun's ray at the tip of the gnomon is found to be subtended by a fiftieth part of the circle. This angle is reckoned equal to the alternate angle formed at the centre of the earth by two straight lines meeting there. One of them, drawn straight, passes through Syene and reaches all the way to the sun; the other passes through Alexandria and, extended to the heavens, makes one straight line with the gnomon. The arc of the earth's circuit between Syene and Alexandria will accordingly be similar to the arc contained in the sky between the zeniths of those places, since the two subtend the same angle at the earth's centre. The stretch between Syene and Alexandria must therefore be a fiftieth part of the earth's greatest circle, and the whole circuit two hundred and fifty thousand stadia.

There is certainly a great discrepancy between the views of Ptolemy and of Eratosthenes — unless, as I suppose, the measure of the stadium was not the same in both. The Arabs too have their own opinions on this matter, which they maintain. However that may be, we shall now follow the authority of Eratosthenes; and assuming from Archimedes the ratio of a circle's circumference to its diameter, we shall find the number of stadia in the earth's radius to be about 39773.

We must note besides that authors differ about the distance from the centre of the earth to the centre of the sun. Ptolemy set it at 1210 units, of which the earth's radius is one, and the sun's radius…

[Margin: Al-Battānī.]

the sun's radius at five and a half. Al-Battānī maintains that the greatest distance is 1146 such units, the mean 1108, and the least 1070. Whichever of the two we adopt, however, nothing worth noting will change in our determination of the portion of the earth the sun illuminates. For neither the beginning of morning twilight nor the end of evening twilight can be examined by eye so exactly that some seconds of time are not lost — and even so, no perceptible difference will arise in the computation of the height of the uppermost vapours. The distance in question could indeed be found each day from the sun's known argument, but it is better always to use the mean distance.

We shall therefore multiply four and a half — the difference of the radii — by 100000, the whole sine, and the result is 450000. We shall divide this number by 1108, the mean distance, and 406 comes out. In the table of right sines an arc of about 14 first minutes corresponds to this, that is, the difference between a quadrant and half the illuminated arc. We shall then subtract these 14 minutes from the 16° 2′ of the arc of the sun's concealment, and 15° 48′ will remain; half of this is 7° 54′. The complement of that half is 82° 6′, whose right sine is 99050. Now let the 39773 stadia of the earth's radius be multiplied by the whole sine: the product is 3977300000. Let this number be divided by 99050, and the division yields 40154 stadia. From these we subtract 39773, and there is left the highest elevation of the vapours, 381 stadia. But if [?] the vapours were to rise to 400 stadia, the arc of concealment would grow to 16° 24′.

[Margin: A passage of Pliny corrected.]

These matters, then, are not undiscoverable and inextricable, as Pliny supposes in Book II, chapter 23 (*Natural History* II.23) — a passage which in my judgment should be read as follows: "Posidonius holds that the height to which mists, winds and clouds reach is not less than 400 stadia; above that the air is pure and clear, of undisturbed light" — not "xl," as the common copies have it. But this is not the only passage that is read in a corrupt state through ignorance of the mathematical arts.

Propositio. XIX.

Given the height of a mountain, to find the arc of the vertical circle by which those who live on the mountain's summit see the sun before those who live at its foot; and further, to determine the interval of time between their two sunrises.

Let the centre of the earth be A, its radius A B, and the point C the topmost ridge of the mountain, the straight line B C being its perpendicular height above the

lowest foot, which is supposed known. Let the straight line A B C be extended as far as D, in the sphere of the sun, and let a great circle be drawn through D and the centre of the sun at the moment when the rising sun itself is seen by those who inhabit C, the summit of the mountain. Let this circle be D G H. It cuts the sphere of the earth along a great circle of that sphere, B e Q — for this was shown by Theodosius in Book I (*Spherics* I) — and it also cuts the mountain itself along the figure f C e. Let e and f be two places inhabited around the foot of the same mountain, e facing east and f facing west.

What we propose is to find by how great an arc of the vertical circle described the rising sun is seen by the inhabitants of C before the inhabitants of e, and also to determine the interval of time between their two sunrises.

From e let the straight line P e R be raised, extended in both directions to the circumference of the vertical circle and touching the circle of the earth at the point e itself. Further, from C let the straight line C t L be drawn, touching the same circle at t and likewise extended in both directions to K and L on the circumference of the same vertical circle. Let the radii A e M and A t N be drawn, and on these let the diameters S A O and V A Z be erected perpendicular from the centre.

The straight line S A O, then, will be the diameter of the horizon of those living at the place e, and the point M their zenith, the pole of that horizon, by Book I of Theodosius; while the straight line P e R lies in the plane of the sensible horizon, standing in place of the apparent horizon [?], and it is from this line that the risings and settings of the stars are observed. Similarly V A Z is the diameter of the horizon of those inhabiting the place t, while K t L is the line of their apparent horizon, from which the risings and settings of the stars are observed, and N is their zenith. The arcs O R and L Z, by contrast, are of imperceptible size, because the whole earth stands as a point and a centre with respect to the heavens.

And so, when the rising sun has lit the place e from the point O, it will be reckoned at R; just as, when it rises at Z and lights the place t with its rays, it will be reckoned at L. But from the same straight line K t L the rising sun is seen alike by the inhabitants of t and by the inhabitants of C, and it is seen earlier by those who live at t than by those who live at e. It must therefore be seen first by those who live on the topmost summit of the mountain,

Diagram. A large woodcut occupying the upper two thirds of the page. A great circle — the vertical circle in the sphere of the sun — is drawn partly in solid and partly in dashed line, with the centre A at the middle of the figure. Around A is

drawn the small circle of the earth, with the mountain shown as a narrow hatched wedge rising from it just above the centre. Straight lines are ruled from A outward to the circumference, and two chords cross the figure — one tangent to the earth's circle at the point marked e, the other tangent at the point marked t and running up toward the upper left and down toward the lower right. Two further diameters cross at A.

Readable labels, with position where it can be fixed on the leaf:

- A (centre of the figure): the centre of the earth.
- B (immediately above A): the foot of the mountain on the earth's circle.
- C (above B, at the apex of the hatched wedge): the mountain's summit.
- e (immediately to the right of B and C): the inhabited place at the foot facing east.
- t (to the right of centre, below N): the point of tangency of the line drawn from C.
- D and M (side by side at the top of the great circle, D to the left of M).
- G (upper right of the great circle).
- N (right side of the great circle, at the end of the radius through t).
- P (left of the great circle, at mid height) and R (right of the great circle, at the same height): the ends of the tangent through e.
- S (left end of the horizontal diameter through A) and O (right end of the same diameter).
- Q (on the earth's circle, below and left of A).
- H (lower part of the figure — position not securely fixed on the scan).
- Z (lower right of the great circle).
- [?] (upper left of the great circle — a single letter is inked here but does not resolve; by the text either V or K stands at this end).
- [?] (lower right, beyond Z — a single letter is inked here but does not resolve; by the text L stands at this end).
- The point f named in the text is not located with certainty on the cut.

rather than by those who live around its foot. The difference between these two appearances will be the arc O Z itself, which we shall determine in the following way.

The straight line B C, the height of the mountain, is supposed known in terms of the measure of the earth's radius; therefore the whole line A C is known. But in the triangle A t C the angle at t is a right angle, by *Elements* III.18. Therefore, by

the lemma of the sixth appendix, as A C is to A t, so is the whole sine to the sine of the arc of the angle A C t. So by the common rule of proportional numbers, three terms of the proportion being known, the fourth will become known, namely the right sine of the arc of the angle A C t; and accordingly that arc will be had as known from the table of right sines. We shall subtract it from the quadrant, and the arc of the remaining angle C A t will be left known. The arc D N, therefore, will become known; and this exceeds the arc M N by an imperceptible difference, since e lies close to B, or at B itself, as we suppose. For that reason the arc M N is known. But when the common arc N O is taken away from the two quadrants M O and N Z, M N and O Z are left equal; and therefore the arc O Z itself will be known, which is what we set out to find.

Again, from the arc O Z once known we shall easily determine the height of the mountain. For the two arcs O Z and D N are reckoned equal, as we have demonstrated. The subtended [?] angle C A t is therefore known, and the remaining angle is left known by the common notion and *Elements* I.32. So now, in the proportion already mentioned, three terms being known — namely the whole sine, the radius A t, and the right sine of the arc of the angle A C t — the side A C will become known; from it we shall subtract the radius A B, and the height of the mountain B C will be left known.

The words are many, to be sure, but the work is finished sooner than the telling. Let the measure of the earth's radius be multiplied by the whole sine. If the product is divided by the distance of the mountain's summit from the centre of the earth — a distance made up of the diameter [?] and the height — the right sine of the complement of the arc below the horizon of those who live around the foot will result; but if it is divided by the right sine itself of the complement of the arc of concealment, the distance of the summit from the centre of the earth will result.

For example: suppose, following the view of certain of the Arabs, that the whole circuit of the earth contains 24000 Italian miles, and let the perpendicular height of some mountain above its lowest foot be eight miles. It is required to find the arc of the vertical circle by which the sun is seen earlier by those who inhabit the mountain's summit than by those who live at that mountain's foot toward the east. Let the 3818 miles which the earth's radius contains be multiplied by the whole sine, and the result is 381800000. Let this number be divided by the 3826 miles reaching from the centre of the earth to the mountain's summit, and the division will yield 99791, the right sine of 86° 16′ 20″ or thereabouts. The complement, therefore, will be 3° 43′ 40″ — that is, nearly a quarter of the 15

degrees. Through these degrees and minutes the sun lies hidden below the horizon of those living at the mountain's foot at the moment when it is already seen by those who inhabit the summit. Moreover, to that arc there correspond on the earth's circumference about 250 Italian miles, across which, with no obstacle whatever, the sight of those who live on the mountain's summit will reach. But since fifteen degrees are not

the measure of the hourly interval on any circle other than the equator and the circles parallel to it, those 3° 43′ 40″ are not on that account a quarter of one hour. Alhazen was therefore quite right, when in a case of this kind he fixed a quarter of an hour between the two sunrises, to warn the reader that this was to be assumed for the sake of those with little practice in geometrical demonstrations.

For the rest, once the latitude of the place set at the mountain's foot is known, together with the sun's declination for the day, it will be evident at once, by the eighth, ninth, or tenth proposition, in how great a space of time the sun rises through the arc of concealment already found.

We shall also demonstrate the converse of this proposition by the same method. For from the known latitude of the place set at the mountain's foot, the sun's declination for the day, and the interval of time before sunrise, the arc of concealment will be known — namely O Z, equal to the arc D N, since we suppose the difference between D and M imperceptible. The angle C A t is therefore known; and when this has been taken from the right angle, the angle A C t is left known. Accordingly, in the proportion already mentioned, the side A C will be found; we shall then subtract from it the radius A B, and the perpendicular height of the summit will be left known.

For example: let the lowest foot of some far-northern mountain have a latitude of 80 degrees, reckoned from the equinoctial circle as is customary; let its topmost peak be lit by the sun's rays until a third of the night, morning and evening, when the sun itself occupies the beginning of Aries or Libra; and let it be required to determine from these data the perpendicular height of the mountain.

Now since, by the ninth proposition, as the whole sine is to the right sine of the altitude of the equator, so the right sine of the arc of time before sunrise is to the right sine of the arc of the sun's concealment corresponding to that same time, we shall accordingly multiply 17364, the right sine of the 10 degrees which the altitude of the equator contains, by 86602, the right sine of 60 degrees, which are contained in four equal hours, a third part of the night; and the product will be 1503757128. We shall divide this number by the whole sine, and 15037 and more

than half of one unit results; to which corresponds an arc of about 8° 39′. Such, then, will be the arc O Z, or D N; and the arc B t will have just as many degrees and minutes. At that distance the mountain's summit will be visible from the place t, if the eye's power of sight is strong enough; for nothing will stand in the way of t being seen from the mountain's topmost peak, and of the same summit being seen from t.

Accordingly we shall subtract from 90 degrees the 8° 39′ — that is, the magnitude of C A t — and 81° 21′ will be left for the magnitude of the angle A C t. We shall then multiply 39773, the earth's radius on Eratosthenes' view, by the whole sine, and divide the product by 98862, the right sine of the angle A C t; and 40230 stadia will result, the distance of the summit from the centre of the earth. From these we shall subtract the radius, and 457 stadia will remain, which the height of the said mountain from its lowest foot to the perpendicular must have, according to the hypotheses laid down. For if the sun were placed outside the equator, something else would necessarily follow, since the same mountain could not be lit on each successive night for one and the same fixed measure of time.

From all this I take it to be established that what Aristotle writes about Mount Caucasus in the first book of the *Meteorology* is a fable. For it stands at a far shorter distance from the equator — being the range that overlooks both seas, the Pontic and the Caspian, and walls the isthmus that divides them, as can readily be gathered from Ptolemy and Strabo. The same may also be conjectured from Aristotle alone: for he says it slopes toward the summer sunrise, and that it is seen by those who sail the mouth of Lake Maeotis, and from the place they call Bathea, that is, the deeps of the sea. But it could not happen that the highest parts of the Caucasus should be lit by the sun's rays for as much as a third of the night unless the range rose to an immense and wholly incredible height — as will be immediately clear to anyone who makes trial of the numbers.

Less credible still is what Pomponius Mela ascribes to the mountain country of Arabia: "where it rises high," he says, "it is so lofty that from its topmost peak it shows the sunrise from the fourth watch" — the latitude of the place, as can be established by computation, not agreeing. Pliny reports the same of Mount Casius, but this too is false and self-contradictory, since he puts its height, measured straight up, at only four thousand paces. Cleomedes affirms that the loftiest mountain rises to a perpendicular height of only fifteen stadia; Alhazen, eight thousand paces. Pliny does not believe Dicaearchus when he says that he found the highest mountain to be 1250 paces by the method of the perpendicular, since he knew that certain peaks of the Alps rise, over a long stretch, by not less

than fifty thousand paces. But these will be lit by the sun for nearly a third of the night if they lie at so great a latitude from the equator…

toward the arctic pole as we assumed in our example: for 457 stadia make up 57 thousand paces.

Here we have brought the little book on twilights to a close. Our remaining short works we shall publish shortly, as we hope: a demonstrative work on the astrolabe; *On Spherical Triangles*; *On the Geometrical Planisphere*; *On Proportion*, on the fifth book of Euclid; *On Delineating the Globe for the Art of Navigation*; and several others that we are at work on today.

¶ Authors cited by us in this little book.

Euclid	Aristotle
Theodosius	Strabo
Menelaus	Pomponius Mela
Archimedes	Pliny
Aristarchus of Samos	Macrobius
Ptolemy	Proclus
al-Battānī	Cleomedes
Geber [Jābir ibn Aflaḥ]	Albertus Magnus
Alhazen	Ioannes de Sacrobusto [Johannes de Sacrobosco]
Witelo	Ioannes Stoflerus [Johannes Stöffler]

FINIS.

ALHAZEN, MOST ANCIENT OF THE ARABS

The Book on Twilights, Gerard of Cremona being the translator

I wish to show in this treatise what twilight is, and what cause it is that necessarily produces its appearance. From there I shall go on to determine the utmost height to which the fine vapours that rise from the surface of the earth are raised above it.

I say, then, that the morning twilight and the evening twilight are of like figure. For the one of them comes about from the approach of the sun's light, the other from its withdrawal. But the colours of the two are different, on account of the difference of the horizons in which the sun is appearing. For when the sun is in the eastern horizon and not much raised, its colour there is other than its colour to the sight when it is at an equal height in the western horizon. And the same holds for its rays that are seen in the twilight, and for what is seen in the ether of its lights. And the ether is itself coloured following upon that, according as the sun is in the one or the other of its parts. For that part of it which is in the east — the colour is whiteness and brightness; and that which is in the west inclines somewhat toward redness. But what thing that illuminating body is, and how it is apparent there, and what cause necessarily produces it — before that we shall set down propositions setting out the thing whose explanation we intend.

One of them is this: that the whole sphere of the heaven is always bright and luminous from the greater luminary, which is the sun, except in so far as the darkness arising from the earth covers it, in the shape of a pyramid, which is night. And in this book I do not mean, by what happens in a reception of light of this kind from the spheres of the stars, anything other than this: that with the sphere, because of the clearness of the air and the fineness of the ether and its thinness, nothing of the sun's light is suspended, as we see it suspended with the high bodies which are the stars — for these are illuminated and carry down to us what they receive of the light, and our sights follow that light in them; and although they differ in the stars, in the moon they do not differ. But sights do not follow what is in them of the lights except in that these bodies are without doubt of a thicker and more forceful corporeity than the ether in which they are. And this is plain from the indications: that some of them cover others from us, because they eclipse them; whereas the air does not cover from us anything of what lies behind it. And for this reason we see that the whole night is according to one and the same condition, in which nothing of the ether is illuminated for us — although we know by our knowledge that the greatest part of it is luminous and not screened from the sun. And we see that the part of it which is exposed to the

sun, and which nothing else covers, is in vision just like the part which the earth covers, the part which the pyramid of darkness contains. And it does not necessarily produce an equality of the two to our sights, except by what we have said of the fineness of the air, and that it does not convey the illumination of the air, and does convey to us its darkness. Then the condition of the shadow does not cease to appear to us after its likeness, until there begins from the east the

brightness of dawn and the scattered light, whose beginning is at first at the surface of the horizon.

And of that beginning there is for us no cause but the sun, since the sun is the cause of illuminations. Yet that beginning is not for us the sun itself, nor its ray alone, since we have already set down that its rays pass through as far as the whole ether — or as we see, up to the greatest part of it — and its condition in that hour is not different from its other condition before it. Nevertheless its rays are then suspended with some body thicker than air; it therefore conveys to us, along with its own thickness, the ray it has put on. And I say that that in which the ray is suspended at that hour is not the earth, nor are the extremities of its regions distinguished by us, since when the seer is on the level of the earth his sight does not reach further than about 23 miles on every side. And since it may happen to him that he is on the highest of mountains that can be, and that mountain does not exceed eight miles — as the sages have said, meaning this — then sight does not pass beyond about 250 miles. And this is manifest from the fact that [?] the shape of the earth does not produce it, but rather the height of the place of sight above the surface of the earth — that is, the interval we have spoken of — cuts off of the orb the amount of a quarter of an hour.

It must therefore be that the sun rises a little after the morning twilight, by a quarter of an hour at the least; but that which lies between the appearance of the twilight and the appearance of the sun is much more than an hour. What we have said, however, is only an approximation, for the sake of one who is not practised in geometry. In truth sight does not reach the point of the earth that is already illuminated by the sun, unless it also reaches and takes in the horn of the sun itself, since two lines touching a point of a circle from two different sides are joined into one straight line. When, therefore, the illuminated part appears to us, then that is not the earth itself, for the reason we have given; nor is it the air filling the whole sphere, since, as we set down before, upon the whole of the air, or the greatest part of it, the sun's ray is always falling by night and by day, and it does not appear in the air on account of the air's fineness.

And above the earth there is no body thicker than air except the ascending vapours, which never fail to be illuminated by the sun. But when the pyramid of shadow is withdrawn from that part of the sphere of the vapours enclosing the earth which our sights follow, and the body of the sun receives them and its rays fall upon them, the ray is suspended with them; and it carries the ray down to us, and our sights follow it, and its light is seen by us — just as we see it appear in the

clouds from the colouring of the ascending moistures, and as we see the colours that are seen in dews in the form of a portion of a circle and in other manners.

When, therefore, we wish to know how great is the utmost elevation of those vapours from the surface of the earth, four things are set down beforehand for that determination, of which none is [?]; and besides these we need no other thing, so that it cannot [?] be done with fewer, nor is more required. Now those four are: the body of the earth; the body of the sun; the distance of the centre of the sun from the centre of the earth in every position; and how great is the depression of the sun below the horizon until the morning twilight appears.

The body of the earth, then, is as the instrument of all the rest; and the size of the great circle enclosing it, according to what the sages have said and have shown by sure propositions, is twenty-four thousand miles. And they said that, on the scale by which the radius of the earth is one part, the radius of the [?] is five parts and a half; and on that same scale the distance of the centre of the sun from the centre of the earth at its mean distance — not in every position — is one thousand one hundred and about ten parts; and that the depression of the sun below the horizon when the twilight arises is 18 degrees, though it is also found to be above 19. And upon this I shall build our computation; for when the reporter of a thing reports it with an addition, his account is the worthier to be accepted, so long as no one else contradicts him. Since indeed the reporter who reports with an addition knows what the other does not know, and attains what the other does not attain. For one who reports of something that he saw it before another saw it is the worthier to attain what he intends, when no suspicion is entertained of him.

I shall therefore set down, before the matter I have in hand, certain propositions of much help.

I say, then, that of any two equal spheres, between which there is no other body to hide the one from the other, the part of each of them that faces the other is exactly half of it. And I mean

by "the face of one turned toward the other," that if one of them is luminous and the other receives light, half of the one receiving light is illuminated and shines.

The example of this is: let there be two equal spheres, A and B; and I set that some plane surface pass through the centre of each. It will therefore cut the two spheres along two equal circles, and in one surface. Let those two circles then be A G H and B D C. And I shall join A with B, and draw the two lines A G, B D perpendicular to the line A B; they are therefore parallel. And I shall join G with

D. And since the two lines A G, B D are equal and parallel, the two lines A B, G D will likewise be equal and parallel; therefore the two angles G and D are right; therefore the line G D is tangent to the two circles.

And when we draw G A and B D in a straight line to the two circumferences of the two circles, as far as the two points E and Z, and then join E with Z, the straight line E Z will be tangent to the two circles. And each of the two portions G H E, D C Z, of which the one faces the other, will be half of a circle, since the diameter of the circle cuts each of them. And the like happens in all plane surfaces that pass through the two centres of the two spheres. It has now therefore been made clear that lines going out from one of two spheres to the other touch them both together, and take in of each of them the half. And that is what we wished to make clear.

Diagram. A woodcut occupying the lower middle of the left column and the page's centre: two circles of the same size, one directly above the other, joined by two straight vertical lines that run down the outside of both and touch each — the whole outline resembling a cylinder capped at each end by a circle. A horizontal line is drawn across each circle where the vertical lines meet it, and a vertical line runs down the middle from the upper circle through the lower one.

- Upper circle, left end of the horizontal line: D
- Upper circle, above the horizontal line, at the top: B
- Upper circle, right end of the horizontal line: Z
- Upper circle, on the central vertical line below B: C
- Lower circle, on the central vertical line, above the centre: H
- Lower circle, at the centre: A
- Lower circle, left end of the horizontal line: G
- Lower circle, right end of the horizontal line: E

But if one of the two spheres is larger than the other, then the part of the smaller that faces the larger is more than half of the smaller; and the part of the larger that faces the smaller is less than half of the larger.

The example of this is: let there be two spheres A and B, and let sphere A be the larger. I shall then draw a plane surface passing through the centre of each; it will therefore cut each of them into two halves along the two circles A G D, B E Z. And I shall join A with B, and draw that line straight on toward the side of H; and I shall make the ratio of the radius of circle A G D to the radius of circle B E Z as the ratio of A H to B H — the taking of this is near at hand from the sixth and the

fifth treatise of Euclid (*Elements* VI and V). And from the point H I shall draw a line tangent to the circle A G D, and let it be the line H C D.

I say, then, that this same line is tangent also to the circle B E Z. Which is plain thus: I shall join A with D by the line A D; it is therefore perpendicular to the line H D. And I shall draw from the point B a perpendicular to the line H C D, and let it be the line B C. And since the two lines B C, A D are perpendicular to the lines H D, they are parallel; and since the line B C is parallel to A D, which is the base of the triangle, the ratio of A D to B C will therefore be as the ratio of A H to H B. And we have already made the ratio of A H to H B as the ratio of the radius of circle A G D to the radius of B E Z; therefore the line B C is the radius of circle B E Z; therefore the point C is on the circumference of circle B E Z. And we made the two angles D, C right; therefore the line H C D is tangent to the smaller circle. But we have already brought it near as tangent to the larger; therefore it is tangent to both

Diagram. A woodcut set in the right column: a large circle at the top and a much smaller circle below it, with two long straight lines running down from the outside of the large circle, past the small circle, and meeting at a point far below — a long narrow tapering figure. A straight line descends from the centre of the large circle through the small circle to that same lower point.

- Large circle, at the centre: A
- Large circle, on the circumference at the upper right: K
- Left of the large circle, at the left-hand line: D
- On the central line, below the large circle: G
- On the central line, lower down: E
- Small circle, at the centre: B
- Small circle, at the left: C
- Small circle, at the right: Z
- Below the small circle, on the central line: I [?]
- At the meeting point of the two long lines, at the foot of the figure: H

both at once. And I shall draw likewise from the point *H* a line touching the two circles likewise on the side *Z*, which let be the line *H Z K*: therefore what, from the circle *A*, the greater, looks with turned face [*versa facie*] toward the circle *B*, the smaller, is the portion *D G K*; and it is less than half the circle, since the angle *H A D* is less than a right angle, since it is itself in one triangle: and the triangle is *D A H*, with the angle *A D H* right: therefore the portion *D G* is less than a quarter of the circle, and likewise the portion *G K* is equal to it: therefore the portion *D G K*

is less than half the circle. And since the line *B C* is parallel to the line *A D*, the angle *C B H* is equal to the angle *D A H*: therefore the portion *C L* will be similar to the portion *D G*; and the whole portion *C L Z* similar to the portion *D G K*: therefore each one of them is less than half the circle: there remains therefore the portion *C E Z*, greater than half the circle: and that is what, from the smaller circle, looks with turned face toward the greater circle: therefore the two portions *C E Z* and *D G K* are from two circles which look at each other with turned face. And I signify indeed by this, that something of the one portion is not covered over by the other circle, and the portion *C E Z* is greater than half the circle, and the portion *D G K* less: and that is what we wish to declare.

Diagram. A large circle above and a small circle below, joined by two long straight lines running down the page from the large circle to the small one, with a further straight line running down the middle between them. Readable labels, with the position at which each is inked:

- *E* — at the top of the large circle
- *A* — at the centre of the large circle
- *N* — inside the large circle, to the right of *A*
- *B* — at the right-hand edge of the large circle
- *D* — at the left of the large circle
- *G* — below *A*, on the lower part of the large circle
- *M* — to the right of *G*
- *K* — at the left of the small circle
- *Z* — at the centre of the small circle
- *L* — at the foot of the small circle
- *H* — at the right of the small circle
- *H* [?] — a second letter of this shape sits inside, at the upper right of the small circle; the glyph does not resolve and may be *T*
- *Q* [?] — a mark at the left of the descending line, midway down; the glyph does not resolve and may be *D*

] I say that when there are two equal circles, and two lines are drawn each one of which touches both circles at once according to the form we set out before, then in each one of the two portions of which the one looks with turned face toward the other, there is no place which veils anything of the one circle from the other circle; and that in the remaining two portions of the two circles, which are not looking at one another face to face, there is no place which appears to the other circle. The example of which is: let there be two circles *A B G D E* and *Z H T K L*;

and let two lines *B H* and *D K* be drawn, touching both circles at once: therefore the two portions *B G D* and *H T K* are those which look at one another face to face; their portions *B E D* and *H L K* are those which are not looking at one another face to face. I say therefore that there is not in the portion *B G D* a point which veils anything of the circle *Z H* from the circle *A B*; and that there is not in the portion *B E D* a point which appears at all to the circle *Z H*, and that

the whole portion itself is veiled from the circle *Z H*: and neither is there in the portion *H L K* a point which appears to the circle *A B*. The demonstration of which is that I shall join *A* with *Z*, by the line *A G Z*; and I shall mark upon the arc *B G D* a point however I please, which let be the point *M*: if therefore the point *M* be from the point *G* toward the side *B*, then I shall draw from the point *M* a line parallel to the line *B H*; and if the point *M* be from the point *G* toward the side *D*, then I shall draw from the point *M* a line parallel to the line *D K*: let it therefore be *M T*. I say therefore that the line *M T* is wholly outside the circle *B M G D E*, and nothing of it falls within it. The demonstration of which is that I shall join *A* with *B*, and shall draw the line *M T* out in a straight line until it meets the line *B A* at the point *N*: therefore of the two angles at *N* each one is right. And I shall join *M* with *A*: the angle *N*, therefore, of the triangle *A N M*, is right; and the side *N M* has already been drawn out straight as far as *T*, and the angle *A M T* comes out outside the triangle, and it is greater than a right angle, namely than the angle *N*. And when a line is drawn from the extremity of the diameter of a circle such that it contains with the diameter more than a right angle, then that line does not cut the circle, nor does anything of it fall within it: therefore of the line *M T* nothing falls within the circle *A M*: therefore the point *M* is looking face to face toward the circle *Z*, and does not veil anything from it: since when nothing veils it from it out of the body of the sphere *A M* itself, then no other thing covers it, since we posited that between the two spheres there should be no body other than themselves which might cover the one of them from the other. And likewise this will be shown at every point upon the arc *H T K*. And I say again that there is not in the arc *B E D* a point which appears to the circle *Z*; nor is it possible that it be joined with anything of the circle

Z, by a line, unless that line cut the circle *A B* and fall within it. But if it is possible, then we shall draw from the point *E* a line arriving at something of the circumference of the circle *H T K L*, and not cutting anything of the circle *A E D*: and if it be possible, let it be the line *E Q L*; and I shall draw the line *D K* out on both sides of its two extremities: it is necessary therefore that it meet the line *E Q L* in two places: since the line *D K*, which we have already posited as touching the two circles, it is not possible that it cut one of the two circles, nor that it fall

between the two of them; and since it does not fall between them, then it will cut the line *E L* in two places: therefore now there are two straight lines containing a surface: but that is contrary and impossible.

What we must do, according to that which we set out before, so that we may find how great is the quantity of the arc of the earth illuminated by the sun — which we have already posited to be greater than half the earth: I shall therefore set down two circles, of the sun and of the earth, upon which one plane surface cuts both, which are *A B G D E*, *Z H T*. Let therefore the circle *A* be the earth's, and the circle *Z* the sun's: and I shall draw two lines touching each one of them, as we said, which let be the two lines *B H* and *E T*: therefore the portion *B G D E* of the earth is the one illuminated by the sun, as we have already shown: and that is more than half the circle. When therefore we wish to know the quantity of it, then we shall join *A* with *B*, and with *Z*, and *Z* with *H*: therefore *B A* and *H Z* are parallel, since both are perpendicular upon the line *B H* touching the two circles. And I shall cut off from the line *H Z* what is equal to the line *B A*, and let it be the line *H K*; and I shall join *A* with *K*: therefore *A K* is perpen-

Diagram. A circle above and a larger circle below, joined by two long straight lines descending the length of the page from the upper circle to the lower one. Readable labels, with the position at which each is inked:

- *A* — at the top of the upper circle
- *B* — at the right of the upper circle
- *E* — well out to the left, outside the upper circle
- *D* — below *A*, on the lower part of the upper circle
- *C* [?] — to the right of *D*, on the lower part of the upper circle; the glyph does not resolve and may be *G*
- *H* — at the left of the lower circle
- *Z* — inside the lower circle, left of its centre
- *K* — inside the lower circle, to the right of *Z*
- *h* [?] — a small letter at the right-hand edge of the lower circle; the glyph does not resolve

dicular upon *H Z*, since it is parallel to *B H* itself, seeing that it contains the whole of what is between the extremities of the two lines *B A* and *H K*, which are equal and parallel: therefore the angle *K* is right. And for this reason, that the line *H Z* is five parts and half a part in the measure by which the line *B A* is one part, there remains the line *K Z* of four parts and a half of one part in that same measure: and by the same measure the line *A Z* is found, 1110, at the mean distances:

therefore in the measure by which the line *A Z*, subtended by a right angle, is 60 degrees, the line *K Z* is 14 minutes and three fifths of one minute: therefore the angle *K A Z* is 14 minutes, less the third part of the fifth part of one minute, in the measure by which a right angle is 90 degrees: and that is the quantity of the arc *G D*. But the arc *B G* is 90 degrees, since the angle *B A G* is right: therefore the arc *B D* is 90 degrees and 14 minutes, less the third part of the fifth part of one minute; and the arc *D E* is equal to the arc *B D*: therefore the whole arc *B G D E* is illuminated to 180 parts, and 27 minutes and four fifths and a third of a fifth of one minute, nearly: and that is what we wish to declare.

Let us begin therefore now from that which we intend concerning the cause of the appearance of the twilight, and the form of its appearance to us, and of the shape of it in the eastern horizon. I shall therefore set down the circle marked upon the sphere of the earth — that along which the plane surface passing through the zenith of our heads and through *Z*, the centre of the earth, and of the sun, cuts off the earth — as the circle *A B*, and the place of sight as *A*: and I shall make a line touching the circle pass through the point *A*; and I shall prolong its two extremities on both sides, upon which let there be *G D*. It is manifest therefore that upon the whole of what falls below the line *G A D*, toward the side *B*, sight is not falling, since the earth veils it from us: because the extension of sight is only along a straight line. And Euclid indeed has already declared that no line goes out from the point of contact between the tangent line and the circle: therefore sight does not fall below the line *G A D*, but falls upon what is raised above it. And I shall set down the form of the pyramid of darkness arising from the shadow of the earth, a little before the twilight, when the depression of the sun is more than 19 degrees by one minute — for instance, or thereabouts — upon which let there be

B E L R [?]: for the whole of what falls within this pyramid marked out, whose head is *L* and whose base is the earth, is covered from the sun, not appearing to it, nor illuminated by it, and is in truth dark; and what falls outside it is appearing to the sun, and upon it its rays and its light are falling. Nevertheless, whatever among bodies is very subtle does not carry through to our sight that which it puts on from the ray, for this reason, that in our sight what of the air is subtle within the pyramid, and what is outside it, are made equal; and the whole aether is seen in the form of light and of darkness. And we indeed know that what of the air holds us in, and what is near to us, is dark, not appearing to the sun; and that what proceeds, in going on high, or to the right, or to the left, and before and behind, is luminous, appearing to the sun: and both of them together with that are with us equally in the whole apprehension of sight. And nothing appears to

our sight before the rising of the sun and after the setting of the sun, unless it be raised above the surface of the horizon, and unless it be outside the pyramid of the shadow, and unless it be denser than the subtle air. It is manifest therefore that nothing appears to our sight in the condition of splendour and illumination except by the gathering of three conditions in it. One of which is that it be not below the line *G A D*: since if it is below it, the sphere of the earth blocks the way between it and sight, because sight does not apprehend it as luminous or as dark. And another is that it be not in the pyramid of the shadow: for if it is in it, it is dark, for this reason, that it is deprived of the face of the sun, and of its illumination from it. And another is that it be denser than the subtle air filling the sphere: since we have already known that the higher air outside the pyramid is falling above the line *G A D*; and even so there does not appear to us in it anything of light, on account of its thinness and its subtlety; and because we see

Diagram. A circle low on the page with a straight line drawn across the top of it, and two long straight lines descending from the upper part of the page and crossing one another above the circle. Readable labels, with the position at which each is inked:

- *H* — high on the page, at the left, at the head of one descending line
- *L* — high on the page, at the right, at the head of the other descending line
- *D* — at the left end of the straight line across the top of the circle
- *A* — at the top of the circle, on that same line
- *E* — on that same line, to the right of *A*
- *G* — at the right, just below *E*
- *B* — at the right, outside the circle
- *K* — at the lower left, outside the circle
- *Z* — at the centre of the circle
- *R* [?] — inside the circle, to the left of *Z*; the glyph does not resolve
- *I* [?] — inside the circle, to the right of *Z*; the glyph does not resolve
- one further letter between *A* and *E* on the top line was not resolved [?]

in this place — and it is a little before the twilight — that what we apprehend of the sphere is covered over, not illuminated; and one part of it is not distinguished from another part. And we know that there is not in it a point nor a single place in which those three conditions are gathered. But the point *E* is where the line *G A D* meets the last position of the pyramid, and we have already posited two conditions in it: since it is not below the line *G A D*, nor is it entering the pyramid: therefore the ray of the sun is falling upon it. Therefore nothing makes its

darkness necessary in our eyes at that time, except its privation of the third condition, which is density. Now therefore it is made certain that the air where the point *E* is, in this place, is subtle, and there do not arrive at it the dense vapours ascending from the earth, which are denser than the air. Then, after the sun is raised a little, and its depression from the horizon becomes 19 degrees only, and the form of the pyramid and the shape of it become like that upon which are *I T H K*, and a luminous thing appears in the horizon, and there was no luminous thing there before, we know that that is the first of the places and lodgings in which the three conditions stated above are gathered: since a little before that, by an amount to which there is no quantity, there was not there anything of light; and the first of the places in which it is gathered that it be not below the line *G A D*, nor be entering the pyramid of the darkness, is the point *T*. Therefore the point *T* is the first of the places in which the third condition is found, and there is the density of the air there: therefore the point *T* is the last position of the vapours, and the highest ascent of them; and they are not cut short of it, nor do they pass beyond it. For if they were cut short of it, the point *T* would be in subtle air, and there would not appear to us in it anything of light, just as it does not appear in that which is after it toward the side *E*; and if they were to pass beyond it, the point *E* would be illuminated for us before this: since we do not posit, in what is between *T* and *E*, in these two places, a perceptible thing. Therefore the point *T* is the last position to which the vapours ascending on high arrive, and it is the meeting of the line *G A D*, touching the sphere of the earth, with the line *H I*. When therefore we wish to know its length from the face of the earth, then we shall describe the circle of altitude passing through the centre of the sun, when its depression from the Horizon is 19 degrees — and that is at the rising of the twilight — upon which let there be *A B G D*: it will therefore cut the sphere of the earth upon the circle *E Z N*; and let the line *A E H* be the one passing through the zenith of our heads and through the centre of the earth, perpendicular to the line

B H D: therefore the line B H D cuts the earth into two halves, the apparent and the hidden. The apparent, therefore, is that which is above it toward the part A, and the hidden is that which is toward the part G — and we do not say this except by way of enlargement and approximation. But the truth is that the apparent is nothing except that which is above the line V E Q K, drawn out touching the sphere above the point of sight. Nevertheless, in this globe of the earth it is not a great quantity.

And I shall set the arc B G at 19 degrees, which is the depression of the sun at the rising of the twilight: upon the point G, therefore, is the centre of the sun. I shall

therefore make there, upon that same point, a circle with a length of five and a half times that which is equal to the line E H, and let it be the circle T I. And upon that same point, namely G, the orb A B G D cuts the sun, and I shall continue the line H G. Then I shall draw out two lines touching the two circles of the sun and of the earth, containing the illumination of the earth by the sun, which are the two lines T L M and I N M, touching the earth upon the two points L and N; and they are the boundaries of the pyramid of the shadow. Therefore the line T L M meets the line E K upon the point Q. Therefore the point Q, according to what we showed in the figure which is before this one, is the luminous place at the rising of the twilight, and it is the last state of the ascension of the vapours.

When therefore we wish to know its length from the surface of the earth, then we shall continue H with Q by the line H Z Q, and I shall continue H with L. Therefore the portion L F N is illuminated, because it looks upon the sun face to face. Now therefore we have shown that it is 180 degrees and 27 minutes and 52 seconds, and the arc F L is the half of it, and is 90 degrees and 13 minutes and 56 seconds: and that is the quantity of the angle L H F. And the angle B H F was already 19 degrees, since it is the depression of the sun; therefore the angle L H B remains 71 degrees, 13 minutes, and 56 seconds. But the angle E H B is 90, because it stands right; therefore the angle E H L remains 18 deg. 46 min. 4 sec. And since the line Q H divides it into two halves — and that is manifest — the angle Q H E is therefore of 9 degrees, 23 min. 2 sec. Therefore the angle H Q B [?] is the complement of the right angle, and that is 80 degrees, 36 min. 58 sec. Its chord, therefore, which is the line E H, is of 59 degrees 11 min. 48 seconds, by the quantity in which the line Q H is 60 degrees; nevertheless, by the quantity in which the line H E is 60 deg., Q Z H will be 60 deg. and 48 min. and five sixths of a minute. But the line H Z, of those, is 60 deg.; therefore Z Q remains 48 min. and 50 sec. And that is [?], in the miles by which the circumference of the earth contains 24000 miles, 51 and 47 min. and 34 sec. and 6 parts out of 11 parts of a second. And that is the utmost to which the vapours ascending from the earth are raised and arrive; and that is what we wish.

This is the end of that which he intends in this letter; for certain things follow in the Arabic which I have passed over, because there is no usefulness in them: for nothing is contained in them except certain matters in which he praises God after the manner of the Saracens, and reproves certain men who were asking what fruit there might be in what he himself has said in this letter. For he says that those men are to be refuted who do not grasp insensible things through sensible ones. And since in the things they say there is no usefulness, I have therefore passed them over.

Diagram. A full-page woodcut occupying the recto, with no caption and no accompanying text: the construction for the height of the vapours from the 19° depression of the sun. A large outer circle (the circle of altitude) carries **A** at the top and **B** at the right; a straight line runs across it from the far left, through the centre **H**, to **B**. A second, obliquely drawn line runs from **V** at the left, through **E** and **Q**, and out to **K** at the right, beyond the circle. A small circle centred on **H** (the sphere of the earth) carries **E** at the top, **N** at its left, and **F** at its foot. From **M**, high on the upper left of the outer circle, two straight lines run down across the figure to the lower right, meeting the outer circle at **T** and passing to **I** at the bottom. A third circle, smaller than the outer one and larger than the earth, is drawn at the lower right, passing through **T**, **G** (at its right) and **I** (at its foot), and cutting the outer circle. A short line marked **L** and a further letter read as **Z** [?] stand in the crowded cluster just above and right of the centre **H**.

- A (top of the outer circle): A
- B (right, on the outer circle, at the end of the diameter through H): B
- D [?] (far left, at the other end of that diameter): D [?]
- G (lower right, on the circle drawn through T and I): G
- H (centre of the figure, centre of the small circle): H
- E (top of the small circle): E
- N (left of the small circle): N
- F (foot of the small circle): F
- V (left, on the oblique line running to K): V
- Q (on the oblique line, just above and right of H): Q
- K (right, at the far end of the oblique line, outside the circle): K
- L [?] (just below and right of Q): L [?]
- Z [?] (in the crowded cluster between Q and H): Z [?]
- M (upper left, on the outer circle): M
- T (right, where the lower circle meets the outer circle): T
- I (foot of the lower circle): I

[The verso of this leaf is blank.]