Fundamentum astronomicum: id est, nova doctrina sinuum et triangulorum

[Foundation of Astronomy: That Is, a New Doctrine of Sines and Triangles]

(1588)

by Nicolaus Reimers Ursus (1551–1600)
Translation by Scott Weisman



Ars Astronomica
2026

"Fundamentum astronomicum: id est, nova doctrina sinuum et triangulorum," published in 1588, is in the public domain. All translations, original additions, including illustrations, commentaries, and summaries, are copyright.

The source text was obtained from Det Kongelige Bibliotek, Copenhagen, digitised for Early European Books ProQuest; distributed through the Internet Archive as `den-kbd-all-130017588436-001` and is available at https://archive.org/details/den-kbd-all-130017588436-001.

Translation © 2026 Scott Weisman. All rights reserved.

Licensed under CC BY-NC-ND 4.0. Share with attribution; non-commercial; no derivatives.



Published by Ars Astronomica

Translator Version 2

Upright reason dictates that the recipients of the good, of whatever type and level of beneficence it may be, must show gratitude and blessing to the beneficent person in every way possible, commensurate with the value of the beneficence. And one who has benefited all the people of the world — for example, one who invented a new instrument for the good of the world, or a good book — it is fitting for every discerning person, out of the obligation of love of fellow beings, to at least purchase it, so that the man will profit and his heart will be encouraged thereby to invent yet more good instruments in the world. And similarly, all other wise-hearted people will likewise strive and exert themselves to invent good things and instruments needed for the repair of the world and its perfection.

And therefore, whoever says: "What do I need this new instrument for?" — he does not act well towards the world. For if not for the man who invented it, where would you be? And what would your city do? And more than this, if the man did not exert himself, what would the world do?

— R. Pinchas Eliyahu Hurwitz, *Sefer HaBris HaShalem* (1797)

If you find this translation useful, consider supporting Ars Astronomica.

ArsAstronomica@protonmail.com

A note on the work

Nicolaus Reimers Ursus, a self-taught Dithmarschen mathematician who later became imperial mathematician to Rudolf II, published this book at Strasbourg in 1588. It sets out a new method of sines and triangles, with tables, and closes with a new hypothesis of the world system: a geo-heliocentric arrangement in which the planets circle the Sun, the Sun circles the Earth, and the Earth turns daily on its axis. Tycho Brahe held that Ursus had taken the system from his papers during a visit to Uraniborg in 1584, and the charge of plagiarism opened one of the bitterest priority disputes of early modern astronomy, to which Kepler's *Apologia pro Tychone* belongs. A German translation exists; this is the first in English.

The three folded plates were photographed separately from the bound text in the Copenhagen copy. They are printed here after the text, in the order photographed.

A note on the translation process

This translation was produced with the translation-pipeline. See the linked repository for details on the process.

This edition has been checked at citation grade. A stratified sample of it was independently re-transcribed from the original page by readers given no sight of the translation, and the two readings were then reconciled letter by letter. Every disagreement in the layer a citing reader depends on — figures, names, headings, and whether a passage is present at all — was settled against the page before publication, and none was left open.

What that does and does not mean. It is a claim about a sample, not about every line. A passage the sample did not reach is neither confirmed nor denied, and silence in an independent reading is not agreement. Where the page itself will not settle a reading, this edition withdraws rather than guesses: a bracketed pair marks a genuine ambiguity the original cannot decide, and a question mark a character that could not be read. Those marks are deliberate refusals, and they are worth more than the confident readings they replace.

A note on the references. A scriptural or bibliographic reference given in brackets or parentheses is, in most cases, an **editorial identification of an allusion rather than a citation printed in the source**. Books of this kind quote and weave scripture constantly without naming it, and the identification is supplied here as a convenience to the reader. Where the source does print a reference of its own, it is given as printed. **An identified allusion is not evidence that the author cited that passage** — look at the original before resting an argument on one.

This text is the product of a highly refined automated translation process. It has undergone only cursory human review and should not be treated as an authoritative or scholarly rendering. It is offered solely as a reference and reading aid — a way to make an otherwise inaccessible work approachable — and any passage of real consequence should be checked against the original source.

Acknowledgements

Translated with the assistance of Claude. The translator thanks the Anthropic team for making this work possible.

Notation and abbreviations

A few conventions have been applied uniformly throughout this translation, so that one quantity is always written one way. They change how a value is *written*, never what it is; where the printed source and this translation differ in notation, the source's own form is recorded in the editorial apparatus.

Numerals are Arabic, even where the source sets Roman ones. Roman numerals are kept only in headings, running headers and signatures, which are transcribed as printed.

Dates are given as year, month, day — *1572 November 11*. Dates are Julian, as the source gives them, and have not been converted to the Gregorian calendar.

Angles are written with the conventional signs — 23° 27′ 30″. Early modern astronomy divides the degree sexagesimally and often runs to places for which modern writing has no common symbol; the orders used in this book are:

sign	order	the source's term
°	degrees	*partes*, *gradus*
′	minutes of arc	*scrupula prima*, *minuta*
″	seconds of arc	*scrupula secunda*

The signs of the zodiac. Longitude is reckoned eastward from the first point of Aries.

	Latin	English	longitude
♈	Aries	the Ram	0°–30°
♉	Taurus	the Bull	30°–60°
♊	Gemini	the Twins	60°–90°
♋	Cancer	the Crab	90°–120°
♌	Leo	the Lion	120°–150°
♍	Virgo	the Virgin	150°–180°
♎	Libra	the Scales	180°–210°
♏	Scorpius	the Scorpion	210°–240°
♐	Sagittarius	the Archer	240°–270°
♑	Capricornus	the Goat	270°–300°
♒	Aquarius	the Water-bearer	300°–330°
♓	Pisces	the Fishes	330°–360°

The Sun, Moon and planets.

	Latin	English
☉	Sol	the Sun
☿	Mercurius	Mercury
♀	Venus	Venus
♂	Mars	Mars
♃	Iuppiter	Jupiter

	Latin	English
♄	Saturnus	Saturn

Abbreviations used in table headings: **RA** right ascension.

Where a table in the source spreads one quantity across separate columns for degrees, minutes and seconds, those columns have been joined into a single column and the value written whole. Where a value runs to thirds and beyond, the columns are kept, since there the columns are the notation.

BY NICOLAUS RAYMARUS URSUS [Nicolaus Reimers Ursus] OF DITHMARSCHEN

FOUNDATION OF ASTRONOMY [Fundamentum astronomicum]:
THAT IS,
A NEW DOCTRINE OF SINES AND TRIANGLES,
and that the most complete and most perfect, together with its use in astronomical calculation and observation.

To which are added:

- I. New and true hypotheses of the motions of the bodies of the world [corporum Mundanorum].
- II. Construction of the canon of sines [Canon sinuum] — the ordinary one, indeed, but easier than the usual way.
- III. Section of an angle in a given ratio, or into any number of parts whatever.
- IV. Quadrature of the circle by a demonstrable method, and its demonstration.
- V. Solution of the more common triangles by a new and very easy method.
- VI. Solution of most triangles by prosthaphaeresis [prosthaphæresis] alone.
- VII. Proof [Apodixis], cause and demonstration of that same prosthaphaeresis.

Problems truly most desired in every age, and very many other paradoxes neither heard nor seen before.

Diagram. Woodcut set between the title matter and the imprint: a laureate bust in profile, facing left, rising from behind a plain rectangular block, inside an architectural strapwork frame. Each side of the frame is a female herm (caryatid) figure ending below in a large scroll; an urn stands on each upper corner, and scrollwork with small foliage fills the top and lower corners. The base of the frame is a tablet carrying the legend.

- Legend on the base tablet: "SAPIENTIA CŌSTĀS" — "Constant Wisdom"[1].
- No other letters or labels are printed on the woodcut.

[Greek: Αγεωμέτρητος οὐδεὶς εἰσίτω (ageōmetrētos oudeis eisitō, "Let no one ignorant of geometry enter")]

STRASBOURG [Argentorati].

Printed by Bernhard Jobin [Bernhardus Iobin].

1588.

[Approval stamp: circular ownership stamp with a crown at the centre, lettered round the rim "BIBLIOTHECA REGIA HAFNIENSIS" — Royal Library, Copenhagen.]

[1] the initial S is partly obscured by the hatching of the tablet; CŌSTĀS is the abbreviated CONSTANS

TO THE MEN EMINENT IN NOBILITY OF BIRTH, PIETY, LEARNING, PRUDENCE, AUTHORITY, and every kind of virtue:

- **Lord Johann Philipp von Kettenheim [Ioanni Philippo a Kettenheim], Praetor;**
- **Lord Abraham Heltius [?] [Abrahamo Heltio], Consular;**
- **Lord Nicolaus Hugo Kniebsius [Nicolao Hugoni Kniebsio], member of the Thirteen [Tredecemviro]:**

Scholarchs of the renowned Republic of Strasbourg [Inclytae Reipublicae Argentoratensis Scholarchis]: his gracious lords and patrons [Mecænatibus], worthy of respect, and to be honoured with the highest and perpetual deference: greetings [S. D.].

They say that Mercurius Hermes [Mercurium Hermetem], that famous Trismegistus — most distinguished men, my lords and most gracious patrons — earnestly begged the friends standing around him, when he was about to die, to remember that ship which he had taught to rise and descend perpetually, hanging in the middle of the aether.

I, for my part, am not thrice greatest but four times least, and the most wretched of all Mercuries. Because of the utterly corrupt morals of this most ulcerated and most demented age of ours, and the perverse little wits of pampered manikins, I sense that I shall soon die. And this ship, as it were, patrons — hanging from the aether, perpetually rising and descending, and presented to the earth from heaven — to you, in no other way than sailors long harried by storm, when at last they put in to the shores they had so longed for, used of old (as the divine poet testifies) to hang up to the Fauns and Satyrs the planks saved from the waves and the wreck, and to dedicate and consecrate them as a most welcome sacrifice: in just that way, I say, I have wished this ship to be dedicated, hung up and consecrated to you — to you as to Fauns, though not rustic ones, but Fauns most cultivated and instructed in every kind of learning, and endowed both with singular wisdom and prudence and with all the heroic virtues. I have wished it as a token [?] of the shipwreck we have suffered, and of the labour exhausted and drained to the dregs [exanthlati] after so many journeys undertaken by land and sea: not merely some plank saved from the waves and billows of so many storms, but the whole ship. And I wish and desire, again and again, that you remember it.

The reason, however, and the rationale of this plan of ours — why I chose to hang up this ship of ours, and to consecrate and dedicate it for ever, to you, as to certain urbane and most humane Fauns, rather than to some rustic one — is neither an ordinary nor a rash one. For when, a year ago now, both the most honourable Republic and the most celebrated

Academy of yours were what I had put in here to see, then — just as once happened to that famous Aristippus of Cyrene [Aristippo illi Cyrenaico], when he was driven out by storms from his wrecked ship and cast ashore at Syracuse — traces of your humanity, and traces far more excellent than those of Syracuse, appeared to me at once, at the very first glance. For what I saw were not geometrical figures drawn in the sand, but things more precious, indeed most

precious, in this most honourable, most powerful and most wealthy city — a city which, by I know not what fate, I have loved so ardently from my very earliest boyhood — things from which I took no small pleasure. Above all, there was that regal and splendid — no, utterly astonishing and magnificent — clock, built at the greatest expense, in the cathedral [in summo templo]; and the Academy, the most celebrated of all, founded and shaped with singular prudence and as wisely as could be.

In that Academy services of singular kindness were shown and rendered to me no less than they were to Aristippus himself by the Syracusans: by your most distinguished Lord Professors, that is. First of all by the Magnificent Lord Rector himself, Lord Melchior Junius [Melchiore Iunio], a man as truly most excellent and most humane as he is most learned and most eloquent, and more humane than humanity itself. Then by Dr Laurentius Tuppius [D. D. Laurentio Tuppio] of Pomerania, a most learned Professor of Law, and no less outstanding and practised in the mathematical arts and in all humane learning. And most of all by Master David Wolkenstein [D. M. Dauide Vvolkenſtenio], the most wise Paedagogus of your College at the Preachers [Collegij ad Prędicatores]. Each of these supreme and excellent men stood in a parent's place to me for as long as I lived in your most celebrated Academy.

Not to add anything here about the generosity and kindness of certain most learned and most humane young students now living in this Academy of yours. During the winter now past, with the permission and on the advice of Master Wolkenstein, they had me [ne [recte: me]] as their teacher in this loftier and subtler mathematics [Mathesis] of ours. As grateful hearers and pupils of mine, they gave a most welcome sign of a grateful mind, and up to now they have generously and most graciously bestowed on me food and clothing and the other things needed to sustain this prison-house [ergastulum] of the body. By their help and effort, too, and at costs liberally borne, this little work of ours, kept suppressed until now by our poverty, has at last come to light and appeared.

And so for more than a year now your nurturing and most celebrated Academy, like a most gracious and most fruitful mother, has fed me, her poor little offspring, most sumptuously and most generously, and has cherished and handled me with a mother's heart. Me — most wretched, drained of all money, stripped and wiped clean of everything, and least worthy of such great benefits bestowed on me — she has taken under her most ample wings, spread most widely with every kindness and largesse. And this not only in respect of food for the body and free sustenance, but also in respect of what must rightly be reckoned [existiman-] of still greater[2] as I hold [-dum eſſe duco]: she has imbued our mind with the liberal arts [Artibus ingenuis], and in her bounty [Alma] she nourishes our breast with the nectar of Minerva.

[2]the page prints "qui ne Præceptore … vſi ſunt" — two sorts, *n* and *e*; the construction ("quique vt grati mei auditores atque diſcipuli") requires *me*.

She also turned aside my mind, previously wild, bear-like [vrſinam] and almost bestial, from a certain boorishness hardened by old roots spreading far and wide, and led it some little way towards the civility of better manners — so far, that is, as could be done where the callus of all wildness had grown over so great and so thoroughly rooted a nature [in tanto tamque radicitùs planè … obducto callo] — and at last to a love of true piety and humanity. Finally she kindled, fired and spurred my mind towards a more perfect and more solid knowledge of all the sciences and good arts, and towards the origin, fount and wellspring of Apodictic or Demonstrable reasoning [Apodicticam ſeu Demonſtrabilem … rationis] — in which she surpasses all others and, as the one Queen and Coryphaea of all, outstrips the Academies of our Germany by many parasangs. She turned me wholly and completely away from the errors and phantasms of certain rather recent little men [nouitiorum quorundam homuncionum], by which I was tossed now this way, now that, like a ship pitching on the waves, and she enticed and drew me to a sounder mind and to the pursuit of truth.

In sum, to speak with the most learned of the Spaniards [Hiſpanorum doctiſſimo], out of a fierce and unbridled beast she has made me an almost perfectly gentle man. Or, if someone should perhaps still deny that this has been done (and I myself could not very well deny it), I have nonetheless conceived a sure and excellent hope that in the course of time she will fully accomplish it and bring it to effect, so that I, a man still imperfect, shall one day turn out a most perfect and most humane animal. That will surely be a Metamorphosis or transmutation not unwelcome to the whole republic of letters, and of the greatest benefit to me.

Most distinguished gentlemen, Lords Scholarchs, best and most gracious Maecenases: when I had turned all this over in my mind with gratitude several times, and lately rather more deeply than usual, words can hardly say what regret had come over me, and with what shame I was covered, that I had received so many, so great and such innumerable benefits from your most celebrated Academy, and how tightly I am bound and obliged on that account to this most honourable matron. From that moment I began to think and ponder some fitting and due kind of repayment, and some token of a grateful mind to be given to the best of my power; and I deliberated with myself by what means I might in some measure make it good, and how I might one day repay you the thanks you have deserved and that are owed to you for it.

I judged that this could be done by no better means or way than by bringing to light some small spark [ſcintillulam] of that most subtle and most rare doctrine — indeed, a doctrine wholly abstruse and hidden from the whole world until now — which I taught privately in your most celebrated Academy this past winter, and in which I instructed, as well as I could, certain most learned Young Men and Masters. By this publication, the students of your Academy — most learned in any case, and most excellent and most practised in every kind of learning — even to the wholly divine and more sublime subti- [diuinarum

prorſus ac ſublimiorum ſubti-] -liorumque [and more subtle] arts — I might invite them to the very fountains of these arts [harum artium ipſos fontes inuitarem].

My aim was that they should by no means go away content with the Euclidean or Elementary mathematics [Euclidea ſeu Elementari … Matheſi], which creeps and crawls along the ground and is the only kind commonly set before and explained to the young learners in our schools and Academies. Rather, taking up mathematical wings — Arithmetic, I mean, and Geometry — they should aspire to and approach the heavenly and far more excellent mathematics, the Ptolemaic and Copernican, that is, true and genuine Astronomy, and should carry their prepared minds up on high, all the way into the heavens.

This true and genuine astronomical doctrine, complete and most perfect in every respect [omnibuſque numeris abſoluta], does not lie (as some commonly suppose) in innumerable wagonloads, hundreds and myriads of computed tables [tabularum ſupputatarum plauſtris, centurijs, atq; myriadibus], nor in the tottering and swaying asylums of lazy fathers — to use the phrase of our M. Plinius [?] [M. Plinij] of pious memory. It lies in the geometrical foundation itself, and in that most abstruse and most recondite doctrine (as Reinholdus [Erasmus Reinhold] and Peucerus [Caspar Peucer] call it) of sines and triangles [doctrina ſinuum & triangulorum]. From this doctrine the computed Tables commonly set before young learners in the schools were indeed built and stitched together; but they now stray very far from their intended mark, and correspond neither to the φαινομένοις [Greek: phainomenois, "appearances, phenomena"] — the appearances in the heavens — nor to the motions themselves. A new correction and emendation of the tables, starting from the foundation itself and from the doctrine of triangles itself, will therefore be needed in future. Therefore the young who are to be instructed in astronomy must be accustomed, not to the most turbid pools [lacunas] of the tables, but to the most limpid fountains of the triangles, and (so to speak) watered at them.

Therefore it seemed to me that some Prince, or some most honourable Commonwealth — above all one that has an Academy and holds the most honourable Muses close to its heart — would deserve best of the republic of letters and of all posterity to come, if he or it should come to the aid of Astronomy, the doctrine of the celestial motions, now on the point of collapse, with some quite small outlay of expense. And above all, effort should be given again and again, and applied to the best of one's ability, so that the foundation of all correction and emendation of all the celestial motions (that is, the doctrine of sines and triangles itself) might be had in as correct and as clear a form as possible and, so far as could be done, μεθοδικ[?]τατον [Greek: methodik[?]taton, "most methodical"] — that is, written and handed down in the best method [optimâ conſcriptum traditumque methodo].

This is what that most learned man Casparus Peucerus always longed for, demanded and sought with so many prayers and sighs from his most learned Teachers, Ioachimus Georgius Rheticus [Georg Joachim Rheticus] and Erasmus

Reinholdus. Each of them did indeed promise it, resolved to apply himself and toil at it, and pledged to bring it to effect; yet neither of them, forestalled perhaps by an untimely death [vel immaturâ morte præuentus], kept his promises and pledges. For it is not within the power of just anyone, even of a most learned and most distinguished Mathematician, to accomplish, correct and amend that Doctrine of Triangles — most profound indeed, secret and arduous — not to say, as hitherto by ancestors it has been handed down and treated — most intricate and obscure. (Not to add anything here about the very many other paradoxes contained in this little work of ours, never before seen or heard; and about mathematical matters that are truly most difficult and very arduous, which exceeded all the grasp and understanding of the whole of antiquity, learned as it was in the highest degree. Many people, long wearied in the investigation of these matters, at last despaired of finding what they were seeking and abandoned the undertaking.)

But to what ease this doctrine — formerly most profound and intricate — has been brought, corrected and emended, by the tireless study, the assiduous meditations and the long-continued reflections of my dearest teacher, the Swiss Iustus Byrgius [Jost Bürgi], and by this little work and pen of mine, and by the Method I present to you here, it is not for me to say. Posterity itself, still to come and late, will say it, see it, judge it and bear witness to it after very many centuries. (For, to be sure, the little men and boys of this pampered and effeminate age of ours mostly pass perverse and unfavourable judgements on such great arts, which they do not understand. Indeed, they even come forward to urge and advise the magistracy itself not to come to the aid of these arts, which are about to collapse and rush to their ruin.)

Therefore neither I nor my dearest Teacher, the builder and restorer of this hidden Doctrine, will ever regret the labour performed or spent [facti impenſiuè laboris], provided we see that these labours of ours, undertaken and toiled through to the end [exanthlatos], will not be unwelcome to Your Lordships [VVV. DDD.]; that the study spent, placed and devoted to this most beautiful and most excellent part of singular and most subtle learning does not in the least displease you, my best and most gracious Lords and Maecenases; and that we may feel in Your Lordships a mind present, ready and inclined to help our astronomical studies and the more-than-Herculean labours we have undertaken.

And especially if you urge, persuade and invite the students of your most celebrated Academy, as well as those of all the others, to the knowledge of these recondite and most delightful Doctrines and loftier arts; and if you by no means allow those raw and ungeometrical youths [ἀγεωμετρήτους (ageōmetrētous, "untrained in geometry")] — from whom risings and settings lie hidden and escape, and over whom one must rightly exclaim with Homer, ὦ φίλοι οὐ γὰρ ἴδμεν ὅπη, ξόφος [recte: ζόφος] οὐδ' ὅπη ἠώς [Greek: "O friends, for we do not know where the darkness is, nor where the dawn"] — to return to their fathers' Lares and go back home, when they have been sent, that is, to your

most celebrated Academy for the sake of cultivating the mind and of a more polished and richer cultivation of it, and have been entrusted to the faithful keeping and discipline of your most learned Teachers and Professors.

How much glory among all nations, and how much praise and honour of true immortality among all learned men and all posterity, this diligence, once applied, will bring to your most celebrated Academy and to the whole most honourable Commonwealth as well, Time — the very daughter of truth — will surely teach and bear witness. Nor, finally, let

Your Lordships take amiss this little admonition of ours, which has come forth and burst out from the best of intentions. Farewell, most distinguished gentlemen, Lords Scholarchs and best and most gracious Maecenases. From your College, July 31 [IL. Calendas Auguſtas][3], in the year of the God-man [Greek: τοῦ Θεανθρώπου (tou Theanthrōpou, "of the God-man")] Jesus Christ 1588.

Your Lordships' [VVV. DDD.]
most devoted and most obedient [Addict. & ſub.]
Nicolaus Raymarus Ursus of Dithmarschen.

To the Author of This Book [AD AVTOREM HVIVS LIBRI.]

> Dear is the SUN, when it lights up all things with its shining flames; and dear is DAY, the faithful handmaid of the earth-born; dear, too, is SLEEP, while it refreshes weary bodies; dear the KING, who bears his fathers' sceptre over the peoples; dear in love the PARENT who cherishes his children; and dear is HOME, since it gives a safe shelter. Dearer than SUN, DAY, SLEEP, KING, PARENT, HOME is URSUS, who solves intricate Triangles.

Another [ALIVD.]

> Savage was the Theban SPHINX with her three bodies; savage the GODDESS who turned men's limbs into a beast's; savage the CYCLOPES, who made war on the gods; nor is the SHE-BEAR, with her greedy claws, any less savage; savage is the LIONESS among the beasts when she rouses her rage; savage too, and of monstrous fierceness, is the DRAGON. MORE SAVAGE is he who carps at this URSUS with a gibe [ſcommate] than are the SPHINX, the GODDESS, the CYCLOPES, the SHE-BEAR, the LIONESS, the DRAGON.

Ioannes Trunccius of Marienburg [Malbork], a Prussian.

[3] the date is printed "IL."; "July 31" reads it as II. Kal. Aug., the only sense the numeral can carry.

Foundation of Astronomy [FVNDAMENTVM ASTRONOMICVM.]

Chapter I. On Astronomical Logistic [Caput I. DE LOGISTICA ASTRONOMICA.]

Astronomy is the art of scrutinizing the stars; in the same sense it is also called Astrology. To scrutinize the stars is to investigate, by a sure method, both the positions of the fixed stars and the motions of the planets; and the foundation of that investigation lies in Calculation, and its exercise in Observation. Accordingly, Astronomy has two parts, Calculation and Observation. This follows the parts of the methodical art and the partition of the art; for according to its species, that is, the distinction of the art, Astronomy can be divided into Methodical and Historical.

Methodical Astronomy, accordingly, is the one that treats the method of calculating and of observing — that is, of the two parts of Astronomy just named.

Historical Astronomy, on the other hand, is the one that hands down the celestial or sidereal history in the stars: drawn, namely, from the observations made by practitioners and from the records left by the ancients.

But we shall discuss this Historical astronomy at sufficient length elsewhere. The Methodical we shall treat as briefly as possible in this little work of ours, and do so according to the two parts of the astronomical art already named: namely Calculation and Observation.

Calculation is the first part of Astronomy, the one that treats astronomical calculation [calculus Astronomicus].

Astronomical calculation, moreover, is carried out either in logistic numbers or in metric matters [rebus Metricis]; and in metric matters it consists in the construction of the Canon of sines [Canon sinuum] and in the treatment of the Doctrine of Triangles [Doctrina Triangulorum].

We shall therefore treat Methodical Astronomy, and complete it, in this order:

- The parts of Astronomy are [Aſtronomiæ partes ſunt]:
  - Calculation, and that in [Calculatio, eaq; in]:
    - logistic computation, which is the subject of this Chapter — I.
    - metric matters [Rebus Meretricis. [recte: Metricis]]:
      - the construction of the Canon of sines — II.
      - the Doctrine of Triangles — III.
  - Observation, and it is [Obſeruatio, eaq; eſt]:
    - of the places of the fixed stars — IV.
    - of the motions of the planets and of the earth — V.

Logistic is a certain compendious method of computing in astronomical number, which does not reach the number sixty and does not grow as far as 60. For whenever 60 have accumulated in any order [ſpecie], the 60 are discarded and in their place a unit is customarily added to the immediately preceding order,

just as in common numbers the number ten [numerus Denarius] is collected from ten units. And therefore that which in nume- ber of the common kind [ro vulgari] is a figure [figura] is, in logistic number, an order [ſpecies]. Every logistic computation, therefore, is carried out by means of a certain inclusive number, contained in another, greater number as many times as that other is itself contained in one greater still.

Logistic computation consists in Numeration and Operation.

Numeration is what expresses the meaning and value of a logistic number; and it consists in Notation and Reduction.

Notation is the method of marking the orders of logistic numbers.

The orders of logistic numbers are of three kinds. Either those orders of logistic numbers are of whole things, and are called integers [integra]; or of things that have arisen from whole things — and that either from gathering together several integers, and these are called sexagenae [ſexagenæ]; or from breaking some integer into its parts or fractions [minutias], and these are named minutes [ſcrupula].

In general, then, as we have said, the orders are three: the integer, the sexagena, and the minute [ſcrupulum ſeu minutum].

An integer is any whole that can fall under calculation. What has arisen [Ortum], on the other hand, is whatever arises from the integer: such as the sexagena and the minute.

A sexagena is a collection of sixty integers, and these are called first sexagenae. Next, a collection or heaping-up of several of these first sexagenae is called second sexagenae; likewise an accumulation of several seconds, thirds; and of several thirds, fourths; and so on, as far as use requires.

A minute is the sixtieth part of some integer, and these are called first minutes [ſcrupula prima]. The sixtieth part of one of them, in turn, is called a second [ſcrupulum ſecundum]; and again, of one second, a third; and of one third, a fourth; and so on.

These orders, however, are placed or arranged in sequence in this way.

The integers occupy the middle place: for there is only a single order of integers, and therefore it always holds the middle place in the arrangement of logistic numbers. The orders of each kind that has arisen from it, by contrast, are various, and can be innumerable. Therefore the sexagenae are arranged in sequence towards the left, so that their first or smallest order is nearest the integer, while their last or greatest order is the farthest from that integer, towards the left.

The minutes, on the other hand, are arranged towards the right, and in such an order that their first or greatest order is nearest the integer, while their last or smallest order, conversely, is the farthest from that integer, towards the right.

The integer is marked with a circle or cipher [Ziphrà] O written above it. The remaining orders are marked with small strokes or apices according to the number of the orders, bent back: for the sexagenae, below to the right

and above to the left; for the minutes the reverse, namely below to the left and above to the right. For example:

two strokes	one stroke	circle (o)	one stroke	two strokes	three strokes
36.	42.	48.	24.	18.	54.

This logistic number is expressed thus:

36 second sexagenae, 42 first, 48 integers; 24 first minutes, 18 seconds, 54 thirds.

And so much for the notation of logistic numbers.

Reduction Follows [Sequitur Reductio.]

Reduction [Reductio] is the method of reducing numbers either from common numbers into logistic ones, or, conversely, from logistic numbers into common ones; and this both in integers and in fractions. Reduction will therefore be of four kinds:

- I. Of common integers into logistic numbers, that is, into sexagenae. And conversely,
- II. Of logistic sexagenae into common integers.
- III. Of common fractions into logistic minutes [Logistica scrupula]. And finally,
- IV. Of logistic minutes into common fractions.

For these four, take the rules and the examples that follow.

I. The Reduction of Integer Numbers into One Another [I. Reductio integrorum numerorum in ſe inuicem.]

The reduction of common integers into logistic sexagenae is done by division by 60. It is done more briefly, however, by first dividing by some larger or largest collection of sexagenae, resolved by multiplication — for instance by 3600, or 216000, or 12960000, or by still larger collections.

For example.

Let this common number, 378945, be reduced to a logistic number. By continued division by 60, then, the first division gives 6315 first sexagenae, with 45 integers remaining. The second division by 60 accordingly gives 105 second sexagenae, with 15 firsts left over. Finally, the third division gives one third sexagena, with 45 seconds left over. That is the common way. It is done more briefly, as was said above, by the larger collections: for example by 3600, for one second sexagena contains that many integers; or by 216000, namely the collection of one third sexagena.[4]

[4]The worked division below is set without rules or column heads. Its digits are given here in the columns in which they stand on the page, counted from the left; every digit is printed struck through with a short horizontal stroke except the "4" and "5" at the right of the fifth row, and the round bracket "(" in that row, which are unstruck.

	1					
	2	3				
	4	8				
1	6	2	3			
3	7	8	9	(	4	5
2	1	6	0		0	0
	3	6	0		0	0
		3	6		0	0
			6		0	0
						6

To the right of the division, on the level of its fifth row, the result is set with its marks above the figures, preceded by a large curved bracket "(":

three strokes	two strokes	one stroke	round dot
1.	45.	15.	45.

II. The reduction of sexagenae, conversely, is done by continued multiplication — as it were by an analysis or resolution of the sexagenae by 60 (or also, just as before, by some larger collection, if one likes). One begins from the greatest, or first, order of sexagenae, towards the left, and proceeds to the right, always adding the number of the sexagenae that come next to the number produced by the multiplication just made. And so, multiplying and adding again, one continues until one has at last arrived at the smallest order of sexagenae, or at the integers.

For example.

The logistic number is set with its marks above the figures:

three strokes	two strokes	one stroke	round dot
1.	45.	15.	4[5?].

5

figures as printed

1.
—
60
45
—
105

[5]Below the "1." the reduction is worked in a single right-aligned column of figures, divided by short horizontal rules. The figures are given here in order from top to bottom; "—" marks the place of each printed rule.

figures as printed
60
—
6300
60
—
378900
45
—
378945

That is enough about the reduction of integers.

The Reduction of Fractions Follows [Sequitur Reductio fractionum.]

III. The reduction of common fractions into logistic minutes is done just as in the common reduction of fractions, namely by seeking the value of the fraction: that is, by multiplying the numerator of the proposed fraction by 60 and dividing the number produced by that multiplication by the denominator [nominator] of the same given fraction. Here it should be noted that if the denominator of the proposed fraction is an "indivisible" number [numerus indiuiduus] with respect to the sexagenary number, that is, to 60, the reduction can never be exact, even if it were repeated to infinity; nor will it ever end exactly in any order of logistic number. The value of that fraction therefore cannot be had in logistic numbers. Yet the more often the reduction is repeated, the more nearly and more exactly the value of the proposed fraction will be had in logistic numbers or kinds — though never precisely.

For example.[6]

fraction	multiplication and division	result
2/3	60. (circle above); 2; rule; 1 2 0 over 3 3 (struck)	(40. (one stroke above)

For the quotient that results from the division is always the number of the next following kinds.

[6]The example is set in three groups across the line. At the left stands the fraction 2 over 3. In the middle, "60." is set with a small circle above it, and below it a "2"; under a rule follows "1 2 0" with "3 3" beneath its last two figures — the "1", the "2" and both "3"s are struck through, the final small "0" is not. At the right a large curved bracket "(" precedes the result "40." with a single stroke above it.

___[7]

3 (curved stroke beneath)	6 0					
7	3					
	———	6 0				
•	~~4~~ 5 ———	——— 5				
	~~1~~ • ~~8~~ ~~0~~					
		———	6 0			
		~~2~~ 6 ———	——— 6			
		~~3~~ ~~0~~ ~~0~~	———	6 0		
			~~1~~ 3 ———	——— 3		
			~~3~~ ~~6~~ ~~0~~	———	6 0	
				~~4~~ 5 ———	——— 5	6 0
				~~1~~ ~~8~~ ~~0~~	———	———
					~~2~~ 6 ———	6
					~~3~~ ~~0~~ ~~0~~	———
						~~1~~ 3
						~~3~~ ~~6~~ ~~0~~
	2 5.	4 2.	5 1.	2 5.	4 2.	5 1.
			7.			

And so, if the reduction is repeated and reiterated to infinity, these same three [hi terni] will always come back, through all the orders of number, even infinitely many, so that there will never be any end at all to the work of reduction, or of reducing. Therefore, once one has at last come down to some very small orders, the prudent calculator [Logista] may rightly disregard the remaining orders that follow, which are smaller still, and lift his hand from an endless labour.

IV. The reduction of logistic minutes [Logisticorum scrupulorum] into common fractions is done by continual multiplication of the orders by 60. One begins with the greatest order, that is, the first toward the left, and always adds

[7]The page opens, directly under the running head and with no caption, with a worked example set as a descending staircase of six groups, each group lower and further right than the one before. At the upper left stands 3 over 7, with a curved stroke under the 3 in place of a straight fraction bar. Each group sets "6 0" with a single figure under its "0", a short rule under that pair, and below the rule two rows: a row of one or two figures, the last of which is joined by a long horizontal line to a figure standing under the "0" of the next group's "6 0", and beneath it a row of three figures. A figure shown here as ~~x~~ is printed struck through by a horizontal stroke; all others are printed plain. Two solid dots are printed: one at the far left of the fourth printed line, below the "7", and one between the struck "1" and the struck "8" of the first group's three-figure row. The six columns of the table below follow the six groups as they stand left to right on the page; the first column holds the fraction at the upper left. A full-width double rule follows the staircase, and between its two rules a row of six numbers stands, each under its own group. Under the lower rule a single "7." is printed, beneath the third of those numbers. The page prints no operator sign and no words in the example.

the number of the order immediately following; then one multiplies again and again just as before, proceeding toward the right, until one has come down to the smallest minutes, that is, the last toward the right. The numerator of the common fraction sought will then be had. The denominator [Nominator], for its part, is found by exactly the same procedure: namely, by continual multiplication of one whole, that is, of 60′ [60. scrupulorum primorum], until one comes down to the smallest order of the proposed logistic number — continually, that is, by the sexagenary number. The denominator of the given fraction will then be had. In this way the logistic minutes will be reduced to a common fraction.

Let the example be this.[8]

Left block:

1	II	III
34.	42.	12.
60		
———		
2040		
42		
———		
2082		
60		
———		
124920		
12		
———		
124932. numerator of the common fraction [vulgaris fractionis Numerator.]		

Right block:

0		1
1	vel [or]	60
		60
———	———	———
		3600

[8]The example is set in two blocks side by side, divided by a heavy vertical rule printed in several segments. The left block works the numerator, the right block the denominator. In each block the figures of every row are set right-aligned under the row above, and a short horizontal rule stands under each pair of rows that is multiplied or added; the page prints no operator sign. In the left block the strokes "I", "II" and "III" stand above the last figure of "34.", of "42." and of "12." respectively; a short faint dash stands above the single stroke and above the double stroke. In the right block a small "0" stands above the "1", and a single stroke "1" stands above the "0" of the "60" in the same row; the "0" of the next row's "60" is printed with its counter filled with ink.

			60
———		———	———
216000.	denominator of the common fraction [Vulgaris fractionis Nominator.]		

Therefore the fraction that first presents itself is
124932. / 216000.

and then, reduced to its lowest terms by the common reduction, it will be
10411. / 18000.[9]

Through these last two kinds of reduction, then, the method of changing and converting the parts of an as [partes assis], or of any common whole, into logistic minutes, and conversely minutes into parts, will not be difficult. So much for reduction and for the whole numeration of logistic numbers; their operation follows.

Operation is the method of working with proposed numbers so as to draw out of them what is sought. And just as what is sought is either simple or proportioned [proportionatum], so the operation itself will also be either simple, like addition and subtraction [Additio & Subductio] — in which both like degrees [gradus] of numbers and like orders are to be placed properly and squarely under like ones — or it will be proportioned, like multiplication and division, and also the extraction of a root [Radicis extractio]: in which only like degrees of numbers must necessarily be placed under like degrees.

Addition and Subtraction [Additio & Subductio.]

In common addition, that is, the addition of common numbers, whenever, through the gathering of the sum or the bringing-in of the parts, so many parts of some lesser thing or value have accumulated that all the parts make up and amount to some as, or the whole value of the whole [assem seu totum valorē Totius], or even exceed it, then one is always added to the immediately preceding order of the greater thing. Just so, in logistic addition, that is, the addition of logistic numbers, whenever 60 have accumulated out of the gathering of the sum once it is made, then one is to be added to the immediately preceding order. Conversely, in common subtraction, whenever more parts cannot be subtracted from fewer like or homogeneous parts, then one whole is resolved into the parts of its value, so that the subtraction can be made from them. Just so, in logistic subtraction, whenever the lower number, or the number to be subtracted, from the upper- per, or greater, number cannot be taken away or subtracted, then the subtraction is to be made by drawing on a gathered sixty [è collectione sexagenarij] of the immediately preceding order, and by resolving that same sixty into parts of the smaller order that immediately follows.

[9]Each fraction is printed inline in the sentence as numerator over a horizontal bar over denominator, with a point after the last figure of each number.

Examples of Addition. [Exempla Additionis.][10]

First example:

	0	1	11
	12.	23.	34.
\	56.	29.	13,
———	———	———	———
1.	8.	52.	47.

Second example:

	42.	36.	54.
	56.	48.	32.
\	35.	27.	18.
———	———	———	———
2	14.	52.	44.

Examples of Subtraction. [Exempla Subductionis.][11]

			1		11	
	0	2	4.		5.	
1	2.	3	5	~~1~~	~~0~~	4.
~~2~~	~~4~~	~~2~~	7	5 ¦	~~4~~	~~4~~

[10] The examples of addition stand in the left half of the page, the examples of subtraction in the right half, divided by a vertical rule printed in several segments. In the left half are two examples, one above the other, separated by a band of linked chain-pattern ornaments between double rules that spans the left half only. Each number is set as two figures followed by a point. In the first example a "0", a single stroke "1" and a double stroke "11" stand above the three numbers of the top row. In each example a short slanted stroke "" is printed at the far left of the third row of figures, and the left-hand figure of the result row ("1." in the first, "2" in the second) stands in that same far-left position. The rule under each pair or trio of rows is printed in several segments and spans the whole example. The page prints no operator sign.

[11] The first example of subtraction is set as four rows of figures in seven figure positions, read here left to right; a figure shown as ~~x~~ is printed struck through by a horizontal stroke. The single stroke "1" and the double stroke "11" stand highest. A row of small figures follows ("0", "2", "4.", "5."), set lower than the strokes; the "0" of that row stands at the same height as the small figures beside it, unlike the "0" of the addition example, which stands level with the strokes. The third row opens with two small unstruck figures, "1" and "2.", and ends with an unstruck "4." carrying a small grave-like mark above it at the right. In the fourth row a broken vertical stroke, with a dot at its foot, stands between the struck "5" and the struck "4" that follows it. Under the fourth row a rule printed in segments spans the example; no figures are printed beneath it.

———	———	———	———	———	———	———	———

Or thus: [Vel ſic:]

3	6	5	9				
4	0.	6	0.	6	8.	3	6.
2	6.	3	6.	4	5.	5	8.
———	———	———	———	———	———	———⌝	———
1	3.	2	3.	2	2.	3	8.

So much for simple numeration [Numeratione simplici]. Proportioned operation follows, such as multiplication, division and the extraction of a root. All of these are carried out with the help of the canon of hexacontads [Greek: τῶν Εξακοντάδων (tōn exakontadōn as printed, without a visible breathing, "of the sixties")], or of sexagesimals, just as in common arithmetic they are carried out by means of the table or abacus of Pythagoras, commonly called *Das ein mal eins* [German: "the one-times-one", the multiplication table]. For this canon of hexacontads [Canon Hexacontadωn] is just like the Pythagorean abacus in common arithmetic. Here, however, because of the longer progression of the numbers, up to 60, it can by no means be grasped by the mind and fixed in the memory in the way it can there. For as in common arithmetic the resolution or reduction of the figures is done by the number ten, so in logistic arithmetic the reduction and resolution of the kinds is done by the number sixty.

Let that logistic canon therefore be built up by logistic addition. For suppose all the numbers below 60 are set out in their successive, natural order, beginning from unity, in the form of a right-angled gnomon [Gnomonis rectanguli], straight down and across; and suppose that, always by continual logistic addition, the number standing immediately before, above, in some proselis [proſelidis] — that is, a common angle — is added to the number that follows immediately next in order below. Then there springs from this the number of the proselis that follows immediately below, the number to be placed in that very proselis next below. This will become clear at first glance, more easily from inspecting the canon employed [adhibiti Canonis] than from little precepts, to those who look at it.

Moreover, in this proportioned logistic operation a reminder will also be needed about the "emerging order" [ſpecie emergente], as they call it. For just as, once the multiplication and division of common fractional numbers, and likewise the extraction of their roots, have been carried out, it is not the same orders of parts that grow out and emerge but always different ones; and again, just as from numbers with ciphers [Ziphris], or figures of nothing, attached and joined to them — o o o. — multiplication once carried out produces numbers with more of them — o o o o. — and, conversely, division and the

extraction of a root produce numbers with fewer — 0 0. — so also in these logistic quasi-fractions and collections it is not the same orders but always different ones that grow out and come forth, according to the varied operation of the various orders fractioned among themselves. For just as, in a common number of common arithmetic, those figures of nothing, or noughts [nullæ], as people commonly call them, serve as indices: so in this logistic arithmetic those little marks [notulæ], or little strokes [virgulæ] written above the orders: 1 11 111. And from this comparison, once made, we judge that the reasons for the following rules about the emerging order are plainly and evidently clear [liquidò ac euidenter patere]. For the emerging order, then, let just these three very brief rules be taken, contained in the following three distichs.

I. On Multiplication [I. De Multiplicatione.]

> Of one kind, add the marks; but of two, take them away: from that comes the last mark of the product.
>
> [Vnius adde notas generis: ſed deme duorum: / Vltima producti prouenit indè nota.]

II. On Division [II. De Diuiſione.]

> By as much as the cutter is greater or less than the cutting, by so much is the last mark from the wholeness of the quotient. [?]
>
> [Quantò ſector erit maiorúe minorúe ſecante, / Tantò extrema nota eſt integritate quoti.]

(For since multiplication and division are contrary kinds [of operation], and the emerging order of the former is evident from the addition of the marks, the emerging order of the latter will necessarily appear, conversely, from the subtraction of the marks. And this holds for marks of one, that is, of the same, order; with marks of different order, however, the contrary happens.)

III. On the Extraction of a Root — Which, However, Can Equally Be Grasped from the Rule of Division [III. De Radicis extractione: quæ tamen æquè ex Diuiſionis Regulâ percipi poteſt.]

> If they are wholes or firsts, the root has wholes; if not, you must halve the even mark.
>
> [Integra vel primæ ſi ſint habet integra Radix: / Sin minùs, eſt tibi par dimidianda nota.]

That is, in the extraction of the square root [Radicis Quadratæ]. In the extraction of the cube root [Radicis Cubicæ], however, let the pentameter of the distich be this:

> Divide the divisible marks into three.
>
> [In tria diuiduas diuide in ipſa notas.]

And so on in the other roots, keeping the natural order and progression of the numbers.

Example of Multiplication. [Exemplum Multiplicationis.]

In multiplication, use the lateral entry [ingreſſu laterali] into the canon of hexacontads; in division and in the extraction of a root, however, the areal entry [ingreſſu areali] must be used. Moreover, logistic multiplication seems to involve almost nothing different from that common multiplication, the multiplication of common arithmetic — provided we notice and recall that in a logistic number the orders will be the same thing that the figures, or the degrees of units, tens, hundreds, thousands, etc., are in a common number. Therefore one must deal here with the orders, in running through and writing beneath, just as one deals there with the figures.[12]

		0	1
		16.	12.
		12.	14.
———	———	———	———
		2.	48.
	3.	44.	
	2.	24.	
3.	12.		
———	———	———	———
0	1	11	111
3.	18.	10.	48.

Example of Division. [Exemplum Diuiſionis.]

In division one must note, and take care, that the divisor is taken in the number to be divided just so many times that what is left after the subtraction has been made remains smaller than the divisor — exactly as is usually done in the division of common arithmetic. How many times this can be done is shown by inspecting the canon of hexacontads [Canonis Hexacontadωn]. So enter the canon with the divisor by the areal entry; the number next smaller than the one from which the subtraction is to be made will then show, from its common proselis [communi proſelide], the quotient written directly above it in the upper side, or margin, of the canon. For it makes no difference in which side or margin of the canon the divisor is taken, since that common proselis

[12] The example of multiplication is set in four figure positions, read here left to right, each holding a number of one or two figures followed by a point. A "0" and a single stroke "1" stand above the two numbers of the top row, and a "0", "1", "11" and "111" stand above the four numbers of the bottom row. Two long rules, each printed in several segments, span all four positions. Between them stand four rows of partial results set in a descending staircase toward the left. The page prints no operator sign.

will in either case show the quotient in the remaining side or margin of the canon.[13]

			~~2~~	~~2~~	~~2~~		
			~~3~~	~~3~~	~~6~~		
			~~6~~	~~7~~	~~0~~		
	o		1		11		111
Dividend [Diuidendus]	~~3~~	~~1~~	~~8~~	~~1~~	~~0~~	~~4~~	~~8~~
					o		1
Quotient [Quotus]				1	6	1	2
Divisor [Diuiſor]		~~1~~	~~2~~	~~1~~	~~4~~	~~1~~	~~4~~
Removed divisor [Diuiſor remotus.]				~~1~~	~~2~~		
To be subtracted [Subducendi.]	~~3~~	~~1~~	~~2~~				
			~~3~~	~~4~~	~~4~~		
			~~2~~	~~2~~	~~4~~		
					~~2~~	~~4~~	~~8~~

Example of the Extraction of a Root. [Exemplum Extractionis Radicis.]

The extraction of a root is carried out in the same way as with common numbers, provided that here the orders [ſpecies] of logistic numbers are understood in place of the figures, or degrees, of a common number, and that every other order — just as is usually done with common numbers, starting from the smallest order, the last one toward the right — is marked and indicated, moving toward the left, by small points written above it. After that the root is drawn out according to the rules of common arithmetic, but with the help of the logistic canon, or canon of hexacontads — and that by the areal entry, just as in division — always taking as the root, around the diagonal [Diagoninum] of the canon, the number next smaller than the proposed number.

[13] The worked example of division is set as labelled rows of figures in four number positions, read here left to right; each position has a tens place and a units place, and each is given its own column below (the first position holds only a units figure). A figure shown as ~~x~~ is printed struck through by a short horizontal stroke: every figure of the dividend, of the figures stacked above it, of the divisor, of the "removed divisor" and of the numbers to be subtracted is so struck; the figures of the quotient are printed without strokes. Above the dividend stand the small marks "o", "1", "11", "111", each over the units figure of its number; the "1" is faint. Above the quotient stand "o" and "1". Three long rules, each printed in several segments, run across the example: below the dividend, below the quotient, and below the "removed divisor". In the rows of the numbers to be subtracted, a rule runs from each single figure on the left to the two-figure number on its right, shown below as "———" after the figure. The page prints no operator sign.

___[14]

			3		ı	
	`		0		1	11
	~~2~~	~~2~~	~~7~~	~~ı~~	~~3~~	~~4~~
———	———	———	———	———	———	———
			0			1
		1	2			8
———	———	———	———	———	———	———
	~~2~~	~~2~~	~~4~~			
The root found, doubled. [Radix inuenta duplicata.]				2.	4	
The new quotient, to be multiplied by the double. [Nouus quotus, inq; duplum ducendus.]					8	
———	———	———	———	———	———	———
Product. [Productum.]			3	ı	~~2~~	
The new quotient multiplied by itself. [Nouus quotus in ſe ductus.]					ı	~~4~~
					———	

End of logistic; the construction of the canon of sines follows. [Finis Logiſtices: ſequitur extructio Canonis ſinuum.]

Chapter II. On the Construction of the Canon of Sines [CAPVT II. DE EXTRVCTIONE Canonis ſinuum.]

The canon of sines is a table that gives the ratio and magnitude, relative to the radius of the circle, of the halves of the straight lines that subtend arcs of the semicircle in a quadrant. These halves are commonly called sines, and the following statements [Enunciata] are to be noted about them.

1. A sine is a straight line drawn from the end of an arc of a quadrant of a circle, perpendicular to the base of that same quadrant and parallel to the other leg [cruri] of the quadrant; this is the very definition of sines. Therefore:
2. Sines stand on the base of the quadrant of a circle at right angles on both sides.

[14] The worked example of the extraction of a root heads the page without a caption of its own; it continues the "Exemplum Extractionis Radicis" of the preceding page. It is set as rows of figures in four number positions, read here left to right; the second and third positions have a tens place and a units place, each given its own column below, and the first and fourth positions hold a single figure. A figure shown as ~~x~~ is printed struck through by a short horizontal stroke. The top row of figures to be rooted carries no label; above it stand small marks: a grave-like stroke "`" over the first position (slanting the opposite way from the strokes "ı"), "0" over the second, "ı" over the third and "11" over the fourth. Above the "0" and the "ı" stand two further struck single figures. Below the first rule the root found is printed without strokes, with "0" and "ı" above it. Below it, and below a second rule, stand struck figures and labelled rows; a third rule separates the last two rows. In the last row a rule runs from the struck "ı" to the struck "4". The page prints no operator sign.

3. All sines are parallel to one another, because they are perpendicular to the same base. For lines perpendicular to the same line are parallel to one another, and conversely.
4. Parallel straight lines that connect parallels are equal to one another.
5. As the subtense [ſubtenſa] is to some whole arc of a circle, so is the sine, or half of the subtense, to half that arc of the same circle.

So much for sines and their canon.

The Construction of the Canon Itself Follows [Sequitur ipſius Canonis extructio.]

The construction of the canon of sines is carried out either in discrete quantity, as in numbers, or in continuous quantity, as in straight lines.

In numbers, further, it is done either in the common way and manner — namely by inscribing in a circle the sides of regular plane figures [laterum planorum ordinatorum] (that is, equilateral and equiangular ones), and that geometrically — or by a certain special method — namely by dividing a right angle, in a given ratio, into any number of parts you please, and that arithmetically. This is the discovery of the Swiss Iustus Byrgius [Jost Bürgi], craftsman [artificis] to the Most Illustrious Prince of Hesse.

In straight lines, on the other hand, the canon is constructed by the quadrature of the circle [Quadrationem Circuli].

This is the discovery of Simon à Quercu [Simon van der Eycke], a Burgundian, citizen of Delft.

Of these three different methods of constructing the canon just mentioned, then, we shall speak in order and as briefly as possible. First, therefore, on the method of constructing the canon by geometrical and infallible inscription.

The regular plane figures [plana ordinata] to be inscribed in the circle, by whose sides the seven primary sines are first investigated, are these seven:

1. Hexagon. 2. Tetragon. 3. Trigon. 4. Decagon. 5. Pentagon. 6. Pentadecagon. 7. Dodecagon. The ratio of the sides of these regular figures to the radius of the circle circumscribed about them is as follows in these statements [Enunciatis]:
2. The radius of the circle equals the side of the inscribed hexagon.
3. The square of the side of the inscribed tetragon is double the square of the radius;
4. while that of the trigon is triple.
5. The square of the radius and the square of the semi-radius, taken together, equal the square of the straight line composed of the semi-radius and the side of the inscribed decagon.
6. The square of the radius and the square of the said side of the inscribed decagon, taken together, equal the square of the side of the inscribed pentagon.
7. The straight line subtended between the bases of a trigon and a pentagon inscribed at the same point — the bases opposite that point — is the side of the inscribed pentadecagon.

8. The straight line subtended between the parallel bases of an inscribed trigon and hexagon is the side of the inscribed dodecagon.

All of these have been demonstrated soundly and evidently on the foundation of geometrical certainty, as by Euclid himself in the Geometrical Elements.

Now therefore let the radius of the circle circumscribed about all the said regular plane figures be assumed, at pleasure, of as many parts as possible — for instance a thousand times a thousand thousand, 1,000,000,000. This being so posited, the side inscribed in that same circle, in the semicircle, will be as follows.

Side of the inscribed	which subtends an arc of degrees	Subtenſes [Subtenſæ]
Hexagon	60° [LX.]	1,000,000,000.
Tetragon	90° [XC.]	1,414,213,562.
Trigon	120° [CXX.]	1,732,050,808.
Decagon	36° [XXXVI.]	618,033,988.
Pentagon	72° [LXXII.]	1,175,570,504.
Pentadecagon	24° [XXIIII.]	415,823,382.
Dodecagon	30° [XXX.]	517,638,090.

[Layout as printed: the seven figure-names (in the genitive, "Hexagoni … Dodecagoni.") stand in a column at the left, joined by one large brace to the words "quod ſubtendit arcum graduum" ("which subtends an arc of degrees"); a second brace opens from those words to the seven arcs, set in Roman numerals. A printed dash-rule runs from each arc to its value; the values stand under the heading "Subtenſæ." at the right of the line above the outline.]

Therefore, in the quadrant:

Arc of degrees [Arcus Graduum]	Halves, or sines [Semiſſes ſeu ſinus]
30° [XXX.]	500,000,000.
45° [XLV.]	707,106,781.
60° [LX.]	866,025,404.
18° [XVIII.]	309,016,994.
36° [XXXVI.]	587,785,252.
12° [XII.]	207,911,691.
15° [XV.]	258,819,045.

[Layout as printed: "Ergo in quadrante." stands as the heading over the arc column and "Semiſſes ſeu ſinus." over the value column; the words "Arcus Graduum" stand at the left, joined by a brace to the seven arcs, each arc followed by a printed dash-rule to its value. The lower part of the page below the outline is blank apart from the signature.]

Diagram of the Inscription, dedicated to D. D. Laurentius Tuppius. [Diagramma Inſcriptionis. D. D. Laurentio Tuppio ſacrum.]

Diagram. A large circle occupying the upper two-thirds of the page. A vertical radius runs from the centre up to the top of the circle and a horizontal

radius from the centre to its right-hand end; both are drawn as hatched double rules, and the quadrant arc between their ends (the upper right quarter of the circle) is likewise drawn as a hatched band. Numbers stand outside that arc at the points where chords and vertical lines meet it. Within the circle are many straight chords, some solid and some dotted: a fan of lines runs from the left-hand point of the circle, a number of vertical lines are dropped across the circle from points on the arc, and further chords cross the lower half. The upright radius carries the word "RADIVS." on its left side and a figure on its right side, both set reading upward.

- A (top of the circle, at the upper end of the upright radius): "A", with "90." beside it
- 60. (on the quadrant arc, upper right)
- 45. (on the arc, below 60.)
- 36. and 30. (two figures together on the arc, below 45.)
- 18., 15. and 12. (three figures together on the arc, below 30., near its lower end)
- B (right-hand end of the horizontal radius)
- C (at the centre, at the foot of the upright radius)
- RADIVS. (along the upright radius, left side, reading upward): "Radius."
- 1,000,000,000. (along the upright radius, right side, reading upward)

[No other letter or figure stands on the diagram: the left-hand point of the circle, the foot of the circle and the upper left quarter were examined and carry none.]

And these seven sines are called primary. From so small a beginning the whole company of all the remaining sines in the quadrant of the circle is afterwards derived and investigated — through 90 degrees, that is, and in each degree through 60 minutes, making up the number 5400. This is done by means of these statements [Enunciata]:

1. In a right-angled triangle, the square on the side subtending the right angle equals the squares on the two remaining sides. Or the same more clearly: in a right-angled triangle, the square described on the base subtending the right angle equals the squares described on the legs [cruribus] enclosing the right angle; and conversely. The penultimate [proposition] of Book 1 of Euclid [penultima 1. Euclidis]. Hence:
2. The radius of the circle is equal in square [poteſt] to the sine of an arc and the sine of the complement of the same arc. And therefore, if the square of the sine of any arc is subtracted from the square of the radius, what remains is the square of the sine of the complement of the same arc.
3. When the sine of the complement of any arc is known, there will at the same time be known also the complement of that same sine of the complement — that is, its shortfall up to the radius — which they call the versed sine [ſinum verſum]. For when the sine of the complement of any arc is subtracted from the radius, what remains is the complement of that same sine of the complement, or the versed sine.

4. The straight line subtended between the right sine and the versed sine [ſinum Rectum & Verſum] of any arc is double the sine of the arc that is half the said arc; and the sines themselves — the right, namely, and the versed — are together equal in square to that same subtense. Therefore, once the sine of any arc is known, the sine of the arc that is half that arc will also become known from this. Conversely:
5. As the radius is to the sine of any arc, so is the sine of the complement of the same arc to half the sine of the arc that is double that arc. And from this, once the sine of any arc is had, the sine of the arc double that arc will likewise be had as well.
6. The difference of two sines and the difference of the complements of those same sines [differentia eorundem ſinuum complementorum] are together equal in square to the subtense of the arc contained between the said sines. And from this the side of the inscribed pentadecagon and of the dodecagon will become known in numbers.

Diagram of the Shortcuts. Dedicated to Cunradus Dasypodius [Konrad Dasypodius], son of my teacher. [Diagramma Compendiorum. Cunrado Daſypodio, mei præceptoris filio ſacratum.]

(See the diagram on the reverse page.)

Now, in turn, all the remaining sines are investigated by these shortcuts [Compendia]: those of the complements and of the half-arcs, and again those of the halves of the complements and of the complements of the halves. This is done continuously, by alternate turns and by mutual alteration and interchange of the shortcuts, until at last one has come down to minutes of an odd number. And so, by continual reiteration, the sines of each and every arc lying 45′ apart from one another may be investigated through these few statements [Enunciata], as appears from the adjoined table, in which the upper derived number signifies the half-arc, and the lower one the complement of that half.

Diagram. The "diagram of the shortcuts" announced on the previous page, filling the upper two-thirds of the page under the running header; it carries letters only, no words or figures. A quadrant of a circle has its centre at C (lower left), an upright radius CB and a horizontal radius running from C along a baseline to A; the solid arc runs from B through L and D down to A. Within the quadrant:

- a solid horizontal from M on the upright radius to L on the arc, and another from N on the upright radius through V to D on the arc;
- a solid perpendicular from L straight down through V and T to O on the baseline, and another from D straight down to I on the baseline;
- solid straight lines from C to L, and from C through R to D; a solid line from L down through R to A; a solid chord from L to D;
- between D and A, besides the arc, the straight chord from D to A, which runs close beside the line from L through R to A;
- a dotted horizontal from T to R and a dotted perpendicular from R down to K on the baseline;

- a dotted semicircle bulging to the left, from L round through the region of N down to the baseline near O; and a dotted arc from R down to the baseline at I.

Outside the quadrant, three dotted figures: a quadrilateral on CD, with its far corners P (left) and Q (top), standing above and to the left of the quadrant; a quadrilateral on DA, with corners E (right of D) and F (right of A); and a quadrilateral on CI below the baseline, with corners G (below C) and H (below I).

Letters printed on the figure, as read:

- B (top of the upright radius)
- M (upright radius, below B; swash form)
- N (upright radius, about mid-height; swash form)
- C (centre, lower left)
- L (on the arc, upper)
- V (above the line N–D, left of the perpendicular from L)
- T (left of the perpendicular from L, at the dotted line to R)
- R (where C–D crosses L–A)
- D (on the arc, right)
- E (right, at the end of the dotted line from D)
- O (baseline, at the foot of the perpendicular from L)
- K (baseline, at the foot of the dotted perpendicular from R)
- I (baseline, at the foot of the perpendicular from D; a small sort)
- A (end of the horizontal radius; swash form)
- F (right, at the end of the dotted line from A)
- G (below C, at the lower left corner of the dotted figure)
- H (below I, at the lower right corner of the dotted figure)
- P (left corner of the upper dotted figure)
- Q (top corner of the upper dotted figure)

[No letter S was found on the figure. A short separate dash stands above the I; it is not read as part of the letter. A ring-shaped mark at the upper left, well away from the figure, is not a letter of the figure.]

The Former Table of Numbers, Marked A. [Tabula numerorum prior, A signata.]

Now the sines of the arcs of three-quarters of one degree and of one degree and a half — that is, of 45′ and 90′ — have been found by the preceding shortcuts and from the preceding table. Next, between them, the sines of the arcs of one degree and of one degree and a quarter — that is, of 60′ and 75′ — are investigated by another, special shortcut. Afterwards, from those once found, the sines of the arcs of 2°, 4°, 8°, 16°, 32°, 64° are found from the former, and from the latter the sines of the arcs of 2½°, 5°, 10°, 20°, 40°, 80°, by the fifth statement of the shortcuts [per quintum Enunciatum compendiorum]. And finally, from the sines of the arcs of 64° and 80° in turn, all the remaining sines are investigated — those of all the arcs lying apart from one another by every

fifteen minutes, that is, by a quarter of a degree — and this according to this second table of numbers, marked B.

The second table of numbers, marked B. [Tabula numerorum poſterior B notata.]

And in this way all the sines in the canon lying fifteen minutes apart will be had. However, that special shortcut which was mentioned just now, and likewise how all the remaining sines between each fifteen minutes are investigated — and that degree by degree and minute by minute — will be taught both by the section of the angle [ſectio anguli] that follows shortly, in its own place and time, and by us, by the common method, in our *Astronomy*. Since this is its foundation, or rather its rudiment and abridgement [compendium], it would take too long to go through everything here. Meanwhile, by way of a corollary and a crowning flourish [coronis], we shall add here another and truly golden shortcut, whose proof [Apodixis], cause and demonstration we shall set out in that same *Astronomy* of ours.

Postulate [Poſtulatum.]

When the sines of any two of the thirds [trientes] in the canon have been found, to investigate the sines of the remaining third by the sole method of addition and subtraction. For from the known sines of the first two thirds, the sines of the last third become known by addition; and from the sines of either of the remaining pairs of thirds, the sines of the remaining third become known by subtraction. On this matter, take these three examples.

I. The first example, and in the first third. [I. Exemplum primum, inque triente primo.]

Take the number of the arc whose sine you wish to find, both above and below 60, and subtract the sine of the smaller number from the sine of the larger; what remains is the required sine of the assumed number, or arc. For example, I wish to find the sine of the arc of 27°. I therefore take 27 below and above 60, and the results will be 33° and 87°. Now, therefore, when the sine of the arc of 33° has been subtracted from the sine of the arc of 87°, what remains is the sine of the required arc of 27°.

II. The second example, and in the middle third. [II. Exemplum ſecundum, inque triente medio.]

Take the shortfall [defectus] below 60 of the number of the arc whose sine you are seeking, and take it above 60. Then subtract the sine of the number of the shortfall from the sine of the arc of the said number above 60, and what remains is the desired sine of the arc of the assumed number. For example, intending to find the sine of the arc of 54°, I take the shortfall of the number 54 up to 60, and this shortfall is 6. Now when the sine of 6°, the shortfall, has been subtracted from the sine of the number lying the same shortfall above 60 — namely from the sine of the arc of 66° — what remains is the sine of the required arc of 54°.

The third example, and in the last. [Exemplum tertium, inque vltimo.]

In the third example, that of the last third, let everything be done contrary to the second. That is, take the excess [exceſſus] above 60 of the number of the arc whose sine you are seeking, and take it below 60. Then add the sine of the number of the excess to the sine of the said number below 60, and the required sine of the number of the proposed arc will be gathered. For example, intending to find the sine of the arc of 66°, I add the sine of the number of the excess above 60, which is 6, to the sine of the number lying the same shortfall below 70[15], namely to 54. The sum gathered from both will show the required sine of the arc of 66°. And so much for this golden shortcut.[16]

- Moreover, [Cæterùm,]
  - the sines of 18° and 30° make up the sine of 54°. [ſinus gradum 18. & 30. conſtituunt ſinum graduum 54.]
  - the squares of the sines of [Quadrata è ſinibus graduum]
    - 18° and
      - 30°
      - 54°
    - 30° and 54°
    - make up the square of the sine of [conſtituunt quadratum è ſinibus graduum]
      - 36°
      - 60°
      - 72°

For its proof, see the *Astronomy*. [Apodixin vide in Aſtronomiâ.]

The sine of one minute — that is, of the first minute — in the canon is 290,888. Therefore that of the penultimate minute, or of the complement, will be 1,999,999,957.[17] Radius. 1,000,000,000.

Difference [Differentia]: 43.[18]

So much, then, for the former way of constructing the canon — or of building it up — in numbers, namely by inscription, and geometrically. But there now follows another method, a special and far easier one, of founding the canon, also in numbers: namely by the section, or division, of a right angle in a given ratio into any number of parts, and that arithmetically.

---

[15] so printed; the rule just stated takes the number below 60, and the number named next, 54, lies 6 below 60

[16] The following is set as a bracketed outline. A tall brace opens after "Cæterùm," and joins two branches. In the second branch a brace after "Quadrata è ſinibus graduum" opens on two rows: the first "18 &" followed by a smaller brace on "30." and "54.", the second "30. & 54."; a single bracket closes these on the right, and from its point the line runs on to "conſtituunt quadratum è ſinibus graduum", which opens a brace on "36.", "60.", "72.".

[17] so printed, with ten figures; a sine cannot exceed the radius printed on the next line, and the difference printed beside the two lines, 43, is the difference between 1,000,000,000 and 999,999,957

[18] In the original the two values 1,999,999,957 and 1,000,000,000 stand one above the other, and a pair of braces to their right joins them to "Differentia" (level with the first) and "43." (level with the second).

On the Section of an Angle in a Given Ratio [De ſectione Anguli datâ ratione.]

The section of an angle in a given ratio is either into two parts only — and this is either astronomical or geometrical — or into any number of parts, and this is arithmetical. The method of cutting an angle is therefore threefold in all: the astronomical, namely, and the geometrical, each of them into two parts only; and the arithmetical, and this into any number of parts.

I. The astronomical, then, is the one made with the help of sines, into two unequal parts according to a given ratio of the legs [crura]; we shall mention this section in its own place, for solving the last spherical triangle.

II. The geometrical is the one made without sines, into two unequal parts according to a given ratio of the legs; on this section and its demonstration, consult the third proposition of the sixth book of Euclid.

III. Finally, the arithmetical section of an angle, according to a given ratio and into any number of parts, is the investigation of the ratios that the segments of a straight line standing perpendicularly on one leg of the angle have to one another — the segments, that is, cut off by certain straight lines drawn across, if the angle itself is divided into certain equal parts by those lines drawn across and dividing it. This section is made first in the right angle, and afterwards, from the right angle itself, in any oblique angle, whether acute or obtuse. In the right angle, then, first of all, this arithmetical section into any number of parts is made by the following invention of Byrgius [Byrgianum inuentum], for which we have composed a statement [Enunciatum] as follows.[19]

„ Having set down at pleasure as many numbers as there are parts into which the right angle is to be cut, in such a way that obuerſum 5. [?] is set down as the difference between the last and the penultimate portion; and that same difference, added to the number set down immediately before it, is set down as the difference between the penultimate and the antepenultimate portion; and so on in order, up to the first number set down; and with the operation itself repeated several times in the same manner: the last, or highest, difference will be the first, or highest, portion of the divided angle; and that same portion, added to the next following difference, will be the second portion. And so on in order down to the lowest difference. And the more often the same operation is repeated, the more exactly the angle will be divided, until at last the ratio of the portions remains almost unvarying.

Diagram of the section of an angle. Hung up, in gratitude, to Iustus Byrgius [Jost Bürgi], my teacher and the inventor of this device.

Diagram. A quadrant of a circle with its centre at the lower left. A vertical radius rises from the centre to the top of the arc, and a horizontal radius runs from the centre to the right end of the arc; the arc joins their ends. From the centre five further straight lines run out to points on the arc; taken from the vertical side downwards they are dotted, solid, dotted, solid, solid. From

[19] Each line of the following statement, from "Poſitis" to "eadem operatio reiteratur", carries a double comma mark („) in the left margin; the last line, "inuariabilis ferè permanebit.", carries none.

the point where each of them meets the arc a perpendicular is dropped to the horizontal radius, dotted for the two dotted lines and solid for the three solid ones. The solid line second from the bottom shows a short break in its stroke about halfway out. To the right of the quadrant stands the block of dots set out below, with the label "I. B." above its left-hand column.

Letters printed on the figure, as read (crop p25quad, native):

- B (top left, above the upper end of the vertical radius)
- C (bottom left, below the centre)
- A (bottom right, at the end of the horizontal radius, beside the foot of the dot block)
- "I. B." (above the left-hand column of the dot block; the "I" carries a dot above it, and the "B." is followed by a small four-pointed diamond mark)

No other letters were seen on the figure. The points where the radii meet the arc, and the feet of the perpendiculars, carry no letters.

[Dot block beside the figure: seven ruled columns of boxed cells, no column heads, no figures. Each cell holds a row of dots only. The columns stand side by side and are staggered: columns 1, 3, 5 and 7 (counting from the left) each have 10 cells; columns 2, 4 and 6 each have 9 cells, set half a cell lower at the top and ending half a cell higher at the foot, so that their cells straddle the cell boundaries of their neighbours. The table below keeps the page's cell contents column by column; its row alignment for columns 2, 4 and 6 is not the page's, and their tenth row is empty because those columns have only nine cells. The column numbers in the header row are editorial.]

[col. 1]	[col. 2]	[col. 3]	[col. 4]	[col. 5]	[col. 6]	[col. 7]
•••••••	•••••••	•••••	••••	•••	••	•
•••••••	•••••••	•••••	••••	•••	••	•
•••••••	•••••••	•••••	••••	•••	••	•
•••••••	•••••••	•••••	••••	•••	••	•
•••••••	•••••••	•••••	••••	•••	••	•
•••••••	•••••••	•••••	••••	•••	••	•
•••••••	•••••••	•••••	••••	•••	••	•
•••••••	•••••••	•••••	••••	•••	••	•
•••••••	•••••••	•••••	••••	•••	••	•
•••••••		•••••		•••		•

And by this golden device the whole canon of sines can be constructed and completed with the greatest ease, in the space of two days, or three, or at most four — something that by laborious and tedious inscription could scarcely, or rather not even scarcely, be done, though so many years were spent on it. So much, then, for both methods of constructing the canon in numbers. The canon itself in numbers, however — both in common numbers, by inscription, and in logistic numbers, by the section of the angle — and in addition the proofs [Apodixes], causes and solid demonstrations of all matters, both those stated so far and those to be stated later, as well as the most evident reasons,

and besides everything that must necessarily be inserted into what has been or will be said: all of this, I say, we shall present (unless the funds fail us) in our great and royal *Astronomy*. In the meantime, however, there follows a still more evident method of constructing the sines in straight lines, namely through the quadrature of the circle [Quadratio circuli]. Let us therefore set out first the quadrature itself, both general and special, or circular, together with the general demonstration of the circular one.

On the Quadrature of the Circle [De Quadratione circuli.]

Quadrature [Quadratio] is the reduction of some plane figure to a square, the quantity remaining the same. And it is:

- of a rectilinear figure — this was known before, from Euclid — and it is of a plane figure that is
  - simple, as of
    - a triangle;
    - a quadrangle, either
      - right-angled, or
      - oblique-angled;
  - compound, as of a polygon [multangulum].
- of a curvilinear figure, as of the circle; and this was
  - hitherto
    - entirely unknown;
    - by many
      - desired with the deepest longing;
      - attempted in vain;
  - but now, through
    - Simon à Quercu,
      - discovered;
      - published — I. / II. — in the year 1584 / 1586.[20]
    - me,
      - reduced to
        - a compendium;
        - a method;
      - translated into
        - Latin
        - German
        - language, from the Dutch.[21]

In the quadrature of the circle, then, we shall have:

- the proposition, or the element [Elementum] of quadrature itself — I.

[20] here a brace joins the two items "I." and "II.", a closing brace joins them to "an-no" (in the year), and a further brace joins that to "1584" and "1586.", set one above the other in the same order.

[21] the words "Idioma, ex Belgico" (language, from the Dutch) stand to the right of "Latinum" and "Germanicum", joined to both of them by a single brace.

- the explanation of the element, by
  - a declaration — II.
  - a demonstration — III.[22]

I. Proposition, or Element, of the Quadrature of the Circle [I. Propoſitio ſeu Elementum Quadrationis circuli.]

If a straight line inscribed in a circle from one end of a diameter is continued through the circumference, outside the circle, to the tangent erected perpendicularly at the other end of the diameter towards the same side, until it cuts off from that tangent a segment equal to itself, then the inscribed line itself, or the segment cut off equal to it, will be equal to the quadrant of the circumference of the circle.

Diagram of the Quadrature. Dedicated to Simon à Quercu, the discoverer of this divine device. [Diagramma Quadrationis. Simoni à Quercu, inuentori huius Diuini artificij conſecratum.]

Diagram. A circle with a solid vertical diameter, A at its top and B at its foot; four further points are lettered down the left side of that diameter, L, K, I and M in order from the top. From A a solid horizontal line runs to the right to E, and a dotted line continues it from E to P. From E a solid vertical line descends to D on the bottom line. A second solid horizontal line runs from L across to N, on the vertical through E, and on to F; from F a solid vertical line descends to G. The bottom line is solid and runs from B through D and G to O. A solid straight line, with a dotted line set close beside it, runs from A down to the right, crosses the circumference at C, and ends at D. H stands on the circumference at the left; from H a dotted line runs horizontally through K across the circle, a dotted arc rises from H and runs over towards the right, and a dotted line runs down from H to B. A dotted line runs from I across to C, and a dotted line runs out from M to the right. Further dotted lines rise from B to the right, cross one another between C and the vertical ED, and one of them reaches N; a dotted line joins N to P. Two dotted arcs come down on the right, one ending at D and the other at G.

- A (top of the diameter; a dot stands above the letter): A
- E (upper right corner of the rectangle): E
- P (right end of the dotted continuation of the top line): P
- L (on the diameter, below A): L
- N (where the line from L meets the vertical through E): N
- F (right end of the line from L): F
- H (on the circumference, left): H
- K (on the diameter, on the dotted horizontal from H): K
- I (on the diameter, below K; dotted I): I
- M (on the diameter, below I): M

[22] in both outlines on this page the numbering is set in italic sorts printed "I.", "II.", "III."; the first item of the second outline, the proposition, matches the section headed "I. Propoſitio…" immediately below.

- C (on the circumference, right, where the line from A crosses it): C
- B (foot of the diameter): B
- D (on the bottom line, foot of the vertical from E): D
- G (on the bottom line, foot of the vertical from F): G
- O (right end of the bottom line; weakly inked)

II. Declaration of the Element, or Proposition [II. Elementi ſeu propoſitionis Declaratio.]

I say that in the accompanying diagram the straight line AC — inscribed in the circle ACBH from A, one end of the diameter AB, and continued through the circumference ACB, outside the circle, as far as the tangent BO, which is drawn from B, the other end of the diameter AB, towards the same side and perpendicularly, up to the point D — or the segment BD, made equal to it by the construction, is equal to the quadrant, or fourth part, of the circumference of the circle. And therefore also, by the mensuration of the circle [per Geodæſian circuli], either the oblong rectangle ABDE, or the square BGFL, equal to that same rectangle by proposition 43 of the first book of Euclid, is equal to the proposed circle ACBH itself.

III. Demonstration of the Same Element [III. Eiuſdem Elementi Demonſtratio.]

It is for us to demonstrate that the quadrant of the circumference of the circle is neither greater nor less than the inscribed straight line AC, or than the segment BD cut off equal to it. Once that is done, it will necessarily be equal to it. This appears clearly and most evidently from the following principles, as well as from the demonstrated elements.

Common Principles. 8. [Principia Communia. 8.]

1. Every whole is equal to all its parts taken together. Therefore every whole is greater than its part; and conversely, every part is less than its whole.
2. As a whole is to another whole, so the homologous part, or the homologous parts, of that whole are to the homologous part, or the homologous parts, of this whole; and conversely. (Commonly: as whole to whole, so part to part.)
3. Whatever is greater than the one that is greater is also greater than the one that is less; and conversely, what is less than the one that is less is also much less than the one that is greater.
4. Any thing whatever is most like itself, and most of all equal to itself.
5. Things that are equal to the same thing, that is, to one third thing, are equal to each other.
6. Nothing greater than the greatest can be given; nor can anything less than the least be given.
7. Whatever can be conceived in the mind to have something greater and something less than it can also be conceived to have something equal to it.

8. Among all quantities there is some ratio of one to another.

Proper Principles. 3. [Principia propria 3.]

1. Between a circle and the greatest regular plane figure [planum ordinatum] inscribed in it, no greater regular plane figure can be given. Conversely, between a circle and the least regular plane figure circumscribed about it, no lesser regular plane figure can be given. And this follows from the sixth common principle, which immediately precedes.
2. The quantity of any figure is produced from its dimensions multiplied into one another. Therefore, from greater dimensions multiplied into one another — whether both together are greater, or only one of the two — a greater quantity of the figure is produced; and from lesser ones, or from only one of the two being less, a lesser. And this holds both by the common mensuration received in practice [Geodæſian] and by the common notion of the mind.
3. A plane rectangle described from the height and half the base, or from the whole base and half the height, both of a triangle and of a sector of a circle, is equal: the former to the triangle itself, the latter to the sector itself.

For brevity's sake, let this stand here as a principle of a kind belonging to mensuration [quaſi Geodæticum], and let it be granted to us, as something demonstrated by the most evident reasoning both by Archimedes himself and by the author or discoverer of the quadrature himself, in his first proposition (which we have left out here for brevity) — indeed by almost all mathematicians, even many centuries before our quadrator — and also received in practice, most constantly and unanimously, through the common and customary mensuration, by the unanimous consent of all surveyors [Geodætarum]. So far, then, the principles, both common and proper, or common notions of the mind. From them there follow the elements that are to be demonstrated and that serve towards the demonstration of the quadrature.

Elements. 5. [Elementa 5.]

I. A straight line subtending an arc, or portion, of a circle is less than the arc it subtends. But a straight line that touches an arc and is enclosed between the radii of the circle to which the arc belongs, continued indefinitely through the ends of the arc, is greater than the arc it touches.

Declaration. I say that in the accompanying diagram the straight line AB, subtending the arc ADB, will be less than the arc ADB itself. On the contrary, the straight line MN, touching the arc ADB at the point D and enclosed between the radii CA and CB of the circle, continued outwards indefinitely through A and B, the ends of that arc, is greater than the arc ADB that it touches.

First Diagram of the Demonstration. Dedicated to David Wolkenstein, table-companion. [Diagramma Demonſtrationis prius. Dauidi Vuolkenſtenio commenſali dedicatum,]

Diagram. A geometrical figure occupying the upper half of the page. A large triangle has its apex O at the top and its base along a horizontal line, with M at the bottom left and N at the bottom right. A vertical line drops from O through F, i, C and E to D at the middle of the base. A circle stands on the base, touching it at D and touching at F the top side of a square drawn about it; the square's corners are lettered Q (top left) and P (top right), and the feet of its sides on the base line R (left) and S (right). The circle is marked at G and H (left and right, joined by a horizontal chord through i), at K and I/J (left and right, joined by a horizontal through C, the centre), and at A and B (lower left and right, joined by a horizontal chord through E). Solid straight lines fan out from F and from D to points on the circle, forming a lattice of crossing chords inside it. Dotted lines run from the corner Q and from the corner P down through the centre region; X stands on the dotted line from Q, near the arc on the upper left, and Y stands on the dotted line from P, near the arc on the upper right. Dotted lines also run from M up through A and from N up through B towards the centre. Between A and D, and between D and B, the arc is shadowed by a dotted chord, lettered T on the left and V on the right.

- O (apex, top centre)
- Q (top-left corner of the square; a swash capital)
- F (top of the circle, on the vertical)
- P (top-right corner of the square)
- X (upper left, on the dotted line from Q)
- Y (upper right, on the dotted line from P; printed with curled arms, in the form Y)
- i (on the vertical, at the chord GH; a dotted letter)
- G (left end of the upper horizontal chord)
- H (right end of the upper horizontal chord)
- K (left, on the circle, at the level of the centre)
- I or J (right, on the circle, at the level of the centre; an upright with a foot turned to the left at the baseline — the form does not separate I from J) [?]
- C (centre, on the vertical; the sort is C-shaped with a descending hook below, not a plain roman C) [?]
- A (lower left, on the circle; a swash capital)
- E (on the vertical, at the chord AB)
- B (lower right, on the circle)
- T (on the lower-left arc, between A and D)
- V (on the lower-right arc, between D and B)
- M (bottom left, base corner of the triangle)
- R (base, foot of the square's left side)
- D (base, point of contact of the circle)

- S (base, foot of the square's right side)
- N (bottom right, base corner of the triangle)

All twenty-two letters listed above were read on the figure; none was supplied from the demonstration. The form of the letter at the centre and of the letter on the right at the level of the centre is uncertain.

Demonstration. The first part of the Element is established by the second proposition of the third book [per 2. tertij], by the twenty-second of the first [per 22. primi], and by our third common principle. The second part is shown as follows. The triangle CMN is greater than the sector CADB enclosed within it, by the first common principle. (For, by the evidence of superposition [ἐφαρμόσεως; Greek: epharmoseōs, "of fitting-together" — the superposition of one figure on another], the triangle CMN itself exceeds the said sector CADB by the quantity of the two triangles AMD and BND.) But one dimension of each of them — namely the height CD — is equal, by the fourth common principle. Therefore the remaining dimension of each, namely the base MN of the triangle, will be greater than the base ADB of the sector, by the second and third proper principles.

Some Consequences from This [Hinc quædam Confectaria.]

1. The quantity of any arc lies between the quantity of the straight line subtending it and the quantity of the straight line that touches it and is enclosed between the radii of the circle of which the arc is a portion, continued indefinitely through the ends of the arc. And this follows from what has just been demonstrated. Therefore:
2. The whole perimeter of a regular rectilinear figure [rectilinei ordinati] inscribed in a circle is less than the circumference of the circle; but the whole perimeter of a regular rectilinear figure circumscribed about a circle is greater than the circumference of the circle. And conversely, the circumference of the circle is greater than the perimeter of the regular figure inscribed in the circle, but less than the peri- -meter of the regular figure circumscribed about the circle. All these follow by the converse [ἀντίστροφον; Greek: antistrophon, "converse"], or conversion, of the second common principle. Therefore:
3. The perimeter of a regular figure circumscribed about a circle is much greater than the perimeter of a regular figure inscribed in the circle; and conversely, the perimeter of a regular figure inscribed in a circle is much less than the perimeter of a regular figure circumscribed about the circle. By the third common principle. Further:
4. The quantity of the whole circumference of the circle lies between the quantities of the perimeters of the regular figures inscribed in the circle and circumscribed about it. And this follows by the first of these consequences, and by the converse of the second common principle.
5. A quadrant of the circumference of a circle is greater than a quadrant of the perimeter of a regular figure inscribed in the circle, but less than a

quadrant of the perimeter of a regular figure circumscribed about the circle. And this follows by the second of these consequences and by the second common principle. And conversely:

6. A quadrant of the perimeter of any regular rectilinear figure inscribed in a circle is less than a quadrant of the circumference of the same circle; and conversely, a quadrant of the perimeter of any regular rectilinear figure circumscribed about a circle is greater than a quadrant of the circumference of the same circle.
7. Of triangles (as also of sectors) that are equal to one another and of equal height, the bases are equal; and of those that are equal to one another and stand on equal bases, the heights are equal. Likewise triangles (as also sectors) with equal bases and equal heights are equal to one another.

And so we have had the first Element: very fruitful, and overflowing most abundantly with the profit of great interest [fænoris]. The second follows.

II. A regular (or equilateral) triangle circumscribed about a circle is four times the regular triangle inscribed in it; and a square circumscribed about a circle is twice the inscribed square.

The declaration is evident from the diagram itself, and the demonstration is evident by the thirty-fourth proposition of the first book [per 34. primi]. We shall therefore be briefer here in declaring and demonstrating.

The Following Consequences Are Deduced from This [Hinc deducuntur ſequentia Conſectaria.]

1. The homologous sides, and therefore also the dimensions, of a regular triangle circumscribed about a circle are double the homologous sides, or dimensions, of a regular triangle inscribed in the circle. And this follows by the nineteenth of the sixth book [per 19. ſexti].
2. But the dimensions of a triangle are the base and the height, or the perpendicular let fall from the vertex, or apex, of the triangle onto the base, or side, of the triangle opposite that vertex or apex. Therefore the base and the height, or perpendicular, of a regular triangle circumscribed about a circle are double the base and the height, or perpendicular, of a regular triangle inscribed in the circle.
3. But the perpendicular of a regular triangle inscribed in a circle is three-quarters [dodrans] of the circle's radius. And this follows by the consequence of the twelfth proposition of the thirteenth book [per conſectarium 12. decimi tertij]. Therefore the perpendicular of a regular triangle circumscribed about a circle is two three-quarters [duorum dodrantium] of the circle's radius, or stands in the sesquialteral ratio [ſeſquialterâ Ratione] — that is, as 3 to 2 — to the radius of the circle.

By reasoning of this kind, then, the ratio of the perpendicular of a regular triangle circumscribed about a circle to the diameter of the circle is now ascertained: the very ratio that was sought as necessary for the demonstration of the Element immediately following. So the Element itself follows.

III. The perimeter of a regular rectilinear figure inscribed in a circle — the fewer sides it has, the smaller it is; conversely, the more sides it has, the greater it is. But on the contrary, the perimeter of a regular rectilinear figure circumscribed about a circle is the greater, the fewer sides it has; conversely, the more sides it has, the smaller it is.

Declaration. I say that in the preceding diagram [præcedente Diagraphâ] the perimeter of the triangle AFB inscribed in the circle, as having fewer sides, will be less than the perimeter of the square FLDK inscribed in the circle, as having more sides; and on the contrary, that the perimeter of the triangle OMN circumscribed about the circle, as having fewer sides, is greater than the perimeter of the square PQRS circumscribed about the circle, as having more sides.

Demonstration. Our author, the discoverer of the quadrature [Quadrationis repertor], demonstrates this Element by a most evident method and geometrically, but at considerable length. For the sake of brevity, therefore, we shall try to demonstrate the same thing in numbers and arithmetically. For by the immediately preceding Element, and with the help of the penultimate proposition of the first book [penultimæ primi], the ratio in numbers of the sides of all regular figures, inscribed as well as circumscribed, is established. Therefore:

Let the radius be taken in numbers as 2; the diameter is therefore 4 (to which the sides PQ, QR, RS, SP of the circumscribed square are equal, by the thirty-third of the first book [per 33. primi]). Therefore the perpendicular FE is 3, and the perpendicular OD is 6; and this by the immediately preceding Element. The sides FK, KD, DL, LF of the inscribed square will be $\sqrt{8}$; the sides FA, AB, BF of the inscribed triangle will be $\sqrt{12}$; and finally the sides OM, MN, NO of the circumscribed triangle will be $\sqrt{48}$; and this by the penultimate proposition of the first book. From the ratio of these numbers the proposition is plainly evident. The same will also become clear and plain, in all the other regular polygons [multangulis ordinatis], inscribed and circumscribed, to anyone not ignorant of the canons [Canones] of Geber [Jābir ibn Aflaḥ]. And finally, by induction over all the species set down, what was intended in the species — that is, through the species and by gathering them together — will be established and concluded in the genus. And exactly the same is to be held of the quadrant of the perimeter of any regular figure, whether inscribed or circumscribed, or indeed of any other portion of the perimeter. By the second common principle.

IV. A straight line equal to a quadrant of the perimeter of a regular triangle circumscribed about a circle cannot be inscribed in the circle; a straight line equal to a quadrant of the perimeter of a square circumscribed about a circle cannot be inscribed in the circle except as the diameter; but a straight line equal to a quadrant of the perimeter of any remaining regular polygon circumscribed about a circle, and likewise a straight line equal to a quadrant of the circumference of the circle, can be inscribed in that circle, and other than as the diameter.

Declaration. The matter is wordy rather than obscure — indeed, it is plain in itself — and seems to need no declaration. Let us look, then, at the demonstration.

Demonstration. First, concerning the circumscribed square. The side, or quadrant of the perimeter (for here they are one and the same), of a square circumscribed about a circle is equal to the diameter of the circle, by the thirty-third of the first book [per 33. primi]. But the diameter itself is the longest of all the straight lines that can be inscribed in a circle, by the fifteenth of the third book [per 15. tertij]. Therefore the side, or quadrant of the perimeter, of a square circumscribed about a circle cannot be inscribed in the circle except as the diameter. Second, concerning the circumscribed regular triangle, it is clear as follows. The side, or quadrant of the perimeter, of a square circumscribed about a circle is the longest of all the straight lines that can be inscribed in the circle, by the fifteenth of the third book. But a straight line equal to a quadrant of the perimeter of a regular triangle circumscribed about a circle is greater than the side, or quadrant of the perimeter, of a square circumscribed about the circle, by the immediately preceding Element. Therefore the straight line just named cannot be inscribed in the circle. Third, concerning the remaining regular polygons, it is clear by the contrary reasoning: since a straight line equal to a quadrant of the perimeter of any remaining regular polygon circumscribed about a circle is less than the side, or quadrant of the perimeter, of a square circumscribed about the circle, by the immediately preceding Element. Therefore the straight line just named can be inscribed in the circle, by the fifteenth of the third book [per 15. tertij]. Fourth and last, concerning the straight line equal to a quadrant of the circumference of the circle, it is clear by the same reasoning. For that line is less than the side, or quadrant of the perimeter, of a square circumscribed about the circle (that is, by the thirty-third of the first book [per 33. primi], than the diameter of the circle itself), by the first of these Elements and its fifth and sixth consequences [eiuſq; 5. & 6. conſectarium]. Therefore a straight line equal to a quadrant of the circumference of the circle can also be inscribed in the circle itself. The proposition is therefore established in every case.

V. If a straight line equal to a quadrant of the perimeter of any regular polygon inscribed in a circle is inscribed in that circle from either end of the diameter, and is continued beyond the circumference of the circle until it cuts, at some point, the tangent drawn perpendicularly (that is, at right angles) from the other end of the diameter and produced indefinitely towards the same side, then the segment cut off from the tangent, enclosed [comprahenſum] between the diameter and the continued inscribed line, will be greater than the inscribed line itself. On the contrary, if a straight line equal to a quadrant of the perimeter of a regular polygon circumscribed about a circle, and homologous (that is, equiangular) to the said inscribed regular polygon, is inscribed in the circle from the same end of the diameter already mentioned and is continued through the circumference of the circle until it cuts the tangent already mentioned at

some point, then the segment just named, cut off from the tangent and enclosed between the diameter and the continued inscribed line just named, will be less than that inscribed line.

Second Diagram of the Demonstration. Offered to D. D. Thomas Fincke, compatriot. [Diagramma Demonſtrationis poſterius. D D. Thomæ Finckio populari oblatum.]

Diagram. A geometrical figure filling the lower half of the page below the caption. A circle with its centre lettered A carries an inscribed regular pentagon with vertices B (top), F (upper right), G (lower right), H (lower left) and i (upper left), and stands inside a circumscribed regular pentagon whose corners are lettered S (top), T (upper right), R (upper left) and Q (lower left); the lower right corner of the circumscribed pentagon falls on the long base line at V. A vertical line runs from B through the centre A to C, where the circle touches the base line. A horizontal line through A runs from D on the left of the circle to the right, where E stands above it and L below it, close to the circle. The base line runs from the corner Q through C and on to the right, carrying in order V, Z, P, X, N and M (at its right end). Several solid straight lines fan from B down to the right: one to F, one passing between E and L and on to N, and one to Z; the right side of the circumscribed pentagon runs from T down to V. A dotted arc runs from B round to the right through O to K. Dotted straight lines run from C up past L to O and from C up to K; a small letter in the form Y or Y stands below L, where the dotted line to K, the line from B to Z, the side TV and the circle cross. A dotted vertical drops from O through the base line at P to the end of a horizontal line below the figure, where a sort set sideways stands; a heavier dotted vertical drops from K through the base line at X to the end of a second, lower horizontal line, lettered Λ. A dotted vertical drops from C below the base line to these two horizontals, whose left ends are lettered Γ (upper) and Δ (lower).

- S (top, apex of the circumscribed pentagon)
- B (top of the circle, vertex of the inscribed pentagon)
- R (upper left corner of the circumscribed pentagon; a swash capital)
- i (upper left vertex of the inscribed pentagon; a dotted lower-case letter)
- T (upper right corner of the circumscribed pentagon)
- F (upper right vertex of the inscribed pentagon)
- O (on the dotted arc, upper right, outside the pentagon)
- E (right, just above the horizontal through A)
- L (right, just below the horizontal through A)
- D (left end of the horizontal through A, on the circle)
- A (centre; a swash capital)
- K (on the dotted arc, right; a swash capital)
- Y or Y (small curled-arm sort, below L, at a crossing of the dotted line from C to K) [?]
- H (lower left vertex of the inscribed pentagon)
- G (lower right vertex of the inscribed pentagon)

- C (bottom of the circle, on the base line)
- Q (lower left corner of the circumscribed pentagon, below the base line; a swash capital)
- V (base line, foot of the pentagon's lower right side)
- Z (base line)
- P (base line, at the dotted vertical from O)
- X (base line, at the dotted vertical from K)
- N (base line; a swash capital)
- M (right end of the base line; a swash capital)
- Γ (left end of the upper horizontal below the figure)
- Δ (left end of the lower horizontal below the figure)
- sideways sort at the right end of the upper horizontal, foot of the dotted vertical from O: M set on its side, top to the left
- Λ (right end of the lower horizontal, foot of the dotted vertical from K)

Twenty-seven labels are listed above, all read on the figure; none was supplied from the text. Two are uncertain in form: the small curled-arm sort below L, and the sideways sort at the end of the upper horizontal.

Declaration. I say, first, that the segment CN — cut off from the tangent CM (drawn perpendicularly from the end C of the diameter BC) by the inscribed line BL continued beyond the circle as BN, and enclosed between the diameter BC and the point of section CN [abſciſsionis puncto C N] — is greater than the said inscribed line BL. I say, second and conversely, that the segment CZ — cut off from the said tangent CM by the inscribed line BY continued beyond the circle as BZ, and enclosed between the diameter BC and the point of section Z — is less than the said inscribed line BY. There follows, therefore, the demonstration of each part of this Element.

Demonstration of the first part, in the regular pentagons ascribed to the circle [in ordinatis circulo adſcriptis quinquangulis]. Let BL be inscribed in the circle, equal to a quadrant of the perimeter of the regular pentagon inscribed in the circle. The straight line CL, which subtends LYC — the complement of the arc BFL up to the semicircle — is continued beyond the circle until it equals the diameter of the circle, as far as O. From the end of the continued line, that is, from the point O, the perpendicular OP is let fall onto the tangent CM lying below; by hypothesis it will meet that tangent perpendicularly, that is, making right angles on both sides, at the point P. This done, two homologous triangles, that is, triangles with equal angles, have been constructed in the accompanying diagram, namely BLC and CPO. For the angle of the triangle BLC at L is right, by the thirty-first of the third book [per 31. tertij]; and the angle of the triangle CPO at P is right by construction; they are therefore equal, by the tenth common notion of the first book [per 10. communem notionem 1.]. Further, the angle of the triangle BLC at C is equal to the angle of the triangle CPO at O, by the twenty-ninth of the first book [per 29. primi]. Therefore the remaining angles of the two triangles, at B and at C, are also equal, by the thirty-second of the first book [per 32. primi]. It is established, then, first, that

the two said triangles are homologous, that is, have equal angles; therefore their sides will be proportional, by the fourth of the sixth book [per 4. ſexti]. But the said line CO, continued beyond the circle, is by construction equal to the diameter of the circle BC, which is the base of the triangle BLC. Therefore the remaining two sides of the two triangles will also be equal, each to each; for all the sides, each to each, stand to one another in the ratio of equality. And so, among the remaining sides, the side of the triangle BLC, that is, the inscribed line BL, will be equal to the side CP of the triangle CPO. But the segment CN, cut off by BN continued beyond the circle, is greater than the side CP itself — the whole, that is, being greater than the part — by the first common principle [per 1. commune principium]. Which was to be demonstrated. In the same way, by the contrary reasoning, the second part of the Element will also be demonstrated, namely in the triangles BYC and CXK. Both parts of the proposition are therefore established.

Conclusion of the Demonstration [Concluſio Demonſtrationis.]

Now, just as this has been demonstrated in the regular pentagons ascribed to the circle, so too it can be demonstrated in all the remaining polygons that follow, and in any two homologous (that is, equiangular) regular polygons ascribed to the circle — either geometrically, in the way just described, in triangles with equal angles, or arithmetically, in algebraic or figurate numbers [in numeris Algebraicis ſeu figuratis] — without limit, up to even the largest inscribed and the smallest circumscribed polygon. Therefore, since a straight line equal to a quadrant of the perimeter of even the smallest regular circumscribed polygon is always greater, and conversely one equal to a quadrant of the perimeter of even the largest inscribed polygon is always less, than the segments cut off in the way described by the continued inscribed lines; and since, even between the very largest regular inscribed polygon of all and the smallest circumscribed one, the magnitude of the circumference of the circle itself always lies between the magnitudes of the perimeters of the largest inscribed and the smallest circumscribed polygon, by the fourth consequence of Element 1 [per 4. conſectarium Elementi 1.]; and since, therefore, the magnitude of the quadrant of the circumference itself also falls between the magnitudes of the quadrants of the perimeters of the largest inscribed and the smallest circumscribed regular polygon, by the first consequence of Element 1 and by the second common principle [per 1. conſectarium Elementi 1. perq; 2. commune principium]: therefore the straight line itself, equal to a quadrant of the circum- ference of the circle, will never be — even if, without limit, the inscribed polygon is posited ever larger or the circumscribed one assumed ever smaller — either greater than the said segment cut off from the tangent itself by that line continued beyond the circle to the tangent, or less than it. For if it were greater, the straight line would not be equal to a quadrant of the circumference of the circle, but would be equal to a quadrant of the perimeter of some regular polygon circumscribed about the circle; and if it were less, the

straight line would again not be equal to a quadrant of the circumference of the circle, but would be equal to a quadrant of the perimeter of some regular polygon inscribed in the circle, by the immediately preceding Element 5 [per proximè præcedens 5. Elementum]. Therefore, necessarily, if the straight line itself, inscribed in the circle, is neither greater nor less than, but exactly equal to, the segment cut off from the tangent itself by that line continued beyond the circle as far as the tangent, then the inscribed line itself will be equal to a quadrant of the circumference of the circle — or so, too, will the segment cut off from the tangent, which is equal to it — by the fifth common principle [per 5. commune principium]. Which was to be demonstrated.

The quadrature of the circle [Quadratio circuli] is thus at last established. But for the manifold and varied use of this most ingenious discovery, and the truly countless benefits that flow from it into all the mathematical arts, see our Astronomy [noſtrâ Aſtronomiâ]: for through this divine discovery and most ingenious device we can do entirely without sines in astronomical calculation. How, however, we shall teach in our Astronomy. For the present, therefore, we shall set the crowning stroke [coronidem] on the Doctrine of Sines [Doctrinæ ſinuum], and from here on we shall show and practise the use of that same most subtle doctrine in one more subtle still, namely in that abstruse and recondite Doctrine of Triangles.

End of the Construction of the Canon; the Doctrine of Triangles follows. [Finis Extructionis Canonis: ſequitur Doctrina Triangulorum.]

Chapter III. On the Doctrine of Triangles [Caput III, DE DOCTRINA TRIANGVLORVM]

The Doctrine of Triangles is a most ingenious method of solving triangles with the aid of sines. To solve triangles, moreover, is to find the remaining three from three of their parts that are given or known. For just as in the rule of proportion [Regula proportionis] a fourth, unknown number is found from three given or known numbers, so too in this doctrine, from three given or known parts, not just one unknown but the remaining three unknowns in the triangle proposed for solution can be found and determined. For in a triangle there are three sides and three angles.

Since, then, we are about to speak of triangles and of their solution, we shall speak in this order:

- On [DE]
  - the sphere [Sphærico.]
  - triangles [Triāgulis]
    - spherical [Sphæricis]
      - right-angled: through six examples [Rectangulis: per ſena Exempla.]
      - oblique-angled [Obliquangulis]
        - the usual ones [vſitatis]
          - first case [Primo. caſu]

- I.
- II.
- second case [Secundo, caſu]
  - I.
  - II.
- the others [De alijs]
  - in general [in genere]
  - in particular [in ſpecie]
    - on the ambiguous [de ambiguo]
      - first [primo.]
      - second [ſecundo.]
    - on the last [de vltimo.]

- plane [planis]
  - in general: all the angles being given [in genere. Datis omnibus angulis.]
  - in particular [in ſpecie]
    - on the right-angled [De Rectangulo]
      - which is twofold [quod eſt duplex]
        - prior [prius.]
        - posterior [poſterius.]
      - which is solved [quod ſoluitur]
        - geometrically [Geometricè.]
        - astronomically [Aſtronomicè.]
    - on the oblique-angled [De obliquangulo]
      - which is threefold [quod eſt triplex]
        - first [primum.]
        - second [ſecundum.]
        - last [vltimum.]
      - which is solved [quod ſoluitur]
        - without a perpendicular [abſque perpendiculari.]
        - with a perpendicular, whose investigation is [cum perpendiculari: cuius inueſtigatio.]
          - astronomical [Aſtronomica.]
          - geometrical [Geometrica.]

I. On the Sphere [I. De Sphærico.]

The sphere [Sphæricum] is a convex surface [ſuperficies gibba], equidistant on every side from the middle or centre of the space it encloses. The greatest circumference of the sphere is the one that bisects the sphere itself and divides it into two equal hemispheres: such are those circles depicted in the heavens. And for brevity's sake we shall call this greatest circumference of the sphere simply the Maxima [Maximam]. Accordingly, **1.** The centre of a Maxima is the same as the centre of the sphere; and so all Maximae bisect one another.

And one Maxima cutting another through its poles bisects it rightly. ("Rightly", that is, at spherical right angles.)

2. A Maxima is distant on every side from its pole by a quadrant of a Maxima. And so every quadrant drawn from the pole of any Maxima to that Maxima falls on it rightly, and therefore makes spherical right angles on either side upon it.

3. Any other circumference of the sphere that is nearer to the Maxima is greater than one more remote; and conversely, one more remote from it is smaller than one nearer. Two circumferences equidistant on either side from the Maxima, or the middle one of all, are equal to each other; and their ratio to one another is the ratio of their diameters, and the ratio of the diameters is the ratio of the sines.

4. Every circumference of the sphere, whether a Maxima or a smaller one, is divided into 360 equal parts or degrees [gradus]. And so all circumferences are homologous to one another in the number of their parts, even though the parts themselves are unequal to one another; and their ratio is the same as before. For the ratio of the parts or segments of the circumferences, that is, of the arcs, is the ratio of the whole circumferences, and the ratio of the angles at the pole and of the bases of those same angles on the Maxima that stands a quadrant away from that pole.

Diagram. A circle drawn as a sphere, set to the left of proposition 4. A horizontal diameter runs from A (left) to B (right), and a vertical diameter, solid in its upper half and dotted in its lower half, runs from the top of the circle (marked 90) down to D at the foot. Two arcs bowed above and below the horizontal diameter join A and B, forming a lens; the upper crosses the vertical diameter at E, the lower at F. Further arcs run from the top of the circle to D: one solid arc to the left of the vertical diameter and one dotted arc to its right. Near the left rim a group of lines, some solid and some dotted, runs between L (upper left, marked 45) and G (lower left), crossing the horizontal diameter at S and P; near the right rim a matching group runs between K (upper right, marked 45) and H (lower right), crossing the diameter at Y and R. Along the horizontal diameter, from left to right, stand the letters S, P, N, C, M, Y and R, at the points where the arcs and lines meet it.

- 90: numeral at the top of the circle, at the upper end of the vertical diameter
- 45: numeral at the upper left of the rim, beside L
- 45: numeral at the upper right of the rim, beside K, set at an angle along the rim
- I: just below the top point, on the vertical diameter
- E: on the vertical diameter, where the upper arc of the lens crosses it
- C: centre, on the horizontal diameter
- F: on the vertical diameter, where the lower arc of the lens crosses it
- D: foot of the circle
- A: left end of the horizontal diameter
- B: right end of the horizontal diameter

- L: upper left on the rim, below the numeral 45
- K: upper right on the rim, below the numeral 45
- G: lower left on the rim
- H: lower right on the rim
- S, P, N: on the horizontal diameter, left of C
- M, Y, R: on the horizontal diameter, right of C (the Y in its curled-arm form)

II. On Triangles in General [II. De Triangulis in genere.]

A triangle is a space enclosed by three lines (or comprehended or bounded by three lines), and that either by three straight lines, as a plane triangle, or by three circular lines or arcs of three Maximae intersecting one another, as a spherical triangle. Accordingly, a plane triangle is rectilinear [Rectilineum], comprehended by three straight lines; a spherical triangle, on the other hand, is a surface of the sphere bounded by three arcs or segments of three Maximae intersecting one another. Accordingly,

5. Segments or arcs of three Maximae, each singly less than a half-sphere [ſemiſpheriâ], intersecting one another, constitute a spherical triangle. And so there is no astronomical spherical triangle unless it is constituted in the manner stated. There can, however, be a geodetic one [Geodæticum] outside the stated condition. The measurement [Dimenſio] of a geodetic triangle consists in investigating the size of its area; but the solution of an astronomical triangle consists in investigating its sides and angles.

6. Straight lines in a triangle parallel to the base cut the legs in the ratio of those parallels, and conversely. Hence the sides of equiangular triangles are proportional. In spherical triangles, however, the sines subtending the arcs are to be taken from the Canon in place of the sides.

7. Any two opposite angles between straight lines cutting one another are equal. And since every Maxima, whether above in the heavens or below on the globe, appears to us as we look at it as a straight line, by 22 of Euclid's *Optics* [per 22. Optices Euclidis], this holds in spherical figures too.

8. The amplitude of an angle depends on [penas] [recte: penes] the arc subtended to it upon equal legs:[23]

- That of a spherical angle [Sphærici] depends on the arc of a Maxima which, described from the angle itself as from the pole of the said Maxima, subtends the angle itself.
- That of a plane angle [plani] depends on the arc of a circle which, described from the angle itself as from the centre of the said circle, subtends the angle itself.

9. In a plane triangle the sides themselves are proportional to the sines of the opposite angles. In a spherical triangle, however, the sines subtending the

[23] The rest of the sentence is set as a braced construction in two rows, read across, the shared words "penes arcum", "qui ex ipſo angulo tanquàm" and "deſcriptus, ipſum angulum ſubtendit" standing between the braces; each row is given here as one branch.

circular sides themselves, that is, the arcs, are proportional to the sines of the opposite angles. For here the sines subtending the sides take the place of the sides themselves.

Diagram. A plane triangle set to the left of proposition 9. Its base runs horizontally from D (left) to A (right); its apex B stands above C on the base and is joined by solid lines to D, to A and, by a vertical, to C. Above and to the right of B stands E, from which a solid vertical runs down to F on the base, crossing the side B–A. A dotted straight line runs from B up to E and appears to continue the side D–B. A dotted arc runs from E, curving down to the right, to H on the base, and is crossed by the side B–A. A second dotted arc runs from a point on D–B just below B down to G on the base. Along the base, from left to right, stand D, G, C, F, H and A.

- E: top of the figure, at the upper end of the vertical E–F and of the dotted line from B
- B: apex of the solid triangle D–B–A
- D: left end of the base
- G: on the base, at the foot of the shorter dotted arc
- C: on the base, at the foot of the vertical from B
- F: on the base, at the foot of the vertical from E
- H: on the base, at the foot of the dotted arc from E
- A: right end of the base

10. The radius of the circle, or greatest sine in the Canon, is subtended to a right angle. For a right angle is of 90 degrees, since the whole periphery of a circle, of 360 degrees, contains within itself four right angles. For any point whatever is surrounded by four right angles, or by some number of oblique angles, all of which taken together are equal to four right angles. Therefore, as four right angles are to the whole periphery, or in the spherical case to the whole Maxima, so is the angle at the centre, or in the spherical case at the pole, to the arc subtended to it on that periphery or Maxima.

General Division of Triangles [Generalis Diſtinctio Triangulorum.]

- Triangle [Triangulum]
  - plane [Planũ]
    - right-angled [Rectangulum]
      - with equal legs [Aequicrurum.]
      - not with equal legs [non æquicrurum.]
    - oblique-angled [Obliquãgulum.]
      - with equal [Aequi]
        - sides [laterum.]
        - legs [crurum]
          - acute-angled [Acutangulum.]
          - obtuse-angled [Obtuſangulum.]
      - unequal [Inæquabile]
        - acute-angled [Acutangulum.]
        - obtuse-angled [Obtuſangulum.]

- spherical [Sphæricum]
  - right-angled [Rectangulum]
    - with equal [Aequi]
      - sides [laterum.]
      - legs [crurum.]
    - the more usual [vſitatius.]
  - oblique-angled [Obliquãgulum]
    - with equal [Aequi]
      - sides [laterum.]
      - legs [crurum.]
    - unequal [Inæquabile]
      - acute-angled [Acutangulum.]
      - obtuse-angled [Obtuſangulum.]

Diagram. A column of fourteen small triangles drawn down the right-hand side of the outline above. A printed rule runs rightward from each of the fourteen final labels of the outline to one drawing, in the same order from top to bottom; the first seven drawings have straight sides, the last seven curved sides. No drawing carries a letter or any other label.

- 1 (rule from "Aequicrurum."): right triangle, right angle at lower left, the vertical and horizontal legs of about equal length
- 2 (rule from "non æquicrurum."): right triangle, right angle at lower left, the horizontal leg much longer than the vertical
- 3 (rule from "laterum.", plane): triangle with its apex above the middle of the base and its sides of about equal length
- 4 (rule from "Acutangulum.", under "crurum"): tall, narrow triangle with its apex above the middle of the base
- 5 (rule from "Obtuſangulum.", under "crurum"): low, wide triangle with its apex above the middle of the base
- 6 (rule from "Acutangulum.", under the plane "Inæquabile"): triangle with its apex to the left of the middle of the base
- 7 (rule from "Obtuſangulum.", under the plane "Inæquabile"): low triangle with its apex well to the left of the middle of the base and a long right side
- 8 (rule from "laterum.", spherical right-angled): curved triangle, all three sides bowed outward, apex above the middle
- 9 (rule from "crurum.", spherical right-angled): narrower curved triangle, sides bowed outward, apex above the middle
- 10 (rule from "vſitatius."): curved figure with a near-vertical left side, a bottom side bowed downward, and an upper side curving down to a sharp point at the lower right
- 11 (rule from "laterum.", spherical oblique-angled): curved triangle, sides bowed outward, apex above the middle
- 12 (rule from "crurum.", spherical oblique-angled): narrower curved triangle, sides bowed outward, apex above the middle

- 13 (rule from "Acutangulum.", spherical): curved triangle with its apex a little left of the middle
- 14 (rule from "Obtuſangulum.", spherical): low, wide curved triangle with its apex left of the middle

III. On Spherical Triangles [III. De Triangulis Sphæricis.]

The sides of a spherical triangle are arcs; and before the quadrature of the circle [Quadrationem circuli], now known, the ratio of arcs to one another, that is, of the curved to the straight, was unknown. Therefore these curvilinear sides, or arcs, are brought back and reduced to straight sides by means of the sines that subtend them, taken from the Canon; and so the whole spherical triangles themselves are brought back and reduced, as it were, to rectilinear or plane ones. Once the reduction to planes has been made, they are solved just as plane triangles themselves are. And the whole use of sines seems to be directed to this end, a use that we can now do without [quo iam carere poſſumus], as will appear in due course.

11. When the legs of an angle of a spherical triangle are extended until they cut each other, they become semicircumferences [ſemiperipherias], by 1, and on the opposite side they enclose an angle equal to the first. The greatest distance of these legs from each other will be at their middle, and that middle will be distant from each of the two angles by a quadrant of a Maxima, since all Maximae bisect one another. By 1 of these [Per 1. horum].

12. When a side or an angle of a spherical triangle is known, the complement of that side or angle is equally known, or its shortfall up to a quadrant or a semicircumference [ſemiperiphæriam].

Diagram. A semicircle set to the left of proposition 13, its diameter horizontal along the foot. Along the diameter, from left to right, stand B (left end), A, C and G (right end). A dotted vertical rises from A to E at the top of the arc. D stands on the arc to the right of E. Three solid straight lines run from D: one to B, one to A, and one straight down to C on the diameter. Labels read: B, A, C, E, D, G; C is printed in a hooked form with a descending tail.

- B (left end of the diameter)
- A (on the diameter, foot of the dotted vertical)
- E (top of the arc, head of the dotted vertical)
- D (on the arc, right of E)
- C (on the diameter, foot of the vertical from D; hooked form with a descending tail)
- G (right end of the diameter)

13. The sine of any arc or angle is also the sine of the complement of that same arc or angle up to the completion of a semicircumference, or up to two right angles. And then what is sought will be the excess over the quadrant or Radius, or over the right angle. Hence the twofold or ambiguous solution [anceps ſeu ambigua ſolutio] of certain triangles, as will be clear in its place.

14. The quantity or value of the three angles of a spherical triangle lies between two and four spherical right angles. For the value of the three angles cannot be that of two right angles, since the spherical triangle would then necessarily turn into a plane triangle, all three of whose angles taken together are, of course, equal to two right angles. Nor can the value of the three angles of a spherical triangle grow all the way to the value of four spherical right angles, for then the spherical triangle would grow into half a hemisphere. That figure is enclosed, like a little boat [ſcaphulã], not by three arcs but by two, which are semicircumferences of a Maxima; and this would be against the definition of a spherical triangle, and also against 5 of these [5. horũ]. Therefore a spherical triangle can have one, or two, or three, and nearly four right angles; but a plane triangle only one, because all three taken together are equal to two right angles. By 32 of the first [Per 32. primi].

But now that the Elements on which we must rely have been laid down, we must at last proceed to the solution of triangles itself: first of spherical right-angled triangles, then of oblique-angled ones, and finally of plane ones. In each kind this is done through six different Examples, according to three different givens and their sixfold permutation.

IV. On the Solution of Triangles [IV. De ſolutione Triangulorum.]

The solution of triangles is primary where, namely, from three known or given parts some fourth, not opposite, is investigated, and this by most of the preceding Elements; or it is secondary, where from four known or given parts, but not with both pairs opposite [ſed non binis vtriſq; oppoſitis], the remaining two are afterwards investigated, and this by the ninth Element alone. Both solutions are carried out and completed sometimes in short form [compendioſè] by prosthaphaeresis [proſthaphæreſin], that is, by Addition and Subtraction [Additionem & Subductionem], but sometimes by the customary operation of the rule of proportion, that is, by Multiplication or division. Let the method [Ratio] of prosthaphaeresis therefore be noted; it is as follows.

Postulate [Poſtulatum.]

With the Radius, or greatest sine, holding the first place in the rule of proportion, to find what is sought by prosthaphaeresis: that is, by way of Addition and subtraction alone, without the tedious operation of Multiplication and division. And this in two different cases and Examples.

Statement [Enunciatum] of the first case, namely when the assumed Given [Datum] is the smaller and its complement the greater.

Diagram of the First Case. Dedicated to Paulo Vuittichio [Paul Wittich] of Breslau. [Diagramma caſus prioris. Paulo Vuittichio Vratiſlauienſi dedicatum.]

Diagram. A large circle filling the lower part of the page below the caption. Its horizontal diameter runs from X (left) through the centre A to V (right), and its vertical diameter from P (top) through A to T (bottom). A solid straight line

runs from N (upper left, on the circle) through O down to A, and continues beyond A down to the right to a looped letter (Q-like) and on to S on the circle at the bottom; the stretch below the looped letter to S is drawn as a double stroke. A second solid straight line runs from B (right, on the circle) through D to A and on to a letter in the form ϒ on the circle at the lower left. M stands on the circle at the upper right. A dotted line runs from O to the vertical diameter and on to M; a second dotted line rises in a curve from A, left of the vertical diameter, crosses it, and also runs to M. A dotted vertical drops from M past R, through L and H (on the horizontal diameter), and on to K below the diameter. A dotted line runs from M down through D, crosses the horizontal diameter near C, and continues to F on the circle at the lower right. A dotted horizontal runs from L through D to i, beside the solid vertical from B. Solid verticals drop from D to E and from B to C on the horizontal diameter; G stands just above the diameter to the right of C, and a dotted vertical drops from the diameter there to F. A dotted arc runs from R round to the left and down towards H; it is broken on the left side. Below the diameter a dotted curve runs from A down to K, and a dotted straight line runs from K to F. A further dotted straight line joins F to the looped letter. Along the horizontal diameter, from left to right: X, A, H, E, C, V, with G above it near V.

- P (top of the circle, head of the vertical diameter)
- N (on the circle, upper left)
- M (on the circle, upper right)
- O (on the line N–A)
- R (at the upper end of the dotted arc, beside the dotted vertical from M)
- L (on the dotted vertical from M, left end of the dotted horizontal)
- D (on the line B–A)
- i (small dotted letter, right end of the dotted horizontal, beside the vertical from B)
- B (on the circle, right)
- X (left end of the horizontal diameter)
- A (centre; swash form)
- H (on the horizontal diameter, foot of the dotted vertical from M)
- E (on the horizontal diameter, foot of the vertical from D)
- C (on the horizontal diameter, foot of the vertical from B; a C-shape with a descending hook)
- G (just above the diameter, right of C)
- V (right end of the horizontal diameter)
- K (below the diameter, on the dotted vertical from M)
- F (on the circle, lower right)
- ϒ (on the circle, lower left; a Y- or upsilon-form letter)
- Q (looped form; on the line from A towards S)
- S (on the circle, bottom, right of T)
- T (bottom of the circle, foot of the vertical diameter)

Subtract the sine of the complement of the two Givens other than the Radius [?] [Sinu complementi duorum præter Radium Datorum] from the sum of the sines [ſummâ ſinuum] of the other Given and of the remaining complement — on condition, however, that taken together they neither make up nor exceed a quadrant: half of the remainder will be the sine of the arc sought. Or let the Given be taken away [tollatur Datum], and subtract the sine of what is left of the complement from the sine of the said sum: half of the remainder will likewise be the sine of the arc sought. Example. As the Radius AB is to the sine BC, so the segment AD of the Radius is to the sought DE. By Element 6 [Per 6. Elementum].

Statement of the second case, namely where the assumed Given is the greater and the complement the smaller.

Diagram of the Second Case of Prosthaphaeresis. Dedicated to Barptolemaeus Scultetus [Bartholomäus Scultetus], senator of Görlitz. [Diagramma poſterioris caſus proſthaphæreſeos. Barptolemæo Sculteto ſenatori Gorlicenſi ſacrum.]

Diagram. A large circle filling the middle of the page below the caption. Its vertical diameter runs from P (top) through the centre A to X (bottom), and its horizontal diameter from T (left) through A to V (right). A solid straight line runs from N (on the circle, left) through O down to A, and continues beyond A down to the right through a looped letter (Q-like) to S on the circle at the lower right. A second solid straight line runs from B (on the circle, upper right) through D to A, and on to a letter in the form ϒ on the circle at the lower left. Solid verticals drop from D to E and from B to C on the horizontal diameter. M (swash form) stands on the circle at the upper left, left of P. From M: a dotted vertical drops through L and H to R on the horizontal diameter; a dotted straight line runs down to the left to O on the line N–A; and a dotted straight line runs down to the right, through D, to F on the circle at the right. A dotted horizontal runs from L through D to i, beside the solid vertical from B. A second dotted horizontal runs from H to F. A dotted vertical drops from F to G on the horizontal diameter, and a dotted straight line runs from F down to the left, across the horizontal diameter, to the looped letter. A dotted curve leaves the circle to the left of M, bows out to the left and comes back round to H; a short dotted stretch also lies along the inside of the circle just left of M. Along the horizontal diameter, from left to right: T, R, A, E, C, G, V. Three small open rings (one right of M, two beside the lower vertical diameter) stand on the figure without letters.

- P (top of the circle, head of the vertical diameter)
- M (on the circle, upper left; swash form)
- B (on the circle, upper right)
- L (on the dotted vertical from M, left end of the upper dotted horizontal)
- D (on the line B–A, where the dotted lines from M and from L cross it)
- i (small dotted letter, right end of the upper dotted horizontal, beside the vertical from B)

- N (on the circle, left)
- H (on the dotted vertical from M, at the left end of the dotted horizontal to F)
- F (on the circle, right)
- O (on the line N–A, at the foot of the dotted line from M)
- T (left end of the horizontal diameter)
- R (on the horizontal diameter, foot of the dotted vertical from M)
- A (centre; swash form)
- E (on the horizontal diameter, foot of the vertical from D)
- C (on the horizontal diameter, foot of the vertical from B; a C-shape with a descending hook)
- G (on the horizontal diameter, foot of the dotted vertical from F)
- V (right end of the horizontal diameter)
- Q (looped form; on the line from A towards S, at the foot of the dotted line from F)
- S (on the circle, lower right)
- Y (on the circle, lower left; a Y- or upsilon-form letter)
- X (bottom of the circle, foot of the vertical diameter)

From the sine of the sum of the other Given besides the Radius and of the remaining complement — which, if taken together, neither make up nor exceed a quadrant — take away the sine of that same complement, added before, so that the remainder of the said sum may be established. Once this is known, add half of it back to the sine subtracted before: the sum will be the sine of the arc sought. Or take half the remainder away from the sine of the whole sum of the Given and the complement: what is left will be the same as before. Example as above.

The cause, however, and the reasoning and Demonstration of these things appear clearly enough from the attached Diagrams. In the solution of triangles, therefore, care must be taken that, whenever it can be done, the triangle to be solved is so fitted and adapted to the proportion that the Radius occupies the first place in the rule of proportion [proportionis Regulâ], so that the solution of the triangle can be made by prosthaphaeresis. With this account of prosthaphaeresis set out first, let us now take up the solution of right-angled triangles [Triangulorum Rectangulorum], which we shall treat and complete in the following order.

[The outline below is set as nested braces; at the right a column of Roman numerals I.–VI. stands, each on the line of one final branch, with the word "Exemplum" (Example) set before I. only. Each numeral is given here at the end of the branch on whose line it is printed.]

- From the Givens [Ex Datis]:
  - all the angles, to find the sides [Omnibus angulis reperire latera.] — Example I.
  - the right angle together with [angulo recto vnâ cum]:
    - the sides [lateribus]:

- the two adjacent [duobus adiacentibus.] — II.
- one adjacent, the other opposite [altero adiacente, altero oppoſito.] — III.

- one of the angles and one side, to the angle [altero angulorum & vno latere angulo]:
  - Right [Recto]:
    - adjacent [adiacente.] — IV.
    - opposite [oppoſito.] — V.
  - lying between both [vtriq; interiacente.] — VI.

[Legend, set in roman within a pair of round brackets: a slanting stroke (/) above and a small circle crossed by a horizontal bar (⊖) below, joined by a brace to "Signi-ſicat"; a brace to "Datum" over "Quæſitum"; a brace to "eruntq́ue ſemper" over "in ipſo medio"; a brace to "Dati." over "Quæſiti.".] The stroke (/) signifies the Given and the barred circle (⊖) the Sought; and they will always stand in the very middle of the Given and of the Sought respectively.

The solution of right-angled triangles is carried out by continuing their arcs, or sides, and the amplitudes of the angles, until they complete a quadrant of a Maxima. In these continuations C, E, F designate the right angle, and this by Element 2 [per 2. Elementum]. So it will be in the first Example.

The Diagrams of the Right-angled Triangles Follow. Dedicated to Gerardus Mercator, with his Sons and Grandson. [Sequuntur Diagrammata Rectangulorum. Gerardo Mercatori cum filijs & nepoti dedicatum.]

Diagram. A spherical right-angled triangle, set at the left of Example I. Solid arcs join A to B, A to C, and B to C, enclosing the triangle, which carries the numeral "i" inside it. Dotted arcs run from B up to D, from D to E, from E back down to B, from E down to F, and from F across to C, completing the construction. Each mark on the figure is a short straight bar with a serif at each end (I-shaped), except at D: bars stand across the dotted arc DE and across the dotted arc EF; one is laid through E, where the dotted arcs cross; at B a single bar is laid through the vertex, broken at it, one half within the angle DBE and the other within the angle ABC; at A a bar is laid through the vertex, one half within the angle BAC and the other projecting beyond the vertex to the left; and a bar is laid through the vertex C. At D the two dotted arcs begin in a short, tapering solid dash pointing up and to the left out of the vertex.

- D (top): D
- E (upper right, on the dotted arc from D): E
- B (middle, where the solid arcs from A and C meet): B
- F (right, foot of the dotted arc from E): F
- C (bottom, beneath the vertex): C
- A (left vertex): A, set as a swash capital with a leading curl
- i (inside the triangle): i

All seven labels read; none unread.

I. The arc FE is known, by 8; ED, by 12. The angle ABC equals the angle DBE, by 7. Therefore, as DBE is to DE, so is DEB to DB, by 9; and once DB is known, BC will at once be established, by 12, and afterwards the two remaining sought quantities, by 9.

Diagram. A second spherical right-angled triangle of the same construction, set at the left of Example II. Solid arcs join A to B, A to C, and B to C, and the triangle carries the numeral "ii" inside it. Dotted arcs run from B up to D, from D to E, from E down to B, from E down to F, and from F across to C. Each mark is a short straight bar with a serif at each end (I-shaped): at D one enters the vertex from above left and runs down to the right into the angle BDE, the first dots of both dotted arcs running over it; bars stand across the dotted arc DB, across the solid arc BC, across the solid arc AC (nearer C than A) and across the dotted arc CF; one lies level through E, where the dotted arcs cross; and one is laid through the vertex C. There is no mark at A, B or F.

- D (top): D
- E (upper right): E
- B (middle): B
- A (left vertex): A, swash capital
- C (bottom): C
- F (right): F
- ii (inside the triangle): ii

All seven labels read; none unread.

The following Examples are solved in several ways, or by different methods. [Sequentia Exempla multiplici viâ ſeu variâ ratione ſoluuntur.]

II. First. The arcs BD and CF are known, by 12. Therefore, as DC is to CF, so is DB to BE, by 6; and once BE is known, BA will at once be known, by 12, and afterwards the two remaining quantities, by 9.

Second. The angle at D is known, by 8. Therefore (just as in the previous Example), as BED is to BD, so is BDE to BE, by 9. From here on, as in the previous case.

Diagram. A third spherical right-angled triangle of the same construction, set at the left of Example III. Solid arcs join A to B, A to C, and B to C, and the triangle carries the numeral "iii" inside it. Dotted arcs run from B up to D, from D to E, from E down to B, from E down to F, and from F across to C. At D the two dotted arcs begin in a short, tapering solid dash pointing up and to the left out of the vertex. The other marks are short straight bars with a serif at each end (I-shaped): across the dotted arc DB, across the dotted arc EB, across the solid arc AB (in its half towards B) and across the solid arc BC; one lies level through E, where the dotted arcs cross; and one is laid through the vertex C. There is no mark at A, B or F.

- D (top): D
- E (upper right): E
- B (middle): B
- A (left vertex): A, swash capital with a leading curl

- C (bottom): C
- F (right): F
- iii (inside the triangle): iii

All seven labels read; none unread.

The following Examples can be adapted to solution in two ways. [Sequentia Exempla duplici modo ſolutioni accommodari poſſunt.]

III. First. The arcs BD and BE are known, by 12. Therefore, as DB is to BE, so is DC to CF, by 6; and once this is known, CA too will become evident, by 12, and afterwards the remaining quantities, by 9.

Second. As DB is to DEB, so is BE to BDE, by 9, that is, to CF, by 8. From here on, as before.

Diagram. A spherical right-angled triangle of the same construction as the figures on the facing page, set at the left of the first "same, arranged differently" passage. Solid arcs join A to B, A to C, and B to C, and the triangle carries the numeral "iii" inside it. Dotted arcs run from B up to D, from D to E, from E down to B, from E down to F, and from F across to C. A slanting stroke stands across the vertex D, with a second short stroke just below it; I-shaped cross-strokes stand at E, across the dotted arc EB, across the solid arc AB, across the solid arc AC and across the dotted arc CF; a further stroke stands at C.

- D (top): D
- E (upper right): E
- B (middle): B
- A (left vertex): A, swash capital with a leading curl
- C (bottom): C
- F (right): F
- iii (inside the triangle): iii

All seven labels read; none unread.

The same, arranged differently. [Idem aliter accommodatum.]

The same. First. The arcs CF and BE are known, by 12. Therefore, as CF is to CD, so is BE to BD, by 6; once this is known, BC too is known in the same way, by 12, and afterwards the remaining quantities, by 9.

Second. Or, just as before, by opposition, or by 9, in the triangle BDE.

Diagram. A spherical right-angled triangle of the same construction, set at the left of Example IV. Solid arcs join A to B, A to C, and B to C, and the triangle carries the numeral "iv" inside it. Dotted arcs run from B up to D, from D to E, from E down to B, from E down to F, and from F across to C. I-shaped cross-strokes stand across the dotted arc DE, across the dotted arc DB, at E, across the dotted arc EF and across the solid arc BC; a serifed bar stands across the vertex A, and a further stroke stands at C.

- D (top): D
- E (upper right): E
- B (middle): B
- A (left vertex): A, swash capital with a leading curl
- C (bottom): C

- F (right): F
- iv (inside the triangle): iv

All seven labels read; none unread.

IV. First. The arc FE is known, by 8; the arcs ED and BD, by 12. Therefore, as BD is to DEB, so is DE to DBE, by 9, that is, to ABC, by 7. Afterwards the remaining quantities will be established, by 9.

Second, and conversely: as BAC is to CB, so is ACB to BA, by 9; from this BA too is established, by 12. Then, as DB is to BE, so is DC to CF, by 6, and from this CA is known, by 12.

Diagram. The same triangle again, set at the left of the second "same, arranged differently" passage. Solid arcs join A to B, A to C, and B to C, and the triangle carries the numeral "iv" inside it. Dotted arcs run from B up to D, from D to E, from E down to B, from E down to F, and from F across to C. A slanting stroke stands across the vertex D, with a second short stroke just below it; an I-shaped cross-stroke stands at E; strokes stand at B, one above and one below the vertex; I-shaped cross-strokes stand across the solid arc AC and across the dotted arc CF; a further stroke stands at C. A small ringed dot stands in the open space to the left of the arc DB.

- D (top): D
- E (upper right): E
- B (middle): B
- A (left vertex): A, swash capital with a leading curl
- C (bottom): C
- F (right): F
- iv (inside the triangle): iv

All seven labels read; none unread.

The same, arranged differently. [Idem aliter accommodatum.]

The same. First. The arc CF is known, by 12, and therefore the angle CDF, by 8. The triangle BDE will thus be one of known angles, from which the sides may be found[24], by the first Example; and when DB and EB are taken away[25] from the quadrants, BA and BC will be left[26], by 12, and afterwards the angle at A, by 9[27].

Second. As ABC is to CA, so is BCA to AB, by 9; and from this BE is likewise known, by 12. Then, as CF (known by 12) or CDF (known by 8) is to BE, so is either BED to DB, or EBD to DE, by 9; and once DE is known, EF will not remain unknown, by 12, that is, the angle at A, by 8. Afterwards the remaining quantities will become evident, by 9.

Diagram. A spherical right-angled triangle of the same construction as the figures of the preceding pages, set at the left of Example V. Solid arcs join A to B, A to C, and B to C, and the triangle carries the numeral "V" inside it.

[24] the page prints "linueniantur"
[25] the page prints "ſub-ſtatoq́;"
[26] the page prints "relinquen-" at the line end and "pur" at the start of the next line
[27] the page prints "er 9.", with no initial p

Dotted arcs run from B up to D, from D to E, from E down to B, from E down to F, and from F across to C. I-shaped cross-strokes stand across the dotted arc DE, at E, across the dotted arc EB, across the dotted arc EF and across the solid arc AB; a serifed bar stands across the vertex A, and a further stroke stands at C.

- D (top): D
- E (upper right, on the dotted arc from D): E
- B (middle, where the solid arcs from A and C meet): B
- A (left vertex): A, swash capital with a leading curl
- C (bottom): C
- F (right, foot of the dotted arc from E): F
- V (inside the triangle): V

All seven labels read; none unread.

V. The arc FE is known, by 8; and hence ED, and likewise EB, by 12. Therefore, as AE is to FE, so is AB to BC, by 6; and once BC is known, BD will be established, by 12. Next. First. As DB is to BE, so is DC to CF, by 6; and once CF is known, CA will not stay hidden, by 12.

Second. As DB is to BED, so is ED to DBE, by 9, that is, to ABC, by 7. And with the four known by either route, the remaining two will be established, by 9.

Diagram. The same triangle again, set at the left of the "same, adapted differently" passage. Solid arcs join A to B, A to C, and B to C, and the triangle carries the numeral "V" inside it. Dotted arcs run from B up to D, from D to E, from E down to B, from E down to F, and from F across to C. A short stroke stands at the vertex D, to the left of the point where the dotted arcs meet; an I-shaped cross-stroke stands at E; an I-shaped cross-stroke stands across the dotted arc EB; slanting strokes stand at B, one above and one below the vertex; an I-shaped cross-stroke stands across the solid arc AB; a further stroke stands at C.

- D (top): D
- E (upper right): E
- B (middle): B
- A (left vertex): A, swash capital with a leading curl
- C (bottom): C
- F (right): F
- V (inside the triangle): V

All seven labels read; none unread.

The same, adapted differently to solution. [Idem aliter solutioni adaptatum.]

The same. But the solution of this latter arrangement is carried out by the following, last Example. You must therefore learn that Example first, before you attempt this other route of solution; and once it has been grasped, the way will be open.

Diagram. The same construction, set at the left of Example VI. Solid arcs join A to B, A to C, and B to C, and the triangle carries the numeral "Vi" inside

it. Dotted arcs run from B up to D, from D to E, from E down to B, from E down to F, and from F across to C. I-shaped cross-strokes stand across the dotted arc DB, at E, and across the solid arc BC; strokes stand at B, one above and one below the vertex; a further stroke stands at C. A thin curled stroke crosses the dotted arc EF a little above F.

- D (top): D
- E (upper right): E
- B (middle): B
- A (left vertex): A, swash capital with a leading curl
- C (bottom): C, with a hooked tail below
- F (right): F
- Vi (inside the triangle): Vi

All seven labels read; none unread.

VI. The arc BD is known, by 12. Therefore, as DEB is to BD, so is ABC — or, by 7, DBE — to ED, by 9. Once ED is known, EF will become evident, by 12, and from it the angle at A, by 8; afterwards the remaining quantities, by 9.

Diagram, dedicated to Albertus Leoninus à Gronevuoude [?]. [Diagramma, Alberto Leonino à Gronevuoude ſacrum.]

Diagram. A spherical triangle with an extended dotted construction, set in the left part of the text block beneath the dedication line and beside the first paragraph. Solid arcs join A to B, A to C, and B to C; no numeral stands inside the triangle. A dotted arc runs from G up through H to the top point; from the top point a long dotted arc descends on the right, crosses another dotted line at E, and runs on to B. A dotted arc comes in from G and a dotted line comes down from H; they cross at D. Below D one dotted line descends steeply to B and the other runs diagonally down to E, crosses there, and continues to F. A dotted arc joins F to C. I-shaped cross-strokes stand across the dotted arc between H and the top point, across the dotted line between H and D, across the dotted arc between D and B, and across the solid arc BC; strokes stand at B, one slanting above and one below the vertex; a further stroke stands at C. Faint short double strokes lie on the dotted arc CF near F, and a small dot stands below and to the right of G.

- I (top, where the arc from H and the long right-hand arc meet), a small, faint letter with a dot printed above it
- H (upper left, on the arc from G): H
- G (left end of the dotted arcs): G
- D (beneath the crossing of the dotted lines from G and H): D
- E (right, at the crossing of the dotted lines): E
- B (middle, where the solid arcs from A and C meet): B
- A (left vertex): A, swash capital with a leading curl
- C (bottom): C
- F (right, at the foot of the dotted line through E): F

Nine labels read; the top label I is uncertain as a letter (see the note at the foot of the page).

The same Example, done differently: by prosthaphaeresis, and by the secondary continuation of the sides. [Idem Exemplum aliter per proſthaphæreſin, perq́; laterum continuationem ſecundariam.]

Let the arcs CD, BH, EG, GI, IE be quadrants by construction. The angles at G, H, I are therefore right angles, by 2. The arc HI is known, by 8. Therefore, as BH is to HI, so will BD be to DE, by 6; and once DE has been found, EF will likewise be established, by 12, and from it the angle at A, by 8; after that the remaining quantities, by 9. Now examine, in order, the remaining Examples set out above, and see whether the same can be done in them. On this, consult our spherical lemma in our *Astronomy*. And so far, this has been the solution of right-angled spherical triangles; the solution of oblique-angled ones follows.

V. On Oblique-angled Spherical Triangles [V. De Triangulis ſphæricis obliquangulis.]

An oblique-angled triangle is one that has all its angles oblique and none right. An oblique angle will be either less than a right angle, and acute — and the triangle is then called acute-angled — or greater than a right angle, and obtuse — and the triangle is then called obtuse-angled (provided it has one obtuse angle). A right angle is one in which one leg stands at right angles upon the other; an oblique angle, one in which it stands obliquely.

VI. On the Two Usual Oblique-angled Triangles [VI. De duobus Triangulis Obliquangulis vſitatis.]

Among the six Examples of oblique-angled triangles there are two whose solution comes up often — indeed very often — and in every observation makes up the whole account [vtramq́; facit paginam]; so we have chosen to call them the usual ones. We shall teach how to solve them in Byrgius's manner, by a technique that breathes of the true Byrgian casket of perfumes [myrothecium] and of the true and genuine Kassel, or Hessian, Astronomy. And we shall do each of the two in two different Examples, or cases, according, that is, as the triangles themselves are acute-angled or oblique-angled, or as their bases are less or greater than a quadrant of a Maxima.

The Six Examples of Oblique-angled Triangles. [Senâ Exempla Triangulorum obliquangulorum.]

- Oblique-angled triangle [Triangulum Obliquangulum]
  - usual [vſitatum]
    - First: given all the sides, to find the angles. Example I.
    - Second: given two sides and the angle between them. II.
  - more unusual [inuſitatius]
    - ambiguous [Ambiguum]
      - I. Given two sides and the angle opposite one of them. III.
      - II. Given two angles and the side opposite one of them. IV.
    - remaining [Reliquum]
      - I. Given two angles and the side lying between them. V.
      - II. Given all the angles, to find the sides. VI.

VII. On the First Usual Triangle, and Its Former Case [VII. De vſitato primo, eiuſque caſu priore.]

First. These arcs are known by hypothesis: [Primùm. noti ſunt ex theſi hi arcus,]	And the complements of the same arcs, by 12. [Eorundemq́; arcuum Complementa, per 12.]	The sines of these and of the arcs. [Horum & arcuum ſinus.]
AB	BC.	BH.
AD Given sides. [Data latera.]	Du.	DL.
BD likewise [item]	DZ. likewise [item]	DM.
AE, the difference of the sides [Differentia laterum]	PC, the remainder from BC and PX. [relictum ex BC & PX.]	PK.
BD and DA.	EG, the sum of EF and FG. [ſumma ex EF & FG.]	ER.

[Layout of the first column as printed: a single brace joins AB, AD and BD on its left and points to "Data latera." set beside AD, with "item" set beneath it beside BD. The entry for AE runs over two printed lines, "AE, Differentia laterum" and "BD & DA.", and each of those lines faces its own entry in the second and third columns. The second entry of the second column is printed "Du.", a capital D followed by a lower-case roman u.]

Diagram of the first usual triangle, and of its former case. Dedicated to Lazarus Schonerus [Lazarus Schöner]. [Diagramma vſitati primi, eiusdemq́ue caſus prioris. Lazaro Schonero ſacratum.]

Diagram. A large circle, drawn solid, fills the upper two-thirds of the page, with straight and curved lines, solid and dotted, inside it. Points on the circumference, going round from the top: A at the top, E just to the right of A, F at the upper right, G at the right end of a horizontal diameter, Γ at the lower right, J at the bottom, X at the lower left, P at the lower left above X, C at the left end of the horizontal diameter, and B at the upper left.

- The horizontal diameter runs from C (a dot stands just right of C, where the line begins) through H, K, V, the sort read as u, L, i and R to G; it is broken by a short gap between R and G, just right of the point where the dotted line from D crosses it.
- A solid vertical line descends from A through i to J.
- A solid curved line runs from A down to D, and a solid curved line runs from B across to D.
- A solid straight line runs from B through T to i and on down to Γ.
- A solid line runs from E under the arc to N; a solid line runs from F through N and down past Z and i towards X.
- Solid perpendiculars descend from B to H, from T to V, and from D to L; a solid vertical descends from K to P.
- A short solid line runs from P to o, which stands beside the line ending at X.
- A solid line runs from D to M.

- Dotted lines: from E down to D, and from D on down past T towards P; a vertical dotted line from E down past Z, M, S and R to Q; a dotted line from D through Z, sloping down to the right, across the diameter between R and G and on to Γ; a dotted horizontal from T through Y to S; a dotted line from D down through the sort read as u and on to J; and a dotted horizontal from P to Q.
- Marks: I-shaped cross-strokes stand across the circle at the upper left between B and A, at the upper right between E and F, on the right side above G, on the left side above C, and on the lower left above P; small strokes stand on the circle between A and E and between P and X; I-shaped cross-strokes stand across the curve AD, across the curve BD, and across the dotted line from D to u, between the labels T and Y. Bold small rings stand on the line from N towards i (about a third of the way down), on the perpendicular TV, and on the line between Z and i. Fainter small dotted rings stand on the circle between B and A, to the right of S, and at the lower right outside the circle; a small dot stands on the line BT.

Labels as read:

- A (top): A, swash capital with a leading curl
- E (top, right of A): E
- F (upper right, on the circle): F
- N (upper right, inside the circle below F): N
- B (upper left): B
- D (left of centre, where the curves from A and B meet): D
- Z (right of centre): Z, a swash form
- M (below Z): M
- S (below M): S
- R (below S, above the diameter): R
- i (centre, above the diameter): i, a lower-case dotted i
- T (left of centre, on the line from B): T
- Y (right of T): Y, with curved arms
- C (left end of the diameter): C
- H (below the diameter, right of C): H
- K (above the diameter): K
- V (below the diameter): V
- the sort between V and L: a V-like left stroke joined at the foot to a vertical right stem, no left stem; read as u [?]
- L (below the diameter): L
- G (right end of the diameter): G
- Γ (lower right): Γ, a stem with its top bar running to the right only
- P (lower left): P
- o (beside the foot of the line from F, above X): o, small and bold
- X (lower left, on the circle): X
- Q (below the dotted vertical, right of centre): Q, a swash form

• J (bottom): J

Twenty-six labels read; one (the sort between V and L) is uncertain as a letter.

Second. DM, TI, EN, PO are equal, by 33 of the first [per 33. primi]; and so are their arcs, DZ, EF, PX. Moreover, by the same, TV, YL, SR are equal; likewise KP, RQ. Third, FI and ET are equal; likewise DT and IZ, by 4. And this DT, or IZ, is to be sought, and in this way:

I. As IB is to BH, so is IT to TV, or YL, by 6. When the YL so found is taken away from DL, DY will remain. And when the same YL is subtracted from ER, ES will remain.

II. As ES is to the radius [Radium] (by 4) ET, so is DY to DT, or (by 4) YZ, by 6. And this IZ will be the prosthaphaeresis for ZF itself — that is, by 8, for the angle ZBF. Since that angle is less than the right angle IBF, the prosthaphaeresis is to be subtracted; but if the angle were greater, it is to be added to it (as in the following second, or later, case), so that the sought angle ZBF is obtained.

Diagram of the latter case of the first usual triangle. Consecrated to the Frisian Edo Hildericus [Edo Hilderich]. [Diagramma Caſus poſterioris vſitati primi. Edoni Hilderico Friſio conſecratum.]

Diagram. A large circle, drawn solid, fills the upper two-thirds of the page, with straight and curved lines, solid and dotted, inside it. Points on the circumference, going round from the top: A at the top, E just to the right of A, F at the upper right, G at the right end of a horizontal diameter, Γ at the lower right, Λ at the bottom, X at the lower left, P at the lower left above X, C at the left end of the horizontal diameter, and B at the upper left.

- The horizontal diameter runs from C through H, K, u, L, the foot of the perpendicular from T (V below it), i and R to G.
- A solid vertical line descends from A through i to Λ.
- A solid curved line runs from A down to D, crossing the diameter at u; a solid curved line runs from B down to D, crossing the diameter between H and K.
- A solid straight line runs from B through T to i and on down to Γ.
- A solid line runs from E under the arc to N; a solid line runs from F through N, i, M and past Z down to the circle just right of X.
- Solid perpendiculars descend from B to H and from L down to D; a solid vertical descends from K to P; a solid vertical descends from T across the diameter (V below it) to Y.
- A short solid line runs from P to the foot of the line from F, beside o.
- A solid line runs from D to M.
- Dotted lines: from E down through T to D, and on down to P; a vertical dotted line from E down past S and R to J and on to Q; a dotted horizontal from T to S, passing above i; a dotted horizontal from D through Y to J; a dotted horizontal from P to Q; a dotted line from D down to Λ; and a dotted curve from D passing between M and Z, sweeping down and round to Γ.

- Marks: I-shaped cross-strokes stand across the circle at the upper left between B and A, between A and E, between E and F, on the right side above G, on the left side above C, on the lower left between C and P, and between P and X; an I-shaped cross-stroke stands across the curve AD at the upper left, and one across the curve BD just left of K; a smaller stroke stands on the curve AD just below the diameter at u. Bold small rings stand on the perpendicular from T just above the diameter, and on the line from i towards M. The circle shows a short break just below C. A small isolated stroke stands outside the circle to the right of F.

Labels as read:

- A (top): A, a swash capital with a large leading curl, the curl standing free of the letter like a C
- E (top, right of A): E
- F (upper right, on the circle): F
- N (upper right, inside the circle below F): N
- B (upper left): B
- C (left end of the diameter): C
- H (below the diameter, right of C): H
- K (above the diameter): K
- u (above the diameter, right of K): u, a lower-case roman u
- L (above the diameter, right of u): L
- T (left of centre, on the line from B): T
- V (below the diameter, right of the perpendicular from T): V
- i (centre, above the diameter): i, a lower-case dotted i
- S (right of the dotted vertical, level with the dotted line from T): S
- R (right of the dotted vertical, above the diameter): R
- G (right end of the diameter): G
- Y (at the foot of the perpendicular from T): Y, with curved arms
- J (right of the dotted vertical, level with the dotted line from D): J, with a dot standing above it
- D (left of centre, where the curves from A and B meet): D
- M (below centre, at the end of the line from D): M
- Z (below M): Z, a swash form set at a slant
- Γ (lower right): Γ, a stem with its top bar running to the right
- P (lower left): P
- o (beside the foot of the line from F, above X): o, small and bold
- X (lower left, on the circle): X, a swash form
- Q (below the dotted vertical, right of centre): Q, a swash form
- Λ (bottom): Λ-shaped, two strokes meeting at the apex on the circle, no crossbar

Twenty-seven labels read; none left unread.

The first quantity sought, TV, is found as before. Once it has been found and added to VY or DL — that is, to the sine of the arc Du (namely, the excess of the arc AD over a quadrant) — the sum will be TY. With this known, it will

then go on just as before: as SE is to ET, or (by 4) to FI, so is TY to the second quantity sought, TD, or (by 4) to IZ — that is, by 8, to the angle ZBI. This angle is the prosthaphaeresis to be added to the radius [Radius] FI — that is, by 8, to the right angle IBF — so that the sought angle ZBF is obtained. After that, the remaining parts will be established in either case, by 9. So much for the first usual oblique-angled triangle; the second follows.

IIX. On the Second Usual Triangle, and Its Former Case [IIX. De vſitato ſecundo, eiuſque priore caſu.]

Here nearly everything is known by hypothesis, and equal to one another, as before — except that here what is sought is the sine DL and its arc Du, whose complement is the arc to be sought, DA. Therefore the sine DL is sought as follows:

I. As IB is to BH, so is IT to TV, or YL, or SR, by 6. And when the SR so found is taken away from ER, ES will remain.

II. As the radius [Radius] (by 4) TE is to ES, so is TD, or (by 4) IZ, to DY, by 6. When the DY now found is added to the YL found earlier, the sum will be the sought sine DL — the sine, namely, of the arc Du. Once that is known, the sought arc AD will also be known in the same way, by 12.

Diagram of the second usual triangle, and of its former case. Dedicated to Philippus Apianus [Philipp Apian], son of Petrus [Peter Apian]. [Diagramma vſitati ſecundi, eiusdemque caſus prioris. Philippo Apiano Petri filio dedicatum.]

Diagram. A large circle, drawn solid, fills the middle of the page, with straight and curved lines, solid and dotted, inside it. Points on the circumference, going round from the top: A at the top, E just to the right of A, F at the upper right, G at the right end of a horizontal diameter, Γ at the lower right, J at the bottom, X at the lower left, P at the lower left above X, C at the left end of the horizontal diameter, and B at the upper left.

- The horizontal diameter runs from C through H, K, V, the sort read as u, L, i and R to G.
- A solid vertical line descends from A through i to J.
- A solid curved line runs from A down to D; from D a solid line continues down through Y to L.
- A solid curved line runs from B across to D, and from D a solid line runs on down to the right, crossing the vertical from A, to the line from F, with M standing below its end.
- A solid straight line runs from B through T to i and on down to Γ.
- A solid line runs from E under the arc to N; a solid line runs from F through N, past Z and on to i, and on down to the circle just right of X.
- Solid perpendiculars descend from B to H and from T to V; a solid vertical descends from K to P.
- A short solid line runs from P to the foot of the line from F, beside o.

- Dotted lines: from E down to D; from D down through T to P; from D down through the sort read as u to J; a vertical dotted line from E down past S and R to Q; a dotted horizontal from T through Y to S; a dotted horizontal from P to Q; and a dotted line from D down to the right, passing close to Z, to Γ.
- Marks: I-shaped cross-strokes stand across the circle at the upper left between B and A, at B itself, between A and E, between E and F, on the right side above G, on the left side above C, on the lower left between C and P, and between P and X; across the curve BD, a little left of its middle; across the line from F, once between N and Z and once just above i. Bold small rings stand on the curve AD about halfway down, on the line DL just below D, and on the perpendicular TV just above the diameter.

Labels as read:

- A (top): A, a swash capital with a leading curl
- E (top, right of A): E
- F (upper right, on the circle): F
- N (upper right, inside the circle below F): N
- B (upper left, on the circle): B
- D (left of centre, where the curves from A and B meet): D
- C (left end of the diameter): C
- H (below the diameter, right of C): H
- K (above the diameter): K
- T (left of centre, on the line from B): T
- Y (right of the line DL, above the dotted line from T): Y, with curved arms
- V (below the diameter, at the foot of the perpendicular from T): V
- the sort between V and L, below the diameter: two stems joined by a foot serif, crossed by the dotted line from D; read as u [?]
- L (below the diameter, at the foot of the line from D): L
- M (below the end of the solid line from D): M
- Z (right of the line from F, above S): Z, a swash form
- S (right of the dotted vertical, level with the dotted line from T): S, set larger and bolder than the other labels
- i (centre, above the diameter): i, a lower-case dotted i
- R (right of the dotted vertical, above the diameter): R
- G (right end of the diameter): G
- Γ (lower right): Γ, a stem with its top bar running to the right
- P (lower left): P
- o (beside the foot of the line from F, above X): o, small and bold
- X (lower left, on the circle): X
- Q (below the dotted vertical, right of centre): Q, a swash form
- J (bottom, below the circle): J

Twenty-six labels read; one (the sort between V and L) is uncertain as a letter.

The Latter Case of the Second Usual Triangle [Caſus poſterior vſitati ſecundi.]

The first quantity sought, TV, is found as before; then the second quantity sought, as follows. As (by 4) the radius [Radius] TP is to PI (that is, the sum of the sine PK and of the first quantity sought, TV or IK), so is TD — by 8, the amplitude of the angle TBD (or, by 4, of IBM) — to DL. When this has been found, and from it the first quantity sought, LY, has been taken away, YD remains: the sine, namely, of the arc DW [D VV], that is, the excess of the known arc AD over a quadrant. With this sine known, the arc DW itself will not remain unknown, from the canon; it is to be added to the quadrant AW [A VV], so that the whole side sought, AD, is established — and afterwards the remaining parts in either case, by 9.

Diagram of the latter case of the second usual triangle. Hung up for Master Michael Maestlinus [Michael Mästlin]. [Diagramma caſus poſterioris vſitati ſecundi. M. Michaeli Mæſtlino ſuſpenſum.]

Diagram. A large circle, drawn solid, fills the middle of the page, with straight and curved lines, solid and dotted, inside it. Points on the circumference, going round from the top: A at the top, E just to the right of A, F at the upper right, G at the right end of a horizontal diameter, Γ at the lower right, Λ at the bottom, X at the lower left, P at the lower left above X, C at the left end of the horizontal diameter, and B at the upper left.

- The horizontal diameter runs from C through H, K, W, Y, V and i to R and G.
- A solid vertical line descends from A through i to Λ.
- A solid curved line runs from A down through W, crossing the diameter there, to D.
- A solid line runs from B down to the right to D, and on from D to M.
- A solid straight line runs from B through T to i and on down to Γ.
- A solid line runs from E under the arc to N; a solid line runs from F through N down to i, and on through M to the foot beside o, just right of X.
- Solid perpendiculars descend from B to H and from T to V; a solid vertical runs from just below L down across the diameter to D.
- A short solid line runs from P to the foot of the line from F, beside o.
- Dotted lines: from E down through T to D; a vertical dotted line from E down past S and R to Q; a dotted horizontal from J through T to S, passing below u and L; a dotted vertical from J down past K to P; a dotted line from D down to P; a dotted horizontal from P to Q; a dotted line from D down to Λ; and a dotted curve from D passing just below M, sweeping down and round to Γ.
- Marks: I-shaped cross-strokes stand across the circle at B itself, at the upper left between B and A, between A and E, between E and F, on the

right side above G, on the left side above C, on the lower left between C and P, and between P and X; one across the line BD, a little above the diameter; and one across the line from i towards M. Bold small rings stand on the curve AD about halfway between A and the diameter, on the vertical to D just below the diameter, and on the perpendicular TV just above the diameter.

Labels as read:

- A (top): A, a swash capital with a leading curl
- E (top, right of A): E
- F (upper right, on the circle): F
- N (upper right, inside the circle below F): N
- B (upper left, on the circle): B
- C (left end of the diameter): C
- H (below the diameter, right of C): H
- J (upper left of the dotted rectangle): J, with a dot standing above it
- K (above the diameter, right of the dotted vertical from J): K
- u (above the dotted horizontal, left of the curve AD): u, two stems joined at the foot, as tall as the L beside it
- L (above the dotted horizontal, right of the curve AD): L
- W (below the diameter, left of the curve AD): W
- Y (above the diameter, between the vertical to D and the perpendicular from T): Y, with curved arms
- T (on the line from B, where the dotted horizontal crosses it): T
- V (below the diameter, at the foot of the perpendicular from T): V
- i (centre, above the diameter): i, a lower-case dotted i
- S (right of the dotted vertical, at the end of the dotted horizontal): S
- R (right of the dotted vertical, above the diameter): R
- G (right end of the diameter): G
- D (left of centre, where the curve from A and the line from B meet): D
- M (at the meeting of the line from D and the line from i): M
- the sort below M, set aslant beside the line from i: two parallel strokes joined by a diagonal, a Z or an N set on its side [?]
- Γ (lower right): Γ, a stem with its top bar running to the right
- P (lower left): P
- o (beside the foot of the line from F, above X): o, small and bold
- X (lower left, on the circle): X, a swash form
- Q (below the dotted vertical, right of centre): Q, a swash form
- Λ (bottom): Λ-shaped, two strokes meeting at the apex on the circle, no crossbar

Twenty-eight labels read; one (the slanted sort below M) is uncertain as a letter.

And so much, thus far, for the two usual oblique-angled spherical triangles, on which the whole task and business of all observation rests, as will become clear in its place. See, then, reader and lover of learning [lector Philomathes];

weigh it well, and admire Byrgius's ingenuity; compare with it the utterly tangled labyrinths [Labyrhinthos] of other authors; and finally recognize from how many troubles and tediums we have been freed by that supreme craftsman himself — and requite this, together with me, with a grateful mind. There now follows in sequ- ence the solution of the remaining oblique-angled spherical triangles — though not such usual ones.

IX. On the Solution of the Remaining Oblique-Angled Triangles in General [IX. De ſolutione reliquorum obliquangulorum in genere.]

But besides the special solution of Byrgius for the two usual oblique-angled triangles, which has now been indicated and shown, the common solution of most oblique-angled triangles — except, namely, the first — depends on the solution of right-angled triangles. For it is carried out and completed by the reduction [reductio] of oblique-angled triangles to right-angled ones. Therefore, once the solution of right-angled triangles is known and understood, the solution of oblique-angled ones will not remain unknown — through their reduction, namely, to right-angled ones. This reduction of oblique-angled triangles to right-angled ones is made either by dropping [Dimiſſio] a perpendicular arc from the vertex of the triangle onto the opposite base — for the perpendicular arc, once dropped, divides and separates the oblique-angled triangle into two right-angled ones, and also an isosceles one [æquicrurum] into two equal ones — or it is made by continuing [continuatio] the arcs of the oblique-angled triangle as far as quadrants or semicircumferences [ſemiperipherias], and also the amplitudes of the angles to one or two right angles, just as was done with right-angled triangles. We shall therefore speak first of reduction and solution by dropping, and afterwards by continuation.

Postulate [Poſtulatum.]

To find the perpendicular arc in an oblique-angled triangle with the help of sines.

As the radius, or greatest sine in the canon, is to the sine of some given angle, so is the sine of some given arc to the sine of the perpendicular arc to be dropped, whether it will fall inside or outside the triangle itself. By 6. (So too, and equally, in plane triangles, but there the sides themselves are taken in place of the sines of the arcs.)

Diagram of Dropping. [Diagramma Dimiſsionis.]

Diagram. A spherical triangle drawn with curved sides, its vertex A above and its base running from B at the left through G and C to D at the right, with dotted construction lines; no circle is drawn round it.

- A solid arc runs from B up to A (the left side); a solid arc runs from A down to D (the right side); a solid arc runs along the bottom from B through G and C to D (the base).
- A solid arc runs from A down to G; a solid line runs from A down to C.
- A dotted line runs from B up through A and on to E, at the upper right; a dotted curve runs from B, arching above A, to E.

- A dotted vertical runs from A down to the level of H; a dotted horizontal runs from B, above the base, past H to D, and on to F.
- A dotted vertical, set with heavier square dots, runs from E down to F.
- Marks: an I-shaped cross-stroke stands at B itself, its ends showing above right and below left of the point; I-shaped cross-strokes stand across the arc BA, across the dotted line BA near A, across the arc AG, across the arc AD, and across the dotted vertical EF. A bold small ring stands on the line AC about halfway down.

Labels as read:

- A (vertex, above centre): A, a swash capital with a leading curl
- E (upper right): E
- B (left end of the base): B
- G (on the base, at the foot of the arc from A): G
- H (left of the dotted vertical from A, just above the dotted horizontal): H
- C (on the base, at the foot of the line from A): C, a swash form with a tail curling below
- D (right end of the base): D
- F (lower right, at the foot of the dotted vertical from E): F

Eight labels read; none left unread.

And in this way, once the perpendicular or normal arc has been found in an oblique-angled triangle, the oblique-angled triangle itself will have been divided [diflocatum] into two right-angled triangles by the arc that has been found, and so reduced [redactum ſeu reductum] to two right-angled ones. After the reduction has been made, let each of these right-angled triangles be solved according to the doctrine of right-angled triangles handed down earlier; and finally let the segments of the arc on which the normal arc falls, at right spherical angles on both sides, be added, so that the whole of that arc is obtained. So much for the reduction of oblique-angled triangles to right-angled ones, and their solution, by dropping. In this, however, it should be noted that if the normal arc falls outside the triangle to be divided, the triangle enclosed between the dropped normal arc and the side of the triangle to be divided is, as it were, a prosthaphaeresis to be subtracted from the triangle to be divided itself.

Next comes the reduction and solution of oblique-angled triangles by continuation. Since this is grasped better and more thoroughly from a very few examples put before the eye than from a great many words, we shall show it not by rules or little precepts but by examples displayed to the eye and the sight.

Continuation of oblique-angled triangles, then, is either primary or secondary. Primary continuation is the extension [reductio] of the legs [crurum] of the oblique-angled triangle until the quadrants are completed; or, to put it another way, to an isosceles right-angled triangle [Rectangulum æquicrurum], whose legs, namely, run out and issue into quadrants of the great circle [Maximæ] — and sometimes, where the condition of the proposed oblique-angled triangle requires it, as far as semicircumferences [ſemiperipherias]. And by this

primary reduction another, doubly right-angled triangle [duplex rectangulum] will be formed against the oblique-angled one, as appears here.

Two different examples of each of the four intermediate oblique-angled triangles, by primary continuation. [Quatuor ſingulorum intermediorum obliquangulorum bina diuerſa Exempla, per continuationem primariam.]

Diagrams of Continuation. Dedicated to Valentinus Otto [Valentin Otho]. [Diagrammata Continuationis. Valentino Ottoni dedicata.]

[The six diagrams stand in two columns and three rows. Each pair in a row carries the same small roman numeral inside the triangle: ii in the top row, iii in the middle row, iv in the bottom row. No diagram numbered i is printed on this page.]

Diagram, top row, left. A narrow spherical triangle ABD drawn with solid curved sides — A at the apex, D at the lower left, B at the lower right — with the numeral ii (two dotted stems) inside it, and dotted construction lines below and to the right.

- Solid arcs: A down to D; A down to B; D across to B.
- Dotted lines: from A, arching to the right, down to E; from E down to F; from B up to E; from D, in heavier dots, out to the left to G; from G down to C; from D down to C; from C along the bottom through H to F; from B down to H.
- Marks: a cross-stroke on the arc AD, about two-thirds of the way down; a cross-stroke on the arc DB; a slanting solid stroke through D, running on below it to the lower left; a short horizontal stroke across the dotted line DC; a cross-stroke at C.

Labels as read:

- A (apex): A, a swash capital with a leading curl
- E (right, on the dotted arc from A): E
- B (lower right corner of the triangle): B
- D (lower left corner of the triangle): D
- G (far left): G
- C (bottom left): C, a swash form with a tail curling below
- H (bottom, at the foot of the dotted line from B): H
- F (bottom right, at the foot of the dotted line from E): F
- numeral inside the triangle: ii

Eight letter labels and the numeral read; none left unread.

Diagram, top row, right. A second narrow spherical triangle ABD of the same shape, with the numeral ii inside it; the lettering of the lower points differs from the left-hand diagram.

- Solid arcs: A down to D; A down to B; D across to B.
- Dotted lines: from A, arching to the right, down to E; from E down to F, in heavier dots; from B up to E; from D out to the left to G; from G down to H; from D down to H; from H along the bottom through C to F; from B down to C.

- Marks: a cross-stroke on the arc AB; a cross-stroke on the arc DB; a cross of two strokes at B; a cross-stroke on the dotted line BC; a cross-stroke at C.

Labels as read:

- A (apex): A, a swash capital with a leading curl
- E (right): E
- B (lower right corner of the triangle): B
- D (lower left corner of the triangle): D
- G (far left): G
- H (bottom left, at the foot of the dotted line from D): H
- C (bottom, at the foot of the dotted line from B): C, a swash form with a tail curling below
- F (bottom right): F
- numeral inside the triangle: ii

Eight letter labels and the numeral read; none left unread.

Diagram, middle row, left. A narrow spherical triangle ABD with the numeral iii (three dotted stems) inside it, lettered and constructed as the top-left diagram.

- Solid arcs: A down to D; A down to B; D across to B.
- Dotted lines: from A, arching to the right, down to E; from E down to F; from B up to E; from D, in heavier dots, out to the left to G; from G down to C; from D down to C; from C along the bottom through H to F; from B down to H.
- Marks: a cross-stroke on the arc AD; a cross-stroke on the arc AB; a slanting solid stroke through D, running on below it to the lower left; a short horizontal stroke across the dotted line DC; a cross-stroke at C; a cross-stroke on the dotted line BH.

Labels as read:

- A (apex): A, a swash capital with a leading curl
- E (right): E
- B (lower right corner of the triangle): B
- D (lower left corner of the triangle): D
- G (far left): G
- C (bottom left): C, a swash form with a tail curling below
- H (bottom, at the foot of the dotted line from B): H
- F (bottom right): F
- numeral inside the triangle: iii

Eight letter labels and the numeral read; none left unread.

Diagram, middle row, right. A narrow spherical triangle ABD with the numeral iii inside it, lettered and constructed as the top-right diagram.

- Solid arcs: A down to D; A down to B; D across to B.
- Dotted lines: from A, arching to the right in heavier dots, down to E; from E down to F, in heavier dots; from B up to E; from D out to the left

to G; from G down to H; from D down to H; from H along the bottom through C to F; from B down to C.
- Marks: a cross-stroke on the arc AD; a cross-stroke on the arc AB; a cross of two strokes at B; a short horizontal stroke across the dotted line DH; a cross-stroke on the dotted line BC; a cross-stroke at C.

Labels as read:
- A (apex): A, a swash capital with a leading curl
- E (right): E
- B (lower right corner of the triangle): B
- D (lower left corner of the triangle): D
- G (far left): G
- H (bottom left): H
- C (bottom, at the foot of the dotted line from B): C, a swash form with a tail curling below
- F (bottom right): F
- numeral inside the triangle: iii

Eight letter labels and the numeral read; none left unread.

Diagram, bottom row, left. A narrow spherical triangle ABD with the numeral iv inside it (a dotted i followed by v), lettered as the other left-hand diagrams; here the line from B to E is drawn solid.
- Solid arcs and lines: A down to D; A down to B; D across to B; B up to E.
- Dotted lines: from A, arching to the right, down to E; from E down to F; from D, in heavier dots, out to the left to G; from G down to C, in heavier dots; from D down to C; from C along the bottom through H to F; from B down to H.
- Marks: a cross-stroke on the arc AD; a cross of two strokes at B; a slanting solid stroke through D, running on below it to the lower left; a short horizontal stroke across the dotted line DC; a cross-stroke at C.

Labels as read:
- A (apex): A, a swash capital with a leading curl
- E (right, at the end of the solid line from B): E
- B (lower right corner of the triangle): B
- D (lower left corner of the triangle): D
- G (far left): G
- C (bottom left): C, a swash form with a tail curling below
- H (bottom, at the foot of the dotted line from B): H
- F (bottom right): F
- numeral inside the triangle: iv, with a dot over the i

Eight letter labels and the numeral read; none left unread.

Diagram, bottom row, right. A narrow spherical triangle ABD with the numeral iv inside it, lettered and constructed as the other right-hand diagrams.
- Solid arcs: A down to D; A down to B; D across to B.

- Dotted lines: from A, arching to the right in heavier dots, down to the point at the right; from that point down to F, in heavier dots; from B up to that point; from D out to the left to G; from G down to H; from D down to H; from H along the bottom through C to F; from B down to C.
- Marks: a cross-stroke on the arc AB; a cross of two strokes at B; a short stroke through D, the solid arc AD running on a little below D to the lower left; a cross-stroke on the dotted line BC; a cross-stroke at C.

Labels as read:

- A (apex): A, a swash capital with a leading curl
- the sort at the right, where the dotted arc from A meets the dotted line from B: heavily inked, its right side closed like a B; read E
- B (lower right corner of the triangle): B
- D (lower left corner of the triangle): D
- G (far left): G
- H (bottom left): H
- C (bottom, at the foot of the dotted line from B): C, a swash form with a tail curling below
- F (bottom right): F, heavily inked
- numeral inside the triangle: iv, with a dot over the i

Seven letter labels and the numeral read; one (the sort at the right) uncertain between E and B.

[Two diagrams stand side by side at the head of the page, above the text. Each carries the numeral V inside the triangle. No caption is printed over them on this page.]

Diagram, left. A narrow spherical triangle ABD drawn with solid curved sides — A at the apex, D at the lower left, B at the lower right — with the numeral V inside it, and dotted construction lines below and to the right.

- Solid arcs: A down to D; A down to B; D across to B.
- Dotted lines: from A, arching to the right, down to E; from E down to F; from B up to E; from D out to the left to G; from G down to C; from D down to C; from C along the bottom through H to F; from B down to H.
- Marks: a slanting solid stroke through A, running on a little above it to the upper left; a short stroke inside the angle at A; a cross-stroke on the arc AD, about two-thirds of the way down; a slanting solid stroke through D, beginning inside the angle at D and running on below it to the lower left; a short horizontal stroke across the dotted line DC; a cross-stroke at C; a cross-stroke on the dotted line between C and H.

Labels as read:

- A (apex): A, a swash capital with a leading curl
- E (right, on the dotted arc from A): E
- B (lower right corner of the triangle): B
- D (lower left corner of the triangle): D

- G (far left): G
- C (bottom left): C, a swash form with a tail curling below
- H (bottom, at the foot of the dotted line from B): H
- F (bottom right, at the foot of the dotted line from E): F
- numeral inside the triangle: V

Eight letter labels and the numeral read; none left unread. The serifed upright between C and H is taken as a cross-stroke on the dotted line, of the same form as the cross-stroke at C, and not as a letter.

Diagram, right. A second narrow spherical triangle ABD of the same shape, with the numeral V inside it; the lettering of the lower points differs from the left-hand diagram.

- Solid arcs: A down to D; A down to B; D across to B.
- Dotted lines: from A, arching to the right, down to E; from E down to F, in heavier dots; from B up to E; from D, in heavier dots, out to the left to G; from G down to H; from D down to H; from H along the bottom through C to F; from B down to C.
- Marks: a slanting solid stroke through A, running on a little above it to the upper left; a short stroke inside the angle at A; a cross-stroke on the arc AB; a cross of two strokes at B; a cross-stroke on the dotted line BC; a cross-stroke on the dotted line between H and C; a cross-stroke at C.

Labels as read:

- A (apex): A, a swash capital with a leading curl
- E (right): E
- B (lower right corner of the triangle, below the crossing): B
- D (lower left corner of the triangle): D
- G (far left): G
- H (bottom left): H
- C (bottom, at the foot of the dotted line from B): C, a swash form with a tail curling below
- F (bottom right): F
- numeral inside the triangle: V

Eight letter labels and the numeral read; none left unread.

Secondary continuation, next, is the extension [Reductio] of the legs [crurum] of the larger right-angled triangle constituted by the primary reduction, up to the completion of the quadrants, or to an isosceles right-angled triangle [Rectangulum æquicrurum] whose legs are quadrants, just as was done above in the continuation of right-angled triangles. Here it should be noted that whenever, in that larger right-angled triangle constituted by the primary reduction, the three Data are not yet established, as in the earlier examples, they are to be found with the help of the known parts in the smaller right-angled triangle so constituted; as in the later examples, in which there is no need for the intervention of the data of that smaller triangle [?]. And so the method of finding will indeed be easier by the later arrangement [accommodationem] than by the earlier.

So much, then, for the continuation of oblique-angled triangles and for both their reduction and their solution in general. Now there follow, in turn, some special matters, and some peculiar to certain triangles [quibuſdam peculiaria quædam].

X. On the Two Ambiguous Oblique-Angled Triangles [X. De duobus obliquangulis Ambiguis.]

In general, then, those four intermediate oblique-angled triangles — that is, all except the first and the last — are solved by a double reduction to right-angled ones. But at this point two monsters [monſtra] among oblique-angled triangles must be considered, which suffer and admit an ambiguous solution, and are therefore called Ambiguous. To remove this ambiguity, therefore, along with the three usual data some further and definite condition must be added and expressly stated as well. For unless this is done, they admit a double answer without any error being committed, just like that sixth Equation [Aequatio] in the Canon of Geber [Canone Gebri], or the Rule of Algebra [Regulâ Algebrâ]. Two kinds of Ambiguous triangles are recognized: the former of these is common to plane triangles as well, while the latter is proper to spherical ones alone.

XI. On the Former Ambiguous Triangle, Common to Plane Triangles as Well [XI. De priore Ambiguo, etiam planis communi.]

In the preceding diagram of dropping [Dimiſſionis Diagrammate], unless it is expressly added, along with the three data, whether the angle at the base is acute at D or obtuse at G — or whether the perpendicular arc AC falls inside or outside the proposed oblique-angled triangle — I am still stuck and in doubt, whether the base sought is BD or BG; and I can answer the questioner ambiguously, and in either direction. Nor is it certain to me whether the prosthaphaeresis — which is either the internal triangle ACD, or the external triangle ACG, equal to it by construction (when, namely, the normal arc falls outside the triangle) — is to be added or subtracted according to the precepts given a little earlier.

This triangle, however, can be ambiguous not always, but under certain definite conditions, of which the first and chief is this:

If the given side opposite the given acute angle is smaller than the given side adjacent to that given angle, the triangle will be ambiguous; if otherwise, not at all, because then the opposite side, on account of its excessive length, cannot fall ambiguously on the base.

The remaining conditions of this ambiguous case, which are subtler — indeed most subtle — see in our *Astronomy* [noſtrâ Aſtronomiâ].

XII. On the Latter Ambiguous Triangle, Proper to Spherical Triangles Alone [XII. De poſteriore Ambiguo, ſoliſque ſphæricis proprio.]

Diagram of the second ambiguous triangle. [Diagramma ambigui ſecundi.]

Diagram. A spherical triangle BAD drawn with solid curved sides — B at the left, A at the top, D at the bottom — with dotted construction arcs and lines running out to the right as far as L.

- Solid arcs: B up to A, with G on it; A down to D, with H on it; B down and round to D.
- Dotted lines: from A, arching to the right, through M and I, down to E; from E down to the right to L; from L down to the left to F; from F along the bottom to the left through N to D; from M straight down to N; from I, curving down to the left, to K; from E down through K to F; from G, curving down, to H.
- Marks: a slanting stroke through A, with a second short stroke below A, the two forming a cross at A; a short horizontal stroke at B; a cross-stroke on the arc AD at H; a cross-stroke at E; a short horizontal cross-stroke on the dotted line EK just above K.

Labels as read:

- A (top of the triangle): A, a swash capital with a leading curl
- M (top, at the head of the vertical dotted line): M, a swash capital with a leading curl
- I (top, right of M, at the head of the dotted arc to K): I, with a dot standing above it
- E (upper right): E, its lower part crossed by a cross-stroke
- L (far right): L
- K (right of centre, low): K
- F (bottom right): F
- N (bottom, at the foot of the vertical dotted line from M): N, a swash form
- D (bottom of the triangle): D
- H (on the arc AD, right): H
- G (on the arc BA): G
- B (far left): B, with a horizontal stroke beneath it

Twelve letter labels read; none left unread. No numeral is printed on this figure.

In the attached diagram the arcs BML and BNL are semicircumferences [ſemiperipheriæ]. The angle GAH, moreover, is equal to the angle IEK by construction. Therefore in this case of ambiguity it must also be given alongside [iuxtà] whether the side to be sought is BA, less than the quadrant BM, or BE, greater than it. The cause of the ambiguity is clear enough from the diagram itself. So much, then, for the four intermediate oblique-angled triangles in general, and also for the two ambiguous ones among them in particular. There follows, also in particular, the last oblique-angled triangle.

XIII. On the Last Oblique-Angled Triangle [XIII. De obliquangulo vltimo.]

So far the twofold way and method of the common solution of most oblique-angled triangles has been shown, demonstrated and indicated: namely, through

a double reduction of the oblique-angled triangle to two right-angled ones — by dropping [dimiſſionem] the normal or perpendicular arc, that is, and by continuing the arcs and the amplitudes of the angles in the manner of right-angled triangles. But there still remains, and is yet to be solved, the sixth and last Example of oblique-angled triangles, the most perplexing of all and truly [reuerà] a kind of Gordian knot, almost indissoluble, in which the sides are investigated from all the angles, given or known. For its solution, indeed, a single method — either one of the said reductions — is by no means sufficient. To untie this most perplexing and intricate knot, the method of both reductions will be needed: both the dropping of the normal arc, and the continuation of the arcs and of the amplitudes of the angles. In addition, the following is also required for the solution of this knot.

Postulate [Poſtulatum.]

To cut a given angle or arc smaller than a semicircumference into two unequal parts in a given ratio, with the help of sines; or also to divide the sum of the two angles at the base of a triangle when the ratio of the legs is known.

Diagram of the astronomical section of an angle. Dedicated to D. D. Henricus Brucaeus [Heinrich Brucaeus]. [Diagramma Aſtronomicæ ſectionis anguli. D. D. Henrico Brucæo Dedicatum.]

Diagram. A semicircle drawn solid, standing on its diameter, with straight lines, solid and dotted, inside it. The diameter runs along the bottom from G at the left through F, A and E to D at the right; A stands at its middle, under the top of the arc.

- Solid lines: the semicircular arc from G up through C, the top at H, and I, down through B to D; the diameter G–F–A–E–D; the upright from A through M and O to H; the chord from C through M and N to B; the lines from A to C and from A to B; the perpendicular from C down to F; the perpendicular from B down to E.
- Dotted lines: from O across to I (horizontal, short); from I down through K, across the chord CB beside N, and through L to A; from K across to B; from C across to L.
- Marks: a serifed cross-stroke on CF, on AM and on BE, each at about the same height; a serifed upright stroke straddling the chord CB between M and the dotted line (the same sort-form as the cross-strokes, set upright); a small circle on the arc between C and H, and another on the arc between I and B.

Labels as read:

- H (top of the arc): H
- I (on the arc just right of H, with a dot above it): I
- O (on the upright just below H, at the left end of the dotted line to I): O
- K (on the dotted line from I, where the dotted line from B meets it): K
- M (on the upright, at its crossing with the chord CB): M
- N (just below the chord CB, right of the dotted line from I): N

- L (on the dotted line from I, where the dotted line from C meets it): L
- C (left end of the chord, on the arc): C, a swash form
- B (right end of the chord, on the arc): B
- G (left end of the diameter): G
- F (foot of the perpendicular from C): F
- A (middle of the diameter): A, a swash capital with a leading curl
- E (foot of the perpendicular from B): E
- D (right end of the diameter): D

Fourteen letter labels read; none left unread. The upright serifed stroke on CB is taken as a mark, not as a letter I; the two small circles on the arc are not letters.

In the adjoined diagram, let the angle BAC, of 100° 37′ [graduum 100.37.], be the one to be cut in the ratio of the given numbers 33 and 43. In their ratio let the subtense BC, which is known from the given angle BAC, be cut at the point N. When this is done, and the intersegment [interſegmento] MN is known (found, that is, from the section just made), and also AM, by the 2nd of the shortcuts of sines and by 33 of the first, AN will also be established, by the penultimate of the first, since the angle at M is right, by the 2nd statement of sines. And once the side AN is found, as AN itself is to NM, so will the radius AI be to IO — the sine, namely, both of the arc IH and, by 8, of the sought angle IAH. This angle will be the prosthaphaeresis, either to be added to half the angle to be cut, so that the larger portion is obtained, or to be taken away from it, so that there is ob- tained the smaller portion of the angle to be cut itself, which in this way will also have been cut in the given ratio.

Or otherwise, thus: the three sides of the right-angled triangle AMN being known, as AN is to the radius, or the sine of the right angle AMN, so is MN to IO — the sine, namely, of the angle to be found, MAN or HAI. And this by 9. This is the same as if I had said: as the segment AN of the radius AI is to the whole radius AI itself, so is NM to IO, by 6, or 2 of the sixth.

And moreover, this lemma or assumption [aſſumptum] must also be set out beforehand for the following solution; it is a corollary [conſectarium] depending on 6.

When parallels are drawn from the ends of one of two straight lines that cut each other obliquely to the other, they are oppositely, or alternately, proportional, the segments of each straight line being enclosed between them.

Diagram of the lemma. [Diagramma lemmatis.]

Diagram. A circle drawn solid, set at the left of the page, with the explanation and the following paragraphs printed beside it on the right. A solid vertical diameter runs from the top of the circle to the bottom.

- Solid lines: the circle; the vertical diameter, carrying D, A, G and C from top to bottom; a short horizontal line from E on the circumference across to D on the diameter; a straight line from E down through A and F to B on the circumference; a horizontal line from C on the diameter across to B.

- Dotted line: from G on the diameter across to F.
- Marks: a small mark on the diameter between G and C.

Labels as read:

- E (on the circumference, upper left): E
- D (just right of the diameter, at the end of the line from E): D
- A (left of the diameter, where the line EB crosses it): A, a swash capital with a leading curl
- G (left of the diameter, at the end of the dotted line): G, a swash form
- F (on the line EB, at the end of the dotted line from G): F
- C (left of the diameter, at the end of the line to B): C
- B (on the circumference, lower right): B

Seven letter labels read; none left unread.

Explanation [Declaratio]: I say that in the adjoined diagram, as AB is to BC, so is AF or AE to FG or ED, by 6.

But now everything whose devices [machinationibus] will be needed for the workhouse [Ergaſtulum] of this perplexing and intricate knot seems to be ready and girded up; and so, with a joyful rowing-call [celeuſmate], let us now attack the very mass of untying the knot.

Delian, io Paean! and io euhoe, Paean! [Delie iô pæan, & iô eüie pæan.]

VI. Given all the angles of a spherical oblique-angled triangle, to find the sides.

Diagram of the last oblique-angled [triangle]. Presented to Christophorus Clavius of Bamberg [Christoph Clavius]. [Diagramma vltimi Obliquanguli. Chriſtophoro Clauio Bambergenſi donatum.]

Hail, venerable priest. [SALVE VENERANDE SACERDOS.]

Diagram. A large figure filling most of the page, drawn with solid and dotted lines. At its centre is a spherical triangle of three solid arcs: from the vertex A near the top, one arc curves down to the left to B and another down to the right to D, and the base arc runs from B through C to D, sagging downward between them. Above A stands a small straight-lined figure with its apex at R; below the base a straight solid line HI crosses the lower part of the figure.

- Upper figure: solid straight lines from R down-left to M, from R down-right to N, and from R down through P to the point A; a solid line from Q across to N; a solid line from M to P. Dotted lines from M to N, from M down to A, and from N down to A.
- Outer dotted arcs: one from M curving down the left side, passing near K, which is joined to B by a short dotted line, and continuing from B down to H; one from N curving down the right side to a point above L, from which a dotted line runs down-left to D; a dotted line runs straight down from D to I.
- Lines through the triangle: a solid line from A down through C, V, S and X to F; a dotted line from A straight down through E and Z to T and G.

- Lower part: the solid straight line from H across to I, passing S and the dotted line near Z; a dotted line from V across to I, carrying a small filled mark between E's line and I; dotted lines from H to X, and from H curving down through F and G and up again to I; a short solid line from F across to T, with a dotted line below it from F through G.
- Marks: short I-shaped strokes are set across the lines RM, RN, QN and MP, across the left outer dotted arc between M and K, and across the right outer dotted arc between N and L; small angle-strokes stand at B, at C (several strokes crossing like a star) and at D; a short stroke crosses the dotted arc between B and H.

Labels as read:

- R (apex of the upper figure): R
- M (upper left, at the end of RM): M, a swash form
- Q (inside the upper figure, at the left end of the line QN): Q
- P (on the line from R to A, at the end of the line from M): P
- N (upper right, at the end of RN): N, a swash form
- V [?] (just above the point A, where the lines from P, M and N meet): a V-shaped sort whose left limb is serifed and whose right limb runs into the line from P; whether it is a letter or a stroke of the drawing is not settled
- A (just below the meeting point): A, a swash capital with a leading curl, the solid and dotted lines passing through it; a further diagonal stroke with a foot, to the right of the dotted line, may be the right limb of the same sort or a separate mark — not settled
- K (left, on the outer dotted arc): K
- B (left vertex of the triangle): B
- C (on the base arc, where the line from A crosses it): C
- E (on the dotted line from A, just below the base arc): E
- D (right vertex of the triangle): D
- L (right, at the foot of the outer dotted arc): L
- V (on the line from C to F, at the left end of the dotted line to I): V
- S (on the same line, just above HI): S, a sloped form
- H (left end of the straight line HI): H
- Z (beside the dotted vertical, just below HI): Z
- X (on the line from C to F, at the end of the dotted line from H): X
- I (right end of the straight line HI): I
- T (at the right end of the short line from F): T
- F (at the foot of the solid line from A): F
- G (at the foot of the dotted line from A, beside F): G

Twenty-two labels listed; the upper V is not settled as a letter, and the extra stroke beside A is not settled. The I-shaped strokes across the lines are not counted as labels.

The triangle ABD is to be solved. Its angles at A, B, D are given. Its sides are continued on both sides as far as the completion of the quadrants — namely,

both to the right, to L, and to the left to K; and both upward, to M and N, and downward, to H and I. The amplitudes of the given angles will therefore lie in the arcs HI, NL, MK, by 8; and their complements MR and NR are known, by 12; and the sines of those complements, MP and NQ, are known from the Canon; and the subtending straight line HI is known, namely the double sine of half the given angle at A.

Now the normal arc let fall from R, the pole of the arc BD — which by the construction is the quadrant RC — falls perpendicularly on the arc BD at the point C, by 2; and therefore it divides the oblique-angled triangle ABD into two right-angled triangles, ACB and ACD, by the same 2. In each right-angled triangle, therefore, two angles are known: namely the angle at the base, given in each, and the angle at C, a right angle in each. It remains, therefore, for us to cut the remaining given angle at A in the ratio of the known sines MP and NQ, by the preceding postulate. This is done as follows.

When the normal arc or quadrant RC, let fall earlier, has now been drawn out further downward as far as the arc HI (that is, the amplitude of the given angle at A), AF too will be a quadrant, by 2 (since A is the pole of the said arc HI, by 8). And therefore the quadrant AF itself will also fall on the arc HI perpendicularly, or at right angles on both sides, and will cut its subtense HI in the ratio we are to find, into the segments, say, HS and IS. The ratio of these segments is evident from the lemma set out before, as follows: as the sum of the sines MP and NQ is to the subtense HI, so, oppositely or alternately, is the sine MP, NQ to the segment IS, HS.[28]

Once the segments IS and HS of the subtense HI that is to be cut have been found, their common intersegment SZ will also be established; and from this known intersegment SZ, by the preceding postulate, the sine FT will also become evident — the sine, that is, of the arc FG, which is to say, by 8, of the intermediate angle FAG. This angle FAG will be the prosthaphaeresis, either to be added to or to be subtracted from half the angle at A that is to be cut, so that its larger or its smaller portion is obtained, just as we directed in the preceding postulate. And in this way the said angle at A will have been cut in the proposed ratio, as desired. And so in each right-angled triangle, ACB and ACD, the three angles will be known; and from these, once known, their sides or arcs will afterwards become known as well, by the first Example of the right-angled triangles. And so, with the solution of spherical triangles, both right-angled and oblique-angled, now complete, we must pass on to the solution of plane triangles.

[28] The page sets "MP" above "NQ" inside one pair of braces and "IS" above "HS" inside a second pair, after "sinus" and after "ad segmentum" respectively; the upper names pair with each other, MP with IS, and the lower, NQ with HS.

XIV. On Plane Triangles, and Their Solution [XIV. De Triangulis planis, eorumque ſolutione.]

Just as the solution of most oblique-angled spherical triangles depends on the solution of the right-angled ones, so in the end the solution of plane triangles seems to depend on the solution of spherical ones. But in spherical triangles, in each kind, there were six different Examples, because of the sixfold permutation of the different givens; here, however, there will be only six different Examples in all: namely, one general, two of right-angled triangles, and the remaining three concerning oblique-angled ones. We shall speak of their solutions in order.

First, then, and in general, this is to be noted in plane triangles:

I. In a plane triangle all three angles taken together are equal to two right angles, by 32 of the First [book] of Euclid's *Elements*. Therefore, when two angles of a plane triangle are known, the remaining one will likewise be established, as the complement of the two known ones up to the value, or completion, of two right angles. And so in plane triangles there are given either all the angles together, as in the first and general example; or all the sides together, as in the last and special oblique-angled one; or only one angle together with the sides, and that in the four intermediate examples.

Secondly — and this too in general, and in the first, general triangle, namely the one in which the sides are to be found from all the angles given — one must know this:

II. When all the sides of a plane triangle are given, then together with their magnitude or quantity their ratio is also given; but not conversely. For when all the angles are given, only the ratio of the sides is given, through the straight lines subtending the angles, or the double sines of the given angles; but not with it a fixed magnitude or definite quantity of the sides themselves, which indeed cannot be given (unless one side is given).

Therefore, in the first and general Example:

Diagram. A circle, set at the left of the paragraph on the first and general example. A plane triangle is inscribed in it, with its vertices A (top), D (left) and B (lower right) on the circumference, its three sides drawn as solid straight lines AD, AB and DB. Inside the triangle a point C stands near the centre of the circle. Dotted lines run from C to each vertex, A, D and B. Three further dotted lines run from C through the sides to the circumference: one up to the left, crossing AD at E and reaching the circle at K; one to the right, crossing AB at F and reaching the circle at i; and one straight down, crossing DB at G and reaching the circle at H. The circumference is broken by a short gap on the arc between B and H, where a short detached stroke stands in the gap.

Labels as read:

- A (top, on the circumference): A, a swash capital
- K (upper left, on the circumference, at the end of the dotted line through E): K
- E (on the side AD, where the dotted line from K crosses it): E

- C (inside the triangle, where the dotted lines meet): C, a small sort with the dotted lines running into its foot
- F (on the side AB, where the dotted line to i crosses it): F
- i (right, on the circumference): i, lower case, with its dot
- D (left vertex, on the circumference): D
- G (on the side DB, where the dotted vertical crosses it): G
- B (lower right vertex, on the circumference): B
- H (bottom, on the circumference, at the foot of the dotted vertical): H

No other letter is printed on the figure.

Given all the angles of a plane triangle, the canon of sines will supply the sines of the given angles; these, when doubled, will be the subtenses of the given angles, and they will show the ratio to one another of the sides sought in the proposed triangle. But if one of the sides happens to be known, the rest will also be known in a definite quantity, by 9, the Element of Triangles [Triangulorum Elementum]. The proof is seen, and is evident to anyone who looks at the attached diagram. So much for the first and general Example. The special ones follow, and first the two of right-angled triangles.

XV. On Plane Right-Angled Triangles and Their Solution [XV. De planis Rectangulis,eorumque ſolutione.]

The solution of a plane right-angled triangle is either geometrical or astronomical. The geometrical solution is made by the penultimate of the first [book] of Euclid. For when two sides of a plane right-angled triangle are known, the remaining one also becomes known by squaring the two known sides: afterwards the greatest side becomes known by addition, and the lesser ones by subtraction of the squares so produced, and finally by the usual finding of the side. After that, however, the acute angles are found astronomically, by 9. It is therefore plainly evident that this geometrical solution is a solution of the sides only, and not of the angles as well, and that it is therefore imperfect. The astronomical solution accordingly follows.

The former Example: where both legs of the right angle are known. Let the statement concerning it be as follows: [Exemplum prius: vbi notum eſt vtrumque crus anguli Recti. De quo Enunciatum tale eſto:]

If a plane right-angled triangle is placed on a circle in such a way that the straight line subtending the right angle becomes a straight line inscribed in the circle, and one or other leg of the right angle becomes a segment of a diameter of that same circle, then it will be so: first, as the leg just mentioned is to the remaining leg, so that remaining leg itself is to the complement of the said segment of the diameter, up to the completion of the diameter, by 8 of the sixth [book] of Euclid — by which, that is, the said remaining leg will be the mean proportional, or in mean proportion, between the segments of the said diameter of the circumscribed circle. And afterwards it will be, secondly:

as the diameter just found, or the sum of its said segments, in the given measure, is to the double radius, or the greatest sine in the canon doubled, in

the assumed measure 2,000,000,000, so is the said remaining leg to the sine of the angle that is double the angle opposite that remaining leg. And this is by 4 of the sixth and by 20 of the third [book] of Euclid.

Diagram. A circle, set at the left below the opening paragraph, with a horizontal diameter running from A (left) to G (right). The diameter is drawn solid from A through D to C, and dotted from C to G; one dot on the dotted stretch is heavier than the rest. From A a solid straight line rises to B on the upper circumference, and from B a solid vertical line falls to C on the diameter, so that ABC is a right-angled triangle with its right angle at C. Dotted lines run from B down to D and from B down to G. A dotted vertical line runs from E on the upper circumference down through D to F on the lower circumference. A round grey spot stands inside the circle between the line AB and the upper arc; it is not taken as a printed point.

Labels as read:

- A (left end of the diameter, on the circumference): A, a swash capital
- E (top, on the circumference, at the head of the dotted vertical): E
- B (upper circumference, right of E, where the lines from A, C, D and G meet): B
- D (on the diameter, where the dotted vertical crosses it): D
- C (on the diameter, at the foot of the vertical from B): C
- G (right end of the diameter, on the circumference): G, a swash form
- F (bottom, on the circumference, at the foot of the dotted vertical): F

No other letter is printed on the figure.

The proof of this, moreover, will be clear and evident from the attached diagram; therefore the matter does not seem to need tedious verbosity.

The same thing, contracted into fewer words, can also be put in this way: Either leg of a right-angled triangle is the mean proportional between the segments of the diameter of a circle, one segment of which diameter is the remaining leg of the said triangle, and in which circle, moreover, the straight line subtending the right angle of the said triangle is inscribed.

Therefore:

The remaining leg itself will be the sine of the angle that is double the angle opposite that remaining leg. So much for the former Example of right-angled triangles — and, so far as I know, no one before us has solved it astronomically. The remaining one follows.

The latter Example: when the line subtending the right angle, and the other leg of the right angle, are known. [Exemplum poſterius: quando nota angulo recto ſubtenſa, & alterum crus anguli recti.]

The solution of this Example is evident by 9, the common Element of Triangles, and by the first observation on plane triangles set out a little before. Therefore, without giving any further instruction about it, let us at last pass on to the solution of plane oblique-angled triangles. Nevertheless, besides the solution of right-angled triangles, it seemed good to add, by way of a corollary

and finishing touch, the following points about right-angled triangles, which are most pleasant to know.

I. When the greatest side of a right-angled triangle, or the side subtending the right angle, is made the radius of a circle circumscribed about the triangle from either of its acute angles, the legs will be the sines of the angles opposite them. (And from this the 9th common Element of Triangles seems to arise.) Accordingly:

II. Either leg of the right angle is the sine of the angle opposite it; and the remaining leg is the sine of the complement of the angle opposite the remaining leg. Therefore:

III. As the radius is to the sine of either angle, so is the side of the right-angled triangle subtending the right angle to the leg opposite the said angle.

Diagram of plane right-angled triangles. Dedicated to Matthaeus Baderus, Rector of the school of Frankfurt. [Diagramma Rectangulorum planorum. Matthæo Badero Rectori ſcholæ Francofurtanæ conſecratum.]

Diagram. Two intersecting circles of equal size, one set higher and to the right, the other lower and to the left, with a right-angled triangle ABC drawn between their centres. The upper circle has its centre at A; the lower circle has its centre at B. Each circle passes through the other's centre: the upper circle's left-hand arc runs down through H to B, and the lower circle's upper arc runs from G across to A. Solid lines: AB (the line joining the two centres), AC (vertical, falling from A), and BC (horizontal, running from B to C), so that the right angle stands at C. Dotted lines: a horizontal through A, running from H on the upper circle's left circumference, through D, through A, to the upper circle's right circumference (no letter at that end); a vertical through A, running from the upper circle's top (no letter) down through A and C to F on the upper circle's lower circumference; a horizontal through B, running from the lower circle's left circumference (no letter) through B and C to E on the lower circle's right-hand arc; and a vertical through B, running from G at the top of the lower circle down through D and B to the lower circle's bottom (no letter). D stands where the dotted line H–A crosses the dotted vertical G–B.

Labels as read:

- G (top of the lower circle, at the head of the dotted vertical through B): G
- H (left, on the upper circle's arc, at the end of the dotted horizontal through A): H
- D (where the dotted horizontal H–A crosses the dotted vertical G–B): D
- A (centre of the upper circle, at the upper vertex of the triangle): A, a swash capital
- B (centre of the lower circle, at the left vertex of the triangle): B
- C (at the foot of the vertical from A, the right-angle vertex): C
- E (right, on the lower circle's arc, at the end of the dotted horizontal through B and C): E

- F (bottom of the upper circle, at the foot of the dotted vertical through A and C): F

No other letter is printed on the figure. Four ends of dotted lines on the circumferences carry no letter.

XVI. On Plane Oblique-Angled Triangles and Their Solution [XVI. De planis Obliquangulis, eorumque ſolutione.]

Of plane oblique-angled triangles, as we said, three Examples remain. The first two of them correspond to the two Examples of right-angled triangles; the last, or remaining one, is the ἀντίστροφον [Greek: antistrophon, "converse"], or converse, of the first and general Example, although it is only special and always oblique-angled. For a right-angled triangle of this kind is not offered for solution (since in that case four things would be given).

I. And the first of these Examples — namely the one in which, together with a given angle, two sides are given, one adjacent to the given angle and the other opposite it — is indeed very easily solved, just as is the right-angled triangle that corresponds to it and immediately precedes it: by 9, the common Element of Triangles, and by the latter part of 32 of the first [book] of Euclid.

Diagram. A plane triangle ABD, set at the left of the narrow column of text, with its base BD horizontal and its apex A above. A solid straight line runs from A down to G on the base, between B and C; a dotted vertical runs from A down to C on the base, between G and D. Beyond D the base is continued as a dotted line through F to H. A dotted line rises from A up to the right to E, above the figure; from E a dotted vertical falls to F, and a second dotted line, slightly curved, runs from E down to H. Short cross-strokes stand across the solid line AG, across the side AD, across the dotted line E–F and across the dotted line E–H. A short stroke stands inside the angle at B, between BA and BD. A small ring stands on the dotted line AC, a little below halfway from A to C, with the dots running into it above and below.

Labels as read:

- A (apex of the triangle): A, a swash capital
- E (top, where the dotted lines from A, F and H meet): E
- B (left end of the base): B
- G (on the base, at the foot of the solid line from A): G, a swash form
- C (on the base, at the foot of the dotted vertical from A): C
- D (right end of the solid base, at the foot of the side AD): D
- F (on the dotted continuation of the base, at the foot of the dotted vertical from E): F
- H (right end of the dotted continuation of the base, at the foot of the dotted line from E): H

No other letter is printed on the figure. The small ring on AC is not read as a letter [?]: it may be a letter O or a mark on the line.

But since this Example, moreover, also corresponds to the earlier Ambiguous [Example] of spherical triangles, what we taught concerning the said ambiguous

Example of spherical triangles must necessarily be considered here as well, and recalled to memory: namely, which of the two angles to be found is acute or obtuse. For this too must be given here, and is required, together with the three usual given things — or beyond the three due given things.

II. The second Example of oblique-angled triangles — namely, when together with a given angle both legs of the given angle, or both sides adjacent to it, are given — is solved either without a perpendicular straight line let fall from the vertex onto the opposite base of the triangle, by the Postulate set out before at the last spherical oblique-angled triangle; or with the help of a perpendicular let fall. And in this Example that perpendicular is found astronomically, or with the help of sines, just as, among the spherical oblique-angled triangles, mention was made of plane ones in passing and it was indicated by us; and so there will be no need to repeat it.

Diagram. A semicircle standing on a horizontal diameter KN, with its centre at A, set at the left of the narrow column of text. The arc runs from K up through I, E, F and B down to N. Solid lines: AI, continued beyond I outside the semicircle to D at the upper left; AB; the chord IB; a long straight line from D down to B; and a short horizontal line from C to F near the top of the arc. Dotted lines: a vertical from E down through C and H to A; a line from A up to F, crossing IB at G; a perpendicular from I down to L on the diameter; and a perpendicular from B down to M on the diameter. Short cross-strokes stand across AI and across AB. Small strokes stand at A between the lines meeting there.

Labels as read:

- D (upper left, outside the arc, at the end of the line through A and I): D
- E (top of the arc, at the head of the dotted vertical): E
- C (on the dotted vertical just below E, at the left end of the short line to F): C
- F (on the arc, right of E): F
- I (left, on the arc, where AD crosses it): I
- H (on IB, where the dotted vertical from E crosses it): H
- G (on IB, where the dotted line from A to F crosses it): G
- B (right, on the arc): B
- K (left end of the diameter): K
- L (on the diameter, at the foot of the perpendicular from I): L
- A (centre, on the diameter): A, a swash capital
- M (on the diameter, at the foot of the perpendicular from B): M, a swash form
- N (right end of the diameter): N

No other letter is printed on the figure.

III. Finally, the last Example of plane oblique-angled triangles — namely, where the angles are to be found from all the sides given — cannot be solved at all without a perpendicular let fall. Moreover, that perpendicular let fall can be found here not astronomically, or with the help of sines, as in the one

immediately preceding (since here, of course, no angle is given or known), but geometrically — though, to be sure, with greater tedium of labour. How this is done, it has seemed good to show in what follows, by a threefold geometrical reasoning and way. And, found by any one of these reasonings, the per- pendicular, the oblique-angled triangle will have been split into two right-angled triangles, and it will easily be solved. Therefore one must first ascertain the segments of the triangle's base, at whose common boundary the perpendicular will fall.

[Outline, set as two braces side by side: the first brace joins ”penultimam” (upper) and ”ſextam” (lower); the second brace, to its right, joins ”tertiÿ” (upper, level with ”penultimam”) and ”ſecundi.” (lower, level with ”ſextam”); a closing brace gathers that pair to ”Euclidis.” A small isolated dot stands just right of ”ſextam”.]

- Geometrical investigation of the segments of the base is made by [Geometrica inueſtigatio ſegmentorum baſeos fit per]
  - the penultimate [penultimam] or the sixth [ſextam]
  - of the third [tertiÿ] or of the second [ſecundi.]
  - of Euclid [Euclidis.]

Diagrams presented to Iohannes Iungen, chief town clerk of Schweidnitz, a consummate arithmetician. [Diagrammata Iohanni Iungen Archigrammatæo Schvuidnitienſi, ſummo Arithmetico donata.]

Diagram. A circle with its centre at A, and a plane triangle EAB whose base EB runs horizontally from E, outside the circle on the left, through D (where the base meets the circle's lower left arc) and C to B on the circle's right. Solid lines: EA, crossing the circle's left arc at I; AB; the base EB; and the vertical AC, falling from A to C on the base. Dotted lines inside and below the circle: a radius from A up to the right to G on the circumference; a line from A down to the left to D; a line from A down, slightly left of AC, crossing the base and the circle's lower arc at K and continuing down to P; and a line from E down to the right through L to K on the lower arc. A dotted line falls from D through L and then curves as an arc down to the left to M. A second dotted arc runs from F at the upper left down to the right to I on the circle. Dotted lines round the outside: E to F, up and slightly to the left; F to H, across the top to the upper right; H down to the circle's upper right arc just above G; E straight down to M; M across to N; B straight down to N; E down and slightly to the left to Q; and Q across to P. Near M the dotted arc from D shows a short gap or stray mark in its dots.

Labels as read:

- F (upper left, at the corner of the dotted lines from E and to H): F
- H (upper right, at the corner of the dotted lines from F and to the circle): H
- G (right, on the circumference, at the end of the dotted radius from A): G
- A (centre of the circle, at the apex of the triangle): A, a swash capital

- I (left, on the circle's arc, where EA crosses it and the dotted arc from F ends): I
- E (left end of the base): E
- D (on the base, where it meets the circle's lower left arc): D
- C (on the base, at the foot of the vertical from A): C, a swash form
- B (right end of the base, on the circle): B
- L (below D, where the dotted line from D crosses the dotted line E–K): L
- K (on the circle's lower arc, on the dotted line from A to P): K
- M (lower left, at the foot of the dotted vertical from E): M, a swash form
- N (lower right, at the foot of the dotted vertical from B): N
- Q (bottom left, at the end of the dotted line from E): Q, a swash form
- P (bottom, where the dotted line from A through K meets the dotted line from Q): P

No other letter is printed on the figure.

I. The First and Easiest Method, by the Penultimate [Proposition] of the Third [Book] [I. Prima ac facillima ratio, per penultimam tertij.]

When the sum of the triangle's legs has been multiplied by their difference, the product divided by the base, and the quotient subtracted from the base, half of what remains will be the smaller segment of the base; and when the said quotient is added to this, the sum will be the greater segment.

Diagram. A plane triangle ABC with its apex A above and its base BC horizontal, with a dotted square set on each of its three sides. Solid lines: AB, AC and the base BC. On AB a dotted square stands outward to the upper left, with corners A, O, N, B. On AC a dotted square stands outward to the upper right, with corners A, P, Q, C. On the base BC a dotted square hangs below, with corners B, E, G, C, its left side running down from B through K to E and its right side running down from C through H to G. A dotted vertical falls from A through D on the base and through I to F on the square's lower side EG; a dotted horizontal runs across the lower square from K through I to H. A dotted horizontal through A runs from L on the left to R on the right; from L a dotted vertical falls to M on the base, just right of B; from R a dotted vertical falls to S, at the right end of a dotted continuation of the base beyond C, crossing the dotted side CQ of the right-hand square on the way. A small isolated dot stands just above A.

Labels as read:

- O (top, the upper corner of the square on AB): O
- N (left, the outer corner of the square on AB): N
- P (upper right, the upper corner of the square on AC): P
- A (apex of the triangle): A
- L (left end of the dotted horizontal through A): L
- R (right end of the dotted horizontal through A): R
- Q (right, the outer corner of the square on AC): Q, a swash form
- B (left end of the base): B

- M (on the base, just right of B, at the foot of the dotted vertical from L): M, a swash form
- D (on the base, at the foot of the dotted vertical from A): D
- C (right end of the base): C, a swash form
- S (on the dotted continuation of the base, at the foot of the dotted vertical from R): S
- K (on the left side of the lower square, at the left end of the dotted horizontal): K
- I (inside the lower square, where the dotted vertical from A crosses the dotted horizontal K–H): I
- H (on the right side of the lower square, at the right end of the dotted horizontal): H
- E (lower left corner of the lower square): E
- F (on the lower side of the lower square, at the foot of the dotted vertical from A): F
- G (lower right corner of the lower square): G, a swash form

No other letter is printed on the figure.

II. The Second and Usual Method, by the Penultimate [Proposition] of the Second [Book] [II. Secunda ac vſitata ratio, per penultimam ſecundi.]

When the square of either leg has been subtracted from the sum of the squares of the other leg and the base, and half of the remainder divided by the base, the quotient will be the segment of the base from the other leg up to the perpendicular; what remains of the base will be the remaining segment.

Diagram. A plane triangle with its apex A above and its base running horizontally from B on the left to C on the right; between them the base carries G, E and D. Solid lines: AB, AC, the base BC, and a fourth solid line from A down to E on the base. A dotted vertical falls from A through D on the base and on down to T. On AB a dotted square stands outward to the upper left, with corners A, K, i, B. On AC a dotted square stands outward to the upper right, with corners A, M, L, C. A dotted horizontal through A runs from H on the left to V on the right; from H a dotted vertical falls through N (where it crosses AB) to G on the base; from V a dotted vertical falls to X, at the right end of a short dotted continuation of the base beyond C. A short dotted line runs from N down to the right to E. Below the base: a dotted vertical falls from B through F to S; a dotted horizontal runs from S to T; a dotted vertical falls from C to R and on to the level of Q; a dotted horizontal runs from F across to R; a dotted vertical falls from E to P, and a dotted horizontal runs from P across to the foot of the line from C, beside Q. A dotted arc curves from E down to the left to F. In the dotted line from H to A, a little right of H, stands a short double stroke followed by a dot (resembling "=•"), set into the line of dots.

Labels as read:

- K (top, the upper corner of the square on AB): K

- i (upper left, the outer corner of the square on AB): a dotted letter, an I-shaped stem with a dot above it
- M (upper right, the upper corner of the square on AC): M, a swash form
- A (apex of the triangle): A, a swash capital
- H (left end of the dotted horizontal through A): H
- V (right end of the dotted horizontal through A): V
- L (right, the outer corner of the square on AC): L
- N (on AB, where the dotted vertical from H crosses it): N
- B (left end of the base): B
- G (on the base, at the foot of the dotted vertical from H): G
- E (on the base, at the foot of the solid line from A): E
- D (on the base, at the foot of the dotted vertical from A): D
- C (right end of the base): C, a swash form
- X (on the dotted continuation of the base, at the foot of the dotted vertical from V): X
- F (on the dotted vertical below B, where the arc from E and the horizontal to R end): F
- R (on the dotted vertical below C, at the right end of the horizontal from F): R
- P (below E, at the foot of the dotted vertical from E): P
- Q (below R, at the right end of the horizontal from P): Q, a swash form
- S (bottom left, at the foot of the dotted vertical from B): S, a swash form
- T (bottom, at the foot of the dotted vertical from A, at the right end of the horizontal from S): T

No other letter is printed on the figure.

III. The Third, and Most Ingenious of All Methods, by the Sixth [Proposition] of the Second [Book] [III. Tertia omniumque ingenioſiſsima ratio, per ſextam ſecundi.]

When the square of the smaller leg has been subtracted from the square of the greater, the remainder divided by the base, and the quotient subtracted from the base, half of what remains will be the smaller segment of the base; and when the said quotient is added to this, the sum will be the greater segment.

Let us therefore make a trial, in numbers, of the given sides of the accompanying triangle, by the first way [per primum modum].

Diagram. A circle with its centre at A, and a plane triangle EAB. The base runs horizontally from E, well outside the circle on the left, through D (where it meets the circle's left arc) and C to B on the circle's right. Solid lines: EA, crossing the circle's left arc at F; AB; the base EB; and the vertical AC, falling from A to C on the base. Dotted lines: a radius from A up to the right to G on the circumference, and a line from A down to the left to D. For a stretch at the upper right, and again just above B, the circle's line is broken into dashes. Numbers are set along the lines: "26" on EF, "15" on FA, "15" along AB (set along the slope of the line), a single numeral beside AC, "28" under ED, "12" under DC and "12" under CB. Faint dotted squares and lines, mirror

show-through of the figure on p. 67, lie behind this figure and are not part of it; a faint mark resembling "7" just right of the circle, above B, is of the same pale tone and is not read as a label.

Labels as read:

- A (centre of the circle, at the apex of the triangle): A, a swash capital
- G (right, on the circumference, at the end of the dotted radius from A): G
- F (on EA, where it crosses the circle's left arc): F
- E (left end of the base): E
- D (on the base, where it meets the circle's left arc): D
- C (on the base, at the foot of the vertical from A): C, a swash form
- B (right end of the base, on the circle): B

Numbers as read:

- on EF: 26
- on FA: 15
- along AB: 15
- beside AC: 9 [?] — a single sort with a closed loop above and a tail below curving to the right
- under ED: 28
- under DC: 12
- under CB: 12

No other letter or number is printed on the figure.

But the proofs and causes of all these appear from the accompanying diagrams, and also from the cited *Elements* of Euclid; and they will be explained more broadly and more fully in our Astronomy. Meanwhile, excellent lover of learning [philomathes], be content with these few things and enjoy them. But now that the segments of the base are known by any one of these [methods], the perpendicular to be let fall will be established and found by the penultimate [proposition] of the first [book] of Euclid.

XVII. On Plane Obtuse-Angled [Triangles], and a Small Caution Concerning Them [XVII. De planis obtuſangulis, inque illis Cautiunculâ quâdam.]

In plane obtuse-angled triangles, one must finally be warned that the sine of an obtuse angle, that is, of an angle greater than a right angle, always indicates an arc or amplitude of that same obtuse angle greater than the semicircumference, or than the straight line subtending a right angle (that is, the very diameter of the circle). And this is by Element 13.

And so much for plane triangles, and for the whole Doctrine of Triangles. This doctrine — arduous, arcane, abstruse, hidden, most ample and most highly artful, indeed profound, secret and most ingenious — has been sought, desired and longed for with the highest prayers by so many eminent men and over so many centuries; no less has it been promised, boasted of and attempted in vain by such great masters, yet it has never been completed by a just and entire method. By us, however, it has been accomplished as if in passing [Greek, as

printed: "cν" with a mark above the ν, then "Γαρόδῳ"; read as ἐν παρόδῳ (en parodō, "in passing"), the standard form having π where the page prints Γ], and as though we were engaged in something else. With it we conclude the Astronomical Calculation.

End of the Doctrine of Triangles; the Observation follows. [Finis Doctrinæ Triangulorum: ſequitur Obſeruatio.]

Chapter IV. On the Observation of the Places of the Fixed Stars [Caput IV. De obſeruatione locorum ſtellarum fixarum.]

So far the Astronomical Calculation has been completed, in the doctrine of sines and in the use of sines in the Doctrine of Triangles. Next there now follows the use of the triangles themselves as well, in the other part of astronomy, namely in observation. And so:

Observation is the latter part of astronomy, and consists in observing the stars. Just as the calculation was geometrical, so the observation will be dioptrical. The dioptrical instruments by which the whole business of every observation is carried out and completed are two, namely the quadrant and the sextant.

A quadrant is the fourth part of the whole circumference of a circle, that is, an arc of 90°.

A sextant is the sixth part of the whole circumference of a circle, that is, an arc of 60°.

The observation of the stars, moreover, consists in investigating either the places of the fixed stars or the motions of the planets. Therefore, before we approach the observation itself, we must introduce, for tracking down the places of the fixed stars, certain great circumferences [peripheriæ Maximæ] of the cosmic sphere (which, in the customary way, we shall call the heaven), that is, celestial circles, which bisect the heaven itself and divide the world into two equal hemispheres. In exploring the motions of the planets, on the other hand, we must establish certain periods, or orbits and tracks, of the planets, along which they advance. They always roll through the air with equal motion, although they appear to us in diverse gyrations, and they are carried toward the east by the unwearied force of the moving mind of the world [vi Mundanæ mẽtis motricis].

There are, then, six great circumferences of the heaven, or celestial circles: three right [Rectæ ſeu Recti], and as many oblique. The right ones are those that bisect the heaven or the world rightly, and part it into two right hemispheres, as:

- I. The Horizon, into upper and lower, or, for one standing on a plain, into visible and invisible.
- II. The Meridian, into left and right, or into eastern and western.
- III. The East circle [Orientalis], into front and back, or into southern and northern.

Next, the oblique ones are those that bisect the heaven or the world obliquely for us, and separate it into two oblique hemispheres, as:

- IV. The Equator, into Arctic and Antarctic, or into ours and the one opposite to ours.
- V. The Polar circle [Polaris], into the hemisphere above and the hemisphere below the poles, both of the Equator and of the world.

Between these two poles, of both the Equator and the world (for they are the same), that straight line, conceived in the mind or constructed, is called the axis both of the world and of the Earth, because it passes through the Earth itself, through its centre. This axis is the diameter of the said polar circle, as also of the meridian. About this axis all the motions of all the bodies of the world take place, namely of the Earth and of the planets. The Earth's motion is by revolution [circumuolutionem] upon that same axis, its centre fixed and remaining in place (as will appear a little later from our new hypotheses [noſtris nouis Hypotheſibus]). The planets' motion, on the other hand, is by circumgyration [circumgyrationem] together with their bodies, their centres being moved; each one, however, moves upon its own particular axis and in a period proper to itself, and about a centre that is either stable in the Sun or movable about the Sun. Of these periods of the planets, the period of the Sun itself, that is, the track described through the air by the solar motion, or the orbit of the Sun, is the leader and coryphaeus [Coryphæa] of all the others. The great circumference (or celestial circle) answering to it in the heaven itself is the sixth and remaining one of the six oblique circles mentioned, and is called the Ecliptic.

- VI. The Ecliptic, then, is a great circumference, or celestial circle, answering in the heaven itself to the path, orbit or track of the Sun; directly beneath it lies that very path and orbit through the air which the so- lar body describes by its motion. This path or orbit of the Sun is called the solar period [ſolaris periodus], and we shall speak of it in its place, among the remaining periods of the remaining planets.

And although the Ecliptic itself is in truth and of itself the most right [rectiſsima] of all the circumferences, and altogether immutable and invariable, it is nevertheless by accident — namely, because the axis of the world, passing through the Earth and reaching to both poles of the world, passes obliquely through the Ecliptic itself, so that all movable bodies are carried round obliquely about that axis of the world with respect to the Ecliptic, and roll through the air — by such an accident, I say, that the Ecliptic itself, as it seems to us, is called oblique and is counted among the oblique circles.

All the right circles bisect one another, and form right spherical angles with one another on either side; so too do the two oblique ones, the Equator and the polar circle. And of any two circles that bisect each other, the diameter of the one is the axis of the other.

The Ecliptic and the Equator, however, intersect each other at an acute spherical angle, of a quantity or amplitude of more or less 23½°. Hence the Ecliptic and the polar circle [intersect] at the complement of the former, 66½°, since all circles that intersect one another rightly are distant from one another

by a quadrant of a great circle, that is, 90°. The obliquity at which the right and the oblique circles intersect one another, on the other hand, varies according to the size of the elevation of the pole at any place on the terrestrial globe. For those living under the Equator itself, however, this obliquity does not appear at all; rather, there the Equator and the East circle, and likewise the Horizon and the polar circle, grow together with one another, coincide, and pass into one and the same great circle. Therefore the sphere is called right [ſphæra Recta] there, but in all other places, on either side away from the Equator, oblique [ſphæra obliqua]. The measure of this obliquity rests with the elevation of one pole or the other.

Moreover, that obliquity, or angle at which the Ecliptic and the Equator intersect each other, is believed to change and vary from century to century, both in quantity and in place. As regards the change in quantity, we may doubt it, not without evident cause and reason, and rightly; this dispute and doubt, which is to be settled and argued out, we shall reserve for our Historical Astronomy [noſtram Aſtronomiam Hiſtoricam], to which it seems properly to belong. Of the other change or variation, however, which occurs as the place is changed and varied, no doubt whatever is left to us, since it plainly agrees with experience itself, from observations that have been made — although it is not yet held to be sufficiently explored. This change is commonly called the precession of the equinoxes [præceſsio Aequinoctiorum]; by us, however, it will be called the creeping forward of the Equator [proreptio Aequatoris]. For the Equator creeps forward and crawls backward beneath the Ecliptic, against the motion of the movable bodies, as do all the remaining periods of the planets as well, as will become clear later in its place.

The points, moreover, at which any two great circles intersect each other, diametrically opposite, are distinguished by various names. For the point in which there intersect each other

- [the outer brace, at "interſecant":]
  - the Horizon and [Horizon &]
    - the Meridian, toward [Meridiana verſus]
      - the south, is called the southern [point] [Meridiem, Auſtrale.]
      - the north, the northern [point] [Septentrionem, Boreale.]
    - the East circle, toward [Orientalis verſus]
      - the rising, the eastern [point] [Ortum, Orientale.]
      - the setting, the western [point] [Occaſum, Occidentale.]
- [the outer brace, continued: a vertical rule at the left of the page, without label or cusp on this page:]
  - the Meridian and [Meridiana &]
    - the East circle [Orientalis]
      - above [?] [ſupr?]
      - below [infra]

- [a closing brace gathers these two to:] the Horizon [Horizontem]
  - the vertical [point] [Verticale.]
  - the foot [point] [Pedale.]
- the polar circle, toward [polaris verſus]
  - the north [Boream]
  - the south [Auſtrum]
  - [a closing brace gathers these two to:] the pole [polus]
    - Arctic [Arcticus.]
    - Antarctic [Antarcticus.]
- the Ecliptic and the Equator, the Sun [Ecliptica & Aequator ſolem]
  - bringing in [aduehens]
  - carrying away [abuehens]
  - [a closing brace gathers these two to:] the equinox [Aequi noctiū]
    - vernal [Vernum.]
    - autumnal [Autumnum.][29]

Finally, the points on the Ecliptic that lie midway between the equinoxes, and are distant from each of them by a quadrant of a great circle, are called the solstices [Solſtitia]: the one toward the north, the summer solstice; the other toward the south, the winter solstice. Through these points three other circumferences, or celestial circles, are commonly drawn. Two are lesser circles, described from the solstitial points themselves, parallel to the Equator and touching the Ecliptic at those same solstitial points; these are called the Tropics [Tropici]. The one toward the north is the summer tropic, or Tropic of Cancer; the other toward the south, the winter tropic, or Tropic of Capricorn — named, that is, from the dodecatemories [dodecatemoria], or twelfths [vnciæ], of the Ecliptic that take their beginning there, into which, as into twelve equal parts (commonly called signs), the Ecliptic is divided. But one is a great circle, passing through the poles both of the Ecliptic and of the Equator, or of the world, which is called the colure of the solstices [colurus ſolſtitiorum]. The other colure, called the colure of the equinoxes [Colorus Aequinoctiorum], which passes through the equinoctial points and through the poles of the Equator, or of the world, only, also intersects it rightly, that is, at right spherical angles on either side — just as each of these two colures also intersects the Equator, and the former intersects the Ecliptic as well.

Moreover, at these same solstitial points there occurs the greatest declination [Declinatio] from the Equator toward either part of the world, both of the Ecliptic and — twice in every year — of the Sun, which lies beneath the Ecliptic itself: namely, of twenty-three and a half degrees, or, as the more exact and more careful observations of the foremost craftsmen living at this time prove and demonstrate, 23° 31′ 25″ [23. gr. 31. 25.]. For the Sun, carried down to

[29] the last sort of the word paired with "infra" is open at the top and does not match the a of "infra" set below it; the page prints "ſupr" and one unresolved sort.

those points by its annual motion, seems as it were to check its course a little, and at once has begun to go back toward the other part of the world; nor does it pass beyond the goals set before it and fixed by divine ordinance, or the limits placed on either side that bound its wandering from the Equator. For, as we have said, the Ecliptic is in truth and of itself the most right of all the great circles, or celestial circles, that is, erected most directly in the infinite air — although to us, looking on from the Earth, which stands obliquely because of the oblique axis of the world, the Sun seems to advance obliquely and to be carried and rolled through the air along an oblique path. Yet in every respect it none the less always moves and proceeds along a path that is of itself most direct.

And all the other great circles (or the remaining celestial circles), even if some of them seem to us most right (such as the Meridian and the East circle), are nevertheless oblique to the Ecliptic, or with respect to it; and this, as we have said, is because of the oblique axis of the world and of the Earth we inhabit, which passes obliquely through the Ecliptic itself. For the very Architect [Architector] of the universe, and founder of all natural things (or of things existing in the nature of things), willed that this obliquity should exist in the very nature of things, so that there should be, every year, alternations of things that return after a year's span has run its course, at a fixed time. This same obliquity, however, lies by no means with the Ecliptic as though it were oblique (as may perhaps seem to us), but with the oblique, or passing obliquely through the Ecliptic itself, axis of the world and of the Earth.

So much, then, as briefly as possible, concerning the great circumferences of the cosmic sphere [ſphæricum mundanum], or heaven — that is, the celestial circles — which were devised and fashioned in the heaven by craftsmen both supreme and most ancient for exploring both the places of the fixed stars and the motions of the planets; and concerning the various points of these same circles at which they intersect one another. With any two of these points that lie diametrically opposite each other, the centre of the terrestrial globe itself falls in one straight line; and the Earth stands in the middle of all the great circles, as the common centre of them all. Of these great circumferences we establish six (for those four common ones, enumerated last and as if in passing, will not be needed in astronomy). Around them, moreover, and along them, as many species of distances [diſtantiarum ſpecies] come up for consideration.

A distance, therefore, in the sphere [in ſphærico] is the arc of some great circle intercepted between two things set apart from each other: for instance, either between two stars set apart from each other, or otherwise between two points set apart from each other and conceived by the mind in the heaven. Of these six distances, two species are explored by the two astronomical instruments already mentioned, and are taken with them in the observation itself, through their sights [Dioptræ]; the remaining four species of distances are afterwards computed by astronomical calculation from these two, observed first. The first

two species of distances, which are to be taken with instruments, are therefore these:

I. The distance of stars or points from one another, to be taken with the instrument of the sextant, which can be carried round, or turned round, to any place whatever [circumducibilis ſeu circumuertbiilis].

II. The meridian altitude [Altitudo meridiana], which is the distance above the Horizon of a star or point passing under the Meridian itself, to be taken with the instrument of the quadrant, set up perpendicularly upon an artificial Horizon (that is, a plane corresponding to the natural Horizon), made fast, and able to be turned round about the axis of the Horizon. For this altitude will be taken most safely, and without any error committed, in the Meridian itself (namely, when the star itself, or a point conceived by the mind in the heaven, passes under and through the Meridian itself).

From these two distances, once observed, the four remaining distances that follow are afterwards computed: either by subtraction alone [ſolâ ſubductione], as the declination, or through the Doctrine of Spherical Triangles, as the remaining three of those that follow.

The remaining species of distances, those to be computed, are therefore these four:

I. Declination [Declinatio] is the distance of a star or point from the Equator, toward one or the other pole of the Equator itself, or of the world.

II. Right ascension [Aſcenſio Recta] is the distance of a star or point from the beginning of the vernal equinox, along the Equator.

III. Longitude [Longitudo] is the distance of a star or point from the same beginning of the vernal equinox, but along the Ecliptic.

IV. Latitude [Latitudo] is the distance of a star or point from the Ecliptic, toward one or the other pole of the Ecliptic itself.

So much, then, by way of preparation and groundwork for observation. There now follows, at last, the practice of observation itself, which will consist in the principle [principium] of observation and in calculation.

The principle of observation is either the first or the second. The first principle of observation lies in the investigation of the pole of the world, or of the Equator. The investigation of this pole, in turn, is by the investigation of two great circles, the Meridian and the polar circle; and in investigating each of these, it is done by means of some northern star of perpetual apparition, that is, one that does not go beneath the horizon — but most safely by means of the star nearest to the pole itself, which for that reason is called the pole star [ſtella polaris]. For every star, by its own motion as it appears to us — or rather by the daily rotation of the earth — describes about the pole of the world (which pole is an invisible point) a certain circumference; and the farther that star is from the pole itself, the greater that circumference is, and conversely the nearer it is, the smaller, by Element 3 of Triangles, or of the spherical [ones] [per 3. Triangulorum ſeu ſphærici Elementum]. And so the pole star itself describes the smallest of all, because it is nearest to the pole. For the radius of the

circumference described will be the star's distance itself from the pole. Whether, then, the circumference described by the star is greater or smaller, nevertheless one diameter of every circumference, set upright to the perpendicular [adq; perpendiculum erecta] [?], is an arc or portion of the Meridian being sought; and the other diameter, which lies across it, standing squarely, or at right angles [rectè ſeu ad angulos rectos], on that first upright one on both sides and passing through it, is an arc or portion of the polar circle being sought. And at the very meeting, or point of intersection, of those two diameters — that is, at the very centre of the circumference described by any star — lies the pole being sought, both of the world and of the Equator. Moreover, once the pole of the Equator has been found, the Equator itself is also easily found, in the sky towards the south and raised above the Horizon on the Meridian itself; for it is distant from its own pole, that is, the pole of the world, by a quadrant or 90 degrees (as all the great circles [Maximæ] are), by our spherical Element 2.

Diagram. Set at the left of the text, which runs beside it on the right. Five concentric circles about a common centre, crossed by two dotted diameters at right angles: a vertical one running from A at the top to B at the foot, and a horizontal one from C at the left to D at the right. At the upper right a single arc, larger than the outermost circle, runs from above the top of the figure down towards the right side and stops short. Seven star-marks (six-rayed asterisks) stand on the curves: two on the outermost circle (one just right of the top, one at the upper right), two on the outer arc, one on the second circle just right of the top, one on the third circle just right of the vertical diameter, one on the fourth circle at the upper left of the centre, and one on the innermost circle at its left, on the horizontal diameter.

- A (top, at the head of the vertical dotted diameter): "A"
- B (foot of the vertical dotted diameter): "B"
- C (left end of the horizontal dotted diameter): "C"
- D (right end of the horizontal dotted diameter): "D"

From all of this one may frame a statement [Enunciatum] such as this:

The midpoint between the highest and the lowest meridian altitude of any northern star of perpetual apparition — that is, one that does not go beneath the Horizon — is the place through which the polar circle passes. And the midpoint between that same star's extreme elongations [egreſsionem] towards the rising and the setting is the place through which the Meridian passes. At the very meeting, or point, at which the polar circle and the Meridian cut each other at right angles, will be the pole itself, both of the world and of the Equator. From that pole the Equator itself will lie, towards the south and on the Meridian itself, at the distance of a quadrant of a circle, or 90 degrees. The arc of the Meridian intercepted between the pole itself and the Horizon towards the north is called the elevation of the pole [Eleuatio poli], and this to distinguish it from other altitudes. The elevation of the pole, therefore, is a particular altitude, or distance of the polar point above the Horizon, and it always remains the same and unmoved, because neither of the poles moves; rather, all moving things

move about the poles and about the axis of the world. The complement of the elevation of the pole, moreover, will be both the distance between the pole itself and the vertical point (which is the pole of the Horizon) and the altitude of the Equator, raised above the Horizon towards the south.

So much for the first principle of observation. From it, too, the first species of the distances to be computed, the declination, is likewise established. For once the Equator is had, the declination of any observed star — its distance from the Equator itself — can likewise easily be found from a measured altitude of that star, namely by subtracting the smaller from the greater: if the remainder is from the greater altitude [è maisre] [recte: maiore] than the Equator has, the declination is boreal, or northern; if from the smaller, it is austral, or southern.

But for investigating the remaining three distances to be computed, the second principle of observation will be needed, which depends on the motion of the Sun under the ecliptic; that is, when the Sun's centre stands at one or the other equinoctial point — that is, at the point of intersection of the ecliptic and the Equator — and on the Equator itself; and this happens twice each year. For the Sun stands near the vernal equinox in the month of March, and near the autumnal in the month of September. But to investigate at what moment of time the Sun's centre stands at one or the other equinoctial point, let the following example be taken: the investigation of the vernal equinox, as the earlier of the two in the very succession of nature [vt ipſius naturæ ſucceſſu prioris].

To Investigate the Vernal Equinox [Inueſtigare Aequinoctium Vernum.]

A little before the vernal equinox, while it is now imminent, let the meridian altitude of the Sun's centre be taken with the instrument, the quadrant [inſtrumẽto Quadrantis]; and likewise a little after the equinox has passed. For that altitude always falls at noon itself. In the earlier observation, let the deficiency [defectus] of the meridian altitude of the Sun's centre below the Equator be noted, and in the later observation the excess [exceſſus] of the same altitude above the Equator. Then, both for the deficiency and for the excess, let the conclusion for the whole days that have elapsed in the meantime be drawn as follows: as the sine of the angle of the Sun's greatest declination is to the radius, or sine of the quadrant of a great circle, so is the sine of

- the deficiency [Defectus] / the excess [Exceſſus] — the two joined by a brace to:
  - of the altitude of the Sun's centre [Altitudinis centri ſolis]
    - below [infra] / above [ſupra] — the two joined by a brace to:
      - the Equator [Aequa-ſtorem]
        - before the impending [ante inſtans] / after the elapsed [poſt elapſum] — the two joined by a brace to:
          - equinox, to the sine of the Sun's distance from the equinox [Aequinoctium ad ſinum Diſtan-ſtiæ ſolis ab Aquinoctio.]

When the portions of the differences have been found, let the time that passed between the two terms of the two observations made now be distributed by the common Rule of Fellowship [Regulam ſocietatis]. One of the terms of numbers found will then be the time still to come after the earlier observation was made; the other, the time already elapsed before the later observation. The proof [Apodixis] appears from this accompanying diagram.

Diagram for investigating the equinox, and its explanation [Diagramma inueſtigandi Aequinoctij, eiuſq; Declaratio.]

Diagram. A long horizontal straight line runs across the figure from B F at the left to G C at the right, passing through A, a little right of centre. A curved arc rises from E, below the left end, passes through A, and climbs to D, above the right end; a straight line also runs from E through A to D, so that arc and chord enclose a thin lens on each side of A. At the left end, a straight line drops from F to E and a curved line from B to E; at the right end, a straight line drops from D to G and a curved line from D to C. A short vertical segment hangs from K, on the horizontal line, down to L, left of A; another short vertical segment rises from H, on the horizontal line, up to I, right of A. Two Sun-marks (a rayed disc with a face) stand on the arc: one below the horizontal line between L and A, one above it over I.

- B (top left, on the horizontal line, outer of the pair): "B"
- F (top left, beside B, at the head of the straight drop to E): "F"
- E (foot of the left end): "E"
- K (on the horizontal line, above the short segment left of A): "K"
- L (foot of that short segment): "L"
- A (intersection of the lines): "A"
- I (top of the short segment right of A): "I"
- H (on the horizontal line, at the foot of that segment): "H"
- D (top right): "D"
- G (bottom right, at the foot of the straight drop from D): "G"
- C (bottom right, beside G, at the end of the horizontal line): "C"

In this figure let the Equator be B A C, the ecliptic E A D, and their point of intersection, or the vernal equinox, A. The deficiency of the meridian altitude of the Sun's centre, from the earlier observation, is K L; the excess of the same, from the later observation, H I. The angle of the Sun's greatest declination is both B A E and C A D, and the sines of each are the straight lines E F and D G. And now the method [ratio] of the investigation is clear, by El. 6 of Triangles. By a like inquiry and reasoning the autumnal equinox [Aequinoctium Autumnum] is also found, only with the order of deficiency and excess reversed. And once each equinox has been found, the places of the solstices will also be established, namely by bisecting the arcs intercepted between the two equinoxes [inter vtraq; bina Aequinoctia]. For we cannot obtain or determine the places of the solstices themselves exactly by a sufficiently evident method, because of the Sun's climb along the ecliptic, perceptible [ſenſibilem] at that time. For about the solstices the Sun ascends and descends with a very slow and almost

imperceptible motion, but near the equinoxes very rapidly and with the swiftest motion; and in any quadrant of the ecliptic it does so in the ratio of the sines, by spherical El. 4 [per 4. El. ſphærici]. For the triangle enclosed between the Equator, the ecliptic and the solstitial colure [Colurum ſolſtitiorum] — each of its legs a quadrant of a great circle, and so having two right angles — is a portion of a spherical quadrant; and therefore, by the common notion of the mind, as whole is to whole, so also is part to part. And now that we have both principles of observation, namely the Equator and the equinox, and have now faithfully laid the first and firm foundation of all observation, we shall from here on proceed to the actual calculation of the remaining three distances, which is carried out by spherical triangles.

II. To Find, from the Sun's Known Daily Altitude or Declination, Its Exact Place in the Ecliptic: That Is, in What Degree and Minute of the Ecliptic It Stands Each Day [II. Ex notâ Altitudine vel Declinatione ſolis quotidianâ reperire certum locum ipſius in Eclipticâ: id eſt, quoto gradu & minuto ſeu ſcrupulo in Eclipticâ quotidiè verſatur.]

To investigate the right ascension [Aſcenſionis Rectæ], then — which was listed above as the second kind of the distances to be computed, though by triangles it is indeed the first — to investigate this, I say, one must first determine the Sun's true place in the ecliptic, that is, in what sign, degree and minute the Sun stands each day. This investigation of the Sun's true place can be made the more exactly, the nearer the Sun stands to one or the other of the equinoxes in the ecliptic [in Ecliqticâ]; and this because of the Sun's climb, more perceptible near the equinoxes than elsewhere, which was mentioned just now. Therefore, once the Sun's meridian altitude is known from observation, and its declination is likewise known by subtraction, from what was said before, the Sun's true place in the ecliptic will next be investigated easily. For as the sine of the angle of the Sun's greatest declination (or of the obliquity of the ecliptic to the Equator) is to the radius, or sine of the quadrant of a great circle, so is the sine of the Sun's daily declination to the sine of the Sun's distance from one or the other equinoctial point: by El. 6 of Triangles. Or by 9: as the sine of the angle of the Sun's greatest declination is to the sine of the Sun's daily declination, so is the radius, or sine of a right angle, or greatest sine in the canon, to the sine of the Sun's distance from one or the other equinoctial point.

(From one or the other, I say: supply [ſuple] by Element 13 of Triangles.)

The cause and reason of both appear from this accompanying diagram; and since everything in it is apparent at first glance to anyone who looks at it, it does not seem to need a fuller explanation. The proposition therefore follows [ſequitur itaq; propoſitum].

Diagram. A long, narrow lens-shaped figure set across the page. Two solid curves meet at a sharp point G at the left and at a sharp point A at the right: the upper curve rises from G through D and runs down past a short stroke crossing it and past B to A; the lower curve runs from G through E down to C and on to A. A solid straight line drops from D to E; a short cross-stroke crosses it

about halfway down. A dotted line runs from D down to F, just right of E on the lower curve; dotted lines run from D and from F, converging at A. At B, on the upper curve, stands a Sun-mark (a small circle with a dot, surrounded by rays); a short solid line runs down from the Sun-mark to C on the lower curve, and a short cross-stroke crosses that line. A small open circle stands on the upper curve between the Sun-mark and A.

- G (left point): "G"
- D (top, at the head of the line D–E): "D"
- E (foot of the line D–E, on the lower curve): "E"
- F (on the lower curve, just right of E, at the foot of the dotted line from D): "F"
- B (upper curve, beside the Sun-mark, at its upper left): "B"
- C (lower curve, at the foot of the short line from the Sun-mark): "C"
- A (right point): "A"

III. To Investigate the Right Ascension of Any Fixed Star [III. Alicuius ſtellæ fixæ Aſcenſionem Rectam inueſtigare.]

Now that the Sun's true place in the ecliptic is known through the immediately preceding [problem], the right ascension of the Sun itself will likewise be known — namely by the fourth example of right-angled spherical triangles [Exemplum Triangulorum Rectãgulorum ſphæricorum]. For in the same triangle A C B, just as its side A B is the dis- tance of the Sun, just sought, from the vernal equinoctial point in the ecliptic (a distance which is otherwise called longitude), so, I say, the side A C of the same triangle A B C will be the sought distance of the Sun from the same equinox; and this distance is called the right ascension. Therefore, once the Sun's right ascension is likewise known, the right ascension of any fixed star will next become known as well — namely from the Sun's right ascension, known and found beforehand, but with Venus intervening, and above all when she is stationary [maximè verò ſtationariâ]: for we sometimes see her appear at perigee while the Sun is still shining and stands above the horizon. And this is done by our first example of oblique-angled spherical triangles [Exemplum Triangulorum obliquangulorum ſphæricorum] — namely by the first usual one — and by such a method and route as appears in the accompanying diagram.

Diagram of the right ascension, and its explanation [Diagramma Aſcenſionis Rectæ, eiuſq́: declaratio.]

Diagram. A quarter-sphere figure. A horizontal straight line runs from G at the left through B to F at the right. A large curve rises from G up to A at the top and comes down on the right to F; on the right-hand side the curve shows a small break just above F. A straight vertical line drops from A to B. Three further curves run down from A to the horizontal line, meeting it at C, D and E respectively. A curve rises from the bottom left, below G, passes through B and climbs to the right to a Sun-mark (a small circle with a dot, surrounded by rays) standing on the curve A–E, and runs on to the outer curve; from G a

curve also drops below the horizontal line and sweeps round to meet the rising curve, closing a lens-shaped area below G–B. A star-mark (an asterisk) stands on the curve A–C, with S to its left; a second star-mark stands on the curve A–D, with the sign ♀ printed below and to its left. A curved line joins the star at S to the star at ♀, and a further curved line joins the star at ♀ to the Sun-mark and runs on past it to the outer curve. Short cross-strokes (each like a capital I set across the line) stand on the curve A–C above S and again below it near C; on the curve A–D high up and again below the star; on the curve A–E high up and again below the Sun-mark; and one across the line S–♀. On the line from ♀ to the Sun-mark stands a small italic i.

- A (top, at the meeting of the curves): "A"
- S (left of the star-mark on the curve A–C): "S"
- ♀ (below and left of the star-mark on the curve A–D): "♀"
- ⊙ (Sun-mark on the curve A–E): drawn as a rayed disc
- G (left end of the horizontal line): "G"
- B (foot of the vertical line A–B): "B"
- C (foot of the curve through S): "C"
- D (foot of the curve through the star at ♀): "D"
- E (foot of the curve through the Sun-mark): "E"
- F (right end of the horizontal line): "F"
- mark on the line ♀–⊙: "i" (italic)

In this figure let ⊙ be the Sun, ♀ Venus, and S some fixed star, whose right ascension B C is to be sought. And let the segments B C, C D, D E, E F of the Equator G F be the amplitudes of the angles at the pole A, by El. 8 of Triang. [per 8. El. Triang.] Then, from what was said before, the Sun's right ascension B E is known; and its declination E ⊙ is known from observation. Its complement ⊙ A is also known, by El. 12 of Triang. The same holds for Venus's declination D ♀ and for its complement ♀ A. Finally, the distance between the Sun and Venus, ⊙ ♀, is known from observation with the instrument of the sextant. All these, I say, being known, all the sides of the oblique-angled spherical triangle A ⊙ ♀ will be known; and from these, once known, let the angle ⊙ A ♀, or its amplitude D E, be sought by our first usual oblique-angled spherical triangle. This angle, when found, will be the prosthaphaeresis, to be either added to or subtracted from the solar right ascension as the case requires — which the method itself will dictate and decide. And let the same then be done between Venus and the fixed star S, namely in the triangle ♀ A S. Finally, once the right ascension of one fixed star is known, the ascensions of all the other fixed stars will thereafter be found with less labour as well, by exactly the same route. And thus one goes to the stars:

Let whoever will creep on the ground.

(The right ascension of the eye of Taurus [oculi ♉], as the fixed star chosen first of all [tanquàm ſtellæ fixæ omnium ex electione primæ], from the most certain and infallible observation of the most illustrious Prince of Hesse [principis Haſsiæ], for this passing year of the world 5558, of Christ the God-man

[Greek: τοῦ Θεανθρώπου (tou Theanthrōpou, "of the God-man")] 1588, with the greatest industry and untiring zeal, and with the prince himself as observer and calculator, was investigated and established as 63° 1[?]′ [63. 1[?]] — sixty-three degrees, with a sixth [ſexagintatrium graduum, cum ſextante]. Once this is known, the certain and indubitable places of the other fixed stars will from now on become evident by calculation, through what immediately follows.

IV. From the Known Right Ascension of Some Fixed Star, to Find Its Latitude and Its Longitude Likewise [IV. Ex notâ Aſcenſione Rectâ alicuius ſtellæ fixæ, eiusdem latitudinem & longitudinem iuxtâ, reperire.]

The right ascension, namely the first of the three kinds of distance to be computed by spherical triangles, is investigated by solving the first usual oblique-angled triangle. In just the same way the remaining two kinds of distance, namely latitude and longitude, are found by solving the second usual oblique-angled triangle. And this is all the easier to do because our prosthaphaeresis, as indicated, is easier and readier than the customary operation of proportion itself (which is carried out by tedious multiplication, or by still more tedious division).

Diagram for investigating the latitude and longitude, and its explanation. Offered to Victorinus Schonfelt of Bautzen [Victorinus Schönfeld]. [Diagramma inueſtigandæ latitudinis & longitudinis, eiusque Declaratio. Victorino Schonfelt Budiſsino immolatum.]

Diagram. A large circle fills the lower part of the page. A solid vertical diameter runs from A at the top through the centre C down to N at the foot; a solid horizontal diameter runs from I at the left through C and E to H at the right. A solid straight line runs from K, on the circle at the lower left, through C up to G, on the circle at the upper right, passing F on the way. A second solid straight line runs from B, on the circle at the upper left, down past M (printed to its left), and on through the region of C to P, on the circle at the lower right. A curve runs from B through a point D, right of the vertical diameter and above the horizontal one; from D lines run down to E on the horizontal diameter and to F on the line C–G. A line runs from A down to D. A dotted straight line runs across the circle through L (printed just left of the vertical diameter) and D, parallel to the horizontal diameter; a second dotted line runs from the circle at the left, just above I, up through D to the circle at the upper right. A small burst of rays surrounds D. Equal-arc marks stand on the lines: short cross-strokes (each like a capital I set across the line) on the circle between B and A, on the line A–D, on the line D–E, and on the horizontal diameter between C and E; small open circles on the curve B–D, on the line D–F, and on the line C–G between C and F and again between F and G.

- A (top of the circle, on the vertical diameter): A, set in an ornamental form
- B (circle, upper left): B
- C (centre): C, set in a small form

- D (inside the circle, right of the vertical diameter): D
- E (horizontal diameter, right of C): E
- F (on the line C–G): F
- G (circle, upper right, end of the line from K): G
- H (circle, right end of the horizontal diameter): H
- I (circle, left end of the horizontal diameter): I, with a dot standing above it
- K (circle, lower left): K
- L (left of the vertical diameter, on the dotted parallel): L
- M (left of the line from B): M, set in an ornamental form
- N (foot of the vertical diameter): N, set in an ornamental form
- P (circle, lower right, end of the line from B): P

In the accompanying diagram, then, let ICH be the equator; KCG the ecliptic; GH, or AB, the greatest declination of the Sun; ACN the beginning of right ascension, and BCP the beginning of longitude. CE is the right ascension itself of the star situated at the point D, and this right ascension is known from the preceding problem. ED is the declination of the same star, known from observation. CF is the longitude to be sought, and FD the latitude to be sought. Therefore ABD is the triangle to be solved. Its side AB is known from the greatest declination of the Sun; its side AD is known from the complement of the declination ED; and the angle BAD is known, being composed, namely, of the right angle IAC and the known right ascension AG. Moreover, MD equals CF, and LD equals CE, by El. 3 of Triang.

Therefore, in the proposed triangle ABD, let the side BD be sought from the three known quantities named. Once it is found, its complement DF will likewise be established, by El. 12 of Triang.; and that complement DF will be the latitude sought. Next, in the same proposed triangle ABD, let the angle at B be sought, by El. 9 of the Triangles. Its amplitude will be GF, by El. 8 of Triangul.; and from it its complement will likewise be established, either as the angle FBC or as its amplitude FC, which will be the longitude sought of the star placed at the point D.

So now, at last, all the distances of the stars stated and to be investigated above are established, together with the places and positions of the stars themselves, scattered across the whole heaven. In this way the whole venerable, august and most beautiful face of the most ample Aplanes, or starry heaven [Aplanes ſeu ſtellati cœli], can in the course of time be duly drawn and described, and also impressed and engraved on a small hand-held celestial globe. Moreover — and this is the one end set before all our calculation and all our observation, and the target fixed in advance, towards which, as the goal of the labour we have undertaken, everything tends and is directed — once the positions of the fixed stars have been properly marked and described, all the motions of all the planets, in longitude as well as in latitude (into which, namely, all the remaining planets besides the Sun are also seen to be carried, as will appear in its place), can be investigated and computed, and certain hypotheses can be

established for their motions. Of these it would now be proper to speak at once and in due sequence, were we not interrupted by the converse [ἀντίστροφον] or reverse [conuerſum] of this investigation of longitude and latitude, which it has pleased us, and seemed good, to insert here as well. It is as follows.

V. Conversely: From the Known Longitude and Latitude of Some Star, to Find Its Declination, Its Right Ascension, and Likewise the Midheaven [V. E contrariò, Ex notâ longitudine & latitudine alicuius ſtellæ, reperire eiusdem Declinationem, Aſcenſionem Rectam, ac pariter medium cœli.]

We have thus shown so far in what order and sequence one must advance in observation and proceed directly: that is, how, from the observed meridian altitude and the distance of the stars from one another, first the declination and right ascension are computed with respect to the equator, and second, from these, the longitude and latitude with respect to the ecliptic. But if anyone should wish, conversely, to advance in the reverse direction — namely, to find, the other way round, the declination and right ascension from the longitude and latitude — let him proceed by the following method.

Diagram of the reciprocal observation, and its explanation. Dedicated to the Lord Doctor Caspar Peucer of Bautzen. [Diagramma reciprocæ obſeruationis, eiusque Explicatio. DD. Caſparo Peucero Budiſsino deuotum.]

Diagram. A large circle fills the middle of the page. A solid vertical diameter runs from A at the top through the centre R down to B at the foot; a solid horizontal diameter runs from C at the left through R to D at the right. Further solid straight lines radiate from R to the circle; as far as they can be followed on the scan they run as diameters through R: from E (upper left) to F (lower right); from L (left) to T (lower right); from I (upper right) to K (lower left), passing on its way the point by Z and the small crossed circle by Y; and from G (right) to H (lower left), passing the point by Q and the mark at P. Two curves come from the left of the circle and meet at M, a point marked with a burst of rays: the upper one from E, the lower one from L. Beyond M two curves continue down to the right: one passes a cross-stroke, crosses the line from I at the point by Z, and ends on the line from G at the point by Q; the other passes through the small crossed circle by Y and ends on the line from G at the cross-stroke by P. A straight line runs from I on the circle down to the point by Z, and a straight line runs from G on the circle to the point by Q. Two dotted straight lines cross at or near M: one from V on the circle, the other from O on the circle; on the lower left, one dotted line ends on the circle at N and the other at X, and S is printed beside the steeper of the two. Equal-arc marks stand on the lines: short cross-strokes (each like a capital I set across the line) on the lower curve between L and M, on the curve between M and Z, on the circle between I and G, on the line from I below the crossed circle, at P, and on two of the lines just above R; small open circles on the line from I and on the line from G, below the point by P.

- A (top of the circle, on the vertical diameter): A, set in an ornamental form
- B (foot of the circle, on the vertical diameter): B
- C (left end of the horizontal diameter): C
- D (right end of the horizontal diameter): D
- E (circle, upper left): E
- F (circle, lower right): F
- G (circle, right): G
- H (circle, lower left, end of the line from G): H
- I (circle, upper right): I, with a dot standing above it
- K (circle, lower left, end of the line from I): K
- L (circle, left): L
- M (beside the rayed point): M, set in an ornamental form
- N (circle, lower left, end of a dotted line): N, set in an ornamental form
- O (circle, upper right, end of a dotted line): O
- P (below the cross-stroke on the line from G): P
- Q (right of the meeting point on the line from G): Q, set in an ornamental form
- R (just below the centre): R
- S (beside a dotted line, left of the vertical diameter): S
- T (circle, lower right): T
- V (circle, top, end of a dotted line): V
- X (circle, lower left, below H, end of a dotted line): X
- Y (left of the small crossed circle): Y, set in an ornamental form with curled arms
- Z (right of the crossing on the line from I): Z
- Not read: one further capital, set aslant across the line from E where the dotted line crosses it (left of the vertical diameter, above S) [?]

In the accompanying diagram, let CD be the horizon; ADBC the meridian; AB the East circle [Orientalis]; GH the equator; IK the ecliptic. Hence the pole of the equator is E, and that of the ecliptic L. The longitude SM is known — or (by El. 4 of Triang.) RZ — and the latitude of the same star, MZ, is also known. To be sought are the right ascension RP, the declination MP, and the mid-heaven RY. Since, therefore, three things are given in the right-angled triangle ZRQ — namely the side RZ, the known longitude; the angle ZRQ, that is, the angle of the Sun's greatest declination, or of the intersection of the ecliptic and the equator; and also the angle RZQ, a right angle — let the remaining three be investigated from these three known ones: namely the angle RQZ and the sides RQ and ZQ. When this last is added to the known side ZM — the known latitude, that is — the whole side MQ of the triangle MQP is obtained, of which also the known angles — namely MQP, just found, and QPM, a right angle. With these three given in the same triangle as well, let the remaining three also be explored: namely the side PM, which will be the declination sought, and the side QP. When QP is subtracted from the side

RQ just found, there remains RP, the right ascension sought. Then, from the right ascension RP thus found, the distance RY will be given; and this same Y will be the mid-heaven, that is, the point of the ecliptic which, together with the said point of the equator, or the right ascension found, mediates the heaven — that is, reaches and climbs to the summit of the heaven. Taken from chapter 4 of book 2 of Copernicus.

And from now on, whenever one likes, the distance of two stars from one another will also be found, with the greatest ease indeed, from their known right ascension and declination: by calculation and without observation, through the solution of the second usual oblique-angled spherical triangle. But since this small matter is in itself very easy and very plain, it has no need at all of particular exposition. So much, then, for the method and procedure of both direct and reciprocal observation, and for the observation of the places of the fixed stars. And so, the face of the heaven having been depicted after a fashion — or rather, the method of depicting it having been indicated — we shall descend a little from the highest aether to those seven lower bodies that hang in the air and wander in various ways, which are called planets, and turn our mind to the circuits they describe through the air.

End of the observation of the places of the fixed stars. There follows the observation of the motions and revolutions of the planets. [Finis Obſeruationis locorum ſtellarum fixarum: Sequitur obſeruatio motuum ac reuolutionum planetarum.]

Chapter V. On the Observation of the Motions of the Planets, Where Our New Hypotheses Are Treated [Caput V. De obſeruatione motuum planetarum: vbi de nouis noſtris Hypotheſibus.]

At last, by the favour of divine clemency and grace, we have reached the goal and the undertaking that we set ourselves above all and fixed before us: namely, our new — and also true and natural — hypotheses of the motions of the planets, or of the world-bodies [corpora Mundana]. We devised them now nearly three years ago in a remote corner of the most extensive kingdom of Poland, and afterwards presented them to the Most Illustrious Prince of Hesse, whose supreme craftsman, the Swiss Iustus Byrgius, built those same hypotheses, as presented, in brass. With the help of these hypotheses any private person can, at home and at very little expense, have built such an instrument — an astrarium [inſtrumentum Aſtrarium], as they call it. It shows and displays not only the mean or equal motions of the planets — which is all that certain showy instruments or clocks, costly rather than skilful, usually produce (and even that barely, scarcely at all) — but also the true and apparent motions of all the world-bodies, quite exactly: indeed far more exactly than all the hundreds and myriads of computed tables of every kind. For we shall demonstrate in due course, in our Astronomy, that those tables are built and stitched together from false and fictitious hypotheses, and therefore (as the observations themselves also prove and teach) are also utterly false.

Since, therefore, we are about to hand down such a device — unheard of before and entirely unknown to all ages — and to put it on show for the whole world to see, a device, that is, for the ease of observing and calculating all the motions of the world-bodies, or of the movable things existing in the frame of the world, it seemed best that we begin the matter itself, in our usual manner, somewhat further back: from the very frame of the world, or the system of the whole of nature, the comprehension of all natural things, and from its definition.

Astronomical Theses: On the World and the Motions of the World-Bodies, and on the New Hypotheses [Theſes Aſtronomicæ: De Mundo ac mundanorum corporum motibus, deque nouis Hypotheſibus.]

I. The World is the system of natural things existing of themselves; or, the frame or comprehension of all things natural and existing in the whole of nature. For everything that exists in nature, comprehended under one name, is customarily called the World by everyone, most consistently and unanimously.

II. And the World itself, or the whole system of nature, is divided into air and the bodies hanging in the air, which are called world-bodies.

III. Air is a most tenuous and most subtle essence. The farther it stands from the terrestrial globe, or the higher it is, the more immobile, tenuous and subtle it is; conversely, the nearer it is to the earth, or the lower, the more mobile, coarse and dense it is.

IV. But whether the air itself is finite or infinite has not yet been explored or demonstrated by any mortal. For what the natural philosophers prattle about the celestial orbs, in which they suppose the stars to be fixed, and about the celestial fire, which they ambiguously affirm to be both fire and not fire — all these things, I say, you are to consider idle: let them be figments and dreams.

V. Yet it is credible and not at all absurd — indeed it agrees with reason itself and with nature — that the air extends and spreads out far beyond the fixed stars; and that the fixed stars themselves therefore exist scattered, like the Cyclades, through an air that is either infinite or extended far further; and that these stars are by no means called fixed because they adhere, as people imagine, to some firmament, as to a vaulted or spherical floor, or are impressed and, as it were, engraved upon it, but rather because, as observation testifies, they always and through all ages keep the same positions and an equal distance from one another.

VI. A world-body is a substance hanging in the air by its own force: such are the fixed stars, the planets, the sun, the moon, and the earth together with the water.

VII. The earth is a world-body that holds the middle place among all the world-bodies and exists in the middle of them all. It is balled together from dust and moisture and holds together by the force of moistening. Each year it grows green again with the sun's colour [colore], sent into it, received and drunk in; and, as if impregnated or made pregnant by the solar rays, it bestows and

imparts to us every year, most kindly, the fruits and offspring it has conceived and brought forth — as the most kindly parent of all living things, and as it were the work of a luxuriant nature. Its cavities are filled with collections of moisture, which are called waters. And so water itself, being as it were a particle of the terrestrial globe, forms and constitutes, together with the earth itself, one round, spherical or globular body.

VIII. Three things, therefore, appear to me to exist in nature as first principles and elements, so to speak, which we can reach and grasp with the senses: namely air, or the liquid; water, or the moist; and earth, or the dense. The differing qualities of these three stand to one another in a certain natural proportion, such that just as air, or the liquid, is by its liquidity (so to speak) to water, or the moist, so is water, or the moist (as a kind of mean proportional between the two extremes), by its moistness to earth, or the dense, in its density. And this is, as it were, their mutual proportion to one another in their quality. As to their quantity, however, we either believe that there will be no proportion between them, because of the infinity of the air, or we do not know it, because of the excessive extent and undiscovered magnitude of that same air.

IX. Concerning the essence, quality and matter of the remaining world-bodies, nothing indeed is established for mortals, except that they appear fiery to us as we look at them. And it seems that from this that fable of the natural philosophers — or rather that fabulous opinion and superstition — about the celestial or elemental fire took its origin. But of material fire, and of flame, smoke, vapour and the other exhalations, this is not the place to speak. So much, then, for the air and the world-bodies hanging in the air; there follows the distinction of those same world-bodies.

X. The world-bodies are either mobile or immobile.

XI. Mobile bodies are those which revolve through the air perpetually and always with a regular and equal motion, and which, after an exact, complete and finished revolution, always return to the same point from which they set out, and come back to the same end-point of their periods or tracks. They therefore always describe the same period, revolved through the air by their motions.

XII. The mobile world-bodies are these. I. The Sun: of all things in the world, and of all things that exist and are visible in the nature of things, the most beautiful, the most excellent and the most useful; the standard-bearer, king, leader, Phoebus and Coryphaeus of all the other bodies existing in the heaven, or in the spherical world; and, finally, the pre-eminent and most illustrious ornament of the whole heaven and of the entire world. It completes and accomplishes its revolution in the space of an annual time, and traverses and travels through its period in 365 97/400 days.

2. The Moon, which appears to us spongy, held together by fibres, and condensed. Since, of course, it is the lowest of all, and therefore also the nearest to the earth, we daily see and experience its wonderful effects in the ebb and flow of the seas, in the perpetual back-and-forth motion of

the Ocean, and, finally, in all things that spring up on land and in the sea. Of its motion and revolutions, a little later.

Besides these, five stars are seen as mobile bodies, whose names are:

3. Saturn, a pale, ghastly-coloured star [ſtella lurida]. Of all the mobile bodies it is seen to be the highest and loftiest, the most remote from the terrestrial globe, and therefore, conversely, the nearest to the fixed stars; and so also the slowest of all in its motion.
4. Jupiter, a bright star, and the nearest to the star of Saturn.
5. Mars, a ruddy star, nearest to the Sun, and placed between the Sun and Jupiter. These three stars encircle both the Sun, which is placed roughly in the middle, and the terrestrial globe, now nearer to it, now farther from it. When nearest the earth they are called perigee [perigææ]; when farthest from it, apogee [apogææ]. So too the remaining two that follow in order, which circle not the earth, it is true, but the Sun, situated in the middle of all, just as the other three do.
6. Venus, the most beautiful star of the whole heaven, the most pleasing to our sight, and appearing exceedingly lovely and gentle. It never departs from the Sun by more than 47 degrees of the ecliptic, and it now goes before the Sun, now follows it. Therefore, when it goes before, it is usually called Lucifer; when it follows, Hesperus.
7. Mercury, the smallest star of all and the most wandering, so much so that its motion (like that of Mars) is thought to be altogether unobservable; and for this reason the motions of Mercury and of Mars have not yet been sufficiently explored. Yet it always circles closest about the Sun and accompanies it like an attendant [ſatelles], and it does not go out from the Sun by more than 27 degrees of the ecliptic, nor any farther in either direction.
8. The earth, however, as far as change of place is concerned, does indeed remain immobile; but as far as its daily rolling-round, or its turning by day and night, is concerned, it is altogether mobile. For every day it is spun round by a rotation of day and night; that is, it is turned once in the space of a day and a night, namely in twenty-four hours, and completes and accomplishes its turning. And so, by this turning of its own by day and night — the turning, that is, by which it revolves toward the Sun — it describes the natural day, which we call the nychthemeron [Nychthemeron].

XIII. A day, therefore, is the span of time in which the earth revolves toward the Sun. Of these days (as was said a little earlier) 365 97/400 make up the mean year [annum medium], in which, that is, the Sun revolves back to the same point that it left the year before and from which it set out. It does so either with respect to some fixed star — and the span of time of this completed revolution is called the sidereal year [Annus ſidereus] — or with respect to some definite point of the ecliptic, or orbit of the Sun — and the span of time of this complete and accomplished revolution is called the true or apparent year [Annus verus

ſeu Apparens]. Let these, then, be the two principal kinds and spans of the measure of time. And to these all the other spans of time, or kinds of measure of time — for instance months, weeks [hebdomades ſeu ſeptimanæ], and the rest — under the astronomical, of time and motions, calculation are recalled and reduced. The method of this reduction will appear in due course in our new tables, which answer to these new hypotheses of ours.

XIV. So much, as briefly as possible, about time. But what is time itself? Time is the succession of the world's age from the past, through the present — as through a point — into the future. For this reason time is commonly defined as the measure of motion according to before and after. For time takes its measure chiefly from the motion of the Sun and of the earth.

XV. It is therefore established that, of all the mobile bodies in the nature of things, there are eight in all. Their names are the Sun, the Moon, Saturn, Jupiter, Mars, Venus and Mercury; and the earth, which always keeps the same place of its own, but does not always keep the same position [ſitum] with respect to the heaven, or the fixed stars. So much, then, for the mobile bodies; there follows the account of the immobile ones.

XVI. Finally, the immobile world-bodies are those which always remain in the same place, or cleave fixed to the same place. Such are all the stars of the heaven, or appearing in the heaven, other than the five already named and the two luminaries (that is, the Sun and the Moon) — all the rest. They are held fixed in definite places in the air, and are therefore, as was made clear earlier, called fixed.

XVII. The whole connection, or comprehension, of the fixed stars, held together and joined by I know not what bond (or rather, all the fixed stars taken together), is called the Aplanes [Aplane], commonly the firmament and the starry heaven, and the primum mobile.

XVIII. Whether the fixed stars are placed and arranged in the air at an equal or an unequal height, or distance from the earth, is not known to us, nor can it be known to mortals or worked out by the acuteness of feeble human reasoning. We conjecture, however, that those which appear larger are nearer the earth, while, conversely, those which appear smaller are more remote from the earth and stand off from it at a longer interval. And if this is granted, all of them perhaps are of nearly the same magnitude. If, however, they are scattered and arranged through the air at the same height, or distance from the terrestrial globe, the sense of the eyes, or sight, proves and shows that they are not of the same magnitude. But each person may believe whichever of these he likes, according to his own disposition [Genio] and talent [ingenio]. And if anyone has clearer and sufficient reasons on this matter, I ask him to bring them forward and share them. Moreover, we believe that very many fixed stars — indeed the greatest part of them — do not appear, either because of their excessive distance from the terrestrial globe or because of their small quantity, or smallness; and, finally, that their number, as Holy Scripture testifies, is altogether infinite, an infinite multitude.

XIX. Up to the sixth magnitude, however (for that many different magnitudes of stars are introduced by the astronomers), we have their number explored and ascertained, and it is namely 1022. These are divided into 48 icons [Icones], or images, for the sake of naming and distinguishing them. For there are, of the first magnitude, 15; of the second, 45; of the third, 208; of the fourth, 474; of the fifth, 216; of the sixth, 50; nebulous, 5; obscure, 9. Their catalogue, which formerly teemed with very many errors and faults, the Most Illustrious Prince of Hesse has corrected and emended with the greatest zeal and a more than Herculean labour. For this supreme and greatest benefit all posterity to come will owe deserved and most worthy thanks to this second star-bearing and orb-bearing Atlas [Athlanti].

XX. For now that this catalogue of the fixed stars has been corrected and emended, and their places and positions properly explored, a very easy approach lies open to exploring and investigating the motions of the planets — something that was not possible before. From this came, perhaps, the many errors and faults with which the whole of astronomy has teemed until now. And so now, more easily and more certainly, we can assign and establish hypotheses agreeing with the motions of the planets [planetnrum] [recte: planetarum] themselves — namely those which now follow.

Diagram of the Hypotheses of the Motions of the Bodies of the World: offered to the Most Illustrious Prince of Hesse, and dedicated to him for ever. [Diagramma Hypotheſium mundanorum corporum motuum: Illuſtriſſimo Principi Haſsiæ oblatum, æternumq; conſecratum.]

Explanation of the Diagram of the Hypotheses [Diagrammatis Hypotheſium Declaratio.]

I. In this appended encyclopedia [Encyclopedia] of the periods, or of the traced paths [delineatorum veſtigiorum] of the motions of the bodies of the world, that greatest and outermost circumference of all represents and stands for the ecliptic in the heavens. With the ecliptic alone rests the measure and reckoning [ratio] of the motion of the sun and of all the planets in their periods. It is therefore divided into twelve twelfths [vncias], or dodecatemories [Dodecatemoria], which are commonly called signs, and these are marked with the twelve customary characters set beside them. Their names are: ♈ Aries, ♉ Taurus, ♊ Gemini, ♋ Cancer, ♌ Leo, ♍ Virgo, ♎ Libra, ♏ Scorpius, ♐ Arcitenens [the Archer], ♑ Caper [the Goat], ♒ Amphora [the Water-jar], ♓ Pisces. Each of these in turn is subdivided into thirty equal parts, or degrees. Hence that whole greatest and outermost circumference of all — or rather the whole circuit of the entire heaven — contains 360 degrees, according to which all the motions, places and positions of all things appearing in the heavens are counted and brought into calculation. And around this circumference are placed certain principal fixed stars, set in their truest places in the ecliptic; according to their positions both the true places of the planets in the heavens, or in the ecliptic, can be indicated, and their true motions computed.

II. Next, the middle circumference ♉ ♋, drawn about the same middle centre of the said ecliptic, indicates, designates and represents that imaginary period of the sun which corresponds entirely to the ecliptic itself. This is the period which the sun traverses and completes by its own annual motion, in the circuit and space of a year, always returning and revolving, after a complete and exact year, to the same point from which it set out before the year had elapsed. For the sun always stands on this period invariably and most directly, and turns aside from it to neither side, neither towards the north nor towards the south — not even, as they say, by a hair's breadth. And so it advances along an altogether most direct path, as was also said before, even though, because of the equator lying obliquely to it, the very opposite seems to us to happen. That space of time which is consumed in such a revolution, or solar turning, is called the year, as was made clear before.

Hence, in connection with this solar period, a twofold obliquity of the world's sphere [ſphærici Mūdani], or rather of the world itself, comes up to be noted and considered in passing. One of these exists on account of the equator that appears oblique to us, and this is by accident: namely because we live under the sphere called (as people commonly say) oblique. For those living under the equator itself this accidental obliquity does not occur at all. The other obliquity, of the world's sphere — or, if you prefer, of the world itself — exists in reality and of itself, and lies in the obliquity of the equator itself with respect to the ecliptic, or solar period, which is oblique. Of this second, and more necessary, obliquity of the world there is no mention whatever among the common astronomers. For they do not say that the equator is oblique with respect to the ecliptic; on the contrary, they assert that the ecliptic itself is oblique to the equator. Nor do they imagine that the equator itself is moved in precedence [in præcedentia], but that the fixed stars themselves are carried in consequence [in conſequentia] by a certain very slow motion — the motion they commonly call the precession of the equinoxes [præceſsionem Aequinoctiorum]. Whereas, how- ever, the contrary can easily be demonstrated and proved: namely, by the latitude of all the fixed stars, which always remains the same, and on the other hand by their declination, which is always changing — as the astronomical observations themselves of ancient and recent astronomers, made in different ages, bear witness.

Further, the little gilded orb [deauratus orbiculus] on this period represents the image of the body of the sun. The five circumferences drawn around it — which are heterocentric, or eccentric [Heterocentræ ſeu Eccentræ], with respect to the previously mentioned centre of the ecliptic, or the earth — designate the periods of the five planets, or wandering stars, which roam to and fro in the air. These five stars, namely, with a perpetual and always equal motion, encircle and go round the body of the sun, which is placed as it were in their midst and fixed for them as their centre; although to us, who view them from the centre of the world, which is a different point, and on account of the eccentricity [Eccentroteta] just mentioned, they seem to be carried and revolved through

the air with an unequal motion. For the sun, by its own proper motion, always carries these five wandering stars along with it εἰς προηγούμενα [Greek: εἰς προηγούμενα (eis proēgoumena, literally "towards the preceding")], that is, in consequence [in conſequentia], or with a direct motion according to the ordinary progression of the signs. And from this, besides the true and natural motion of the planets, there also arises a certain other motion of theirs, by accident. Therefore the common centre of all these periods lies either in the body of the sun itself or not very far removed from it. Which of these two it is — that dispute will be settled in our Astronomy [noſtrâ Aſtronomiâ]. Since the present work is its foundation, that is, its beginning or summary, we shall not discourse at great length on such great subtleties, and on the innermost sanctuaries of Urania, in so rough and childish a little work (with respect to what follows).

III. The smallest and innermost circumference of all signifies the period, or path, of the lunar body. This lunar circumference is oblique to the solar circumference or period, just as the period of the sun itself is oblique to the equator — although that obliquity is not found to be of so great an interval; rather, it is held to be inclined from the solar period towards either side by an interval of only five degrees, by which the moon goes out and runs off towards either region of the world, now to the north, now to the south, from the orbit or path of the sun, which is most direct. The periods of the sun and the moon intersect each other at two diametrically opposite points, marked with these signs ☊ ☋. These points of intersection are commonly called nodes; of them, the one that brings the moon up [lunam aduehens] is called the head, and the other, which carries it away [ipſam abuehens], the tail of the dragon. By the more learned, however, the former is usually called ἀναβιβάζων [Greek: ἀναβιβάζων (anabibazōn, "the one that raises")], the latter κα[σ?]αβιβάζων [recte: καταβιβάζων] [Greek: καταβιβάζων (katabibazōn, "the one that lowers")]. These points themselves move εἰς ἑπόμενα [Greek: εἰς ἑπόμενα (eis hepomena, literally "towards the following")], or with a retrograde motion against the ordinary progression of the signs, daily by a little less than 3′ of one degree [tribus minutis ſeu ſcrupulis primis vnius gradus]; so that they complete and accomplish their period, or entire revolution, in about 6798⅓ days. And from this arise the cause, and the method of calculating, of the eclipses, or failings [deliquiorum], of both luminaries, the sun as well as the moon, as we shall explain in our Astronomy.

Meanwhile, let this suffice concerning the lunar period, its deflection from the ecliptic or solar orbit, and hence the departure [exorbitatione] of the lunar body into latitude on either side, both northern and southern. Concerning the lunar body itself, however, it should further be noted briefly that it has no light of its own, but receives that light, poured into it from outside, from the sun as its bestower, and draws it in and drinks it up just as a sponge does water. And so, with a light that is as it were adventitious, received from outside from the sun as from a most abundant and inexhaustible fountain of light, it lights up

and illuminates the terrestrial globe; and it is subject to various vicissitudes and changes. For since it has a body that is not diaphanous or translucent, but dense, and opaque with a thick density, the solar rays sent into it and poured upon it cannot penetrate and illuminate its whole body, but only a little more than half. (And how far beyond half — the penultimate [proposition] of Alhaceni [Ibn al-Haytham], and also 59 of the tenth [book] of Wi- telo, shows.) And so the part of it turned toward the sun appears illuminated to us as we look from the earth. The remaining part, however, turned away from the sun, is not illuminated, and is therefore less visible to us, and usually shows itself to our sight only with the greatest difficulty. And hence it is seen by us sometimes swelling with its full body, sometimes semicircular, and often bounded on one side by an oblong [ob longo] and arched edge; when new, however, or soon to vanish and about to return to the sun, it is horned, that is, rising into horns at either end; and sometimes, lying hidden under the sun's rays and turning toward us its dark and unilluminated part, it is not seen or perceived on any side at all. All this depends, of course, on how far the part of it illuminated by the solar rays comes and falls into our view, or on how far it is seen from the side-on aspect of the sun, while the sun itself bestows and imparts its light to it.

IV. The middle point, or centre, of the whole encyclopedia [Encyclopedia] of celestial harmony, and of the whole system of the world and of nature, represents the image of our little globe, which we tiny little creatures inhabit and which we call the earth. Beyond all doubt it stands in the middle of all things; and compared with the almost infinite magnitude and immense bulk of the Aplanes, or heaven, it serves as a little point, or exists as a point. On this little earthly point of ours, granted to us by the gods [Diuis], most minute and somewhat squalid, we tiny and wretched little creatures called men carry on our "popas" ("pomps," I mean) and our haughty, childish and ridiculous airs. Every day, partly by dragging I know not what earthly and putrid dung from one another, partly by our vain and empty little disputes, by quarrels often over swine's wool, and often by quarrels carried from words to blows, we offer and present laughable spectacles to the inhabitants of heaven. But of these things more broadly and at greater length in Erasmus's most learned *Moria* [Moriâ — the *Praise of Folly*], and in More's most ingenious *Utopia*; and not a little also in our own little wheel [rotulâ] of folly and wickedness, rolled through all the estates of this ridiculous kind. Let us therefore return to ourselves for a little.

This little earthly point, then, is placed and set in the middle of the Aplanes, or heaven (which is upward on every side), always clinging to the same place of the surrounding air and persisting always in its own same, fixed place. By a certain hidden and imperceptible force, unknown to us, of a moving mind diffused through the infinite air, it is turned round and rolled round with a perpetual whirling and circling motion daily, that is, in the space of one natural day. (And the space of time consumed in this earthly turning is called the natural day, or Νυχθημέρινον [Greek: Νυχθημέρινον (nychthēmerinon, "of a night and a day")], as it were a "Night-day" [Noctidium].) Yet it by no means

leaves its fixed place, divinely appointed for it, and the stable centre of nature; nor does it ever run out to any side, neither upward nor downward, neither to the right nor to the left. Moreover, by a certain hidden and magnetic force divinely implanted in it, it seizes all heavy bodies, endowed with perceptible weight, and draws them with the greatest impetus to its centre. For with so great an avidity, divinely implanted, inborn and set in it, does it seek back its own lost pledges, removed from it even a little, and draw them back into itself most greedily; and it by no means suffers them to be separated or torn away from it.

With these hypotheses thus laid down and granted — hypotheses that by no means conflict with the rules of nature itself, and much less break through its barriers — there will follow at once a wonderful agreement of all moving things and of the whole celestial harmony [Harmaniæ] [recte: Harmoniæ]. Indeed, an altogether new and most beautiful sympathy [Sympathia] of the whole system of nature, and a bond of the world's structure and of a stable, concordant nature, could be built from this. Moreover, by the applause of all philosophers and by a unanimous and ratified consent approved through all ages, each thing that is movable of itself, that is, each body of the world, will have one motion only, εἰς τὰ προηγούμενα [Greek: εἰς τὰ προηγούμενα (eis ta proēgoumena, literally "towards the preceding")]; and besides this none at all by accident, and none carried off into the preceding [in præcedẽtia] (εἰς ἑπόμενα [Greek: εἰς ἑπόμενα (eis hepomena, literally "towards the following")]) (which the common astronomers invent) by the primum mobile [primi mobilis]. And no primum mobile, or firmament (that is, a vaulted heaven, as it were), and no orbs fashioned beyond that primum mobile, and no other fictions at all of the ancients or of more recent writers for saving the celestial motions, will be retained. And once this scrupulous and tangled apparatus has been removed from the nature of things, and no less those ravings about a threefold motion of the earth in the aether have been rejected and sent away, we shall without doubt have an astronomy stripped of that barbarous amphibious [ἀμφιβία] — or "physico-mathematical," as they call it — astronomy, torn out from all superstition and scrupulosity at the root and wholly freed from them: a plainly pure mathematical astronomy, and therefore also a genuine one, most perfect and most complete in all its parts [omnibúsq; numeris]. This will appear shortly, with the gods above favouring us and assenting to our wish and purpose — unless perhaps in the meantime this palm is snatched from us by others (of course). But if by chance anyone attempts this, let him, after our manner and example, begin his own web from the very principles and from the first foundation of astronomy, that is, from the doctrine of triangles itself, while nonetheless leaving ours untouched for us [relinquiens].

V. The little machine of driving wheels [rotularum motricum] on the back of the instrument, which move the whole instrument [organon] on the front, is of this small size. In it the single solar wheel carries and moves along with itself all the other periods together, those of all the other planets. The number

set beside the wheels indicates how many teeth each of the wheels has. This little machine is therefore to be fixed to the back of the instrument.

Diagram of the driving wheels. Dedicated to Ioannes Dee [John Dee], an Englishman. [Diagramma rotularum motricum. Ioanni Dee Anglo dedicatum.]

Diagram. A woodcut of the gear-train: two systems of circles, left and right, overlapping at the middle of the figure, crossed by a dotted horizontal line running the whole width and a dotted vertical line through the middle point. At the crossing of the dotted lines stands a small circle carrying a crescent moon on its lower right. Above it, near the top of the middle, a small drawn sun (a ring with a central circle, ringed with dots) stands on one of the arcs. Each system is a set of concentric rings, labelled with numerals and planetary characters, all set radially along their circles; the numerals are the tooth counts named in item V. Labels as read:

- Left system, numerals: 156 (outermost, top); 133; 48; 58 (on an inner ring, top); 58 (above the small inner circle); 60 (inside the left edge of the middle-sized circle, set sideways).
- Left system, characters on the outer rings, top to bottom: ♂; ♄; ♃; ♀. On the small inner circle, left, at the horizontal line: ☿.
- Right system, numerals: 83 (top); 73; 44; 56; 183 (outermost, upper right).
- Right system, characters: ♀ (inner ring, right, just below the horizontal line); ♂ (inner ring, lower); ♄ (lower right); ♃ (bottom); ☿ (on the outermost circle, right).
- Middle: the small sun (drawn, no type sort) and the crescent moon on the central circle (drawn).

No letter labels are printed on the figure. Unread: none of the labels above was left unread; the reading of "60" depends on the orientation taken for a sideways numeral (see the note below).

End of the Foundation of Astronomy. [Finis Fundamenti Astronomici.]

Diagram of the System of Nature, representing the hypotheses of the motions of the bodies of the world. Dedicated and consecrated to the Most Illustrious Prince of Hesse, etc., William [Landgrave William IV of Hesse-Kassel], in token of obedience and respect, by Nicolaus Raymarus Ursus [Nicolaus Reimers Ursus] of Dithmarschen. [DIAGRAMMA SYSTEMATIS NATVRAE, REPRAESENTANS HYPOTHESES MOTVVM CORPORVM MVNDANORVM, ILLVSTRISSIMO PRINCIPI HASSIAE &c. GVILIELMO: OBSEQVII OBSERVANTIAEQVE ERGO DEDICATVM CONSECRATVMQVE PER NICOLAVM RAYMARVM VRSVM DITHMARSVM.]

Diagram. A large circular figure fills the plate. Its outer boundary is a narrow band, hatched and divided by tick marks, with small six-pointed stars drawn along its outer and inner edges; the twelve zodiac characters stand just inside it. A straight line runs across the figure from side to side, nearly horizontal, and a second line runs from top to bottom; both pass through a

small ring at the centre. Many dotted lines radiate from the centre to the outer band. Inside the outer band are three large concentric rings. Near the centre a smaller circle is drawn about the central ring and passes through the centre of a shaded disc (the Sun, drawn, not labelled) that stands up and to the left of the centre; two circles are drawn about that disc. A very small circle surrounds the central ring itself. A hatched, tapering cone runs from the centre down and to the right.

Characters and labels printed on the figure, as read:

- Outer band, zodiac characters, reading anticlockwise from the right: ♈ (right, above the horizontal line, set sideways); ♉ (upper right, set sideways); ♊ (top, right of the vertical line); ♋ (top, left of the vertical line, set sideways); ♌ (upper left); ♍ (left, above the horizontal line); ♎ (left, below the horizontal line, set sideways); ♏ (lower left); ♐ (bottom, left of the vertical line); ♑ (bottom, right of the vertical line); ♒ (lower right); ♓ (right, below the horizontal line).
- Just outside the outer band, at the right end of the horizontal line (the line runs into the band, and a short stub crosses the band to its outer edge): a cross mark with splayed arms (a cross pattée), its left arm in line with the horizontal line; not a letter.
- "OC." at the outer end of a dotted line, upper right, between ♉ and ♈.
- "OR." at the outer end of a dotted line, lower left, beside ♏.
- On the three large rings: ♄ (outermost ring, upper right); ♃ (middle ring, upper left); ♂ (innermost ring, top, left of the vertical line).
- About the Sun: ☿ (inner circle about the Sun, left); ♀ (outer circle about the Sun, below).
- Beside the circle about the centre that passes through the Sun: ☊, just outside its left arc, between the two circles about the Sun, where a dotted line from the centre ends on the arc immediately to the right of the sign. On the same circle: ☋ (right).
- On the very small circle about the centre: ☾ (crescent, lower left of the centre).
- At the centre: a small ring, unlabelled.

Apart from "OC." and "OR.", no letters are printed on the figure. Unread: none of the characters listed above was left unread; ♋, ♎ and ♈ are set sideways and were read with that orientation.

Urania to the Spectator: S. & S. [VRANIA SPECTATORI S. & S.]

> You marvel at the work, and at the ship hanging in the air, the ship of our Mercury, dying and seeking the stars, which he taught to climb upward, and downward in turn. You marvel at the astonishing work, friendly spectator: and not rashly; but you rightly marvel that the lights of highest Olympus return through the air in circuits so varied, and that the stars go back perpetually in swift rotations. Now the hidden cause of the stars and of things lies open, a cause known neither to the ancients nor to our own ages:

why the bodies of heaven are carried in varied motion through the aether, why the heavenly lights go back in their course, and why they appear to us larger and smaller, and how they are carried toward the earth, and conversely away from it. Now they are seen to roll in the aether near the Earth; now they appear far removed from the Earth at the centre. Since the ancients could not in the least see the causes of these things, a vain and tangled furniture was fashioned: so great a store of eccentrics, so many orbs beyond the visible stars, and spindles fitter for women's tasks, than which Nature herself abhors and hates nothing more; trifles which the Maker of Olympus also mocks, whom they rashly accuse of having created rash things in the world: and a rash order charges God with levity. So the ancients erred; and he too erred who laid down a threefold motion of the Earth in the aether, and, on the other hand, that the Titan rests in the middle of the world, although every thing has one motion only. Not so, O men, not so are they carried round: but O how beautifully they are carried round in a circle by a lawful motion! By Nature's law they leap into charming dances. Always from the west toward the east they strive toward one goal, never arriving at it, in no way varying Nature's accustomed steps, and not overstepping the bounds set for them; always far from the Sun, and they do not refuse to obey it. And so, in a system and a just order, this highest throne of Jove, and the dwelling of the gods is composed.

To the Foundation of Astronomy of Nicolaus Raymarus Ursus of Dithmarschen: ΣΩΜΑΤΩΠΟΙΗΣΙΣ ΤΗΣ ΑΣΤΡΟΝΟΜΙΑΣ [Greek: Σωματοποίησις τῆς ἀστρονομίας (sōmatopoiēsis tēs astronomias, "the embodiment of astronomy", that is, astronomy given the shape of a body)]. [AD FVNDAMENTVM ASTRONOMICVM NICOLAI RAYMARI VRSI DITHMARSI, ΣΩΜΑΤΩΠΟΙΗΣΙΣ ΤΗΣ ΑΣΤΡΟΝΟΜΙΑΣ.]

[The outline is set as nested braces, reading from left to right. Each item below is one printed label; the items indented under it are the branches joined by the brace that follows it, in the order printed from top to bottom. A closing brace that gathers two branches into a single label on its right is noted where it occurs.]

- *Of astronomy [*ASTRONOMIAE]
  - Kinds [Species]
    - Methodical, whose parts are [Methodica, cuius partes]
      - Calculation, and it lies in [Calculatio, eaq; eſt in]
        - Logistic numbers, in which are considered [Numeris Logiſticis, in quibus ſpectatur]
          - Numeration [Numeratio]
            - Notation [Notatio]
              - The whole number [Integrum]

- The derived, such as [Ortum, vt]
  - the minute [ſcrupulum]
  - the sexagesimal unit [ſexagena]
- Reduction [Reductio]
- Operation [Operatio]
  - Simple [ſimplex]
    - Addition [Additio]
    - Subtraction [Subductio]
  - Proportioned [Proportionata]
    - Common [Vulgaris]
      - Multiplication [Multiplicatio]
      - Division [Diuiſio]
    - Figurate [Figurata]
      - Common: extraction of the root [Communis, Extractio Radicis]
        - square [Quadratæ]
        - cube [Cubicæ]
        - and of others [Aliarumq;]
      - Progressive, commonly Phronica [Progreſſiua, vulgò Phronica]
- Metrical things, in which are [Rebus Metricis in quibus ſunt]
  - The construction of the canon of sines, and that in [Extructio Canonis: ſinuum eaq; in]
    - Numbers, by [Numeris per]
      - the inscription of a regular [polygon] [Inſcriptionem Ordinati]
      - the section of an angle [Sectionem Anguli]
    - Lines, by the quadrature of the circle [Lineis, per Quadrationem Circuli]
  - The Doctrine of Triangles: the triangle [Doctrina Triangulorum: Triangulum]
    - Spherical [Sphæricum]
      - Right-angled [Rectangulum]
      - Oblique-angled [Obliquangulum]
        - Usual [vſitatum]
          - The former [Prius]
          - the latter [poſterius]
        - Less usual [inuſitatius]
          - First [Primum]
          - second [ſecundum]
          - third [tertium]
          - fourth [quartum]

- [A brace to the right of "second" and "third" alone joins them to:] These two ambiguous [Hæc duo Ambigua]
  - Plane [Planum]
    - Right-angled [Rectangulum]
    - Oblique-angled [Obliquangulum]
      - Acute-angled [Acutangulum]
      - Obtuse-angled [Obtuſangulum]
- Observation, which is [Obſeruatio, quæ eſt]
  - Of the fixed sphere, in which [Aplanes, in quâ]
    - The heavenly circles, or great imaginary circumferences in the heavens [Circuli cœleſtes, ſeu Maximæ in cœlo imaginariæ peripheriæ]
      - Upright [Rectæ]
        - The horizon [Horizon]
        - The vertical [Verticalis]
          - Cardinal [Cardinalis]
            - the meridian [Meridiana]
            - the East circle [Orientalis]
          - Declining, such as the hour circles [Declinans, vt Horarij]
      - Oblique [Obliquæ]
        - Inclined [Decliuæ]
          - the equator [Aeq[[u/n]]ator]
          - the polar circle [Polaris]
        - The ecliptic [Ecliptica]
    - Distances [Diſtantiæ]
      - of the stars from one another [ſtellarum inter ſe]
      - along the circles, or great circles [ſecus Circulos ſeu Maximas]
        - the equator [Aequatorem]
          - Declination [Declinatio]
          - Right ascension [Aſcenſio Recta]
        - the ecliptic [Eclipticam]
          - Latitude [Latitudo]
          - Longitude [Longitudo]
  - Of the planets [Planetarum]
    - System [ſyſtema]
      - The circumvolution of the Earth, or its daily rotation [Telluris circumuolutatio, ſeu quotidiana conuerſio]
      - Of the luminaries [Luminarium]
        - of the Sun [Solis]
        - of the Moon [Lunæ]

- [A closing brace gathers "of the Sun" and "of the Moon" to:] where [the matter is] of eclipses [vbi de Eclipſibus]
- Hypostasis, the Earth [hypoſtaſis, terram]
  - of those surrounding [Ambientium]
    - of Saturn [Saturni]
    - of Jupiter [Iouis]
    - of Mars [Martis]
  - of those not surrounding [non ambientium]
    - of Venus [Veneris]
    - of Mercury [Mercurij]
- Historical, which is [Hiſtorica: quæ eſt]
  - Ναυτικὴ [Greek: ναυτική (nautikē, "nautical", the seafarer's art)], observer of the celestial motions, and it is [Obſeruatrix motuum cœleſtium: eaq;]
    - of the authors [Auctorum]
      - of the ancients, from Ptolemy [veterum, ex Ptolemæo]
      - of the recent ones [recentium]
        - from the Arabs [ex Arabibus]
        - from the Germans [ex Germanis]
    - ours [noſtra]
  - Μαντικὴ [Greek: μαντική (mantikē, "divination")], diviner of natural effects, commonly called astrology, whose parts are [Diuinatrix effectuum naturalium, vulgò Aſtrologia dicta: cuius partes]
    - The construction of the figure of the heavens [Conſtitutio thematis cœli]
      - Kinds [ſpecies]
        - common [vulgaris]
        - rational [rationalis]
      - Parts [partes]
        - the inscription of the houses [inſcriptio Domorum]
        - the placing of the planets [impoſitio Planetarum]
    - Judgement from the position of the heavens, and that according to [Iudicium ex poſitu cœli: idq; ſecundum]
      - Ptolemy [Ptolemæum]
      - the Arabs [Arabes]
- Consequences, and that by reason [Conſectaria, eaq; ratione]
  - Of places [Locorum]
    - Total: cosmography [Totalis, Coſmographia]
    - Partial [partialis]
      - Geography [Geographia]
      - Chorography or topography [Chorographia ſeu Topographia]

- Of times [Temporum]
  - Of years: chronology or chronography. From this [Annorum, Chronologia ſeu Chronographia: Hinc]
    - The reckoning [Ratio]
      - of years [Annorum]
      - of the calendar [Kalendarij]
    - History [Hiſtoria]
      - Ecclesiastical [Eccleſiaſtica]
        - before [ante]
        - after [poſt]
        - [A closing brace gathers "before" and "after" to:] the birth of Christ [natum Chriſtum]
      - Political [politica]
        - Universal [Vniuerſalis]
        - Particular, of some [Particularis, alicuius]
          - place [Loci]
          - time [Temporis]
  - Of shadows [Vmbrarum]
    - Sciography [Sciographia]
    - Gnomonics [Gnomonica]
    - [A closing brace gathers "Sciography" and "Gnomonics" to:] From this [Hinc]
      - the making of the sundial [fabricatio Horologij]
      - the description of the planisphere [deſcriptio Planiſphærij]

Nature remakes some shapes from others. [Ex alijs alias reparat Natura figuras.]

[Folded plate, unfolded: two frames within a double rule-and-chain border. The left frame holds a bracketed outline headed by the letter A and, at its right, a two-part table; the right frame holds a bracketed outline headed by the letter B and, at its upper right, a short note. No running head, folio, signature or catchword.]

First cohort of sines derived from the said primary sines. [Prima cohors deductorum ſinuum è dictis ſinibus primarijs.]

A

[In the outline below each number is an arc in degrees, with its fraction of a degree as printed. A brace joins each number to the two numbers it branches into. The printer sets a point after most values; the points are not carried. A fraction whose numerator will not separate between 2 and 3 on the leaf is given as [?]/4.]

- They are derived from [Deducuntur è]
  - the tetragon, of degrees [Tetragono, graduum]
    - 45
      - 22 ½
        - 11 ¼

        - 78 ¾
      - 67 ½
        - 33 ¾
        - 56 ¼
    - 45.
- the hexagon, of degrees [Hexagono, graduum]
  - 30
    - 15
      - 7 ½
        - 3 ¾
        - 86 ¼
      - 82 ½
        - 41 ¼
        - 48 ¾
    - 75
      - 37 ½
        - 18 ¾
        - 71 ¼
      - 52 ½
        - 26 ¼
        - 63 ¾
  - 60.
- the pentadecagon, of degrees [Pētadecagono, graduum.]
  - 12
    - 6
      - 3
        - 1 ½
          - 0 ¾
          - 89 ¼
        - 88 ½
          - 44 ¼
          - 45 [?]/4
      - 87
        - 43 ½
          - 21 ¾
          - 68 ¼
        - 46 ½
          - 23 ¼
          - 66 ¾
    - 84
      - 42
        - 21
          - 10 ½
            - 5 ¼

- 84 ¾
- 79 ½
- 39 ¾
- 50 ¼
- 69
- 34 ½
- 17 ¼
- 72 ¾
- 55 ½
- 27 ¾
- 62 ¼
- 48
- 24.
- 66
- 33
- 16 ½
- 8 ¼
- 81 ¾
- 73 ½
- 36 ¾
- 53 ¼
- 57
- 28 ½
- 14 ¼
- 75 ¾
- 61 ½
- 30 [?]/4
- 59 ¼
- 78
- 39
- 19 ½
- 9 ¾
- 80 ¼
- 70 ½
- 35 ¼
- 54 ¾
- 51
- 25 ½
- 12 [?]/4
- 77 ¼
- 64 ½
- 32 ¼
- 57 [?]/4
- the decagon, of degrees [Decagono, graduum.]

- 18
  - 9
    - 4 ½
      - 2 ¼
      - 87 ¾
    - 85 ½
      - 42 [?]/4
      - 47 ¼
  - 81
    - 40 ½
      - 20 ¼
      - 69 ¾
    - 49 ½
      - 24 ¾
      - 65 ¼
- 72
  - 36.
  - 54
    - 27
      - 13 ½
        - 6 [?]/4
        - 83 ¼
      - 76 ½
        - 38 ¼
        - 51 ¾
    - 63
      - 31 ½
        - 15 ¾
        - 74 ¼
      - 58 ½
        - 29 ¼
        - 60 [?]/4

The numbers set in order. [Numeri in ordinem redacti.]

Degrees, minutes [Gradus, minuta.]	Degrees, minutes [Gradus, minuta.]
0° 45′	45° 45′
1° 30′	46° 30′
2° 15′	47° 15′
3° 0′	48° 0′
3° 45′	48° 45′
4° 30′	49° 30′
5° 15′	50° 15′
6° 0′	51° 0′
6° 45′	51° 45′
7° 30′	52° 30′

Degrees, minutes [Gradus, minuta.]	Degrees, minutes [Gradus, minuta.]
8° 15′	53° 15′
9° 0′	54° 0′
9° 45′	54° 45′
10° 30′	55° 30′
11° 15′	56° 15′
12° 0′	57° 0′
12° 45′	57° 45′
13° 30′	58° 30′
14° 15′	59° 15′
15° 0′	60° 0′
15° 45′	60° 45′
16° 30′	61° 30′
17° 15′	62° 15′
18° 0′	63° 0′
18° 45′	63° 45′
19° 30′	64° 30′
20° 15′	65° 15′
21° 0′	66° 0′
21° 45′	66° 45′
22° 30′	67° 30′
23° 15′	68° 15′
24° 0′	69° 0′
24° 45′	69° 45′
25° 30′	70° 30′
26° 15′	71° 15′
27° 0′	72° 0′
27° 45′	72° 45′
28° 30′	73° 30′
29° 15′	74° 15′
30° 0′	75° 0′
30° 45′	75° 45′
31° 30′	76° 30′
32° 15′	77° 15′
33° 0′	78° 0′
33° 45′	78° 45′
34° 30′	79° 30′
35° 15′	80° 15′
36° 0′	81° 0′
36° 45′	81° 45′
37° 30′	82° 30′
38° 15′	83° 15′
39° 0′	84° 0′
39° 45′	84° 45′
40° 30′	85° 30′
41° 15′	86° 15′
42° 0′	87° 0′
42° 45′	87° 45′
43° 30′	88° 30′
44° 15′	89° 15′

Degrees, minutes [Gradus, minuta.]	Degrees, minutes [Gradus, minuta.]
45° 0′	90° 0′

Second cohort of sines derived from the known sines of the arcs of 64 and 80 degrees. [Secunda cohors deductorum ſinuum è notis ſinibus arcuum 64. & 80. graduum.]

B

[The right-hand frame sets two outlines side by side, divided by a vertical rule. The first opens with "Ex" and a brace joining 64 and 80. Its branch "4 *" carries an asterisk, and the second outline opens with the same number marked "* 4", from which its brace runs to 2 and 88. Numbers and fractions as in outline A; a fraction whose numerator will not separate between 2 and 3 is given as `[?]/4`.]

- From [Ex]
  - 64
    - 32
      - 16
        - 8
          - 4 * [continued in the second outline]
          - 86
            - 43
              - 21 ½
                - 10 ¾
                - 79 ¼
              - 68 ½
                - 34 ¼
                - 55 ¾
            - 47
              - 23 ½
                - 11 [?]/4
                - 78 ¼
              - 66 ½
                - 33 ¼
                - 56 [?]/4
        - 82
          - 41
            - 20 ½
              - 10 ¼
              - 79 ¾
            - 69 ½
              - 39 ¾
              - 50 ¼
          - 49
            - 24 ½
              - 12 ¼

- 77 [?]/4
- 65 ½
- 32 [?]/4
- 57 ¼
- 74
- 37
- 18 ½
- 9 ¼
- 80 ¾
- 71 ½
- 35 [?]/4
- 54 ¼
- 53
- 26 ½
- 13 ¼
- 76 [?]/4
- 63 ½
- 31 ¾
- 58 ¼
- 58
- 29
- 14 ½
- 7 ¼
- 82 ¾
- 75 ½
- 37 [?]/4
- 52 ¼
- 61
- 30 ½
- 15 ¼
- 74 ¾
- 59 ½
- 29 [?]/4
- 60 ¼
- 80
- 40
- 20
- 10
- 5
- 2 ½
- 1 [?]/4
- 88 ¾
- 87 ½
- 43 ¾

- 46 ¼
- 85
- 42 ½
- 21 ¼
- 68 ¾
- 47 ½
- 23 ¾
- 66 ¼
- 80, as a check [so, proba.]
- 70
- 35
- 17 ½
- 8 [?]/4
- 81 ¼
- 72 ½
- 36 ¼
- 53 ¾
- 55
- 27 ½
- 13 [?]/4
- 76 ¼
- 62 ½
- 31 ¼
- 58 ¾
- 50
- 25
- 12 ½
- 6 ¼
- 83 ¾
- 77 ½
- 38 [?]/4
- 51 ¼
- 65
- 32 ½
- 16 ¼
- 73 ¾
- 57 ½
- 28 [?]/4
- 61 ¼
- * 4
- 2
- 1
- 0 ½ [?]
- 0 ¼

- 89 ¾
- 89 ½
- 44 ¾
- 45 ¼

- 86
- 44 ½
- 22 ¼
- 67 [?]/4
- 45 ½
- 22 [?]/4
- 67 ¼

- 88
- 44
- 22
- 11
- 5 ½
- 2 [?]/4
- 87 ¼
- 84 ½
- 42 ¼
- 47 ¾
- 79
- 39 ½
- 19 [?]/4
- 70 ¼
- 50 ½
- 25 ¼
- 64 [?]/4
- 68
- 34
- 17
- 8 ½
- 4 ¼
- 85 [?]/4
- 81 ½
- 40 ¾
- 49 ¼
- 73
- 36 ½
- 18 ¼
- 71 [?]/4
- 53 ½
- 26 [?]/4
- 63 ¼

- 56
  - 28
    - 14
      - 7
        - 3 ½
          - 1 [?]/4
          - 88 ¼
        - 86 ½
          - 43 ¼
          - 46 ¾
      - 83
        - 41 ½
          - 20 [?]/4
          - 69 ¼
        - 48 ½
          - 24 ¼
          - 65 ¾
    - 76
      - 38
        - 19
          - 9 ½
            - 4 ¾
            - 85 ¼
          - 80 ½
            - 40 ¼
            - 49 ¾
        - 71
          - 35 ½
            - 17 ¾
            - 72 ¼
          - 54 ½
            - 27 ¼
            - 62 ¾
      - 52
        - 26
          - 13
            - 6 ½
              - 3 ¼
              - 86 [?]/4
            - 83 ½
              - 41 [?]/4
              - 48 ¼
          - 77
            - 38 ½

- 19 ¼
- 70 [?]/4
- 51 ½
- 25 [?]/4
- 64 ¼

64.

- 62
  - 31
    - 15 ½
      - 7 ¾
      - 82 ¼
    - 74 ½
      - 37 ¼
      - 52 [?]/4
  - 59
    - 29 ½
      - 14 [?]/4
      - 75 ¼
    - 60 ½
      - 30 ¼
      - 59 ¾
- 46
  - 23
    - 11 ½
      - 5 ¾
      - 84 ¼
    - 78 ½
      - 39 ¼
      - 50 ¾
  - 67
    - 33 ½
      - 16 [?]/4
      - 73 ¼
    - 56 ½
      - 28 ¼
      - 61 ¾

Note. [Notandum.]

- ½, ¼, ¾ [a brace joins the three fractions to:] is worth [valet] [a brace joins the three numbers:] 30, 15, 45 minutes, or *scrupula* [minnta ſeu ſcrupula.] [recte: minuta]

[A short horizontal stroke stands alone in the blank space below the note, at the right of the frame; it carries no figure or letter.]