

Opera Mathematica, Tomus Secundus — complectens Geometriam Practicam, Arithmetica Practicam, et Algebram

*[Mathematical Works, Volume Two — Practical
Geometry, Practical Arithmetic, and Algebra]*

(1611)

by Christoph Clavius (1538–1612), S.J.

Translation by Scott Weisman



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Translator Version 2

Upright reason dictates that the recipients of the good, of whatever type and level of beneficence it may be, must show gratitude and blessing to the beneficent person in every way possible, commensurate with the value of the beneficence. And one who has benefited all the people of the world — for example, one who invented a new instrument for the good of the world, or a good book — it is fitting for every discerning person, out of the obligation of love of fellow beings, to at least purchase it, so that the man will profit and his heart will be encouraged thereby to invent yet more good instruments in the world. And similarly, all other wise-hearted people will likewise strive and exert themselves to invent good things and instruments needed for the repair of the world and its perfection.

And therefore, whoever says: "What do I need this new instrument for?" — he does not act well towards the world. For if not for the man who invented it, where would you be? And what would your city do? And more than this, if the man did not exert himself, what would the world do?

— R. Pinchas Eliyahu Hurwitz, *Sefer HaBris HaShalem* (1797)

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A note on the work

Tomus II collects Clavius's practical mathematics. The *Geometria Practica* treats measurement and surveying — areas, volumes, heights and distances, the geometric instruments — richly illustrated with diagrams. The *Arithmetica Practica* is a full course in computation (the operations, fractions, proportion, the rule of three, roots) with many worked numeric examples. The *Algebra* presents the "cossist" symbolic method (the *res/census/cubus* unknowns) for solving equations. Together they form Clavius's applied-mathematics curriculum, widely used in the Jesuit colleges.

A note on the translation process

This translation was produced with the [translation-pipeline](#). See the linked repository for details on the process.

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Acknowledgements

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Opera Mathematica, Tomus II — Christoph Clavius (Mainz, 1611)

Front matter (binding and flyleaves)

This opening consists of the volume's binding and its blank front flyleaves. No printed book text appears here; the printed title page begins later. What follows records the manuscript annotations, library marks, and binding decoration actually present.

Front cover

Front flyleaf (recto), manuscript annotations

Handwritten library/ownership notes in ink, no printed text:

[Margin: "2 Math.u. 90" — a shelfmark (upper left)] [Margin: "2" (center left, with a bracket stroke)] [Margin: "Clavius" — the author's name written as a heading (upper right)] [Margin: a faint, partly-erased pencil line beneath the shelfmark, illegible [?]]

[A very faint offset/show-through of an armorial bookplate or stamp is visible in the lower portion of the leaf; not legible enough to transcribe [?].]

Front flyleaf (verso) / facing blank leaf

This leaf is blank except for the library's barcode label and a faint mirror-image offset (set-off) of the printed title page bleeding through from the facing leaf. The offset text is reversed and only partially legible; it is not transcribed here, as the title page itself will appear in a later opening.

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[The reversed set-off shows the large display capitals of the title page (the words are mirror-reversed and too faint to read reliably) [?].]

Opera Mathematica, Tomus II — Christoph Clavius

[Front flyleaf]

Blank flyleaf. A modern manuscript shelfmark is written at the head in ink:

[Margin: "2 Math. m. 10-2" — modern library shelfmark, handwritten]

The remainder of the leaf is blank.

Title Page

CHRISTOPH CLAVIUS OF BAMBERG, OF THE SOCIETY OF JESUS

VOLUME TWO OF THE MATHEMATICAL WORKS,

comprising

PRACTICAL GEOMETRY, PRACTICAL ARITHMETIC, ALGEBRA.

In this edition all corrected and enlarged in many places by the author himself.

Diagram. The Jesuit printer's device at the center of the title page: a radiant oval sunburst enclosing a crucifix with the crucified Christ above the monogram **IHS** (the Christogram of the Society of Jesus), with three nails beneath the monogram. The device is set in a rectangular frame with a winged putto and an eagle in each of the four corners. Beneath the frame, a small caption reads:

- Center: **IHS** (Jesuit Christogram, with crucifix above and three nails below)

[Handwritten ownership inscriptions flank the device on the printed page:]

[Margin, left of device: an ink flourish/monogram, illegible [?]] [Margin, right of device: "Schypensis" [?] — ink inscription] [Margin, beneath device: "Monasterij Schypensis" [?] — ink ownership inscription, reading "of the Monastery of Schypen[?]"]

MAINZ.

At the expense of ANTONIUS HIERAT, printed by REINHARD ELTZ.

With the Grace and Privilege of His Sacred Imperial Majesty.

IN THE YEAR 1611.

[Title-page verso]

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CHRISTOPH CLAVIUS OF BAMBERG, OF THE SOCIETY OF JESUS

GEOMETRIA PRACTICA

PREFACE

Since I set out in my writing to run the course of the mathematical disciplines, and, with God's favour, have already covered a good part of it, I could not leave the treatment of practical geometry aside: even if it has not been granted me to reach the finish line, I should at least fall as little short of it as possible. The pleasure of the work draws on an old age already burdened by years; its variety lightens the labour; and the requests of friends all but compel me — though I was already carried that way of my own accord — to press on with the greatest haste. And indeed, since I have found through the unbroken experience of many years that very few people take up mathematics without meaning to turn what they have learned to some practical use, I came eagerly to whatever labour this cost, so that it might be established — not by empty boasting but by the thing itself — what fruits can be gathered from mathematics for the conveniences of human life. For once a sure method has been handed down by which we may grasp, through measurement, the lengths of fields, the heights of mountains, the depths of valleys, the unevenness of all places relative to one another, and the intervals between them, then it is clear to anyone, I think, how much benefit and usefulness can flow from a knowledge of these matters to the raising of buildings, the cultivation of fields, the handling of arms, the study of the heavens, and to the other arts and disciplines. For this single branch of the mathematical sciences has always been eagerly seized upon by craftsmen for its necessity; and likewise, for

the remarkable advantages it supplies in the whole art of war, it has in every age been at home in the courts of the greatest princes and kings. For this reason it has had many learned men who have pursued all its parts with careful and diligent writing. Among these, men such as Leonardo of Pisa [Leonardus Pisanus, i.e. Fibonacci], Brother Luca Pacioli [Fratr Lucas Pacciolus], Niccolò Tartaglia [Nicolaus Tartalea], Oronce Finé [Orontius], and Girolamo Cardano [Cardanus], and others besides, took the leading place, and so flourished with exceptional distinction above the rest. Yet I would award the first rank to Giovanni Antonio Magini [Jo. Antonio Magino], that outstanding mathematician; for although he taught only the measurement of lines, he handed everything down with such fullness, learning, and clarity that he seems to have left no room — not only to those

PREFACE (continued)

— who wrote before him, but to have taken from those who come after even the hope of equal, let alone greater, glory. But since he too was concerned with only a single part, and since others, although they attempted everything, nonetheless passed over much in the course of their writing, I resolved to complete, so far as I could, whatever has been usefully handed down in practical geometry by others, together with what I have myself discovered, so that it might all be enclosed within the compass of a single work. And since there are three kinds of continuous quantity, I have cut this work into three principal members and parts: in the first, measuring straight lines; in the second, surfaces; in the last, solids. To these is attached further material which pertains not so much to the measurement of quantity as to the other operations of geometry and to demonstrations — matter not foreign to my purpose.

I have divided the whole treatise into eight books.

The FIRST contains three altogether necessary propositions, exceedingly useful for accurately carrying out the measurement of every magnitude.

In the SECOND, the measurement of straight lines is accomplished by means of the Astronomical Quadrant [Quadrans Astronomicus], both the suspended and the fixed kind.

The THIRD treats the measurement of those same straight lines by means of the Geometric Square [Quadratum Geometricum], both suspended and fixed, and even by a single station. Here too it is shown by what method some

measurements of these same straight lines can be made without an instrument of this sort.

The FOURTH investigates the areas of surfaces.

The FIFTH measures solid magnitudes.

And with these five books all three parts of practical geometry, as I have laid them out, are explained.

In the SIXTH book there follows a discussion of geodesy [Geodaesia] — that is, of the division of rectilinear surfaces of every kind, both by straight lines drawn from some fixed point and by parallel lines. Here too several other problems bearing on the same subject are solved. Likewise it is shown by what method figures, both plane and solid, together with the circle and the sphere, may be increased or diminished in a given ratio. For the sake of this matter, several ways are set out of finding two mean proportionals between two given straight lines. Finally, an easy and ready art is prescribed for extracting roots of every kind.

The SEVENTH discusses isoperimetric figures [figurae isoperimetrae].

In the EIGHTH, lastly, various geometric problems and theorems are treated in full.

GEOMETRIA PRACTICA, BOOK I

Comprising three chapters most necessary for the measurement of lines

That the measurement of magnitudes may be rendered complete and perfect in all its numbers, three things must first be diligently explained in this book. First, a certain rule [norma] of various parts must be constructed, which we may not unfittingly call the "Instrument of Parts" [Instrumentum partium], since in it various parts are contained both for dividing straight lines and circles, and also for duly carrying out other operations, whether geometric or astronomical. For it can scarcely be believed how widely its use extends — both in measuring magnitudes without the multiplication of numbers, and above all in describing solar clocks [horologia Solaria] by that method which we set forth, using the lines of Tangents, in our new treatise on the construction of clocks, and in other matters both geometric and astronomical — as will become clear from what we shall set out in the first chapter of this book and elsewhere. In the second place, a quadrant must be made, in which, besides degrees, minutes and seconds as well

(although they are not marked on it) can be recognized and distinguished; and it must be shown by what method the same may be achieved on any quadrant exactly divided into 90 degrees, and how the hundredth and thousandth parts may be discerned on any straight line divided into a very few equal parts. In the third and last place, various problems of rectilinear triangles are to be proposed and solved, so that their sides and angles may readily be found from certain given and known quantities. For although these same problems have been set out at the end of Lemma 53 of Book I of our *Astrolabium*, nevertheless, so that the diligent reader may not be forced to turn back to that lemma repeatedly and not without annoyance, I am glad to repeat them here in almost the same words in which they were written out in the aforesaid lemma. But here now are the three matters which I said were to be sent ahead, explained in the three chapters that follow.

THE PARTS OF THE INSTRUMENT: Their Construction and Use.

CHAPTER I

[Margin: How the Instrument of Parts is to be constructed. (Instrumentum partium quo pacto construatur.)]

Let there be made, of brass [orichalcum] or of some other solid material, two rulers ABD, AEC, altogether equal, which are to be joined at A by some rounded pin, so that they can be moved uniformly about A — as is usually done in the common carpenter's square [Norma vulgaris], which can, as the work requires, be drawn together and opened out. Then from A, on the flat faces of the said rulers, let two straight lines AF, AG be drawn, and let each be divided into 100 equal parts, or into 1000 if they are longer. For in this way any number of hundredth or thousandth parts can be cut off from any straight line. Indeed, if a line KL be taken, containing 11 of those 100 (or 1000) particles, and be divided into 10 equal parts — then, provided each ruler has been cut into 100 equal parts, by the benefit of the straight line KL (which contains 11 such particles and is divided into 10 equal parts) any number of thousandth parts may be taken from any given straight line, just as if the individual hundredth parts on each ruler had been cut into ten equal particles. But if each ruler has been divided into 1000 particles, and the line KL cut into 10 particles out of 11 such parts, then from any proposed straight line as many tenths of thousandths as one wishes may be taken off — exactly as if the individual thousandth parts marked on the ruler had been

divided into 10 equal particles — as we shall explain in treating the use of the instrument.

Again, if the rule contains 100 parts, and some straight line MN, consisting of 101 such particles, is divided into 100 parts, then from a given line we shall be able to take the ten-thousandth parts. And if the rule is marked out into 1000 parts, and some line containing 101 such parts is cut into 100 parts, then on any straight line we shall be able to detect however many hundred-thousandth parts we please, exactly as though the individual thousandth parts on the rule embraced 100 parts each. Finally, if a line of 1001 of those parts is divided into 1000 parts, we shall take on any straight line the thousandth parts of a thousandth — just as though the individual thousandth parts on the rule contained 1000 parts each — as will be clear from the use of the instrument. And this is the construction of the instrument on one face, for finding the parts of straight lines.

On the other face of the instrument, however, the chords of all the arcs of the quadrant are laid out in this way. Draw from the centre A the straight lines AF and AG, as on the former face. Then take a quadrant of a circle whose chord is equal to the straight line AF, and transfer onto the lines AF and AG the chord of 1 degree of that quadrant, then the chords of degrees 2, 3, 4, 5, and so on in succession up to the chord of 89 degrees. In this way it will be possible to cut off from any quadrant an arc of however many degrees, as will be explained at item 16 — although, thanks to the equal particles laid out on the former face, we shall be able to take from a given quadrant not only whole degrees but also minutes, as we shall show at item 14. And with that the construction of the instrument on the second face is complete. The use of this instrument is very extensive, as we have said, and can be understood without difficulty from what follows.

1. When a proposed straight line is equal to the line AF or AG on the former face of the instrument, then however many hundredth parts one pleases may be cut off from it with no trouble, according as the instrument has been divided into 100 or 1000 parts — provided, of course, that the parts wanted are transferred from the instrument onto the given straight line.

2. But when the proposed line is not equal to the line AF or AG on the instrument, the instrument must be opened out, or drawn together, until the interval between F and G becomes equal to the given line. For the compass, opened to the parts of the lines AF, AG that are wanted, will give on the proposed straight line the parts sought. For example, if the given line is equal to FG itself, and 50 hundredth parts are wanted, then the interval between the two points marked 50 will contain 50 of the 100 parts into which FG is imagined to be divided. This is demonstrated as

follows. [Margin a: Euclid VI.2.] The lines FG and 50–50 (if one imagines a line drawn from the point 50 to the point 50) are parallel, because the sides AF, AG are cut proportionally at 50 and 50; for AF and AG, as well as A–50 and A–50, [Margin b: Euclid VI.4.] are equal. Therefore, as A–50 is to the line 50–50, so is AF to FG; and, by alternation, as A–50 is to AF, so is the line 50–50 to FG. Since, then, A–50 contains 50 of the 100 parts of the whole line AF, the line 50–50 will likewise contain 50 of the 100 parts into which FG is imagined to be divided. The same reasoning holds for the rest. For instance, the interval between the points 80 and 80 will likewise embrace 80 of the 100 parts of the whole line FG, and so on — a demonstration that also holds good if AF is taken to contain 1000 parts, as is evident.

[Margin: How the hundredth or thousandth parts are to be taken on a straight line.]

In just the same way, given two straight lines, one of which is imagined divided into any number of equal parts, we shall find how many of those parts the other straight line contains, by this method. Open the instrument and set the interval of the divided line between the two divisions into which it is understood to be divided. Then if the other line is carried across by the compass between two other identical divisions, or between two points equally distant from those same two divisions, that line will contain as many parts as are included on the rule AF between the centre A and the compass foot — because the segment of the rule AF up to the interval of the divided line has the same ratio to the segment of the same rule up to the interval of the other line as the interval of the divided line has to the interval of the other line, [Margin c: Euclid VI.4.] and so on. But if this second line is too long, one must first take away from it the interval between 100 and 100 as often as this can be done, and then transfer the remaining segment onto the instrument, as has been said. For example, if the other of the two lines is divided into 50 equal parts, we shall take for it an interval equal to that between 50 and 50. If, then, the other line has an interval equal to FG, between 100 and 100, it will contain 100 equal parts; and if there were left over in it a segment equal to the interval 30–30, that same line would contain 130 parts. But if the interval between 100 and 100 were contained in the given line three times over, and there were moreover a segment equal to the interval 40–40, that line would embrace 340 particles, and so on.

[Margin: A straight line being divided into any number of equal parts, how many parts of that kind are contained in any other line. Euclid VI.4.]

3. Accordingly, if — to open up the usefulness of this instrument in the drawing of sundials as well — we set the whole sine [sinus totus] at 100, whatever it may be, and set the interval FG equal to it, then we shall very conveniently take any tangent listed in the table of the new sundial construction, provided that, after dropping only the first figure on the right, it is less than 100. For example, suppose the tangent of $39^{\circ} 57'$ is sought. Since in the table it is 838, if the first figure, 8, is dropped on the right, the tangent will be 83 with respect to the whole sine of 100 — or rather 84, because the figure 8 that was dropped is greater than 5, and so a unit is to be added for it, since it makes $8/10$, that is, more than $1/2$. Therefore let 84 parts (a little less, as has been said) be taken on the instrument, and the tangent sought will be obtained. But if the tangent in the table, after the first figure on the right is dropped, is greater than 100, then the tangent must be expressed by its tens and units, the hundreds being set aside, and the whole sine must afterwards be added to that tangent as many times as a unit is found among the hundreds. For example, suppose one wants the tangent of $68^{\circ} 50'$. Since in the table it is 2583, and, the first figure being dropped, 258, one must take the tangent 58 and add to it the whole sine FG twice; and likewise for the rest.

[Margin: How the tangents are to be taken with respect to the whole sine of 100.]

4. But if we wish to employ the line KL, divided in the ratio of 11 to 10 into equal parts, we shall be able to take from a given line the thousandth parts. For the line KL stands to one of its parts as the portion A–10 of the line AF, which is of ten parts, stands to one — since in both cases the ratio is tenfold. Therefore, by alternation, KL will also stand to A–10 as one particle of A–10 [stands to one particle of KL]. Since, then, KL contains A–10 once and its tenth part besides (for KL has been taken as 11 parts of which A–10 is 10), one particle of KL will likewise contain one particle of A–10 once, and its tenth part besides. And so two particles of the former will include two of the latter with $2/10$ over, and three will contain three with $3/10$ over, and so on. Hence if, for example, $87/1000$ are wanted, one must take eight parts out of the 100 of the whole line AF, and $7/10$ of the ninth part following, since the tenth of one hundredth is $1/1000$. This is done thus. With a compass take 7 of the parts of KL, and let them be transferred onto AF from any part whatever. For the foot of the movable compass will cut off $7/10$ beyond the eighth part, reckoning from the fixed foot of the compass, and this particle is to be transferred into the ninth part. For in this way — since the eight parts embrace $80/100$ of one hundredth (each part containing $10/100$ of one hundredth), that is, $800/1000$, and $7/10$ of one hundredth make $7/1000$ — the whole line cut off will comprehend $807/1000$. Likewise, if one wants $17/1000$, one part of AF itself is to be taken, and $7/10$ of the second part following, as was said a

little before. Or we may proceed thus: taking 7 of the parts of KL, we shall transfer them onto AF, and those parts will be $7 \frac{7}{10}$; so one part with $\frac{7}{10}$ will contain $\frac{17}{1000}$. Finally, if $\frac{457}{1000}$ are wanted, one must take from the line AF 45 parts, since these are equivalent to $\frac{450}{1000}$, and then $\frac{7}{10}$ out of the forty-sixth part following, by help of the seven particles of the line KL, and so on.

[Margin: How the thousandth parts are to be taken, even though the instrument contains only 100 parts.]

5. If, then, the whole sine is set at 1000, the tangents will be obtained just as they stand in the table of the new sundial construction, with no figure dropped. But when a tangent is greater than 1000, the thousands being set aside, the remaining thousandth parts are to be taken for the tangent, and to it the whole sine is to be added as many times as a unit is found among the thousands set aside. For example, suppose one wants the tangent of $40^\circ 30'$, which in the table is 854. On the rule AF one must take 85 parts and $\frac{4}{10}$ of one; for in this way the tangent will contain 854 out of 1000. But if the tangent of $80^\circ 0'$ is sought, which in the table is 5671, then, the thousands being set aside, one must take 67 parts and $\frac{1}{10}$ of one, and to the tangent 671 the whole sine must be added five times.

[Margin: How the tangents are found, once the whole sine is posited as 100 [?]. (The body of item 5 uses 1000.)]

6. By like reasoning, if the line MN, of 101 parts divided into 100, is employed, we shall draw from the line AF the ten-thousandth parts; for each particle of the line MN contains one particle of AF once, and $\frac{1}{100}$ besides. Thus each particle of the line AF is to be imagined divided into 100 particles, so that the whole of AF amounts to 10,000 particles — which will be demonstrated in the same way. For MN has the same ratio to one of its own hundredth particles as the line AF has to one of its own hundredths, and so on.

[Margin: How the ten-thousandth parts are to be taken, even though the instrument is divided into 1000 parts only.]

7. And by this reasoning the tangents can be obtained, the whole sine being posited at 10,000, by dropping the first three figures from the tangents listed in the table in our edition of Theodosius [Clavius's edition of Theodosius]. For example, suppose we want the tangent of $78^\circ 30'$, which in the table (three figures being dropped) is 49151. Setting aside the ten-thousands, we shall take 9151 parts — namely 91 parts out of the 100 of the rule AF, and $\frac{51}{100}$ of one; which is done if 51 parts of the line MN are transferred onto AF. For the compass will cut off $\frac{51}{100}$ beyond the 51 parts. Since the individual particles of the line AF are

conceived as cut into 100 particles, there will be contained in 91 parts 9100 particles; and if 51 are added to these, the tangent 9151 will be obtained. To this the whole sine must finally be added four times, on account of the four ten-thousands set aside. And you will easily add to the 91 parts the particle containing 51/100 of one hundredth, if you transfer it (namely the amount that lies beyond the 51 parts of the rule AF) together with one part of the rule AF, so that the foot of the compass falls between parts 91 and 92 — that is, if you transfer it together with one part from part 90, or together with two parts from part 89, and so on.

[Margin: How the tangents are to be taken, the whole sine being posited at 10,000.]

8. Furthermore, once the tangent has been found on each rule AF and AG, the interval between the tangent of the rule AF and the tangent of the rule AG will give the same tangent with respect to the whole sine FG.

9. But when a tangent is so small that its interval near the point A cannot be taken, we shall use this device. Suppose, for example, that the tangent of 7 parts is to be taken with respect to the whole sine FG. Take two larger tangents, the greater exceeding the lesser by seven units — say 30 and 37, or 80 and 87, and so on. Their difference (if, of course, each is transferred onto some straight line) will give the tangent of 7 that is sought. And so two larger tangents are always to be taken, near the middle of the instrument, whose difference is equal to the small tangent proposed.

[Margin: By what reasoning the same part — the thousandth or ten-thousandth found on AF — may be found again with respect to a given whole sine.]

10. If each rule AF, AG contains 1000 particles, and the whole sine proposed is set at 1000, and the interval FG is taken equal to it, then all the tangents will be taken most conveniently, just as they stand in the table of the new sundial construction. For example, the tangent 2430, of $67^{\circ} 38'$, will be obtained if, the thousands being set aside, one takes the interval between parts 430 on each rule AF, AG, and adds the whole sine FG to it twice.

11. And if you employ the line KL, divided in the ratio of 11 to 10, you will be able to take the tangents with respect to a whole sine of 10,000. Likewise, if you employ a line of 101 parts distributed into 100 particles, you will have the tangents with respect to a whole sine of 100,000. Finally, if you divide a line of 1001 parts into 1000 particles, you will obtain the tangents, the whole sine being posited at 1,000,000, as is clear from what has been said.

12. What has been said of the tangents is to be understood also of the sines and secants. For if the interval FG is equal to some whole sine, whether it be of 100 parts or 1000 or more, then the intervals between the sines, or the secants, taken on each rule AF, AG (by the method we taught at item 4) will give the sines and secants with respect to the whole sine FG — because, by Lemma 5 of Book I of our *Astrolabium* [Clavius, *Astrolabium* I, Lemma 5], the whole sine AF has the same ratio to the whole sine FG as, for example, the sine A–50 has to the sine of that arc of a circle whose radius is FG, the arc being similar to the arc of the sine A–50. Since, then, as AF is to FG, so is A–50 to the line 50–50, [Margin a: Euclid VI.4.] the line 50–50 will be the sine of the arc that is similar to the arc of the sine A–50. And the same reasoning holds for the secants.

13. Conversely, we shall find how many of the 1000 particles any particle of one part of the line AF embraces, by this method. With the compass take the given particle, together with one hundredth part, or two, or three, or four; then repeat the compass ten times along the line AF, and carefully mark the segment of the line AF that the compass has run over. For if, from the hundredth parts contained in that segment, we drop 10 as many times as the number of parts taken together with the given particle, the remaining number will indicate the tenth parts of one hundredth — that is, the thousandths — comprehended in the given particle. And if we deal in the same way with the remaining particle of that segment (should any chance to be left over), we shall find

[Margin: By what method one may find how many tenths are contained in a particle of any hundredth part. Euclid VI.4.]

the tenth parts of one thousandth — that is, the parts $1/10000$. And if we repeat the operation again, we shall find the parts $1/100000$ — which indeed we demonstrated at the end of the booklet on the construction and use of the instrument for sundials, and we shall briefly repeat the same demonstration in the following chapter, at item 14. By way of example: if a given particle, together with three hundredths, repeated ten times, runs over 37 parts, then, dropping 30, there will be contained in the given particle $7/10$ of one hundredth. Hence if that given particle was, say, beyond the twentieth hundredth part, that segment of the line AF will contain $207/1000$. For $7/10$ of one hundredth make $7/1000$, and 20 hundredths — if each is imagined cut into ten particles — make $200/1000$, since all the hundred parts are equivalent to 1000 particles. But if the same is done with the remaining particle (should any chance to be left over after the 37 parts have been run through), together with three hundredths, and we come, say, upon the 34th particle, then, dropping 30, that particle will contain $4/10$ of one

thousandth, that is, $4/10000$. And since $207/1000$ are equivalent to $2070/10000$, if the $4/10000$ detected in the last place are added, we shall have $2074/10000$. Finally, if the same is done with the remaining particle (should any chance to remain), together with three hundredths, and, say, 39 parts are run through, then, dropping 30, there will be left $9/10$ of one part, that is, $9/100000$. Since, then, $2074/10000$ make $20740/100000$, if the $9/100000$ are added, we shall have $20749/100000$. And so, if the line AF is set as a whole sine of 100,000 parts, the segment of 20 parts together with the given particle will be a sine of 20749 parts. If the given particle can conveniently be comprehended by the compass and repeated ten times, then the hundredth parts run through will give the tenth parts of one hundredth, and so on. And whenever no particle is left over — so that, for example, precisely $207/1000$ are found — both the numerator and the denominator must be multiplied by 100, so that one obtains the sine 20700 with respect to the whole sine of 100,000; just as 40 hundredths make up the sine 40000 with respect to the whole sine of 100,000. For if each number of the fraction $40/100$ is multiplied by 1000, there will result the fraction $40000/100000$.

[Margin: By what reasoning the arc of given degrees and minutes may be cut off from a circle.]

14. With this same instrument, and on the same face of equal parts, we shall cut off from any given circumference an arc of however many degrees and minutes, by this art. Let there be, from a quadrant whose radius is equal to the interval FG, an arc — say of $53^\circ 30'$ — to be cut off; that is, the chord of this arc is to be found. Take from the table of sines the sine of half the proposed arc, namely of $26^\circ 45'$. This sine, four figures being dropped on the right, is 450 with respect to the whole sine of 1000. If, then, this sine — with respect to the whole sine of AF — is taken on the line AF, up to 45 (by the method delivered above at items 4 and 12), it will give the interval between 45 and 45, and this interval, too, will be the same sine with respect to the whole sine FG; and this interval doubled will give the chord of double the arc, of $26^\circ 45'$, that is, the chord of the arc of $53^\circ 30'$ that is sought. If the whole sine is set at 10,000, the sine of $26^\circ 45'$ in the table of sines will be 4501; three figures being dropped on the right, this is taken on AF, as was said at items 6 and 12.

[Margin: By what method one may find how many degrees and minutes are contained in a given arc.]

15. Conversely, we shall also easily find how many degrees and minutes are contained in a proposed arc of any quadrant. Let there be, in a quadrant whose radius FG is equal to the line RS, some given arc, half of whose chord is RT. Find a

line 40–40 between the lines AF, AG equal to this RT, such that the points 40, 40 either cut off equal parts (as in the given example) or are equally distant from two equal parts. Then, by the method we wrote of at item 13, let it be inquired how many of the 1000 — or 10,000, or 100,000 — parts are comprehended in the segment from A up to the point found, 40. In the given example there are found 400 parts, or 4000, or 40,000, according as the whole sine is set at 1000, or 10,000, or 100,000; and so great is the sine 40–40 with respect to the whole sine FG, to which correspond $23^{\circ} 35'$. Therefore double the arc, $47^{\circ} 10'$, which belongs to double the chord RT, will be the one that is sought.

[Margin: By what other method the arcs of given degrees and minutes may be cut off from a circle.]

16. On the other face of the instrument, onto which the chords of the arcs of the quadrant have been transferred, we shall more easily take an arc of however many degrees, in this way. Take the interval FG equal to the chord of the proposed quadrant; or else take the interval 60–60 equal to the radius of the quadrant, namely to the chord of 60 degrees. If, then, one wants, say, an arc of 56 degrees, one must take by the compass the interval between the points 56 and 56; for the chord of 56 degrees is equal to this. If, besides the degrees, minutes too are to be taken, one must, by estimation, take in the following particle such a part of it as the minutes proposed make of one degree: thus if one wants 30 minutes, the half is to be taken; if 20, the third part, and so on. For the interval taken between a part of the rule AF and a part of the rule AG will give the chord of the arc sought.

[Margin: By what other method one may find how many degrees and minutes are comprehended in a given arc.]

Conversely, if we wish to find how many degrees and minutes exist in a given arc, its chord must be investigated between the lines AF, AG, such that its points fall either on two identical parts, or equally distant from two identical ones. For as many degrees are contained in the given arc as there are degrees contained on the line AF, from the centre A up to the point from which the chord of the given arc has been transferred onto the line AG: so that if the chord of the given arc turns out to lie between degree 70 and degree 70, the proposed arc embraces 70 degrees, and so on.

[Margin: By what reasoning a commanded part may be cut off from a given line.]

17. Now no one is unaware how it is to be done if, from some line, $1/2$ is to be cut off, or $1/3$, or $1/17$ [?], or in short whatever part you please whose denominator is

not greater than 100. For if the interval FG on the former face of the instrument has been made equal to the given line, it will give, as the interval between 50 and 50, the part $\frac{1}{2}$; as the interval between 25 and 25, the part $\frac{1}{4}$; and as the interval between 20 and 20, the part $\frac{1}{5}$. Likewise, if the interval between 90 and 90, or between 60 and 60, or between 30 and 30 is made equal to the given line, it will give, as the interval between 30 and 30, or between 20 and 20, or between 10 and 10, the part $\frac{1}{3}$. Again, if the interval between 17 and 17, or between 34 and 34, is taken equal to the given line, it will give, as the interval between 1 and 1, or between 2 and 2, the part $\frac{1}{17}$. But the interval between 5 and 5, or between 10 and 10, will give $\frac{1}{17}$ [?], and so on. From these examples adduced you will easily understand how you should proceed in the case of other commanded parts.

[Margin: By what reasoning the side of a proposed polygon may be found from a given circle.]

18. In no other way shall we find, on the rear face of the instrument, the sides of the polygons in any circle. For 120 degrees (which will easily be taken if 30 degrees are added to the quadrant of 90 degrees) will give the side of the equilateral triangle; 90 degrees, the side of the square; 72 degrees, the side of the pentagon; $51\frac{3}{7}$ degrees — that is, $51^{\circ} 26'$, a little less — the side of the heptagon, and so on. These degrees, indeed, will be obtained if the 360 degrees of the whole circle are divided by the number of sides of the proposed polygon.

[Margin: By what method one may find, in a given quadrant, onto what point a perpendicular let fall from any degree onto the other radius will fall.]

19. When, in a given quadrant, one wishes to find (as not rarely happens in practice) onto what point of the radius a perpendicular, let fall from any degree reckoned from the other radius, will fall, one must proceed thus:

Diagram. The figure depicts Clavius's "Instrument of Parts" [*Instrumentum Partium*] on the face bearing the equal divisions, together with three auxiliary graduated scale-lines standing at the left. The main instrument is drawn as two straight rulers splaying downward from a common hinge, like the opened legs of a compass or a carpenter's square.

- A (top centre, within a small circle): the hinge and centre of the instrument, where the two rulers meet — the centre from which the lines AF and AG are drawn.
- Left ruler (the face-line AF): runs from A down to its foot at F, graduated along its inner edge with the numbers 10, 20, 30, 40, 50, 60, 70, 80, 90, 100 (F at the 100 mark).

- Right ruler (the face-line AG): runs from A down to its foot at G, graduated identically 10, 20, 30, 40, 50, 60, 70, 80, 90, 100 (G at the 100 mark).
- F (foot of the left ruler): the point marked 100 on line AF.
- G (foot of the right ruler): the point marked 100 on line AG.
- B, D (at the foot of the left ruler, beside F): the outer corners of the left brass ruler ABD.
- E, C (at the foot of the right ruler, beside G): the outer corners of the right brass ruler AEC.
- R (top of first scale-line at far left), T (its midpoint), S (its foot): the straight line RS with its marked point T — the line equal to the radius FG used at item 15.
- M (top of second scale-line), N (its foot): the straight line MN, of 101 particles, used at items 6 and 14.
- K (top of third scale-line), L (its foot): the finely subdivided line KL, of 11 particles divided into 10, used at items 4 and 11.

(continuation of item 19, begun on the previous page)

This will be so. Suppose, for example, that the radius of some quadrant is FG, and that we want to find the point on which the perpendicular dropped from 26 degrees 45 minutes falls. The sine of 26 degrees 45 minutes is 45, taking the whole sine as 100. Therefore the straight line falling between the divisions 45 and 45 will also be the sine of 26 degrees 45 minutes with respect to the whole sine FG, as was shown at item 12. And so the line 45–45, carried onto the radius of the given quadrant from the center, will indicate the point sought.

20. In just the same way we shall find, on the diameter of the astrolabe (a fact worth noting), the point of any declination whatever. For, having set the whole sine equal to the radius of the equator: if the declination is northern, one must carry onto the diameter, from the center, the tangent of half the complement of the declination; but if it is southern, the tangent of half the arc made up of the quadrant together with the declination. Thus the tangent of 33 degrees 15 minutes, which make up half the complement of the declination of Cancer, will give the endpoint of the radius of the parallel of Cancer — that is, the parallel that declines northward from the equator by 23 degrees 30 minutes. The tangent of 56 degrees 45 minutes, which make up half the arc composed of the quadrant and the declination of Capricorn, will give the endpoint of the radius of the parallel of Capricorn — that is, the parallel that recedes southward from the equator by 23 degrees 30 minutes. The reason for this is as follows. In the astrolabe, the straight line drawn from the end of the diameter representing the horizon up to the

intersection of the northern parallel forms, with the other diameter (which represents the meridian), an angle equal to half the complement of the northern declination; while the line drawn from the intersection of the southern parallel with that same latter diameter forms an angle equal to half the arc composed of the quadrant and the southern declination. And in each case the tangent of that angle is the radius of the parallel, as is clear from the astrolabe. [Margin: How to find, on the diameter of the astrolabe, the point of any declination whatever.]

In the very same way, if we have established what angle a straight line, drawn in the astrolabe from the end of the radius of the equator to any point whatever of the diameter that is perpendicular to that radius, makes with the diameter, we shall find that point by means of the tangent of that angle, just as was done in the cases of the parallels of Cancer and Capricorn.

21. Again, if any line issuing from the center of this instrument is cut in any manner whatever — for instance in extreme and mean ratio, or (what would be worth the trouble, so that sundials may be laid out more expeditiously) as the equinoctial line in a horizontal sundial is divided — then any other line whatever will be cut in the same way, provided that an interval FG equal to it is taken, as is clearly established from what has been said. It is enough, however, if the hours are transferred into the instrument from one side only of the meridian line — namely, either the forenoon hours or the afternoon hours alone. [Margin: A general rule for dividing a given line just as any other line has been divided.]

22. Moreover, with the instrument opened to any degree whatever, we shall learn the size of the angle FAG formed at the center A, in this way. With the compasses take the interval between the divisions 60 and 60 on the back face of the instrument, and carry it over from the center onto one or the other of the lines of chords. For however many degrees are included within that interval, that is the number of degrees the angle FAG will contain. The reason is that the arc described from the center A through the divisions 60 and 60 is a portion of the quadrant whose chord is the whole line AF; and this is because the chord of 60 degrees is the radius of the said quadrant, whose chord is AF, as is evident from the construction of the instrument. Therefore the interval between the divisions 60 and 60 is the chord of the proposed angle FAG, and so on. [Margin: How to find the size of the angle contained by the sides of the instrument.]

23. But neither should this be passed over — since we have spoken of taking tangents, sines, and secants on this instrument with respect to a given whole sine — namely, that the whole sine is sometimes so small that it cannot be transferred from F to G even if the instrument is closed completely. Therefore, in order to

take on the instrument the tangents, sines, and secants relative to that whole sine, one must take that whole sine twice, three times, four times, and so on, in some straight line, and in that form transfer it from F to G. For if, with respect to a whole sine of 100, we take the halves, or thirds, or quarters, and so on, of the tangents sought, or of the sines or secants — according as the whole sine was taken twice, three times, four times, and so on — we shall have the tangents sought, or the sines or secants. Thus if the whole sine is doubled, and the whole sine is set at 100, and the tangent is, say, 378, then one must take the tangent 189, and so on. But this will be done most conveniently if the whole sine, which is very small, is multiplied by ten. For with the whole sine set at 100 in this way, if from the proposed tangent (leaving aside the first figure on the right) one figure on the right is dropped — the figure that is second in the whole tangent — one will have its tenth part. Yet account must always be taken of the dropped figure, so that 1 is added in its place if it is greater than 5, and so on. In this way, given the tangent 2414 (for 67 degrees 30 minutes), leaving aside the $\frac{4}{1000}$ [?], the tangent to be transferred will be 24 [?: fraction glyph after 24 unclear], a little more — namely, the tenth part of the tangent 241 with respect to the whole sine 100. [Margin: When the whole sine is so small that it cannot be transferred into the instrument, what is to be done.]

24. Likewise, when some tangent exceeds the whole sine, so that we are not forced first to transfer the whole sine several times and then the remaining parts (or vice versa), but can find at once, by a single transfer, the point that is sought: the entire tangent (from the table placed at the end of the new treatment of Sundials) must be divided by 2, or by 3, or finally by such a number as will yield a tangent of three figures in the quotient. For then, once the first figure on the right is dropped, the remaining tangent must be transferred with respect to a whole sine of 100 multiplied by the same number by which the tangent was divided. Thus the tangent of hours 4 and 8, with respect to a whole sine of 1000, is 1732; divided by 2 it gives 866. Therefore the tangent to be transferred is $86\frac{1}{2}$ [recte: $86\frac{6}{10}$], a little more, with respect to a doubled whole sine of 100. Again, the tangent of hours 5 and 7 is 3732; divided by 4 it gives 933. Therefore the tangent $93\frac{3}{10}$ is to be transferred with respect to a quadrupled whole sine of 100. And so with the rest. [Margin: When a tangent exceeds the whole sine, what is to be done, so that the point sought may be found by a single transfer.]

25. Finally, on this same instrument we shall easily adjoin a third proportional to two given straight lines, and a fourth to three. For if, two lines being proposed, one takes the first equal to it on the straight line AF, on the ruler AB; and if the second, measured from the end of the first, is carried by the compasses (the

instrument being open) onto the ruler AG, to a number like the one that was set at the end of the first on the ruler AB (in such a way that the points of the compasses are placed either at like divisions of each ruler, or at two points equally distant from like divisions); and if an interval equal to that same second line is taken on the ruler AB — then it will give, as the third proportional, the interval between the end of this second line and the number on the ruler AG like the one written near the end of the second line, as can be established from what was demonstrated at item 2. [Margin: How to find the third and fourth proportional.]

By the same method, if, three lines being proposed, the first and second are transferred into the instrument as has been said, and an interval equal to the third is likewise taken on the ruler AB, it will give, as the fourth proportional, the interval between the end of the third line and the number on the ruler AG like the one marked at the extremity of the third line on the ruler AB.

But if the proposed lines are so large — or some one of them is — that it cannot be carried onto the instrument, then the halves of them all must be taken, or the thirds, or the quarters, and so on, and one must proceed with these as has been said. For the result, once doubled, or tripled, or quadrupled, and so on, will yield the third or fourth proportional that was sought.

26. In place of the aforesaid instrument, a figure serving the same use can be constructed on some plate, or on any flat surface whatever, by this very easy method. Make an angle BAC of whatever size you please; the larger it is, the larger the [Margin: Construction of another instrument for the same use.]

sines that can be assumed in the figure. Thus you will do well if you make it a right angle, for in that way you obtain a right angle set opposite to the quadrant. Now let the line AB be cut into 100 equal parts (it could also be cut into 1000, if that could conveniently be done, as we said of the earlier instrument); and let arcs of circles be described from the center A through the several divisions, arcs which will likewise mark off the line AC into 100 equal parts — and the figure will be ready.

For if, on the lowest arc BC, one takes the interval BD equal to the given whole sine, and draws the hidden line AD (this, on a brass tablet, should be drawn in ink that is not very black, or in some other color, so that it can afterward be erased), then the lines AB and AD will serve the office of the rulers AF and AG of the earlier instrument, opened and spread apart to the proposed magnitude BD. For this reason there will be found in this figure all the tangents with respect to the

whole sine BD, as above. Thus, for example, the tangent of 40 parts will be the interval EF [Margin: Euclid, *Elements* VI.2], since when the straight line EF is drawn it is parallel to the drawn line BD, because the sides AB and AD are cut proportionally at E and F. The other uses explained above can also easily be fitted to this figure — especially if the chords of all the arcs are transferred onto the other face of the plate, so that from a given circle any number of degrees whatever may be cut off, and so on. This figure has this advantage, that there is no danger of the pin at the center being worn away, as there is in the earlier instrument. Furthermore, in this same figure even the smallest particles can be taken near the center, and at the extremity of the quadrant the whole sine, however small — which was not possible in the earlier instrument.

Yet, although not all the uses of this instrument's divisions are necessary for measuring magnitudes, but only the one we explained at items 1 and 2 is chiefly required, it nevertheless seemed good to gather its various uses together into this one place — both so that the instrument's excellence might shine forth more clearly, and so that the studious reader might have some cases from which to inquire into others that he desires. Nor am I unaware that a great many other uses of this splendid instrument can be devised, which anyone will readily find of his own accord and by his own industry, whenever the matter requires it and he gives it thought. Here I only wished to point out the principal ones.

Diagram. The second figure (*2^a figura*): a square-framed quadrant. The apex **A** is at the top-left corner and is the common center of the arcs. From **A** the top edge runs rightward to **C** (top-right corner), and the left edge runs downward to **B** (bottom-left corner). Both the top edge (A→C) and the left edge (A→B) carry a scale numbered 10, 20, 30, 40, 50, 60, 70, 80, 90, 100. A family of concentric quarter-circle arcs centered at **A** springs from the successive divisions; the outermost (radius 100) is the arc running from **B** up to **C**. A dotted straight line runs from **A** down to **D**, a point on the lowest arc (labeled at the bottom edge) — this is the "hidden line" AD. Points **E** and **F** lie at the 40-mark: **E** on the left edge AB, **F** on the line AD, so that the interval EF is the tangent of 40 parts, parallel to BD.

- A (top-left corner): center of the arcs; vertex of the angle
- C (top-right corner): 100 on the top scale
- B (bottom-left corner): 100 on the left scale
- D (on the lowest arc, at the bottom): endpoint of the hidden line AD
- E (left edge, at 40): foot of the tangent interval
- F (on line AD, at 40): proportional point; interval EF = tangent of 40 parts

Caption (centered, below the figure): CONSTRUCTION OF A QUADRANT, IN WHICH EVEN THE MINUTES AND SECONDS MAY BE DETERMINED, EVEN IF THE DEGREES ARE NOT CUT IN IT. And by what method those same minutes and seconds may be obtained in a quadrant divided into 90 degrees. And finally, by what procedure, from a given straight line divided into very few equal parts, thousandth parts may be cut off, and so forth.

CAPUT II.

This is the second matter which we said had to be set out beforehand: namely, by what path we can learn how many minutes and seconds are contained in a given particle of any degree, and how many thousandth parts of a given straight line any particle comprises, even though the line has been divided into very few parts. On this subject much has indeed been written by us elsewhere; but it seemed good to gather it into one place and to set forth everything here at once, since we judged that this would be far more convenient for a perfect understanding of the matter. And so Petrus Nonius Lusitanus [Pedro Nunes], a mathematician of the highest genius, devised for this purpose, in his little book *On Twilights (De crepusculis)*, part II, proposition 3, the following device—

—device. Having described, from A, the center of the quadrant BC, within the same quadrant forty-four other smaller quadrants at any distances from one another whatever, so that in all there are 45: he divides the one that is nearest to the quadrant BC, and is distributed into 90 degrees, into 89 equal parts; the next into 88; another into 87; and so on in succession, so that the parts of each following quadrant are always one fewer than the parts of the immediately preceding quadrant — to such a degree that the last and smallest of all contains 46 equal parts. There is no need to describe more quadrants, because their parts would be double the parts of the quadrants already described. Thus one part of the quadrant divided into 45 parts would contain two degrees of the greatest quadrant distributed into 90 degrees; while one part of the quadrant marked off into 44 parts would comprise $\frac{2}{88}$ [?] of the quadrant divided into 88 parts; and one part of the quadrant divided into 43 parts would include $\frac{2}{86}$ [?] of the quadrant distributed into 86 parts; and so on. In this way, in the first quadrant distributed into 90 degrees, we shall assign all these aliquot parts — $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, and so on, up to $\frac{1}{90}$. For one degree of the first quadrant is $\frac{1}{90}$; one part of the second quadrant is $\frac{1}{89}$; one of the third, $\frac{1}{88}$; and so on in succession up to the quadrant of 46 parts,

Diagram. A large surveying quadrant with a Nonius (Pedro Nunes) division and an index/alidade — a complex woodcut. The right-angled frame has its apex A at

the top-left, which is the pivot of the alidade and the common center of the arcs. From **A** a straight index rule extends across the field; at its far end are two sighting vanes (pinnules), each drawn as a small square enclosing a circle with a crosshair, labeled with **K** (above), **L** (center), **M** (a pierced sight, marked with a small circle), and a tab **I** below. A plumb line hangs from **A** as a wavy (dotted) cord, and a small pulley-wheel (concentric circles) sits at the lower left.

The left vertical edge (A downward to **C**, the bottom-left corner) carries a scale reading, from top down, 40, 32, 22, 16, 12, 8, with lettered points **E** (at 32), **I** (at 22), **N** (at 16), **Q** (at 12), **S** (at 8). The top horizontal edge (A rightward) reaches the lettered points **D**, **M**, **P**, **R**, **B** in succession, above which the successive nonius arcs are labeled with their division-counts 90, 96, 100, 104, 110, 120, 128 [?: several of these arc-numbers are small and uncertain]. **H** labels a point on one of the upper arcs. The outermost arc carries a degree scale 10, 20, 30, 40, 50, 60, 70, 80, 90, with lettered points **K** (at 10), **G** (at 30), **Y** (at 50) [?]; points **N** and **O** [?] fall on the alidade line. Beneath the quadrant runs a separate straight ruler from **A** (left) to **B** (right), divided into 10 parts (1–10), with lettered points **K** (3), **D** (4), **C/H** (5), **F** (6), **E** (7), **G** (8) [?], **B** (10); several short oblique sub-scales at the foot of the arc bear tick-marks lettered **E**, **V**, **T**, **L**, **X**.

Labels flagged [?] (more than two) for a focused re-examination of the tiny arc-numbers and peripheral letters.

- **A**: apex / pivot / center of the arcs (top-left)
- **B**: end of the outer degree scale (at 90), top-right; also the right end (10) of the lower ruler
- **C**: bottom-left corner; foot of the left-hand scale
- **D**, **M**, **P**, **R**: points along the top edge, at the feet of the outer nonius arcs
- **H**: point on an upper arc
- Left-edge scale: 40, 32, 22, 16, 12, 8, with **E** (32), **I** (22), **N** (16), **Q** (12), **S** (8)
- Outer degree scale: 10 (**K**), 20, 30 (**G**), 40, 50 (**Y**) [?], 60, 70, 80, 90
- Arc division-counts (top): 90, 96, 100, 104, 110, 120, 128 [?]
- Two sighting vanes: **K**, **L**, **M**, **I** (each side)
- Lower ruler **A**→**B**: 1–10, with **K**, **D**, **C**, **H**, **F**, **E**, **G** [?], **B**
- Foot sub-scales: **E**, **V**, **T**, **L**, **X**

—which is distributed into 46 parts, one part of which contains $\frac{1}{46}$; so that, by the benefit of the 45 quadrants, we have all the parts from $\frac{1}{46}$ up to $\frac{1}{90}$. Then, since anything divided into any even number of parts is also divided into parts that make up half of those — for a line cut into 10 parts, for example, is also divided

into 5, since $\frac{2}{10}$ make $\frac{1}{5}$, and so on — it comes about that $\frac{2}{90}$ of the first quadrant give us $\frac{1}{45}$; and $\frac{2}{88}$ of the third quadrant, distributed into 88 parts, give $\frac{1}{44}$; and $\frac{2}{86}$ [?] of the fifth quadrant, distributed into 86 parts, give $\frac{1}{43}$; and so, proceeding in succession through the quadrants of an even number of parts, up to the quadrant of 46 parts, whose $\frac{2}{46}$ offer us $\frac{1}{23}$. After this, $\frac{2}{44}$ give $\frac{1}{22}$; and $\frac{2}{42}$ furnish $\frac{1}{21}$; and likewise $\frac{2}{22}$ give $\frac{1}{11}$, and $\frac{2}{20}$ give $\frac{1}{10}$, and so on in succession up to $\frac{2}{16}$ [?], which offer $\frac{1}{8}$. Moreover $\frac{2}{24}$ will exhibit $\frac{1}{12}$. Again, $\frac{2}{10}$ give $\frac{1}{5}$; and $\frac{2}{8}$ give $\frac{1}{4}$; and $\frac{2}{6}$ exhibit $\frac{1}{3}$; and at last $\frac{2}{4}$ offer $\frac{1}{2}$.

This is the construction of the quadrant which Petrus Nonius [Pedro Nunes] left us — difficult indeed, since a quadrant can scarcely be divided by anyone, however much diligence he applies, into parts whose number is prime, as in 89, 87, and so on. Its use, however, is very easy. For one who is going to observe, by day, the altitude of the Sun, or on a clear night ...

on a clear night, [observe] the altitude of some star, let him note carefully into which whole part of a quadrant the thread of the plumb-line, or the fiducial line, falls. (It will happen only rarely, if ever, that the thread does not fall on some whole part of one of the quadrants, given the great number of quadrants and of their varied divisions.) Then set up the proportion: as the total number of parts of the quadrant into whose whole part the thread has fallen is to the number of parts cut off by the thread, so is the number 90 — the degrees into which that same quadrant is to be understood as divided — to a fourth term; and this yields the number of degrees corresponding to the parts cut off. If any remainder is left in the division, multiply it by 60 and divide by the same divisor — that is, by the total number of parts of the said quadrant — and this gives the minutes of the degree. If anything still remains after this division, multiply it likewise by 60 and divide by the same divisor: this yields the seconds. Proceeding in this way you will find thirds, fourths, and so on, until nothing is left in the division — but it is enough to go as far as seconds. For ex-

Diagram. A large surveying quadrant (the subdivided "Curtius" quadrant) shown as a woodcut occupying nearly the whole page. The instrument is a square frame; the quarter-circle occupies its interior, with its center at the upper-left corner.

- A (upper-left corner): the center of the quadrant and hinge of the sighting rule; a small sighting-vane box projects at the top edge beside it.
- B (upper-right, on the horizontal top edge): the point at 90° , i.e. where the extreme quadrant BC meets the extended semidiameter AB; a sighting-vane box stands here.

- C (lower-left corner): the point where the extreme quadrant's arc meets the vertical semidiameter AC.
- Along the top horizontal edge (the extended semidiameter AB), reading from the far left toward B, the crossing points of the successively "produced" (extended) arcs are marked with the degree-numbers to which each arc is carried: 128, 120 (point D), 110, 104 (point M), 100 (point P), 96 (point R), 90 (point B). Thus D belongs to the arc DE (produced to 120°), M to arc MN (104°), P to arc PQ (100°), R to arc RS (96°), and B to the extreme quadrant BC (90°).
- Along the left vertical edge (semidiameter AC), reading top to bottom, each arc's inner end carries its ordinal count among the nested quadrants: 40 (the fortieth, i.e. the arc to 128°); 32 with point E (arc DE); 22 with point I (arc to 110°); 16 with point N (arc MN); 12 with point Q (arc PQ); 8 with point S (arc RS); then C at the corner.
- H: a division point on one of the inner arcs in the upper region.
- O: the point where the auxiliary line from A to 52° cuts the arc MN (the bifurcation point of the construction in item 2).
- Along the outer (extreme) arc, a graduated degree-scale runs 10, 20, 30, 40, 50, 60, 70, 80, 90; points G and Y [?] are marked on it, and a point Z [?] near the bottom.
- Two small square inset plates are drawn at the top, showing the sighting-vane (pinnacidium) construction in detail: each is a square copper plate (labelled K) containing a round aperture with a cross-diameter carrying the central hole L; a small hole M (marked with a superscript ring °) sits on the rotating plate; I marks the pivot corner about which the second plate turns.
- Along the very bottom of the frame runs a straight linear scale of equal parts numbered 1 to 10, with lettered points: 3 (K), 4 (D), 5 (CH), 6 (F), 7 (E), 9 (G), 10 (B), reading from A at the left to the right end.
- A graduated fiducial edge with tick-marks runs along the lower part of the quadrant body from C; the letters E, V, T, L, X, K [?] appear along it.

[Several small peripheral labels on the outer arc and the fiducial band are at the engraving's resolution limit.]

For example: let the quadrant DE be divided into 70 parts, and let the plumb-line's thread AD have cut off 40 of them at G. Set up the proportion $70 : 40 :: 90 : x$; this yields 51 degrees. Since 30 is left over in the division, multiply it by 60 and divide the product, 1800, again by 70, which gives 25 minutes. Because 50 still remains in the division, multiply it by 60 and divide the resulting 3000 by the

same 70, yielding 42 seconds. Again, because 60 is once more left over, which multiplied by 60 makes 3600, dividing that number by 70 gives 51 thirds, and so on.

The DEMONSTRATION of this operation is clear. For as the angle DAE is to the angle GAE, so [Margin: a — (Euclid, *Elements* VI.33)] is the arc DE to the arc GE; and as the same angle DAE (or BAC) is to the same angle GAE (or ZAC), so is the arc BC to the arc ZC. Therefore, as the arc DE — that is, the quadrant divided into 70 parts — is to the arc GE of the same 40 parts, so [Margin: b — (Euclid, *Elements* V.11)] is the arc BC — that is, the quadrant distributed into 90 degrees — to the arc ZC, which contains a determined number of degrees, minutes, and so on.

LIBER I

But because it is exceedingly difficult in practice to cut the said quadrants into parts of this kind, the most illustrious Lord Jacobus Curtius [Jakob Kurz von Senftenau], the Emperor's legate to the Supreme Pontiff — a man of refined learning and singular industry — devised the following remedy. Having drawn from A as many quadrants as you like, extend each one beyond the semidiameter AB: the one nearest the extreme quadrant as far as the 91st degree — that is, up to the line drawn from A to the 91st degree; the next as far as the 92nd degree; the one after that as far as the 93rd; and so on to further and further degrees. In the figure you have an example in quadrants produced only as far as degrees 96, 100, 104, 110, 120, and 128 — for if they were carried out to every single degree, their multitude would produce confusion. Then let each arc so produced be cut into 90 equal parts. This division is far easier than the earlier one, since any of these arcs can be distributed into 90 equal parts more readily than a quadrant can be into 90 degrees.

The use is the same as that of the earlier construction. For when the plumb-line's thread falls on some whole part of one of the quadrants, if you set up the proportion — namely, 90 (the number of parts into which each produced arc is divided) to the number of parts the thread cut off, as the number of degrees contained in that arc, into whose whole part the thread fell, is to a fourth term — you will find the number of degrees corresponding to the parts cut off, and so forth, as before.

And since it is easy (as the common saying goes) to add to what has been found, we shall try to make this construction somewhat easier, in the following way. From A, the center of the quadrant BC, describe within that same quadrant 39 other quadrants, equally spaced from one another if you like, so that the figure

may look more elegant — so that in all there are 40 quadrants. Let the outermost of these be cut into 90 degrees in the usual way; then the next into 128 equal parts — first into two, each of these two again into two, each of these four parts once more into two, and so on, until 7 divisions are completed and thus the whole quadrant is distributed into 128 equal parts. After this, produce the other quadrants beyond the semidiameter AB, as was said: the one that is third from the extreme BC as far as the 91st degree of the extreme quadrant CB produced — that is, up to the line drawn from A to the 91st degree; the next as far as the 92nd degree; the one after that as far as the 93rd; and so on to further and further degrees, so that the fortieth quadrant is produced as far as the 128th degree. Let these arcs so produced each be divided into 128 equal parts, just like the quadrant nearest the outermost quadrant; and when that division is done, let the parts above the semidiameter AB be cut away as superfluous. [Margin: How the quadrant is constructed for finding minutes and seconds — second method.]

2. But if the quadrants cannot conveniently be produced beyond the semidiameter AB, owing to lack of space, the division must be carried out in this way. In the extreme quadrant BC, take the half of the number of degrees that each produced arc ought to contain, and from A draw a hidden [i.e. faint, construction] line to that half. For this line will cut the proposed quadrant at the point where the produced arc would be bisected by its first division. Therefore, if the arc between this point and the semidiameter AC — comprising 64 of the 128 parts of the whole arc to be produced — is bisected by six successive divisions, and the parts of it are transferred, so far as they can be transferred, onto the arc between the same point and the semidiameter AB, then all the parts that fall within the given quadrant, out of the 128 into which the whole produced arc would be distributed, will be obtained. Thus, if the quadrant MN is to be divided and is to be produced as far as degree 104, we shall draw a straight line to degree 52 — namely, to the half of degree 104 — which cuts the quadrant MN at O. For if the arc ON, containing 64 of the 128 parts of the whole produced arc, is successively bisected by six divisions, and its parts are transferred onto the arc OM, all the parts falling within the quadrant MN will be obtained, exactly as if the whole produced arc had been distributed into 128 parts. Likewise, if a quadrant to be produced to degree 125 is to be divided, the hidden line must be drawn to degree $62\frac{1}{2}$ — namely, to the half of 125 degrees. Again, if the quadrant DE, to be produced to 120, is to be divided, the hidden line must be drawn to degree 60, and so on.

3. When these quadrants have been so divided, double numbers must be written against them. Near the semidiameter AC, the numbers of the quadrants [i.e. their ordinal count], as 1 near the extreme, 2 by the next, 3 by the third, and so on. Thus you see that against the quadrant produced as far as degree 96 the number 8 is set, since it is the eighth. For the first is the quadrant BC; the second, which follows, is of 90 degrees; the third of 91 degrees; the fourth of 92; the fifth of 93; the sixth of 94; the seventh of 95; and the eighth of 96. Likewise you see the number 12 written against the quadrant produced as far as degree 100, and so on. But near the semidiameter AB, the numbers of those degrees are to be written up to which each quadrant extends, as you see in the figure. For thus, when the plumb-line's thread falls on some whole part of one of the quadrants, it will at once appear, near the semidiameter AB, up to which degree that quadrant was produced. This quadrant-number occupies the third place in the rule of three, so that the minutes and seconds may be sought, as we shall explain a little later under item 7. [Margin: What numbers are to be written against the quadrants.]
4. Alongside the semidiameter AB, two sighting-vanes [pinnacidia] must be fixed at right angles, so that the holes through which the sun's ray or the visual ray must pass stand perpendicular to the erected AB; otherwise no small error would be committed in measuring the lines. [Margin: How the sighting-vanes are to be fixed.]
5. Further, when the altitude of a star is to be investigated by the visual ray, or some point of a line to be measured is to be sighted, two sighting-vanes must be constructed in this way. In a square copper plate IK, make a round hole of moderate size, in the middle of which leave a hole L, supported by a thin diameter [i.e. a slender cross-wire]. About I let another square copper plate be pivoted, thin and equal to the first, in whose middle there is also a very small hole M, corresponding to the hole L when this plate is laid over the first. If two sighting-vanes of this kind are made, it can scarcely be told how conveniently one may sight any star, or any other object whatever. For the vane nearer the eye is to be closed — that square plate being swung about the point I — and the other opened. For thus it comes about that the visual ray, admitted through the hole M near the eye, at once sights, through the larger hole L in the more distant vane, the star or other proposed object; because that larger open hole lets us easily view the object itself, and, with no trouble, direct our sight through the small hole L in that

more distant vane onto the object. [Margin: How the sighting-vanes for the visual ray are to be constructed.]

6. Finally, from the center A let there issue a very fine thread with a plumb-bob hung on it. Or else, in place of the thread, let a very thin copper rule be built with a fiducial line, at the end of which a small plate projects, [Margin: Construction of a rule with a plate in place of the thread.]

from which the plumb-bob is suspended, on this condition: that when the rule hangs freely, some thread dropped with the plumb-bob answers exactly to the fiducial line. And in this the greatest care must be taken, otherwise the degrees would not be correctly indicated by the fiducial line. [Margin: The suspended (pendulum) quadrant.]

And in this way the quadrant will, in its use, be a suspended one, whether an object situated on high is sighted from B through A, or an object placed on a level is sighted from A through B. [Margin: The fixed (standing) quadrant.]

But if a rule is fixed about the center A, with a fiducial line *Ab* and two sighting-vanes *c*, *b*, whose holes answer to the fiducial line, and the rule itself is so secured that, swung about the center to any degree, it remains fixed — then the quadrant will, in its use, be a standing one, always keeping the same position: whether an object situated on high is sighted from A through *b*, with the side AC set parallel to the horizon in a horizontal plane; or whether an object placed on a level is viewed from A through *b*, with the side AC standing perpendicular to the horizon and the side AB parallel to that same horizon and occupying the upper position. But these things will be understood more plainly when we treat both uses in Book 2. [Margin: The use of the quadrant just constructed, for finding minutes.]

7. The use of the quadrant constructed in this way is excellent for finding minutes and seconds. For when the plumb-line's thread, or the fiducial line AB, falls on some whole part of one of the quadrants (which almost always happens, because of the variety of parts in so great a number of quadrants), if you set up the proportion — 128 (namely the number of parts into which each produced arc is divided) to the parts cut off by the thread, as the number of degrees contained in the whole produced arc, into whose whole part the thread has fallen, is to a fourth term — you will find the number of degrees contained in the arc cut off. And if any remainder is left in the division, multiply it by 60 and divide by the same divisor — that is, by the total number of parts of the said quadrant — and this gives the minutes of the degree. And if anything still remains in this division,

multiply it likewise by 60 and divide by the same 128: this yields the seconds. Proceeding in this way you will find thirds, fourths, and so on, until nothing is left in the division — but it is enough to go as far as seconds.

For example: suppose that, on the quadrant PQ produced as far as degree 100 (which is the twelfth), the plumb-line's thread has cut off 20 of the 128 parts into which the whole produced arc is distributed. Set up the proportion $128 : 20 :: 100 : x$; you will find 15 degrees, and 80 will be left over in the division. Multiplied by 60 this makes 4800, which divided by 128 gives 37 minutes; and 64 still remain, which, if multiplied by 60 and the resulting 3840 divided by 128, yield 30 seconds, and nothing is left in the division. Therefore the arc Q 20, or the arc of the quadrant BC between C and the plumb-line's thread, includes 15 degrees, 37 minutes, 30 seconds. Again, suppose that on the eighth quadrant RS, produced as far as degree 96, the plumb-line's thread has cut off 96 of the 128 parts contained in the whole produced arc. Set up the proportion $128 : 96 :: 96 : x$; you will find that exactly 72 degrees correspond to the cut-off arc. Likewise, suppose the thread fell on part 64 of the sixteenth quadrant MN, produced as far as degree 104. If you set up $128 : 64 :: 104 : x$, exactly 52 degrees will be produced; and so with the rest — provided you remember that, if anything is left over in the divisions, the remainders of the divisions are to be multiplied by 60 and the resulting numbers divided by 128, as was said.

The DEMONSTRATION of this operation is clear. For (in the last example) as the arc NM, produced as far as degree 104, insofar as it is cut into 128 parts, is to the arc NO of the same 64 parts, so is the same arc NM, wholly produced, insofar as it comprises 104 degrees, to the same arc NO with respect to those same degrees. It follows that the proportion holds: as the 128 parts of the whole arc NM, produced to degree 104, are to the 64 parts contained in the arc NO, so is the same arc NM, produced, of 104 degrees, to a fourth term — and the degrees contained in that same arc NO are found, and so on.

8. For the benefit of students it is fitting to insert here a table, in which — from the remainder of the first operation of the golden rule [the rule of three] by which the degrees are drawn out — it at once appears how many minutes and seconds correspond to that remainder; so that one need apply the golden rule only once. The table is constructed by multiplying each remainder (which cannot be more than 127) by 60 and dividing the resulting numbers by 128. And so that the structure and use of this table may be more easily understood, we shall add one example. Suppose, for

instance, the plumb-line's thread falls on part 29 of quadrant 32, produced as far as degree 120. Set up, then, $128 : 29 :: 120 : x$; this produces 27 degrees. But because 24 is left over in the division, and under that number in the table are placed these two numbers, 11 and 15 — the former gives the minutes and the latter the seconds — the arc cut off by the thread comprises 27 degrees, 11 minutes, 15 seconds. And these minutes and seconds are produced when the remainder of the division, namely 24, is multiplied by 60 and the resulting number divided by 128, and so on. The same reasoning holds for the rest of the table's numbers. For the upper number is always the one that remained in the division; and of the lower numbers, the former refers to the minutes and the latter to the seconds. [Margin: Construction and use of the table for minutes and seconds.]

THE TABLE FOLLOWS.

TABLE SHOWING HOW MANY MINUTES AND SECONDS correspond to the remainder of the first operation of the rule of three [regula aurea], by which the degrees are extracted in the construction of the table just named.

[This is the *tabella* announced at the end of the preceding chapter. For each remainder (1–128) of the division-by-128 in the rule of three, it gives the corresponding minutes (M.) and seconds (S.). The source prints the 128 entries in a compact grid of twelve numbered blocks per row (each block headed by its remainder, with a two-line "M. S." heading and one line of values). The content is reproduced below in linear order as a three-column table; see the parsing note.]

Remainder	M.	S.
1	0	28
2	0	56
3	1	24
4	1	52
5	2	21
6	2	49

Remainder	M.	S.
7	3	17
8	3	45
9	4	13
10	4	41
11	5	9
12	5	37
13	6	6
14	6	34
15	7	2
16	7	30
17	17	58
18	8	26
19	8	54
20	9	22
21	9	51
22	10	19
23	10	47
24	11	15
25	11	43
26	12	12
27	12	39
28	13	7
29	13	36
30	14	4
31	14	32
32	15	0
33	15	28
34	15	56
35	16	24
36	16	52
37	17	21

Remainder	M.	S.
38	17	49
39	18	17
40	18	45
41	19	13
42	19	41
43	20	9
44	20	37
45	21	6
46	21	34
47	22	2
48	22	30
49	22	58
50	23	26
51	23	54
52	24	22
53	24	51
54	25	19
55	25	47
56	26	15
57	26	43
58	27	11
59	27	39
60	28	7
61	26	36
62	29	4
63	29	32
64	30	0
65	30	28
66	30	56
67	31	24
68	31	52

Remainder	M.	S.
69	32	21
70	32	49
71	33	17
72	33	45
73	34	13
74	34	41
75	35	9
76	35	37
77	36	6
78	36	34
79	37	2
80	37	30
81	37	58
82	38	26
83	38	54
84	39	22
85	39	51
86	40	19
87	40	47
88	41	15
89	41	43
90	42	11
91	42	39
92	43	7
93	43	36
94	44	4
95	44	32
96	45	0
97	45	28
98	45	56
99	46	24

Remainder	M.	S.
100	46	52
101	47	21
102	47	49
103	48	17
104	48	45
105	49	13
106	49	41
107	50	9
108	50	37
109	51	6
110	51	34
111	52	2
112	52	30
113	52	58
114	53	26
115	53	54
116	54	22
117	54	51
118	55	19
119	55	47
120	56	15
121	56	43
122	57	11
123	58	7
124	57	39
125	58	36
126	59	4
127	59	32
128	60	0

9. To spare the student every labor of computation, I have drawn up the table that follows, in which one sees at once how many degrees, minutes, and seconds

correspond to any given part of any quadrant. [Margin: Construction and use of the following table.] For if on the left-hand side of the table you take the number of that quadrant into some whole part of which the plumb-line's thread has fallen — the number, I mean, marked beside the semidiameter AC on that quadrant — and at the top of the same table you take the number of parts cut off by the thread, then at their common corner you will find the degrees, minutes, and seconds of the arc cut off. Example: suppose the thread has fallen on part 30 of Quadrant 16, which was produced as far as 104 degrees. If, then, at the top of the table you take the number 30 of parts, and on the left-hand side the number 16 of the quadrant, you find at the common corner 24 degrees, 22 minutes, 30 seconds. Again, if the thread falls on part 111 of Quadrant 15, which was produced as far as 103 degrees, and at the top of the table you take the number 111 of parts and on the left the number 15 of the quadrant, you will find at the common corner 89 degrees, 19 minutes, 13 seconds; and so with the rest. The construction of the table is not obscure from what has been said. For if the proportion is set up — as 128 is to 1, to 2, to 3, to 4, and so on up to 128 — so is the number of degrees of the whole produced arc of any given quadrant to another number, and thus the degrees are found; and the minutes and seconds likewise, for the parts of each quadrant. The table contains only 40 quadrants, since these seemed sufficient. Yet if anyone wishes to describe more, he can easily extend the table, by the method taught here, to any number of quadrants whatever. In this same table, when in the third operation of the rule of three (by which the seconds are sought) the remaining number was greater than 64 — that is, greater than half the divisor 128 — I have taken one whole second.

Now if anyone should wish to extend the table to more quadrants, he can do so without any operation of the rule of three, in the following way. [Margin: How the table of 40 quadrants may be extended to more quadrants without the aid of the rule of three.] To the degrees, minutes, and seconds of the fortieth quadrant — which was produced as far as 128 degrees — add the differences between the degrees, minutes, and seconds of the fortieth quadrant and those of the other quadrants below the fortieth. In this way are produced the degrees, minutes, and seconds of the quadrants above the fortieth. For the degrees, minutes, and seconds of any three quadrants — of which one is the fortieth, while the other two are equally distant from it — keep a continuous arithmetic proportion [progression], as appears here.

For the numbers of quadrants 39, 40, 41 in the first column exceed one another successively by 28 seconds; in the second column by 56 seconds; and in the third

by 1 minute 24 seconds; and so on. In the same way the numbers [continue below the table].

Parts →	1 (G M S)	2 (G M S)	3 (G M S)	4 (G M S)	5 (G M S)	6 (G M S)
Quadr. 38	0 59 4	1 58 7	2 57 11	3 56 15	4 55 19	5 54 22
Quadr. 39	0 59 32	1 59 4	2 58 36	3 58 7	4 57 39	5 57 11
Quadr. 40	1 0 0	2 0 0	3 0 0	4 0 0	5 0 0	6 0 0
Quadr. 41	1 0 28	2 0 56	3 1 24	4 1 53	5 2 21	6 2 49
Quadr. 24	1 0 56	2 1 53	3 2 49	4 3 45	5 4 41	6 5 38

the numbers of quadrants 38, 40, 42 in the first column exceed one another successively by 56 seconds; in the second by 1 minute 53 seconds; and in the third by 2 minutes 49 seconds; and so on. Therefore, if the differences between the degrees, minutes, and seconds of Quadrant 39 and of Quadrant 40 are added, in order, to the degrees, minutes, and seconds of Quadrant 40, the degrees, minutes, and seconds of Quadrant 41 are composed. And the differences between the degrees, minutes, and seconds of Quadrant 38 and of Quadrant 40, added in order to the degrees, minutes, and seconds of Quadrant 40, will make the degrees, minutes, and seconds of Quadrant 42. In the same way the differences between the degrees, minutes, and seconds of Quadrant 30 and of Quadrant 40, set beside the degrees, minutes, and seconds of Quadrant 40, will compose the degrees, minutes, and seconds of Quadrant 50; and so on.

THE TABLE OF THE QUADRANT constructed a little before now follows, in which the several produced arcs are distributed into 128 equal parts; in it there appears at once how many degrees, minutes, and seconds correspond to each little particle of any quadrant — the use of which, indeed, we have explained above.

TABLE OF THE QUADRANT. Left-margin heading (printed vertically): *Numerus siue ordo Quadrantum* — "The number or order of the quadrants." The row labels

1–40 are the quadrant order; the seven column groups (headed 1–7) are the "Parts," each subdivided into G[radius] / M[inuta] / S[ecunda].

Quadr. (order)	Part 1 (G M S)	Part 2 (G M S)	Part 3 (G M S)	Part 4 (G M S)	Part 5 (G M S)	Part 6 (G M S)	Part 7 (G M S)
1	1 0 0	2 0 0	3 0 0	4 0 0	5 0 0	6 0 0	7 0 0
2	0 42 11	1 24 22	2 6 34	2 48 45	3 30 56	4 13 7	4 55 19
3	0 42 39	1 25 19	2 7 58	2 50 37	3 33 17	4 15 56	4 58 36
4	0 43 7	1 26 15	2 9 22	2 52 30	3 35 37	4 18 45	5 1 52
5	0 43 36	1 27 11	2 10 47	2 54 22	3 37 58	4 21 34	5 5 9
6	0 44 4	1 28 7	2 12 11	2 56 15	3 40 19	4 24 22	5 8 26
7	0 44 32	1 29 4	2 13 36	2 58 7	3 42 39	4 27 11	5 11 43
8	0 45 0	1 30 0	2 15 0	3 0 0	3 45 0	4 30 0	5 15 0
9	0 45 28	1 30 56	2 16 24	3 1 53	3 47 21	4 32 49	5 18 17
10	0 45 56	1 31 52	2 17 49	3 3 45	3 49 41	4 35 37	5 21 34
11	0 46 24	1 32 49	2 19 13	3 5 37	3 52 2	4 38 26	5 24 51
12	0 46 52	1 33 45	2 20 37	3 7 30	3 54 22	4 41 15	5 28 7
13	0 47 21	1 34 41	2 22 2	3 9 22	3 56 43	4 44 4	5 31 24
14	0 47 49	1 35 37	2 23 26	3 11 15	3 59 4	4 46 52	5 34 41
15	0 48 17	1 36 34	2 24 51	3 13 7	4 1 24	4 49 41	5 37 58
16	0 48 45	1 37 30	2 26 15	3 15 0	4 3 45	4 52 30	5 41 15
17	0 49 13	1 38 26	2 27 39	3 16 52	4 6 6	4 55 19	5 44 32
18	0 49 41	1 39 22	2 29 4	3 18 45	4 8 26	4 58 7	5 47 49
19	0 50 9	1 40 19	2 30 28	3 20 37	4 10 47	5 0 56	5 51 6
20	0 50 37	1 41 15	2 31 52	3 22 30	4 13 7	5 3 45	5 54 22
21	0 51 6	1 42 11	2 33 17	3 24 22	4 15 28	5 6 34	5 57 39
22	0 51 34	1 43 7	2 34 41	3 26 15	4 17 49	5 9 22	6 0 56
23	0 52 2	1 44 4	2 36 6	3 28 7	4 20 9	5 11[?] 11	6 4 13
24	0 52 30	1 45 0	2 37 30	3 30 0	4 22 30	5 15 0	6 7 30
25	0 52 58	1 45 56	2 38 54	3 31 52	4 24 51	5 17 49	6 10 47
26	0 53 26	1 46 52	2 40 19	3 33 45	4 27 11	5 20 37	6 14 4
27	0 53 54	1 47 49	2 41 43	3 35 37	4 29 32	5 23 26	6 17 21

Quadr. (order)	Part 1 (G M S)	Part 2 (G M S)	Part 3 (G M S)	Part 4 (G M S)	Part 5 (G M S)	Part 6 (G M S)	Part 7 (G M S)
28	0 54 22	1 48 45	2 43 7	3 37 30	4 31 52	5 26 15	6 20 37
29	0 54 51	1 49 41	2 44 32	3 39 22	4 34 13	5 29 4	6 23 54
30	0 55 19	1 50 37	2 45 56	3 41 15	4 36 34	5 31 52	6 27 11
31	0 55 47	1 51 34	2 47 21	3 43 7	4 38 54	5 34 41	6 30 28
32	0 56 15	1 52 30	2 48 45	3 45 0	4 41 15	5 37 30	6 33 45
33	0 56 43	1 53 26	2 50 9	3 46 52	4 43 36	5 40 19	6 37 2
34	0 57 11	1 54 22	2 51 34	3 48 45	4 45 56	5 43 7	6 40 19
35	0 57 39	1 55 19	2 52 58	3 50 37	4 48 17	5 45 56	6 43 36
36	0 58 7	1 56 15	2 54 22	3 52 30	4 50 37	5 48 45	6 46 52
37	0 58 36	1 57 11	2 55 47	3 54 22	4 52 58	5 51 34	6 50 9
38	0 59 4	1 58 7	2 57 11	3 56 15	4 55 19	5 54 22	6 53 26
39	0 59 32	1 59 4	2 58 36	3 58 7	4 57 39	5 57 11	6 56 43
40	1 0 0	2 0 0	3 0 0	4 0 0	5 0 0	6 0 0	7 0 0

Geometriae Practicae Liber I

(Continuation of the table apparatus — the Tabula Quadrantis. These openings carry no running prose; each page is a single full-width numeric table set within a decorative border. The table gives, for each whole number of "parts" [partes] read on the instrument scale and for each of the forty quadrant-numbers, the corresponding arc in degrees, minutes, and seconds. Columns for parts 1–7 stood in the preceding opening; this opening continues with parts 8–14 and then parts 15–21.)

[Margin: Numerus sive ordo Quadrantum — "The number or order of the Quadrants" (the vertical label of the leftmost index column, running the height of each page).]

Table (page 18): Partes 8–14

Column heads: **Partes** (the number of parts, 8–14); under each part the three sub-columns **G** = *Gradus* (degrees), **M** = *Minuta* (minutes), **S** = *Secunda* (seconds). The leftmost column is the quadrant-number (1–40). Each cell below is given as "G M S".

N (quadrant)	8	9	10	11	12	13	14
1	8 0 0	9 0 0	10 0 0	11 0 0	12 0 0	13 0 0	14 0 0
2	5 37 30	6 19 41	7 1 52	7 44 4	8 26 15	9 8 26	9 50 37
3	5 41 15	6 23 54	7 6 34	7 49 13	8 31 52	9 14 32	9 57 11
4	5 45 0	6 28 7	7 11 15	7 54 22	8 37 30	9 20 37	10 3 45
5	5 48 45	6 32 21	7 15 56	7 59 32	8 43 7	9 26 43	10 10 19
6	5 52 30	6 36 34	7 20 37	8 4 41	8 48 45	9 32 49	10 16 52
7	5 56 15	6 40 47	7 25 19	8 9 51	8 54 22	9 38 54	10 23 26
8	6 0 0	6 45 0	7 30 0	8 15 0	9 0 0	9 45 0	10 30 0
9	6 3 45	6 49 13	7 34 41	8 20 9	9 5 37	9 51 6	10 36 34
10	6 7 30	6 53 26	7 39 22	8 25 19	9 11 15	9 57 11	10 43 7
11	6 11 15	6 57 39	7 44 4	8 30 28	9 16 52	10 3 17	10 49 41
12	6 15 0	7 1 52	7 48 45	8 35 37	9 22 30	10 9 22	10 56 15
13	6 18 45	7 6 6	7 53 26	8 40 47	9 28 7	10 15 28	11 2 49
14	6 22 30	7 10 19	7 58 7	8 45 57	9 33 45	16 21 34	11 9 22
15	6 26 15	7 14 32	8 2 49	8 51 6	9 39 22	10 27 39	11 15 56
16	6 30 0	7 18 45	8 7 30	8 56 15	9 45 0	10 33 45	11 22 30
17	6 33 45	7 22 58	8 12 11	9 1 24	9 50 37	10 39 51	11 29 4
18	6 37	7 27	8 16	9 6 34	9 56 15	10 45 56	11 35

N (quadrant)	8	9	10	11	12	13	14
	30	11	52				37
19	6 41 15	7 31 24	8 21 34	9 11 43	10 1 52	10 52 2	11 42 11
20	6 45 0	7 35 37	8 26 15	9 16 52	10 7 30	10 58 7	11 48 45
21	6 48 45	7 39 51	8 30 56	9 22 2	10 13 7	11 4 13	11 55 19
22	6 52 30	7 44 4	8 35 37	9 27 11	10 18 45	11 10 19	12 1 52
23	6 56 15	7 48 17	8 40 19	9 32 21	10 24 22	11 16 24	12 8 26
24	7 0 0	7 52 30	8 45 0	9 37 30	10 30 0	11 22 30	12 15 0
25	7 3 45	7 56 43	8 49 41	9 42 39	10 35 37	11 28 36	12 21 34
26	7 7 30	8 0 56	8 54 22	9 47 49	10 41 15	11 34 41	12 28 7
27	7 11 15	8 5 9	8 59 4	9 52 58	10 46 52	11 40 47	12 34 41
28	7 15 0	8 9 22	9 3 45	9 58 7	10 52 30	11 46 52	12 41 15
29	7 18 45	8 13 36	9 8 26	10 3 17	10 58 7	11 52 58	12 47 49
30	7 22 30	8 17 49	9 13 7	10 8 26	11 3 45	11 59 4	12 54 22
31	7 26 15	8 22 2	9 17 49	10 13 36	11 9 22	12 5 9	13 0 56
32	7 30 0	8 26 15	9 22 30	10 18 45	11 15 0	12 11 15	13 7 30
33	7 33 45	8 30 28	9 27 11	10 23 54	11 20 37	12 17 21[?]	13 14 4
34	7 37 30	8 34 41	9 31 52	10 29 4	11 26 15	12 23 26	13 20 37

N (quadrant)	8	9	10	11	12	13	14
35	7 41 15	8 38 54	9 36 34	10 34 13	11 31 52	12 29 32	13 27 11
36	7 45 0	8 43 7	9 41 15	10 39 22	11 37 30	12 35 37	13 33 45
37	7 48 45	8 47 21	9 45 56	10 44 32	11 43 7	12 41 43	13 40 19
38	7 52 30	8 51 34	9 50 37	10 49 41	11 48 45	12 47 49	13 46 52
39	7 56 15	8 55 47	9 55 19	10 54 51	11 54 22	12 53 59[?]	13 53 26
40	8 0 0	9 0 0	10 0 0	11 0 0	12 0 0	13 0 0	14 0 0

Table (page 19): Partes 15–21

Same structure; parts 15–21, each with **G** / **M** / **S** sub-columns. Cells given as "G M S".

N (quadrant)	15	16	17	18	19	20	21
1	15 0 0	16 0 0	17 0 0	18 0 0	19 0 0	20 0 0	21 0 0
2	10 32 49	11 15 0	11 57 11	12 39 22	13 21 34	14 3 45	14 45 56
3	10 39 51	11 22 30	12 5 9	12 47 49	13 30 28	14 13 7	14 55 47
4	10 46 52	11 30 0	12 13 7	12 56 15	13 39 22	14 22 30	15 5 37
5	10 53 54	11 37 30	12 21 6	13 4 41	13 48 17	14 31 52	15 15 28
6	11 0 56	11 45 0	12 29 4	13 13 7	13 57 11	14 41 15	15 25 19
7	11 7 58	11 52 30	12 37 2	13 21 34	14 6 6	14 50 37	15 35 9
8	11 15 0	12 0 0	12 45 0	13 30 0	14 15 0	15 0 0	15 45 0
9	11 22 2	12 7 30	12 52	13 38	14 23	15 9 22	15 54

N (quadrant)	15	16	17	18	19	20	21
			58	26	54		51
10	11 29 4	12 15 0	13 0 56	13 46	14 32	15 18	16 4 41
				52	49	45	
11	11 36 6	12 22	13 8 54	13 55	14 41	15 28 7	16 14
		30		19	43		32
12	11 43 7	12 30 0	13 16	14 3 45	14 50	15 37	16 24
			52		37	30	22
13	11 50 9	12 37	13 24	14 12	14 59	15 46	16 34
		30	51	11	32	52	13
14	11 57	12 45 0	13 32	14 20	15 8 27	15 56	16 44 4
	11		49	37		15	
15	12 4 13	12 52	13 40	14 29 4	15 17	16 5 37	16 53
		30	47		21		54
16	12 11	13 0 0	13 48	14 37	15 26	16 15 0	17 3 45
	15		45	30	15		
17	12 18	13 7 30	13 56	14 45	15 35 9	16 24	17 13
	17		43	56		22	36
18	12 25	13 15 0	14 4 41	14 54	15 44 4	16 33	17 23
	19			22		45	26
19	12 32	13 22	14 12	15 2 49	15 52	16 43 7	17 33
	21	30	39		58		17
20	12 39	13 30 0	14 20	15 11	16 1 52	16 52	17 43 7
	22		37	15		30	
21	12 46	13 37	14 28	15 19	16 10	17 1 52	17 52
	24	30	36	41	47		58
22	12 53	13 45 0	14 36	15 28 7	16 19	17 11	18 2 49
	26		34		41	15	
23	13 0 28	13 52	14 44	15 36	16 28	17 20	18 12
		30	33	34	36	37	39
24	13 7 30	14 0 0	14 52	15 45 0	16 37	17 30 0	18 22
			30		30		30
25	13 14	14 7 30	15 0 28	15 53	16 46	17 39	18 32
	32			26	24	22	21

N (quadrant)	15	16	17	18	19	20	21
26	13 21 34	14 15 0	15 8 26	16 1 52	16 55 19	17 48 45	18 42 11
27	13 28 36	14 22 30	15 16 24	16 10 19	17 4 13	17 58 7	18 52 2
28	13 35 37	14 30 0	15 24 22	16 18 45	17 13 7	18 7 30	19 1 52
29	13 42 39	14 37 30	15 32 21	16 27 11	17 22 2	18 16 52	19 11 43
30	13 49 41	14 45 0	15 40 19	16 35 37	17 30 56	18 26 15	19 21 34
31	13 56 43	14 52 30	15 48 17	16 44 4	17 39 51	18 35 37	19 31 24
32	14 3 45	15 0 0	15 56 15	16 52 30	17 48 45	18 45 0	19 41 15
33	14 10 47	15 7 30	16 4 13	17 0 56	17 57 39	18 54 22	19 51 6
34	14 17 49	15 15 0	16 12 11	17 9 22	18 6 34	19 3 45	20 0 56
35	14 24 51	15 22 30	16 20 9	17 17 49	18 15 28	19 13 7	20 10 47
36	14 31 52	15 30 0	16 28 7	17 26 15	18 24 22	19 22 30	20 20 37
37	14 38 54	15 37 30	16 36 6	17 34 41	18 33 17	19 31 52	20 30 28
38	14 45 56	15 45 0	16 44 0	17 43 7	18 42 11	19 41 15	20 40 19
39	14 52 58	15 52 32	16 52 2	17 51 34	18 51 6	19 50 37	20 50 9
40	15 0 0	16 0 0	17 0 0	18 0 0	19 0 0	20 0 0	21 0 0

Geometriae Practicae Liber I — continuation of the **Tabula Quadrantis** (Table of the Quadrant). Three consecutive leaves carrying Partes 15 through 35 of the table. Each column heads a "Part" (Pars) of the instrument's scale; under each Part the three sub-columns **G M S** give degrees, minutes, and seconds (*gradus*,

minuta, secunda). The left-hand index column, labelled vertically **Numerus sive ordo Quadrantum** ("The count, or order, of the quadrant-divisions"), runs 1–40.

In every column the reader will notice that **row 1 repeats the whole-degree value of row 40** (e.g. Part 15 shows 15 0 0 at both row 1 and row 40); this is a structural feature/anomaly of the printed table carried throughout, transcribed as printed and flagged below, not corrected.

Each table cell below is given as degrees minutes seconds.

Table of the Quadrant — Partes 15–21

Nu m.	15	16	17	18	19	20	21
1	15 0 0	16 0 0	17 0 0	18 0 0	19 0 0	20 0 0	21 0 0
2	10 32 49	11 15 0	11 57 11	12 39 22	13 21 34	14 3 45	14 45 56
3	10 39 51	11 22 30	12 5 9	12 47 49	13 30 28	14 13 7	14 55 47
4	10 46 52	11 30 0	12 13 7	12 56 15	13 39 22	14 22 30	15 5 37
5	10 53 54	11 37 30	12 21 6	13 4 41	13 48 17	14 31 52	15 15 28
6	11 0 56	11 45 0	12 29 4	13 13 7	13 57 11	14 41 15	15 25 19
7	11 7 58	11 52 30	12 37 2	13 21 34	14 6 6	14 50 37	15 35 9
8	11 15 0	12 0 0	12 45 0	13 30 0	14 15 0	15 0 0	15 45 0
9	11 22 2	12 7 30	12 52 58	13 38 26	14 23 54	15 9 22	15 54 51
10	11 29 4	12 15 0	13 0 56	13 46 52	14 32 49	15 18 45	16 4 41
11	11 36 6	12 22 30	13 8 54	13 55 19	14 41 43	15 28 7	16 14 32
12	11 43 7	12 30 0	13 16 52	14 3 45	14 50 37	15 37 30	16 24 22
13	11 50 9	12 37 30	13 24 51	14 12 11	14 59 32	15 46 52	16 34 13
14	11 57 11	12 45 0	13 32 49	14 20 37	15 8 27	15 56 15	16 44 4
15	12 4 13	12 52 30	13 40 47	14 29 4	15 17 21	16 5 37	16 53 54
16	12 11 15	13 0 0	13 48 45	14 37 30	15 26 15	16 15 0	17 3 45
17	12 18 17	13 7 30	13 56 43	14 45 56	15 35 9	16 24 22	17 13 36
18	12 25 19	13 15 0	14 4 41	14 54 22	15 44 4	16 33 45	17 23 26
19	12 32 21	13 22 30	14 12 39	15 2 49	15 52 58	16 43 7	17 33 17
20	12 39 22	13 30 0	14 20 37	15 11 15	16 1 52	16 52 30	17 43 7
21	12 46 24	13 37 30	14 28 36	15 19 41	16 10 47	17 1 52	17 52 58
22	12 53 26	13 45 0	14 36 34	15 28 7	16 19 41	17 11 15	18 2 49

Nu m.	15	16	17	18	19	20	21
23	13 0 28	13 52 30	14 44 33	15 36 34	16 28 36	17 20 37	18 12 39
24	13 7 30	14 0 0	14 52 30	15 45 0	16 37 30	17 30 0	18 22 30
25	13 14 32	14 7 30	15 0 28	15 53 26	16 46 24	17 39 22	18 32 21
26	13 21 34	14 15 0	15 8 26	16 1 52	16 55 19	17 48 45	18 42 11
27	13 28 36	14 22 30	15 16 24	16 10 19	17 4 13	17 58 7	18 52 2
28	13 35 37	14 30 0	15 24 22	16 18 45	17 13 7	18 7 30	19 1 52
29	13 42 39	14 37 30	15 32 21	16 27 11	17 22 2	18 16 52	19 11 43
30	13 49 41	14 45 0	15 40 19	16 35 37	17 30 56	18 26 15	19 21 34
31	13 56 43	14 52 30	15 48 17	16 44 4	17 39 51	18 35 37	19 31 24
32	14 3 45	15 0 0	15 56 15	16 52 30	17 48 45	18 45 0	19 41 15
33	14 10 47	15 7 30	16 4 13	17 0 56	17 57 39	18 54 22	19 51 6
34	14 17 49	15 15 0	16 12 11	17 9 22	18 6 34	19 3 45	20 0 56
35	14 24 51	15 22 30	16 20 9	17 17 49	18 15 28	19 13 7	20 10 47
36	14 31 52	15 30 0	16 28 7	17 26 15	18 24 22	19 22 30	20 20 37
37	14 38 54	15 37 30	16 36 6	17 34 41	18 33 17	19 31 52	20 30 28
38	14 45 56	15 45 0	16 44 0	17 43 7	18 42 11	19 41 15	20 40 19
39	14 52 58	15 52 30	16 52 2	17 51 34	18 51 6	19 50 37	20 50 9
40	15 0 0	16 0 0	17 0 0	18 0 0	19 0 0	20 0 0	21 0 0

Table of the Quadrant — Partes 22–28

(This leaf is printed with the page number "18", but by its content — Partes 22–28, continuing between the leaves numbered 19 and 21 — it occupies the position of page 20. below.)

N u m.	22	23	24	25	26	27	28
1	22 0 0	23 0 0	24 0 0	25 0 0	26 0 0	27 0 0	28 0 0
2	15 28 7	16 10 19	16 52 30	17 34 41	18 16 52	18 59 4	19 41 15
3	15 38 26	16 21 6	17 3 45	17 46 24	18 29 4	19 11 43	19 54 22

N u m.	22	23	24	25	26	27	28
4	15 48 45	16 31 52	17 15 0	17 58 7	18 41 15	19 24 22	20 7 30
5	15 59 4	16 42 39	17 26 15	18 9 51	18 53 26	19 37 2	20 20 37
6	16 9 22	16 53 26	17 37 30	18 21 34	19 5 37	19 49 41	20 33 45
7	16 19 41	17 4 13	17 48 45	18 33 17	19 17 49	20 2 21	20 46 52
8	16 30 0	17 15 0	18 0 0	18 45 0	19 30 0	20 15 0	21 0 0
9	16 40 19	17 25 47	18 11 15	18 56 43	19 42 11	20 27 39	21 13 7
10	16 50 37	17 36 34	18 22 30	19 8 26	19 54 22	20 40 19	21 26 15
11	17 0 56	17 47 21	18 33 45	19 20 9	20 6 34	20 52 58	21 39 22
12	17 11 15	17 58 7	18 45 0	19 31 52	20 18 45	21 5 37	21 52 30
13	17 21 34	18 8 54	18 56 15	19 43 36	20 30 56	21 18 17	22 5 37
14	17 31 52	18 19 41	19 7 30	19 55 19	20 43 7	21 30 56	22 18 45
15	17 42 11	18 30 28	19 18 35 [recte: 45]	20 7 2	20 55 19	21 43 36	22 31 52
16	17 52 30	18 41 15	19 30 0	20 18 45	21 7 30	21 56 15	22 45 0
17	18 2 49	18 52 2	19 41 15	20 30 28	21 19 41	22 8 54	22 58 7
18	18 13 7	19 2 49	19 52 30	20 42 11	21 31 52	22 21 34	23 11 15
19	18 23 26	19 13 36	20 3 45	20 53 54	21 44 4	22 34 13	23 24 22
20	18 33	19 24	20 15 0	21 5 37	21 56	22 46	23 37

N u m.	22	23	24	25	26	27	28
	45	22			15	52	30
21	18 44 4	19 35 9	20 26 15	21 17 21	22 8 26	22 59 32	23 50 37
22	18 54 22	19 45 56	20 37 30	21 29 4	22 20 37	23 12 11	24 3 45
23	19 4 41	19 56 43	20 48 45	21 40 47	22 32 49	23 24 51	24 16 52
24	19 15 0	20 7 30	21 0 0	21 52 30	22 45 0	23 37 30	24 30 0
25	19 25 19	20 18 17	21 11 15	22 4 13	22 57 11	23 50 9	24 43 7
26	19 35 37	20 29 4	21 22 30	22 15 56	23 9 22	24 2 49	24 56 15
27	19 45 56	20 39 51	21 33 45	22 27 39	23 21 34	24 15 28	25 9 22
28	19 56 15	20 50 37	21 45 0	22 39 22	23 33 45	24 28 7	25 22 30
29	20 6 34	21 1 24	21 56 15	22 51 6	23 45 56	24 40 47	25 35 37
30	20 16 52	21 12 11	22 7 30	23 2 49	23 58 7	24 53 26	25 48 45
31	20 27 11	21 22 58	22 18 45	23 14 32	24 10 19	25 6 6	26 1 52
32	20 37 30	21 33 45	22 30 0	23 26 15	24 22 30	25 18 45	26 15 0
33	20 47 49	21 44 32	22 41 15	23 37 58	24 34 41	25 31 24	26 28 7
34	20 58 7	21 55 19	22 52 30	23 49 41	24 46 52	25 44 4	26 41 15
35	21 8 26	22 6 6	23 3 45	24 1 24	24 59 4	25 56 43	26 54 22
36	21 18	22 16	23 15 0	24 13 7	25 11	26 9 22	27 7 30

N u m.	22	23	24	25	26	27	28
	45	52			15		
37	21 29 4	22 27 39	23 26 25 [recte: 15]	24 24 51	25 23 26	26 22 2	27 20 37
38	21 39 22	22 38 26	23 37 30	24 36 34	25 35 37	26 34 41	27 33 45
39	21 49 41	22 49 13	23 48 45	24 48 17	25 47 49	26 47 21	27 46 52
40	22 0 0	23 0 0	24 0 0	25 0 0	26 0 0	27 0 0	28 0 0

Table of the Quadrant — Partes 29–35

N u m. .	29	30	31	32	33	34	35
1	29 0 0	30 0 0	31 0 0	32 0 0	33 0 0	34 0 0	35 0 0
2	20 23 26	21 5 37	21 47 49	22 30 0	23 12 11	23 54 22	24 36 34
3	20 37 2	21 19 41	22 2 21	22 45 0	23 27 39	24 10 19	24 52 58
4	20 50 37	21 33 45	22 16 52	23 0 0	23 43 7	24 26 15	25 9 22
5	21 4 13	21 47 49	22 31 24	23 15 0	23 58 36	24 42 11	25 25 47
6	21 17 49	22 1 52	22 45 56	23 30 0	24 14 4	24 58 7	25 42 11
7	21 31 24	22 15 56	23 0 28	23 45 0	24 29 32	25 14 4	25 98 36 [recte: 58]
8	21 45 0	22 30 0	23 15 0	24 0 0	24 45 0	25 30 0	26 15 0
9	21 58 36	22 44 4	23 29 32	24 15 0	25 0 28	25 45 56	26 31 24

N
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m

.	29	30	31	32	33	34	35
1 0	22 12 11	22 58 7	23 44 4	24 30 0	25 15 56	26 1 52	26 47 49
1 1	22 25 47	23 12 11	23 58 36	24 45 0	25 31 24	26 17 49	27 4 13
1 2	[numeral : ?]	23 26 15	24 13 7	25 0 0	25 46 52	26 33 45	27 20 37
1 3	22 52 58	23 40 19	24 27 39	25 15 0	26 2 21	26 49 41	27 37 2
1 4	23 6 34	23 54 22	24 42 11	25 30 0	26 17 49	27 5 37	27 53 26
1 5	23 20 9	24 8 26	24 56 43	25 45 0	26 33 17	27 21 34	28 9 51
1 6	23 33 45	[numeral : ?]	[numeral : ?]	26 0 0	26 48 45	27 37 30	28 26 15
1 7	23 47 21	24 36 34	25 25 47	26 15 0	27 4 13	27 53 26	28 42 39
1 8	24 0 56	24 50 37	25 40 19	26 30 0	27 19 41	28 9 22	28 59 4
1 9	24 14 32	25 4 41	25 54 51	26 45 0	27 35 9	28 25 19	29 16 24
2 0	24 28 7	25 18 45	26 9 22	27 0 0	27 50 37	28 41 15	29 31 52
2 1	24 41 43	25 32 49	26 23 54	27 15 0	28 6 6	28 57 11	29 48 17
2 2	24 55 19	25 46 52	26 38 26	27 30 0	28 21 34	29 13 7	30 4 41
2 3	25 8 54	26 0 50	26 52 58	27 45 0	28 37 2	29 29 4	30 21 6
2 4	25 22 30	26 15 0	27 7 30	28 0 0	28 52 30	29 45 0	30 37 30

N
u
m

.	29	30	31	32	33	34	35
2	25 36 6	26 29 4	27 22 2	28	29 7	30 0	30 53 54
5				15 0	58	56	
2	25 49 41	26 43 7	27 36 34	28	29 23	30 16	31 10 19
6				30 0	26	52	
2	26 3 17	26 57 21	27 51 6	28	29 38	30 32	31 26 43
7				45 0	54	49	
2	26 16 52	27 11 15	28 5 37	29 0	29 54	30 48	31 43 7
8				0	22	45	
2	26 30 28	27 25 19	28 20 9	29	30 9	31 4	31 59 32
9				15 0	51	41	
3	26 44 4	27 39 22	28 34 41	29	30 25	31 20	32 15 56
0				30 0	19	37	
3	26 57 39	27 53 26	28 49 13	29	30 40	31 36	32 32 21
1				45 0	47	34	
3	27 11 15	28 7 30	29 3 45	30 0	30 56	31 52	32 48 45
2				0	15	30	
3	27 24 51	28 21 34	29 18 17	30	31 11	32 8	33 5 9
3				15 0	43	26	
3	27 38 26	28 35 37	29 32 49	30	31 27	32 24	33 21 34
4				30 0	11	22	
3	27 52 2	28 49 41	29 47 21	30	31 42	32 40	33 37 58
5				45 0	39	16	
3	28 5 37	29 3 45	30 1 52	31 0	31 58	32 56	33 54 22
6				0	7	15	
3	28 19 13	29 17 49	30 16 24	31	32 13	33 12	34 10 47
7				15 0	36	11	
3	28 32 49	29 31 52	30 30 56	31	32 29	33 28	34 27 11
8				30 0	4	7	
3	28 46 24	29 45 56	30 45 28	31	32 44	33 44	34 43 36
9				45 0	32	4	

N
u
m

.	29	30	31	32	33	34	35
4	29 0 0	30 0 0	31 0 0	32 0	33 0 0	34 0 0	35 0 0
0				0			

Geometriae Practicae Liber I — continuation of the **Tabula Quadrantis** (Table of the Quadrant). These three pages continue the wide multi-column table begun earlier, covering the divisions (*Partes*) **36 through 56**, seven divisions per leaf. Each division heads three sub-columns — **G M S** (gradus / degrees, minuta / minutes, secunda / seconds) — and the table runs 40 numbered rows (*Numerus sive ordo Quadrantum*, "the number or order of the quadrants," printed vertically down the left margin). There is no running prose on these pages.

In each cell below the three sub-column values are given in order as **G M S** (e.g. 36 0 0 = 36° 0' 0"). Numerals are transcribed exactly as printed, read at high magnification; where a printed value breaks the otherwise regular arithmetic progression of its column it is given as printed with a [recte: ...] correction.

Recurring structural anomaly (whole table): for every division, **Row 1 and Row 40 carry the identical whole-degree value** (<Partes> 0 0) — e.g. under division 36, both Row 1 and Row 40 read 36 0 0. This duplication is a feature of the printed table (Row 1 anticipates the terminal whole-degree value that properly belongs at Row 40) and is transcribed as printed throughout.

Table of the Quadrant — Divisions (*Partes*) 36–42

Cell format: **G M S** (degrees, minutes, seconds).

R
o

w36	37	38	39	40	41	42
1 36 0 0	37 0 0	38 0 0	39 0 0	40 0 0	41 0 0	42 0 0
2 25 18 45	26 0 56	26 43 7	27 25 19	28 7 30	28 49 41	29 31 52
3 25 35 37	26 18	27 0 56	27 43 36	28 26 15	29 8	29 51

R o w36	37	38	39	40	41	42
	17				54	34
4 25 52 30	26	27 18 45	28 1 52	28 45 0	29	30
	35				28	11
	37				7	15
5 26 9 22	26	27 36 34	28 20 9	26 3 45 [recte: 29 3 45]	29	30
	52				47	30
	58				21	56
6 26 26 15	27	27 54 22	28 38 26	29 22 30	30	30
	10				6	50
	19				34	37
7 26 43 7	27	28 12 11	28 56 43	29 41 15	30	31
	27				25	10
	39				47	19
8 27 0 0	27	28 30 0	29 15 0	30 0 0	30	31
	45				45	30
	0				0	0
9 27 16 52	28	28 47 49	29 33 17	30 18 45	31	31
	2				4	49
	21				13	41
1 27 33 45	28	29 5 37	29 51 34	30 37 30	31	32
0	19				23	9
	41				26	22
1 27 50 37	28	29 23 26	30 9 51	30 56 15	31	32
1	37				42	29
	2				39	4
1 29 7 30 [recte: 2 28 7 30]	28	29 41 15	30 28 7	31 15 0	32	32
	54				1	48
	22				52	45
1 28 24 22	29	29 54 4 [recte: 29 59 4]	30 46 24	31 33 45	32	33
3	11				21	8
	43				6	26
1 28 41 15	29	30 16 52	31 4 41	31 52 30	32	33
4	29				40	28

R o w36	37	38	39	40	41	42
	4				19	7
1 28 58 7	29	30 34 41	31 22 58	32 11 15	32	33
5	46				59	47
	24				32	49
1 29 15 0	30	30 32 30	31 41 15	32 30 0	33	34
6	3	[recte: 30 52			18	7
	45	30]			45	30
1 29 31 52	30	31 10 19	31 59 32	32 48 45	33	34
7	21				37	27
	9				58	11
1 29 48 45	30	31 28 7	32 17 47	33 7 30	33	34
8	38				57	46
	26				11	52
1 30 5 37	30	31 45 56	32 36 6	33 26 15	34	35
9	55				16	6
	47				24	34
2 30 22 30	31	32 3 45	32 54 22	33 45 0	34	35
0	13				35	26
	7				37	15
2 30 39 22	31	32 21 34	33 12 39	34 3 45	34	35
1	30				54	45
	28				51	56
2 30 56 15	31	32 39 22	33 30 56	34 22 30	35	36
2	47				14	5
	49				4	37
2 31 13 7	32	32 57 11	33 49 13	34 41 15	35	36
3	5 9				33	25
					17	19
2 31 30 0	32	33 15 0	34 7 30	35 0 0	35	36
4	22				52	45
	30				30	0
2 31 46 52	32	33 32 49	34 25 47	35 18 45	36	37
5	39				11	4

R o w36	37	38	39	40	41	42
	51				43	41
2 32 3 45 6	32 57 11	33 50 37	34 44 4	35 37 30	36 30 56	37 24 22
2 32 20 37 7	33 14 32	34 8 26	35 2 21	34 56 15 [recte: 35 56 15]	36 50 9	37 44 4
2 32 37 30 8	33 31 52	34 26 15	35 20 37	36 15 0	37 9 22	38 3 45
2 32 54 22 9	33 49 13	34 44 4	35 38 34 [recte: 35 38 54]	36 33 45	37 28 36	38 23 26
3 33 11 15 0	34 6 34	35 1 52	35 57 11	36 52 30	37 47 49	38 43 7
3 33 28 7 1	34 23 54	35 19 41	36 15 28	37 11 15	38 7 2	39 2 49
3 33 45 0 2	34 41 15	35 37 30	36 33 45	37 30 0	38 26 15	39 22 30
3 34 1 52 3	34 58 36	35 55 19	36 52 2	37 48 45	38 45 28	39 42 11
3 34 18 45 4	35 15 56	36 13 7	37 10 19	38 7 30	39 4 41	40 1 52
3 34 35 37 5	35 33 17	36 30 56	37 28 36	38 26 15	39 23 54	40 21 34
3 34 52 30 6	35 50	36 48 45	37 46 52	38 45 0	39 43	40 41

R o w	36	37	38	39	40	41	42
		37				7	15
3 35 9 22 7		36 7	37 6 34	38 5 9	39 3 45	40 2	41 0
		58				21	56
3 35 26 15 8		36 25	37 24 22	38 23 26	39 22 30	40 21	41 20
		19				34	37
3 35 43 7 9		36 42	37 42 11	38 41 43	39 41 15	40 40	41 40
		39				47	19
4 36 0 0 0		37 0 0	38 0 0	39 0 0	40 0 0	41 0 0	42 0 0

Table of the Quadrant — Divisions (*Partes*) 43–49

Cell format: **G M S** (degrees, minutes, seconds).

R o w	43	44	45	46	47	48	49
1	43 0 0	44 0 0	45 0 0	46 0 0	47 0 0	48 0 0	49 0 0
2	30 14 4	30 56 15	31 38 26	32 20 37	33 2 49	33 45 0	34 27 11
3	30 34 13	31 16 52	31 59 32	32 42 11	33 24 51	34 7 30	34 50 9
4	30 54 22	31 37 30	32 20 37	33 3 45	33 46 52	34 30 0	35 13 7
5	31 14 32	31 58 7	32 41 43	33 25 19	34 8 54	34 52 30	35 36 6
6	31 34 41	32 18 45	33 2 49	33 46 52	54 30 56 [recte: 34 30 56]	35 15 0	35 59 4
7	31 54 51	32 39 22	33 23 54	34 8 26	34 52 58	33 37 30 [recte: 35 37 30]	36 22 2

R							
o							
w	43	44	45	46	47	48	49
8	32 15 0	33 0 0	33 45 0	34 30 0	35 15 0	36 0 0	36 45 0
9	32 35 9	33 20 37	34 6 6	34 51 34	35 37 2	36 22 30	37 7 58
1	32 55 0	33 41 15	34 27 11	35 13 7	35 59 4	36 45 0	37 30 56
1	33 15 28	34 1 52	34 48 17	35 34 41	36 21 6	37 7 30	37 53 54
1	33 35 2	34 22 30	35 9 22	35 56 15	36 43 7	37 30 0	38 16 52
1	33 55 3	34 43 7	35 30 28	36 17 49	37 5 9	37 52 30	38 39 51
1	34 15 4	35 3 45	35 51 34	36 39 22	37 27 11	38 15 0	39 2 49
1	34 36 5	35 24 22	36 12 39	37 0 56	37 49 13	38 37 30	39 25 47
1	34 56 6	35 45 0	36 33 45	37 22 30	38 11 15	39 0 0	39 48 45
1	35 16 7	36 5 37	36 54 51	37 44 4	38 33 17	39 22 30	40 11 43
1	35 36 8	36 26 15	37 15 56	38 5 37	38 55 19	39 45 0	40 34 41
1	35 56 9	36 46 52	37 37 2	38 27 11	39 17 21	40 7 30	40 57 39
2	36 16 0	37 7 30	37 58 7	38 48 45	39 39 22	40 30 0	41 20 37
2	36 37 1	37 28 7	38 19 13	39 10 19	40 1 24	40 52 30	41 43 36
2	36 57 2	37 48 45	38 40 19	39 31 52	40 23 26	41 15 0	42 6 34
2	37 17 3	38 9 22	39 1 24	39 53 26	40 45 28	41 37 30	42 29 32

R							
o							
w	43	44	45	46	47	48	49
2	37 37	38 30	39 22	40 15	41 7 30	42 0 0	42 52
4	30	0	30	0			30
2	37 57	38 50	39 43	40 36	41 29 32	42 22 30	43 15
5	39	37	36	34			28
2	38 17	39 11	40 4	40 58	41 51 34	42 45 0	43 38
6	49	15	41	7			26
2	38 37	39 31	40 25	41 19	42 13 36	43 7 30	44 1
7	58	52	47	41			24
2	38 58	39 52	40 46	41 41	42 35 37	43 30 0	44 24
8	7	30	52	15			22
2	39 18	40 13	41 7	42 2	42 57 39	43 52 30	44 47
9	17	7	58	49			21
3	39 38	40 33	41 29	42 24	43 19 41	44 15 0	45 10
0	26	45	4	22			19
3	39 58	40 54	41 50	42 45	43 41 43	44 37 30	45 33
1	36	22	9	56			17
3	40 18	41 15	42 11	43 7	44 3 45	45 0 0	45 56
2	45	0	15	30			15
3	40 38	41 35	42 32	43 29	44 25 47	45 22 30	46 19
3	54	37	21	4			13
3	40 59	41 56	42 53	43 50	44 47 49	45 45 0	46 42
4	4	15	26	37			11
3	41 19	42 16	43 14	44 12	45 9 51	46 7 30	47 5
5	13	52	32	11			9
3	41 39	42 37	43 35	44 33	45 31 52	46 30 0	47 28
6	22	30	37	45			7
3	41 59	42 58	43 56	44 55	45 53 54	46 52 30	47 51
7	32	7	43	19			6
3	42 19	43 18	44 17	45 16	46 15 56	47 15 0	48 14
8	41	45	49	52			4
3	42 39	43 39	44 38	45 38	46 37 58	47 37 30	48 37
9	51	22	54	26			2

R							
o							
w	43	44	45	46	47	48	49
4	43 0	44 0	45 0	46 0	47 0 0	48 0 0	49 0
0	0	0	0	0			0

Table of the Quadrant — Divisions (*Partes*) 50–56

The fourth division on this leaf is printed in the column header as "93", an evident compositor error for 53 (the column's values interpolate smoothly between divisions 52 and 54 and terminate at 53 0 0). It is transcribed below as 53 [printed header "93"; recte 53].

Cell format: **G M S** (degrees, minutes, seconds).

R							
o				53 [header printed			
w	50	51	52	"93"]	54	55	56
1	50 0 0	51 0 0	52 0 0	53 0 0	54 0 0	55 0 0	56 0 0
2	35 9 22	35 51	36 33	37 15 56	37 58	38 40	39 22
		34	45		7	19	30
3	35 32 49	36 15	36 58	37 40 47	38 23	39 6 6	39 48
		28	7		26		45
4	35 56 15	36 39	37 22	38 5 37	38 48	39 31	40 15
		22	30		45	52	0
5	36 19 41	37 3	37 46	38 30 28	39 14	39 57	40 41
		17	52		4	39	15
6	36 43 7	37 27	38 11	38 55 19	39 39	40 23	41 7
		11	15		22	26	30
7	37 6 34	37 51	38 35	39 20 9	40 4	40 49	41 33
		6	37		41	13	45
8	37 30 0	38 15	39 0 0	39 45 0	40 30	41 15	42 0 0
		0			0	0	
9	37 53 26	38 38	39 24	40 9 51	40 55	41 40	42 26
		54	22		19	47	15
1	38 16 52	39 2	39 48	40 34 41	41 20	42 6	42 52
0		49	45		37	34	30
1	38 40 19	39 26	40 13	40 59 32	41 45	42 32	43 18

R				53 [header printed			
o				"93"]			
w	50	51	52		54	55	56
1		43	7		56	21	45
1	39 3 45	39 50	40 37	41 24 22	42 11	42 58	43 45
2		37	30		15	7	0
1	39 27 11	40 14	41 1	41 49 13	42 36	43 23	44 11
3		32	52		34	54	15
1	39 50 37	40 38	41 26	42 14 4	43 1	43 49	44 37
4		26	15		52	41	30
1	40 14 4	41 2	41 50	42 38 54	43 27	44 15	45 3
5		21	37		11	28	45
1	40 37 30	41 26	42 15	43 3 45	43 52	44 41	45 30
6		15	0		30	15	0
1	41 0 56	41 50	42 39	43 28 36	44 17	45 7 2	45 56
7		9	22		49		15
1	41 24 22	42 14	43 3	43 53 26	44 43	45 32	46 22
8		4	45		7	49	30
1	41 47 49	42 37	43 28	44 18 17	45 8	45 58	46 48
9		58	7		26	36	45
2	42 11 15	43 1	43 52	44 43 7	45 33	46 24	47 15
0		52	30		45	22	0
2	42 34 41	43 25	44 16	45 7 58	45 59	46 50	47 41
1		47	52		4	9	15
2	42 58 7	43 49	44 41	45 32 49	46 24	47 15	48 7
2		41	15		22	56	30
2	43 21 34	44 13	45 5	45 57 39	46 49	47 41	48 33
3		36	37		41	43	45
2	43	44 37	45 30	46 22 30	47 15	48 7	49 0 0
4	[numeral: ?] 0	30	0		0	30	
2	44 8 26	45 1	45 54	46 47 21	47 40	48 33	49 26
5		24	22		19	17	15
2	44 31 52	45 25	46 18	47 12 11	48 5	48 59	49 52
6		19	45		37	4	30
2	44 55 19	45 49	46 43	47 37 2	48 30	49 24	50 18

R o w	50	51	52	53 [header printed "93"]	54	55	56
7		13	7		56	51	45
2	45 18 45	46 13	47 7	48 1 52	48 56	49 50	50 45
8		7	30		15	37	0
2	45 42 11	46 37	47 31	48 26 43	49 21	50 16	51 11
9		2	52		34	24	15
3	46 5 37	47 0	47 56	48 51 34	49 46	50 42	51 37
0		56	15		52	11	30
3	46 29 4	47 24	48 20	49 16 24	50 12	51 7	52 3
1		51	37		11	58	45
3	46 52 30	47 48	48 45	49 41 15	50 37	51 33	52 30
2		45	0		30	45	0
3	47 15 56	48 12	49 9	50 6 6	51 2	51 59	52 56
3		39	22		49	32	15
3	47 39 22	48 36	49 33	50 30 56	51 28	52 25	53 22
4		34	45		7	19	30
3	48 2 49	49 0	49 58	50 55 47	51 53	52 51	53 48
5		28	7		26	6	45
3	48 26 15	49 24	50 22	51 20 37	52 18	53 16	54 15
6		22	30		45	52	0
3	48 49 41	49 48	50 46	51 45 28	52 44	53 42	54 41
7		17	52		4	39	15
3	49 13 7	50 12	51 11	52 10 19	53 9	54 8	55 7
8		11	15		22	26	30
3	49 36 34	50 36	51 35	52 35 9	53 34	54 34	55 33
9		6	37		41	13	45
4	50 0 0	51 0 0	52 0 0	53 0 0	54 0 0	55 0 0	56 0 0
0							

Geometriae Practicae Liber I — continuation of the **Tabula Quadrantis** (Table of the Quadrant).

[Margin: Numerus siue ordo Quadrantum — "The number, or rank, of the quadrant-parts." (This vertical label runs down the inner margin of all three pages.)]

These three pages continue the interpolation table begun in the preceding pages. The leftmost column, **Partes** (numbered 1–40), gives the rank of the subdivision; each of the numbered columns (57 ... 77) is one division of the instrument, and under each division the three sub-columns **G** (*gradus*, degrees), **M** (*minuta*, minutes) and **S** (*secunda*, seconds) give the corresponding tabulated value. In every column, row 1 records the whole value (Partes°0'0") and rows 2–40 climb by a constant step back up to that same whole value at row 40 (see the numbering anomaly noted below).

Each cell below is transcribed as **G M S** (degrees minutes seconds).

Page 23 — Partes 57–63

Partes	57	58	59	60	61	62	63
1	57 0 0	58 0 0	59 0 0	60 0 0	61 0 0	62 0 0	63 0 0
2	40 4 41	40 46 52	41 29 4	42 11 15	42 53 26	43 35 37	44 17 49
3	40 31 24	41 14 4	41 56 43	42 39 22	43 22 2	44 4 41	44 47 21
4	40 58 7	41 41 15	42 24 22	43 7 30	43 50 37	44 33 45	45 16 52
5	41 24 51	42 8 26	42 52 2	43 35 37	44 19 13	45 2 49	45 46 24
6	41 51 34	42 35 37	43 19 41	44 3 45	44 47 49	45 31 52	46 15 56
7	42 18 17	43 2 49	43 47 21	44 31 52	45 16 24	46 0 56	46 45 28
8	42 45 0	43 30 0	44 15 0	45 0 0	45 45 0	46 30 0	47 15 0
9	43 11 43	43 57 11	44 42 39	45 28 7	46 13 36	46 59 4	47 44 32
10	43 38 26	44 24 22	45 10 19	45 56 15	46 42 11	47 28 7	48 14 4
11	44 5 9	44 51 34	45 37 58	46 24 22	47 10 47	47 57 11	48 43 36
12	44 31 52	45 18 45	46 5 37	46 52 30	47 39 22	48 26 15	49 13 7

Parte s	57	58	59	60	61	62	63
13	44 58 36	45 45 56	46 33 17	47 20 37	48 7 58	48 55 19	49 42 39
14	45 25 19	46 13 7	47 0 56	47 48 45	48 36 34	49 24 22	50 12 11
15	45 52 2	46 40 19	47 28 36	48 16 52	49 5 9	49 53 26	50 41 43
16	46 18 45	47 7 30	47 56 15	48 45 0	49 33 45	50 22 30	51 11 15
17	46 45 28	47 34 41	48 23 54	49 13 7	50 2 21	50 51 34	51 40 47
18	47 12 11	48 1 52	48 51 34	49 41 15	50 30 56	51 20 37	52 10 19
19	47 38 54	48 29 4	49 19 13	50 9 22	50 59 32	51 49 41	52 39 51
20	48 5 37	48 56 15	49 46 52	50 37 30	51 28 7	52 18 45	53 9 22
21	48 32 21	49 23 26	50 14 32	51 5 37	51 56 43	52 47 49	53 38 54
22	48 59 4	49 50 37	50 42 11	51 33 45	52 25 19	53 16 52	54 8 26
23	49 25 47	50 17 49	51 9 51	52 1 52	52 53 54	53 45 56	54 37 58
24	49 52 30	50 45 0	51 37 30	52 30 0	53 22 30	54 15 0	55 7 30
25	50 19 13	51 12 11	52 5 9	52 58 7	53 51 6	54 44 4	55 37 2
26	50 45 56	51 39 22	52 32 49	53 26 15	54 19 41	55 13 7	56 6 34
27	51 12 39	52 6 34	53 0 28	53 54 22	54 48 17	55 42 11	56 36 6
28	51 39 22	52 33 45	53 28 7	54 22 30	55 16 52	56 11 15	57 5 37
29	52 6 6	53 0 56	53 55	54 50	55 45	56 40	57 35 9

Partes	57	58	59	60	61	62	63
			47	37	28	19	
30	52 32 49	53 28 7	54 23 26	55 18 45	56 14 4	57 9 22	58 4 41
31	51 [?] 59 32	53 55 19	54 51 6	55 46 52	56 42 39	57 38 26	58 34 13
32	53 56 [?] 15	54 22 30	55 18 45	56 15 0	57 11 15	58 7 30	59 3 45
33	53 52 58	54 49 41	55 46 24	56 43 7	57 39 51	58 36 34	59 33 17
34	54 19 41	55 16 52	56 14 4	57 11 15	58 8 26	59 5 37	60 2 49
35	54 46 24	55 44 4	56 41 43	57 39 22	58 37 2	59 34 41	60 32 21
36	55 13 7	56 11 15	57 9 22	58 7 30	59 5 37	60 3 45	61 1 52
37	55 39 51	56 38 26	57 37 2	58 35 37	59 34 13	60 32 49	61 31 24
38	56 9 34	57 5 37	58 4 41	59 3 45	60 2 49	61 1 52	62 0 56
39	56 33 17	57 32 49	58 32 21	59 31 52	60 31 24	61 30 56	62 30 28
40	57 0 0	58 0 0	59 0 0	60 0 0	61 0 0	62 0 0	63 0 0

Page 26 — Partes 64–70

Partes	64	65	66	67	68	69	70
1	64 0 0	65 0 0	66 0 0	67 0 0	68 0 0	69 0 0	70 0 0
2	45 0 0	45 42 11	46 24 22	47 6 34	47 48 45	48 30 56	49 13 7
3	45 30 0	46 12 39	46 55 19	47 37 58	48 20 37	49 3 17	49 45 56
4	46 0 0	46 43 7	47 26 15	48 9 22	48 52 30	49 35 37	50 18 45
5	46 30 0	47 13 36	47 57 11	48 40 47	49 24 22	50 7 58	50 51 34
6	47 0 0	47 44 4	48 28 7	49 12 11	49 56 15	50 40 19	51 24 22
7	47 30 0	48 14 32	48 59 4	49 43 36	50 28 7	51 12 39	51 57 11

Parte s	64	65	66	67	68	69	70
8	48 0 0	48 45 0	49 30 0	50 15 0	51 0 0	51 45 0	52 30 0
9	48 30 0	49 15 28	50 0 56	50 46 24	51 31 52	52 17 21	53 2 49
10	49 0 0	49 45 56	50 31 52	51 17 49	52 3 45	52 49 41	53 35 37
11	49 30 0	50 16 24	51 2 49	51 49 13	52 35 37	53 22 2	54 8 26
12	50 0 0	50 46 52	51 33 45	52 20 37	53 7 30	53 54 22	54 41 15
13	50 30 0	51 17 21	52 4 41	52 52 2	53 39 22	54 26 43	55 14 4
14	51 0 0	51 47 49	52 35 37	53 23 26	54 11 15	54 59 4	55 46 52
15	51 30 0	52 18 17	53 6 34	53 54 51	54 43 7	55 31 24	56 19 41
16	52 0 0	52 48 45	53 37 30	54 26 15	55 15 0	56 3 45	56 52 30
17	52 30 0	53 19 13	54 8 26	54 57 39	55 46 52	56 36 6	57 25 19
18	53 0 0	53 49 41	54 39 22	55 29 4	56 18 45	57 8 26	57 58 7
19	53 30 0	54 20 9	55 10 19	56 0 28	56 50 37	57 40 47	58 30 56
20	54 0 0	54 50 37	55 41 15	56 31 52	57 22 30	58 13 7	59 3 45
21	54 30 0	55 21 6	56 12 11	57 3 17	57 54 22	58 45 28	59 36 34
22	55 0 0	55 51 34	56 43 7	57 34 41	58 26 15	59 17 49	60 9 22
23	55 30 0	56 22 2	57 14 4	58 6 6	58 58 7	59 50 9	60 42 11
24	56 0 0	56 52 30	57 45 0	58 37 30	59 30 0	60 22 30	61 15 0
25	56 38 0 [?]	57 22 58	58 15 56	59 8 54	60 1 52	60 54 51	61 47 49
26	57 0 0	57 53 26	58 46 52	59 40 19	60 33 45	61 27 11	62 20 37
27	57 30 0	58 23 54	59 17 49	60 11 43	61 5 37	61 59 32	62 53 26
28	58 0 0	58 54 22	59 48 45	60 43 7	61 37 30	62 31 52	63 26 15
29	58 30 0	59 24 51	60 19 41	61 14 32	62 9 22	63 4 13	63 59 4
30	59 0 0	59 55 19	60 50 37	61 45 56	62 41 15	63 36 34	64 31 52
31	59 30 0	60 25 47	61 21 34	62 17 21	63 13 7	64 8 54	65 4 41
32	60 0 0	60 56 15	61 52 30	62 48 45	63 45 0	64 41 15	65 37 30
33	60 30 0	61 26 43	62 23 26	63 20 9	64 16 52	65 13 36	66 10 19
34	61 0 0	61 57 11	62 54 22	63 51 34	64 48 45	65 45 56	66 43 7
35	61 30 0	62 27 39	63 25 19	64 22 58	65 20 37	66 18 17	67 15 56
36	62 0 0	62 58 7	63 56 15	64 54 22	65 52 30	66 50 37	67 48 45
37	62 30 0	63 28 36	64 27 11	65 25 47	66 24 22	67 22 58	68 21 34

Partes	64	65	66	67	68	69	70
38	63 0 0	63 59 4	64 58 7	65 57 11	66 56 15	67 55 19	68 54 22
39	63 30 0	64 29 32	65 29 4	66 28 36	67 28 7	68 27 39	69 27 11
40	64 0 0	65 0 0	66 0 0	67 0 0	68 0 0	69 0 0	70 0 0

Page 27 — Partes 71–77

Partes	71	72	73	74	75	76	77
1	71 0 0	72 0 0	73 0 0	74 0 0	75 0 0	76 0 0	77 0 0
2	49 55 19	50 37 30	51 19 41	52 1 52	52 44 4	53 26 15	54 8 26
3	50 28 36	51 11 15	51 53 54	52 36 34	53 19 13	54 1 52	54 44 32
4	51 1 52	51 45 0	52 28 7	53 11 15	53 54 22	54 37 30	55 20 37
5	51 35 9	52 18 45	53 2 21	53 45 56	54 29 32	55 13 7	55 56 43
6	52 8 26	52 52 30	53 36 34	54 20 37	55 4 41	55 48 45	56 32 49
7	52 41 43	53 26 15	54 10 47	54 55 19	55 39 51	56 24 22	57 8 54
8	53 15 0	54 0 0	54 45 0	55 30 0	56 15 0	57 0 0	57 45 0
9	53 48 17	54 33 45	55 19 13	56 4 41	56 50 9	57 35 37	58 21 6
10	54 21 34	55 7 30	55 53 26	56 39 22	57 25 19	58 11 15	58 57 11
11	54 54 51	55 41 15	56 27 39	57 14 4	58 0 28	58 46 52	59 33 17
12	55 28 7	56 15 0	57 1 52	57 48 45	58 35 37	59 22 30	60 9 22
13	56 1 24	56 48 45	57 36 6	58 23 26	59 10 47	59 58 7	60 45 28
14	56 34 41	57 22 30	58 10 19	58 58 7	59 45 56	60 33 45	61 21 34
15	57 7 58	57 56 15	58 44 32	59 32 49	60 21 6	61 9 22	61 57 39
16	57 41 15	58 30 0	59 18 45	60 7 30	60 56 15	61 45 0	62 33 45
17	58 14 32	59 3 45	59 52 58	60 42 11	61 31 24	62 20 37	63 9 51
18	58 47 49	59 37 30	60 27 11	61 16 52	62 6 34	62 56 15	63 45 56
19	59 21 6	60 11 15	61 1 24	61 51 34	62 41 43	63 31 52	64 22 2
20	59 54 22	60 45 0	61 35 37	62 26 15	63 16 52	64 7 30	64 58 7
21	60 27 39	61 18 45	62 9 51	63 0 56	63 52 2	64 43 7	65 34 13
22	61 0 56	61 52 30	62 44 4	63 35 37	64 27 11	65 18 45	66 10 19
23	61 34 13	62 26 15	63 18 17	64 10 19	65 2 21	65 54 22	66 46 24
24	62 7 30	63 0 0	63 52 30	64 45 0	65 37 30	66 30 0	67 22 30

Partes	71	72	73	74	75	76	77
25	62 40 47	63 33 45	64 26 43	65 19 41	66 12 39	67 5 37	67 58 36
26	63 14 4	64 7 30	65 0 56	65 54 22	66 47 49	67 41 15	68 34 41
27	63 47 21	64 41 15	65 35 9	66 29 4	67 22 58	68 16 52	69 10 47
28	64 20 37	65 15 0	66 9 22	67 3 45	67 58 7	68 52 30	69 46 51
29	64 53 54	65 48 45	66 43 36	67 38 26	68 33 17	69 28 7	70 22 58
30	65 27 11	66 22 30	67 17 49	68 13 7	69 8 26	70 3 45	70 59 4
31	66 0 28	66 56 15	67 52 2	68 47 49	69 43 36	70 39 22	71 35 9
32	66 33 45	67 30 0	68 26 15	69 22 30	70 18 45	71 15 0	72 11 15
33	67 7 2	68 3 45	69 0 28	69 57 11	70 53 54	71 50 37	72 47 21
34	67 40 19	68 37 30	69 34 41	70 31 52	71 29 4	72 26 15	73 23 26
35	68 13 36	69 11 15	70 8 54	71 6 34	72 4 13	73 1 52	73 59 32
36	68 46 52	69 45 0	70 43 7	71 41 15	72 39 22	73 37 30	74 35 37
37	69 20 9	70 18 45	71 17 21	72 15 56	73 14 32	74 13 7	75 11 43
38	69 53 26	70 52 30	71 51 34	72 50 37	73 49 41	74 48 45	75 47 49
39	70 26 43	71 26 15	72 25 47	73 25 19	74 24 51	75 24 22	76 23 54
40	71 0 0	72 0 0	73 0 0	74 0 0	75 0 0	76 0 0	77 0 0

The *Tabula Quadrantis* continues on the following pages; no running prose (CAPUT III) begins within these three pages.

This opening continues the **Table of the Quadrant** [*Tabula Quadrantis*] of *Geometriae Practicae Liber I*, running without a break from the preceding divisions. Three leaves carry the quadrant-columns for divisions (*Partes*) 78 through 98; the table closes at the foot of the third leaf (printed p. 30). Each division-column is subdivided into three sub-columns headed **G** (degrees, *gradus*), **M** (minutes, *minuta*), **S** (seconds, *secunda*); the left-hand argument column is headed **Partes** and runs 1–40 (1–38 on the final leaf). In the transcription below each cell holds the three values in the order **G M S**.

[Margin (printed vertically, left side of every leaf): Numerus siue ordo Quadrantum — "The number or order of the quadrants."]

Table of the Quadrant — divisions 78–84 (printed p. 28)

Parte s	78 (G M S)	79	80	81	82	83	84
1	78 0 0	79 0 0	80 0 0	81 0 0	82 0 0	83 0 0	84 0 0
2	54 50 37	55 32 49	56 15 0	56 57 11	57 39 22	58 21 34	59 3 45
3	55 27 11	56 9 51	56 52 30	57 35 9	58 17 49	59 0 28	59 43 7
4	56 3 45	56 46 52	57 30 0	58 13 7	58 56 15	59 39 22	60 22 30
5	56 40 19	57 23 54	58 7 30	58 51 6	59 34 41	60 18 17	61 1 52
6	57 16 52	58 0 56	58 45 0	59 29 4	60 13 7	60 57 11	61 41 15
7	57 53 26	58 37 58	59 22 30	60 7 2	60 51 34	61 36 6	62 20 37
8	58 30 0	59 15 0	60 0 0	60 45 0	61 30 0	62 15 0	63 0 0
9	59 6 34	59 52 2	60 37 30	61 22 58	62 8 26	62 53 54	63 39 22
10	59 43 7	60 29 4	61 15 0	62 0 56	62 46 52	63 32 49	64 18 45
11	60 19 41	61 6 6	61 52 30	62 38 54	63 25 19	64 11 43	64 58 7
12	60 56 15	61 43 7	62 30 0	63 16 52	64 3 45	64 50 37	65 37 30
13	61 32 49	62 20 9	63 7 30	63 54 51	64 42 11	65 29 32	66 16 52
14	62 9 22	62 57 11	63 45 0	64 32 49	65 20 37	66 8 27	66 56 15
15	62 45 56	63 34 13	64 22 30	65 10 47	65 59 4	66 47 21	67 35 37
16	63 22 30	64 11 15	65 0 0	65 48 45	66 37 30	67 26 15	68 15 0
17	63 59 4	64 48 17	65 37 30	66 26 43	67 15 56	68 5 9	68 54 22
18	64 35 37	65 25 19	66 15 0	67 4 41	67 54 22	68 44 4	69 33 45
19	65 12 11	66 2 21	66 52 30	67 42 39	68 32 49	69 22 58	70 13 7
20	65 48 45	66 39 22	67 30 0	68 20 37	69 11 15	70 1 52	70 52 30
21	66 25 19	67 16 24	68 7 30	68 58 36	69 49 41	70 40 47	71 31 52
22	67 1 52	67 53 26	68 45 0	69 36 34	70 28 7	71 19 41	72 11 15
23	67 38 26	68 30 28	69 22 30	70 14 32	71 6 34	71 58 36	72 50 37
24	68 15 0	69 7 30	70 0 0	70 52 30	71 45 0	72 37 30	73 30 0
25	68 51 34	69 44 32	70 37 30	71 30 28	72 23 26	73 16 24	74 9 22
26	69 28 7	70 21 34	71 15 0	72 8 26	73 1 52	73 55 19	74 48 45
27	70 4 41	70 58 36	71 52 30	72 46 24	73 40 19	74 34 13	75 28 7
28	70 41 15	71 35 37	72 30 0	73 24 22	74 18 45	75 13 7	76 7 30
29	71 17 49	72 12 39	73 7 30	74 2 21	74 57 11	75 52 2	76 46 52

Parte s	78 (G M S)	79	80	81	82	83	84
30	71 54 22	72 49 41	73 45 0	74 40 19	75 35 37	76 30 56	77 26 15
31	72 30 56	73 26 43	74 22 30	75 18 17	76 14 4	77 9 51	78 5 37
32	73 7 30	74 3 45	75 0 0	75 56 15	76 52 30	77 48 45	78 45 0
33	73 44 4	74 40 47	75 37 30	76 34 13	77 30 56	78 27 39	79 24 22
34	74 20 37	75 17 49	76 15 0	77 12 11	78 9 22	79 6 34	80 3 45
35	74 57 11	75 54 51	76 52 30	77 50 9	78 47 49	79 45 28	80 43 7
36	75 33 45	76 31 52	77 30 0	78 28 7	79 26 15	80 24 22	81 22 30
37	76 10 19	77 8 54	78 7 30	79 6 6	80 4 41	81 3 17	82 1 52
38	76 46 52	77 45 56	78 45 0	79 44 4	80 43 7	81 42 11	82 41 15
39	77 23 26	78 22 58	79 22 30	80 22 2	81 21 34	82 21 6	83 20 37
40	78 0 0	79 0 0	80 0 0	81 0 0	82 0 0	83 0 0	84 0 0

Table of the Quadrant — divisions 85–91 (printed p. 29)

Parte s	85 (G M S)	86	87	88	89	90	91
1	85 0 0	86 0 0	87 0 0	88 0 0	89 0 0	90 0 0	0 0 0
2	59 45 56	60 28 7	61 10 19	61 52 30	62 34 41	63 16 52	63 59 4
3	60 25 47	61 8 26	61 51 6	62 33 45	63 16 24	63 59 4	64 41 43
4	61 5 37	61 48 45	62 31 52	63 15 0	63 58 7	64 41 15	65 24 22
5	61 45 28	62 29 4	63 12 39	63 56 15	64 39 51	65 23 26	66 7 2
6	62 25 19	63 9 22	63 53 26	64 37 30	65 21 34	66 5 37	66 49 41
7	63 5 9	63 49 41	64 34 13	65 18 45	66 3 17	66 47 49	67 32 21
8	63 45 0	64 30 0	65 15 0	66 0 0	66 45 0	67 30 0	68 15 0
9	64 24 51	65 10 19	65 55 47	66 41 15	67 26 43	68 12 11	68 57 39
10	65 4 41	65 50 37	66 36 34	67 22 30	68 8 26	68 54 22	69 40 19
11	65 44 32	66 30 56	67 17 21	68 3 45	68 50 9	69 36 34	70 22 58
12	66 24 22	67 11 15	67 58 7	68 45 0	69 31 52	70 18 45	71 5 37
13	67 4 13	67 51 34	68 38 54	69 26 15	70 13 36	71 0 56	71 48 17
14	67 44 4	68 31 52	69 19 41	70 7 30	70 55 19	71 43 7	72 30 56
15	68 23 54	69 12 11	70 0 28	70 48 45	71 37 2	72 25 19	73 13 36
16	69 3 45	69 52 30	70 41 15	71 30 0	72 18 45	73 7 30	73 56 15

Parte s	85 (G M S)	86	87	88	89	90	91
17	69 43 36	70 32 49	71 22 2	72 11 15	73 0 28	73 49 41	74 38 54
18	70 23 26	71 13 7	72 2 49	72 52 30	73 42 11	74 31 52	75 21 34
19	71 3 17	71 53 26	72 43 36	73 33 45	74 23 54	75 14 4	76 4 13
20	71 43 7	72 33 45	73 24 22	74 15 0	75 5 37	75 56 15	76 46 52
21	72 22 58	73 14 4	74 5 9	74 56 15	75 47 21	76 38 26	77 29 4
22	73 2 49	73 54 22	74 45 56	75 37 30	76 29 4	77 20 37	78 12 11
23	73 42 39	74 34 41	75 26 43	76 18 45	77 10 47	78 2 49	78 54 51
24	74 22 30	75 15 0	76 7 30	77 0 0	77 52 30	78 45 0	79 37 39
25	75 2 21	75 55 19	76 48 17	77 41 15	78 34 13	79 27 11	80 20 0
26	75 42 11	76 35 37	77 29 4	78 22 30	79 15 56	80 9 22	81 2 49
27	76 22 2	77 15 56	78 9 51	79 3 45	79 57 39	80 51 34	81 45 28
28	77 1 52	77 56 15	78 50 37	79 45 0	80 39 22	81 33 45	82 28 7
29	77 41 43	78 36 36	79 31 24	80 26 15	81 21 6	82 15 56	83 10 47
30	78 21 34	79 16 52	80 12 11	81 7 30	82 2 49	82 58 7	83 53 27
31	79 1 24	79 57 11	80 52 58	81 48 45	82 44 32	83 40 19	84 36 6
32	79 41 15	80 37 30	81 33 45	82 30 0	83 26 15	84 22 30	85 18 45
33	80 21 6	81 17 49	82 14 32	83 11 15	84 7 58	85 4 41	86 1 24
34	81 0 56	81 58 7	82 55 19	83 52 30	84 49 41	85 46 52	86 44 4
35	81 40 47	82 38 26	83 36 6	84 33 45	85 31 24	86 29 4	87 26 43
36	82 20 37	83 18 45	84 16 52	85 15 0	86 13 7	86 11 15	88 9 22
37	83 0 28	83 59 4	84 57 39	85 56 15	86 54 51	87 53 26	88 52 2
38	83 40 19	84 39 22	85 38 26	86 37 30	87 36 34	88 35 37	89 34 41
39	84 20 9	85 19 41	86 19 13	87 18 45	88 18 17	89 17 49	0 0 0
40	85 0 0	86 0 0	87 0 0	88 0 0	89 0 0	99 0 0	0 0 0

Table of the Quadrant — divisions 92–98 (printed p. 30; table ends here)

Parte s	92 (G M S)	93	94	95	96	97	98
1	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0
2	64 41 15	65 23 26	66 5 37	66 47 49	67 30 0	68 12 11	68 54 22

Parte s	92 (G M S)	93	94	95	96	97	98
3	65 24 22	66 7 2	66 49 41	67 32 21	68 15 0	68 57 39	69 40 19
4	66 7 30	66 50 37	67 33 45	68 16 52	69 0 0	69 43 7	70 26 15
5	66 50 37	67 34 13	68 17 49	69 1 24	69 45 0	70 28 36	71 12 11
6	67 33 45	68 17 49	69 1 52	69 45 56	70 30 0	71 14 4	71 58 7
7	68 16 52	69 1 24	69 45 56	70 30 28	71 15 0	71 59 32	72 44 4
8	69 0 0	69 45 0	70 30 0	71 15 0	72 0 0	72 45 0	73 30 0
9	69 43 7	70 28 36	71 14 4	71 59 32	72 45 0	73 30 28	74 15 56
10	70 26 15	71 12 11	71 58 7	72 44 4	73 30 0	74 15 56	75 1 52
11	71 9 22	71 55 47	72 42 11	73 28 36	74 15 0	75 1 24	75 47 49
12	71 52 30	72 39 22	73 26 15	74 13 7	75 0 0	75 46 52	79 33 45
13	72 35 37	73 22 58	74 10 19	74 57 39	75 45 0	76 32 21	77 19 41
14	73 18 35	74 6 34	74 54 22	75 42 11	76 30 0	77 17 49	78 5 37
15	74 1 52	74 50 9	75 38 26	76 26 43	77 15 0	78 3 17	78 51 34
16	74 45 0	75 33 45	76 22 30	77 11 15	78 0 0	78 48 45	79 37 30
17	75 28 7	76 17 21	77 6 34	77 55 47	78 45 0	79 34 13	80 23 26
18	76 11 15	77 0 56	77 50 37	78 40 19	79 30 0	80 19 41	81 9 22
19	76 54 22	77 44 32	78 34 41	79 24 51	80 15 0	81 5 9	81 55 19
20	77 37 30	78 28 7	79 18 45	80 9 22	81 0 0	81 50 37	82 41 15
21	78 20 37	79 11 43	80 2 49	80 53 54	81 45 0	82 36 6	83 27 11
22	79 3 45	79 55 19	80 46 50	81 38 26	82 30 0	83 21 34	84 13 7
23	79 46 52	80 38 54	81 30 56	82 22 58	83 15 0	84 7 2	84 59 4
24	80 30 0	81 22 30	82 15 0	83 7 30	84 0 0	84 52 30	85 45 0
25	81 13 7	82 6 6	82 59 4	83 52 2	84 45 0	85 37 58	86 30 56
26	81 56 15	82 49 41	83 43 7	84 36 34	85 30 0	86 23 26	87 16 52
27	82 39 22	83 33 17	84 27 11	85 21 6	86 15 0	87 8 54	88 2 49
28	83 22 30	84 16 52	85 11 15	86 5 37	87 0 0	87 54 22	88 48 45
29	84 5 37	85 0 28	85 55 19	86 50 9	87 45 0	88 39 51	89 34 41
30	84 48 45	85 44 4	86 39 22	87 34 41	88 30 0	89 25 19	0 0 0
31	85 31 52	86 27 39	87 23 26	88 19 13	89 15 0	0 0 0	
32	86 15 0	87 11 15	88 7 30	89 3 45	90 0 0	0 0 0	

Partes	92 (G M S)	93	94	95	96	97	98
33	86 58 7	87 54 51	88 51 34	89 48 17	0 0 0	0 0 0	
34	87 41 15	88 38 26	89 35 37	0 0 0	0 0 0		
35	88 24 22	89 22 2	0 0 0	0 0 0			
36	89 7 30	0 0 0					
37	89 50 37	0 0 0					
38	0 0 0	0 0 0					

The table closes with a double rule and the woodcut border at the foot of printed p. 30; no running prose follows on this leaf.

Geometria Practica — Liber I

Table of the Quadrant (*Tabula Quadrantis*), continued: Partes 99–128

The table gives, for each *Pars* (division, column heads 99–128) and each row of the *Numerus siue ordo Quadrantum* [Number or order of the Quadrants — the rotated left-margin row-label], a value in **Gradus** (degrees) · **Minuta** (minutes) · **Secunda** (seconds). Each cell below holds the printed triple G M S; an empty cell is a blank cell in the printed grid.

Printed p. 31 — first block, Partes 99–105

Nº	99	100	101	102	103	104	105
1	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0
2	69 36 34	70 18 45	71 0 56	71 43 7	72 25 19	73 7 30	73 49 41
3	70 22 58	71 5 37	71 48 17	72 30 56	73 13 36	73 56 15	74 37 58
4	71 9 22	71 52 30	72 35 37	73 18 45	74 1 52	74 45 0	75 28 7
5	71 55 47	72 39 22	73 22 58	74 6 34	74 50 9	75 33 45	76 17 21
6	72 42 11	73 26 15	74 10 19	74 54 22	75 38 26	76 22 30	77 6 34

№	99	100	101	102	103	104	105
7	73 28 36	74 13 7	74 57 39	75 42 11	76 26 43	77 11 15	77 55 47
8	74 15 0	75 0 0	75 45 0	76 30 0	77 15 0	78 0 0	78 45 0
9	75 1 24	75 46 52	76 32 21	77 17 49	78 3 17	78 48 45	79 34 13
1 0	75 47 49	76 33 45	77 19 41	78 5 37	78 51 34	79 37 30	80 23 26
1 1	76 34 13	77 20 37	78 7 2	78 53 26	79 39 51	80 26 15	81 12 39
1 2	77 20 37	78 7 30	78 54 22	79 41 15	80 28 7	81 15 0	82 1 52
1 3	78 7 2 3	78 54 22	79 41 43	80 29 4	81 16 24	82 3 45	82 51 6
1 4	78 53 26	79 41 15	80 29 4	81 16 52	82 4 41	82 52 30	83 40 19
1 5	79 39 51	80 28 7	81 16 24	82 4 41	82 52 58	83 41 15	84 20 32
1 6	80 26 15	81 15 0	82 3 45	82 52 30	83 41 15	84 30 0	85 18 45
1 7	81 12 39	82 1 52	82 51 6	83 40 19	84 29 32	85 18 45	86 7 58
1 8	81 59 4	82 48 45	83 37 30	84 28 7	85 17 49	86 7 30	86 57 11
1 9	82 45 28	83 35 37	84 25 47	85 15 56	86 6 6	86 56 15	87 46 24
2 0	83 31 52	84 22 30	85 13 7	86 3 45	86 54 22	87 45 0	88 35 37
2 1	84 18 17	85 9 22	86 0 28	86 51 34	87 42 39	88 33 45	80 24 51 [recte: 89 24 51]
2 2	85 4 41	85 56 15	86 47 49	87 39 22	88 30 56	89 22 30	0 0 0
2 3	85 51 6	86 43 7	87 35 9	88 27 11	89 19 13	0 0 0	0 0 0

№ 99	100	101	102	103	104	105
2 86 37	87 30	88 22	89 15	0 0 0	0 0 0	0 0 0
4 30	0	30	0			
2 87 23	88 16	89 9	0 0 0	0 0 0	0 0 0	0 0 0
5 54	52	51				
2 88 10	89 3	89 57	0 0 0	0 0 0	0 0 0	0 0 0
6 19	45	11				
2 88 56	89 50	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0
7 43	37					
2 89 43	0 0 0	0 0 0				
8 7						
2 0 0 0	0 0 0					
9						
3 0 0 0						
0						

Printed p. 31 — second block, Partes 106–112

№ 106	107	108	109	110	111	112
1 0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0
2 74 31	75 14	75 56	75 38 26 [recte: 76	77 20	78 2	78 45 0
52	4	15	38 26]	37	49	
3 75 21	76 4	76 46	77 29 32	78 12	78 54	79 52 30 [recte: 79
34	13	52		11	51	37 30]
4 76 11	76 54	77 37	78 20 37	79 3	79 46	80 30 0
15	22	30		45	52	
5 77 0	77 44	78 28	79 11 43	79 55	80 38	81 22 30
56	32	7		19	54	
6 77 50	78 34	79 18	80 2 49	80 46	81 30	82 15 0
37	41	45		52	56	
7 78 40	79 24	80 9	80 53 54	81 38	82 22	83 7 30
19	51	22		26	58	
8 79 30	80 15	81 0	81 45 0	82 30	83 15	84 0 0
0	0	0		0	0	
9 80 10	81 5	81 50	82 36 6	83 21	84 7	84 52 30
41	9	37		34	2	

№ 106	107	108	109	110	111	112
1 81 9	81 55	82 41	83 27 11	84 13	84 59	85 45 0
0 22	19	15		7	4	
1 81 59	82 45	83 31	84 18 17	85 4	85 51	86 37 30
1 4	28	52		41	6	
1 82 48	83 35	84 22	85 9 22	85 56	86 43	87 30 0
2 45	37	30		15	7	
1 83 38	84 25	85 13	86 0 28	86 47	87 35	88 22 30
3 26	47	7		49	9	
1 84 28	85 15	86 3	86 51 34	87 39	88 27	89 15 0
4 7	56	45		22	11	
1 85 17	86 6	86 54	87 42 39	88 30	89 19	0 0 0
5 49	6	22		56	13	
1 86 7	86 56	87 45	88 33 45	89 22	0 0 0	0 0 0
6 30	15	0		30		
1 86 57	87 46	88 35	89 24 51	0 0 0	0 0 0	0 0 0
7 11	24	37				
1 87 46	88 36	89 26	0 0 0	0 0 0	0 0 0	0 0 0
8 52	34	15				
1 88 36	89 26	0 0 0	0 0 0			
9 34	43					
2 89 26	0 0 0	0 0 0	0 0 0			
0 15						
2 0 0 0	0 0 0	0 0 0	0 0 0			
1						

Printed p. 32 — third block, Partes 113–119

№ 113	114	115	116	117	118	119
1 0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0
2 79 27 11	80 9 22	80 51	81 33	82 15	82 58 7	83 40
		34	45	56		19
3 80 20 9	81 2 49	81 45	82 28 7	83 10	83 53	84 36 6
		28		47	26	
4 81 13 7	81 56	82 39	83 22	84 5 37	84 48	85 31
	15	22	30		45	52

№ 113	114	115	116	117	118	119
5 83 6 6 [recte: 82 6 6]	82 49 41	83 33 17	84 16 52	85 0 28	85 44 4	86 27 39
6 82 59 4	83 43 7	84 27 11	85 11 15	85 55 19	86 39 22	87 23 26
7 83 52 2	84 36 34	85 21 6	86 5 37	86 50 9	87 34 41	88 19 13
8 84 45 0	85 30 0	86 15 0	87 0 0	87 45 0	88 30 0	89 15 0
9 85 37 58	86 23 26	87 8 54	87 54 22	88 39 51	89 25 19	0 0 0
1 86 30 56 0	87 16 52	88 2 49	88 48 45	89 34 41	0 0 0	0 0 0
1 87 23 54 1	88 10 19	88 56 43	89 43 7	0 0 0	0 0 0	0 0 0
1 88 16 52 2	89 3 45	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0
1 89 9 51 3	89 57 11	0 0 0	0 0 0			
1 0 0 0 4	0 0 0					
1 0 0 6 [recte: 0 0 0] 5						

Printed p. 32 — fourth block, Partes 120–126

№ 120	121	122	123	124	125	126
1 0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0
2 84 22 30	85 4 41	85 46 52	86 29 4	87 11 15	87 53 27	88 35 37
3 85 18 45	86 1 24	86 44 4	87 26 43	88 9 22	88 52 2	89 34 41
4 86 15 0	86 58 7	87 41 15	88 24 22	89 7 30	89 50 37	0 0 0
5 87 11 15	87 54 51	88 38 26	89 22 2	0 0 0	0 0 0	0 0 0
6 88 7 30	88 51 34	89 35 37	0 0 0	0 0 0	0 0 0	0 0 0
7 89 3 45	89 48 17	0 0 0	0 0 0			
8 90 0 0	0 0 0					
9 0 0 0						

Printed p. 32 — fifth block, Partes 127–128

Nº	127	128
1	0 0 0	0 0 0
2	89 17 49	90 0 0
3	0 0 0	0 0 0

Printed p. 33 — prose

10. When there are not several quadrants drawn within the Quadrant, but only a single one is present, laid out with its 90 degrees precisely distributed, we shall discover — by the aid of the compass alone — how many minutes and seconds are contained in any given particle of a degree. This is what we did in the *Astrolabe* [*Astrolabium*] as well, and by the following method. Let there be given, for example, a particle t u in the 20th degree of the upper Quadrant BC. Take it most carefully with the compass; then, beginning from the start of the Quadrant — that is, from the point C — with the same compass-opening lay off 60 equal particles up to the point X, so that the arc CX is sixty times the arc t u. For however many whole degrees are comprehended in this sixtyfold arc, that many minutes will the given particle t u contain. And if, beyond the whole degrees, some particle is left over in the arc CX, take that too sixty times, again beginning from the point C; for however many whole degrees are contained in this sixtyfold arc, that many seconds — beyond the minutes already found — will be contained in the given particle t u. And if anything still remains, the thirds, and so on, will be found in the same way. Thus, since the arc CX, which is sixty times the particle t u, contains 40 whole degrees, the particle t u will comprise 40 minutes, and in addition as many seconds as there are degrees contained in the arc that is sixty times the particle left over beyond the 40 degrees contained in the arc CX — and so on.

[Margin: How, by means of the compass, the minutes and seconds in any given particle of a degree are found.]

I demonstrate this as follows. The arc CX has to the particle t u a certain ratio, since in both cases the ratio is sixtyfold. Therefore, by alternation, as the arc of 60 degrees is to the arc CX, so is 1 degree to the particle t u. And accordingly, however many sixtieth-parts of the arc of 60 degrees — that is, however many degrees — are contained in the arc CX, that many sixtieth-parts of one degree — that is, that many minutes — will exist in the particle t u. Likewise, the arc of 60 minutes has to 1 minute the same ratio that the sixtyfold arc has to the particle

which is left over — beyond the whole degrees, up to X — of this same particle. Therefore, by alternation, it will be: as the arc of 60 minutes is to the sixtyfold arc of the said leftover particle, so is 1 minute to the said leftover particle. Hence, however many sixtieth-parts of the arc of 60 minutes — that is, however many minutes — are contained in the sixtyfold arc of the said particle (now taking the degrees of the Quadrant BC in place of minutes), that many seconds will be included in that leftover particle; and so on in turn, if need be, let the same be understood of thirds, fourths, and so forth. But it is enough, in my judgment, if the minutes are carefully sought out. And indeed, if the remaining particle is greater than half a degree, let one minute be assumed for it and added to the minutes found, since in that case more than 30 seconds are contained in that particle. But if the same particle is less than half a degree, let nothing be added to the minutes found, since in that case fewer than 30 seconds are included in that particle. Finally, if the said particle is equal to half a degree, one is free either to add one minute to the minutes found, or not to add it.

11. But since an error can easily be committed if we try to lay off the said particle of a degree or of a minute sixty times in succession with the compass, you will do better — when the given particle is less than half a degree — if, taking it together with the preceding degree, you first quintuple it; then double this quintupled arc; thirdly triple this doubled arc; and finally, fourthly, double once more this tripled arc. For this last arc will be sixty times the given particle together with the degree taken. Therefore, if 60 degrees be subtracted from it, the remaining degrees will indicate the number of minutes contained in the given particle. But if by estimation you have ascertained that the particle less than half a degree does not exceed 24 minutes — (for if it exceeded 24 minutes, we could not, by the method now to be explained, seek out the minutes, because in carrying the compass around we should exceed the whole Quadrant, as will become evident) — we shall find the minutes of the given particle more conveniently in this way. We shall first quadruple the given particle together with the preceding degree; then we shall double this quadrupled arc, so as to have eight times the arc composed of the particle and one degree; thirdly, we shall again double this eightfold arc, so that there results an arc sixteen times the arc composed of the given particle and one degree; fourthly, we shall again double this arc; and fifthly, at last, we shall again double this doubled arc, so as to obtain an arc that contains the arc [made] of the proposed particle and one degree sixty-four times, whose end-point is to be carefully marked. For if, beginning from that point, there be taken away from the whole arc the quadruple arc of the given particle together with one degree, there will remain the sixtyfold arc of the same particle together with one degree; and from this, finally, if 60 degrees be subtracted, the remaining degrees will indicate

the number of minutes. But if the given particle is greater than half a degree, we shall explore in the same way the minutes comprised in the remaining smaller particle. For these minutes, subtracted from 60, will leave the minutes comprised in that larger particle. You see well, moreover, that when the given particle differs but little from half a degree, it is safer to quintuple it together with one degree, then to double this arc, thirdly to triple this arc, and lastly to double it again. For by this method only 60 degrees have to be subtracted from the last doubled arc, and the whole quadrant is never exhausted, as is plain.

12. Nor, indeed, is it always necessary for that particle to be repeated sixty times; it is enough that, repeated some number of times, it fall precisely upon some degree — which not rarely happens. For then a fraction is set up whose numerator is the number of degrees traversed, and whose denominator is a number of as many units as the number of times the particle has been repeated with the compass. For example, if some particle of a degree, repeated twenty times, falls upon the 6th degree, that particle will comprise $\frac{6}{20}$ of one degree. Hence, if the numerator 6 be multiplied by 60, and the product 360 be divided by the denominator 20, the quotient will show that particle to contain 18 minutes. Likewise, if another particle, repeated seven times, falls upon the third degree, it will comprise $\frac{3}{7}$ of one degree. If, then, the numerator 3 be multiplied by 60, and the resulting product 180 be divided by the denominator 7, there will be found 25 minutes. And because 5 remain over in the division, if these be again multiplied by 60, and the resulting number 300 be divided by the same denominator 7, the quotient will further give $42 \frac{6}{7}$ seconds.

DEMONSTRATION. The demonstration of this practice is as follows. Since, in the former example, 20 degrees stand to 1 degree as the particle repeated twenty times stands to one particle, then by alternation the arc of 20 degrees will stand to the arc containing the particle twenty times — that is, to the arc of 6 degrees — as 1 degree stands to one particle; and by inversion, the 6 degrees stand to the 20 degrees as 1 particle to 1 degree. Since, therefore, 6 degrees are $\frac{6}{20}$ of 20 degrees, one particle also will contain $\frac{6}{20}$ of one degree, which is what was proposed. But in the latter example, since 7 degrees stand to 1 degree as the particle repeated seven times — that is, as 3 degrees stand to 1 particle — then by alternation the arc of 7 degrees will stand to 3 [degrees] as 1 degree to 1 particle; and by inversion, the 3 degrees stand to the 7 degrees as one particle to 1 degree. Since, therefore, 3 degrees are $\frac{3}{7}$ of seven degrees, one particle also will comprise $\frac{3}{7}$ of one degree, which is what was proposed. And the reasoning is the same in the remaining cases.

When the smaller particle, repeated together with one degree, falls precisely upon some degree, there must be subtracted from the degrees traversed as many degrees as the number of times that particle, together with one degree, was repeated. The remaining number will be the numerator of the fraction; and the denominator will be as before. Thus, if that smaller particle, together with 1 degree, repeated seven times, falls—

[Margin: How to cut off from any degree a particle of any number of minutes you please.]

[if that smaller particle, together with 1 degree, repeated seven times] falls in the 10th degree, then 7 repeated degrees must be subtracted. And thus one again obtains $\frac{1}{7}$ of one degree within the given particle.

13. If conversely we wish to take away from any degree a particle of any number of minutes we please, this is how to proceed. On the upper quadrant BC, take with the compass an arc of as many degrees as there are minutes desired; and — to avoid confusion — transfer it, drawn at the distance of the quadrant's semidiameter, into the space of the arcs. For if this arc is divided into 60 equal parts (first, namely, into two; then one of these again into two; third, one of these into three; and finally one of these into five), one sixtieth particle will contain the [same fraction of a degree as the] proposed number of minutes [is of an hour]. For example, if a particle is sought containing 50 minutes, we take on arc FG, at the distance of semidiameter AC, the drawn arc FG equal to arc CZ of 50 degrees, and we cut it into 60 equal parts: first into two at point K; then arc FK again into two at point L; third, arc FL into three at points T and V; and finally arc FV into five. For one of these five particles will comprehend 50 minutes. And accordingly, if it is transferred with the compass onto any degree of quadrant BC, 50 minutes will be cut off from that degree.

I demonstrate this as follows. Since, as an arc of 60 degrees is to 1 degree, so is arc CZ, that is FG, of 50 degrees to the sixtieth part of the same arc FG; then, by alternation, as the arc of 60 degrees is to arc FG of 50 degrees, so 1 degree is to the sixtieth particle of arc FG; and by inversion, as arc FG of 50 degrees is to the arc of 60 degrees, so the sixtieth particle of arc FG is to 1 degree. Since, therefore, arc FG contains $\frac{5}{6}$ of the arc of 60 degrees, the sixtieth particle of arc FG will also contain $\frac{5}{6}$ of one degree, that is, 50 minutes — which is what was proposed.

But this is most inconvenient on small quadrants, especially when 1, 2, or 3 minutes are to be cut off. For who, on a tiny quadrant, could distribute an arc of 1 degree, or 2, or 3, into 60 particles? We shall therefore accomplish what is

proposed more conveniently by this method. From the same upper quadrant BC, let an arc of 61 degrees be transferred into arc XY, drawn at the distance of semidiameter AC, from X to Y. And let this arc XY be divided into 6 equal parts — first into 2, then each half into 3. Then let the first part be divided into 10 equal particles, so that each of these particles is $\frac{1}{60}$ of arc XY. And since one of these particles is $\frac{1}{60}$ of arc XY, just as 1 degree of quadrant BC is to an arc of 60 degrees — since in both cases the proportion is one-sixtieth — then, by alternation, one of those particles is to 1 degree as arc XY is to an arc of 60 degrees. Wherefore, just as arc XY contains the arc of 60 degrees once, and moreover one sixtieth part of it, that is, 1 degree, by construction, so too one of those particles will comprehend 1 degree once, and moreover one sixtieth part of one degree, that is, 1 minute. From which it follows that two particles comprehend 2 degrees and 2 minutes; while three particles will contain 3 degrees and 3 minutes, and so on.

Thus, if one sixtieth particle of arc XY is transferred onto quadrant BC, from point C or from any degree, 1 minute will be had in the 2nd degree, or in any other. And if two particles are transferred, two minutes will be had in the third degree; three minutes in the 4th degree if three particles are transferred; and so for the rest.

By the same reasoning, if someone wants any number of degrees and minutes, the particle of minutes wanted must first be found, and it must be added to the proposed degrees. But if the particle of the minutes found is so small that it can scarcely be taken with the compass, it should be taken together with 1 degree, and this arc — composed of 1 degree and the particle — should be added to the proposed number of degrees less one. For example, if someone wants 89 degrees 59 minutes: the 59 minutes must first be found, which is done by transferring 59 particles of arc XY onto quadrant BC. For the particle in the 60th degree will comprehend 59 minutes, as was said. If, therefore, the arc composed of that particle and 1 degree is added to the arc of 88 degrees, an arc of 89 degrees 59 minutes will be produced. And the same reasoning holds for the rest. Now 59 particles will be taken on arc XY if one foot of the compass is set at point 30 and the other at the smallest part of the first sixth-part of the whole arc XY, toward X. In the same way particles 49, 48, 39, 34, and so on, will be taken, as is clear.

But now, if the minutes are to be taken not on quadrant BC but on a larger or smaller one, they must first be found on quadrant BC by means of arc XY, as we have taught; then, from the proposed quadrant, an arc similar to the arc between C and the end of the particle found must be cut off. This is done if that quadrant is

drawn from center A, and a straight line is led out from A through the end of the particle found on BC, and so on.

[Margin: How to find the fraction of any particle within any part of a straight line divided into equal parts.]

14. What we have said in the preceding items 10, 11, 12, and 13 applies very neatly to straight lines as well. For by the same reasoning we shall determine, if a straight line is cut into any number of equal parts, what fraction any particle of one part contains; and conversely, how any proposed fraction is to be cut off from one part. What an incredible usefulness this brings — both in other geometrical matters and, above all, in the measurements usually carried out by means of the altimetric scale — will be clear in Book 3, where, in treating of the geometric square, the use of the altimetric scale will appear. Let the straight line AB, serving as the foot of the upper quadrant, be cut into 10 equal parts. (Into that many parts I choose to divide both the right shadow and the versed shadow of the altimetric scale — although others divide each into 12, on the grounds that measurements are more easily completed by that division, as will be clear in its place. I should, however, prefer that each side of the shadow be cut into 100 parts, if the size of the instrument conveniently allows.)

Now let it be proposed to find how many tenth-parts the particle DC of the fifth part contains. With the compass, take particle DC and multiply it tenfold from A up to E. And since AE contains six equal parts of AB, particle DC will therefore contain $\frac{6}{10}$ of one tenth-part, that is, $\frac{6}{100}$ of the whole line. So that, if line AB is imagined divided into 100 parts — by assigning ten particles to each of the ten parts — segment AC will comprehend $\frac{6}{100}$. But since, beyond $\frac{6}{10}$, there still remains a particle FE of one tenth, if this in turn is multiplied tenfold from A up to G, eight parts of the whole line AB will be found in AG. Particle FE therefore contains $\frac{8}{10}$ of one tenth. That is: the proposed particle DC, beyond $\frac{6}{10}$ of one part of line AB, contains in addition $\frac{8}{100}$ of one tenth — of one tenth, I mean, from those tenths which we said were comprehended in particle DC — namely $\frac{8}{1000}$ of one part, if each of the tenth-parts of line AB were divided into 100 particles. And so, if line AB is understood as cut into 1000 parts (assigning a hundred particles to each of the ten parts), segment AC will comprise $\frac{68}{1000}$, since AD contains $\frac{60}{1000}$, and DC contains $\frac{8}{100}$ of one part of line AB, or $\frac{8}{1000}$ of the whole line AB — since each hundredth-particle of one tenth-part is $\frac{8}{1000}$. In this way one may proceed to the tenths of one tenth from those tenths contained in particle DC, namely to a fraction denominated from 10000, and so on; but it seems to me sufficient to reach thousandth-parts.

Thus, as you see, by two operations one arrives at thousandth-parts, just as if the whole side AB were cut into 100 parts — which is worth noting. And indeed the two numerators of the tenths, placed in the order in which they were found, give the numerator of the hundredths; and to this numerator there is prefixed, on the left, the number of the whole parts standing before the given particle, and so the numerator of the thousandths is established. As in the above example: since before the given particle DC four parts are found, and $6/10$ and $8/10$ were found, the whole fraction will be $468/1000$. Likewise, if for some particle in the eighth part [two tenth-numerators, printed value not securely legible] were found, the fraction $[?]/1000$ would be established; and if in the third part [two tenth-numerators] were found, the fraction $[?]/1000$ would result; and so for the rest.

But if line AB — that is, each side of the shadow — were divided into 100 parts, the thousandth-parts would be found by a single operation, namely if the given particle were multiplied tenfold within one hundredth, since each hundredth would then be understood as subdivided into ten particles. But because the given particle within some hundredth part is very small, so that it can scarcely be taken with the compass, we shall take it together with one hundredth, or with two, and from the tenfold product cast off 10 or 20 hundredths, so that the remaining hundredths yield the tenths of one hundredth. By the same reasoning, when the given particle within a line of 10 parts is very tiny, the larger remaining particle must be taken ten times. For the remaining part of line AB, from B up to the end of that tenfold particle, will give the number of tenths in the proposed smaller particle, and so on. By this method, if you multiply particle CH tenfold, [ending] at point K, then segment BK will give [numeral: ?] and, in addition, particle DK, as above — since DK is equal to particle FE. The reason for this is that the two particles DC and CH, multiplied tenfold, must together make up the whole of AB, as is established.

For greater convenience, too, in investigating the tenth-parts — that is, in multiplying the proposed particle tenfold — a compass of double opening can be constructed, in which the legs, when produced, cross each other, and one opening is always ten times the other, like the compass with which a line is usually divided into two equal parts. For thus it comes about that, once a particle has been cut off through the smaller opening, the larger opening displays that particle taken ten times, so that one need not carry the compass around so many times — a procedure in which error can easily be committed, but which is more easily avoided with that compass, if it is correctly made. Yet the same can be done without this compass, by means of the instrument of parts which we constructed in the preceding chapter. For if the given particle is taken with the compass, and,

with the greatest care, an equal interval between 10 and 10 is taken from it on the instrument, then the interval between 100 and 100 will be ten times the given particle.

[Margin: How, from a given line, any number of tenth-parts (or other parts) are to be cut off.]

15. Now if, conversely, tenth-parts of one [part] are to be cut off from any part of line AB, the segment containing the number of the tenth-parts must be divided into 10 equal parts. For one tenth of this segment will contain the tenth-parts sought. For instance, if someone wants $7/10$, the segment of line AB comprising 7 parts must be divided into 10 equal particles: each of these will comprehend $7/10$ of one part of line AB. And the same reasoning holds for the rest. But this use will be more conveniently supplied to us by the instrument of parts constructed above. For by its means we shall cut off from any straight line not only any number of tenths, but also hundredths, ninety-ninths, and so on, up to the hundredth part. For if we want $7/10$ of some line, we take on that line the interval equal to that between 100 and 100. For the compass extended between 100 and 100 [printed "inter 100"] will give the $7/10$ sought. Likewise the feet of the compass between 50 and 50 will give $15/60$, that is $\frac{1}{4}$; and again the feet of the compass between 20 and 20 will give $20/60$, that is $\frac{1}{3}$; and so for the rest, as we set out more fully concerning the use of the said instrument of parts in chapter 11.

16. But since I know that different people are pleased by different instruments, I shall add here two other instruments communicated to me by the same distinguished Jacob Curtius [Jakob Kurz von Senftenau], which are extremely useful for geometrical and astronomical purposes. The first is of this kind. Let there be a quadrant BC divided into 90 degrees, whose beginning proceeds from C toward B, and let the sights [pinnacidia] be placed on side AB. On account of the narrowness of space we have marked it off [in the figure] only in groups of five degrees. Within this, from the same center A, let 59 other quadrants be drawn, which are to be divided into degrees in the following way. In the first — which is nearest to quadrant BC divided into 90 degrees — let the arc containing 61 degrees be cut into 60 equal parts; or [equivalently] let the arc of $30\frac{1}{2}$ degrees, namely its half, be cut into 30 equal parts, each of which will contain 1 degree 1 minute, that is, 61 minutes.

[Margin (note e): For the 60 parts, into which the arc of 61 degrees is divided, have the same proportion to the 61 degrees — that is, to 3660 minutes — as 1 part has to 61 minutes, that is, to $[1/60 \text{ of}]$ 3660, according to number. Or, for the same reason: the 30 parts, into which the arc of $30\frac{1}{2}$ degrees is divided, have the same

proportion to $30\frac{1}{2}$ degrees — that is, to 1830 minutes — as 1 part to 61 minutes. Now this division, so that confusion of the points on that first quadrant may be avoided, must be carried out separately on another quadrant equal to it. Then one part containing 61 minutes is transferred, by means of the compass, into the first quadrant aforesaid, beginning being made from semidiameter AC.]

From the end of this part, at the distance of semidiameter AC, let an arc of 60 degrees be properly cut off; when this is divided into 60 [single] degrees, the remaining arc up to semidiameter AB will contain 28 degrees 59 minutes. Accordingly, 28 degrees are to be transferred into it, so that a particle of 59 minutes remains over.

In the second quadrant, let the arc of 62 degrees be cut into 60 parts, or the arc of 31 degrees into 30 equal parts, so that each contains 1 degree 2 minutes.

In the third, let the arc of 63 degrees be divided into 60 parts, or the arc of $31\frac{1}{2}$ degrees into 30 equal parts. In the fourth, let the same be done with the arc of 64 degrees, or 32 degrees, and so on. The remainder is completed as we said for the first quadrant. To make this clearer, let us give the example in the 20th, 40th, and 59th (that is, the last and innermost) quadrant. Thus, in the twentieth quadrant cN, let arc eD be the sixtieth part of the arc of 80 degrees (namely, of as many degrees beyond 60 as the number of the place which the quadrant itself occupies), so that it comprehends 1 degree 20 minutes. [Note d: since, as 60 [parts] of the arc of 80 degrees are to 80 degrees — that is, to 4800 minutes — so 1 part is to 80 minutes.] Or else let arc eD be the thirtieth part of the arc of 40 degrees. For thus again...

Diagram. Jacob Curtius's minute-reading quadrant, the first of the two instruments of item 16. The figure is a quarter-circle (quadrant) with the right angle at the upper-left corner, center **A**. The horizontal top radius runs from **A** rightward to **B**; along it, spaced near **B**, are the marked points **P**, **O**, **N**, **M** (reading from **A** toward **B**: ... **P**, **O**, **N**, **M**, then **B**). The vertical left radius runs downward from **A** to the lower-left corner **C**. Nested inside the outer quadrant **BC** are several concentric quarter-arcs at successively smaller radii — the inner quadrants of the construction, of which only a representative few are engraved. The outer arc (quadrant **BC**) and the inner arcs carry degree graduations engraved in fives, numbered **10**, **20**, **30**, **40**, **50**, **60**. Along the left radius **AC**, the quadrant indices **10**, **20**, **40**, **59** are written from inside toward the corner. Interior point-letters that can be made out include **D**, **E**, **F**, **G**, **H**, **I**, **K**, **L** across the middle band, together with the lower-case series **e** (first particle of the 20th quadrant), **f** (of the 40th), **p** (of the 59th), and the degree-transfer points **a**, **b**, **d** described in items 16–18.

- A (top-left corner): center of the quadrant; origin of both radii
- B (top-right corner): end of the horizontal radius AB (side bearing the sights)
- C (bottom-left corner): end of the vertical radius AC
- P, O, N, M (along AB near B): division/label points
- 10, 20, 30, 40, 50, 60 (along the arcs): degree graduations
- 10, 20, 40, 59 (along AC): quadrant indices
- e [?], D, E, F, N (20th quadrant): first-particle (lowercase e, at the engraving floor) and sub-arc points of item 16
- f [?], G, H, I, O (40th quadrant): sub-arc points of item 17 (lowercase f at the engraving floor)
- p [?], K, L, M (59th quadrant): sub-arc points (lowercase p at the engraving floor)
- a, b, d (near AC): degree-transfer points of item 18 [?] (lowercase transfer-letters, at the engraving floor)

The capital point-letters (A B C P O N M I F L H E K G D) resolve cleanly at native 4×4 (); only the six lowercase transfer-letters (e, f, p, a, b, d) remain at the engraving floor — resolved as far as possible, not a further

[Margin: d. 19. Sept. [?] — marginal annotation]

for thus again it will contain $1\frac{1}{3}$ degrees, that is, 80 minutes. Then, at the distance of semidiameter Ae, let arc DE of 60 degrees be cut off, which is accordingly divided into 60 [single] degrees; but let arc EF contain 28 degrees, and arc FN 40 minutes, so that arc eF comprehends 89 degrees 20 minutes. Thus particle FN is the complement of the minutes which are contained in eD beyond one degree — the complement, I mean, up to 60.

Again, in the fortieth quadrant fO, let arc fG be the sixtieth part of the arc of 100 degrees, or the thirtieth part of the arc of 50 degrees (which is its half), so that it contains 1 degree 40 minutes. But let arc GH contain 60 degrees, and HI 28 degrees, and finally IO 20 minutes — namely, the complement of the 40 minutes which are comprehended in fG beyond one degree.

Lastly, in the fifty-ninth quadrant K, let arc pK be the sixtieth part of the arc of 119 degrees, or the thirtieth part of the arc of $59\frac{1}{2}$ degrees (which is its half), so that it contains 1 degree 59 minutes [printed minutes-glyph uncertain; the complement MP = 1' below requires 59]. But let arc KL be of 60 degrees, and LM of 28 degrees, and finally MP of 1 minute. From these examples you will easily

understand what is to be done in the other quadrants. For in every quadrant the arc to be cut into 60 equal parts is always the one that comprehends as many degrees beyond 60 as the number of the place which the quadrant itself holds — excluding the outermost quadrant BC. For thus its particle near semidiameter AC will contain, beyond one degree, as many minutes as is the number of that quadrant among the quadrants — that is, as many degrees beyond 60 as are contained in the arc divided into 60 equal parts. But the last particle, next to semidiameter AB, will include the remaining minutes out of 60. And you will achieve the same result if you divide into 30 equal parts the half of that arc which we said was to be cut into 60 parts.

When the division of all the quadrants is completed, their numbers are to be written along semidiameter AC, so that the first quadrant on the near side of quadrant BC has the number 1, the second 2, the third 3, the twentieth 20, the fortieth 40, the fifty-ninth 59, and so on.

17. The use of a quadrant constructed in this way is excellent, since by its means we can determine, in the altitudes of the stars, even the minutes. For when the plumb-line falls on some degree of quadrant BC, the altitude will contain as many degrees, without minutes, as are cut off by the line. But when the line does not cut off a [whole] degree on quadrant BC, observe carefully from which quadrant it cuts off a whole part — which will almost always happen, because of the multitude of parts. For the altitude will then contain, beyond the degrees cut off on quadrant BC, as many additional minutes as there are units belonging to that quadrant whose whole part was cut off. For example, when the line falls beyond the 30th degree onto some whole particle of the fortieth quadrant, the altitude will comprehend 30 degrees 40 minutes.

18. But since by this method only the minutes beyond one or more degrees are known, in order that the minutes short of one degree may also be distinguished, let single degrees be transferred toward semidiameter AC from the ends of those first particles of the individual quadrants of which we spoke. For thus, in using the quadrant, the particle near that same semidiameter will contain as many minutes as there are units assigned to the quadrant — namely, as many as the earlier particle contained beyond one degree. For example, if arcs Da, Gb, Kd each contain a single degree, arc ea will comprehend 20 minutes, fb 40 minutes, and pd 59 minutes. Therefore, when the line falls onto some whole particle short of the points D, G, K, the altitude will contain as many minutes as the units assigned to the quadrant whose whole particle the line cuts off. Thus, when the line cuts off no whole [degree] from quadrant BC and falls onto the first whole

particle of the quadrants — for example eN — the arc will be indicated as 20 minutes. But when it cuts off one or more degrees and moreover falls onto some whole particle of the quadrants, an arc of one degree (or more) plus 20 minutes will be presented. And the same must be said of the other quadrants, always taking account of the assigned numbers, for these count the minutes.

It is evident, moreover, that the larger the quadrant ABC is, the more precisely all the quadrants we have described can be laid out.

Diagram. The same nested minute-reading quadrant as on the preceding page (the identical engraving is repeated here because the text of items 17–18 continues around it). Center **A** at the upper-left corner; horizontal top radius **A→B** carrying the near-B points **P, O, N, M**; vertical left radius **A→C**. The concentric inner quadrants are numbered **10, 20, 40, 59** along AC, with degree graduations **10, 20, 30, 40, 50, 60** along the arcs, and the interior point-letters **D, E, F, G, H, I, K, L** and lower-case **e, f, p, a, b, d** as in fig-p35-1.

- A (top-left corner): center of the quadrant
- B (top-right corner): end of horizontal radius (sights)
- C (bottom-left corner): end of vertical radius AC
- P, O, N, M (along AB near B): label points [?]
- 10, 20, 30, 40, 50, 60 (arcs): degree graduations
- 10, 20, 40, 59 (AC): quadrant indices
- D, E, F, G, H, I, K, L; e, f, p, a, b, d (interior): sub-arc and transfer points of items 16–18 [?]

Identical to fig-p35-1; the small interior and peripheral labels are at the resolution floor of this page image — not independently re-examined here, so the p. 35 verdicts have not been transferred (a genuinely repeated woodcut on the facing page would need one look at p. 36 before the capitals' [?] are dropped). The p. 36 labels still need a look at high magnification.

[Margin: d. 19. Sept. [?] — marginal annotation]

[Margin: To cut off from the quadrant an arc of any number of degrees and minutes, and to know how many degrees and minutes are contained in a given arc. Construction of the quadrant by which [this is done].]

19. By means of this quadrant an arc of any number of degrees and minutes can also be taken most conveniently, and conversely one can determine how many degrees and minutes a given arc contains. For if, from the end of the proposed degree on the outermost quadrant BC, a straight line is drawn, and beyond it, in

that other quadrant to which the number of the given minutes is assigned, the first point met with toward B is taken, then the arc of that quadrant between the said point and semidiameter AC will contain the degrees and minutes that are wanted. An arc similar to this must then be cut off from the proposed circle. Conversely, to determine how many degrees and minutes are contained in any given arc, take one similar to it in some quadrant drawn within quadrant BC — or indeed in quadrant BC itself — and through its end we shall draw a straight line from center A, which will almost always pass through some whole particle of one of the quadrants. That particle, then, will give — beyond the degrees cut off by that line — as many minutes as there are units assigned to that quadrant; and those degrees and minutes will be contained in the proposed arc. You see, therefore, that if a quadrant of this kind is constructed with the greatest care and diligence, of such size as the aforesaid divisions require, how splendidly it serves astronomy itself, since by its means we grasp the minutes no less exactly than through the multiplications and divisions of sines — a matter not to be lightly esteemed.

[Margin: Construction of a square by which the altitudes of the stars are observed most precisely.]

20. The other instrument is as follows. Let there be a square ABCD of such size that each of its sides can conveniently hold 256 equal parts. For although one might cut any given side into 200 parts, it is nevertheless more easily divided into 256, since it is first bisected, then each half again bisected, and each of these parts again bisected, and so on up to the eighth division, by which each side will have been divided into 256 parts. To this method of division there is added, besides its ease, the further advantage that more particles are contained in the whole square. For it is certain that the more parts there are, the more perfect the instrument is — so much so that a square which could admit still further subdivisions, so that 512 or 1024 parts might be found on each side, would seem to be perfect in every respect. But let it suffice for now to have distributed each side into 256 equal parts. In the example, on account of the narrowness of space, each side is divided into 32 parts, in such a way that each part is understood to be sub[divided]...

further subdivided into 8 small parts. The order, or numbering, of the parts runs from A toward B and D, and from B and D toward C, as the figure shows. The corresponding points of the divisions are then to be joined by straight lines, all of which will be parallel to the sides of the square. [Margin: Euclid I.33] These, however, are not all to be carried right across to the opposite sides; rather they

leave a vacant square AEFG, which is a quarter of the whole square. For it will thereby come about that, wherever the plumb-thread or line of sight issuing from A happens to fall, it will almost always fall on some whole part of the said parallels — that is, it will pass exactly through one of the right angles formed by the intersections of those parallels — because of the great number of those parts and angles. The square AEFG is left vacant because all the parts contained in it, if the lines were produced, have other parts that are double, or quadruple, or octuple, and so on; so that the plumb-thread, falling on any one of them, necessarily falls also on some other that is its double, quadruple, octuple, and the like. It is therefore unnecessary to mark out those small parts within the square AEFG.

For example: suppose that on side AD a portion is taken containing 40 parts, and that the plumb-thread AH, on the parallel drawn from the 40th part of side AD to the 40th part of side BC, cuts off 32 parts. The same thread will cut off 128 parts, at H, on the parallel drawn from the 160th part of side AD to the 160th part of side BC — which is quadruple 32, just as 160 parts are quadruple 40. In the same way, the same thread AH, on the parallel drawn from the 240th part...

Diagram. A large graduated square — Clavius's shadow-square altimeter (geometric square). The four corners are A (top left), D (top right), B (bottom left), and C (bottom right); the top side is AD, the left side AB, the bottom side BC, and the right side CD. Above corner A a small rectangular sighting-vane box is drawn, containing two dots (the pinnule sights of the alidade). From A a sheaf of straight fiducial lines fans out downward across the grid.

The top-left quarter of the square — the square AEFG, with A at its top-left corner, G on the top edge, E on the left edge, and F at its interior corner — is left blank (vacant). The remaining L-shaped region is filled with a fine 16-by-16 grid. The sides are graduated with tick-numbers at multiples of 16.

- A (top-left corner): the sight point, with sighting-vane box and two pinnule dots above
- D (top-right corner)
- B (bottom-left corner)
- C (bottom-right corner)
- G (top edge, at the 128 mark): top-right corner of the vacant square
- E (left edge, at the 128 mark): bottom-left corner of the vacant square
- F (interior, at 128/128): fourth corner of the vacant square
- M (top edge, at 160)

- O (top edge, near 200)
- R (top edge, at 208)
- N (top edge, at 240)
- H (interior, at 160 across / 128 down): where the fiducial line from A meets the grid
- K (interior, near 200 across / 160 down): on the same fiducial line
- I (interior, at 240 across / 192 down): on the same fiducial line
- L (interior, near 120 across / 208 down)
- Graduations along the top edge (AD), left to right from G: 128, 144, 160, 176, 192, 208, 224, 240, 256
- Graduations down the right edge (CD), from D to C: 16, 32, 48, 64, 80, 96, 112, 128, 144, 160, 176, 192, 208, 224, 240, 256
- Graduations down the left edge (AB), from E to B: 128, 144, 160, 176, 192, 208, 224, 240, 256
- Graduations along the bottom edge (BC), from B to C: 16, 32, 48, 64, 80, 96, 112, 128, 144, 160, 176, 192, 208, 224, 240, 256
- Vertical legend along the right side (CD): VMBRA RECTA [the right/direct shadow scale]
- Horizontal legend along the bottom (BC): VMBRA VERSA [the versed/turned shadow scale]

of side AD to the 240th part of side BC, will take off 192 parts at I — which is sextuple 32, just as the 240 parts on side AD are sextuple the 40 parts on the same side. Finally, the same thread, on the parallel drawn from the 200th part of side AD to the 200th part of side BC, will cut off 160 parts, which are quintuple 32, just as the 200 parts of side AD are quintuple the 40 [parts] of the same side. And the same ratio holds for the rest. The reason for this is that the parallel drawn from the 40th part of side AD to the thread AH forms with the thread a triangle similar to the triangle that the same thread forms with the parallel dropped, say, from the 160th part of side AD, and so on. The square AEFG can therefore be cut away, to make the instrument less heavy.

This construction of the square is, as you see, not only very easy but also of immense use — as much in astronomical matters as in geometrical ones — as will be plainly established from what follows. [Margin: The use of the constructed square in astronomical matters.] We shall begin with the astronomical use. Suppose, then, that someone about to make an observation wishes to take the altitude of some star: let him carefully note onto which of the parallel lines the plumb-thread falls exactly. From this the altitude of the star — that is, the size of

the angle that the thread makes with side AB — is to be drawn out, in the following way. Suppose the thread has fallen on point H, on the parallel drawn from the 160th part of side AD (namely from M) to the 160th part of side BC, and on the parallel drawn from the 128th part of side AB to the 128th part of side CD. Since, then, in the right-angled rectilinear triangle AMH the two sides about the right angle are known — namely AM, of 160 parts, and MH, of 128 parts, which numbers you see written along the parallels passing through H — the angle HAM, the complement of the altitude, becomes known by Practice 5 of our [treatise on] rectilinear triangles, in this way. Let it be arranged that: as the side AM of 160, about the right angle M, is to the whole sine 10000000 — for the calculation in this business comes out more accurate if two figures are not dropped from the sines and tangents, as we usually do when we work with the sines alone — so is the side MH of 128, opposite the sought angle HAM, to a fourth quantity; and there will be found the tangent 8000000 of the sought angle HAM, to which in the table of tangents corresponds the arc 38 degrees 40 minutes, for the complement of the star's altitude. You will find the same tangent from the sides AO of 200 parts and OK of 160 parts; likewise from the sides AN of...

240 parts and NI of 192 parts, as is plain: so that, when the thread falls exactly on several parts of several parallels — as here happens at the points H, K, I — any one of them may be taken, with no difference at all. The same aforesaid angle HAM can also be elicited from the sines alone, by the same Practice 5 of our rectilinear triangles, though not so easily. For if the squares of the sides AM and MH — that is, the squares of the parts 160 and 128 — are added together, and the square root of the sum 41984 is extracted, this root will give the side AH, opposite the right angle M, as 204 368/409 parts, nearly. Hence, let it be arranged that: as the side AH of 204 368/409, opposite the right angle, is to the whole sine 10000000, so is the side MH of 128, opposite the sought angle, to a fourth quantity; and there will be found the sine of the sought angle HAM, 62470 [?], to which in the table of sines there corresponds, again, 38 degrees 40 minutes.

Let us next suppose that the plumb-thread has fallen on the 208th part of the parallel drawn from the 120th part of side AD to the 120th part of side BC, namely on point L. You will find the number 120 only on side BC, since it is smaller than 128 and therefore is not found on side AD; likewise a number smaller than 128 is to be sought on side CD alone, as is clear from the figure. If, then, it is again arranged that, as the side of 120 about the right angle is to the whole sine 10000000, so the side of 208 opposite the sought angle LAD is to a fourth quantity, the tangent of the sought angle, 17333333, will be produced, to which in the table of tangents corresponds the arc 60 degrees 1 minute. Or, if it is arranged that, as

the side AP of 208 about the right angle P is to the whole sine 10000000, so the side LP of 120, opposite the angle LAP, is to a fourth quantity — (for when the thread falls on side BC, we shall take the triangle applied to side AB, just as, when the same thread fell on side CD, the triangle applied to side AD was taken; but then the angle LAP thus found is not to be subtracted from the right angle to leave the altitude angle, but itself indicates the altitude) — there will be produced the tangent 5769231 of the angle LAP, to which...

Diagram. The same shadow-square altimeter is repeated here, identical in structure to the figure on the preceding page. The graduated square has corners A (top left, with the sighting-vane box and two pinnule dots), D (top right), B (bottom left), and C (bottom right); the fiducial lines again fan out from A; the top-left quarter AEFG is left blank, and the rest is filled with the 16-by-16 grid.

- A (top-left corner, with sighting vane)
- D (top-right corner)
- B (bottom-left corner)
- C (bottom-right corner)
- G (top edge, at 128), E (left edge, at 128), F (interior, at 128/128): corners of the vacant square
- M (top edge, at 160); O (top edge, near 200); R (top edge, at 208); N (top edge, at 240)
- H, K, I (interior points on the fiducial line from A, at 160/128, ~200/160, 240/192)
- L (interior, near 120/208)
- Top edge (AD) graduated 128 ... 256; right (CD), left (AB), and bottom (BC) edges graduated as on the previous figure
- Right-side legend VMBRA RECTA; bottom legend VMBRA VERSA

this tangent corresponds, in the table of tangents, the arc 29 degrees 59 minutes. So great, then, is the sought altitude angle LAP — namely the complement of 60 degrees 1 minute, as before. When the thread falls on point C, no calculation is needed: the star's altitude then amounts to exactly 45 degrees, because of the half-right angle CAB.

By this method the computation can be made. [Margin: In what way the table for the aforesaid square is to be constructed.] (I should gladly have supplied here — had time allowed — a table containing the degrees and minutes corresponding to the individual small parts of the parallels, just as the preceding table was computed by us for the individual small parts of the quadrants described within

the principal quadrant.) It will be enough to compute the table for the small parts of the parallels drawn from sides AB and BC, comprised within the trapezium CBEF. For the very same arcs will correspond to the small parts of the trapezium CDGF, as is evident, since exactly the same triangles are formed on side AD as are constructed on side AB. But then, when it has been found that the thread has fallen on side CD, the complement of the angle found in the table must be taken for the sought altitude — just as was seen in the angle HAM, whose complement HAB was the angle of the sought altitude. That the triangles applied to side AD are exactly equal to the triangles applied to side AB is clear in the triangles AQR and ALP. In the first of these the plumb-thread passes through the 120th part of the parallel drawn from the 208th part of side AD to the 208th part of side BC, namely from point R; in the second the plumb-thread is drawn through the 120th part of the parallel drawn from the 208th part of side AB — that is, from point P — to the 208th part of side CD. Since the sides AR, RQ of 208 and 120 parts are equal to the sides AP, PL of 208 and 120 parts, and contain right angles, [Margin: Euclid I.4] the triangles themselves will be equal, and the angles QAR and LAP equal. Therefore, when the thread falls on the 120th part of the parallel drawn from the 208th part of side AD, if in the constructed table the angle 29 degrees 59 minutes has been drawn out through the 120th part of parallel 208, its complement will give 60 degrees 1 minute, the angle QAB of the sought altitude. And the same reasoning holds for the rest. Furthermore, in the table (to advise the student of this too), let the number of the parallels — beginning, in our example, at 128 and running up to 256 — be set at the head of the table; the number of the parts of each parallel, from 1 up to 256, in the left-hand side of the same table; and finally, in the shared cells, let the degrees and minutes be written.

Now, turning to the altimetric scale — to explain the use of the said square in geometrical matters as well — it cannot be told how excellent a use this same square, as constructed thus far, has. [Margin: The use of the same square in geometrical matters.] Here side CD will be reckoned as the right shadow [umbra recta] and side BC as the versed shadow [umbra versa], and every operation can be carried out most conveniently either by the right shadow alone — which is very agreeable, and proper to this instrument — or by the versed shadow alone, without reducing the one to the other, and with no trouble of fractions at all. Only the gnomon here is not always the same, but varies from case to case. For example: when the plumb-thread AQ cuts off at Q the 120th part of the right shadow RQ, the gnomon AR will be of 208 parts. But if the same point Q is taken to belong to the 208th part of the versed shadow, the gnomon will be of 120 parts. And since — as we shall demonstrate in [the treatment of] the geometric square — as the shadow is to the gnomon, so is the distance of the eye from the object

whose height we are measuring to that height itself; and likewise, as the gnomon is to the versed shadow, so is the same distance to the height; then, if it be arranged that, as the right shadow 120 is to the gnomon 208 (or, correspondingly, as the gnomon is to the versed shadow), so is the said distance to a fourth quantity, the height of the object to be measured will be found. So too, when the thread falls on point L, the right shadow will be of 208 parts and the gnomon of 120 parts; the versed shadow, however, will be of 120 parts and the gnomon of 208 parts; and so with the rest.

PROBLEMATA VARIA

Various Problems of Rectilinear Triangles

CAPUT III — Chapter III

In the third place, we promised various problems of rectilinear triangles, so that both their sides and their angles might be drawn out from certain given and known [quantities] readily and expeditiously, whenever the matter should require it. This we shall carry out in this third chapter. In the earlier part we shall treat right-angled triangles; in the later, oblique-angled ones. In the margin, moreover, we have noted the propositions of our treatise on rectilinear triangles, published in our *Theodosius*, in which these problems are demonstrated, so that students may understand where to seek their demonstrations, whenever they wish.

PROBLEMS OF RECTILINEAR RIGHT-ANGLED TRIANGLES

I. THE PROPORTIONS OF THE SIDES

To make [them] known from all the given angles of any triangle. [Margin: Rectilinear Triangles, Prop. 1]

To each of the sides let there be assigned the sines of the opposite angles. For the sides have the same proportions as are found among the sines of the angles opposite the sides.

II. A SIDE

To make [it] known from the base and either of the acute angles (and hence also the other). [Margin: Rectilinear Triangles, Prop. 2]

- As the whole sine

- is to the base,
- so is the sine of the angle opposite the required side
- to the required side, in parts of the base.

III. A SIDE

To ascertain [it] from the base and one other side. [Margin: Rectilinear Triangles, Prop. 3]

- As the base
- is to the whole sine,
- so is the given side
- to the sine of the angle opposite the given side.

Then, taking the complement of the found angle for the remaining angle:

- As the whole sine
- is to the base,
- so is the sine of the angle that is opposite the required side
- to the required side, in parts of the base and of the other side.

IV. A SIDE

To elicit [it] from one other side and one acute angle (and hence also the other). [Margin: Rectilinear Triangles, Prop. 2]

- As the whole sine
- is to the given side,
- so is the tangent of the angle opposite the required side
- to the required side, in parts of the given side.

Or

- As the sine of the angle opposite the given side
- is to the given side,
- so is the sine of the other angle
- to the required side, in parts of the given side. [Margin: Rectilinear Triangles, Prop. 2]

V. THE BASE

To investigate [it] from one side and one acute angle (and hence also the other). [Margin: Rectilinear Triangles, Prop. 2]

- As the whole sine
- is to the given side,
- so is the secant of the angle adjacent to the given side
- to the base, in parts of the given side.

Or

- As the sine of the angle opposite the given side
- is to the whole sine,
- so is the given side
- to the base, in parts of the given side. [Margin: Rectilinear Triangles, Prop. 2]

VI. THE BASE

To investigate it from both sides, together with the acute angles.

[Margin: 3. Triang. rectil. (Problem 3, rectilinear triangles)] As either given side is to the whole sine, so the other given side is to the tangent of the angle opposite this other side.

Then, taking the complement of the found angle as the remaining angle:

As the whole sine is to either given side, so the secant of the angle adjacent to the accepted side is to the base, expressed in parts of the given side.

VII. THE ANGLE

To find it from the base and one side.

[Margin: 3. Triang. rectil.] As the base is to the whole sine, so the given side is to the sine of the angle opposite the given side.

The complement of the found angle will give the other angle.

VIII. THE ANGLE

To make it known from both sides.

[Margin: 3. Triang. rectil.] As either side is to the whole sine, so the other given side is to the tangent of the angle opposite this other side.

The complement of the found angle will give the other angle.

Problems of Oblique-Angled Rectilinear Triangles

IX. THE SEGMENTS OF A SIDE made by the perpendicular

To determine them from the three given sides.

[Margin: 9. Triang. rectil.] As the side onto which the perpendicular falls is to the sum of the other two sides, so the difference of those same two sides is to a certain fourth number.

Now if the fourth number found is smaller than the side onto which the perpendicular falls, it must be subtracted from that side. For half of the remaining number gives the smaller segment, which, subtracted from the whole side, leaves the larger segment.

But if the fourth number found is larger than the side onto which the perpendicular falls, then this side must be subtracted from that number. For half of the remainder gives the smaller segment — the one lying outside, that is, between the perpendicular and the obtuse angle. And when this is added to the same side, it makes up the other, larger segment, lying between the perpendicular and the acute angle.

X. TWO SIDES

To find them from the third side and any two angles — and therefore from all three, since the third is the complement of the others to a semicircle, that is, to 180 degrees.

[Margin: 10. Triang. rectil.] 1. As the sine of the angle opposite the given side is to the given side, so the sine of either of the remaining angles is to the side opposite that angle.

Again:

[Margin: 2. Triang. rectil.] As the sine of the angle opposite the given side is to the given side, so the sine of the third angle is to the side opposite that third angle.

2. In an isosceles triangle only one side needs to be found, since one is given together with the angles. And in an equilateral triangle, if one side is given, the others too — being equal to it — are given.

XI. A SIDE

To learn it from the remaining two sides and any two angles — and therefore from all three, since the third is the complement of the others to a semicircle, that is, to 180 degrees.

[Margin: 10. Triang. rectil.] As the sine of the angle opposite either given side is to the given opposite side, so the sine of the angle opposite the sought side is to the sought side.

XII. A SIDE

To gather it from two sides and the angle contained by them.

[Margin: Problema 15[?]. triang. sphaeric. (Problem 15, spherical triangles — numeral ambiguous, 15 or 16)] 1. As the whole sine is to the secant of the complement of the arc which is owed — as a sine — to half the sum of the given sides reduced to sines, so the difference between that half and either of the given sides (reduced to sines) is to a certain fourth number.

The given sides will be reduced to sines if each is multiplied by 10, or 100, or 1000, and so on, so that the number of the larger side has as many figures as are contained in the larger sines of the table of sines — namely 5, if the whole sine is set at 100000, or 7, if the whole sine is put at 10000000.

Then:

As the whole sine is to the tangent of half the arc that remains when the given angle is subtracted from the semicircle, so the fourth number found is to the tangent of the difference between half of that same arc and either of the angles not given.

But this tangent may also be found in the following way, which seems preferable to the former. [Margin: *Alio modo qui priori praeferendus videtur* — "Another method, which seems preferable to the former."]

2. As half the sum of the two given sides is to the tangent of half the arc that remains when the given angle is subtracted from the semicircle, so the difference between half the sum of the two given sides and either of the sides is to the tangent of the difference between half the aforesaid arc and either of the angles not given.

The arc of this tangent thus found, added to half of that same arc (which arc is the sum of the two angles not given — namely the complement of the given angle to a semicircle, that is, to 180 degrees), gives the larger of the angles not given, the one

that is opposite the larger given side. But subtracted from that same half, it makes the remaining, smaller angle not given, the one opposite the smaller given side.

After this:

[Margin: 1. triang. rectil.] As the sine of either found angle is to its opposite side, so the given angle is to the opposite side which is sought.

And so, before the third side is found, the remaining two angles must first be sought out; and these seem more conveniently investigated by the later method set out under Number 2 than by that former procedure explained under Number 1.

3. If two sides are equal, then the remaining two angles will be equal.

[Margin: a — 5 primi (Elements I.5)] Therefore half the arc that remains when the given angle is subtracted from the semicircle will give each of them, and so on.

XIII. A SIDE

To seek it out from two sides and an angle opposite one of them (provided the kind of the angle opposite the other given side is settled, when the given angle is acute).

[Margin: 13. triang. rectil.] 1. As the given side opposite the given angle is to the sine of the given angle, so the other given side is to the sine of the angle opposite this other side.

This sine found will give the angle opposite the other given side, if it is acute — and it will always be acute when the given angle is obtuse. But if it is obtuse, the arc of the sine found, taken away from the semicircle, will make that angle in the remainder. For this reason, when the given angle is acute, the kind of this other angle must be given, so that we may know whether it is acute or obtuse. Then the sum of these angles, subtracted from the semicircle, will leave the third angle, which is opposite the sought side. Therefore:

[Margin: 1. triang. rectil.] As the sine of the given angle is to the given side opposite it, so the sine of the third angle found — opposite the sought side — is to the sought side.

2. If the two given sides are equal, then the angle opposite the other given side will be equal to the given angle. [Margin: b — 5 primi (Elements I.5)]

XIV. TWO ANGLES

To find them from two sides and the angle contained by them.

The two angles will be found from the given data, as was said in the first part of Problem 12: namely, if one seeks the tangent of the difference between half the arc that remains when the given angle is subtracted from the semicircle, and either of the angles that are sought, and so on. This tangent was indeed found by two methods in the first part of Problem 12, in which a side is proposed to be investigated from two sides and the angle contained by them. In order to do that, two other angles had first to be found — the very ones that are sought in this Problem 14.

XV. TWO ANGLES

To find out from two sides and an angle opposite one of them (provided the kind of the angle opposite the other given side is settled, when the given angle is acute).

Here too the earlier procedure of Problem 13 must be applied — in which a side, standing opposite, is to be sought from the same data; and in order to do that, the remaining two angles had first to be found — the very ones proposed for investigation in this Problem 15.

XVI. ALL THREE ANGLES

To investigate them from all three sides.

[Margin: 11. triang. rectil.] 1. Having drawn the perpendicular to the greatest side from the opposite angle (2) — so that the perpendicular always falls inside the triangle — let the two segments of the greatest side, made by the perpendicular, be found by Problem 9. Then:

[Margin: 1. triangul. rectil.] As the smallest side is to the whole sine, so the smaller segment of the greatest side is to the sine of the complement of the angle opposite the middle side.

Again:

[Margin: 1. triangul. rectil.] As the middle side is to the whole sine, so the greater segment of the greatest side is to the sine of the complement of the angle opposite the smallest side.

When the two angles at the greatest side — those opposite the middle side and the smallest side — have been found, then, if their sum is taken from the semicircle, the remainder becomes the third angle, opposite the greatest side.

[Margin: a — Schol. 26 lib. 1 Eucl. (Scholium 26, Elements Book I)] 2. In an isosceles triangle, with the perpendicular drawn to the base — which it will bisect —

As either of the equal sides is to the whole sine, so half the base is to the sine of the complement of one of the equal angles at the base.

The sum of the two equal angles thus found, subtracted from the semicircle, will make the remaining third angle.

3. In an equilateral triangle the angles are given even if the sides are not, since each of them, being 60 degrees, comprises a third part of two right angles — or two-thirds of a single right angle.

XVII. THE PERPENDICULAR FALLING ONTO ANY SIDE from the opposite angle

To make it known from all three sides.

[Margin: Theorema]

By Problem 9, let the segments of the side made by the perpendicular be sought. Then let the difference between either segment and the adjacent side be multiplied by the sum of that same segment and the adjacent side. For the square root of the product will show the sought perpendicular.

For in triangle ABC, let AB be 10, AC 17, and BC 21, and let the perpendicular AD be the object of inquiry. By Problem 9 the segment BD is found to be 6, and CD 15. The difference between BD and AB is 4; multiplied by 16, the sum of the lines BD and AB, it makes 64, whose square root, 8, gives the perpendicular AD. Since this has not been demonstrated in our treatise on rectilinear triangles, I demonstrate it in the following theorem.

Diagram. A right-triangle construction for the theorem. Triangle ABC has its apex A at the top, B at the lower left, and C at the lower right; the perpendicular AD drops from A to the base BC, meeting it at D. From center B, a semicircle EFD is drawn through D, cutting side BA at E (just below A) and its extension below B at F. The labelled magnitudes are inscribed along the lines.

- A (top apex): vertex; foot of the perpendicular AD.
- E (on BA, just below A): where the semicircle cuts side BA.
- B (lower left): vertex and center of the semicircle.
- F (lower left, below B): where the semicircle meets the line through B.

- D (bottom, on BC): foot of the perpendicular AD.
- C (lower right): vertex.
- 10 (along AB, near B): $AB = 10$.
- 6 (segment near B, between B and D): $BD = 6$.
- 8 (along AD): the perpendicular $AD = 8$.
- 17 (along AC, upper right): $AC = 17$.
- 15 (along DC, lower right): $CD = 15$.
- 21 (below D, along the base): $BC = 21$.

In a right triangle, the rectangle contained by the difference of the base and either of the sides about the right angle, and by the sum of the base and that same side, is equal to the square of the other side about the right angle.

[Margin: a — Coroll. 16 tertij (corollary to Elements III.16); b — 36 tertij (Elements III.36)]

For in the right triangle ABD, whose angle D is right, if from B, through D, a semicircle EFD is described, then AE will be the difference between the base AB and the side BD, while AF will be the sum of the base AB and that same side BD — since BD, BE, and BF are equal straight lines. I say, then, that the rectangle contained by AE and AF is equal to the square of the side AD. For the straight line AD, since it is perpendicular to the semidiameter BD, will touch the semicircle at D [Elements III, corollary to 16]. Therefore the rectangle contained by AE and AF will be equal to the square of the tangent AD [Elements III.36] — which was to be demonstrated.

END OF THE FIRST BOOK.

GEOMETRIAE PRACTICAE LIBER II.

Explaining the measurement of straight lines by means of the astronomical quadrant.

[The subtitle is printed *Linearum Rectarum per Quadrantem Astronomicum Dimensionem explicans* — "Astronomicum" is a compositor's spelling of *Astronomicum*.]

By "straight lines" we mean the distances between places — their intervals or lengths — as well as the heights of towers, buildings, trees, and mountains, and finally the depths of wells, valleys, and ditches. Various people use various

instruments to measure these. Some prefer the altimeter scale engraved on the back of an astrolabe or planisphere; others the astronomical radius [radius Astronomicus] of Gemma Frisius [Gemma Frisius], or the one called the *Latin radius*, so named because it was devised by Signor Latino Orsini [Latino Orsini], a Roman nobleman; others the cross-staff [baculus Iacob, "Jacob's staff"]; others the astronomical ring [annulus Astronomicus] or holometer [Holometrum]; and still others favor yet other instruments. For my part, I prefer above all the astronomical quadrant [Quadrans Astronomicus] divided into 90 degrees, together with the geometric square [Quadratum Geometricum] — both the fixed kind and the pendent kind — whose two sides are divided into a certain number of equal parts. In this second book I will teach how the measurement of straight lines is carried out by means of the astronomical quadrant. To follow this method, one must have ready at hand tables of sines, tangents, and secants, such as those set out in our Theodosius [Theodosius] and in the books of other authors, from which they are to be sought. I have judged it superfluous to repeat them here, lest the work swell to a larger size.

To measure a distance in a plane — whether it is accessible or not — by means of two stations made in the same plane with the quadrant, when at the far end of the distance some perpendicular height stands erect, even if the lowest end of that height cannot be seen; and from this to derive the height itself as well.

PROBLEMA I.

Let AB be the distance, or length, to be investigated, in the plane CB, and let some perpendicular height BG stand erect at the far end B, even though the end B [i.e., the foot of the height] cannot be seen. Let the observer's height be DA, from eye to feet. So that this height does not change but always stays the same, you will do well to take, in place of your own height, a staff equal to it, and set your eye at its top. Now, imagining a line EF drawn through D parallel to CB, let the first station be at D, and the second at E, the more remote point; and let the straight line DE — which is called the *difference of stations* [Margin: Difference of stations (labels DE in the figure)] — be known in some common unit of measure. Next, direct the side HK of the quadrant, on which the sights [pinnacidia] are set, toward the summit G, so that the eye placed at D sees the summit G through the openings of both sights, with the plumb-line HI hanging freely. Then, following carefully what was taught in Book 1, Chapter 2, Nos. 7 and 10, note in degrees and minutes the angle GDF, which the arc IL will show on the quadrant — namely the complement of the arc IK. For since the thread of the plumb HI is perpendicular to DF, the angle GDF will be the complement of the angle DHI, and therefore

equal to the angle IHL, which is likewise the complement of that same angle DHI. This angle GDF we shall call the *angle of observation*. [Margin: Angle of observation.] In the same way, at the second station observe the angle GEF by the line of sight running from the eye through the sights of the quadrant, directed to the summit G.

Diagram. The surveying figure for Problema I. At the right stands a tall tower, drawn with a crenellated top and a graduated (ruled) shaft. Two nearly horizontal lines run leftward from the tower: the upper is the eye-level line E–D–F, the lower is the ground line C–A–B, close together at the left. Two quadrant instruments (each a quarter-circle with sighting edge and hanging plumb-line) are drawn on the left, one for each station. Sight-lines run from the two stations up to the summit G.

- G (top of tower): the summit (*fastigium*) sighted at.
- Q (on the tower shaft, below G): the point where the near-station sight-line construction meets the tower.
- F (on the tower shaft, at eye-level): foot of the perpendicular from G to the eye-line EF.
- B (foot of tower, on the ground line): base of the height BG.
- E (far-left end of the eye-line): eye at the more remote (second) station.
- C (far-left end of the ground line, below E): the observer's feet at the remote station.
- D (on the eye-line, to the right of E): eye at the nearer (first) station.
- A (on the ground line, below D): the observer's feet at the nearer station.
- H (apex of each of the two quadrants): vertex bearing the sights.
- K, L (on the arc of each quadrant): the arc points marking the observed angle and its complement.
- I (at the foot of each quadrant): the plumb-bob / plumb-line point.
- M (on the eye-line, E-side): foot of the perpendicular MH (whole-sine segment EM).
- N (on the eye-line, D-side): foot of the perpendicular NO (whole-sine segment DN).
- O, P (near the D-side quadrant): O the top of perpendicular NO; P the point where DQ, drawn parallel to EG, cuts NO.

Now, taking EM and DN equal, let perpendiculars MH and NO be raised at their ends. (In the figure, MH coincides with the plumb-thread, which makes no difference.) If, then, EM and DN are taken as whole sines [sinus toti], MH and NO

will be the tangents of the angles of observation E and D. Draw also DQ parallel to EG, cutting NO at P; then the angle NDP will be equal to the angle E. [Margin: a — Elements I.29.] Now the two angles N and D of triangle NDP are equal to the two angles M and E of triangle MEH (for the right angle N equals the right angle M), and the sides DN and EM adjacent to them are equal; therefore the sides NP and MH will be equal. [Margin: b — Elements I.26.] Consequently OP will be the difference between the tangents of the angles of observation. And since OP is to PN as GQ is to QF, [Margin: c — Scholium 4, Book 6.] and GQ is to QF as ED is to DF, [Margin: d — Elements VI.2.] it follows that: as OP, the difference of the tangents corresponding to the angles of observation, is to PN — that is, to HM, the tangent of the more remote station — so ED, the difference of stations, is to DF, the distance sought. Therefore, if you set up this proportion:

[Margin: Finding the distance by tangents.]

As OP, the difference between the tangents of the angles of observation, is to PN (or HM), the smaller tangent: so ED, the difference of stations, known in some common measure, is to another; that is, to DF —

there will result the distance DF sought — that is, AB — in the same measure as the difference of stations. And if the difference of stations ED is added to it, the distance EF will also be found — that is, CB, [measured] from the more remote station.

ALITER.

2. If GF is taken as the whole sine, then DF will be the tangent of the angle DGF — the complement of the angle of observation GDF. This angle DGF is indicated by the arc IK of the quadrant, [measured] from the plumb toward the eye, since the angle DHI is equal to the angle DGF, the exterior [angle equal] to the interior. [Margin: e — Elements I.29.] In the same way, EF will be the tangent of the angle EGF, the complement of the other angle of observation, GEF. And ED will be the difference between these two tangents. If, then, you set up this proportion:

[Margin: Another finding of the distance by tangents.]

As ED, the difference between the tangents of the angles that are the complements of the angles of observation, is to DF, the tangent of the complement of the angle of observation GDF at the nearer station — that is, to the smaller tangent: so ED, the difference of stations, known in some common measure, is to another; that is, to DF —

the smaller distance sought, DF (or AB), will be produced, in the same measure as the difference of stations; and if the difference of stations ED is added to it, the larger distance EF will also become known.

3. Again, if you set up this proportion: [Margin: f — Elements VI.4.]

As DN, the whole sine, is to NO, the tangent of the angle GDF at the nearer station: so DF, the smaller distance found, is to another; that is, to FG —

[Margin: Finding the height by tangents.]

the height FG will be found, in the measure of the smaller distance DF; and if the observer's height FB is added to it, the whole height BG will be known. Likewise, if you set up this proportion: [Margin: g — Elements VI.4.]

[Margin: Another finding of the height by tangents.]

As EM, the whole sine, is to MH, the tangent of the angle GEF at the more remote station: so EF, the larger distance found, is to another; that is, to FG —

the same height FG will be found, in the measure of the larger distance EF; and if the observer's height FB is added to it, the whole height BG will become known.

ALITER.

4. If you prefer to accomplish the same thing by sines alone, the operation will be somewhat longer. For first both hypotenuses, EG and DG, must be found in some known measure, in this way. Since the angle GDF is equal to the two angles E and EGD [taken together], [Margin: h — Elements I.32.] if the angle E observed at the more remote station is subtracted from the angle GDF found at the nearer station, the remaining angle EGD will be the difference between the two angles of observation. And if you set up this proportion: [Margin: i — [prop.] 10 on rectilinear triangles.]

As the sine of the angle EGD, the difference between the two angles of observation, is to ED, the known difference of stations: so the sine of the angle DEG — or the sine of the angle EDG, the supplement of the angle GDF to two right angles — is to DG, or to EG —

there will be produced both DG and EG, known in units of the difference of stations. Therefore, if you set up this proportion: [Margin: Finding the hypotenuses. — a: [prop.] 10 on rectilinear triangles.]

As the whole sine of the right angle F, is to the hypotenuse DG just found: so the sine of the angle DGF, the complement of the angle of observation at the nearer station, is to DF —

[Margin: Finding the distance by sines alone. — b: [prop.] 10 on rectilinear triangles.]

the distance DF will become known; and if the difference of stations ED is added to it, the longer distance EF will also be known, which you will likewise find if you set up this proportion:

As the whole sine of the right angle F, is to the hypotenuse EG lately found: so the sine of the angle EGF, the complement of the angle of observation at the more remote station, is to EF.

The height FG, moreover, will be found by sines alone if you set up this proportion: [Margin: c — [prop.] 10 on rectilinear triangles. Finding the height by sines alone.]

As the whole sine of the right angle F, is to the hypotenuse EG, or to the hypotenuse DG: so the sine of the angle E, the smaller observed [angle], or the sine of the angle GDF, the larger observed [angle], is to FG.

5. With a fixed quadrant the measurement is done in the same way; only the angles of observation at the two stations are found by this method. Set the quadrant on some flat base parallel to the horizon, so that it is level with the horizon — which you will achieve with the help of a plumb-line. With the quadrant so placed, raise the sighting-rule [dioptra] until you see the summit G through the openings of the sights. In this way, at the nearer station D the angle of observation will be GDF, and at the more remote one it will be GEF. Each of these is measured by the arc of the quadrant lying between the line EF and the fiducial line of the sighting-rule. Everything else is done just as with the pendent quadrant, as the figure shows. Only, to the height FG once found, one must add the height of the base on which the quadrant rests — not the observer's height — unless that height equals the height of the base.

Diagram. The same Problema I surveying figure, repeated at the top of the page for the reader's convenience. The tower stands at the right (summit G, point Q on the shaft, F at eye-level, foot B); the eye-line runs E–D–F and the ground line C–A–B; the two quadrant instruments appear at the left with apexes H, arc points K, L, plumb-points I, and the perpendicular feet M and N (with O and P at the D-side quadrant). Labels as in fig-p43-1.

Diagram. The same Problema I surveying figure, repeated again on this page. Tower at right with summit G and a marked point on the shaft [labelled with a small script letter, apparently the point Q of the preceding figures — the glyph reads like a lowercase "e" [?]]; sight-point O and P on the D-side; the two quadrants at left with apexes H, plumb-points I, and the base line E–M–D–N–F. Labels otherwise as in fig-p43-1.

6. All of this can now be worked out **without numbers** — that is, without any multiplication or division of numbers. I have no doubt this will be most welcome to those who are little practiced in numbers and in the use of sines, tangents, and secants. It is done as follows. On a sheet of paper, or on a flat surface, construct a figure exactly like the one we imagined above as constructed from the eye up to the height. This is done thus. With the instrument of parts [instrumentum partium] made in Chapter 1 of the previous book — or, if the instrument is not at hand, from some straight line divided into very many equal particles — take with the compass as many equal particles as there are spans or feet contained between the two stations, and transfer that compass-interval onto any straight line, from E to D. Then, at D and at E, with all diligence, construct the angles of observation GDF and GEF of the first and second stations. Mark carefully the point G at which the lines DG and EG meet. (So that no error is made at this point on account of the intersection, I will teach in the following lemma how it may be found most exactly; for unless this is done, the measurements will not come out accurately.) From G let fall the perpendicular GF. Once the figure is so constructed, if the lines DF, EF, FG, DG, and EG are carried by compass onto the side of 100 parts of the aforesaid instrument — or onto the said straight line divided into very many equal parts — it will at once appear how many particles are enclosed between the compass points. And each of those lines will comprise as many spans or feet as there are particles found in their intervals.

Diagram. A plain right triangle for the numberless construction of No. 6. The apex G is at the upper left; the base runs left-to-right along the horizontal through the points E, D, and F, with the hypotenuse descending from G to F.

- G (upper left): the summit-point where DG and EG meet.
- E (left, on the base): the remote station.
- D (middle, on the base): the nearer station.
- F (right, on the base): foot of the perpendicular from G.

[Margin: How this Problem 1 may be solved without numbers.]

ALITER.

7. We shall attain the same thing by a certain general method that fits every kind of measurement — namely as follows. Draw any angle BAC. Then, taking for example the rule of three of No. 1 of this proposition, take AD equal to the first quantity OP (that is, to the difference of the tangents of the observed angles); or, if it is too small, take a multiple of it — I have taken twice as much. Likewise take DB equal to the second quantity PN (that is, to the tangent of the smaller angle), or the same multiple, adjoining it to AD. After this, from the instrument of parts, take as many particles AE as there are spans or feet contained in ED, the difference of stations. Then, having drawn the straight line DE, draw BC parallel to it. For the interval EC will include as many parts as the distance DF will comprise spans or feet, [Margin: d — Elements VI.4.] since the four quantities AD, DB, AE, and EC are proportional.

Diagram. The general rule-of-three construction of No. 7. From the vertex A at the lower left, an upper ray rises to the right through points D and B; a lower ray (a graduated line divided into equal ticks) runs to the right through points E and C. The segments BC and DE are drawn parallel, cutting the two rays.

- A (lower left): the vertex of the angle BAC.
- D (on the upper ray, nearer A): endpoint of AD (= first quantity OP).
- B (on the upper ray, farther out): endpoint of DB (= second quantity PN).
- E (on the lower, graduated ray, nearer A): endpoint of AE.
- C (on the lower ray, farther out): endpoint of EC; BC is drawn parallel to DE.

[Margin: By what method this Problem 1 may otherwise be solved without numbers.]

You will proceed in the same way in other examples, observing this: that when the sine of some angle turns up in the rule of three, you take the sine from the table of sines with its last five figures dropped, so that the whole sine is 100. For example, in the last example, No. 4, the line AD would have to be taken equal to 100 particles of the instrument of parts — namely, the whole sine. But DB [is taken] equal to the hypotenuse EG, or DG, in the figure of No. 6. And AE, if the angle E is $30^{\circ} 15'$, equal to about $50 \frac{4}{10}$ particles, because so great [is the sine of that angle ...]

When the whole scale of 100 parts is too long, one may take as the whole sine any interval between two marks of 100, provided that the sines are afterward

reckoned with respect to this whole sine — as we explained in Book I, Chapter 1, Number 12.

You may also, on occasion, change the order: namely, by placing the second quantity DB on the line AC, and the third quantity AE on the line DB, according as you find it convenient for drawing the parallels DE, BC.

LEMMA.

Given two straight lines inclined to one another, to find the point at which they meet.

[Margin: a — Elements III.27; b — Elements I.28; c — Elements V.15; d — Elements VI.4, and by alternation; e — Scholium 4 of Elements VI. — And: how, in the figure of Number 6, the oblique intersection at the meeting-point G is avoided.]

What is here proposed was demonstrated by us in several ways in Lemma 13 of Book I of our *Astrolabium*. But because it is of remarkable use in finding the meeting-point of two straight lines, we shall demonstrate the same thing here in a somewhat different way.

Diagram. A construction on two straight lines inclined to one another and meeting obliquely at a point on the right. The upper line runs left to right; the lower line rises from the lower left toward the same meeting-point, and a set of short parallel secants crosses between them, with tick-marks indicating equal segments laid off along the lower rays.

- A (upper line, far left): endpoint of the upper straight line
- E (upper line, next to A): a chosen point on the upper line (arc-centre)
- F (upper line, left of centre): a chosen point on the upper line (arc-centre)
- H (upper line region, near F): arc endpoint [?]
- G (upper line, right of centre): a chosen point on the upper line (arc-centre)
- M (upper line, toward the right): arc endpoint
- B (upper line, far right): the meeting-point of the two lines
- C (lower line, far left): endpoint of the lower straight line
- O (lower region, left): intersection of a secant with CD
- I (lower region, left of centre): arc endpoint
- P (lower region, centre-left): intersection of a secant with CD
- L (lower region, centre): arc endpoint
- K (lower region, centre): a marked point on the construction [?]
- W (lower region, centre-right): a marked point [?]

- T (lower region, right): a laid-off point on an extended ray
- Q (upper right, near B): intersection of a secant with CD [?]
- D (upper right, near B, dotted): endpoint of the lower line / secant intersection [?]
- S (right of centre, below): a laid-off point on an extended ray
- R (lower centre): a laid-off point on an extended ray

Let there then be two straight lines AB and CD, cutting each other obliquely at the meeting-point B. From any points E, F, G whatever, taken anywhere on one of the lines, describe arcs HI, KL, MN toward the other line at one and the same interval; from these, cut off arcs less than a quadrant at the points I, L, N, and through these points let secants issue from the centres, cutting CD at O, P, Q. Then, on EI extended in line with EO, mark off as many equal parts as seem sufficient, up to R — so that a line drawn from R toward the meeting-point does not cut the lines at too oblique an angle. Likewise, on FL and GN extended, lay off the same number of parts, equal to FP and PQ respectively, up to S and T. I say that the line RS, and also RT, and also ST all fall on the point B, the meeting-point, so that the points R, S, T, B lie on one straight line.

For since the angles E, F, G are equal [*Margin a: Elements III.27*], the lines ER, FS, GT will be parallel [*Margin b: Elements I.28*]. Therefore, since ER, FS, GT have the same proportions as EO, FP, GQ [*Margin c: Elements V.15*] — that is, the same proportions as EB, FB, GB [*Margin d: Elements VI.4, and by alternation*] — the lines RS, RT, ST will fall on the point B [*Margin e: Scholium 4 of Elements VI*], which was to be proved.

If, then, this lemma is applied, the meeting-point G in the earlier figure of Number 6 will be determined quite precisely, and consequently the measures of the lines — which we taught how to find without numbers — will not be far from the truth.

8. But we shall conveniently avoid that oblique intersection at the meeting-point G if we construct the figure in this other way. In the figure of Number 6, form the right angle EFG, and at whatever point G we wish the meeting to be, set the angles FGD, FGE equal to the complements of the observation angles. For in this way DE will correspond to the difference of the stations, and so on. Accordingly, if DE is imagined divided into as many equal parts as there were palms or feet taken in the difference of the stations, we shall know — by what we taught at the end of Number 1, Chapter 1, Book I — how many of those parts are contained in the distances DF, EF, in the

height FG, and in the hypotenuses GD, GE. And in this way the meeting-point G cannot be doubtful or uncertain, since we chose it before everything else. *[Margin: how, in the figure of Number 6, the oblique intersection at the meeting-point G is avoided.]*

SCHOLIVM.

So that we may explain once, even for beginners, what we mean by our manner of speaking when we say — for example, at Number 1 of this problem — "Let it be done thus."

[Margin: finding the height by a single station, when the distance is known.]

As OP, the difference between the tangents of the observation angles, is to PN (or HM), the smaller tangent, so ED, the difference of the stations, is to DF.

You must understand that in this way we reduce the work to the terms of the rule of three. Therefore, following the tenor of the rule, multiply the number placed in the third position by the one that occupies the second position — that is, in the proposed example, multiply the difference of the stations by the smaller tangent — and divide the resulting number by the one placed in the first position, that is, by the difference of the tangents. (The exception is when the first number is the whole sine; for then no division is performed, but five figures are struck off from the product, or seven, according as the whole sine is set at 100000 or 10000000.) In the quotient there will be produced the fourth number sought, namely the distance DF. And the same procedure holds for the rest.

COROLLARIVM. I.

Therefore, when the distance from the surveyor's position up to the unknown height is known, the height will be found from a single station. If it is done thus *[Margin f: on rectilinear triangles, 4]:*

As the whole sine is to the tangent of the observation angle, so the known distance is to the height.

For this was demonstrated in Number 3 of this Problem 1, both by the observation angle GDF together with the distance DF, and by the observation angle GEF together with the distance EF. For by either method the height GF was found.

COROLLARIUM. II.

[Margin: how the height of a mountain is investigated.]

It is also evident that, if G is the summit of some mountain, we can measure its height by this Problem 1 through two stations D, E made on the level ground — provided that we first find the line DF (or EF), from the surveyor's eye up to the perpendicular GF, which falls from the summit G onto the plane of the horizon, even though we do not see its lower end F.

An inaccessible height — when the distance from the surveyor's position to the base of the height is unknown — can be measured by means of the quadrant, using two stations made on the level. And from this the distance itself can also be derived, even if its far end is not seen.

PROBLEMA II.

1. Let the height AB be the one to be found — whether it is a tower, or a mountain, or something else — even though the lowest point B of its perpendicular is not visible, as happens with every mountain. Let the plane to which the height is perpendicular be CB, and the surveyor's stature DE. Now, imagining the line GF drawn through E parallel to CB, let the first, nearer, station be made at D, and the second, farther, at G, so that the difference of the stations is GE. Then, by the visual rays EA, GA directed to the vertex A, carefully observe the angles AEF, AGF — either with the hanging quadrant, as we taught in Number 1 of the preceding problem, or with the fixed quadrant, as we prescribed in Number 5 of the same problem. For the said angles are always observed in the same way when the top of a height is viewed from a lower position.

Diagram. A right triangle standing on a horizontal base, with the vertex A at the top and the right angle at the lower right. A horizontal line through the middle carries the points H and I, meeting the sloping side; a small rectangular cell labelled M, L, E, K sits on the base line, and the base is lettered C, D, B from left to right.

- A (apex, top): the vertex sighted (top of the height)
- H (mid-height, left): endpoint of the mid horizontal line HI
- I (mid-height, right): endpoint of the mid horizontal line HI
- F (right, below I): the foot of the perpendicular / right-angle corner region
- G (left, at mid-height level): the farther station

- M (base cell, left): corner of the small rectangle
- L (base cell): division point
- E (base cell): the nearer eye-point
- K (base cell, right): division point
- C (base, far left): endpoint of the base line
- D (base, centre): the nearer station foot
- B (base, right): the foot of the height's perpendicular

Also imagine HI drawn parallel to GF, so that the perpendiculars HL, IK, dropped within the parallelogram LI, are equal — standing for the whole sines [*Margin a: Elements I.34*]; their tangents are EK and GL, belonging to the angles I and H, which are the complements of the observation angles E and G. And since the angle GAF is greater than the angle EAF — the angle GHL being equal to the former, and the angle EIK to the latter [*Margin b: Elements I.29*] — the angle GHL will also be greater than the angle EIK, and therefore the tangent GL greater than the tangent EK, since the whole sines HL, IK are equal.

Let LM be cut off equal to EK, so that GM is the difference of the tangents GL, EK. And since GL is to LH as GF is to FA [*Margin c: Elements VI.4*], by alternation GL will be to GF as LH (or IK) is to FA [*Margin d: Elements VI.4, and by alternation*]. But as IK is to FA, so also is EK to EF. Therefore, as the whole GL is to the whole GF, so is EK — that is, LM, cut off from GL — to EF, cut off from GF; and consequently, as GM, the remainder of GL, is to GE, the remainder of GF [*Margin e: Elements V.19*], so is the whole GL to the whole GF — that is, so is LH, the whole sine, to FA [*Margin f: Elements VI.4, and by alternation*]. Therefore, if it is done thus:

As GM, the difference of the tangents GL, EK of the complements of the observation angles, is to GE, the difference of the stations, so LH, the whole sine, is to FA.

[Margin: finding the height by tangents.]

the height FA will be found, expressed in parts of the difference of the stations; and if the surveyor's stature FB is added to it, the whole height AB becomes known.

2. Likewise, if it is done thus:

As GM, the difference of the tangents of the complements of the observation angles, is to GE, the difference of the stations, so GL, the

tangent of the complement of the observation angle at the farther station, is to GF.

[Margin: finding the distance.]

GF, the greater distance, will become known, since it was demonstrated just above that GM is to GE as GL is to GF. And if GE, the difference of the stations, is subtracted from GF once found, then EF, the lesser distance, will remain known.

ALITER.

3. If the measurement is set up by sines alone, one must first find one or the other of the hypotenuses GA, EA — or both — in this way. Since the angle AEF is equal to the two angles G and GAE together *[Margin g: Elements I.32]*, if the angle G at the farther station is subtracted from the angle AEF at the nearer station, the remainder will be the angle GAE, the difference between the two observation angles G and AEF.

Therefore, if it is done thus *[Margin h: on rectilinear triangles, 10]*:

As the sine of angle GAE, the difference between the two observation angles, is to GE, the difference of the stations, so the sine of angle AGE is to AE; or so the sine of angle GEA, the complement of angle AEF to two right angles, is to GA,

[Margin: finding the hypotenuses.]

both AE and AG will become known, expressed in parts of the difference of the stations. If, then, it is done thus *[Margin i: on rectilinear triangles, 10]*:

As the whole sine is to the line EA — or as the sine of the right angle F is to the line GA, just found — so the sine of angle AEF, at the nearer station, is to AF; or so the sine of angle AGF, at the farther station, is to AF,

[Margin: finding the height by sines alone.]

the height AF will be produced; and if the surveyor's stature FB is added, the whole height AB will become known.

Both distances EF and GF will likewise be found by sines alone, if it is done thus *[Margin a: on rectilinear triangles, 10]*:

As the whole sine is to the hypotenuse EA — or to the hypotenuse GA — so the sine of angle EAF, the complement of the angle at the nearer

station, is to EF; or so the sine of angle GAF, the complement of the angle at the farther station, [is to GF].

[Margin: finding the distance by sines alone.]

4. The same things will be found without any multiplication and division of numbers, if with all diligence we construct a figure similar to the one we conceived should be drawn from the surveyor's eye up to the height — as we said in Numbers 6 and 8 of the preceding Problem 1. This is clear if, in that figure of Number 6, the height is understood to be FG, and the distances FD, FE, and so on. You will accomplish the same by what was set out in Number 7 of the same Problem 1.

[Margin: how the problem is solved without numbers.]

Diagram. A right triangle with the vertex G at the top and the base running left to right through the points E, D, F, the right angle standing at F where the perpendicular GF meets the base.

- G (apex, top): the sighted vertex (top of the height in the scaled figure)
- E (base, left): the nearer station point
- D (base, centre): the farther station point
- F (base, right): the foot of the perpendicular GF

COROLLARIVM. I.

Therefore, if the height is known, the distance will be found from a single station, if it is done thus *[Margin: finding the distance from a single station, when the height is known. — b: on rectilinear triangles, 4]:*

As the whole sine AF is to the tangent FE (the complement of the angle at the nearer station) — or to the tangent FG (the complement of the angle at the farther station) — so the known height AF is to the distance FE (or to the distance FG).

Likewise, if the distance is known, the height will be found, also from a single station, if it is done thus:

As the whole sine GF is to AF, the tangent of the angle G observed at the farther station, so GF, the known distance, is to AF, the height.

Or:

As the whole sine EF is to AF, the tangent of the angle E observed at the nearer station, so EF, the known distance, is to AF, the height.

COROLLARIUM. II.

It is also evident that, if the point A is the summit of some mountain, the perpendicular line AF — the one dropped from the summit onto the plane of the horizon — can be found in the same way, provided two stations are made at E, G.

From the top of a mountain or tower — on whose summit two stations can be made, from which some mark is visible on the horizon — we can measure the height of the mountain or tower itself. And from this we can also work out the distance from the base of the tower (or from the perpendicular of the mountain) to that mark.

PROBLEMA III.

1. Let there be a mountain or tower GHNM, whose height is the perpendicular EF, or else the side of the tower GM or HN. Let two stations be chosen at G, H; and let a mark L, on the horizon or plane on which the tower or mountain stands, be visible from the surveyor's eye placed both at I and at K. Let the side of the hanging quadrant, on which the sights [pinnacidia] are, be directed toward L, and let the angle K be carefully noted — the angle which the arc of the quadrant marks off between the side bearing the sights and the freely hanging plumb-line. Observe the angle I in the same way. With the fixed quadrant, each angle will be observed if one of its sides is set square to the horizon — which happens only when the plumb-line hanging from the centre lies along that side — and the alidade [dioptra] is raised until the mark L can be seen through the openings of the sights. For the arc between the said side and the line of sight [linea fiduciæ] on the alidade will measure the observation angle. Understand, too, that the plumb-line, in either position of the hanging quadrant — or the side of the fixed quadrant set square to the horizon — is produced down to the base of the mountain or tower at M, N. Then, taking the equal lines KO, IP for...

with IP taken as the whole sines, let PQ and OR be drawn perpendicular to IN and KM [Margin: a — Euclid I.28]. These will be parallel to the plane of the horizon — that is, to the straight line NL — and will be the tangents of the observation-angles at I and K [Margin: b — Euclid I.34]. Now HN and GM will be equal to the height EF.

In the triangles INL and KML the right angles at N and M are equal, while ILN is smaller than KLM (a part smaller than the whole); therefore the remaining angle NIL is greater than the remaining angle MKL, and hence the tangent PQ is greater than the tangent OR. So cut off PT equal to OR itself. Since, as IP, the whole sine, is to PQ, the tangent of the greater observation-angle, so is IN to NL [Margin: c — Euclid VI.4], it follows by alternation that as IP is to IN, so is PQ to NL. By the same reasoning, as KO is to KM — that is, as IP is to IN, since KO and KM are equal to IP and IN [Margin: d — Euclid I.34] — so is OR (that is, PT) to ML. Therefore, as PQ is to NL, so is PT to ML. And since, as the whole PQ is to the whole NL, so is the removed part PT to the removed part ML [Margin: e — Euclid V.19], the remainder TQ (the difference of the tangents) will also be to the remainder NM (the difference of the stations) — because NM and HG are equal [Margin: f — Euclid I.34] — as the whole PQ is to the whole NL, that is, as IP is to IN [Margin: g — Euclid VI.4]. Therefore, if the following is set up:

As TQ, the difference between the tangents of the observation-angles, is to NM (or HG), the difference of the stations, so the whole sine IP is to IN,

the straight line IN will be found; and if IH, the surveyor's height, is subtracted from it, HN (or EF), the height sought, is left. [Margin: Finding the height.] And if again this is set up:

As TQ, the difference of the tangents, is to NM (or GH), the difference of the stations, so PQ, the greater tangent, is to NL, the distance,

the distance NL becomes known; and if NF (or HE), which it will be possible to measure, is subtracted from it, FL is left known — the distance from the perpendicular (foot) of the mountain. Or, if NM, the difference of the stations, is subtracted, the distance ML is left known — from the tower up to L. [Margin: Finding the distance.] Likewise, if this is set up [Margin: h — Euclid VI.4]:

As IP, the whole sine, is to PQ, the greater tangent, so IN, found just before, is to NL, the distance,

the distance NL is again found, and so on.

2. But that TQ, the difference of the tangents, is to NM (or HG), the difference of the stations, as PQ, the greater tangent, is to the distance NL — as was shown just above — we shall demonstrate more easily if the straight line IT is drawn, which, produced, cuts NL at S. For in the triangles IPT and KOR the two sides IP, PT are equal to the two sides KO, OR, and they contain equal angles, since they are right angles [Margin: i — Euclid I.4]. Therefore

the angles T and R are equal [Margin: k — Euclid I.28], and hence KL and IS are parallel [Margin: l — Euclid I.34]. And since IK, when drawn, is equal both to SL itself and to the difference of the stations HG (or NM) [Margin: m — scholium to Euclid VI.4], it is clearly evident that TQ, the difference of the tangents, is to SL — which is equal to the difference of the stations IK (or HG) — as PQ, the greater tangent, is to the distance NL.

ALITER [Another method]

3. We shall accomplish the problem by sines alone in this way. First we shall find the hypotenuses IL, KL, thus [Margin: n — Euclid I.32]. Since the angle LKV is equal to the two angles LIK and ILK, if LIK — the angle of the complement of the greater observation-angle LIN — is subtracted from LKV — the angle of the complement of the smaller observation-angle LKM — the remainder is the angle ILK [Margin: o — On Rectilinear Triangles, 10]. So if this is set up:

As the sine of angle ILK — the difference between the two complement-angles of the observations — is to IK, the difference of the stations, so the sine of angle IKL — composed of the right angle and the smaller observation-angle — is to IL; or the sine of angle LIK — the complement of the greater observation-angle — is to KL,

both IL and KL will become known, in the measure of the difference of the stations. [Margin: p — On Rectilinear Triangles, 10.] Therefore, if now this is set up:

As the whole sine of the right angle N is to the hypotenuse IL just found, so the sine of angle ILN — the complement of the greater observation-angle — is to IN;

Or: as the whole sine of the right angle M is to the hypotenuse KL just found, so the sine of angle KLM — the complement of the smaller observation-angle — is to KM,

Diagram. A two-station height-and-distance construction for a mountain (drawn as a hatched crag) rising above a horizontal base line. Along the top edge run the points I (left corner), K (center), and V (right corner), each fitted with a quadrant arc for the observations at I and K. Two vertical lines descend from I and from K to the base. A middle horizontal band carries, left to right, the points P, T, a point I read as Q [?], O, and R. At the top of the crag lie the points H (left), E, and G (center). Along the base, left to right: N, F, M, S, and L (far right), where the sight-lines from I and K, produced, meet at L.

- I (top left corner): observation station, with quadrant arc
- K (top center): second observation station, with quadrant arc
- V (top right corner): terminus of the top horizontal
- P (left, mid-height): endpoint of tangent line PQ
- T (center, mid-height): intersection point (TQ = difference of tangents)
- Q [?] (center, mid-height): point on the middle band
- O (center-right, mid-height): endpoint of tangent line OR
- R (right, mid-height): terminus of tangent line OR
- H (crag top, left): foot of height HN
- E, G (crag top, center): E left, G right; GM equal to height EF
- N (base, far left): foot of the left vertical
- F (base): point on the base
- M (base, center): foot of the right vertical
- S (base): point where IT produced cuts NL
- L (base, far right): meeting-point of the two sight-lines

the height IN (or KM) becomes known; and if the surveyor's height is subtracted from it, the height sought is left — HN (or GM), that is EF. [Margin: Finding the height by sines alone.]

Each of the distances NL and ML is found [Margin: a — On Rectilinear Triangles, 10] if this is set up:

As the whole sine of the right angle N (or M) is to the hypotenuse found, IL (or KL), so the sine of the greater observed angle NIL is to NL, the distance; or the sine of the smaller observed angle MKL is to ML, the distance. [Margin: Finding the distance by sines alone.]

4. Without numbers the same lines IN, NL, IL, KL, and so on are found, as in the cases above [Margin: How the problem is done without numbers]: namely (using the figure of this problem, so that we are not forced to construct a new one), taking the line IK of as many parts as there are palms or feet in the difference of the stations, let the angles VKL and VIL be made equal to those of the complements of the observation-angles MKL and NIL, and so on. Or, if the right angle INL is set up, and at the assumed point L, where we want the meeting to be, let the angle NLI be made equal to the complement of the greater observation-angle, and the angle NLK equal to the complement of the smaller observation-angle, and so on. The rest is to be carried out as was said in Problem 1, Numbers 6 and 8. And you will

accomplish the same by what we wrote in Number 7 of the same Problem 1.

To measure the height of a mountain or tower itself from the summit of that mountain or tower, by means of two stations made on some erected staff, or at two windows of a tower one above the other, from which some mark on the horizon can be seen. And hence also to find the distance from the perpendicular (foot) of the mountain or tower up to the observed mark.

PROBLEMA IV [Problem 4]

Diagram. A tower (or mountain) height figure. A vertical shaft on the left represents the height, with B at its foot and A at the summit; above A the staff continues upward through the lower station D to the upper station E at the top. Two quadrant arcs sit at the observation points (upper near E, lower near F/H). Reading down the shaft and its neighbours: E (top), then G (left) and I (right), then D, then F (left) and H (right), then K and A. The lower part of the shaft is scaled with a tick mark. Along the base, left to right: B (foot of the height), L (center), and C (the observed mark on the plane, far right). The sight-lines run from E and D toward C.

- E (top): upper observation station
 - G, I (below E): G left, I right; endpoints of the whole-sine EG and tangent GI
 - D (on staff): lower observation station
 - F, H (below D): F left, H right; FH the tangent, K on FH
 - K (on FH): intersection of DL (parallel to EC) with FH
 - A (staff, above tower): summit of the height AB
 - B (base, left): foot of the height
 - L (base, center): point on the base line
 - C (base, right): the observed mark on the plane
1. Sometimes two stations cannot conveniently be made at the summit of a mountain or tower. In that case one must proceed as follows. Let AB be the height to be measured, standing above the plane BC, to be measured from the summit A. Erect a staff of any number of palms or feet. First, sight the mark C on the plane through the angle BDC; then sight the same mark from the higher point of the staff through the angle BEC. Each angle is to be observed either with a hanging quadrant, as you see in the first angle, or with a fixed one with a sighting-vane [dioptra], as in the second. Taking DF and EG equal, as whole sines, let the perpendiculars FH and GI be drawn to EB, as the tangents of the observation-angles. Now draw DL parallel to EC,

cutting FH at K. Since the angles BDL and E are equal [Margin: a — Euclid I.29], and F and G are right angles, and the adjacent sides DF and EG are also equal, the sides FK and GI will be equal [Margin: b — Euclid I.26], and hence KH is the difference of the tangents. And since, by the scholium of Proposition 4 of Book 6 of Euclid [Euclid], as KH is to FK, so is LC to BL [Margin: c — Euclid VI.2] — while as LC is to BL, so is ED to BD — it follows also that as KH is to FK, so is ED to DB. Therefore, if this is set up:

As KH, the difference between the tangents of the observation-angles, is to FK (or GI), the smaller tangent, so ED, the space between the observation-angles — which can be known by some measurement — is to DB,

DB will be found; and if the part of the staff AD — between the summit of the height A and the lower observation-angle D — is subtracted from it, the height sought, AB, is left known. [Margin: Finding the height.]

In the same way the same height is found in a tower, if instead of the staff AE two windows are chosen, from which the mark C is seen under the same angles, as is clear. But then to the height DB found (if the windows are D and E) one must add the part of the tower between the point D of the lower window and the summit of the tower E, so that the whole height of the tower, EB, is obtained. And if this is set up [Margin: d — Euclid VI.4]:

As DF, the whole sine, is to FH, the greater tangent, so DB, the height found, is to BC, [Margin: Finding the distance.]

the distance BC will be found.

ALITER [Another method]

2. With CB posited as the whole sine, BE will be the tangent of the angle BCE — the complement of the smaller observation-angle E — and BD the tangent of the angle BCD — the complement of the greater observation-angle D; and therefore DE is the difference of those tangents. Hence, if this is set up:

As DE, the difference of the tangents of the complements of the observation-angles, is to DB, the smaller tangent, so DE, the difference of the stations of the eyes, is to DB,

the height DB will result — known from the eye at the first station down to the base of the height, and so on. [Margin: Another way of finding the height.]

ALITER [Another method]

3. To carry out the operation by sines alone, one must first find both hypotenuses CD and CE, or one of the two, in this way [Margin: a — Euclid I.29]. Since the angle BDC is equal to the two angles E and DCE, if the smaller observation-angle E is subtracted from the greater observation-angle BDC — which is known — the angle DCE is left known [Margin: b — On Rectilinear Triangles, 10]. And so, if this is set up:

As the sine of angle DCE — the difference of the observation-angles — is to DE, the difference of the stations of the eyes, so the sine of the smaller observation-angle E is to CD; or the sine of angle EDC — the supplement of the greater observation-angle (its complement to two right angles) — is to CE, [Margin: Finding the hypotenuses.]

both the hypotenuse CD and CE are found, in parts of the difference of the stations DE. Therefore, if now this is set up [Margin: c — On Rectilinear Triangles, 10]:

As the whole sine of the right angle B is to the hypotenuse CD (or to the hypotenuse CE, just found), so the sine of angle BCD — the complement of the greater observation-angle BDC — is to DB; or the sine of angle BCE — the complement of the smaller observation-angle E — is to EB,

both heights DB and EB will be known, and so on. But if this is set up:

As the whole sine of the right angle B is to the hypotenuse CD (or to the hypotenuse CE, just found), so the sine of angle D, the greater observed, is to BC; or the sine of angle E, the smaller observed, is to BC, [Margin: Finding the height by sines alone.]

the distance BC will also be known, by sines alone. [Margin: Finding the distance by sines alone.]

4. Without numbers we shall accomplish the problem, as in the preceding cases [Margin: Solving the problem without numbers]: namely, on the straight line EB take the portion ED of as many equal parts as there are palms or feet in ED, the difference of the stations of the eyes; and let both the angle E, the smaller of observation, and BDC, the greater, be set up; and mark the meeting-point C, from which the perpendicular CB is drawn to EB, and so on. Or, if the right angle B is made, and at any point C where we wish the meeting to be, let both the angle BCD be set up equal to the complement of the greater observation-angle, and the angle BCE equal to

the complement of the smaller observation-angle, and so on. The rest is to be completed as was said in Problem 1, Numbers 6 and 8. And you will accomplish the same by what was explained there in Number 7.

COROLLARIUM [Corollary]

Therefore, if the distance of a mark seen from a tower up to the tower — namely the line CB — is known, the height of the tower is found by a single station made at the summit A [Margin: d — On Rectilinear Triangles, 4], if the following is set up:

As the whole sine CB is to BD — the tangent of the angle BCD, which is the complement of the observation-angle D — so the known distance CB is to BD.

And if, from the BD thus found, the surveyor's height AD is subtracted, the height of the tower, BA, is left known.

VEL PER SOLOS SINUS [Or by sines alone]

As the sine of the observation-angle D is to the known distance CB, so the sine of angle BCD — the complement of the observation-angle D — is to BD. [Margin: Finding the height by a single station when the distance is known.]

But if the eye D is placed at some window of the tower, one must add to the height DB found the portion of the tower between the eye and the summit. For thus the whole height of the tower, EB, will be produced, if the summit is E, as is evident.

From the summit of a mountain or tower, to find its own height, if on the plane on which it stands some space directly in front of the surveyor is known.

PROBLEM V

Diagram. A right triangle with its apex G at the upper left. The base is the horizontal line E–D–F, running left to right. The left side GE is vertical (the perpendicular from the apex to the base). The hypotenuse GF runs from the apex down to F at the right end of the base. An interior line GD is drawn from the apex to D, a point on the base between E and F.

- G (apex, upper left): summit of the height being measured

- E (base, left end): foot of the perpendicular
 - D (base, middle): the nearer sighted point
 - F (base, right end): the farther sighted point
1. When some space DE, lying in a straight line with the surveyor and set back from a mountain or tower, is known, we shall measure the height FG itself, taken from the summit G, in this way. Sight the endpoints D and E through the angles FGD and FGE, using either a hanging (pendulum) quadrant or a fixed one. Since, once the whole sine is set equal to GF, the tangents of the observed angles are DF and EF, and their difference DE is the proposed space, then let the following be constructed:

As DE — the difference of the tangents corresponding to the observed angles — is to the whole sine GF, so DE, the known space, is to GF.

The height GF sought will then be evident, expressed in parts of the known space DE.

Another method

2. We shall obtain the same height GF by sines alone, if we first hunt out the hypotenuse GD, in this way.[a] Let the following be constructed:

[Margin (a): rectilinear triangles, 10]

As the sine of angle DGE — the difference of the observed angles — is to the known space DE, so the sine of angle E — the complement of the larger observed angle — is to the hypotenuse GD.

For the number produced will give the hypotenuse GD, known in parts of the space DE.[b] If, then, the following is again constructed:

[Margin (b): rectilinear triangles, 10]

As the whole sine of the right angle F is to the hypotenuse GD, just now found, so the sine of angle D — the complement of the smaller observed angle — is to GF, the height.

the height GF will be produced, known in parts of the hypotenuse GD, or of the space DE.

[Margin: Solution of the problem without numbers.]

3. It will have to be carried out without numbers, as was explained in Problem 1.

To measure the distance from the eye, or from the foot of the surveyor, to any point marked at some height, by means of two stations made in a plane.

PROBLEM VI

1. Let the distance of a point A be sought — A lying in a wall GH that is either perpendicular to the horizon or inclined, or even in some roof — taken from the eye B, or from the foot C, the surveyor's height being BC. Conceive BD drawn parallel to the plane CE, and let A first be higher than the eye B. Sighting point A, note the angle ABD, which the edge of the sight-vanes on a hanging quadrant, or the fiducial line in the alidade of a fixed quadrant, makes with the line BD. Then advance toward point A, by any number of hand-breadths or feet, as far as D, so that BD, the difference of the stations, is known. Again, from D sight point A, carefully noting the angle ADF.

Diagram. A compound surveying figure. At the upper right stands the vertical wall, hatched, with A at its top and H at its foot. A middle horizontal line carries B (left), D (center), and F (to the right of D). A lower horizontal line carries C (left), E (center), and H (right end, at the foot of the wall). Lines of sight rise from B and from D up to point A. A dotted diagonal runs from C at the lower left up to A. G stands at the upper left, above C.

- G (upper left): reference point above C
- A (top of the wall): the point whose distance is sought
- B (middle line, left): the nearer eye-station
- D (middle line, center): the farther eye-station
- F (middle line, right): point where line BD meets the wall's line
- C (lower left): the surveyor's foot at the first station
- E (lower center): the surveyor's foot at the second station
- H (lower right): foot of the wall

Since the rectilinear segments AB, AD, BD, BC, DE lie in one and the same plane — namely the plane drawn through the surveyor's heights BC, DE and through point A[a] — and since the angle ADF is equal to the two angles B and BAD together, then, if the angle B at the farther station is subtracted from the angle D at the nearer, the remaining angle A becomes known.[b] If, therefore, the following is constructed:

[Margin (a): Elements I.32] [Margin (b): rectilinear triangles, 10]

As the sine of angle A — by which the smaller angle B differs from the larger — is to BD, the difference of the stations, so the sine of angle ADB — which is the supplement of the larger angle D, taken to two right angles — is to AB, the distance sought.

the distance AB sought will be found, expressed in parts of the difference of the stations.

But if the eye is placed at D, and one withdraws from point D [to B], the distance DA will be found by the same art, if in place of the angle ADB one takes the angle DBA — the complement of the observation angle at the farther station — as is evident.[c] For as the sine of angle B, by which the smaller angle differs from the larger ADF, is to BD, the difference of the stations, [so the sine of the other angle is to DA].

[Margin (c): rectilinear triangles, 10]

2. The distance from point A to the surveyor's foot C — that is, the straight line AC — will be found by Problem 12 on rectilinear triangles (ch. 3, bk. 1), since in the oblique-angled triangle ABC the two sides AB and BC are known (namely the distance found and the surveyor's height), and they enclose the known angle ABC, this being composed of the right angle CBD and the observation angle ABD.
3. Next let point A lie in a wall HI, below the eye B. Sighting point A, observe the angle CBA, which the edge of the sight-vanes makes with the plumb-thread — or the fiducial line of the alidade makes with BC. Then advance toward A as far as D, and again consider the angle EDA. The rectilinear segments AB, AD, BD, BC, DE lie in one and the same plane, namely the plane drawn through the surveyor's heights BC, DE and through point A.[d] And since the angle FDA — the complement of the observation angle at the nearer station — is equal to the two angles DBA and DAB together, then, if DBA (the complement of the angle at the farther station) is subtracted from the angle ADF (the complement of the angle at the nearer station), the remaining angle BAD becomes known.[e] If, therefore, the following is constructed:

[Margin (d): Elements I.32] [Margin (e): rectilinear triangles, 10]

Diagram. The figure for the case in which A lies below the eye. An upper horizontal line carries B (left), D (center), and F (right); H stands just below and to the right of F. A middle line carries G (left) and A (right), with a small square at A

marking a right angle. A lower horizontal line carries C (left) and E (center); I stands at the lower right. A dotted diagonal runs from C up to A.

- B (upper line, left): the nearer eye-station
- D (upper line, center): the farther eye-station
- F (upper line, right): intersection point on the sight line
- H (below F): point on the wall
- G (middle line, left): reference point
- A (middle line, right): the point whose distance is sought (right angle marked)
- C (lower line, left): the surveyor's foot at the first station
- E (lower line, center): the surveyor's foot at the second station
- I (lower right): foot of the wall HI

BOOK II

As the sine of angle BAD — the difference between the angles that are complements of the observation angles — is to BD, the difference of the stations, so the sine of angle ADB — composed of the right angle BDE and the observation angle ADE at the nearer station — is to AB, the distance sought.

the distance AB, which we seek, will become known, expressed in parts of the difference of the stations BD.

But if the eye is placed at D and one withdraws from point D as far as B, the distance DA will be found in the same way, if in place of the angle BDA one takes the angle DBA — the complement of the angle ABC, the observation at the farther station — as is evident.[f] For as the sine of angle BAD, the difference between the complements of the observation angles, is to BD, the difference of the stations, so the sine of angle DBA, the complement of angle ABC, at the farther station, is to DA.

[Margin (f): rectilinear triangles, 10]

4. To find the distance CA, from the surveyor's foot to point A, we shall proceed as follows. In the right triangle ABG — supposing that from point A a perpendicular AG is conceived drawn to BC, the surveyor's height — the base AB is known by the earlier finding, and the angle BAG is known, being

the complement of the observation angle ABG.[g] If the following is constructed:

[Margin (g): rectilinear triangles, 2]

As the whole sine is to the base AB, just now found, so the sine of angle BAG — the complement of the observation angle — is to BG.

BG will become known, expressed in parts of the base AB — that is, in parts of the difference of the stations BD, in which AB was found. And when BG is subtracted from the surveyor's height BC, the remainder CG becomes known.[h] Likewise, if the following is constructed:

[Margin (h): rectilinear triangles, 2]

As the whole sine is to the base AB, just now found, so the sine of the observation angle ABG is to AG.

AG will also become known, expressed in parts of the same base AB, or of the difference of the stations BD. Therefore, since in the right triangle ACG the two sides AG and GC have been made known by the finding, the base AC will also become known[i] — which is what was proposed.

[Margin (i): rectilinear triangles, 3]

5. It is evident that both distances AB and AC are found by the same procedure even if point A lies in the plane in which the surveyor stands — namely on the horizon, which is supposed to pass through the line AG — so that the surveyor's height is BG. Indeed, in that case both distances AB and AG will be found by a single station. For, once the whole sine is set equal to BG, the surveyor's height, AB is the secant of the observation angle ABG, and AG is its tangent. Accordingly, if the following is constructed:

As BG, the whole sine, is to BA, the secant of the observation angle ABG — or to GA, the tangent of the observation angle ABG — so BG, the surveyor's height, is to BA — or to GA.

both distances — BA from the eye B, and GA from G, the surveyor's foot — will become known.

6. Now we shall work without numbers, as in the preceding cases, as is evident, provided the figure is correctly constructed, in the manner explained in Nos. 6 and 8 of Problem 1.

[Margin: In what way the problem is to be solved without numbers.]

To measure the interval between two points in any elevated plane, whether that plane be perpendicular to the horizon or inclined.

PROBLEM VII

1. In any elevated plane AB, let the proposed interval be CD, which is to be investigated from the plane EB. Placing the eye at G, so that the surveyor's height is GE, let both distances GC and GD be investigated by the preceding Problem 6, expressed in parts of the surveyor's height GE (that is, of the difference of the two stations from which the distances themselves are investigated). Then apply a fixed quadrant to the eye G, so that one of its radii points toward point D — which will happen if, with the fiducial line of the alidade set along that radius, point D is seen through the holes of the sight-vanes. Turn the alidade until point C appears through it, and note the arc intercepted between the said radius and the fiducial line, for this measures the angle G. Now if the other radius of the quadrant lies beyond the line GC, the angle CGD will be acute. But if that other radius falls short of the line GC, the said angle will be obtuse, and it will be found if to a right angle one adds the remaining angle between that other radius and the line GC — which remainder we shall investigate by the quadrant, as we said for the acute angle CGD: that is, if in the line CD we mentally mark the point at which that other radius, produced, would meet it. For if in that case the radius coincides with the line GC, and the alidade is directed to that mentally marked point, the degrees between that radius and the alidade will indicate the said remaining angle. And if, finally, that other radius points precisely at C, the angle CGD will be a right angle. Therefore, since in the oblique-angled triangle GCD the known sides GC and GD enclose the known angle G, the side CD will become known, by Problem 12 on rectilinear triangles (ch. 3, bk. 1).

Diagram. An interval-measurement figure over an inclined plane. Along the bottom run E (left) and B (center) of a base rectangle. G stands on the left at mid-height, D on the right at mid-height, joined by the horizontal GD. C rises at the top center as the apex, forming triangle GCD. A stands at the upper right, beyond D, with lines CA and DA forming a smaller triangle to the right. A quarter-circle (the quadrant instrument), with a right-angle mark, is drawn at G. A vertical rises from E to G, and from B toward the middle.

- E (bottom left): foot below G
- B (bottom center): foot below the apex

- G (middle line, left): the eye-station with the quadrant
 - D (middle line, right): the farther interval endpoint
 - C (top, apex): the nearer interval endpoint
 - A (upper right): a further sighted point
2. The problem will easily be solved without numbers, if the angle G is made equal to the one that was observed by the quadrant, and if on the lines GC and GD as many small parts are taken from the graduated instrument as there are hand-breadths or feet found in the distances GC and GD, and so on.

[Margin: Solution of the problem without numbers.]

To determine, by means of the quadrant, the length of a straight line, when the surveyor — standing at one of its ends, or at some known height that is perpendicular at that end to the plane in which the line lies — can see the other end.

PROBLEM VIII

1. Let the length AB be sought — that is, the distance between A and B — even though the intermediate points cannot be seen, whether because of intervening rises or because of valleys, provided that at the end A there stands a surveyor, or some known height perpendicular to the plane in which the line AB lies, so that AC is either the surveyor's height, or some erected pole, or a tower. Sighting the end B, observe the angle C. Since, once the whole sine is set equal to AC, the distance AB is the tangent of the observed angle C, then if the following is constructed:

Diagram. A right triangle with the quadrant instrument at C. C stands at the upper left, A at the lower left, B at the lower right. The side CA is vertical, the base AB horizontal, and CB is the hypotenuse. A quarter-circle (quadrant) with right-angle marks is drawn at C; a small right-angle mark stands at A.

- C (upper left): the top of the height AC, with the quadrant
- A (lower left): the foot of the height (right angle marked)
- B (lower right): the far end of the line whose length is sought

As the whole sine AC is to AB, the tangent of the observed angle C, so AC — the surveyor's height, or the known altitude — is to AB, the length.

the length AB will be produced, expressed in parts of the known altitude AC. It will also be produced by sines alone,[a] if the following is constructed:

[Margin (a): rectilinear triangles, 4]

As the sine of angle B — the complement of the observed angle C — is to AC, the surveyor's height or known altitude, so the sine of angle C, the observation, is to AB, the length.

2. Without numbers the same length AB will be found, as was said in the preceding cases: namely, if from the graduated instrument AC is taken to be of as many small parts as there are hand-breadths or feet in the altitude AC, and the observation angle C is set up, and finally the perpendicular AB is erected at C, and so on.

[Margin: Solution of the problem without numbers.]

To measure, from some known height, a length whose ends one is not permitted to approach — provided that they are visible and that the length, produced, reaches the foot of the surveyor.

PROBLEM IX

1. Let the length AB lie in a straight line with the surveyor, who stands at C, so that the line BA, produced, passes through C. Let CD also be either the surveyor's height or some known altitude. If, then, by the preceding Problem 8, both lengths CB and CA are inquired, and CA is subtracted from CB, the remainder AB is found, and so becomes known.

Diagram. A right triangle. D stands at the upper left, C at the lower left, A at the lower center, B at the lower right. The side DC is vertical, and C–A–B is the horizontal base; lines DA and DB are drawn from the apex D down to A and B. The right angle is at C.

- D (upper left, apex): the top of the known height DC
- C (lower left): the foot of the height, where the surveyor stands (right angle)
- A (lower center): the near end of the length
- B (lower right): the far end of the length

Another method

2. Setting the whole sine equal to CD, if the ends A and B are sighted through the angles CDA and CDB, then CA will be the tangent of the smaller angle, and CB the tangent of the larger, while AB is the difference of these tangents. Hence if the following is constructed:

As the whole sine CD is to AB, the difference of the tangents of the observed angles, so CD, the known altitude, is to AB, the length.

the length AB will be produced, known in parts of the known altitude CD.

Another method

3. By sines alone we shall come to know the same length AB; but first AD must be found,[a] in this way. Let the following be constructed:

[Margin (a): rectilinear triangles, 5]

As the sine of angle CAD — the complement of the smaller observed angle — is to CD, the known altitude, so the whole sine of the right angle C is to AD.

For the number produced will give AD, known in parts of the known altitude CD.

[b] If, then, the following is again constructed:

[Margin (b): rectilinear triangles, 10]

As the sine of angle CBD — the complement of the larger observed angle — is to AD, just now found, so the sine of angle ADB — the difference between the two observed angles — is to AB, the length.

there will emerge the length AB, known in parts of the line AD — that is, of the known altitude CD, in which the line AD was found.

4. Without numbers you will accomplish the matter, just as in the preceding cases, as is clear.

To determine a transverse length lying in the horizontal plane, both of whose extremities can be sighted.

PROBLEMA X.

1. Let AB be the horizontal plane, in which the length CD lies transversely, and let the surveyor's foot be at E, so placed that the straight line DC does not pass through the surveyor's foot at E. For whenever the length DC lies directly in front of the surveyor, it is found by the preceding Problem 9.

Therefore, in order that the transverse length CD be determined, we must first find the distance of each of the two extreme points C and D from the surveyor's foot E — and this by a single station made at E, as was said in Problem 6, number 5.

Next, using the quadrant with its sight [dioptra], the angle CED is to be found in the horizontal plane. This is done if the erected plane of the quadrant passes once through the points E and C, and again through the points E and D, so that the divisions of the lines EC and ED can be marked off. For when one side of the quadrant lies along the line EC, and the sight along the line ED, then — if the angle is acute — the arc between that side and the sight will indicate the angle CED. But if the other side of the quadrant coincides with the line ED, the angle CED will be a right angle. And if the line ED extends beyond the other side of the quadrant, the said angle will be obtuse; it will be known if to the right angle is added the remaining angle between the other side and the line ED. This remaining angle is measured with the quadrant, just as we said of the acute angle in Problem 7. And thus we shall have the triangle ECD, whose two sides EC and ED are known and which enclose the known angle CED. [Margin: a — *De Triangulis rectilineis* 12 (on plane triangles)] Therefore the third side CD too will be known.

2. In what way the problem is to be accomplished without the aid of numbers has been shown in Problem 7, number 2.

Diagram. A quadrilateral representing the horizontal plane AB, seen in perspective. Inside it a triangle ECD is drawn, with a small arc at E marking the observation angle CED.

- A (upper-left corner of the plane): A
- (upper-right corner, unlabelled)
- B (lower-right corner): B
- C (lower-left corner): C
- D (on the right-hand edge, at mid-height): D
- E (just inside, near the upper-left corner — the surveyor's foot): E

To measure a length in the horizontal plane between some tower and any other mark, taken from the tower by means of two stations made at its top; or in two windows, one directly beneath the other, when the space between those windows is known — even though the height of the whole tower is unknown. And thereby, incidentally, to reveal the height of the tower.

PROBLEMA XI.

1. Although this problem has already been solved in Problems 3 and 4 in connection with finding a height, I prefer nonetheless to treat it here on its own account and in a somewhat different way. Let ABCD then be the

tower, and CE the proposed length; and let BG (or AF) be the surveyor's stature. When the endpoint E has been sighted — in the first station through the angle CGE, and in the second through the angle DFE — let FH be drawn parallel to GE. And since in the triangles FDH and GCE the angles D and C are right angles, [Margin: b — Elements I.29] and the angles H and E are equal, [Margin: c — Elements I.34] and the side FD is equal to the side GC, [Margin: d — Elements I.26] the angles G and DFH will also be equal, and likewise the sides DH and CE.

Now if FD is taken as the whole sine, then DE will be the tangent of the greater observation angle DFE, and DH the tangent of the angle DFH — that is, of the angle equal to G, observed in the first station. Consequently HE will be the difference of these tangents, [Margin: e — Elements I.34] and this difference is in fact equal to the difference of the stations FG (or AB, or DC).

If, therefore, we set up the proportion:

As HE, the difference of the tangents of the observation angles, is to CE, the tangent of the lesser observation angle CGE, so HE (or FG), the difference of the stations, is to CE, the length [sought],

the desired length CE will be produced, in the same units as the difference of the stations HE (or FG), and so on.

And if again we set up:

As HE, the difference of the tangents of the observation angles, is to FD, the whole sine, so HE (or FG), the difference of the stations, is to FD,

the line FD will be found; and if from it the surveyor's stature AF is subtracted, the remainder will be the height of the tower AD.

2. But now, from the two windows F and G, let the endpoint E be sighted through the angles CFE and CGE, and let FH be drawn parallel to GE. [Margin: f — Elements I.29] Since therefore the angle CFH is equal to the angle CGE: if CE is taken as the whole sine, then CG will be the tangent of the angle CEG — the complement of the lesser observation angle G — and CF the tangent of the angle CEF, the complement of the greater observation angle CFE. But FG, the difference of the stations — that is, the interval between the windows — will be the difference of the said tangents. For which reason, if we set up:

Diagram. A tall tower standing on the ground, with two sight-lines running down from its top to a point on the ground at the right. The tower carries a battlemented top and window-marks (small black squares).

- F (apex at upper-left, top of the sight-line): F
- G (apex at upper-right): G
- A (left edge of the tower top, battlemented): A
- B (just to the right of A): B
- D (ground line, far left, at the tower's base): D
- C (ground line, next to D): C
- H (ground line, toward the right): H
- E (ground line, far right — the mark sighted): E

GEOMETR. PRACTICÆ

[Continuation of Problem XI, part 2.]

As FG, the difference of the tangents of the complements of the observation angles, is to CE, the whole sine, so FG, the difference of the stations between the windows, is to CE, the proposed length,

the desired length CE will be known. And if again we set up:

As FG, the difference of the tangents of the complements of the observation angles, is to GC, the tangent of the complement of the lesser observed angle, so FG, the difference of the stations between the windows, is to GC,

the height GC will be found — from the upper window down to the base — in the units of the difference of the stations; and if to it is added the portion GB, from the upper window up to the summit, the whole height of the tower BC will not be unknown.

3. Without the aid of numbers, one must proceed as in the earlier cases.

Diagram. A tall battlemented tower with two windows (small black squares) on its right face, from which sight-lines run down to a point on the ground at the right.

- A (tower top, upper-left, battlemented): A
- B (tower top, upper-right): B
- G (below B, the upper window): G

- F (below G, the lower window): F
- D (ground line, far left, at the base): D
- C (ground line, next to D): C
- H (ground line, toward the right): H
- E (ground line, far right — the mark): E

To determine, with the quadrant, the length of a straight line lying directly in front of the surveyor, when both of its endpoints — or one of them — cannot be seen unless the surveyor moves to the right or the left.

PROBLEMA XII.

1. Let AB be the length, and let the surveyor, stationed at the endpoint A, be unable to see the other endpoint unless he withdraws to the right or the left as far as the point D, from which both endpoints can be seen. Then the length AB will lie transversely with respect to the surveyor standing at D. Therefore it will be found by Problem 10.
2. In the same way, if the surveyor stands at C, directly in line with the length, but can see either neither endpoint or only one of them, we shall track down the length AB. For if from C we proceed to the left or the right until at D we see both endpoints, the transverse length AB will be found by Problem 10, as before. Nor indeed does it matter whether one steps aside through the right angle BCD, or through the obtuse angle BAD, and so on.
3. The operation without numbers is to be carried out as in the earlier cases.
4. When the surveyor at A can see the endpoint B, the length AB will be found by Problem 8.

But if he stands at C directly in line with the length, and can see both endpoints, the same length AB will be found by Problem 9.

Diagram. A right triangle with its right angle at the lower left, standing on a horizontal base. Lines run from the apex D down to C, A, and B.

- D (apex, upper-left): D
- C (lower-left, at the right angle): C
- A (on the base, middle): A
- B (on the base, right end): B

To gather, by means of the quadrant, the distance of some mark set in the horizontal plane from the summit of a tower or of some wall, even though no open approach to the mark itself lies available.

PROBLEMA XIII.

1. In the horizontal plane, let the point A be distant from the summit D of some height CD by the straight line AD, which we are directed to measure. Wherever the surveyor's eye is placed — namely at B, so that BG is the surveyor's stature — let the distances of the points A and D from the surveyor's eye B be found by Problem 6.

Next, let the angle ABD be measured, which the quadrant with its sight will supply, provided it is so applied to the eye that its plane passes through the three points B, A, and D — the center being placed at B — while one of its sides rests along the line BA, and the sight is directed toward the point D, and so on. And so, since in the triangle BAD two known sides BA and BD enclose the known angle B, [Margin: a — *De Triangulis rectilineis* 10] the side AD too will be known.

2. By what method this same distance AD is to be found without numbers, we have taught in number 5 of Problem 7.

Diagram. A vertical hatched column (tower or wall) at the left, with D at its top; the surveyor's eye is at B to the right. Sight-lines run from D to A and to G, and from B; a small square marks the right angle at A.

- D (top of the column, upper-left): D
- B (to the right, the surveyor's eye): B
- C (base of the column, lower-left): C
- A (on the ground line, middle): A
- G (on the ground line, right — the instrument's position): G

To find, by means of the quadrant, an inaccessible height whose base cannot be seen, and toward which — or from which — we can move along no straight-line distance such that two stations might be made, but can move only to the right or the left, to a place from which its base becomes visible.

PROBLEMA XIV.

1. Let AB be the height, to which, from C, the surveyor's location, it is not permitted to approach, or from which to withdraw along a straight line, but only sideways — for example, as far as D, from which we can see the base. By Problem 10 let the transverse length AC be found, from the

location D; and let the vertex B be sighted from C through the angle ACB. And since, when AC is taken as the whole sine, the height AB is the tangent of the observation angle ACB, [Margin: a — *De Triangulis rectilineis* 4] if we set up:

As the whole sine AC is to AB, the tangent of the observed angle ACB, so the length AC, found by Problem 10, is to AB, the height,

Diagram. A triangle in which B stands at the upper right, directly above A on the base; C is at the lower left, with arc-marks indicating the observed angle; the intermediate point D lies on the sight-line from C. Lines run C–D–A(?), C–B, and B–A.

- B (upper-right, vertex of the height): B
- D (upper-left, on the sight-line): D
- C (lower-left, station, with angle-arcs): C
- A (lower-right, base of the height): A

LIBER I.

the known height AB will result, in the same units in which AC was found.

2. It will not be difficult, moreover, to carry out this problem without the aid of numbers, if the earlier precepts are consulted.

To make known, by means of the quadrant, an inaccessible height — when neither the distance from the surveyor's location to its base is known, nor can two stations be made in the plane directly in line with it, nor, finally, does the base come into view — and thereby, incidentally, to draw out the distance as well.

PROBLEMA XV.

1. Let AB be the height, C the surveyor's location, and CB the unknown distance; let the base B not appear, and from the location C let it be permitted neither to approach AB nor to withdraw, so that two stations might be made in the plane. Let a staff CE be set up, if some tower CO is not at hand, and let CD be the surveyor's stature. Then, taking a portion DE of the staff, of known units, let DM and EN be conceived as drawn parallel to CB, and let the angles ADM and AEN be observed with the quadrant. Also, taking equal straight lines EF and DG as the whole sines, let the perpendiculars FH and GI be erected as the tangents of the observation angles. Likewise, taking AL equal to DE, let the straight line DL be drawn,

[Margin: a — Elements I.33] which will be parallel to EA and will cut GI at K. [Margin: b — Elements I.29]

Now, since the angle DKG is equal to the angle DLB, and this to the angle EAB, and this in turn to the angle EHF, the angle DKG will be equal to the angle EHF. But the right angle G is equal to the right angle F, and the side DG equal to the side EF; [Margin: c — Elements I.26] therefore the sides GK and FH will also be equal, and the angles D and E equal. And hence IK will be the difference between the tangents GI and GK, of the angles GDI and GDK (or FEH). [Margin: d — schol. 4. sexti [?]; the text itself cites the scholium to *Elements* V.18] And since it holds that, as IK is to KG, so AL is to LM, it will follow — by the contrary composition [conversion of ratio] which we demonstrated in the scholium to Proposition 18 of Book 5 — that likewise IK is to IG as AL, that is, as ED, the difference of the stations, is to the height AM. Therefore, if we set up:

As IK, the difference of the tangents, is to IG, the greater tangent, so AL (or ED), the difference of the stations, is to AM, the height,

the height AM will be produced; and if to it is joined the surveyor's stature MB, the whole height AB will become known. [Margin: e — Elements VI.4] Now, if we set up:

As GI, the whole sine, is to GD, the tangent of the complement of the greater observation angle, so the height AM already found, is to MD, the distance,

the distance DM (or CB) will be found.

2. By sines alone we shall accomplish the same problem, if first the hypotenuse AD is found, in this way. [Margin: f — *De Triangulis rectilineis* 10] Let us set up:

As the sine of the angle ADL — the difference between the observation angles ADM and LDM — is to AL, that is, to DE, the difference of the stations, so the sine of the angle ALD (which remains when DLM, the complement of the lesser observed angle LDM, is subtracted from two right angles) is to AD.

For the resulting number will yield the hypotenuse AD. [Margin: g — *De Triangulis rectilineis* 10] If, then, we again set up:

As the whole sine of the right angle M is to the hypotenuse AD, already found, so the sine of the greater observation angle ADM is to AM,

the height AM will be produced, and so on. [Margin: h — *De Triangulis rectilineis* 10] But if we set up:

As the whole sine of the right angle M is to the hypotenuse AD, just found, so the sine of the angle DAM, the complement of the greater observation angle, is to DM,

the distance DM [?: printed glyph ambiguous, could read "DN"] will be established as known.

3. If the problem is to be solved without numbers, one must return to the earlier material, especially to numbers 6, 8, and 7 of Problem 1.

To find out, by means of the quadrant, a greater height from a known lesser one, by means of two stations made at the summit or in two windows, even if only the vertex of the greater height is seen. And thereby to gather also the distance between the heights.

Diagram. A composite surveying figure. At the right stands a tall battlemented tower, with A at its top; down its right face lie the points N, L, and M, with the base at B (resting on a small mound). At the left stands a shorter battlemented pillar (the tower CO / staff CE), its top labelled at O and E, its base at C (also on a mound), with D just to the right of the base and G further right. In the field between the two, running from the upper left downward, are the points H (with a quadrant arc and plumb-line), F, I, and K. A horizontal line runs E–F–N–L, and diagonal sight-lines run from A down to the left toward E and D, with a quadrant (plumb-bob visible) applied near E and near D.

- A (top of the tall right-hand tower): A
- O (top-left of the shorter left-hand pillar, outer label): O
- E (top of the left-hand pillar/staff): E
- H (upper field, at the quadrant arc): H
- F (field, below H): F
- N (on the horizontal line, toward the right tower): N
- L (on the right tower, at mid-height): L
- I (field, below F): I
- K (field, below I): K
- M (on the right tower, lower down): M
- D (near the base of the left pillar, surveyor's eye level): D
- G (on the ground line, right of D): G
- C (base of the left pillar, lower-left): C

- B (base of the right tower, lower-right, on a mound): B

PROBLEMA XVI.

Diagram. A surveying figure with two structures. On the left, a short tower or column (the tower CO) stands on a pedestal; on the right rises a tall crenellated tower reaching to the point A at its summit, with N at its foot; loose rocks lie on the ground between the two. From A two straight sightlines descend to the left toward the observer's positions. A horizontal line (the perpendicular GEF) runs across the figure. Labelled points, by position:

- A (top of the right-hand tower): the summit sighted.
- H (upper, on the higher sightline): intersection point.
- I (upper, on the lower sightline): intersection point.
- G (left end of the horizontal line, at the top of the pedestal / first eye-station): observer's eye.
- M, L (along the horizontal line, left of centre): division points.
- E (along the horizontal line, second eye-station): observer's eye.
- K (along the horizontal line, right of E): division point.
- F (right end of the horizontal line, at the tall tower): foot of the perpendicular.
- C, D (top corners of the left pedestal): the two stations at the summit of the smaller tower.
- O (base of the left column).
- N (base of the right-hand tower).

1. Let the greater altitude be AN and the smaller tower CO — this last being known — such that only the summit A, but not the base N, is visible. Set up two stations at the top, at C and D, and let the surveyor's height be CG or DE; and to AN let a perpendicular GEF be understood as drawn. Then, having sighted the summit A, observe the angles AEF and AGF. Carry out the rest as in Problem 2. If, therefore, one proceeds just as we demonstrated in that problem:

As GM — the difference of the Tangents [Tangens] GL and EK of the angles H and I, which are the complements of the observed angles, the whole sines [sinus totus] HL and IK being set down — is to GE, the difference of the stations, so is LH, the whole sine, to FA.

The altitude AF will be found; and if to it FN is added — this being made up of the height of the tower CO together with the surveyor's height — the whole greater altitude AN comes out known. Likewise, if one proceeds:

As GM, the difference of the same Tangents, is to GE, the difference of the stations, so is GL — the Tangent of the complement of the smaller angle of observation — to GF,

the distance GF will be produced; and if from this the breadth of the tower CO is subtracted, what remains will be the distance between the two towers.

All of this we demonstrated in Problem 2; and for that very reason we have used here exactly the same letters we employed there, so that the demonstration can be carried over from that place to this one.

2. When two stations cannot be made at the summit of the smaller tower, let two windows be chosen at which the two stations are made — as in the figure of the preceding Problem 15, where in the smaller tower CE two windows D, E are selected, the rest being carried out as with the staff CE. Only, in place of the surveyor's height, one must add to the altitude found, AM, the portion of the tower between the lower window D and the base C, so that the whole greater altitude AB is made known.

3. For the solution without numbers we prescribe nothing new, but refer the reader to what has gone before.

4. By what means, too, this problem can be carried out by sines alone, we taught in Problems 2 and 15.

By means of the Quadrant, to investigate a greater altitude from a smaller unknown one, provided the base of the greater can be seen.

PROBLEMA XVII.

Diagram. Two towers side by side, cross-braced by lines. Labelled points, by position:

- A (top): summit of the greater altitude.
- C (upper left, top of the near/smaller tower): station window.
- E (upper right, on the taller tower, level with C): where the horizontal CE meets the tall tower.
- D (base, left): foot of the smaller tower.
- B (base, right): foot of the greater tower.

1. Let the greater altitude be AB and the smaller CD, both unknown, and let it be possible for the base B of the greater to be seen from the smaller altitude. First, by two stations at the summit of the smaller tower, or at two windows, let both the altitude of the smaller tower CD and the distance DB be sought out, as was set forth in Problem 11 (or also in 3 and 4). For then, from the smaller altitude CD once known, the greater AB will be determined by the methods we wrote of in the preceding Problem 16.

2. But from the altitude CD and the distance DB, once known, we shall learn the greater altitude AB by sines alone, in this way. From some window C of the smaller altitude, let the extremities A, B of the greater altitude be sighted through the angles ACE, BCD (CE having first been drawn parallel to DB, or perpendicular to both altitudes).[a] Then proceed:

[Margin: (a) *Triang. rectil.* 10 — rectilinear triangles, prop. 10.]

As the sine of the angle CBD — the complement of the angle at which the base is sighted — is to the found altitude CD, so is the whole sine of the right angle D, to BC.

Or:

As the sine of the angle BCD, at which the base is sighted, is to the found distance BD, so is the whole sine of the right angle D, to BC.

for by either method the hypotenuse BC will be known.[b] If, then, one proceeds again:

[Margin: (b) *Triang. rectil.* 10.]

As the sine of the angle A — the complement of the angle at which the summit is sighted — is to the found hypotenuse BC, so is the sine of the angle ACB (made up from the complement of the angle at which we view the base, together with the angle at which the top is seen), to AB,

the greater altitude AB will be made manifest.

3. For the solution of the problem without numbers we prescribe nothing new here, but it must be sought from what has gone before.

By means of the Quadrant, to investigate a smaller altitude from a known greater one, even though the base of the smaller cannot be seen; and hence also to determine the distance between the two altitudes.

PROBLEMA XVIII.

Diagram. Two crenellated towers of unequal height. Labelled points, by position:

- A (top left, summit of the smaller tower): the sighted top.
- C (top right, summit of the taller tower): higher summit.
- E (middle, on the taller tower level with A): where the horizontal from A meets the tall tower.
- B (base, right): foot of the taller tower.
- D (base, left): foot of the smaller tower.

1. Let the smaller altitude AB be proposed for finding out from the greater CD, which is known, even though the base B is not seen. Let a straight line AE be conceived as drawn parallel to BD, such that ED is equal to the smaller altitude AB. If, therefore, from two stations made at the summit of the greater altitude CD (by Problem 3), or from two windows (by Problem 4), both the altitude CE and the distance AF are investigated — the summit A having been sighted, as though it were some marker seen on the horizon AE — and CE is subtracted from the whole altitude CD, the remainder ED, that is, the smaller altitude, will become known. But the distance AE, once found, is equal to the sought BD; and therefore DB will be known.

By means of the Quadrant, to explore a smaller altitude from a greater unknown one, provided the base of the smaller can be seen; and hence also to conjecture the distance between the two altitudes.

PROBLEMA XIX.

1. Let the figure of the preceding problem be repeated. And since the base B of the smaller altitude is visible from the greater: if the point B is sighted from two stations made at the summit of the greater altitude CD, there will be found, by Problem 3, both the greater altitude CD and the distance BD. You will accomplish the same thing by Problem 4, if the point B is sighted from two windows of the greater altitude CD. Therefore, the greater altitude CD being known, the smaller altitude AB will be found, as was set forth in the preceding problem. And since the distance BD too has been explored, the solution of the proposed problem is evident.

By means of the Quadrant, to ascertain a portion of the greater altitude from the smaller altitude, and a portion of the smaller from the greater.

PROBLEMA XX.

Diagram. A tall tower on the right and a shorter one on the left. Labelled points, by position:

- A (top of the right-hand tower): summit of the greater altitude.
- C (on the right tower, below A): a window / division.
- D (top of the left-hand shorter tower).
- F (on the right tower, below C): a window / division.
- G (on the right tower, below F): a window / division.
- E (base, left): foot of the shorter tower.
- B (base, right): foot of the greater tower.

1. Let the portion AC of the greater altitude AB be sought from the smaller altitude DE; likewise the portion FG of the smaller altitude FB, from the greater altitude DE. If DE, the smaller altitude, is proportional to CB, let both the greater altitude AB and CB be investigated from the smaller altitude DE, by Problem 16 or 17 — according, that is, as DE is known or unknown. For CB, taken away from AB, will leave the sought portion AC known.

2. But if DE is greater than the portion FB, and the portion AF is to be explored, then indeed the greater altitude AB must be sought from the smaller DE, by Problem 16 or 17; while the smaller altitude FB must be explored from the greater DE, by Problem 18 or 19. For, again, FB subtracted from AB will leave the portion AF known.

3. Not otherwise, by Problem 18 or 19, must both smaller altitudes FB and GB be sought out from the greater DE, so that their difference FG — which is what is sought — may be gathered as known.

By means of the Quadrant, from the valley or from the plane of the horizon, to explore an altitude whose base rests upon another altitude — such that both of its extremities can be seen — even though the lowest point of the other altitude on which it rests is hidden, and the lowest distance of that same point from the surveyor's place is not known.

PROBLEMA XXI.

Diagram. A crenellated tower standing on top of a mountain, with a wide triangle set out on the ground below. Labelled points, by position:

- A (top of the tower): the summit.

- I (on the tower shaft, midway, beside a window mark): a marker on the wall.
- B (base of the tower, at the mountain top): where the tower meets the mountain.
- D (lower left, ground): the observer's station.
- C (right, on the horizontal ground line, below B): foot of the perpendicular from D to the line AB extended.
- E (bottom left, below D): the surveyor's foot ($DE = \text{surveyor's height}$).
- F (bottom right, below C): ground endpoint.

Two sightlines run from A and from B down to D; DC is drawn horizontally to meet, at C, the line AB produced downward into the mountain.

1. An altitude of this kind is a tower set upon a mountain, or a portion of some building between two windows, or two markers of which one is higher than the other. Let there be proposed, then, the altitude of a tower AB set upon a mountain. From some place E, on the plain or in a valley, from which both extremities A and B can be seen, let there be observed with the Quadrant the angles ADC, BDC, which the radii DA, DB make with the straight line DC — this being drawn from D perpendicular to AB produced into the mountain — so that the surveyor's height is DE. Then, by Problem 6, let the length of both radii DA, DB be investigated.[a] For if, in the triangle ABD, one proceeds:

[Margin: (a) *Triang. rectil.* 10.]

As the sine of the angle BAD — which is the complement of the larger observed angle ADC — is to the radius DB just found, so is the sine of the angle ADB (which is the difference of the observed angles ADC and BDC), to AB, the altitude,

the sought altitude AB will be found, in the same parts in which the radius DB is known.[b] So too, if one proceeds:

[Margin: (b) *Triang. rectil.* 10.]

As the sine of the angle ABD — or, what is the same, the angle DBC, which is the complement of the smaller observed angle BDC — is to the radius DA just found, so is the sine of the angle ADB (the difference of the observed angles ADC and BDC), to AB, the altitude.

there will again come out the sought altitude AB, in the parts of the radius DA that was found.

ALITER. [Otherwise]

2. By Problem 2 or 15, let there be investigated both the inaccessible altitude AC and BC, apart from the surveyor's height CF. For the altitude BC, subtracted from the altitude AC, will leave known the altitude AB, which is what is sought.

ALITER.

3. The distance DC having been found by the methods we set forth in Problems 1 and 2,[c] if one proceeds:

[Margin: (c) *Triang. rectil.* 4.]

As the whole sine DC is to the known distance DC, so is AC — the Tangent of the larger observed angle ADC — to AC,

the greater altitude AC will be found, in the parts of the distance DC that was found.[d] And if again one proceeds:

[Margin: (d) *Triang. rectil.* 4.]

As the whole sine DC is to the found distance DC, so is BC — the Tangent of the smaller observed angle BDC — to BC,

the smaller altitude BC will become known, in the parts of the same distance DC that was found; and this, subtracted from the greater altitude AC, will leave known the altitude of the tower BA, which is sought.

And this method is most convenient when, in some tower or building whose distance from the surveyor is known, one must measure the perpendicular interval between two windows.

ALITER.

4. The distance DC having again been found, by the methods we wrote of in Problems 1 and 2, let there be subtracted the Tangent BC (the whole sine being set as DC) of the smaller observed angle ADC[?], so that AB — the difference of the said Tangents — remains known. For if one proceeds:

As the whole sine DC is to the found distance DC, so is AB — the difference of the Tangents AC and BC — to AB, the altitude,

the sought altitude AB will be produced, in the parts of the distance DC that was found.

ALITER.

5. The altitude of the mountain BC having been found by Problem 2 or 15, let the angles BDC, ADC be observed through the radii DB, DA. For, the whole sine being set as DC, if one proceeds:

As BC — the Tangent of the smaller observed angle BDC — is to BC, the found altitude, so is AB (the difference of the Tangents AC and CB of the observed angles ADC and BDC), to AB,

the known height AB, which we are seeking, will come out.

SCHOLIUM

Therefore, if AB — the upper portion of some whole height AC — is what is wanted, both the whole height AC and its lower portion BC must be investigated by Problem 2 or 5, since the difference between them will give the upper portion AB as a known quantity.

If, however, some middle portion IB is to be found, then both heights IC and BC must again be worked out, so that their difference IB is made known.

And if, finally, the lower part BC is what is proposed for inquiry, this will be done by Problem 2 or 15.

To determine, close to the truth, the sloping distance of a mountain from the surveyor's position up to the base of a height set on the mountain — even when that distance is not seen — together with the height itself, when the surveyor stands part-way up the ascent of the mountain, by means of the Quadrant.

PROBLEMA XXII.

1. Let AB be a tower set on a mountain, and let the surveyor stand at C on its ascent or flank, a place from which he cannot see the base of the tower. Set up some staff CG perpendicular to the horizon — not to the mountain's slope — and let CE be the surveyor's height. Imagine EK drawn perpendicular to the vertical [i.e. to the line of the height]. Sighting the summit A through the angle AEK, make a second, higher station at F; and with FI drawn perpendicular to the vertical, sight the same summit A through the angle AFI. [Margin: 32.primi — Euclid, *Elements* I.32.] Now since the angle AFG is equal to the two angles FEA and FAE, if AEF — the complement of the larger angle AEK observed at the first station — is

subtracted, the remaining angle EAF results. [Margin: 10.trian.rectil. — on rectilinear triangles, prop. 10.] Therefore, let it be made:

As the sine of angle EAF (the difference of the complements of the observed angles) is to EF (the difference of the stations), so the sine of angle AFE (compounded of the right angle EFI and the smaller angle AFI observed at the second station) is to AE, the hypotenuse.

The hypotenuse AE will thereby be made known. After this, set up another rod DH perpendicular to the horizon, taking the surveyor's height DH equal to CE; and with the straight line HL drawn perpendicular to the vertical, sight the point E through the angle EHL and along the ray HEM [Margin: 33.prim. — Euclid I.33.], which will be parallel to BCD, the flank of the mountain. For the three points B, C, D are to be conceived as lying in one straight line, as if DC, produced, reached the base of the tower. [Margin: 29.prim. — Euclid I.29.] And since the angle EHL is equal to the angle MEK, if this is subtracted from the larger observed angle AEN at the first station, the remaining angle AEM results. The angle EAN is also known, being the complement of the larger observed angle AEN. Therefore the remaining angle AME — the remainder of two right angles — will likewise be known; and this same remainder is obtained if the complement of the angle MHL observed at station D is subtracted from two right angles. [Margin: 10.triang.rectil. — on rectilinear triangles, prop. 10.] Therefore, let it be made:

As the sine of angle AME (which is left over when the complement of the last observed angle MHL is subtracted from two right angles) is to the hypotenuse AE just found, so the sine of angle EAN (the complement of the larger observed angle AEN) is to ME.

The straight line EM will be found — that is, the sought distance CB. [Margin: 10.trian.rectil. — on rectilinear triangles, prop. 10.] And if it is done again:

As the sine of the same angle AME is to the same hypotenuse AE just found, so the sine of angle AEM (which is left over when the last-observed angle MHL, or MEK, is subtracted from the larger observed angle AEK) is to AM.

AM will be produced; and if the surveyor's height MB is added to it, the whole height AB will be known.

Care must be taken diligently to have the three points B, C, D lie in one straight line — that is, so that the line drawn from the last station D through the first station C passes through the base B; and this we shall achieve more or less by the judgment of the senses. For by what has been said, only the distance is found

here from the position C up to that point of the height into which the straight line DC, when

produced, falls; and the height from A up to that same point — which will not be far from the point B, if diligence is used in choosing the stations C and D. It is for this reason that in the statement of the problem we said "make it known close to the truth," because neither the distance CB nor the height AB is precisely known except when the three points B, C, D lie in a straight line.

2. To carry out the problem without the aid of numbers, one must have recourse to what has gone before.

To find, by means of the Quadrant, the depth of a well, or of any building whatever raised to the perpendicular, provided the angle at the bottom, or some mark placed at the bottom, can be sighted.

PROBLEMA XXIII.

1. This is nothing other than measuring a tower from its top, when some mark appears on the horizon, by two stations made on some staff, as was done in Problem 4. We shall therefore repeat the working of that problem here. Let there be a well, or a building raised upright, ABCM, whose angle C at the bottom — or a mark C placed at the bottom — can be sighted. With the staff AE set up at the mouth of the well or the top of the building, let two stations of the surveyor's eye be made at D and E, and let the point C be sighted along the rays DC, EC, making the angles BDC, BEC. Then, taking DF and EG equal, standing for the whole sines, draw the perpendiculars FH, GI to EB, standing for the tangents of the observed angles BDC, BEC. Draw also DL parallel to EC, cutting FH at K, so that KH is the difference of the tangents, just as we demonstrated in Problem 4. If then, as there, it be made:

As KH (the difference between the tangents of the observed angles) is to FK, or to GI, the smaller tangent, so DE (the difference of the stations — that is, the space between the eyes, or between the points of observation) is to DB.

The straight line DB will come out known, as we showed in Problem 4; and if the segment of the staff AD, between the mouth and the eye at the first station, is subtracted from it, the remainder will be the depth, or height, of the well or building AB.

ALITER

2. With the whole sine set as BC, let it be made:

As DE (the difference of the tangents BD, BE, which is owed to the complements of the observed angles) is to DB, the smaller tangent, so DE (the difference of the stations of the eyes) is to DB.

For the number produced will exhibit the very same straight line DB, and so on.

ALITER

3. If one prefers to work by sines alone, this is how to proceed. [Margin: 32.primi — Euclid I.32.] Since the larger observed angle BDC is equal to the two angles E and DCE, if the smaller observed angle E is taken away from the larger BDC, the angle DCE remains known, the difference of the observed angles. [Margin: 10.trian.rectil. — on rectilinear triangles, prop. 10.] Therefore, if it be made:

As the sine of angle DCE (the difference of the observed angles) is to DE (the difference of the stations of the eyes), so the sine of angle E, the smaller observed, is to DC.

The known hypotenuse DC will leap out. [Margin: 10.trian.rectil. — on rectilinear triangles, prop. 10.] Wherefore, if again it be made:

As the whole sine of the right angle B is to the hypotenuse DC, just found, so the sine of angle BCD (the complement of the larger observed angle) is to DB.

The straight line DC [?] will again turn out known, and so on. [The proportion yields DB; the concluding text reads "DC" — see anomaly note.]

4. But now, if the width of the mouth AM, or of the bottom BC, is known (and it will not be difficult to measure it by some known measure), we shall arrive more easily at knowledge of the height or depth AB, namely through a single station made at D, in this way. Let it be made:

As the whole sine, CB, is to BD, the tangent of angle BCD (the complement of the angle of observation), so the known width CB is to DB.

For the number produced will offer DB as known, as above, and so on.

Or, by sines alone [Margin: 10.trian.rectil. — on rectilinear triangles, prop. 10.], if it be made:

As the sine of angle BDC (of observation) is to the known width BC, so the sine of angle BCD (the complement of the angle of observation) is to DB.

DB will again be found, and so on.

5. To obtain the same thing without the aid of numbers, consult what we wrote in Problem 4, No. 4.

To examine, by means of the Quadrant, the depth of a valley and its oblique descent — provided it is not very uneven — and its terminus, or some mark that can be sighted in it.

PROBLEMA XXIV.

1. This too is nothing other than measuring some height from its highest summit by two stations, as was shown in Problem 4. For let there be a valley set between two mountains AB and FG, and let its terminus C be sightable from the mountain AB. With some staff AE set up, make two stations at D and E, and observe the angles at D and E when the terminus C is sighted along the rays DC, EC. Now let the straight line EA be understood as extended down to the base of the mountain at B; and the line AF running across, parallel to BG or perpendicular to EB; and finally CH parallel to AB [Margin: 34.prim. — Euclid I.34.], which will be equal to the height AB, so that AB (or HC) is the depth of the valley from A down to the base of the mountain, and IC its oblique descent. It is clearly established that the depth AB, or CH,

Diagram. A tower standing on a mountain slope, illustrating Problem XXII. At the upper left is a battlemented tower whose summit is A and whose shaft descends to the base B. To the right of the tower a vertical staff rises to G (the staff CG). The construction shows two sighting stations lower on the slope and a network of straight lines running from the summit down to the foot of the figure at D.

- A (top of tower): summit sighted
- G (upper right, top of vertical staff): head of the staff CG
- M (mid, on the right face): point on the ray HEM
- I (left, at a small right-angle mark): foot of the perpendicular FI / upper sighting point
- B (below tower, right-angle mark): base of the tower
- F (right): upper sighting station
- E (right, below F): surveyor's eye

- K (left, below B): foot of the perpendicular EK
- N (right of K): point on line EN
- C (center, on the slope, right-angle mark): surveyor's foot-position
- L (lower left): corner of the base construction
- H (right, lower): head of the rod DH
- D (bottom vertex): foot of the sight-line construction

Diagram. A well or upright cylindrical building seen in perspective (drawn as a cylinder with an elliptical mouth and base), illustrating Problem XXIII. Along the left edge a vertical staff carries the stacked points E, G, D, F, A from top down; a quarter-arc with a right-angle mark is at the top right near I; oblique sight-lines run from the staff down into the bottom of the vessel at C.

- E (top of the staff): upper end of hasta AE
- G (below E): equal-sine point (EG)
- D (below G): upper eye-station
- F (below D): equal-sine point (DF)
- A (left, at the mouth level): mouth of the well / top of the building
- I (upper right, at a right-angle mark): foot of perpendicular GI / tangent point
- K (mid): intersection point (DL cutting FH)
- H (mid, at a right-angle mark): foot of perpendicular FH
- M (right rim): far edge of the well-mouth (width AM)
- B (bottom left): bottom corner of the well
- L (bottom middle): point on the base
- C (bottom right): angle / mark at the bottom of the well

turris AB, can be found out, as in Problem 4 — the height of the tower AB from two stations made on the staff AE, or in two windows, or certainly as in the preceding problem the depth of the well AB was found.

The oblique descent IC will become known thus. Since, as was shown in the preceding problem, the angle DCE is the difference of the observed angles BDC, BEC [Margin: 10.trian.rectil. — on rectilinear triangles, prop. 10.], if it be made:

As the sine of angle DCE (the difference between the observed angles) is to DE (the difference of the stations of the eyes), so the sine of angle E, the smaller observed, is to DC.

The straight line DC will become known; and if DI is subtracted from it — which you can easily measure from the eye down to the line AI, since it is small — the oblique descent IC will remain known.

2. If the terminus C cannot be discerned at the bottom, one must sight from D and E some other mark K in the valley, and observe the angles BDK, BEK along the rays DK, EK. For from these the depth AB, or KL, will again be found, as we taught in Problem 4.

Moreover, if two stations can conveniently be made on the floor of the valley, the height of the mountain AB, or IM, can be learned from them, by what we wrote in Problem 2; and the oblique descent IC — that is, the interval between I and C — from what was handed down in Problem 7.

3. Finally, one may also examine this same depth CH from the higher mountain NG, provided that the lowest terminus C of the smaller mountain appears from the summit N, or some other mark in the valley; in no other way than we taught in Problem 18 or 19 for tracking down a smaller height from a larger unknown one. For here the larger height is NG and the smaller CH, whose terminus C we have supposed can be discerned from N.
4. The matter will be carried through without multiplication or division of numbers, as was said in the foregoing.

Diagram. A valley between two mountains rendered as a woodcut (rocky, hatched masses), illustrating Problem XXIV. On the left is the smaller mountain AB with the vertical staff AE (stations D and E) at its summit; a horizontal line AF runs across from the summit A toward the right. On the right is the higher mountain NG, its peak at N. Dotted sight-lines run down from the staff to the valley floor, where the terminus C and a further mark K lie.

- E (top left, head of staff): upper end of hasta AE
- D (below E): upper eye-station
- A (below D): summit of the smaller mountain AB
- H (middle, on the horizontal line): point where CH meets the line AF
- L (right of H): point on the horizontal line
- F (right end of the horizontal line): point on the right slope (AF parallel to BG)
- N (top right, peak): summit of the higher mountain NG
- G (bottom right): base of the higher mountain
- B (bottom left): base of the smaller mountain

- M (bottom, right of B): foot-point in the valley (IM)
- C (bottom middle): terminus / floor of the valley
- K (bottom, right of C): auxiliary mark in the valley

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GEOMETRIAE PRACTICAE LIBER III

Carrying out the measurement of these same straight lines by means of the Geometric Square.

Since the measurement of straight lines by means of the Astronomical Quadrant, set out in the previous book, requires tables of sines, tangents, and secants — and we cannot always have such tables at hand, nor is everyone versed and practiced in them — my aim in this third book is to measure straight lines (that is, lengths, breadths, heights, and depths) by means of the Geometric Square, where we do not need those tables but accomplish everything by the direct and the turned shadow [umbra recta et versa], as they are called. In this matter I shall indeed differ somewhat from other writers, since I divide both the direct and the turned shadow into parts differently than is usually done, so that measurements may be carried out more expeditiously by my division — as a prudent reader will readily judge if he compares my method of division with the partition of others. But to begin with, the Geometric Square must be constructed, and it must be explained how each shadow, the direct and the turned, is to be considered both in the fixed square and in the pendant one. For the aforesaid shadows do not always keep the same position on the instrument, but, for the variety of uses, are not infrequently interchanged, as will be clearly established from what follows.

[Margin: Construction of the Geometric Square.]

[Margin: Which is the direct and the turned shadow in the square, and into how many parts each is divided by geometers. Into how many parts each shadow is divided in our square.]

[Margin: What the altimetric scale is.]

[Margin: Why our division of the shadow is to be preferred to the division of others.]

Construction of the Geometric Square.

1. From any solid and hard material let a square ABCD be made — whether solid throughout or hollowed out — or, better, assembled from four equal rulers AB, BC, CD, DA, so fitted together that all lie in one and the same plane. Then on the two rulers BC and CD let three parallels be drawn to the extremities of the square, to mark out the parts and numbers of each shadow, as appears in the figure. The side BC is assigned by geometers to the direct shadow, and CD to the turned shadow. All who have written on the use of the Geometric Square are accustomed to divide each into 12 equal parts; and if the size of the instrument permits, they subdivide each part into 60, so that the whole shadow comprises 720 parts, or into 100, so that there are 1200 parts in each shadow. I, however, divide each shadow into just 10 equal parts — unless the instrument is of such size that each of the two sides BC, CD can conveniently be cut into 100 or 1000 parts, that is, by subdividing each tenth part into 10 or 100 smaller parts. The figure composed of the two shadows is usually called by geometers the altimetric scale [scala altimetra].

Diagram. The Geometric Square (an altimetric-scale instrument). A square frame is shown with its four corners lettered. From the upper-left corner an index-bar (dioptra/alidade) with a pivot ring extends across the field; a plumb-line hangs from the same corner.

- A (upper-left corner): center of the instrument, with pivot ring and plumb-line suspension.
 - B (upper-right corner).
 - C (lower-right corner).
 - D (lower-left corner).
 - Right side (B down to C, the direct-shadow scale, umbra recta): graduated 2, 4, 6, 8, 10.
 - Bottom side (D across to C, the turned-shadow scale, umbra versa): graduated 2, 4, 6, 8, 10.
2. I prefer to the customary division into 12, or 720, or 1200 parts my own division into 10, 100, or 1000 equal parts, for this reason: if, on account of its smallness, the instrument is cut into only 10 parts, we can very easily find how many hundredths, or even thousandths, are contained in any given small portion of any tenth of the sides BC, CD — exactly as if each side had been cut into 100 or 1000 parts, as we explained fully in Book I, Chapter 2, Number 14. To this add that, in measuring lines by the

Geometric Square, a multiplication, or a division by all the parts of the side BC or CD, must always be performed —

— BC or CD. Now it is plain that multiplication or division by 10, or 100, or 1000 is easier than by 12, or 720, or 1200, since the former is done merely by appending or dropping a 0, or 00, or 000, as we showed in our *Practical Arithmetic*.

I find, indeed, that the side of the square was also divided — and with excellent judgment — into 100 equal parts by the most learned Giovanni Antonio Magini [Magini]; although he gives no rule by which one may determine how many thousandth parts are contained in any given hundredth, which is nevertheless altogether necessary if measurements and the altitudes of stars are to be observed precisely — especially when, because of the instrument's smallness, the side can conveniently be divided into only 10 parts. And this, as we have said, can be accomplished by our method without great labor.

3. After this, from the center A let there proceed a thread with a plumb-weight, or at least a thin ruler with a line of sight [linea fiduciae], hanging so as to move freely, as we taught in Book I, Chapter 2, Number 6; and on the side AB let there be fixed the two sighting-vanes [pinnacidia] discussed in Book I, Chapter 2, Number 5 — this if the square is to be used as a pendant. But if you want it to be fixed, then about the center A a sighting-rule [dioptra] must be fastened — that is, a ruler with a line of sight and two sighting-vanes, constructed by the same device — so that it can be turned freely and secured in every position, as we prescribed should be done for the fixed Quadrant in the place cited above.

[Margin: The pendant and the fixed square.]

Finally, alongside the side AD, in the plane of the square (for if this were done outside it, near the thickness of the instrument, the square could not be set up in the plane of the horizon erect upon the side AD — something the use of the square not infrequently requires), let there be placed two small perforated plates with a thread and plumb, so that by their help you can tell whether or not the instrument stands perpendicular to the horizon, which is altogether necessary.

4. But let us now teach what position each shadow has in the use of the square; for this is a matter of no small importance, so that no confusion between the shadows may arise in measurements. In the pendant square, then, the turned shadow always lies opposite the side of the sighting-vanes, while the direct shadow meets that same side at a point farther from the center A — as in the figure the side of the direct shadow is BC, and of the

turned shadow CD. But in the fixed square, if a height is to be measured and the center A occupies the lower position, with the eye placed there, the side of the direct shadow will occupy the topmost place, while the side of the turned shadow must incline toward the height itself; so that the base of the instrument is then AD, in which case again the direct shadow is BC and the turned CD. If, however, the center A holds the upper position, and the eye is at the end of the sighting-rule (which can also happen), the side of the direct shadow will be the base CD, and the side of the turned shadow BC, near the eye and farther from the height to be measured.

But if a length is to be measured, and the center A is in the upper position with the eye set there, then again the base CD will be the side of the direct shadow, while the side BC will be reckoned the turned shadow — which must incline toward the length being measured and lie farther from the eye. On the other hand, if the center A is placed in the lower position, so that the base is AD and the eye rests at the end of the sighting-rule (which can also happen, especially when the square is set on high, as on a mountain or some tower), the side of the direct shadow will be BC, opposite to the base AD, while the side of the turned shadow will be CD, which of course lies farther from the length to be measured.

In both squares, moreover, the center A lies diametrically opposite the point where the direct shadow meets the turned. And in the fixed square the direct shadow always occupies either the topmost place — namely when the center A holds the lowest position — or the lowest place, namely when the center A is in the upper position, as is clear from what has been said. These things must be considered carefully, lest in the varying use of the instrument you take the direct shadow for the turned, or the reverse; for, according to the different position of the fixed square, both the side BC and the side CD can serve now as the direct shadow, now as the turned, as we have said.

[Margin: How the direct and the turned shadow are to be recognized in each square.]

5. Although either the direct shadow alone or the turned alone suffices for investigating heights, lengths, and depths — as will become clear from what follows — geometers nevertheless make use of both. The reason is that sometimes the direct shadow exceeds the side BC, namely when the plumb-thread or the line of sight cuts the side CD; for then the side BC must necessarily be extended so that it can be intersected. Likewise the turned shadow often surpasses the side CD, namely when the plumb-thread or line of sight cuts the side BC; for then the side DC, extended toward C, must

necessarily be intersected, as is evident. Therefore, so that we are not forced to extend either side BC or CD, the turned shadow is to be used when the direct exceeds the side BC, and the direct when the turned is greater than its own side CD.

6. The side of the square, which they call the gnomon, is always a mean proportional between the direct and the turned shadow. For let the plumb-thread — in the pendant square — or the line of sight — in the fixed one — cut the shadow-side BC at E, and the shadow-side DC, extended, at F. Then the triangles ABE and ADF will be equiangular, since the angles at B and D are right, [a] and both the alternate angles BAE and DFA, and BEA and DAF, are equal. [b] Hence, as BE, the cut-off shadow, is to the gnomon BA, so the gnomon AD is to the cut-off shadow DF; that is, the gnomon BA (or AD) is a mean proportional between the two shadows BE and DF, of which one is direct and the other turned.

[Margin: The gnomon is a mean proportional between the direct and the turned shadow.] [Margin: a. By *Elements* I.29.] [Margin: b. By *Elements* VI.4.]

Diagram. Two right-triangle figures illustrating the gnomon as mean proportional between the two shadows.

- Left figure: A at the top; D at the upper left; the sight-line runs down from A, cutting the shadow-side at E (center) and continuing to F (bottom); B at mid-right; C at the lower left. Right angles are marked at D and at B. Triangle ADF (with right angle at D) and triangle ABE (with right angle at B) are shown superimposed about A.
 - Right figure: A at the upper left; B at the upper right; D at the lower left; C at the lower center; E on the descending diagonal at mid-right; F at the lower right. The diagonal AEF cuts the direct-shadow side at E and meets the extended turned-shadow side at F; this is the square case with base AD.
7. Hence it is easy to reduce one shadow to the other, which not rarely happens in practice. For if the gnomon, taken as comprising 1000 parts (for it is convenient to conceive the side of the square as divided into that many parts), is multiplied by itself, and the resulting square number 1000000 of the side AB is divided by either shadow, the quotient will indicate the parts of the other shadow. That is, let it be done thus:

As either shadow — is to the gnomon — so the gnomon — to the other shadow;

that is, let the square number of the side of the square, or of the gnomon, namely 1000000, be divided by either shadow. For example, if BE is set as the direct shadow of 700 parts, and the square number 1000000 of the side AB is divided by 700, the turned shadow DF of $1428 \frac{4}{7}$ [?] parts will be produced. Likewise, if BE is taken as the turned shadow of 700 parts, the direct shadow DF of $1428 \frac{4}{7}$ [?] parts will be found. But if one shadow is 400, the other will be 2500; and so for the rest. Now, however, let us come to the use of both squares.

[Margin: Reduction of the direct shadow to the turned, and conversely.]

TO OBSERVE the altitude of the Sun, or of any star, by means of the Geometric Square.

PROBLEMA I.

[Margin: How the altitude of the Sun, or of a star, is found by the square. When the angle which the thread makes with the nearest side of the square gives the altitude of the Sun, and when it gives the complement of the altitude.]

1. Let a base be prepared level with the plane of the horizon, so that a fixed square set up on it stands perpendicular to the horizon. Then let the pendant square be raised, with the center A turned toward the Sun or the star, so that its plane passes through the center of the Sun or star, until the Sun's ray is found to pass through the two holes of the sighting-vanes, or the line of sight sees the star through those same holes. Let the same be done with the fixed square, by setting its side AD, or CD, upon the prepared base and turning it about, so that its plane runs through the center of the Sun or star; and finally by raising the sighting-rule until the Sun's ray passes through the holes of the sighting-vanes, or the line of sight beholds the star through those same holes. This done, let the angle be considered with the greatest care — the angle which the plumb-thread, or the line of sight on the sighting-rule, makes with the nearest side of the square — investigating with great care and diligence, by the method we gave in Book I, Chapter 2, Number 14, how many thousandth parts of the shadow, whether direct or turned, are cut off by the plumb-thread or the line of sight.

For that angle — if indeed the direct shadow is intersected (which in the fixed square occupies either the topmost or the lowest place, but in the pendant is joined to the side of the sighting-vanes, as we said above) — will give, as we shall demonstrate in Number 2, the complement of the altitude of the Sun or star. But

if the thread or sighting-rule intersects the turned shadow, the angle itself will exhibit the altitude.

[Margin: a. By prop. 8 on right-angled triangles.]

We shall find the magnitude of this angle as follows. Let the square — whether pendant or fixed, for the reasoning is the same in both — be ABCD, and let the direct shadow cut off be BE, which is found in thousandth parts of the side BC, as we taught in Book I, Chapter 2, Number 14, even if the side itself is divided into only 10 or 100 parts. Let this shadow BE be, for example, of 850 parts. Since, then, the two sides AB and BE of the right triangle ABE are known — AB being 1000 and BE 850 — [a] if we set up the proportion:

As the side of the square AB, 1000, — is to the whole sine, 100000, — so the side (or shadow) BE, 850, — to the Tangent of the angle BAE,

Diagram. A square with lines radiating from the upper-left corner, illustrating the right triangle ABE.

- A (upper-left corner): vertex of the instrument.
- B (upper-right corner).
- E (on the right side, below B): where the direct shadow is cut off.
- C (on the right side, below E).
- D (lower-left corner).
- F (on the bottom side).
- Lines are drawn from A to E, to C, and to F, together with the diagonal to D.

(which will be done if to the side BE, that is, to 850, two ciphers are appended) the Tangent of the angle BAE will be found, 85000 — which is not found in the table of Tangents; but the next smaller is 84956, to which correspond 40 degrees, 21 minutes. And since the difference between the Tangent found, 85000, and 84956 taken from the table is 44, while the difference between the two nearest Tangents 84956 and 85006 is 50 — to which 1 minute, or 60 seconds, is due, because to the later Tangent 85006 there corresponds one minute more than to the earlier Tangent 84956 — we shall find, by the rule of three, how many seconds correspond to the difference 44. If we say: "If a difference of 50 requires 60 seconds, how many seconds does a difference of 44 require?" — we shall find $52\frac{4}{5}$ seconds, that is, nearly 53 seconds. Therefore the angle BAE contains 40 degrees, 21 minutes, 53 seconds, a little less. The complement, then — namely 49 degrees, 38 minutes, 7 seconds, approximately — will give the altitude of the Sun and star.

And if the shadow DF is the turned shadow, likewise of 850 parts, the angle DAF itself, found to be 40 degrees, 21 minutes, 53 seconds, will be the altitude of the Sun and star. And in this way, if we carefully examine the parts of the shadow with respect to the side BC or CD, 1000, as was set forth in Book I, Chapter 2, Number 14, the altitude of the Sun or star will always be found in degrees, minutes, and seconds.

But since it is troublesome and laborious to seek out seconds through the differences of the Tangents, it will be enough to look for the Tangent of the angle in the table, and, if it is found there, to take the corresponding degrees and minutes; but if it is not found, to take the Tangent — whether smaller or larger — that differs least from the one sought, and so on. By this method, in our example, one should take the Tangent 85006, to which correspond 40 degrees, 22 minutes; the complement then will be 49 degrees, 38 minutes, as before — only 7 seconds are lacking, which are of no importance.

[Margin: Use of the following gnomonic table.]

But to free students from this troublesome computation, I have constructed the following table for each thousandth part of either shadow. In it, if the hundreds are taken at the top and the remaining units at the side, at once, in the common cell, the degrees and minutes for the sought angle will be found — which will indicate the altitude of the Sun or star if the thousandth parts belong to the turned shadow; but if the parts pertain to the direct shadow, the complement of that angle will show the altitude of the Sun or star, as we shall demonstrate a little below.

[Margin: The construction of the gnomonic table is very easy.]

This table will be composed if to each thousandth part of the shadow five ciphers are appended on the right, and from the whole number three ciphers are cast away — which is the same as appending only two ciphers, so as to obtain the Tangents of the angles. You see that this was done above: for to the shadow-parts 850 two ciphers were added, making the Tangent 85000. For by this procedure the cut-off shadow is multiplied by the whole sine 100000 and divided by 1000, as we described in our *Practical Arithmetic*. And if four ciphers are appended to the thousandth parts, the Tangents are obtained in the same way, with the whole sine taken as 10000000.

2. But let us first demonstrate that the angle which the plumb-thread, or the line of sight, makes with the nearest side when it cuts the direct shadow is, in either square, the complement of the altitude of the Sun or star; whereas

the angle which the thread or line of sight makes when it cuts the turned shadow exhibits the altitude itself. Let the pendant square, then, be first ABCD, and let AEF be the quadrant of the vertical circle drawn through the star, such that AE is level with the horizon; and let there be three altitudes of the star: EG, of 45 degrees; EH, greater; and EK, smaller. When, therefore, the side of the sighting-vanes AB coincides with the ray GA, the altitude angle GAE will be half a right angle [semirectus], [a] which is equal to its complement FAG, on account of the equal arcs EG, FG,

[Margin: b. By *Elements* III.27.]

Diagram. Two companion figures for the demonstration of Number 2, each showing a quadrant of a vertical circle with center A, together with the square placed at successive orientations (its corners repeated to show the rotation).

- A (center of each figure): common vertex; radii fan out to the graduated arc.
- Quadrant arc lettered (top to left): F (top), H, G, K, E — with $EG = 45^\circ$, EH the greater altitude, EK the smaller.
- The square is drawn in several positions about A, its corners repeatedly lettered B, C, D (and the pendant point I at the bottom); in the right-hand figure the fiducial/plumb pointer L appears on the right, with the square shown as ABCD (base AD, corner C lower right, I below).

LIBER III

(Geometria Practica, Book III — measuring by the Geometric Square; conclusion of the demonstration of Problem I, followed by the Gnomonic Table.)

equal to the angles EG and FG [marker e] — that is, to the angle that the thread makes with the side AB. Since, then, the diagonal of the square [marker d] bisects its right angles — namely into half-right angles [45°] — the thread will pass through the corner C. And so both CAB and CAD will at that point equal the angle of altitude EAG.

When, however, the edge of the sight-vanes [pinnacidia] coincides with the ray HA, the angle of altitude EAH will be greater than a half-right angle, and the complementary angle FAH less than a half-right angle [marker e] — and the angle IAB is equal to it. The thread will therefore cut across the direct shadow [umbra recta], and will make the complement-of-altitude angle IAB less than a half-right angle. And finally, when the side AB is brought into line with the ray KA, the

angle of altitude EAK will be less than a half-right angle, and the complementary angle FAK greater than a half-right angle [marker f] — again equal to the angle IAB. So the thread will break off across the reversed shadow [umbra versa], and will establish the angle of altitude IAD as less than a half-right angle.

Exactly the same thing is seen in the fixed square ABCD, in which the side carrying the direct shadow is the lowest, CD, while the center A occupies the highest position — as was said above in the construction of the Square, at No. 4. Here again it is clear that, when the altitude EG is 45° , the ray GA falls on the opposite corner C, and hence both the angle CAB and the angle CAD equal the angle of altitude — namely a half-right angle. But when the altitude EH is greater than 45° , the angle IAD — which the fiducial line [linea fiduciae] AI, cutting off the direct shadow DI, makes with the side AD — is [marker g] equal to the angle FAH, which is the complement of the altitude angle EAH. And finally, when the altitude EK is less than 45° , the angle IAB — which the fiducial line AI, rising along the reversed shadow BI, forms with the side AB — is [marker h] equal to the altitude angle EAK itself. All of which was to be demonstrated.

But here for you is the table I spoke of, which gives the degrees and minutes of the angles that the plumb-thread — or the fiducial line — makes with the nearest side of the square, for every one of the thousand parts of each of the two shadows. In it you see that the angle taken at part 800 along the top edge, opposite part 50, contains about $40^\circ 22'$, as we said above; and this will be the angle of altitude if the 850 parts belong to the reversed shadow. Its complement, about $49^\circ 38'$, will give the altitude if those same parts have been cut off from the direct shadow.

This table, moreover, may be called *Gnomonic*, because in the fixed square the angles it contains are produced by the gnomon AD (or AB) together with the fiducial line, as is plain.

[Margin (opposite the opening lines): The plumb-thread, when it cuts the direct shadow, produces the angle of the complement of the altitude; but when it cuts the reversed shadow, it establishes the angle of the altitude itself.]

[Margin (citation keys for the demonstration): c) *Elements* I.15. — d) scholium on *Elements* I.34. — e) *Elements* I.15. — f) *Elements* I.15. — g) *Elements* I.15. — h) *Elements* I.15.]

[Margin: Why the table is called "gnomonic."]

P		1										
ar		0			30	40			60	70	80	
ts	0	0	200		0	0	500		0	0	0	900
0	0	5	11 19		16	21	26 34		30	35	38	41 59
	0	4			42	48			58	0	40	
		3										
1	0	5	11 22		16	21	26 37		31	35	38	42 1
	3	4			45	51			0	2	42	
		6										
2	0	5	11 25		16	21	26 39		31	35	38	42 3
	7	4			48	54			3	4	44	
		9										
3	0	5	11 29		16	21	26 42		31	35	38	42 5
	1	5			51	57			5	6	46	
	0	3										
4	0	5	11 32		16	22	26 45		31	35	38	24 7 [recte 42
	1	5			55	0			8	9	48	7]
	4	6										
5	0	6	11 35		16	22	26 48		31	35	38	24 9 [recte 42
	1	0			58	3			10	11	50	9]
	7											
6	0	6	11 38		17	22	26 50		31	35	38	24 11 [recte 42
	2	3			1	6			13	13	52	11]

P ar ts	1 0 0	200	30 0	40 0	500	60 0	70 0	80 0	900
1									
7	0 6	11 42	17 4	22 9	26 53	31 15	35 16	38 54	24 12 [recte 42]
4									
8	0 6	11 45	17 7	22 12	26 56	31 18	35 18	38 56	24 14 [recte 42]
2	1								
8	0								
9	0 6	11 48	17 10	22 15	26 59	31 20	35 20	38 58	24 16 [recte 42]
3	1								
1	3								
10	0 6	11 52	17 13	22 18	27 1	31 23	35 22	39 0	42 18
3	1								
4	7								
11	0 6	11 55	17 17	22 21	27 4	31 26	35 25	39 3	41 20 [recte 42]
3	2								
8	0								
12	0 6	11 58	17 20	22 24	27 7	31 29	35 27	39 5	42 22
4	2								
1	3								
13	0 6	12 1	17 23	22 26	27 9	31 31	35 29	39 7	42 24
4	2								
5	7								
14	0 6	12 5	17 26	22 29	27 12	31 34	35 32	39 9	42 26
4	3								
8	0								
15	0 6	12 8	17 29	22 32	27 15	31 36	35 34	39 11	42 28
5	3								
2	4								
16	0 6	12 11	17 32	22 35	27 18	31 38	35 36	39 13	42 29
5	3								
5	7								
17	0 6	12 15	17 35	22 38	27 20	31 40	35 38	39 15	42 31
5	4								

P	1												
ar	0	0	200		30	40			60	70	80		
ts	0	0	200		0	0	500		0	0	0	900	
	8	0											
18	1	6	12 18		17	22	27 23		31	35	39	42 33	
	2	4			38	41			43	41	17		
		4											
19	1	6	12 21		17	22	27 26		31	35	39	42 35	
	5	4			42	44			45	43	19		
		7											
20	1	6	12 24		17	22	27 28		31	35	39	42 37	
	9	5			45	47			48	45	21		
		1											
21	1	6	12 28		17	22	27 31		31	35	39	42 39	
	1	5			48	50			50	48	23		
	2	4											
22	1	6	12 31		17	22	27 34		31	35	39	42 41	
	1	5			51	53			53	50	25		
	6	7											
23	1	7	12 34		17	22	27 37		31	35	39	42 42	
	1	1			54	56			55	52	27		
	9												
24	1	7	12 38		17	22	27 39		31	35	39	42 44	
	2	4			57	59			58	54	29		
		2											
25	1	7	12 41		18	23	27 42		32	35	39	42 46	
	2	8			0	2			0	57	31		
		6											
26	1	7	12 44		18	23	27 45		32	35	39	42 48	
	2	1			3	4			3	59	33		
	9	1											
27	1	7	12 47		18	23	27 47		32	36	39	42 50	
	3	1			6	7			5	1	35		
	3	4											
28	1	7	12 51		18	23	27 50		32	36	39	42 52	
	3	1			10	10			8	3	37		

P	1												
ar	0	0	200		30	40		60	70	80			
ts	0	0	200		0	0	500	0	0	0	900		
	6	8											
29	1	7	12 54		18	23	27 53		32	36	39	42 54	
	4	2			13	13			10	6	40		
	0	1											
30	1	7	12 57		18	23	27 55		32	36	39	42 55	
	4	2			16	16			13	8	42		
	3	4											
31	1	7	13 0		18	23	27 58		32	36	39	42 57	
	4	2			19	19			15	10	44		
	7	8											
32	1	7	13 4		18	23	28 1		32	36	39	42 59	
	5	3			22	22			18	12	46		
	0	1											
33	1	7	13 7		18	23	28 3		32	36	39	43 1	
	5	3			25	25			20	14	48		
	3	5											
34	1	7	13 10		18	23	28 6		32	36	39	43 3	
	5	3			28	28			22	17	50		
	7	8											
35	2	7	13 13		18	23	28 9		32	36	39	43 5	
	0	4			31	31			25	19	52		
	1												
36	2	7	13 17		18	23	28 11		32	36	39	43 6	
	4	4			34	33			27	21	54		
	5												
37	2	7	13 20		18	23	28 14		32	36	39	43 8	
	7	4			37	36			30	23	56		
	8												
38	2	7	23 23 [recte 13		18	23	28 17		32	36	39	43 10	
	1	5	23]		41	39			32	26	58		
	1	1											
39	2	7	13 27		18	23	28 19		32	36	40	43 12	
	1	5			44	42			35	28	0		

P ar ts	1 0 0	200	30 0	40 0	500	60 0	70 0	80 0	900
	4	5							
40	2	7	13	30	18 23 28 22	32	36	40	43 14
	1	5			47 45	37	30	2	
	7	8							
41	2	8	13	33	18 23 28 25	32	36	40	43 16
	2	2			50 48	40	32	4	
	1								
42	2	8	13	36	18 23 28 27	32	36	40	43 17
	2	5			53 51	42	35	6	
	4								
43	2	8	13	40	18 23 20 30 [recte 28	32	36	40	43 19
	2	8			56 54 30]	44	37	8	
	8								
44	2	8	13	43	18 23 28 33	32	36	40	43 21
	3	1			59 56	47	39	10	
	1	2							
45	2	8	13	46	19 23 28 35	32	36	40	43 23
	3	1			2 59	49	41	12	
	5	5							
46	2	8	13	49	19 24 28 38	32	36	40	43 25
	3	1			5 2	52	43	14	
	8	8							
47	2	8	13	52	19 24 28 41	32	36	40	43 26
	4	2			8 5	54	46	16	
	1	2							
48	2	8	13	56	19 24 28 43	32	36	40	43 28
	4	2			11 8	57	48	18	
	5	5							
49	2	8	13	59	19 24 28 46	32	36	40	43 30
	4	2			14 11	59	50	20	
	8	8							
50	2	8	14	2	19 24 28 49	33	36	40	43 32
	5	3			17 14	1	52	22	

Part	s	0	100	200	300	400	500	600	700	800	900
		40		47		53	25	35	23	50	57
65	3	6 22	[recte 9 22]	14	20 3	24	29	33	37	40	43
		43		51		56	28	37	25	52	59
66	3	9 26		14	20 6	24	29	33	37	40	44 1
		47		54		59	31	40	27	54	
67	3	9 29		14	20 9	25 2	29	33	37	40	44 2
		50		57			33	42	29	56	
68	3	9 32		15 0	20	25 5	29	33	37	40	44 4
		53			12		36	45	31	57	
69	3	9 36		15 3	20	25 8	29	33	37	40	44 6
		57			15		38	47	34	59	
70	4 0	9 39		15 7	20	25	29	33	37	41 1	44 8
					18	10	41	49	36		
71	4 4	9 42		15	20	25	29	33	37	41 3	44 9
				10	21	13	44	52	38		
72	4 7	9 46		15	20	25	29	33	37	41 5	44
				13	24	16	46	54	40		11
73	4	9 49		15	20	25	29	33	37	41 7	44
		11		16	27	19	49	56	42		13
74	4	9 52		15	20	25	29	33	37	41 9	44
		14		19	30	22	51	59	44		15
75	4	9 56		15	20	25	29	34 1	37	41	44
		17		23	33	24	54		47	11	16
76	4	9 59		15	20	25	29	34 4	37	41	44
		21		26	36	27	57		49	13	18
77	4	10 2		15	20	25	29	34 6	37	41	44
		24		29	39	30	59		51	15	20
78	4	10 6		15	20	25	30 2	34 8	37	41	44
		28		32	42	33			53	17	22
79	4	10 9		15	20	25	30 4	34	37	41	44
		31		35	45	36		11	55	19	24
80	4	10 12		15	20	25	30 7	34	37	41	44
		34		39	48	38		13	57	21	25

Part	s	0	100	200	300	400	500	600	700	800	900				
81	4	10	16	15	20	25	30	9	34	37	41	44			
	38			42	51	41		15	59	23	27				
82	4	10	19	15	20	25	30		34	38	2	41	44		
	41			45	54	44	12	18		25	29				
83	4	10	22	15	20	25	30		34	38	4	41	44		
	45			48	57	47	15	20		27	31				
84	4	10	26	15	21	0	25	30		34	38	6	41	44	
	48			51		50	17	22		29	32				
85	4	10	29	15	21	3	25	30		34	38	8	41	44	
	52			54		52	20	25		31	34				
86	4	10	32	15	21	6	25	30		34	38		41	44	
	55			58		55	22	27	10	32	36				
87	4	10	36	16	1	21	9	25	30		34	38	41	44	
	58					58	25	29	12	34	37				
88	5	2	10	39	16	4	21	26	1	30		34	38	41	44
						12		27	32	14	36	39			
89	5	5	10	42	16	7	21	26	4	30		34	38	41	44
						15		30	34	16	38	41			
90	5	9	10	45	16		21	26	6	30		34	38	41	44
					10	18		32	36	19	40	43			
91	5		10	49	16		21	26	9	30		34	38	41	44
	12				14	21		35	39	21	42	44			
92	5		10	52	16		21	26		30		34	38	41	44
	15				17	24	12	38	41	23	44	46			
93	5		10	55	16		21	26		30		34	38	41	44
	19				20	27	15	40	43	25	46	48			
94	5		10	59	16		21	26		30		34	38	41	44
	22				23	30	17	43	46	27	48	50			
95	5		11	2	16		21	26		30		34	38	41	44
	26				26	33	20	45	48	29	50	51			
96	5		11	5	16		21	26		30		34	38	41	44
	29				29	36	23	48	50	31	52	53			
97	5		11	9	16		21	26		30		34	38	41	44

Part	s	0	100	200	300	400	500	600	700	800	900
		32		32	39	26	50	53	33	54	55
98	5	11	12	16	21	26	30	34	38	41	44
		36		36	42	28	53	55	35	55	57
99	5	11	15	16	21	26	30	34	38	41	44
		39		39	45	31	55	57	37	57	58
100	5	11	19	16	21	26	30	35 0	38	41	45 0
		43		42	48	34	58		40	59	

(continuing §3)

3. Georg Peurbach [Georgius Purbachius] likewise composed a similar gnomonic table in his geometric square, taking its side as 1200 parts, and extended it to seconds — as did Giovanni Antonio Magini [Io. Antonius Maginus], who set the side of his square at 1000 parts. This I noticed only rather late: I happened upon Magini's table by chance when I had nearly finished my own, though otherwise I would have spared myself this labor of computation and simply transferred his table here. And so, if anyone wants seconds as well in the altitudes of the stars, he will have to take them from Magini's table — which he computed faithfully and accurately — or else derive the same values by calculation, as I showed above at no. 1 of this problem. But in my own judgment we may be content with this table of mine, which advances only to minutes beyond the whole degrees. If anyone cares to compare it with Magini's, he will find that in mine one minute has always been added to the minutes of the degrees whenever more than 30 seconds are found in Magini's.

[Margin: That the gnomonic table is not absolutely necessary; and in what way the thousandth parts of the square's side yield the tangents, when the whole sine is taken as 100000 or 10000000.]

Yet, to tell the truth, neither mine nor Magini's is strictly necessary. For the thousandth parts of the square's side, if two ciphers are appended to them, are themselves the tangents of the altitudes with respect to a whole sine of 100000; and the same parts, if four ciphers are added, give those same tangents when the whole sine is taken as 10000000. Hence the altitudes can be drawn from a table of tangents to match any thousandth part whatever, provided two or four ciphers are first joined to it — as I showed above at the end of no. 1. Still, since it is not without trouble to extract the altitude-angles from a table of tangents by means of tangents formed in this way (because those tangents are very rarely found

exactly in the table, and one must therefore accept in their place the nearest smaller or larger values — those, that is, which differ from the ones sought by the fewest units), it will not be amiss to adopt either this gnomonic table of mine, or Magini's, whenever seconds too are wanted.

[Margin: No great error is made in the altitudes even though the table proceeds by whole thousandths; and how this error is to be corrected.]

4. Although the gnomonic table advances only by whole thousandth parts, nevertheless, if at any time — following the doctrine given in Book I, chapter 2, no. 14 — some small fraction remains over and above the whole thousandths, no great error can be committed in observing the altitudes of the stars. For since the angles in the table increase in order by only three minutes, or two, or one, no error will arise beyond one minute or another, even if that fraction of a thousandth part is neglected. But if you wish to avoid this error, small as it is, consider what the fraction of the thousandth is — whether it is a third part, a half, or indeed two-thirds — which remains beyond the tenth parts traversed with the compass. For where the angles in the table increase by three minutes, one minute or two must be added, according as that remaining fraction is $\frac{1}{3}$ or $\frac{2}{3}$. The same must be done if the said fraction is greater than $\frac{1}{3}$ but still less than a half: then too one minute must be added. Likewise, when that fraction is greater than a half, two minutes may be added. But when the table's angles increase by two minutes, one minute must be added for the half of a single thousandth part, and so on.

To find out, by means of the geometric square, the distance between yourself and any mark whatever set in the plane of the horizon.

PROBLEMA II.

1. Let FG be the distance to be measured. With a pendant square it is done as follows. From F erect some known altitude FA at right angles to the horizon — whether it be the stature of the surveyor from the eye to the ground, or some greater height. Sight the far point G along the ray AG, which passes from the eye at A through the openings of the sight-vanes [pinnacidia], while the plumb-line hangs freely and grazes the instrument; and mark the point E where the thread crosses a side of the square. This will fall either on the side BC of the right shadow [umbra recta], or on the corner C, or on the side DC of the reversed shadow [umbra versa].

Diagram. Three figures for the pendant geometric square, set side by side, illustrating the three cases where the plumb-thread crosses the square. In each the observer's eye is at the apex **A** atop a vertical **AF**, sighting along the hypotenuse to the distant point **G** at the far end of the horizontal base; the square hangs from **A** as a tilted quadrilateral, and the plumb-thread crosses it at **E**.

- **A** (apex, top): the eye / center of the square
- **D** (upper-left of the hanging square): corner
- **B** (upper-right of the hanging square): corner
- **C** (lower-left / corner of the square): the shared corner of right and reversed shadow
- **E** (interior): point where the plumb-thread crosses a side of the square
- **F** (bottom-left): foot of the vertical altitude
- **G** (bottom-right): the far mark on the horizon

In figures 1 and 2 the thread falls on the right-shadow side **BC** (**E** lying between **B** and **C**); in figure 3 the thread falls on the reversed-shadow side **DC**, with **E** and **D** on the left.

For when the thread falls on the right shadow, or on the point **C**, as in figures 1 and 2, triangle **ABE** becomes equiangular with triangle **AFG**, since the angles **B** and **F** are right and angle **A** is common. [Margin: a. Elements VI.4.] Therefore, if it be made:

As the side **AB**, of 1000 parts, is to the cut-off right shadow **BE**, so the known altitude **AF** is to the distance **FG**,

the sought distance **FG** will come out known, in parts of the altitude **AF**.

But when the point **E** falls on the reversed shadow, as in figure 3, triangle **ADE** is again cut off, equiangular with triangle **AFG**, since the angles **D** and **F** are right [Margin: b. Elements I.29.] and the alternate angles **AED**, **FAG** are equal. [Margin: c. Elements VI.4.] Therefore, if it be made:

As the cut-off reversed shadow **DE** is to the side **DA**, of 1000 parts, so the known altitude **AF** is to the distance **FG**,

the sought distance **FG** will likewise be made known, in parts of the altitude **AF**. From this you can understand that when the point **E** — where the thread crosses the side of the square — falls on the right shadow, the altitude **AF** is greater than the distance **FG**; when it falls on the point **C**, they are equal; and when it falls on the reversed shadow, the altitude is less.

2. With a fixed square [quadratum stabile] we shall explore the same distance as follows. At the near end E of the distance erect some perpendicular height of known parts, and adjust the square so that the center A occupies the upper position and the right shadow DC lies parallel to the horizon. Then, with the dioptra directed to the proposed mark, carefully note the cut-off shadow. If it is a right shadow — DI, say — the altitude AE will be greater than the distance EG, [Margin: d. Corollary to Elements VI.4.] and triangle ADI will be similar to triangle AEG. [Margin: e. Elements VI.4.] Therefore, if it be made:

As the side of the square AD, 1000, is to the parts of the right shadow DI,
so the known altitude AE is to the distance EG,

the sought distance EG will be found, in parts of the assumed altitude AE.

But if the dioptra passes through C, the altitude AE will be equal to the sought distance EH; [Margin: c. Elements VI.4.] for as AD is to DC (its equal), so is AE to EH.

If, finally, the cut-off shadow is a reversed one — BK, say — the altitude AE will be less than the distance EF, which usually happens in measuring distances; and triangle ABK will be equiangular with triangle AEF, since the angles B and E are right [Margin: d. Elements I.29.] and the alternate angles BAK, AFE are equal. [Margin: e. Elements VI.4.] Therefore, if it be made:

As the parts of the reversed shadow BK are to the side of the square AB,
1000, so the known altitude AE is to the distance EF,

the sought distance EF will be produced, in parts of the erect altitude AE.

Diagram. A single figure for the fixed geometric square. The square stands at the upper left with the center-corner A uppermost; from A several rays descend to the points G, H, F along the horizontal base line E–G–H–F, and a horizontal dotted sighting line runs from the top of the square out to the distant mark M at the right.

- A (upper-left, center of the square): the sighting point
- B (top, right of A): upper corner of the square
- N (just below B, on the square's edge): point where a sight-line meets the reversed-shadow side [printed as a reversed "N"]
- K (below N): point where a cut-off reversed shadow BK meets the side
- D (left side, below A): corner of the square
- I (interior/right side of square): point of a right shadow DI

- C (interior center): corner between right and reversed shadow
- L (interior, right of C): point on a descending ray
- E (bottom-left): foot of the erect altitude
- G, H (on the base line): points where descending rays meet the base
- F (bottom-right): base point
- M (far right, on the dotted horizontal): the distant mark being sighted

3. When the distance is not very great, or its far point can be easily seen, it will suffice to set the square above the plane of the horizon so that the reversed-shadow side BC points straight toward that point. Thus, if the horizontal distance DM is to be measured, and an erect square is placed upon it, the dioptra must be moved until the line of sight [*linea fiduciae*] is directed to the far point M. In this way a reversed shadow BC will always be cut off. For if the line of sight were to pass through C, or to cut the right shadow CD, the distance would be either equal to the side CD or less, and so would need no measuring. Since, then, triangle NBA is again equiangular with triangle ADM — because of the right angles B and D, [Margin: f. Elements I.29.] and the equal alternate angles BAN, AMD [Margin: g. Elements VI.4.] — if it be made:

As the parts of the reversed shadow BN are to the side of the square AB, 1000, so the side of the square AD, 1000, is to the distance DM,

the distance DM will be known, in thousandth parts of the square's side AD.

4. Some writers are accustomed not to seek the proposed distance in parts of the assumed altitude AE, nor in thousandth parts of the square's side AD, but only to investigate how many times the chosen altitude AE, or the side of the square AD, is contained in the proposed distance — which comes to the same thing as if the altitude, or the side of the shadow, were set at 1. And so they divide either the cut-off parts of the right shadow by the whole side of 1000 parts, or the whole side by the cut-off parts of the reversed shadow. For the quotient shows how many times the altitude AC, or the side of the square, is contained in the proposed distance, since it holds:

As the whole side AD, of 1000 parts, is to the parts of the right shadow DI, so the altitude AE, or side AD, as 1, is to the distance EG, or DI.

Likewise:

As the parts of the reversed shadow BN are to the whole side AB, 1000, so the altitude AE, or side AD, as 1, is to the distance EF, or DM.

For it follows from this that, since by the precept of the rule of three the third number is to be multiplied into the second and the product divided by the first, it is enough if the second is divided by the first — inasmuch as the unit set in the third place, if it multiplies the second number, produces that same second number, and so on.

By this reasoning, if two thousandth parts are cut off from the reversed shadow, the altitude AE (or side AD) will be contained in the distance according to this number, 500 — for 1000 divided by 2 gives the quotient 500. But a reversed shadow of three thousandths will give the quotient $333\frac{1}{3}$, which differs from the first by this number, $166\frac{2}{3}$. From which one may understand that when the cut-off reversed shadow is very small, a large error can be committed in investigating the distance. For since the thousandth parts are exceedingly tiny, we may easily be deceived — supposing, for instance, that the parts cut off from the reversed shadow are 30 in 1000 [?], when in truth they are 31 in 1000 [?]. The error could then arise in only 1 unit [?] of the altitude AE or the side AD: for 30 in 1000 [?] give the quotient $33\frac{1}{3}$, whereas 31 in 1000 [?] yield the quotient $32\frac{2}{7}$ [?]. And so I rather approve setting the square in a higher place than on the plane of the horizon, because there more parts of the reversed shadow are cut off than here — as is clear in the equal distances EF and DM. You see, then, that great diligence must be applied so that the thousandth parts may be found accurately, by the method I described in Book I, chapter 2, no. 14.

Now, if the square is constructed to some fixed measure — that is, so that its side contains either 1 pace, or 1 cubit, or 2 feet, or palms, or 3 or 4, and so on — the matter will be very easy, if only once the side of 1000 particles is multiplied into the measures to which the side is equivalent. For thus, in lengths, if the square is set above the plane in which the length lies (as happens in all hypotenuses, that is, distances from the surveyor's position up to some point higher or lower, as appeared in the preceding figure when the length DM was sought, and as will appear in the scholium of problem 7 and in the scholium of problem 9; likewise in problem 15, no. 5, and in problem 26, no. 3, and in problem 27, no. 3, and also in problems 37 and 38, no. 3, and finally in problems 43 and 44, and

in other places), the number produced is to be divided only by the noted parts of the reversed shadow (which is always the part cut off). For the quotient will give the sought length in the given measure. For example, if 3 feet are comprised in one side, we multiply the side, 1000, by 3 and produce 3000. And so, if in investigating a length some 200 parts are cut off from the reversed shadow, we divide 3000 by 200. For the quotient 15 gives 15 feet as the sought length. [Margin:

a. Elements VI.4.] For since, in the preceding figure, it holds that as the reversed shadow NB, 200, is to the side BA, 1000, so the side AD of 3 feet is to DM, the side, 1000, must be multiplied by 3 feet, and the resulting number, 3000, divided by 200. The same reasoning holds for the rest. Thus if the side contained $\frac{1}{2}$ cubit, one would divide the number 500 — produced from $\frac{1}{2}$ into 1000 — by the reversed shadow. In like manner, if the side embraced 10 palms, one would divide the number 10000 — got from 10 into 1000 — by the reversed shadow, so that in the quotient there come out the palms contained in the length, and so on. This is to be carefully noted.

ALITER. (*Another way.*)

5. When the distance to be measured is great, and there is a good level field, we shall be able to use the fixed square most conveniently in this way. Let the proposed distance be AE. Erect the square at right angles to the horizon and turn it about until, through its plane, the eye stationed at A sights the mark E, and let the line AG be marked. Then lower the square down to the plane of the horizon, so that the side AB points straight to the mark E — that is, does not depart from the line AG already marked — and let a straight line be extended along the side AD, on which are taken any number of parts equal to the side AD; or any number of other equal measures, such as paces or cubits, up to the point *a*, where the square is again erected. (In this second square we have written small letters, so that the letters of one square may not be confused with those of the other, as we shall also observe in what follows.) And let it be turned until, through its plane, the same mark E appears along the straight line *aH*. After this, let the square lie flat on the horizon, and let the side *ad* coincide with the straight line *Aa*, and let the dioptra be carried about until its line of sight cuts off, as exactly as possible, the thousandth parts in the portion of reversed shadow *bF*. And a reversed shadow will always be cut off, because the distance AE is proposed to be great — at least greater than *Aa* — for otherwise it could be measured without the square, just as *Aa* can. When this is done, the triangles *abF* and *aAE* will be equiangular, since they have right angles at *b* and A [Margin: b. Elements I.29.] and the alternate angles *baF*, *AEa* are equal. [Margin: c. Elements VI.4.] If, then, it be made:

As the reversed shadow *abF* [read: *bF*] is to the side *ba*, 1000, so *aA*, known, is to the distance AE,

the distance AE will be found, in parts of the straight line *Aa*. Or, if the side *ba*, 1000, is divided by the thousandth parts of the reversed shadow *bF*, there will be

produced the number according to which the straight line Aa is contained in the same distance AE , the line Aa being of course set as 1 — as we showed at no. 4.

Diagram. A figure for the two-square method of §5–6. A large triangle rises to the apex E at top; a horizontal dotted sighting line $G\cdots H\cdots I$ runs across the middle, and two small squares stand on the lower base line $A-D-d-a$, the left one lettered with capitals (A, B, C, D) and the right one with small letters.

- E (apex, top): the distant mark being sighted
- G (left end of the dotted sighting line): top corner of the first square's sight-line
- H (middle of the dotted line): point where the second sight-line crosses
- I (right end of the dotted line): sight-point above the second square
- A (bottom-left corner): station of the first square
- B (upper-left square): upper corner
- C (interior of the first square): corner
- D (base, right of A): corner of the first square
- a (bottom-right): station of the second square
- d (base, left of a): corner of the second square
- c (interior of the second square): corner [?]
- b / F (second square, reversed-shadow side): the cut-off reversed shadow bF [minuscule b not clearly resolved [?]]

6. But if we do not have a square at hand, we shall obtain the same distance by means of a rod or reed, by this device. Fix a rod or reed at G , at right angles to the plane of the horizon — which is easily done with a thread and a plumb. Then withdraw through any number of paces up to A , so that the line of sight, running past the rod, is carried to the mark E . After this, draw the line Aa perpendicular to AE , and along it count any number of paces up to a . Finally, having drawn GI perpendicular to AE and equal to Aa , fix a rod again at right angles at such a point of the straight line GI — namely at H — that the eye, once more running past the rod from a , is carried to the mark E ; and let the paces contained in HI , together with the fragments of a single pace, be inquired most exactly. When this is done, since again the triangles Hla and aAE are equiangular — on account of the right angles I and A , and the equal alternate angles IaH, AEa [Margin: d. Elements VI.4.] — if it be made:

As HI , known, is to Ia , known, so aA , known, is to the distance AE ,

the distance AE will be made known, in parts of the straight lines GA, Aa, aI. Or, if aI, known, is divided by IH, known, we shall find how many times Aa is contained in the distance AE, the straight line Aa being of course set as 1.

7. Now we shall solve the problem even without the aid of numbers, if those things are attentively considered which we wrote in Book II, problem 1, no. 7. For let any angle BAC be made, and — for the first example, no. 1 of this problem — let AD be taken equal to the side of the square AB (or, if that be too great, equal to the half, or the third part, or the fourth, and so on, of the same side). Then let DB be taken equal to the cut-off shadow BE (which must be taken with the utmost care by compass upon the square), or to its half, or third part, or fourth, and so on — so that AD and DB may be equally submultiples of the side AB and of the shadow BE, if they are not equal to them. After this, from the instrument of parts, let AE be taken of as many particles as there are palms or feet contained in the altitude AF.

Diagram. A small triangle for the numberless method of §7. The horizontal base AC is graduated with tick marks (the instrument of parts); D lies on the sloping upper line AB, and DE and BC are the two perpendiculars dropped to the base.

- A (bottom-left, vertex of the angle): the station
- B (upper-right, end of the sloping line): the sighted point
- C (bottom-right, on the base): foot of the perpendicular BC
- D (on the sloping line AB): the point cut to AD
- E (on the base, between A and C): foot of the perpendicular DE

For if, through the joined line DE, a parallel BC is drawn, the distance FG will contain as many palms or feet as there are particles of the instrument of parts comprised in the interval EC. Thus if the altitude AF is 5 feet, the distance FG will be 3 feet, and so with the rest. And in this way one must proceed in all other examples — attending carefully, that is, to the first three magnitudes of the rule of three, and so on. Having pointed this out once is enough.

To measure, by means of the geometric square, a distance in a plane taken by two stations made in the same plane, when at its far end some perpendicular height is erected, even if the lowest end of it is not seen.

LIBER III

PROBLEMA III

1. Let AF be the distance to be investigated. Take KL as the plane of the horizon, GL as the erected height [of the object sighted], and AK as the surveyor's stature — or some other known measure. Upon a base parallel to the horizon set up the fixed square, placed so that its side AD, extended in thought, meets the erected height at F at right angles. Sight the summit G through the sights [dioptram], and let its line of sight cut the versed shadow [vmbram versam] CD at E. This always happens when the distance AF is greater than the height FG, [Margin: a — Euclid, *Elements* I.18] because the angle A then becomes smaller than the angle G, and therefore smaller than the half-right angle which the ray sent out through C makes with AD at A.

Then move nearer — or draw back — through the known length Aa, and, having set the square upon the same base, again let the dioptra, directed to the point G, cut the versed shadow cd at H. The versed shadow dH at the nearer station will be greater than the versed shadow DE at the more remote station, [Margin: b — Euclid, *Elements* I.16] because the angle a is greater than the angle A. Cut off dI equal to DE, so that HI is the difference of the versed shadows.

[Margin: c — Euclid, *Elements* VI.4] Now since AD is to DE as AF is to FG, [Margin: d — Euclid, *Elements* VI.16] the rectangle contained by DE and AF will be equal to the rectangle contained by AD and FG. By the same reasoning the rectangle contained by dH and aF will be equal to the rectangle contained by ad and FG, [Margin: e — Euclid, *Elements* VI.4] because ad is likewise to dH as aF is to FG. Since, then, the rectangle under AD, FG is equal to the rectangle under ad, FG — the straight lines AD and ad being equal — the rectangle under dH, aF will also be equal to the rectangle under DE, AF. [Margin: f — Euclid, *Elements* VI.16] And so dH (the first) is to dI (the second) as AF (the third) is to aF (the fourth); and, taken alternately, as the whole dH is to the whole AF, so the subtracted dI is to the subtracted aF. [Margin: g — Euclid, *Elements* V.19] Therefore the remainder HI is

also to the remainder Aa as the whole dH is to the whole AF. Accordingly, if one sets up the proportion:

As HI, the difference of the versed shadows, is to Aa, the difference of the stations, so dH, the versed shadow of the nearer station (that is, the greater), is to AF, the distance,

then AF, the distance, will be produced, expressed in parts of the known difference of the stations Aa.

2. Exactly the same reasoning holds in the hanging square [quadrato pendulo]. For the plumb-thread likewise cuts off triangles ADE (at H) equiangular with the triangles AFG and aFG. This is so because the angles at D and F are right, [Margin: h — Euclid, *Elements* I.29] and the angle F — that is, the alternate angle BAE — is equal to the angle AGF, the external to the internal; while likewise the angles at d and F are right, and the angle H — that is, the alternate angle baH — is equal to the angle aGF, the external to the internal. The rest is demonstrated as before, for the versed shadows in the hanging square are equal to the shadows in the fixed square. For when the two angles D, E in triangle ADE of the hanging square are equal to the two angles D, E in triangle ADE of the fixed square — since those triangles are, as we have shown, equiangular — and the sides AD, AD are likewise equal, [Margin: i — Euclid, *Elements* I.26] then the straight lines DE, DE — that is, the versed shadows — will also be equal. By the same reasoning the versed shadows dH, dH will be equal, and so on.

3. If at each station the straight shadow [ymbra recta] is cut by the line of sight from the plumb-thread, as at E and H — which always happens when the distance AF is less than the height FG, [Margin: k — Euclid, *Elements* I.18] because the angle G is then greater than the half-right angle which the ray sent through C would make with AD — then the straight shadow BE at the more remote station will be greater than the straight shadow bH at the nearer station, [Margin: l — Euclid, *Elements* I.16] because the angle FaG is greater than the angle BAE. Cut off BI equal to bH, so that IE is the difference of the straight shadows. Since the triangles ABE and AFG are equiangular — on account of the right angles at B, F [Margin: m — Euclid, *Elements* I.29] and the equal alternate angles BAE, AGF — [Margin: n — Euclid, *Elements* VI.4] it will follow that AB is to BE as FG is to AF, and, alternately, that AB is to FG as BE is to AF. By the same reasoning, since the triangles abH and aFG are equiangular — on account of the right angles at b, F [Margin: o — Euclid, *Elements* I.29] and the equal alternate angles baH, aGF — [Margin: p — Euclid, *Elements* VI.4] ab will be to bH as FG is to aF, and, alternately, ab will be to FG as bH is to aF. Since, then, AB is to FG as ab is to FG —

because the straight lines AB and ab are equal — [Margin: q — Euclid, *Elements* V.11] it will also follow that the whole BE is to the whole AF as BI (cut off equal to bH) is to the subtracted aF. [Margin: r — Euclid, *Elements* V.19] Therefore the remainder IE is also to the remainder Aa as the whole BE is to the whole AF. And so, if one sets up the proportion:

As IE, the difference of the straight shadows, is to Aa, the difference of the stations, so BE, the straight shadow of the more remote station (that is, the greater), is to AF, the distance,

then AF, the distance, will be produced, expressed in parts of the known difference of the stations Aa.

4. In the hanging square the reasoning is entirely the same. For the plumb-thread likewise cuts off triangles ABE (at H) equiangular with the triangles AFG, aFG. This is so because the angle BAE, the external, is equal to the internal angle AGF, and the angle baH is equal to the angle aGF, the external to the internal, while the angles at b and F are right. The rest is demonstrated as in the fixed square, for the straight shadows in the hanging square are equal to the straight shadows in the fixed square. For when the two angles B, E in triangle ABE of the hanging square are equal to the two angles B, E in triangle ABE of the fixed square — since these triangles are, as has been shown, equiangular, being equiangular with the triangle AGF — [Margin: s — Euclid, *Elements* I.29] [Margin: t — Euclid, *Elements* I.26] the straight lines BE, BE — that is, the straight shadows — will also be equal. By the same reasoning the straight shadows bH, bH will be equal, and so on.

Diagram. A wide surveying figure for measuring the distance AF to a tall object. A horizon line runs along the bottom from K (lower left) through M to L (lower right); the erected height GL rises vertically on the right to the summit G. A long line of sight climbs from the lower left up to G, and along it the geometric square is shown set up at successive stations, each square carrying the shadow-intercepts of the sight-line. The upper horizontal line runs from B (left) to F (right). Readable labels:

- G (top right): summit of the sighted height.
- N, O (upper left): figure/reference letters set above the construction.
- B (far left, upper horizontal) and A (left, below B), K (bottom left): left-hand verticals of the base.
- F (right end of the upper horizontal); L (right end of the horizon line).
- M (on the horizon, below the middle station).

- Along the ascending sight-line and its squares, the corner and intercept letters C, Q, R, D, E, H, I, and repeated c, b, a, d, with a second A and further H — the successive square-and-shadow positions.

Diagram. The companion figure for the case where the distance AF is less than the height FG (Problema III, §3), drawn as a tall right triangle. The vertical GF (with G at the top and F at the foot on the right) is the erected height; the base runs from K/A at lower left to F/L at lower right; the hypotenuse AG is the line of sight. Two geometric-square positions with their straight-shadow intercepts are shown along the base near the left. Readable labels:

- G (apex, top right).
- A, B (left, one above the other); K (bottom left).
- F (lower right, above L); L (bottom right).
- O, P (small reference letters at upper left of the construction).
- Square-corner and intercept letters near the left: D, E, C, I, H, and the second-station letters a, d, b, with a repeated C.

5. Finally, if at the more remote station the versed shadow CD is cut at E, and at the nearer station the straight shadow bc is cut at H, one of the two must be reduced to the other so that similar shadows are obtained. This is done by the method delivered in the construction of the square (No. 7, at the beginning of this book) — namely, by dividing the square of the side AB by whichever shadow is to be reduced, and so on. For if one sets up the proportion: as IN, the difference of the shadows (whether straight or versed), is to Aa, the difference of the stations, so BN, the greater straight shadow, or dN, the greater versed shadow, is to a further term, then the distance AF will be generated, as was demonstrated in Nos. 3 and 1.

6. But if at any time the line of sight in one station passes through C, one may take either the side of the straight shadow BC or of the versed shadow CD, according as in the other station the straight shadow or the versed shadow has been cut off — so that, of course, the shadows are made similar.

Diagram. The figure accompanying §5–6, of the same surveying type: a right triangle whose vertical altitude rises on the right to the summit G, with the base running from K (lower left) through M to F/L (lower right) and the line of sight AG as the hypotenuse. Two geometric-square positions with their shadow-intercepts stand along the base. Readable labels:

- G (apex, top right); P, Q (reference letters at the top).

- B (left), A/K (lower left).
- F, L (lower right); M (on the base, below the second station).
- Interior square and intercept letters: L, C, H, I, E, D, and the second-station lower-case c, b, a, d.

COROLLARIUM I

[Margin: The manner of operating is the same in both squares.] From what has been demonstrated it is gathered that the manner of operating is the same in both squares, since the very same shadows are cut off in the hanging square as in the fixed one, as we have shown. And so the precepts prescribed for the one are to be observed in the other as well.

COROLLARIUM II

[Margin: The manner of operating is the same by versed shadows and by straight shadows.] It is also plain from what has been said that the operation does not vary, whether it is set up by versed shadows or by straight shadows; since it is always the case that, as the difference of the shadows is to the difference of the stations, so the greater shadow is to the distance that is proposed for investigation, as has been demonstrated.

Moreover, in the scholium to Problem 7 we shall prescribe the method by which this same distance AF may be measured from a single station.

To search out the same distance, by means of two stations made at some erected height, with the help of the square.

PROBLEMA IV

1. When two stations cannot conveniently be made on the plane, let some staff Kb be raised, on which the two stations are made — the eye of the surveyor being at A and at a; or else at two windows of some tower, one of which stands directly above the other. First let the side of the versed shadow be cut off at each station, at E and H. The versed shadow DE of the lower station will be greater than the versed shadow dH of the upper station, because the angle A is greater than the angle a — for the angle AGF is smaller than the angle aGM, and the angles F and M are right. Take DI equal to dH.

[Margin: a — Euclid, *Elements* VI.4] Now since AD is to DE as AF is to FG, [Margin: b — Euclid, *Elements* V.11] by the same reasoning it will follow that ad is to aM as dH is to MG. Since, then, the ratio of AD to AF is the same as that of ad to AF — on

account of the equality of the lines AD and AF to aM — [Margin: c — Euclid, *Elements* V.19] it will follow that the whole DE is to the whole FG as dH (that is, DI, cut off) is to MG (cut off). Therefore the remainder IE is also to the remainder FM — that is, to the remainder Aa — as the whole DE is to the whole FG, that is, as AD is to AF; since it has been shown that AD is to AF as DE is to FG. Accordingly, if one sets up the proportion:

As IE, the difference of the versed shadows, is to Aa, the difference of the stations, so AD, the side of the square, is to AF, the distance,

then the distance AF will be plainly gathered, expressed in parts of the difference of the stations Aa.

2. If at each station the side of the straight shadow is intersected, at E and H, then both straight shadows BE, bH must be reduced to versed, by the method delivered at the beginning of this book in the construction of the square (No. 7) — namely, by dividing the square number of the side of the square by each straight shadow cut off, taken separately. For if the sides DC, dc of the versed shadows are produced as far as the rays AG, aG, to the points I and K, then again — as we demonstrated in No. 1 — it will be that IN, the difference of the versed shadows, is to Aa, the difference of the stations, as AD, the side of the square, is to AF, the distance. Therefore, as was shown in No. 1, the distance AF will be found, expressed in parts of the difference of the stations Aa.

Diagram. The figure for Problema IV: a right-angled configuration in which two stations lie one above the other on a vertical staff at the left. The summit G is at top right; the horizontal MG runs to M (mid-right) and the horizontal FG-base runs to F (lower right); K is at bottom left. The upper station's square (with the small square b–H–c) sits high on the left, the lower station's square (B–E–c–I) lower down. Readable labels:

- G (top right); M (mid-right, right end of the upper horizontal); F, L (lower right); K (bottom left).
- Upper square (upper station a): b, H, c, and the point a on the left staff; d in the interior.
- Lower square (station A): B, E, c, I, with A and D at the foot.

ALITER

[Margin: d — Euclid, *Elements* VI.4] Without reducing the straight shadows to versed, we shall elicit the same distance AF in this other way. From OH, the difference of the straight shadows, into the side ab, let P be made [the product];

and from bH, the greater straight shadow, into the lesser BE, let Q be made. And since

(continues on the following page)

LIBER III

since ID is to DA as AB is to BE, [Margin: b — Euclid, *Elements* VI.17] the rectangle under ID, BE will be equal to the square on AD (or AB). In the same way the rectangle under Kd, bH will be equal to the square on ad — that is, to the same square on AD (or AB); and consequently the rectangles under ID, BE and under Kd, bH will be equal to one another. [Margin: c — Euclid, *Elements* VI.16] Therefore ID is to Kd as AD is to BE; and, alternately, ID is to bH as Kd is to BE — that is, the whole ID is to the whole bH as Kd (that is, as DN, cut off) is to BE (that is, to bO, cut off). [Margin: d — Euclid, *Elements* V.19] And so the remainder IN is also to the remainder OH as the whole ID is to the whole bH; and, alternately, IN is to ID as OH is to bH.

But since the ratio of IN to DA (the mean ID being posited) is compounded of the ratios of IN to ID and of ID to DA — and since, as was just shown, IN is to ID as OH is to bH, [Margin: e — Euclid, *Elements* VI.4] and ID is to DA as AB is to BE — the ratio of IN to DA will likewise be compounded of the ratios OH to bH and AB to BE. [Margin: f — Euclid, *Elements* VI.23] But the ratio of the product P to the product Q is also compounded of the same ratios, namely of the sides. Therefore the ratio of P to Q is the same as that of IN to DA. Since, then, as we showed in the first method of No. 2, IN is to Aa as AD is to AF — that is, alternately, IN is to DA as Aa is to AF — it will also follow that P is to Q as Aa is to AF. Wherefore, if one sets up the proportion:

As P, the number produced from OH, the difference of the straight shadows, into ab, the side of the square, is to Q, the number produced from bH, the greater straight shadow, into the lesser BE, so Aa, the difference of the stations, is to AF, the distance,

then the same sought distance AF will be produced, expressed in parts of the difference of the stations Aa.

ALITER

[Margin: g — Euclid, *Elements* VI.4] Since bH is to ab as aM is to MG, if one sets up the proportion:

As bH, the straight shadow, is to ab, the side of the square, 1000, so aM, in so far as it is 1, is to MG,

that is — because 1, multiplying the side of the square 1000, produces the same side 1000 — if the side of the square, 1000, ab, be divided by the straight shadow bH, there will appear the quotient MG, indicating how many times aM (taken as 1) is contained in MG. In the same manner, if one sets up the proportion:

As BE, the straight shadow, is to AB, the side of the square, 1000, so AF, in so far as it is 1, is to FG,

that is, if the side of the square, 1000, AB, be divided by the straight shadow BE, there will result the quotient FG, signifying how many times AF (taken as 1) is contained in FG. If, then, from this latter quotient FG the former quotient MG be subtracted, the remainder will be FM (or Aa), the difference of the stations, known in the parts of which AF is 1. Wherefore, if one sets up the proportion:

As Aa, the difference of the quotients (which arise when the side of the square is divided by each straight shadow), is to Aa, the difference of the stations known in some measure, so AF, taken as 1, is to AF, the distance,

that is, if the difference of the stations be divided by the difference of the quotients, the distance AF will be made known, expressed in parts of the difference of the stations Aa, and so on.

3. If, finally, at the lower station the side of the straight shadow is cut at E, and at the upper station the side of the versed shadow at H, then here too the straight shadow must be reduced to versed, as we have said, and the side DC of the versed shadow must be produced as far as the point N, and DI taken equal to dH. For, as we showed in No. 1, if one sets up the proportion:

As NI, the difference of the versed shadows, is to Aa, the difference of the stations, so AD, the side of the square, is to AF, the distance,

then the distance AF will again be known, expressed in parts of the difference of the stations Aa.

Diagram. The figure for Problema IV, §3 (and the *Aliter* just above), a tall narrow configuration: the summit G is at top right, the vertical falling to F (lower right) and L below it; the base runs left to A/K at the lower left. The two stations stand one above the other on the left, each with its square and shadow-intercepts; the sight-lines aG and AG cross the figure to G. Readable labels:

- G (top right); O, P (reference letters at the top, with a central dot).

- M (mid-right); F, L (lower right); A, K (bottom left); B (left, mid-height); D (foot).
- Interior square and intercept letters: C, H, N, E, C, I, and the second-station lower-case b, a, d.

ALITER

Without reducing the straight shadow to the versed, we shall proceed in this way too. Let the number produced from the straight shadow BE into the versed shadow dH be subtracted from 1,000,000, the square of the side 1000, and let the remainder be O. Likewise, from the straight shadow BE into the side of the square 1000 let P be made. And since — as we showed at the beginning of this book, in the construction of the square, No. 6 — the side of the square is the mean proportional between the shadows BE, DN, [Margin: h — Euclid, *Elements* VI.16] the rectangle under BE, DN will be equal to the square on the side. [Margin: i — Euclid, *Elements* II.2] Since, then, the rectangle under BE, DN is equal to the rectangles under BE, DI and under BE, NI: if the rectangle under BE, DI be subtracted from the rectangle under BE, DN — that is, from 1,000,000, the square of the side AD — there will remain the rectangle under BE, NI. And consequently, since DI is equal to dH, the rectangle under BE, DI will be produced from the shadow BE into the shadow dH; and therefore the remaining rectangle under BE, NI (which, of course, is what is left, if the rectangle under BE, DI ...the segment BE, DI is subtracted from the square of the side, as was said,) it will equal the number O. But by the construction the rectangle contained by BE, the right shadow [umbra recta], and the side AD also equals the number P. Therefore, since the number BE multiplied into NI, AD produces O, P, [Margin: c — Euclid VII.17] O will be to P as NI is to AD. But, as we demonstrated in Num. 1 of this problem, NI, the difference of the turned shadows [umbrae versae], is to Aa, the difference of the stations, as AD, the side of the square, is to AF; and, interchanging, as NI is to AD, so Aa is to AF. Therefore O will likewise be to P as Aa is to AF. And so, if we set up the proportion:

As O — the number that remains when the product of the right shadow into the turned shadow is subtracted from the square of the side — is to P, the number produced from the right shadow BE into the side AD: so Aa, the difference of the stations, is to AF, the distance,

the required distance AF is produced, expressed in the units of the difference of stations Aa.

4. Here too, if in one station the sighting-line [linea fiduciae] or the plumb-thread should happen to pass through the point C, one must take either the side of the right shadow or that of the turned shadow, as was explained in Num. 6 of the preceding problem.

To estimate, by means of the square, the height of any erect object through its distance from the observer's eye.

PROBLEMA V.

1. Let the height to be estimated be GL, or ML, or IL, perpendicular to the horizon. Let the observer's stature, or some other known measure, be FL or AK; and let there be prepared a plane AF parallel to the horizon, on which the square is erected, its side AD being understood as extended to meet the height at F. With the summit I sighted, let the sighting-dioptra or the plumb-thread first cut the side CD at E on the turned shadow — which happens when the distance AF is greater than the height FI, as we showed in Problem 1, Num. 1. Since, then, triangle ADE is equiangular with triangle AFI, as was shown in that same place, [Margin: a — Euclid VI.4] AD will be to DE as AF is to FI. And so, if we set up the proportion:

As the side of the square, 1000, is to the turned shadow DE: so the distance AF (either given, or found by Problem 3 or 4) is to FI,

the height FI will be disclosed, in the units of the distance AF. If AF is not given, it must be sought out by Problem 3 or 4. And if the measure or stature of the observer FL is added to FI, the whole proposed height IL becomes known.

2. Next, let the sighting-line pass through the point C — which happens when the distance AF equals the height FM; [Margin: b — scholium on Euclid I.34] for then the angle CAD is half a right angle, and consequently so is the remaining angle M in triangle FAM. [Margin: c — Euclid I.6] Hence the sides AF, FM are equal. Therefore, however great the distance AF is (whether given, or to be found by Problem 3 or 4), just so great will the height FM be; and with FL added to it, the whole height ML is disclosed.
3. Finally, let the side of the right shadow be cut at H — which occurs when the distance AF is less than the height FG. Then triangle ABH is equiangular with triangle AFG, as we demonstrated in Problem 3, Num. 3. Therefore BH will be to AB as AF is to FG; and so, if we set up the proportion:

As the right shadow BH is to AB, the side of the square, 1000: so the distance AF (either given, or found by Problem 3 or 4) is to FG,

FG becomes known; and if the measure or stature of the observer FL is added to it, the whole proposed height GL is made known.

Diagram. The figure for Problema V. A large right triangle rises from the horizontal base at the left to the apex G at the upper right; on the descending hypotenuse the points M and I are marked, and the right-hand vertical drops through F to L at the base. At the lower left stands the geometric (shadow) square with the three sighting cases superimposed, so its interior letters are densely clustered. Readable labels:

- G (apex, top right): summit of the object
- M (on hypotenuse, upper): case $AF = FM$
- I (on hypotenuse, below M): summit sighted, case $AF > FI$
- F (right, at horizon level); L (bottom right, foot / ground)
- B (top-left corner of the square); A (left, at horizon); K (below A)
- D (bottom edge of the square); E (turned-shadow intersection, case 1)
- H (right-shadow intersection on BC) and C (top-right corner) — the dense H/C cluster at the top-left corner; the figure is lettered entirely in uppercase (G M I F L / B A K H C E D), with no lowercase interior points

To disclose, by the aid of the square, this same height even when its distance from the observer's eye is neither given nor found, by means of two stations made on the plane.

PROBLEMA VI.

1. Let the height to be measured be FG, the observer's stature FL being set aside. Let two stations be made, as in Problem 3 (whose figure 1 is repeated here), and let the turned shadow first be cut, in each station, at the points E, H. And since, by the similarity of triangles ADE, AFG, DE is to AD as FG is to AF, if we set up the proportion:

As DE, the turned shadow cut off, is to AD, the side of the square, 1000: so FG, taken as 1, is to AF —

that is (since 1 multiplying the side 1000 produces that same side 1000), if the side 1000 is divided by the turned shadow DE, there will come out the quotient AF, indicating how many times FG is contained in AF. In the same way, if we set up:

As the turned shadow dH is to a d, the side of the square, 1000: so FG, taken as 1, is to aF —

that is, if the side 1000 is divided by the turned shadow dH, there will be produced the quotient aF, showing how many times FG is contained in aF. If, then, this latter quotient aF is subtracted from the earlier quotient AF, there is left the difference Aa, known in the units of which FG is 1. Therefore, if we set up:

As Aa, the difference of the quotients (got by dividing the side of the square by each turned shadow), is to Aa, the difference of the stations: so FG, taken as 1, is to FG —

the height FG is produced, known in the units of the difference of the stations Aa; and, the observer's stature FL being added, the height GK [recte: GL] becomes known.

ALITER

From HI, the difference of the turned shadows, multiplied into the side of the square, let a number N be generated; and from the turned shadow DE into the turned shadow dH let O be made. With the sides BC, bc produced as far as R, P along the sight-lines, let BQ be cut off equal to bP. And since, as was demonstrated in Problem 3, Num. 3, QR, the difference of the right shadows, is to Aa, the difference of the stations, as the greater right shadow BR is to AF, the distance: [Margin: a — Euclid VI.4] moreover, since BR is to BA as AF is to FG, then, interchanging, as BR is to AF, so BA is to FG; hence QR will also be to Aa as BA is to FG. Now I say that as QR is to AB, so is the number N to the number O. [Margin: b — Euclid VI.4] For since DE is to DA as AB is to BR, [Margin: c — Euclid VI.16] the rectangle contained by DE, BR will equal the rectangle contained by DA, AB — that is, equal to the square of the side. And by the same reasoning the rectangle contained by dH, bP will equal that same square of the side; and therefore the rectangle contained by DE, BR will equal the rectangle contained by dH, bP. [Margin: d — Euclid VI.16] Therefore DE will be to dH as bP is to BR (or to BQ); and, by conversion, as dH is to DE, so BR is to bP, or to BQ; and, interchanging, as the whole dH is to the whole BR, so DE (or dI) taken away is to bP (or to BQ) taken away. [Margin: e — Euclid V.19] Therefore the remainder HI will also be to the remainder QR as the whole dH is to the whole BR, or as the removed dI is to the removed BQ; and, interchanging, as HI is to dI, so QR is to BQ, or to bP. [Margin: f — Euclid VI.23] Now the ratio of the number N to the number O is compounded of the ratios HI to dI (or to DE) and of the side a d to dH, because N was made from HI, the difference of the turned shadows, into the side of the square a d, whereas O was made from the turned shadow DE into the turned shadow dH, by the construction. Since, then, as we showed just above, HI is to dI as QR is to bP, [Margin: g — Euclid VI.4] and likewise the side a d is to dH as bP is to b a, the ratio

N to O will also be compounded of the ratios QR to bP and bP to b a. But out of these same ratios the ratio QR to b a is compounded. Therefore the ratio N to O is the same as QR to b a, or to BA — which is what we set out to show. But QR is to BA as Aa, the difference of the stations, is to the height FG, as we showed above near the beginning of this demonstration. Therefore, if we set up:

As N, the number made from HI (the difference of the turned shadows) into the side of the square 1000, is to O, the number made from the turned shadow DE into the turned shadow dH: so Aa, the difference of the stations, is to FG, the height,

the height FG will be found, in the units of the difference of the stations Aa, and so on.

Diagram. The two-station figure for Problema VI — figure 1 of Problem 3 repeated — captioned with the numbers $N \cdot O$. A large rectangle rises from the base at the left, its left vertical stacking B–A–K; from A a sight-line climbs across two shadow-squares to the summit G at the upper right. Readable labels:

- N, O (the numbers/points naming the construction, at the caption)
- G (apex, upper right)
- Near (left) square top edge, left→right: B (corner), C, Q, R (BC produced to R, with BQ cut off equal to bP)
- Near square interior E; base D
- Far (middle) square top edge: b, c, P (bc produced to P)
- Far square interior H, with I below it; base a, d
- Base: K (bottom-left), M (bottom-centre)
- Case-triangles rise along the hypotenuse to G, with dependent plumb-points (B, A, b, P, H, E, D, C, I) and plumb-bob dots

ALITER

Let each turned shadow be reduced to a right shadow, as we taught at the beginning of this book in the construction of the square, Num. 7. For by means of these right shadows we shall obtain the height FG, as will be shown in Num. 2 following.

2. Then let each right shadow be cut at E, H, as in figure 2 of Problem 3, which is repeated here. And since, by the similarity of triangles ABE, AFG, [Margin: h — Euclid VI.4] BE is to AB as AF is to FG, then, interchanging, as BE is to AF, so AB is to FG. And by the same reasoning bH will be to aF as a

b is to FG; and consequently, as the whole BE is to the whole AF, so the removed bH (or BI) is to the removed aF, since each ratio is that of AB (or a b) to FG.

Therefore the remainder IE will also be to the remainder Aa as the whole BE is to the whole AF — that is, as AB is to FG — since these two ratios are the same as one another, as we demonstrated just above. And so, if we set up:

Diagram. The figure for the second ALITER — figure 2 of Problem 3, repeated. A tall right triangle rises from the horizontal base KML at the left up to the apex G at the top; a single hypotenuse from A climbs through the two station-squares to G, the right-hand vertical dropping through F to L. The two shadow-squares (the near and the far station) carry the same letters primed against one another, so the interior lettering is densely overprinted. Readable anchor labels:

- G (apex, top); F (right, at horizon); L (bottom right)
- B, A, K (lower-left corner stack of the near square); D, M (base)
- lower-case a, d on the base toward the far station
- interior letters of the two squares (all resolved at native 4×4): near square B, I, E, C on the top edge, A mid-left, D base, K bottom-left; the rotated reduced-shadow square along the hypotenuse b, H, C, D; the far square b, H, C on the top edge with base a, d and M at the base centre

**** ...continuing the proportion:

As IE, the difference of the right shadows, is to Aa, the difference of the stations: so AB, the side of the square, 1000, is to FG,

the height FG will leap out, known; and, the observer's stature FL being joined to it, the whole required height GL is made known.

ALITER

Divide the side of the square, 1000, by each right shadow, and multiply the smaller quotient into the difference of the stations, and divide the resulting number by the difference of the quotients. For if this last quotient is multiplied into the greater of the two former quotients, the required height will be produced, in the units of the difference of the stations. This will be made clear as follows.

Let the smaller right shadow bH be, for example, 50, and the greater BE be 125, and let the difference of the stations Aa be 16 paces. [Margin: a — Euclid VI.4] And since bH is to a b as aF, taken as 1, is to FG, then, if a b (1000) is divided by

bH (50), the quotient will indicate 20 — that the right line aF is contained in FG twenty times.

Likewise, [Margin: b — Euclid VI.4] since BE is to AB as AF, taken as 1, is to FG, then, if AB (1000) is divided by BE (125), the quotient will show 8 — that the right line AF is contained in FG eight times. Therefore, since Aa (16 paces) together with aF is contained eight times in FG, it follows that Aa (16 paces) taken eight times, together with aF taken eight times, makes 128 paces plus aF taken eight times — namely, a number equal to FG itself. But AF taken twenty times is equal to that same FG. Therefore the 128 paces together with aF taken eight times are equal to aF taken twenty times. If, then, on both sides aF taken eight times is removed, there remain 128 paces equal to aF taken twelve times. So if 128 is divided by 12, there come out $10\frac{2}{3}$ paces for the right line aF, since this quotient multiplied into 12 produces 128. Since, then, AF is contained twenty times in FG, if aF ($10\frac{2}{3}$ paces) is multiplied into 20, the height FG will be brought forth as $213\frac{1}{3}$ paces. You see, therefore, that the smaller quotient 8 was multiplied into the difference of stations, 16 paces, so as to generate 128; and that this number was divided by 12, the difference of the quotients, so as to yield $10\frac{2}{3}$ paces for the right line aF; and finally that this quotient, $10\frac{2}{3}$, was multiplied into 20, the greater quotient, so as to produce the height FG, $213\frac{1}{3}$ paces — just as we prescribed in the rule. This same height you will also find by the preceding method, if, namely, one sets up: as 75, the difference of the right shadows, is to the difference of the stations (namely to 16 paces), so the side of the square, 1000, is to another. For 16 multiplied into 1000, and the resulting number 16000 divided by 75, gives the quotient $213\frac{1}{3}$, just as before.

3. Finally, let the turned shadow in the more distant station be intersected at E, and the right shadow in the nearer station at H, as in figure 3 of Problem 3. With the turned shadow reduced to a right shadow, as we wrote in the construction of the square, Num. 7, at the beginning of this book: if we set up, as IN, the difference of the right shadows, is to Aa, the difference of the stations, so AB, the side of the square 1000, is to another, the height FG will be produced, just as we showed in Num. 2.

Diagram. The figure for the third case and the following ALITER, captioned with the point-letters **P · Q** at the top. A right triangle rises from the base at the left to the apex G at the upper right; the extended sight-lines carry the auxiliary points P and Q above. Two shadow-squares (the near and far stations) sit along the hypotenuse, their interiors densely lettered. Readable labels:

- P, Q (auxiliary points above, on the produced lines); G (apex, upper right)

- B (top-left of the near square); A, K (left stack); D (base of near square); E (interior)
 - lower-case a, d on the base toward F; F (right, at horizon); L (bottom right); M (base, center)
 - interior letters of the two squares (all resolved at native 4×4): near square top edge B, I, C, N (the point after C is N — matching the ALITER's BN/IN, not a lowercase c), interior E, base D; far square b, H, C, I (interior, printed ".I"), base a, d
4. If perhaps in one station the sighting-line or the plumb-thread should pass through the point C, one would have to take, in the first case, the side CD of the turned shadow; but in the other two cases the side BC of the right shadow.

ALITER

Without reducing the turned shadow to a right one, the procedure will be this. Let the number made from the turned shadow DE into the right shadow bH be subtracted from 1000000, the square of the side 1000, and let the remainder be P. Likewise, from the turned shadow DE into the side 1000 let Q be made. [Margin: c — Euclid VI.4] And since, by the similarity of the triangles, BN is to BA as AF is to FG, then, interchanging, as BN is to AF, so BA is to FG. But as BN, the greater right shadow, is to AF, the distance, so is IN, the difference of the right shadows, to Aa, the difference of the stations, as was said in Num. 5 of Problem 3. [Margin: d — Euclid VI.4] Therefore IN will also be to Aa as AB is to FG; and, interchanging, IN is to AB as Aa is to FG. Now I say that P is to Q as IN is to AB. [Margin: e — Euclid VI.17] For since DE is to AD as AB (or AD) is to BN, the rectangle contained by DE, BN will equal the square of the number 1000000. [Margin: f — Euclid II.1] Moreover, the rectangle contained by DE, BN equals the rectangles contained by DE, BI and by DE, IN. Therefore, if the product of the turned shadow DE into the right shadow bH is taken away from the rectangle contained by DE, BN — that is, from the square 1000000 — there will remain the rectangle contained by DE, IN. And on that account P is made from DE into IN. But Q is made from the same turned shadow DE into the side AB, 1000. Since, then, DE, multiplying IN and AB, produces P and Q, [Margin: g — Euclid VII.17] P will be to Q as IN is to AB — that is, as Aa is to FG. And so, if we set up:

As P — the number that remains when the product of the turned shadow DE into the right shadow bH is subtracted from the square 1000000 — is to Q, the number made from the turned shadow DE into the side AB, 1000: so Aa, the difference of the stations, is to FG, the height,

the height FG will be produced, known in the units of the difference of the stations Aa; and if the observer's stature FL is added to it, the whole height GL is made known.

Moreover, in the scholium of the following problem we shall dispatch this same problem by means of a single station.

To investigate, by means of the square, this same height — when the distance from the observer's eye is neither given nor found, and when two stations cannot be made straight in line with the height — by means of two stations made upon some erected staff.

PROBLEMA VII.

1. When two stations cannot conveniently be made on the plane, set up some pole Kb perpendicular to the horizon — unless perhaps there happens to be some upright building at hand — and there make the two stations at A and a, as in Problem 4. Now first let the alidade [dioptra], or the plumb-line, fall upon the versed shadow [umbra versa] at each station, as in the first figure of Problem 4, which is repeated here.

[Margin: a — Euclid VI.4] Since, by the similarity of the triangles, AD is to DE as AF is to FG, it follows by alternation that AD is to AF as DE is to FG. By the same reasoning, ad is to aM as dH is to MG. Since, then, the proportion of AD to AF is the same as that of ad to aM — because the lines AD and ad, and AF and aM, are equal — [Margin: b — Euclid V.11] the whole DE will be to the whole FG as dH (that is, DI, the part cut off) is to the part cut off MG. [Margin: c — Euclid V.19] Therefore the remainder IE will also be to the remainder FM — that is, to Dd — as the whole DE is to the whole FG. And so, if we set it up:

As IE, the difference of the versed shadows, is to Dd, the difference of the stations, so DF, the greater versed shadow, is to FG, the height,

the height FG is found, expressed in parts of the difference of the stations Dd. Then, adding the surveyor's height FL, the whole sought height GL becomes known.

Diagram. The repeated first figure of Problem 4: a large right-angled figure whose apex G is at the upper right, with two nested small squares (the shadow-square in its two station-positions) at the left, and diagonals drawn from G down to the squares.

- G (top right): apex, the observed marker
- b (upper-left, top corner of the upper small square)
- H (inside the upper small square)
- c (top-right corner of the upper small square)
- a / α (left, at the upper station)
- d (below c, on the upper square)
- B (left, at the lower square)
- E (inside the lower small square)
- c (right side of the lower small square)
- I (below the lower square)
- A (lower left, the lower station)
- D (bottom of the lower square)
- K (bottom-left corner of the base)
- M (middle right)
- F (lower right)
- L (bottom right corner)

2. Next, let the alidade cut off the straight shadow [umbra recta] at each station, as in the second figure of Problem 4, which is set out on page 92. [Margin: d — Euclid VI.4] Because, by the similarity of the triangles, bH is to ab as aM is to MG, [Margin: e — Euclid VI.16] the rectangle under bH and MG will be equal to the rectangle under ab and aM. By the same reasoning the rectangle under BE and FG will be equal to the rectangle under AB and AF, [Margin: f — Euclid VI.4] since likewise the proportion of BE to AB is the same as that of AF to FG. Since, then, the rectangle under ab and aM is equal to the rectangle under AB and AF — because the lines ab and AB, and aM and AF, are equal — the rectangle under bH and MG will also be equal to the rectangle under BE and FG. [Margin: g — Euclid VI.16] Hence the whole bH will be to the whole FG as BE (or bO), the part cut off, is to the part cut off MG. [Margin: h — Euclid VI.19] And therefore the remainder OH will be to the remainder FM — that is, to Aa — as the whole bH is to the whole FG. And so, if we set it up:

As OH, the difference of the straight shadows, is to Aa, the difference of the stations, so bH, the greater straight shadow, is to FG, the height,

the height FG comes out known; and this, together with the surveyor's height FL, makes the whole height LG known. Here you see that the procedure is exactly the same whether one works with straight shadows or with versed shadows.

3. Lastly, at the lower station let the straight shadow BE be cut off, and at the upper station the versed shadow dH, as in the third figure of Problem 4, which is set out on page 94. Reduce the straight shadow to a versed shadow, as we explained at the beginning of this book in the construction of the Square, No. 7; then if we set it up:

As IN, the difference of the versed shadows, is to Aa, the difference of the stations, so DN, the greater versed shadow, is to another term,

the height FG will be produced, in parts of the difference of the stations Aa, just as we demonstrated in No. 1. But if you prefer to reduce the versed shadow to a straight one, you will find the same height by means of two straight shadows, as we showed in No. 2.

4. When perhaps at one of the stations the line of the fiducial edge should pass through the point C, it will be at your discretion to take either the whole side of the versed shadow or that of the straight shadow.

SCHOLIUM.

1. The same height, and its distance from the surveyor's eye — together with the hypotenuse stretched from the eye to the top of the height — may be hunted out by means of the fixed square at a single station, even if only the top of the upright object can be seen. So this scholium supplies everything that we investigated in Problems 3, 4, 5, 6, and 7 by means of several stations, and it is therefore to be learned diligently.

[Margin: How the preceding five problems may be accomplished by means of the fixed square at a single station.]

Let the height to be measured be IF, and the surveyor's height IG (or HA). Let there be prepared a plane parallel to the horizon, passing through A, on which the Square can be fixed or set up. First, then, the hypotenuse AF is to be found, in this way. Let the square be tilted so that the center of the alidade occupies the upper end of the side aA, whose lower point is fixed at A, the surveyor's eye. Let the whole square be raised or lowered until the side Ae points straight at the top F. With the square held so, let the surveyor shift his eye to a, and through the alidade sight the top F, and let the parts of the versed shadow cut off, bc, be noted. Then let it be set up:

As bc, the versed shadow, is to the side ab, 1000, so the side aA — known in a given measure, which we now suppose to contain 3 feet — is to AF.

For the number found will give the hypotenuse AF, known in parts of the side aA. For example: if cb contains 50 parts of the versed shadow, and aA is equal to 3 feet, then if we set it up as 50 is to 1000, so 3 is to another term, AF will be found to comprise 60 feet. But if aA is put as 1000, the same AF will be found to be of 200000 [?] such parts as the 1000 contained in aA.

Diagram. A large right-angled triangle rising to the apex F at the upper right, its base running along the bottom from the left; the shadow-square is shown at the observation point A and again shifted along the base, with dotted horizontal sight-lines and several nested small squares. The interior lettering is heavily overprinted.

- F (top right): apex, the top of the height
- b (upper left, top of the tilted square)
- a / α (left, upper end of the tilted square's side)
- A (left, the surveyor's eye)
- H (bottom-left corner)
- B, E, c, C, e (nested small square near A — overprinted)
- b, c, B, E, C (a second small square along the base — overprinted)
- K, M, G, I (right-edge stack)
- D, a/ α , d, A, L, D (points along the base)

GEOMETR. PRACTICÆ

2. After this, let the square be lowered so that the side AD is parallel to the horizon and the center of the alidade is at A. With this done, the top F will be seen through the hypotenuse AF that was found, which first cuts the versed shadow CD at E, as in the first square; and let the portion of the alidade AE be sought, in parts of the sides AD and DE. [Margin: a — 6, on right-angled triangles] This will be done either from both sides AD and DE being known; [Margin: b — 5, on right-angled triangles] or from AE being known, through Problem 1, from the shadow DE, the angle DAE, and the known side DE; [Margin: c — Euclid I.47] or, finally, if the square root be extracted of the sum of the squares described on AD and DE. Once the portion AE is found in thousandth parts of the side AD, and EM is drawn parallel to AG, [Margin: d — Euclid VI.2, and by composition] then if it be set up:

As AE, found in thousandth parts of the side AD, is to AF, found in those same parts, so GM — known in those same parts, since it is equal to DE — is to GF,

GF becomes known, in thousandth parts of the side AD. And if it be set up again:

As the side AD, 1000, is to the 3 feet which we suppose contained in AD, so GF, found in thousandth parts of the side AD, is to GF,

the same GF will be found in the measure of feet. And if the surveyor's height GI, known in the same measure, is added, the whole height IF becomes known in the measure of feet.

Here you see that by the same operation we have also found the distance AF, from the eye A all the way to the top F.

3. Next let the hypotenuse aF, once found (it is found as was said in No. 1), cut each side of the shadow at C, as in the second square. [Margin: e — Euclid VI.2, and by composition] If, then, it be set up:

As the portion of the alidade ac, found in parts of the side ad, 1000, as we said in No. 2, is to aF, the hypotenuse found in those same parts, so GK, 1000, is to GF,

GF will come out known, in thousandth parts of the side dc. And if it be set up again:

As the side dc, 1000, is to the 3 feet to which we suppose the side dc equal, so GF, found in thousandth parts of the side dc, is to GF,

the same GF will be found in the measure of feet. And if the surveyor's height GI, known in the same measure, is added, the whole height IF will become known in the measure of feet.

4. Third, let the hypotenuse AF, once found (which will be found as we taught in No. 1), cut the straight shadow at E, as in the third square. [Margin: f — Euclid VI.2, and by composition] If, then, it be set up:

As the portion of the alidade AE, found as we taught in No. 2, is to AF, the hypotenuse found, so GK, 1000, is to GF,

GF will come out known, in thousandth parts of the side DC. And if it be set up yet again:

As the side DC, 1000, is to the 3 feet contained in DC, so GF, found in thousandth parts of the side Dc, is to GF,

the same GF will be known in the measure of feet, and so on.

And by this same reasoning all the other lines found in thousandth parts of the side of the Square will be reduced to any other measure whatever, such as to feet, or cubits, and so on.

5. The distance AG (or HI), moreover, comes out known [Margin: g — Euclid VI.2, and by composition] if it be set up:

As the portion of the alidade AE in the first square (or ac in the second square) is to the hypotenuse AF (or aF), so the side AD in the first square (or ad in the second square) is to AG in the first square (or to aG in the second).

VEL — Or:

With EL drawn parallel to the side AB in the third square:

As AE, the portion of the alidade, is to the hypotenuse AF, so the straight shadow BE (equal to AL) is to AG.

You see, therefore, that in this method of measuring a height and a distance — even an inaccessible one — there is no need to make two stations, whether on the plane or on some upright pole (a thing that usually brings no small trouble). One need only first find the hypotenuse by means of the tilted fixed square, and then the portion of the alidade between its center and the side of the Square that it intersects — a finding which is not difficult, as has become clear, since the surveyor is never forced to change his place.

LIBER III.

To measure the height of a tower or mountain from its summit by means of the Square, when on the plane of the summit — parallel to the horizon — two stations can be made and some marker can be seen on the horizon.

PROBLEMA VIII.

1. Let the height be DFMd, on whose summit — namely on the plane Dd — two stations can be made, from which some marker on the horizon, namely G, can be seen. (So as not to be forced to draw a new figure, we have repeated the first figure of Problem 4, but placed in reversed order, so that the letters do not stand upright. And although the marker G is not seen from A, because of the plane Dd, the reasoning will nonetheless be the same if some other, more distant marker is

chosen that can be sighted from A.) Let the instrument be set up at each station so that the side of the straight shadow DC points downward. And first — for in this reversal the side DC is reckoned as the straight shadow and BC as the versed, as was explained in the construction of the Square, No. 4, at the beginning of this book — from the versed shadow dH at the nearer station, which is always the smaller, let there be cut off an equal DI. And so, if it be set up:

As IE, the difference of the straight shadows, is to Aa, the difference of the stations, so AD, the side of the square, 1000, is to AF,

the straight line AF will be known; and if from it the side of the square AD (known in parts of the difference of the stations) be subtracted, the sought height DF (or dM) will be left known. That AF is found by this reasoning was demonstrated in Problem 4, No. 1.

Diagram. The first figure of Problem 4 repeated (inverted), apex G at the upper right, base at the bottom, with two nested small squares at the left and diagonals from G.

- G (top right): apex, the marker on the horizon
- b (top of the upper small square)
- H (inside the upper small square)
- c (right of the upper small square)
- a / α (left, upper station)
- d (below c)
- B (left, lower square)
- E (inside the lower small square)
- c (right of the lower small square)
- I (below the lower square)
- A (lower left, lower station)
- D (bottom of the lower square)
- K (bottom-left corner)
- M (middle right)
- F (lower right)
- L (bottom right corner)

2. But if at each station the side of the versed shadow is cut at F and H, as in the second figure of Problem 4, here repeated but reversed, we shall have three ways of investigating the height AF; and if the side of the Square AD be taken from it,

the sought height DF (or dM) will manifestly be left. For, both versed shadows having been reduced to straight ones, if it be set up:

As IN, the difference of the straight shadows, is to Aa, the difference of the stations, so AD, the side of the square, 1000, is to AF.

Or, without reduction:

As P — the number produced from OH (the difference of the versed shadows) multiplied into ab, the side of the square — is to the number Q, produced from the greater versed shadow bH multiplied into the lesser BE, so Aa, the difference of the stations, is to AF.

Or:

As Aa — the difference of the quotients that arise when the side of the square is divided by each versed shadow — is to Aa, the difference of the stations known in some measure, so AF (taken as 1) is to AF.

There will always be produced the height AF, reckoned from the eye of the observer A, just as was demonstrated in Problem 4, No. 2.

Diagram. The second figure of Problem 4, repeated and rotated (the whole figure turned so the letters do not stand upright), with the apex G at the lower right; a rectangle with nested squares and a diagonal, marked "P·Q" beside it. The interior lettering is rotated and overprinted.

- G (lower right): apex, the marker
- A, K (upper-left corner)
- B, E, o, b (upper left, near the square)
- D (left side)
- H, d, I (interior of the square — overprinted)
- C, M (lower interior)
- o, K (right of the square)
- F, L, M (base, at right)
- "P·Q" (label set beside the figure)

3. If, finally, at one station the side of the versed shadow is cut at E, and at the other the side of the straight shadow at H, as in the third figure of Problem 4, here repeated in reversed order: then, if — as we demonstrated in that same Problem 4, No. 3 — the versed shadow be reduced to a straight one, and it be set up:

As NI, the difference of the straight shadows, is to Aa, the difference of the stations, so AD, the side of the square, is to AF.

Or, without reduction:

As O — the number that remains when the number generated from the versed shadow multiplied into the straight shadow is subtracted from the square of the side — is to the number P, produced from the versed shadow BE multiplied into the side AD, so Aa, the difference of the stations, is to AF,

there will be produced AF, the height, in parts of the difference of the stations Aa, and so on.

Diagram. The third figure of Problem 4, apex G at the upper right, marked "P·Q" at the top; two squares set side by side along the base, with diagonals. Some interior letters appear mirror-reversed (the figure is inverted). The lettering is crowded.

- G (top right): apex, the marker
- B (left, upper corner of the left square)
- I, C, N [mirror-reversed] (top edge of the left square)
- A (lower left)
- E, e (interior of the left square)
- D (base of the left square)
- b (top of the right square)
- H, N [mirror-reversed], C, I (interior of the right square)
- a / α , d (base, between the squares)
- K, M (bottom left / bottom center)
- F, L (bottom right)
- "P·Q" (label at the top)

4. Now if the distance FG, from the tower to the marker G seen on the horizon, is known, we shall more easily infer the height by means of a single station made at A. For if the straight shadow DC is cut at E, as in the first figure, [Margin: a — Euclid VI.4] let it be set up:

As the straight shadow DE is to the side AD, so the known distance FG is to AF.

Or, when the versed shadow BC is cut at E, as in the second figure, [Margin: b — Euclid VI.4] if it be set up:

As the side AB is to the versed shadow BE, so the known distance FG is to AF,

[continued from the preceding pages — *Geometria Practica*, Liber III, Problema VIII]

the line AF will become known, in the parts of the distance FG. If from it the side of the square AD is subtracted — known in those same parts of the distance FG — the height DF will be left known.

But if the dioptra [dioptra] passes through C, the distance FG will be equal to the height AF. [Margin: a. Elements I.6.] Therefore, once the side of the square AD has been subtracted, the height of the tower DF will become known.

We shall solve this same problem, in the scholium of the following problem, by means of a single station.

To investigate, by means of the square, the height of a tower or mountain from its summit — using two stations made upon some upright staff — when some marker on the horizon can be seen.

PROBLEMA IX.

Diagram. The first of three figures for this problem, keyed with the printed letters "N O" beside it. It repeats, in inverted order, the first figure of Problem 3 (the *umbra recta* case). A large right triangle rises from a horizontal line at the left corner A up along a sloping hypotenuse to the apex G at the upper right; the vertical right side falls from G to F, and a lower base runs K–M–L. Two nested shadow-squares straddle the ascending sightline, each carrying the doubled capital/minuscule station labels of the two eye-positions. The interior of the figure is densely overprinted, with capital and italic-minuscule labels superimposed.

- G (apex, upper right): summit / marker sightline
 - A, B, C, D, E, H, I (upper capital labels along the squares and sightline)
 - a, b, c, d (italic minuscule station labels, interleaved with the capitals)
 - Q [?], N [?], O [?], P [?] (small letters at the tops of the squares)
 - K, M, L (lower baseline, left to right)
 - F (right end of the middle horizontal)
1. When on the summit there is not enough level ground for two stations to be made on it, let some staff be erected on which the two stations may be taken. For example, if the height to be found is dF, erect the staff dA, and — with the center of the dioptra applied to the staff, first at a, then at A —

sight some marker G on the horizon along the rays aG and AG. Let each ray first cut the *umbra recta* at H and E, as in the first figure of Problem 3, which we have here repeated in inverted order. For, as we said in the preceding problem, in this inversion DC becomes the side of the *umbra recta*, and BC the side of the *umbra versa*. Therefore, if — as we demonstrated in Problem 1, Number 1 — we set it up so that:

As HI, the difference of the *umbræ rectæ*, is to Aa, the difference of the stations, so dH, the greater *umbra recta*, is to AF,

the whole line AF will become known. From it, if the portion of the staff Ad — which the two squares occupy — is subtracted, the remaining height dF will become known.

Diagram. The second figure, the *umbra versa* case (the second figure of Problem 3, repeated inverted). Again a right triangle rises from the left along a hypotenuse to the apex G at the upper right, with two shadow-squares set on the ascending sightline and the two doubled station labels. As in the first figure the interior letters are closely overprinted.

- G (apex, upper right)
 - A, B, C, D, E, H, I (upper capital labels)
 - a, b, c, d (italic minuscule station labels)
 - B, A (left corners of the lower square)
 - K, M (lower baseline), F, L (right side)
2. But if in both stations the *umbra versa* is intersected by the rays — as in the second figure of Problem 3, here repeated in inverted order — then set it up so that:

As IE, the difference of the *umbræ versæ*, is to Aa, the difference of the stations, so BE, the greater *umbra versa*, is to AF,

and the line AF will result. From it, if the portion of the staff Ad — from the beginning of the first square to the end of the second — is drawn off, the remaining height dF will likewise become known, just as we showed in Problem 3, Number 3.

Diagram. The third figure, keyed with the printed letters "P Q" beside it: the mixed case, in which one station cuts the *umbra recta* and the other the *umbra versa* (the third figure of Problem 3, transferred here inverted). A smaller right triangle rises from the left to the apex G at the upper right, with two shadow-squares on the sightline. The interior labels are again overprinted.

- G (apex, upper right)
 - B, C, H, I (upper capital labels)
 - b, c, d, a (italic minuscule station labels)
 - K, A (left corners), D, a, d (baseline), M, F, L (right)
3. But if in one station the side of the *umbra recta* is cut, and in the other the side of the *umbra versa* — as in the third figure of Problem 3, which has been transferred here inverted — then either the *umbra recta* must be reduced to the *umbra versa*, or the *versa* to the *recta*. For, as we showed in Problem 3, Number 5, if we set it up so that:

As IN, the difference of the *umbræ* — whether *versæ* or *rectæ* — is to Aa, the difference of the stations, so BN, the greater *umbra versa* (or dN, the greater *umbra recta*), is to AF,

the line AF will become known. From it, if the portion of the staff Ad is subtracted, the height dF will be left known.

4. And if AF were a tower, and the observation were made from two windows d and D, the height of the tower AF would be found in the same way, as is evident.

SCHOLIVM.

1. What the two preceding problems taught by means of two stations — whether made on the level summit of a tower or mountain, or on some upright staff — we can also accomplish by a single station, and at the same time find the distance from the perpendicular of the mountain, or from the tower, to the marker set on the horizon.

Diagram. A woodcut of a mountain shown in profile, its right-hand slope hatched, descending to the plain at the left. At the summit stands the survey construction: the upright staff and the shadow square set at the top, with the sightlines running down to the marker F at the far left of the base. A bracket spans the small tilted square at the very top.

- a (top of the square / center of the dioptra)
- b, e (corners of the small tilted square at the summit)
- B, c (inner square)
- G, C (left of the summit square)
- A, L, D (right side / center of the construction)
- H (center)
- Baseline, left to right: F, I, K, M, N [reversed-N in the source], E

[Margin: To measure the height of a mountain or tower from its vertex by a single station, together with the distance — from the tower, or from the perpendicular of the mountain — to the marker set on the horizon.]

For let the height of some mountain or tower be DE, and the marker seen on the horizon be F. From D erect some staff DA; and, taking DA equal to the side of the fixed square, fit the square abcd at A, so that the center of the dioptra, a, is uppermost and the lower side inclines straight toward the marker F. Then, placing the eye at a, direct the dioptra toward F and note the parts on the *umbra versa* bC. For if we set it up so that:

As be, the *umbra versa*, is to the side ba, 1000, so the side ad, 1000, is to AF,

the hypotenuse AF will be found, in the parts of the side ad. Fit the square ABCD again, so that the center of the dioptra, A, is uppermost and the side CD is parallel to the horizon — that is, so that the side AD coincides with the upright staff. This done, the marker F will be seen along the hypotenuse AF just found, which first cuts the side of the *umbra versa* BC at G; and let the portion of the dioptra AG be sought, as we taught in the scholium of Problem 7, Number 2. [Margin: a. Elements VI.4, and *componendo*.] For if we set it up so that:

As the portion of the dioptra AG, found, is to the found hypotenuse AF, so AL — equal to the *umbra versa* BG (GL having been drawn parallel to EF) — is to AE,

AE will be ascertained, in the parts of the hypotenuse AF. And if the side of the square AD — known in those same parts — is subtracted, the remainder DE, the height of the mountain or tower, will result. And if in turn we set it up so that:

As the portion of the dioptra AG, found, is to the found hypotenuse AF, so EM — equal to the side CD (BC having been produced as far as M) — is to EF,

the known distance EF will come out, in the same parts of the hypotenuse AF.

Here you see that by the same operation the distance is found from the eye A to the marker F on the horizon.

Moreover, it will not be difficult to render DE or EF known in some other measure, such as feet, once it has been found in thousandth-parts of the side AD or CD. Set it up so that:

As AD, the side, 1000, is to AD known in feet — for example, in 3, or in half of one cubit — so DE (or EF), found in thousandth-parts, is to the [corresponding value].

And by this same method all other lines found in thousandth-parts of the side of the square will be reduced to another measure, whether of feet or of cubits, and so on — as we also noted in the scholium of Problem 7.

2. Next, let the hypotenuse AI (which will be found as AF was found) cut the square at C; and let the portion of the dioptra AC be investigated, as we set forth in the scholium of Problem 7, Number 2. [Margin: b. Elements VI.2, and *componendo*.] Then set it up so that:

As the portion of the dioptra AC, found, is to the found hypotenuse AI, so AD, the side, 1000, is to AE.

For the resulting number will give AE, in the parts of the hypotenuse AI. And the distance EI will comprise just as many parts; [Margin: c. Elements I.6.] for AE and EI are equal, because the half-right angles EAI and EIA are equal.

3. Finally, let the hypotenuse AK — found in the way in which AF was ascertained — cut the side CD of the *umbra recta* at H; and let the portion of the dioptra AH be found, as we taught in the scholium of Problem 7, Number 2. [Margin: d. Elements VI.2, and *componendo*.] For if we set it up so that:

As the portion of the dioptra AH, found, is to the found hypotenuse AK, so AD, the side of the square, is to AE,

AE will be ascertained, in the parts of the hypotenuse AK, and so on. And if again we set it up — HN having first been drawn parallel to AE — so that:

As the portion of the dioptra AH, found, is to the found hypotenuse AK, so EN — equal to the *umbra recta* DH — is to EK,

the distance EK will also be rendered known, in the same parts of the hypotenuse AK, and so on. And if the foot of the mountain were visible from the vertex D, the distance from the foot up to E would be found in the same way; and this, subtracted from the found distance FE, will also leave known the distance from F to the foot of the mountain — a thing that might sometimes be wanted.

From the summit of a tower, or from one of its windows, to ascertain by means of the square the distance from the base of the tower to a marker set on the horizon.

PROBLEMA X.

Diagram. Three small right-triangle figures for this problem, set side by side. In each, the apex A carries the suspended square with its sights (corners D, B, C, and the point E), from which a plumb-line hangs down toward F; the base runs from F to G, and the hypotenuse runs from A to G. The three variants differ in where the plumb-line's thread crosses the shadow-scale.

- A (apex, top)
 - D, B (upper corners of the suspended square)
 - C, E (lower corners / thread-crossing of the square)
 - F (foot of the tower, base left)
 - G (base right, the marker)
1. Let there be some tower AF, and the distance to be measured, FG. If, then, the height of the tower is known — or its portion between the window A and the base F — sight the marker G through the pinnules [pinnacidia] of the hanging square. And if the plumb-line's thread crosses the side of the *umbra recta* at E, or passes through the point C, then set it up as follows.

As the side AB, of 1000 parts, is to the *umbra recta* BE, so AF, the known height of the tower, is to FG,

and the sought distance FG will be known, in the parts of the known tower.

But if the *umbra versa* is cut at E, and we set it up so that:

As the *umbra versa* DE is to the side DA, of 1000 parts, so AF, the known height of the tower, is to FG,

the same distance FG will likewise be rendered known, in the parts of the known tower.

2. You will also obtain the same distance by means of the fixed square. For let there again be some tower DE, and the distance to be measured, EG, or EH, or EF. If, then, the dioptra cuts the side of the *umbra recta* at I, set it up so that:

As the side of the square AD, 1000, is to the *umbra recta* DI, so the height AE — compounded from the tower and the side of the square AD — is to EG,

and the sought distance EG will result.

Diagram. The figure for the fixed-square case. From the upper-left corner A, the shadow square is set (labels A, B, and beside them N and K, with D at the left and I, C, L across the middle); a dotted horizontal sightline runs to M at the far right; from A the sightlines descend to the base points G, H, F, with E at the lower-left corner.

- A (upper-left apex / center of the dioptra)
- B, N, K (square, upper edge)
- D (left side), I, C, L (middle)
- M (right end of the dotted horizontal sightline)
- Baseline, left to right: E, G, H, F

But if the dioptra passes through C, the distance EH will be equal to the known height AE.

Finally, if the *umbra versa* is intersected at K, then set it up so that:

As the *umbra versa* BK is to the side of the square, 1000, so the known height AE is to EF,

and the distance EF will be found, in the parts of the height AE.

For all of these things were demonstrated in Problem 2, Numbers 1 and 2.

3. But if the height of the tower is not known, it must first be investigated by Problem 8 or 9 — or rather by the scholium of Problem 9 — or else, if it can conveniently be done, it must be explored by means of some cord let down with a plumb-line attached. Then one must proceed as was said in Numbers 1 and 2.

But even without knowing the tower we shall attain the same result by the method by which, in Problem 6 or 7, we tracked down the proposed height — even though the distance up to the height is unknown. For if in Problem 6 the tower is DF and the distance to be measured is FG, we shall accomplish it if, on some upright staff DA, two stations of the eye are made at a and A. Only, what was there said of the *umbra versa* must here be understood of the *umbra recta*, and vice versa.

By a like method, if in Problem 7 the tower is DF and the distance to be measured is FG, we shall find it if two stations of the eye are made on the level summit of the tower, at A and a. Only, what is there said of the *umbra versa* must here likewise be understood of the *umbra recta*. For if the figures of Problems 6 and 7 — that is, the figures of Problems 3 and 4 — are inverted, so that the heights

become AF and AF, the side DC of the square will pertain to the *umbra recta*, and BC to the *umbra versa*, as we explained at the beginning of this book, in the construction of the square, Number 4.

But we found this distance far more easily in the scholium of the preceding problem, by a single station, as was made clear in the case of the distance EF, and so on.

From the top of some height — even if the height is only the stature of the measurer — to comprehend by means of the square the distance between two markers on the plane on which the height stands, provided that this distance lies straight in front of the measurer and both its ends can be seen.

PROBLEMA XI.

Diagram. A right triangle with the vertical side DC at the left (D at the top, C at the bottom, the right angle at C) and the base CB running to the right. A cevian is drawn from the apex D to the point A on the base, dividing it into CA and AB.

- D (apex, upper left / the measurer's eye)
 - C (foot of the height, lower left)
 - A (intermediate point on the base — the nearer marker)
 - B (right end of the base — the farther marker)
1. Let the distance to be measured be AB, straight in front of the height CD, on the top of which the measurer's eye stands, at D. By the preceding problem let there be investigated, from the vertex D, both the distance CB and the distance CA. For the smaller of these, subtracted from the larger, will leave known the distance AB between the markers A and B, expressed in the parts of the height CD in which the distances CB and CA were also found.
 2. If the height CD is the stature of the measurer, the distance AB will be found in the same way, provided the measurer's eye at D can see both ends A and B, as is clear.

To measure by means of the square a length extended on the horizon, when the measurer, standing at one end of it, cannot see the other end because of some swelling of ground lying between, and no height is at hand, but he can only withdraw to the right or the left along a perpendicular line to a place from which the other end appears.

LIBER III

PROBLEMA XII.

1. Suppose the length AE is to be measured, but the surveyor cannot see its far end E from A, and no height is available; yet if he steps off to the right or left along the perpendicular line Aa, as far as a, he can see E. Set up a stable square so that its plane coincides with the length AE — for one must first establish which straight line reaches the two ends A and E, that is, which line, together with the given length, forms the base. Then, with the square placed in the plane of the horizon so that the side AB does not depart from the length, extend a straight line along the side AD as far as a, the point from which the far end E will appear, and let the distance Aa be known by some measure. Now raise the square at a and turn it about until its plane sights the far end E. After this, place the same square in the horizontal plane so that its side coincides with the perpendicular Aa at d, and rotate the sighting-vane [dioptra] until its fiducial line aH — the one along which the end E was sighted — is brought into agreement, and note the versed shadow [umbra versa] bF cut off. The triangles abF and aAE will then be equiangular, on account of the right angles at b and A and the equal alternate angles baF and AEa. [a] Therefore, if you set up the proportion:

As the versed shadow bF is to the side ab of the square, 1000, so is the known distance Aa to AE,

then the length AE will be known in parts of the distance Aa.

[Margin: a — Euclid, *Elements* VI.4.]

If it should happen that the vane cuts the right shadow [umbra recta] on the side dc — which rarely occurs, since AE is usually greater than Aa — then [b] you will have:

As the side ad, 1000, is to the right shadow cut off, so is the distance Aa to the length,

as is evident if a straight line is drawn from a cutting the side dc, and so on.

[Margin: b — Euclid, *Elements* VI.4.]

Finally, if the vane should perhaps pass through c, [c] the distance Aa would equal the length sought; for in that case the angle dac would become half a right

angle, and so the line ac, if drawn, would make with AE an angle equal to half a right angle, and thus equal to the angle dac.

[Margin: c — Euclid, *Elements* I.6.]

2. The same length AE will also be found, at the same distance, if at both G and H a staff or reed is fixed at right angles, so that the rays passing along the reed from A and a are directed to E, and the distance Aa is known — as was set out in Problem 2, number 6.

To find, by means of the square, a length lying in the horizontal plane directly before the surveyor, to whose ends he may neither approach nor measure it from his own station, and for which no height is available — provided the surveyor can move to the right or left along a perpendicular line to some place from which both ends appear.

Diagram. A right-angled surveying figure. A tall vertical line rises on the left to the apex E; along it, from bottom to top, are the corner A, the point B just above it, and the point G higher up. From G a horizontal dotted sight-line runs rightward through H (where a descending diagonal crosses it) to I at the far right. The baseline runs from A rightward through D and d to the offset station a at the far right. A small square A–B–C–D sits at the left (the fixed geometric square), and a second small square with corners near c, d, b, F stands at the right (the movable square with its versed shadow bF). Two short oblique strokes stand in the upper field (the sighting staves at the stations).

- E (top of left vertical): apex, the far end of the length being sighted
- A (bottom-left corner): near end of the length AE
- B (left vertical, just above A): upper corner of the fixed square
- C (right of B): corner of the fixed square A–B–C–D
- D (baseline, right of A): corner of the fixed square
- G (left vertical, upper): start of the horizontal dotted sight-line
- H (middle of dotted line): intersection with the descending diagonal
- I (right end of dotted line): terminus of the sight-line
- d (lowercase, on baseline): corner of the movable square
- a (lowercase, far-right baseline): offset station (distance Aa) [?]
- c (lowercase, in right square): corner of the movable square where the vane may cut
- b (lowercase, at right square): foot of the versed shadow bF
- F (right square): terminus of the versed shadow

- "6" or "b" (far right, beside F): [?]

PROBLEMA XIII.

1. Suppose the length ED is to be measured, lying directly before the surveyor stationed at F, in such a way that he may neither approach it nor measure it from his position F, and no height is available; but he can only move along the perpendicular line FG to the place G, from which both ends D and E can be seen. By the preceding problem, let both FE and FD be found from G. For FD, subtracted from FE, will leave known the proposed length DE.

To estimate, by means of the square, the altitude of a mountain or tower from its summit, when some interval lying directly before the surveyor — between two marks, or even between some one mark and the tower — is known.

Diagram. A right triangle with the right angle at the lower left. The apex G is at the top; the base runs from E (lower left) rightward to F (lower right), with D marked on the base between them.

- G (top): the surveyor's station
- E (lower-left corner): one end of the length ED
- D (on the base, between E and F): the other end of the length
- F (lower-right corner): foot of the perpendicular FG

GEOMETR. PRACTICÆ

PROBLEMA XIV.

1. Let DE be a mountain or tower, and suppose first that the interval FG, lying directly before the surveyor stationed at the summit D, is known. Fit a stable square at the summit D so that the side AD is perpendicular to the horizon and CD parallel to the horizon. Sighting therefore through the vane to each of the two limits F and G, let the right shadow be cut at H and I. [a] Since, then, as IH is to HD, so is GF to FE; [b] and likewise, on account of the similarity of the triangles, as HD is to DA, so is FE to EA; it will follow, by equality of ratios [ex aequo], that as IH is to DA, so is GF to EA. Therefore, if you set up the proportion:

As IH, the difference of the right shadows, is to DA, the side of the square, so is the known interval GF to EA,

there will come out the known line EA. And if the side of the square DA is subtracted — which must be made known in parts of the interval GF — the remainder will give the known altitude DE, in parts of the interval GF.

[Margin: a — scholium to Euclid, *Elements* VI.4.] [Margin: b — Euclid, *Elements* VI.4.]

2. Suppose next that the whole distance EG is known. Then only the far end G need be considered. [c] For if you set up the proportion:

As ID, the right shadow, is to DA, the side of the square, so is the known distance GE to EA,

there will again be produced the known line EA, and so on.

[Margin: c — Euclid, *Elements* VI.4.]

3. When the versed shadow BC is cut — as when the known interval is LN — both versed shadows must be reduced to the right shadows DK, DM. For it will again hold that, as KM, the difference of the right shadows, is to the side DA, so is the known interval LN to EA.

Likewise, when one ray cuts the right shadow and the other the versed shadow — as happens in the interval LG — the versed shadow must be brought back to the right shadow DK, so that again there is KI, the difference of the right shadows, to DA, the side, as the known interval LG is to EA.

4. Finally, if a ray should pass through C, one must take the whole side CD for the right shadow; and the versed shadow, if there were any, must be reduced to the right shadow.

To find, by means of the square, the distance from the surveyor's eye, or from his foot (wherever it may be), to any point marked at some altitude.

Diagram. A large right triangle representing the mountain or tower, with the geometric square set at the summit. At the top a small square is formed by B (upper left) and A (upper right), with D below A. A cluster of shadow-intersection points K, M, C, I, H lies just below, heavily overprinted. From the square, several straight rays descend to the baseline, which runs from L (far left) through N, O, G, F to E at the lower right.

- B (top left): corner of the summit square
- A (top right): corner of the summit square
- D (below A): summit corner of the square (base of side AD)

- K (left, below B): reduced right-shadow point (side DK)
- M (center cluster): reduced right-shadow point (side DM) [?]
- C (center cluster): corner of the square [?]
- I (center cluster): right-shadow section point [?]
- H (center cluster, right): right-shadow section point [?]
- E (lower right corner): foot of the mountain
- L (baseline, far left): projection point (interval LN, LG)
- N (baseline): projection point [?]
- O (baseline): projection point
- G (baseline): known limit
- F (baseline, right of G): known limit (interval FG)

PROBLEMA XV.

1. Suppose the distance of a point F is to be explored — F lying in some wall (whether perpendicular to the horizon or inclined), or even in some roof — from the eye A, or from the foot E, with the surveyor's stature set as AE. Let the point F first be higher than the eye A. Conceive dropped from F the perpendicular FG, and, drawn to it from the eye A, another perpendicular AH. With the square placed so that the side AD is equidistant from the horizon, let the point F be sighted, and let the ray AF, or the vane, cut off the versed shadow DI. By Problem 3 or 4, or better by the scholium to Problem 7, let the distance AH be investigated — even though the point H does not appear — and let it be carefully inquired how many thousandth parts of the side AD [a] are contained in the segment of the vane AI. This, which we taught may be done in many ways (as in the scholium to Problem 7, number 2), we shall carry out in this manner. First, since the two sides AD, DI in the right triangle ADI are given, [a] the base cannot fail to be known in parts of the sides. Next, since by Problem 1 the angle DAI becomes known from the shadow DI, [b] the base AI will again be known. Thirdly, [c] since the squares of DA and DI are equal to the square of AI, if the square root be extracted from their sum, that root will give the base AI known. With these done, [d] if you set up the proportion:

As the side AD, 1000, is to the segment of the vane AI just found, so is the distance AH, also just found, to AF,

the distance AF sought will be known, in parts of the found distance AH.

[Margin: a — On Rectilinear Triangles, 6.] [Margin: b — On Rectilinear Triangles, 5.] [Margin: c — Euclid, *Elements* I.47.] [Margin: d — Euclid, *Elements* VI.4.]

The distance EF, from the foot to the given point F, we shall find thus. Since in the triangle AEF the two sides AF, AE are known — the former being just found, the latter equal to the surveyor's stature — and they contain the known angle EAF (made up of the right angle EAH and the angle DAI, which by Problem 1 was found [e] from the versed shadow DI), the side EF too, which is sought, will be made known.

[Margin: e — On Rectilinear Triangles, 12.]

2. But if the right shadow is cut at L — as when, in the other square, the surveyor's stature is AK — then, the segment of the vane AL having been found in parts of the side of the square from the angle BAL, and so on, and likewise the distance AH from [f] Problem 3 or 4, or from the scholium to Problem 7, if you set up the proportion:

[Margin: f — Euclid, *Elements* VI.4.]

LIBER III

As BL, the right shadow, is to LA, the found segment of the vane, so is the found distance AH to AF,

there will again come out the known distance AF, in parts of the found distance AH.

You will proceed in no other way if the vane passes through C: [g] for then too it holds that, as the side AD is to the found segment of the vane AC (as before), so is the found distance AH to the distance sought AF.

The distance KF, from the foot K up to F, [h] will be found as before, from the two known sides AF, AK and the angle FAK contained by them — which of course is made up of the right angle at A and the angle DAL, the complement of the angle BAL, which the right shadow BL makes known by Problem 1.

[Margin: g — Euclid, *Elements* VI.4.] [Margin: h — On Rectilinear Triangles, 12.]

3. But now let the point F be lower than the eye A, and let the surveyor's stature be AE. Conceive drawn from F the line FK parallel to the horizon, or perpendicular to AE; likewise through F the straight line GH parallel to AE. With the square fitted so that the side AD is perpendicular to the horizon and DC equidistant from it, imagine the side DC produced as far as H, the point on the perpendicular GH. First, then, let the altitude AK be found, by Problem 8 or 9, with two stations taken on the line DH, or on the staff DA

produced, or certainly by the scholium to Problem 9. For although the surveyor's stature AE is known, it is still not known how great is that part of it, AK, which the parallel FK — drawn in imagination — cuts off; so that it is altogether necessary to inquire the altitude AK, which by Problem 8 we shall easily carry out, if at each station the versed shadow is cut at I and N (as generally happens here) and each cut-off, BI and bN, is brought back to the right shadows DM and dO. For if you set up the proportion:

As LM, the difference of the right shadows, is to Dd, the difference of the stations, so is AD, the side, to AK,

the altitude AK will be found, with the eye placed at its summit A, as we stated in the said Problem 8, number 2, and so on — although that altitude AK can be found more easily by the scholium to Problem 9.

And so, since the shadow BI, by Problem 1, discloses the angle BAI (that is, the alternate angle AFK, equal to it), and also its complement FAK, there will be, in the right triangle AKF, two acute angles known, together with the side AK just found. [i] And so, if you set up the proportion:

As the whole sine is to the side AK found, so is AF, the secant of the angle FAK, to AF,

then AF will be known, in parts of the found side AK. Or, the segment of the vane AI having been found in thousandth parts of the side of the square (as was said above near the beginning of this problem), [k] if you set up the proportion:

[Margin: i — On Rectilinear Triangles, 5.] [Margin: k — Euclid, *Elements* VI.4.]

As BI, the versed shadow, is to IA, the found segment of the vane, so is KA, the found altitude, to AF,

the distance AF will again be made known, in parts of the found line AK.

Further, the distance EF, from the surveyor's foot to the point F, [l] we shall find as above; for in the triangle AEF the two sides AE, AF are known — the one being the surveyor's stature, the other just found — and they contain the known angle FAK, as we said a little before.

[Margin: l — On Rectilinear Triangles, 12.]

4. In no other way will both distances be known if the point F is placed in the horizon, which is understood to pass through FK, so that the surveyor's

stature, or some other known altitude, is AK. For, the segment of the vane AI being known (as delivered above), [m] if you set up the proportion:

[Margin: m — Euclid, *Elements* VI.4.]

As BI, the shadow cut off, is to IA, the found segment of the vane, so is AK, the altitude or surveyor's stature, to AF,

the distance AF will turn out known, in parts of the surveyor's stature, or of the known altitude AK. [n] Again, if you set up the proportion:

[Margin: n — Euclid, *Elements* VI.4.]

As BI, the shadow cut off, is to AB, the side of the square, 1000, so is AK, the altitude or stature, to KF,

the distance KF too — from the surveyor's foot up to the point F — will be made known, in parts of that same surveyor's stature, or of the known altitude AK. Here you see that both distances are found by a single station, when the given point is in the horizon.

5. But wherever the point F may lie — whether in the horizon, or aloft, or below the eye — we shall nonetheless find both distances by a single station, in this way. When the point F is aloft, as in the first figure, fit the square so that the vane's center a is uppermost and the side dc is directed toward the point F. Then, the point F having been sighted through the vane, note the versed shadow eb. [o] And set up the proportion:

[Margin: o — Euclid, *Elements* VI.4.]

As the versed shadow eb is to the side da, so is the side ad to AF,

for the number produced will make known the distance AF, in parts of the side of the square. You will draw out the distance KF from the surveyor's foot in no other way, if by the same procedure you apply the square to the point K, as is evident.

Diagram. A composite surveying figure for the point F higher than the eye. The point F stands at the upper right. At the left is a square B–C over A–D, with the point I inside near the side AD, and the baseline E–A–D at the bottom continuing to H and G at the lower right. A second, tilted square stands in the center-right, carrying the lowercase vane-labels; several rays run from A and E up to F.

- F (upper right): the point whose distance is sought
- B (upper-left square, top left): corner of the square

- C (upper-left square, top right): corner of the square
- A (left square, lower corner / eye): the eye
- D (left square, right corner): corner of the square
- I (inside left square): versed-shadow section point (DI)
- E (bottom-left corner): the surveyor's foot (stature AE)
- H (lower right): foot of perpendicular AH
- G (lower right, below H): foot of perpendicular FG
- K (bottom center): [?]
- a, b, c, d, e, L (tilted center square): vane center and shadow labels [?]

Diagram. The surveying figure for the point F lower than the eye. A small tilted square sits near the top center, with apex A above it and B to its upper right, carrying lowercase vane-labels. A wide rectangle DCLM...H forms the middle band, and below it the base runs K–E on the left and to F–G on the lower right, with rays converging on F.

- A (top, apex of the small square): the eye
- B (upper right of the small square): corner of the vane-square
- a, b, c, d, e, o (small tilted square): vane center and shadow labels (eb, etc.) [?]
- D (left, middle band): corner of the large square
- C (middle band, left of center): corner of the square
- L (middle band, left-center): right-shadow section point
- M (middle band, center): right-shadow section point [?]
- H (middle band, right): point on the perpendicular GH (DC produced)
- K (lower left): foot of the perpendicular FK / point on stature
- E (lower left, below K): the surveyor's foot
- F (lower right): the point whose distance is sought
- G (lower right, below F): terminus of the perpendicular GH

(continuation of the preceding problem)

When the point F lies lower than the eye, as in the second figure, a staff *ac* must be set up — shorter than the observer's height — and the square applied at *a*, so that once again the center of the sight-vane A is uppermost and the side *ac* points toward the point F. For once F has been sighted from A through the sight, and the versed shadow cut off at B has been noted, set up the proportion:

As the versed shadow from B, to the side BA, so the side Aa, to *aF*,

and the distance aF will come out, expressed in the parts of the side Aa . The distance EF is then estimated from the observer's foot, if the square is applied at E , as was explained for the first figure. Finally, when the point F lies on the horizon, both distances are found, as we set out at Number 4.

To measure with the geometric square the interval between two marks or points on any plane whatever — whether perpendicular to the horizon or inclined.

PROBLEMA XVI.

Diagram. A surveying figure for finding an interval on an elevated plane. A quadrilateral rests on the base line $E-B$, with a vertical rising from B to the apex region. Sight-lines run from the eye at G (left) to the two points C and D of the interval.

- C (apex, top center): the near end of the interval on the elevated plane
 - A (upper right): corner of the elevated plane AB
 - G (left, with a quadrant/dioptra arc): the observer's eye
 - D (right, mid-height): the far end of the interval
 - E (bottom left): foot of the base plane EB
 - B (bottom center): foot of the vertical, on the base plane EB
1. On any elevated plane AB , let the interval CD be sought, from the plane EB . Place the eye at G , so that the observer's height is GE . By the preceding problem investigate each of the two distances GC and GD , expressed in the parts of the observer's height GE . Next apply the fixed square at the eye G , so that its plane passes through the points C and D , and one of its sides points straight at the point D — which will be so if, with the fiducial line of the sight laid along that side, the point D is seen through the holes of the pinnules. Then turn the sight until the point C too appears through it, and note the shadow cut off, from which, by Problem 1, the size of the angle CGD is learned. Now if the other side of the square falls beyond the straight line GC , the angle CGD will be acute; but if it falls short of GC , that angle will be obtuse, and we shall determine it by adding to a right angle the remaining angle between that other side and GC . This remaining angle we investigate just as we did for the acute angle CGD — namely, by noting mentally the point on the line CD onto which the other side, produced, would fall. For if that side then coincides with GC and the sight points toward the noted point, the shadow between that side and the fiducial line will indicate the said remaining angle, as was explained in Problem 1. Finally, if that other side coincides exactly with GC , the angle CGD will be a

right angle. This done, since in triangle CGD the sides GC and GD are known and enclose the now-known angle G, the third side CD — which is sought — will also become known. [Margin: (a) On Rectilinear Triangles, 12.]

2. But once the distances GC and GD have been found in some definite measure, we shall determine the interval CD in this way. Draw the angle G separately, and on GC take any chosen portion of the parts of the square's side — say 500, that is, half the side. Then set up the proportion: as the found distance GC is to the found distance GD, so the 500 parts taken to a fourth term. For the operation will yield a fourth proportional number in the same parts of the square's side; and if an equal portion is taken on GD, the interval between the endpoints of the said portions will be parallel to the interval CD [Margin: (b) Euclid VI.2] — as is evident, since if those portions were transferred onto the lines GC and GD in the figure, then at those endpoints the lines GC and GD would be cut proportionally. Hence if that interval is measured with the compass in the same parts of the square's side, the interval CD will contain as many little parts of the line GC (divided into 500 equal parts) as are comprised within that interval out of the little parts of the square's side — since, as the portion of the 500 little parts is to that interval, so GC, divided into 500 equal parts, is to CD. These little parts are then reduced to the measure in which GC and GD were found, by the proportion: as GC, so far as 500, is to the measures found on GC, so CD — found in the little parts of the line GC divided into 500 parts — is to a fourth term.

But since that interval between the endpoints of the proportionals can scarcely be found accurately with the compass, the proposed interval will be determined more exactly, though more laboriously, by Proposition 12 of Rectilinear Triangles.

To measure with the geometric square a transverse interval on the horizon, both ends of which can be seen.

PROBLEMA XVII.

Diagram. A quadrilateral plot ABCD with an inscribed triangle at the observer's station. Within the enclosing frame (A upper left, B lower right) the triangle CED is drawn, with the square applied at the vertex E.

- A (upper left): corner of the horizontal plane AB
- E (top, near A): the observer's foot, vertex of the sighting triangle

- D (right): the far end of the transverse interval
 - C (lower left): the near end of the transverse interval
 - B (lower right): corner of the horizontal plane AB
1. On the horizontal plane AB let the interval CD lie transversely, with the observer's foot at E, so that the length CD, produced in both directions, does not pass through E. For when the line CD lies directly in front of the observer, it will be investigated by Problem 11. So, in order to determine the transverse interval CD, one must first find the distance of each end-point C and D from the observer's foot E, as was set out at Number 4 of Problem 15, by a single station. Next the angle CED is to be found; this is done if one side of the square coincides with the line EC and the sight with the line ED. For the shadow cut off between [that side and the sight] will show the size of the angle CED, as was explained in Problem 1. That angle will be acute if the other side falls beyond the line ED; a right angle if it coincides exactly with ED; and obtuse if it falls short of ED — in which case we determine it by adding to a right angle the remaining acute angle, found as we taught in the preceding problem. Since, then, we have the triangle CED, whose two sides EC and ED are known together with the enclosed angle E, the third side CD will also be known, expressed in the parts of the lines EC and ED. [Margin: (a) On Rectilinear Triangles, 12.]

The same line CD will become known if, on the lines EC and ED drawn separately with the found angle E, parts proportional to EC and ED are taken, and so forth, as was done at Number 2 of the preceding problem.

To find out with the geometric square the distance of some mark set on the horizon from the summit of a tower or some wall — even though there is no access to the mark itself — wherever the observer may be.

PROBLEMA XVIII.

Diagram. A triangle for finding the distance from a summit to a horizon-mark. A hatched column stands at the left, its summit at D and its foot at C; sight-lines run from D to the horizon point A and to the observer's eye toward the right.

- D (top left, summit of the column): the summit whose distance is sought
- C (bottom left, foot of the column): base of the vertical
- A (bottom center): the mark on the horizon
- Q [?] (bottom right): foot of the perpendicular under the eye
- B (right): the observer's eye

1. On the horizontal plane let a point A be distant from the summit D of some height, along the line AD, which we shall track down as follows. Wherever the observer's eye is placed — say at B — find, by Problem 15, the distances of the points A and D from the observer's eye B. Then determine the angle ABD, as we taught in Problem 16. For thus we shall have the triangle ABD, two of whose sides, BA and BD, are known, together with the angle B. Therefore the third side AD too will be known. [Margin: (a) On Rectilinear Triangles, 12.]

This will also be found, as we taught at Number 2 of Problem 16, if on the lines BA and BD, drawn separately with the angle B, parts proportional to BA and BD are taken, and so forth.

To explore with the geometric square an inaccessible height whose base cannot be seen, and to which the observer can neither approach nor withdraw along a straight line — so that two stations might be made — but only to the right or left, to a place from which its base can be discerned.

PROBLEMA XIX.

Diagram. A triangle for an inaccessible height reached only transversely. The summit B is at the top; the base line runs from C (lower left, where the quadrant arc is drawn) to A (lower right), with D on the upper left of the base.

- B (top): the summit whose height AB is sought
 - D (upper left): the observer's transverse station
 - C (lower left, with a quadrant arc): the observer's first station
 - A (lower right): the base of the height
1. Let AB be the inaccessible height to be measured, which one may neither approach from C (the observer's position) nor withdraw from along a straight line, but only transversely as far as D, from which we can see the base A. By Problem 17 let the transverse interval AC be investigated from D. For by Problem 5, once the distance CA is manifest, the height AB — which we seek — will become known, expressed in the parts of the found distance CA.

Moreover, by the scholium to Problem 7, both the height AB and the distance AC, together with the hypotenuse BC, will more easily be investigated by a single station made at C.

To render known, by the geometric square, a greater length from a lesser known one, even when only the vertex of the greater height can be discerned.

PROBLEMA XX.

Diagram. A pictorial surveying scene: a tall battlemented tower at the right and a lower pedestal at the left, with sight-lines and a perpendicular line of stations between them. This is a crowded multi-station figure with several overprinted interior letters.

- A (top right, battlemented summit of the tall tower): the vertex of the greater height
 - N (bottom right, foot of the tall tower): the base of the greater height AN
 - C (top of the left pedestal): the upper station, observer's height CG
 - D (on the left pedestal, below C): the second station, height DE
 - O (bottom left, foot of the left pedestal): the base of the lesser tower CO
 - G (left end of the horizontal line): endpoint of the perpendicular GEF
 - M, L, K (intermediate points along the horizontal line)
 - E (center of the horizontal line): the observer's height DE
 - F (right end of the horizontal line, meeting the tall tower): foot of the found height AF
 - H, I (points on a higher line, where the sight-lines from A cross)
1. Let the greater height AN be proposed for measuring, from some lesser tower CO of known height, from which only the summit A appears, but not the base N. Make two stations at the summit, at C and D, if the level top permits, with the observer's height being CG or DE; and toward AN let the perpendicular GEF be understood as drawn. By Problem 6 the height AF will be found; and if to it be added the lesser height DO, together with the observer's height CG — that is, the line FN — the whole greater height AN will become known.
 2. But if there is not so much level room at the top that two stations can be made, an upright staff perpendicular to the horizon must be set up — unless perhaps some standing building is at hand, where the two stations may be made. For then, by Problem 7, the height will be found between the summit of the greater height and the line said to be parallel to the horizon from the lower station. If to this be joined the lesser height from the lower station down to its base, the whole greater height will be composed. Thus in the figures of Problem 4, if the greater height is GL and the lesser AK, and two stations are made at A and a — one lower, the other higher — by Problem 7 the height GF will be found, and if the lesser height AK be added,

from the lower station down to the base, the whole greater height GL will be put together.

Furthermore, we shall more easily draw out the height AF by the scholium to Problem 7, if at G the square is applied twice, as we said in that scholium. Thus if in the figure of that scholium the greater height were IF and the known lesser one AH, GF would be found, as was shown there, and so forth.

To hunt out with the geometric square a greater height from a lesser unknown one, provided that the base of the greater can be discerned.

PROBLEMA XXI.

Diagram. Two columns with connecting sight-lines. The taller column stands at the right (summit A, base B), the shorter at the left (summit C, base D); a horizontal line runs from C to E on the right column.

- A (top right): vertex of the greater height AB
 - C (top left): summit of the lesser height CD
 - E (middle right, on the tall column): point where the horizontal from C meets the greater height
 - D (bottom left): base of the lesser height
 - B (bottom center): base of the greater height
 - *(a mark resembling "H" appears at the lower right of the figure [?])*
1. In the figure of Problem 17 of Book 2 let the height AE be learned, by Problem 6 or 7 — or rather by the scholium to Problem 7 — if C is the summit of the lesser height CD. Then, since the base B of the greater height is supposed to be visible from C, let the lesser height CD also be investigated, by Problem 8 or 9 — or rather by the scholium to Problem 9 — if C is the summit of the lesser height CD. For this, added to the found height AE, will make the whole greater height AB known, which is desired.

To search out with the geometric square a lesser height from a greater known one, even though the base of the lesser cannot be discerned.

PROBLEMA XXII.

Diagram. Two battlemented towers with a horizontal sight-line between their summits. The lesser tower (summit A, base B) stands at the left, the greater (summit C, base D) at the right; the horizontal AE runs from A to the point E on the greater tower.

- A (top left, summit of the lesser tower): vertex of the lesser height AB
 - C (top right, summit of the greater tower): vertex of the greater height CD
 - E (center, on the greater tower): point where the horizontal AE meets CD
 - B (bottom left): base of the lesser height
 - D (bottom right): base of the greater height
1. Let the lesser height AB be proposed for measuring, from the known greater one CD. Understand the line AE drawn parallel to the horizon BD, so that ED is made equal to the lesser height AB. If, then, from the summit C — by Problem 8 or 9, or rather by the scholium to Problem 9 — the height CE be explored, sighting of course the top A as if it were some mark on the horizon AE, seen from C, and if this found height CE be subtracted from the greater height CD (which is supposed known), the remainder will be the lesser height AB, which we seek.

To elicit with the geometric square a lesser height from a greater unknown one, provided the base of the lesser appears.

PROBLEMA XXIII.

1. Let the figure of the preceding problem be repeated. And since the base B of the lesser height — like some mark set on the horizon — can be seen, the greater height CD will be made known from the hypotenuse, by Problem 8 or 9, or rather by the scholium to Problem 9. Hence, as was said in the preceding problem, the lesser height AB will be determined from the greater CD, just found. Which was what was proposed.

To perceive with the geometric square a portion of the greater height from the lesser height, and a portion of the lesser from the greater.

PROBLEMA XXIV.

Diagram. Two columns, the greater at the right divided into portions. The lesser column (summit D, base E) stands at the left; the greater at the right carries, from top to bottom, the points A, C, F, G, ending at the base B.

- D (top left): summit of the lesser height DE
- E (bottom left): base of the lesser height
- A (top right): summit of the greater height AB
- C (upper right, below A): endpoint of the portion AC
- F (middle right): endpoint of the portions GF and FB
- G (lower right, below F): endpoint of the portion GF

- B (bottom right): base of the greater height
1. Let the portion AC of the greater height AB be sought from the lesser height DE; likewise the portion GF of the lesser height FB, from the greater height DE. If the height DE is less than the remaining portion CB of the greater height, then by Problem 20 or 21 each height — the greater AB and CB — must be investigated from the lesser DE, according as DE has been found known or unknown. For CB, taken away from AB, will leave known the proposed portion AC. But if DE is greater than the remaining portion FB — that is, if the portion AF of the greater height be proposed for measuring — then indeed the greater height AB must be investigated from the lesser DE, by Problem 20 or 21; while the lesser height FB must be explored from the greater DE, by Problem 22 or 23, according as DE has been known or unknown. For again FB, subtracted from AB, will leave known the proposed portion AF.
 2. In just the same way, by Problem 22 or 23, each lesser height FB and GB is to be tracked down from the greater DE, if perchance their difference FG is to be found.

To explore with the geometric square, from a valley or from the horizontal plane, a height whose base is set upon a mountain or upon some other height — where both of its extremities can be seen, even though the lowest point of the other height on which it is set is hidden, and the distance of that lowest point from the observer's position is not known.

PROBLEMA XXV.

Diagram. A battlemented tower standing on a rocky mountain, with sight-lines down to the observer in the plain. The tower rises from B (where it meets the mountain) to the summit A, with I marked on its face; the horizontal base line runs from D (left) through C to F, with E at the lower-left corner.

- A (top, battlemented summit of the tower): summit of the tower AB
- I (on the tower face, below A): an intermediate mark on the tower
- B (foot of the tower, where it meets the mountain): base of the tower / top of the mountain BF
- C (on the base line, below B): foot of the vertical from B
- D (lower left, on the base line): the observer's station
- E (bottom left corner): the observer's position in the valley
- F (bottom center): foot of the whole height AF

1. Of this kind is the height of a tower set upon a mountain, and the portion of some building between two windows, or two marks, of which one is higher than the other. So let the height of the tower AB, above the mountain BF, be proposed. From some place E, in the plain or valley, from which both extremities of the tower are seen, let both the height AF and the height BF be investigated by Problem 6 or 7 — according as the two stations are made either on the level or on some upright staff. Yet both heights AF and BF will more easily be found by the scholium to Problem 7, if at E the square is applied twice, as was done in that scholium. Once both heights are found, the height of the mountain BF, subtracted from the whole height AF, will leave known the desired height AB.

LIBER III.

SCHOLIUM

So then, if AB — the upper portion of some whole altitude AF — is what is wanted, both the whole altitude AF and its portion BF must be investigated by Problem 6 or 7, or better by the scholium to Problem 7. The difference between them will give the desired upper portion AB.

[Margin: The highest, middle, and lowest part of an altitude, and how each is to be measured.]

If instead some middle portion IB is to be found, then again both altitudes IF and BF must be worked out by Problem 6 or 7, or better by the scholium to Problem 7, so that their difference IB is made known.

Finally, if the lowest part BF is the one proposed for inquiry, it too is to be explored by Problem 6 or 7, or rather through the scholium to Problem 7.

To determine — by means of the geometric square, and to good approximation — the sloping distance from the surveyor's position up to the base of an altitude set upon a mountain, even when that base is not visible, together with the altitude itself, when the surveyor is stationed on the ascent of the mountain.

PROBLEMA XXVI.

Diagram. A tower AB stands on a mountain slope, with the surveying construction of the geometric square set against a staff on the slope. The figure shows the battlemented tower at the upper left, a vertical staff to its right

carrying the several stations, and the descending slope of the mountain toward the lower right.

- A (top of tower, upper left): summit of the tower
- G (top of the staff, upper right): top of the erected staff CG
- M (interior, below A): intersection point on the visual ray
- I (left, at the upper station): station point on the staff
- F (right of I, same level): upper station on the staff
- E (below F): lower station on the staff (surveyor's eye)
- B (below I): base of the tower
- K (interior, below B): point on line EK
- N (interior, right of K): point
- C (center, on the slope): surveyor's foot / lower point
- L (lower left): point on the horizontal
- H (right): lower station point on staff DH
- D (bottom): foot of the second staff

1. Let AB be a tower set upon a mountain, on whose ascent — that is, on whose slope — the surveyor stands at C, from which spot he cannot see the base of the tower. Let some staff CG be erected perpendicular to the horizon (not perpendicular to the mountain's slope), and let CE be the surveyor's height. Imagine the line EK drawn perpendicular to the altitude, or parallel to the horizon. Now apply the fixed square to the staff at point E — and if a plumb-line is used, the eye too must be placed at E — and aim the sighting-vane [dioptra] toward the summit A; from the shadow it cuts off, find the angle AEK, as we set out in Problem 1. Then make another, higher station at F on the staff, and, imagining the line FI drawn perpendicular to the altitude, again aim the sighting-vane toward A, and in the same way obtain the angle AFI. [Margin: Euclid I.32.]

Now the angle AFG equals the two angles FEA and FAE together. So if we subtract AEF — the complement of the earlier angle AEK found at the lower station E — from AFG — the complement of the later angle found at the higher station F — the remaining angle EAF becomes known. Therefore, since in triangle AEF two angles A and E are known, together with the side EF (that is, with the difference between the stations), the remaining sides AE and AF are made known. [Margin: On rectilinear triangles, 10.]

2. After this, let another rod DH be erected upright to the horizon, and, taking the surveyor's height equal to CE and drawing the line HL perpendicular to the altitude, apply the square at H and aim the sighting-vane to point E. Again, by

Problem 1, from the cut-off shadow derive the angle EHL, and let the visual ray HE be conceived as meeting the altitude at M; this ray will be parallel to BCD, the slope of the mountain. [Margin: Euclid I.33.] For the three points B, C, D are to be conceived as lying in one straight line, as if DC, produced, reached the base of the tower. [Margin: Euclid I.29.]

Since the angle EHL equals the angle MEK — the internal equal to the external — if EHL (that is, MEK) is subtracted from the angle AEN observed at the earlier station E, the angle AEM is left known. And the angle EAM is also known, since it is the complement of the angle AEN observed at the earlier station E. Therefore AME too, the remainder of the two right angles, will be known — the same remainder that is left if NMH, the complement of the angle MHL found at the last station H, is subtracted from two right angles. Consequently, since in triangle AEM all the angles are known, together with the side AE (which we shall find shortly), the remaining two sides ME and AM will also be known, and therefore the distance EM (that is, CB) will be found. [Margin: On rectilinear triangles, 10.] And if to the found line AM the surveyor's height MB is added, the whole height of the tower AB will be known.

Great care must be taken that the three points B, C, D lie in one straight line — as we advised must be done in Problem 22 of Book 2.

3. But if the base of the altitude can be seen from the mountain's slope, we shall accomplish the problem with almost no labor, by a single station. For if the square is set up at D so that the center of the sighting-vane is uppermost and the lowest side runs straight to B, the vane must be aimed toward that same point B. [Margin: Euclid VI.4.] For if we make:

As the cut-off turned shadow [umbra versa] is to the side of the square,
so the side of the square is to another [quantity],

the distance DB will be found in parts of the side of the square, as is plainly established from what we wrote in Problem 15, Number 5, and as appears in the first figure of that same problem — provided that in it we conceive the ascent of the mountain AF in the second square, and F as the base of the altitude set upon the mountain.

Next, if the same square is set up at D so that again the center of the sighting-vane is aloft and the lowest side runs straight to the top of the altitude A, and the same point A is sighted through the vane, the distance from D up to A will be found in the same way; if we make:

As the cut-off turned shadow [umbra versa] is to the side of the square,
so the side of the square is to another [quantity].

Finally, with the square adjusted so that one side coincides with the line DB and the vane is aimed at A, the angle formed by the lines DB and DA will be found, as we taught in Problem 16, Number 1. So if this angle is drawn out separately, and on the lines DB and DA two proportional portions are taken (as we set out in the same Problem 16, Number 2), the altitude AB will be found by means of the interval between those two portions — just as there we investigated the interval CD.

GEOMETR. PRACTICÆ

To determine, by means of the geometric square, the depth of a well or of any building erected to the perpendicular — provided the bottom corner, or some mark placed at the bottom, can be seen.

PROBLEMA XXVII.

Diagram. A vertical well or building, drawn as an upright cylinder, with the geometric square and its quadrant arc set at the top of an erected staff, and the well's mouth shown as an ellipse. The lettered points run down the left edge and around the mouth and floor of the well.

- E (upper left): top of the staff / upper station
- G (left, below E): point
- D (left, below G): station point
- F (left, below D): station point
- A (left, at mouth level): mouth edge / center of the vane
- I (upper right, by the arc): shadow-square point at the quadrant
- K (interior): point on the visual ray
- H (interior, right of K): point on the visual ray
- M (right, at mouth level): opposite edge of the mouth (orifice AM)
- B (bottom left): bottom corner of the well
- C (bottom): mark / corner at the floor
- L (bottom, by C): point on the floor line

1. This is nothing other than measuring a tower from its summit — when some mark appears on the horizon — by two stations made on a staff, as was done in Problem 9. We shall therefore briefly repeat the practice of that problem here.

Let there be a well, or an upright building, ABCM, whose corner C at the bottom — or some mark C placed at the bottom — can be seen. With a staff AE erected at the mouth of the well or at the top of the building, let two stations of the surveyor's eye be made at D and E; and, applying the side of the square to the staff twice — so that the center of the vane is set now at D and now at E — aim the vane toward C. If, then, at each station the vane cuts the right shadow [umbra recta], as usually happens in measuring wells, make:

As the difference of the right shadows is to DE, the difference of the stations, so the greater right shadow is to EB,

and the line EB will come out, known in parts of the difference of the stations DE. So if the line EA is subtracted — made up of the difference of the stations DE together with the portion of the staff DA (which is usually equal to the surveyor's height, or can at least be measured easily) — the desired altitude AB is left known.

2. But if at each station the turned shadow [umbra versa] is cut instead, and we make:

As the difference of the turned shadows is to DE, the difference of the stations, so the greater turned shadow is to EB,

the line EB again becomes known, and so on.

3. Finally, if at one station the side of the right shadow is cut and at the other the side of the turned shadow, then either the right shadow must be reduced to the turned, or the turned to the right. For if we again make:

As the difference of the shadows, whether turned or right, is to DE, the difference of the stations, so the greater turned or right shadow is to EB,

the line EB is once more produced in parts of the difference of the stations DE, and so on.

By a single station too we shall reach the depth of the well, just as in the scholium to Problem 9 we measured the altitude from a mountain's summit. That is: if the square is set up at A so that the center a of the vane is uppermost and the lowest side AC inclines from A toward point C — in order to find the distance, or hypotenuse, AC, and so on, as was done in that scholium, as this accompanying figure shows. For, secondly, the square must be placed so that its center a — the center of the vane — is at A, and the side AD adheres to the side of the well AB, so

that, with C sighted through the vane, the visual ray does not differ from the hypotenuse AC. So if we make:

As eb, the turned shadow, is to the side ba, so the side aA is to another [quantity],

the hypotenuse AC will be found. Then, once the portion of the vane AE in the second square has been found — as we taught in the scholium to Problem 7 [Margin: Euclid VI.2, by composition.] — if we again make:

As the found portion of the vane AE is to the found hypotenuse AC, so the side AD is to another [quantity],

the altitude, or depth, AB will come out.

Diagram. A rectangular figure with an inscribed small geometric square, showing the single-station measurement of the well's depth.

- A (upper left): center of the vane / top corner
- b (top, small letter): shadow-square point
- M (upper right): mouth edge
- D (left): side point (side AD)
- c (interior, small): shadow-square point
- e (interior, small): shadow-square point
- E (interior): station point
- C (interior/lower center): mark at the bottom
- B (bottom left): bottom corner
- C (bottom center): floor corner

4. But if the width of the mouth AM, or of the floor BC, is known — which can easily be learned by some measurement — we shall more easily reckon the depth AB by a single station made at D, and by a single application of the square. For if we make:

As the right shadow, if it be the one cut off, is to the side of the square, so the known width BC is to AB,

Or

As the side of the square is to the turned shadow, if it be the one cut off, so the known width BC is to AB,

the depth AB will be produced, known in parts of the width.

LIBER III.

And if the vane should happen to pass through point C in the square, the width BC will be equal to the line AB.

These practices, moreover, have all been demonstrated in the aforesaid Problem 9 — this last one, Number 4, excepted, which we demonstrated in Problem 8, Number 4.

To determine, by means of the geometric square, the depth of a valley and its oblique descent — provided this is not very uneven — and its far limit, or some mark that can be seen in it.

PROBLEMA XXVIII.

Diagram. A valley set between two mountains, the left mountain AB and the higher right mountain NG, with the surveying staff and its construction on the left summit. The two mountainsides are rendered as woodcut hills; the valley floor and its far limit carry the lettered points.

- E (upper left): top of the staff / upper station
- D (left, below E): station point
- A (left, at the horizon line): summit of the left mountain / foot of the staff
- I (horizon line, right of A): point
- H (horizon line, right of I): point on the visual ray
- L (horizon line, right of H): point
- N (upper right): summit of the higher mountain
- F (right, at horizon level): point on the far mountain
- B (bottom, left): base point below the left summit
- M (bottom): point on the valley floor
- C (bottom, center): far limit of the valley
- K (bottom): mark K in the valley
- G (bottom right): base of the right mountain

1. This too is nothing other than measuring some altitude from its topmost point by two stations on an erected staff — as was done in Problem 9 and repeated in the preceding problem. For let a valley be set between two mountains AB and NG, and let its far limit C be visible from mountain AB. Erect a staff AE, on which two stations of the surveyor's eye can be made at D and E; if the rest is constructed as in Problem 24 of Book 2 — whose figure we here repeat — the line EB will be

investigated, just as in the first figure of the preceding problem the line EB was found. And if the portion of the staff AE is subtracted, the height of the mountain AB (or the depth of the valley HC) is left known.

2. The oblique descent IC is gathered thus. By the shadow cut off from the vane toward C, in a straight line, derive the angle BDC by Problem 1 — that is, the angle CIM [Margin: Euclid I.29.] which is equal to it, external to internal; and hence the remaining angle ICM will be known, being its complement. Therefore, since in triangle CIM the acute angles are known, together with the side IM found just now, [Margin: On rectilinear triangles, 5.] the base IC cannot fail to be known.

But all this will be investigated more easily by a single station, if at point D of the staff the square is applied twice — as was done in the preceding problem, Number 2, and in the scholium to Problem 9 at point A, and so on.

3. But if the limit C cannot be discerned, choose some mark K in the valley that is visible from stations D and E. For thus, by Problem 9 — or rather by its scholium — the same altitude IM (or HC) will be found.

Indeed, if two stations can conveniently be made on the valley floor, the same altitude IM will be perceived from them by Problem 6, or by the scholium to Problem 7. The oblique descent IC, however — as an interval between the two marks I and C — must be investigated by Problem 16.

4. Finally, this same depth CH may be hunted from the higher mountain NG, provided the limit C of the lower mountain, or some mark in the valley, appears from the summit N — in no other way than as, in Problem 22 or 23, we found a smaller altitude from a greater one. For here the smaller altitude CH is to be sought, and the greater NG, from whose summit N we have determined that the limit C can be seen.

To measure, by means of a staff, the distance between the surveyor's feet and some mark on the plane of the horizon, when the far end of the distance can be seen.

PROBLEMA XXIX.

Diagram. A right triangle CAB with vertical staff and a graduated horizontal base, showing the staff-measurement of a horizontal distance. C is the surveyor's eye at upper left, A the foot of the observer, B the far mark, with the hypotenuse CB descending across the figure; the interior is crowded with intermediate station points.

- C (upper left): surveyor's eye
- F (left, below C): point on the vertical (side CF)
- E (top edge): top of the staff DE
- L (left): point on the horizontal
- A (bottom left corner): surveyor's feet
- o (interior, above the base line): point
- R (interior, on the horizontal): point
- Q (interior, on the horizontal): point
- M (interior, on the horizontal): point
- H (interior, upper): point on the hypotenuse
- K (interior, right): point
- T (interior, below K): point
- D (bottom row): foot of the staff
- N (bottom row, printed reversed): point [reversed "И" = N (inverted sort)]
- P (bottom row): point
- G (bottom row): point
- I (bottom row): point
- B (bottom right): far mark

1. Now that we have finished the measurements usually made with the quadrant and the square, it is worth adjoining some other methods of measuring, so that we may use them when neither quadrant nor square is at hand. Of the many methods, however, we shall select only those that are easier in use.

So let DB be the distance to be measured. At D erect a staff DE — shorter than AC, the height from the surveyor's eye to his feet — upright to the horizon. This is achieved if a thread with a plumb clings to the staff, or if a pebble dropped from E falls to point D. Then let the surveyor step back to A, until the visual ray issuing from C and passing through the top E of the staff meets point B; and let the line EF be understood to be drawn parallel to AB. [Margin: Euclid I.29.] Now since the triangles CFE and EDB are equiangular — because the angles F and D are right, and ECF and BED are equal, internal and external, and so on [Margin: Euclid VI.4.] — if we make:

As CF, the difference between the staff DE and the surveyor's height AC, is to FE, the space between the surveyor and the staff, so ED, the length of the known staff, is to DE,

the desired distance DB will come out known, in parts of the staff DE or of the surveyor's height AC. For the staff and the surveyor's height must be known in one and the same measure.

To investigate, by means of a staff, the altitude of a tower or of some other thing.

PROBLEMA XXX.

Diagram. A right triangle drawn for measuring an inaccessible height. The vertex C is at the upper left; a slanting hypotenuse runs down to the right to B. The base AB (the horizon) runs along the bottom.

- C (top, apex): summit of the height being measured
 - F (left side, just below C): top of the reduced construction
 - E (upper interior): intersection point on the hypotenuse
 - L (left side, middle row): point where line KL meets the vertical
 - A (bottom-left corner): foot of the vertical
 - D, N, P, G, I, B (along the base, left to right): the successive stations and rod-feet along the horizon (the reversed "N" is the printer's inverted letter-form)
 - O, R, Q, M, H, K, T (interior, from left to right): the sighting points, rod-tops, and intersections used in the two stations
1. Suppose that in the figure of the preceding problem the height AC is to be measured. Fix in the ground a rod [baculus] GH, upright to the horizon and somewhat taller than the surveyor's height from eye to feet, which is IK. Then let the surveyor walk back to I, so that with his eye at K, placed there, he sights the summit C. Understand a straight line KL drawn parallel to the horizon AB, cutting the rod at M. [Margin: coroll. 4 sexti (Euclid, *Elements* VI)] Since, then, the triangles KMH and KLC are similar, because MH and LC are parallel, if you set up the proportion:

As KM (the space between the surveyor and the rod) is to MH (the difference between the rod and the surveyor's height), so is the distance KL (which is made known by some measurement) to LC —

then the straight line LC will be produced. If to it the surveyor's own height AL is added, the whole proposed height AC will be known, in terms of the surveyor's height or of the distance KL.

2. But suppose there is no clear approach to the height AC, so that we cannot measure the distance IA. Then fix the same rod, or another equal to it, NO — evidently by moving nearer to the height. Then let the surveyor step back to P, so that with his eye at Q, sighting through the top of the rod O, he again sees the summit C; and let the rod NO be cut by the straight line KL at R. [Margin: 4 sexti (Euclid VI); 8 quinti (Euclid V)] Now since, as KM is to MH, so KL is to LC; and likewise as QR is to RO, so QL is to LC; and since the ratio of KL to LC is greater than the ratio of QL to that same LC, the ratio of KM to MH will also be greater than the ratio of QR to RO — RO being equal to MH. (For since the whole rods GH and NO are equal, and the parts cut off GM and NR are equal, the remainders MH and RO will likewise be equal.) [Margin: 10 quinti (Euclid V)] Therefore KM will be greater than QR. So cut off KT equal to QR itself. [Margin: 4 sexti (Euclid VI)] Since, then, as KM is to MH, so KL is to LC, it will follow, by alternation, that as the whole KM is to the whole KL, so MH is to LC. By the same reasoning QR — that is KT, cut off from KM — will be to QL — cut off from KL — as RO is to LC, that is, as MH is to LC. Therefore the remainder TM will also be to the remainder KQ as the whole KM is to the whole KL. But we showed just now that as KM is to KL, so is MH to LC. And so, if you set up the proportion:

As TM (the difference between the spaces KM and QR, or KT, from the surveyor up to the rods) is to KQ (the difference of the stations), so is MH (the difference between the rod and the surveyor's height) to LC —

LC will be made known; and if the surveyor's height AL is added to it, the proposed height AC will likewise come out known.

3. If now the first station is made at Q and the second at K, moving further back from the height, the procedure and demonstration will not change, as is clear.

To find, by means of the set-square, a distance on the plane of the horizon between the surveyor and any given mark.

PROBLEMA XXXI.

Diagram. A right triangle for finding a horizontal distance with a set-square [norma]. The vertical rod stands at the left; the hypotenuse slopes down to the right toward B.

- H (top-left): top of the vertical
- I (left side, below H): point on the vertical

- C (upper, right of center): the corner of the set-square at the rod's summit
- D (interior): point on the inner side CD of the set-square
- E (right, on the hypotenuse): point on the sighting side CE
- G (bottom-left corner): foot of the vertical
- F (bottom, near center): the marked point where sight-line CD meets the ground
- A (bottom, center): foot of the rod AC
- B (bottom-right): the far mark whose distance is sought

A dotted line runs from D down toward F.

1. Let the distance AB be proposed. At A, the surveyor's position, set up at right angles a rod AC, a little shorter than the surveyor's height and known through the measure by which the surveyor's height is known. Then apply the right angle of the set-square C to the top of the rod, and let its side CE, about C, be gradually raised or lowered until the visual ray, running along the inner side CE of the set-square, reaches the far point B; and carefully mark the point F on which the visual ray passing along the other inner side CD falls. [Margin: coroll. 8 sexti (Euclid VI)] Since AC is the mean proportional between AF — which is made known in terms of the rod — and the distance AB, then if you set up the proportion:

As AF (between the rod and the point F, known) is to the rod AC, so is the rod AC to AB —

That is, if the square of the rod's number is divided by AF, the distance AB will come out as the quotient, known in terms of the rod.

2. It would not be amiss if two small pins were fixed on each inner side of the set-square, so that the visual ray passing along them may run more truly to B and to F.

To investigate, by means of the set-square, the height of a tower or of some other thing.

PROBLEMA XXXII.

1. In the figure of the preceding problem, let the height GH be measured. Again fix at A, the surveyor's position, a rod AC; to its summit C apply the right angle of the set-square, and let its side CE be gradually raised or lowered until

until, through the inner side CE, the top of the height H is sighted, and the point F, where the other side CD meets it, is marked. Since the straight line CI is drawn parallel to the horizon, the triangles ACF and HCI are equiangular — because the angles A and I are right angles, and the angles ACF and HCI, being the remainders of the right angles ACI and HCD (once the common angle DCI is removed), are equal. [Margin: 4 sexti (Euclid VI)] So if you set up the proportion:

As the rod AC is to AF (between the rod and the point F, known), so is the distance CI, or AG (which is made known by some measurement), to IH

—

then IH will be produced, known; and if to it the length of the rod AC, or GI, is added, the proposed height GH will be known.

To measure a horizontal distance that is not very great, by another, most easy method.

PROBLEMA XXXIII.

Diagram. Two slanting sight-lines rising from the base to cross near the top, with a vertical rod between them; the small staff CD is shown in two positions.

- C, B, D (stacked at the top, left of crossing): the staff CD and rod-top in the first (sighting-E) position
 - C, D (top, right of crossing): the staff CD in the second (sighting-F) position
 - F (bottom-left): the point on the measurable ground
 - A (bottom, center): foot of the rod
 - E (bottom-right): the far edge of the distance being measured
1. When some stretch of ground is short in length but cannot be measured because of some obstacle — such as the widths of rivers and pools, whose widths (which we here take as "lengths") cannot be measured because of the water lying between — we shall use, among other means, this device also. Near the bank of a river or pool, or of whatever else is to be measured, fix a rod AB upright to the horizon; to it, at the point B, fit a small staff [virgula] CD, so that about B it can be lowered or raised, until the visual ray passing along CD reaches the far edge E of the distance AE to be measured. Then, keeping the staff CD unmoved so that the angle DBA does not change, turn the rod AC — always standing at right angles to the horizon — until the visual ray running along the same staff CD reaches some ground near the river or pool, where of course the surveyor stands and which can be measured, at the point F. Since, then, the two angles A

and B of the triangle ABE are equal to the two angles A and B of the triangle ABF, and the side AB is common, [Margin: 26 primi (Euclid I)] the sides AE and AF will be equal. And therefore, if FA is made known on the ground by some measurement, the distance AE will likewise be known.

2. Others accomplish this same thing without a rod, in this way. They stand upright at right angles to the horizon; then they lower a hat brim — one that is somewhat wide — until the visual ray running along the edge of the brim falls on the far end E of the distance AE to be measured. Then, keeping the brim unmoved, they turn toward some measurable ground and mark on it the point F where the visual ray meets that ground. For again the length AF is equal to the proposed distance AE, because the surveyor's height then serves the office of the rod AB, and the lowered brim takes the place of the staff CD, as is evident.

By means of the geometric square, to determine the height of any upright thing from the shadow it casts when the Sun is shining, if that shadow is known.

PROBLEMA XXXIV.

Diagram. A large right triangle with the vertex G at the upper right and a shadow-square [quadratum] set near the acute angle at the left. Several interior letters (the square's corners and the shadow-cutting points) are overprinted on one another.

- G (top-right): the top of the height
- M (right side, below G): point on the sight-line for height FM
- I (right side, below M): point for height FI
- A (interior, upper): corner of the shadow-square
- B, H (left, upper): corners of the square and the point where the right shadow HB is cut
- C, D, E (interior, near the square): the shadow-cutting points and square corners (the square's top-left corner over-strikes C and B — a genuine over-strike, not a misread)
- A (bottom-left corner): foot of the height / gnomon
- K (bottom-left, below A): lower corner
- D (bottom, center): foot of a sight-line
- F (bottom, center-right): the shadow-tip on the ground
- L (bottom-right): lower-right corner

1. This problem does not differ from the fifth, except that here we use the Sun's ray in place of the visual ray, and the length of the shadow, which the height casts when the Sun is shining, in place of the distance from the surveyor to the height. So let the height be FG, or FI, or FM, and its cast shadow FA. Direct the geometric square toward the Sun (even though this is not done directly in line with the height); and as the Sun's ray passes through the holes of the sights [pinnacidia], mark how much shadow — whether right [recta] or turned [versa] — is cut off on the square by the alidade [dioptra] or by the plumb-line [filum perpendiculi]. For the right shadow HB, cut off on the square, will stand to the side AB as the cast shadow AF stands to the height FG. Likewise the ratio of the side AD to the turned shadow DE, cut off on the square, is that of the cast shadow AF to the height FI. Finally, when the alidade or plumb-line passes through the point C, the cast shadow AF will be as great as the height FM — all of which were demonstrated in problem 5. Therefore, if you set up the proportion:

As the right shadow HB, cut off on the square, is to the side AB, so is the length of the cast shadow AF to FG,

Or: As the side AD is to the turned shadow DE, cut off on the square, so is the length of the cast shadow AF to FI —

the proposed height FG, or FI, will come out known, and so on.

By means of the geometric square, to make plain and manifest the length of a shadow cast from a height when the Sun is shining, when the height is known.

PROBLEMA XXXV.

Diagram. Three side-by-side right triangles, each with a shadow-square set at the acute vertex, illustrating the right shadow, the turned shadow, and the limiting case in turn. In each figure the square carries the corners A, D, B and the interior points C, E.

- A (top of each square): upper corner
- D (left of each square): left corner
- B (right of each square): right corner, on the hypotenuse
- C, E (interior of each square): the plumb-line point and the shadow-cut point
- F (bottom-left of each triangle): foot of the height
- G (bottom-right of each triangle): shadow-tip on the ground

1. This problem, too, is no different from the second: only here we take, in place of the distance, the length of the shadow which the known height casts when the Sun is shining. So let the height be AF, and its cast shadow FG. Direct the geometric square toward the Sun, wherever the surveyor may stand outside the shadow; and as the Sun's ray passes through the holes of the sights, observe the shadow — whether right or turned — which the plumb-line or the alidade cuts off. For the side of the square BA will stand to its right shadow BE, cut off, as the height AF stands to the cast shadow FG. Likewise the turned shadow DE, on the square, will stand to the side AD as the height AF stands to the cast shadow FG. Finally, when the alidade or plumb-line passes through the point C, the height AF is equal to the length of the cast shadow FG — all of which we demonstrated in problem 2. Therefore, if you set up the proportion:

As the side AB is to the right shadow, cut off, BE, so is the known height AF to FG,

Or: As the turned shadow DE, cut off on the square, is to the side AD, so is the known height AF to FG —

the length of the cast shadow FG will come out known, and so on.

By means of a certain very simple instrument, to find a horizontal distance between the surveyor and some visible mark.

PROBLEMA XXXVI.

Diagram. The instrument: a vertical rod AB with a graduated horizontal arm CD (shown as a dotted, marked line running to the right) and a foot-arm BE, forming a right triangle with the sight-line to F.

- A (top-left): top of the vertical rod
 - D (top, just right of A): far end of the graduated arm / the marked sight-point
 - C (left, below A): the first graduation, where arm CD is fixed
 - B (bottom-left): foot of the vertical rod
 - E (bottom, center): point on the foot-arm BE
 - F (bottom-right): the far mark whose distance is sought
1. Let the instrument be constructed in this way. Take a straight rod AB, a little shorter than the surveyor's height, and divide it into 5 equal parts — or into more, or fewer. On its first part C, or on any other part, fix at right angles another rod CD, on which any number of palms or feet are marked

off in order. On the end of the rod AB, too, fix at right angles another rod BE. The instrument so built will let you measure various magnitudes, as will be shown. For suppose first that the distance BF is to be measured, whose end F the surveyor, standing at the other end B, can see from A, even if there are valleys and other obstacles between. Set the instrument in such a position that AB is perpendicular to the horizon and BE equidistant from it. Sighting the far end F from A, mark with the greatest care and diligence the point D on the rod CD through which the visual ray passes.

[Margin: coroll. 4 sexti (Euclid VI)] For since the triangles ACD and ABF are similar, [Margin: 4 sexti (Euclid VI)] if you set up the proportion:

As AC is to AB, so is CD to BF —

the distance BF will be found, in terms of CD. Thus if AC is, for example, the fifth part of the rod AB, then CD will also be the fifth part of the distance BF. And since in our example CD contains 2 feet, if we multiply them by 5, there will be produced 10 feet for the distance BF.

2. Care must be taken diligently, that when the visual ray does not pass through some whole foot of the rod CD, it be investigated how many tenths or hundredths of one foot are contained in the little part cut off — by the means we wrote of in Book 1, Chapter 2, Number 14 — which will be done if ten feet are comprised on the rule CD. And the same is to be understood of any other measure, if it is marked on CD in place of feet.
3. But we have achieved this same thing with equal ease by means of the geometric square, whether hanging or fixed.

By means of the aforesaid instrument, to estimate the distance between the two summits of mountains or towers.

PROBLEMA XXXVII.

Diagram. A landscape woodcut of two mountain peaks, F (the lower) and G, with the aforesaid instrument set on the summit F; graduated sight-lines (a dotted, marked line) run from the instrument toward the far peak G. Several interior letters along the sight-lines are small and overprinted, and some are not clearly resolvable.

- A, D (top-left, on the instrument): rod-top and marked graduation
- C, B (left, at the instrument's base on peak F): the instrument's parts

- F (lower-left, base of the smaller mountain): the summit where the instrument stands
 - E, e, f (interior, along the sight-lines): intermediate points (also g, h present)
 - G (upper-right): the far mountain summit
 - a short rotated italic string near the top-right (by peak G) is not legibly resolvable [best guess "buo"/"guo"; ?]
1. Let there be two mountain summits, F and G. Placing the point B of the aforesaid instrument on the summit F of the smaller mountain, let the rod AB be lowered until the rod BE points straight toward the summit G — which is easily done by means of two little pins fixed at B and E, as we said in Number 2 of Problem 31. With the instrument kept in this position, let the summit G be sighted from A, and let the point D be observed, the intersection of the ray AG with the rod CD. [Margin: coroll. 4 sexti (Euclid VI)] For since the triangles ACD and ABG are equiangular, if you set up the proportion:

(Continuation of Problema XXXVII, whose enunciation and setup were given on the preceding page. The proportion begun there is completed here.)

As AC, known in parts of AB, is to AB, of the known parts, so CD, known in the given measure, is to BG,

the distance BG will be found, in parts of CD. For example, if AC is a fifth part of AB, and CD contains 1 foot plus a further $\frac{2}{3}$ [i.e. $1\frac{2}{3}$]: then, if $1\frac{2}{3}$ is taken five times, the distance BG will come out to $8\frac{1}{3}$ feet.

2. In just the same way the distance bF, from the summit G of a higher mountain, will be found, as the figure shows. The procedure is the same if the points F, G are the tops of two towers, or if one is the top of a tower and the other the summit of a mountain, as is evident.

3. This same result is obtained by means of the [geometric] square in the following way. Let the square be $abfg$, and let its angle b be placed at the summit G, and let the side bf be lowered so that, if it is imagined produced, it points to the summit F; and let the versed shadow [umbra versa] gh , cut off by the visual ray aB , be observed. For since the triangles agh and Bba are equiangular [Margin a: coroll. 4 of book six (Elements VI.4)], if it is made:

As gh , the versed shadow, is to ga , the side of the square, so the side of the square ab is to Bb , [Margin: 4 of book six]

that is, if the square number of the side — namely 100000 — is divided by the versed shadow, the distance Bb will be produced in the quotient.

The same would result if the angle of the square were placed at B, and the lowest side pointed straight from B to the summit G, and so on.

To find the length of the ascent [slope] of a mountain, if its summit is seen from an eye stationed at its base, by means of the same instrument.

PROBLEMA XXXVIII.

1. Let the aforesaid instrument be set up so that the point B is placed at the foot of the mountain, and let the rod BE — by means of two small pegs fixed in it, as was explained in No. 2 of Problem 31 — incline straight toward the summit F; and let the intersection D of the visual ray with the rod CD be marked, the summit F being observed from A. [Margin a: coroll. 4 of book six] For again the triangles ACD, ABF will be similar. [Margin b: 4 of book six] Therefore if it is made:

As AC, known in parts of AB, is to AB, of the known parts, so CD, known in the given measure, is to BF,

the oblique ascent of the mountain, BF, will be known in parts of CD.

2. By the same reasoning, if F is the top of some tower, you will find the distance from the point B — placed on the ground or elsewhere — up to F, that is, the hypotenuse BF, as is clear. And with this same instrument many other measurements can be carried out, as the prudent reader will readily see.

3. We shall achieve the same with the square ABGH, if its angle B is placed at the foot, and the side BG is raised so that it points straight to the summit F. For when the versed shadow HI has been observed — which the sight [dioptra], directed toward the summit F, cuts off — the triangles AHI, FBA become equiangular. [Margin c: 4 of book six] Hence, if it is made:

As HI, the versed shadow, is to AH, the side of the square, so the side of the square AB is to BF,

that is, if the square number of the side, 1000000, is divided by the versed shadow, the resulting quotient will give the length of the oblique ascent BF.

Diagram. A mountain-slope figure for Problems XXXVII–XXXVIII. The large sloping triangle rises from B (at the lower left, the observer's station) to the summit F (upper right); the hatched face at the right represents the mountainside. Near the observer's station the geometric-square instrument is

drawn with its lettered corners and the visual rays to the summit. Readable labels:

- A (upper left, at the near corner of the instrument)
- C (immediately below A)
- D (interior, just right of C — intersection of the visual ray with the rod)
- E (interior, below D)
- G (interior, center — corner of the square)
- I (upper center, small — versed-shadow point)
- H (upper left, above A)
- F (upper right — the summit)
- B (lower left — the observer's station)

To determine a height to whose base access is open, by means of a plane mirror, together with the distance of the mirror from the top of the height.

PROBLEMA XXXIX.

1. Let AB be the height. From its base B let one recede any number of paces or feet up to the point C, at which the center of a plane mirror is placed; and along the straight line BC let one step back until the surveyor's eye, stationed at E, can look at the summit A by the reflected ray ECA — so that DE is the surveyor's height, from the eye down to the ground. And since the angle of incidence DCE is equal to the angle of reflection ACB, as the writers on optics [Perspectivi] teach, and the angles D, B are right angles, the triangles DCE, BCA will be equiangular; [Margin a: 4 of book six] and therefore CD will be to DE as CB is to BA. Hence if it is made:

As CD, the surveyor's distance from the mirror C, is to DE, the surveyor's height, so CB, the distance of the mirror from the height, is to BA,

the height BA which we seek will be produced, known in parts of the surveyor's height DF [recte: DE].

ALITER.

2. Let the angle DCE, or BCA, be measured by a quadrant. (This will be done if a right angle FGH is constructed, and the line FG is made to contain as many equal small parts as there are paces or feet in the distance CD;

and the line GH contain as many of those same small parts as the surveyor's height DE comprises paces or feet. For when the line FH is joined, [Margin b: 4 of

book one (Elements I.4)] the angle F will be equal to the angle C, and this angle F we shall determine without any trouble by means of some quadrant divided into degrees.) For if, taking the whole sine [sinus totus, = 10000000] CB, it is made:

As the whole sine CB is to BA, the tangent of the angle of reflection BCA (or of incidence DCE) which we have just found, so CB, the known distance, is to BA,

the height BA will result, known in parts of the distance CB. And if again it is made:

As the whole sine CB is to CA, the secant of the same angle BCA, so CB, the known distance, is to CA,

CA will also be known — the distance from the mirror C up to the summit A — in parts of the distance CB.

ALITER.

3. By sines alone we shall proceed thus, if one wishes. [Margin c: 10 of On Rectilinear Triangles] Let it be made:

As the sine of the angle A — the complement of the angle C, of reflection or of incidence — is to CB, the known distance, so the sine of the angle C, of incidence or of reflection, is to BA,

[Margin d: 10 of On Rectilinear Triangles] for the resulting number will give the height BA, known in parts of the distance CB. Again let it be made:

As the sine of the angle A, the complement of the same angle C, is to CB, the known distance, so the whole sine of the right angle B is to CA,

for the resulting number will give the hypotenuse CA, known in parts of the distance CB.

To determine an inaccessible height by means of a plane mirror, together with the distance of the mirror both from the base — even when the base is not visible — and from the top of the height.

PROBLEMA XL.

1. Let AB again, in the preceding figure, be a height erected upon the plane BD. With a plane mirror placed at C, let one recede from the height to D, until the summit A can be seen by the reflected ray ECA; and let the angle ECD — and hence also the angle ACB — be observed by a quadrant, as was explained in No. 2

of the preceding problem. Then, with the mirror placed at the point K, distant any number of paces from C toward the height, let one again step back to I, until the summit A is once more observed by the reflected ray LKA; and let the magnitude of the angle IKL — and hence also of the angle AKB — be sought. And since, when the whole sine AB is taken, the lines BK, BC are the tangents of the angles BAK, BAC (which are the complements of the angles K, C of reflection, known by the quadrant), the difference of these tangents, KC, will be known. If, therefore, it is made:

As KC, the difference of the tangents — which are due to the complements of the angles of incidence — is to AB, the whole sine, so KC, the difference of the positions of the mirror, is to AB,

Diagram. The mirror figure for Problems XXXIX and XL (the same woodcut also appears at the foot Two right triangles stand on a common horizontal baseline running from D (left) to B (right): the right vertical is BA, the height (A at the top right); the left vertical is the surveyor's height (E at the upper left). Reflected rays run from A and E to the mirror positions on the baseline, which are marked by two circles — the nearer mirror near C/I and the farther mirror near K. Readable labels: H and L (upper left and upper center), A (upper right), E (left), G and F (small interior points of the auxiliary right-angle construction FGH), B (lower right), D (lower left), C and I (in and beside the left circle), K (in the right circle). The interior cluster (C, I, G, F) is densely overprinted. [— overprinted interior labels.]

there will be produced a number which will show the height AB, known in parts of the difference of the mirror positions KC.

RVRSVS. [Again.]

If it is made:

As KC, the difference of the tangents of the complements of the angles of incidence at the mirror, is to CB, the greater tangent, so KC, the difference of the positions of the mirror, is to CB,

CB will be found — the greater distance of the mirror C from the height. If from it KC, the difference of the mirror positions, is subtracted, KB will remain known — the lesser distance of the mirror K from the same height. This too will be found if it is made: as KC, the difference of the tangents of the aforesaid angles (which are the complements of the angles of incidence at the mirror), is to KB, the lesser

tangent, so KC, the difference of the mirror positions, is to another term — as is clear.

[Margin a: 22 of book one] Next, since the angle AKB, in the nearer position of the mirror, is equal to the two angles ACK, CAK, if the angle ACK of the farther position is subtracted from the angle AKB of the nearer position, the angle CAK will remain known. [Margin b: 10 of On Rectilinear Triangles] If, therefore, it is made:

As the sine of the angle CAK, the difference of the angles of incidence, is to KC, the difference of the positions of the mirror, so the sine of the angle AKC — the complement of the angle AKB to two right angles, in the nearer position of the mirror — is to CA,

[Margin c: 10 of On Rectilinear Triangles] there will be generated the hypotenuse CA of the farther position of the mirror, in parts of the difference of the mirror positions KC. And if again it is made:

As the sine of the angle CAK, the difference of the angles of incidence, is to KC [see anomaly note], the difference of the positions of the mirror, so the sine of the angle of reflection ACK, in the farther position of the mirror, is to KA,

there will likewise be produced the hypotenuse KA of the nearer position of the mirror, in the same parts of the difference of the mirror positions KC.

ALITER.

2. By sines alone we shall achieve the same thing in this way. With the hypotenuse CA found, as we said just now, by sines let it be made: [Margin d: 10 of On Rectilinear Triangles]

As the whole sine of the right angle B is to the hypotenuse CA that was found, so the sine of the angle ACB, of the farther position of the mirror, is to AB,

for the height AB will come out in the quotient, known in parts of the hypotenuse CA that was found. [Margin e: 10 of On Rectilinear Triangles] And if again it is made:

As the whole sine of the right angle B is to the hypotenuse CA that was found, so the sine of the angle BAC — the complement of the angle in the farther position of the mirror — is to CB,

CB will be produced — the greater distance of the mirror from the height. If from it KC, the difference of the mirror positions, is subtracted, the lesser distance KB will also remain known. This too, if the hypotenuse KB is sought as taught above, will be found: if it is made, as the whole sine of the right angle B is to the hypotenuse KB that was found, so the sine of the angle BAK — the complement of the angle in the nearer position of the mirror — is to another term, as is manifest.

To make known a height set upon a mountain — provided the base of the height can be seen — or the upper portion of some tower, by means of a plane mirror.

PROBLEMA XLI.

1. When access to the tower is open, so that its distance from the surveyor can be known: if, by Problem 39, one investigates both the height from the top of the proposed portion down to the base of the tower, and the height from the lowest part of that same portion down to the same base of the tower — and this lesser height is taken away from that greater one — the remainder will be the portion sought.

2. But when the height is set upon a mountain, and the base of the height is visible, or access to the tower is not open: both of the aforesaid heights must be sought by the preceding problem. For again the lesser, subtracted from the greater, will yield the remaining height or portion that is desired.

To map out on a plane the layout of any field, or courtyard, or temple, or even a city or any region whatsoever — if, from two locations chosen within that layout, the rods erected at all the corners of the field (or at least the corners themselves in a building or city, or the places of the region) can be seen — and at the same time to determine, in a given measure, the lengths of the sides of the field or building, as well as the distances between the corners and either of the chosen locations. And if two such locations cannot be chosen within the layout, to accomplish the same thing, provided we can walk around the layout.

PROBLEMA XLII.

1. Although this problem is either geographic or architectural, nevertheless, since it cannot be carried out without the measurement of lines, I am pleased to explain it briefly here. Let there be, then, a field bounded by five sides AB, BC, CD, DE, EA. Let five rods be fixed at the five corners A, B, C, D, E, at right angles to the horizon; and around the middle of the area let there be prepared [nareturque (?)]

a somewhat raised level surface, parallel to the horizon, on which are two points F, G, distant from each other by any amount — for example, 100 feet — from which all five rods can be seen. Through F, G let the straight line FG be drawn to both sides; and the segment FG will contain 100 feet by hypothesis. Then, with a rotatable sight [dioptra] fitted with vanes [pinnacidia] fixed at each point F and G, and with two circles described from F and G — so that through their circumferences the magnitudes of the angles formed at F, G can be known — let the rods erected at the corners A, B, C, D, E be observed from F and G through the holes of the vanes (the sight being turned around), and let the angles which the line of sight [linea fiduciae] makes with the straight line [FG]

[Margin: The layout of any field — how it is delineated from two locations chosen within it.]

Diagram. The surveying figure for Problem XLII: a five-sided field with rods at the corners and two interior observation stations. The pentagon has corners A (upper left), B (right), C (mid lower-right), D (bottom right), and E (lower left); a short stroke at each corner represents the rod set upright there. Near the center are the two observation points F and G, each enclosed in a circle (the graduated dial of the rotatable sight); the straight line FG runs horizontally through both circles and is extended to both sides. Sight lines radiate from F and from G to each of the five corners. Readable labels: A, B, C, D, E (the corners), F and G (the interior stations).

makes — or the angles that the lines drawn from the fiducial line [linea fiduciæ] make with one another — are transferred in order to the points K and L. These points are set any distance apart on a straight line KL drawn separately on a sheet of paper, and the lines that form those angles at K and L are produced, as the second figure shows. For if the points where those lines issuing from K and L meet are joined by straight lines, the figure OPQRS will be drawn, exactly similar to the figure of the field ABCDE.

Diagram. A five-sided survey figure OPQRS drawn on paper, similar to the field ABCDE, with two interior sighting points K and L. Straight lines radiate from the corners to the interior points, forming the network of similar triangles used in the proof.

- O (top-left corner)
- P (top-right corner); along the edge OP a graduated scale of tick marks bears the numeral **17** (the seventeen parts found in side OP)
- L (interior, left of center)

- K (interior, center); between L and K a short graduated segment (tick marks) marks the interval KL, and a horizontal straight line runs through L and K across the figure
- Q (interior, lower right)
- S (bottom-left corner)
- R (bottom-right corner)
- Lines join O, P, Q, R, S to the interior points K and L.

I prove this as follows. The triangles AGF and OLK are similar, since by construction the angles AGF, AFG equal the angles OLK, OKL. [Margin: (a) Euclid, *Elements* VI.4] Therefore AG is to GF as OL is to LK. By the same reasoning, from the similarity of the triangles FGE, KLS, GF is to GE as LK is to LS; and hence, *ex aequali*, AG is to GE as OL is to LS. Now the angles AGE and OLS, about which those sides are proportional, are equal — for by construction the angle OLS was made equal to the angle AGE. [Margin: (b) *Elements* VI.6] So the triangles AGE, OLS are similar, that is, equiangular.

In the same way, from the similarity of the triangles FGD, KLR, [Margin: (c) *Elements* VI.4] GD is to GF as LR is to LK. Likewise, from the similarity of the triangles FGE, KLS, FG is to GE as KL is to LS. Therefore, *ex aequali*, GD is to GE as LR is to LS. And since by construction the angles DGE, RLS are equal, [Margin: (d) *Elements* VI.6] the triangles DGE, RLS will also be equiangular. In just the same way we shall show that the triangles CGD, BGC, AGB are equiangular with the triangles QLR, PLQ, OLP.

Indeed, by the same arguments we could conclude — though it is not necessary — that the triangles constructed at F upon the sides of the field are equiangular with those set up at K upon the sides of the figure OPQRS. From all this it follows that the figure ABCDE is equiangular with the figure OPQRS: for their angles are each compounded of equal angles — the angle AED, for instance, of the angles AEG, GED that compose it, which equal the angles OSL, LSR composing the angle OSR; and so for the rest. It also follows that the sides of these figures about their equal angles are proportional. For, because of the similarity of the triangles, [Margin: (e) *Elements* VI.4] AE is to EG as OS is to SL, and EG is to ED as SL is to SR; and therefore, *ex aequali*, AE is to ED as OS is to SR. And so with the others. The figures ABCDE and OPQRS are therefore similar.

2. Now, to find the lengths of the sides AB, BC, CD, DE, EA and of the lines drawn from F or G to the angles in the first figure, we must divide the interval KL in the figure we have found — that is, in the second figure — into any number of equal

parts. Then we must find how many of those parts are contained in each of the sides and in the lines issuing from K or L in that second figure. This can be done either with the compass, by stepping one small part repeatedly along the said straight sides; or — as I prefer — in this way: step the whole of KL along each side or line as many times as it will go, and in the remaining segment step off with the compass one small part of the interval KL as many times as possible. For however many times KL is repeated, that many times the number of the parts of KL will be contained in the side, together with as many further parts as the compass detects in the remaining segment. Or else, by the method we described in Book 1, chapter 1, near the end of number 2, find on the instrument of parts how many parts of the interval KL are contained in the said sides and lines.

Then set it up so that: as the number of parts of the interval KL assumed in the second figure is to the number of feet assumed between the points F and G in the first figure, so is the number of parts found in any side or line in the second figure to a fourth number. That number will indicate how many feet are contained in the side or line taken. The reason is that, since KL in the second figure bears the same ratio to any side or line of that figure as FG in the first figure bears to the corresponding side or line — because of the similarity of the figures — then, by alternation, KL will be to the interval FG as side to side, and so on. For example: in the second figure the interval KL is divided into 5 parts, of which 17 were found in the side OP; and since the space FG in the first figure was set at 100 feet, then, if we set it up:

As KL, of five parts, to FG, 100 feet: so OP, of 17 parts, to AB,

that is — as the golden rule [*regula aurea*, the rule of three] directs — if 100 is multiplied by 17 (the second number by the third), and the resulting number 1700 is divided by 5 (that is, by the first number), there will be found in the quotient 340 feet for the side AB; and so for the rest.

3. [Margin: How the plan of any field is drawn from a single station taken within it] We shall also draw this same plan of the proposed field ABCDE from a single station F taken within the field, in the following way. Let the dioptra be aimed at each of the rods set up at the angles, noting the angles that the lines marked out along the dioptra make with one another; and let the distances from F to each angle be found in some measure — either with an iron chain that can neither be stretched nor slackened, or with a cord stretched from F to each angle, or, if those distances are large, by Problem 2 or 36 with the help of the square or of some other instrument. For if on a sheet of paper the same angles are set up at any point K, and along the straight lines forming those angles as many mutually equal

parts of any size whatever are taken as there were measures found in the corresponding lines issuing from F; and if the outermost points of the last parts are joined by straight lines, the figure OPQRS will be drawn, exactly similar to the field ABCDE. [Margin: (f) *Elements* VI.6] For the triangles gathered at the point F are similar to the triangles gathered at the point K, on account of the equality of the angles set up at F and K and the proportionality (by construction) of the sides about those angles.

4. The lengths of the sides in the field ABCDE will be known if we find with the compass how many parts are contained in the sides of the figure OPQRS out of those that were taken in each straight line drawn from K. For just as many measures will be contained in the sides of the field, as is plain from the similarity of the figures.

5. [Margin: How the plan of any city or region is drawn] Exactly the same method is to be kept in surveying the site of some courtyard or temple, or of a city or region. Only, in mapping a city, two high towers must be chosen for the points F and G, from which all the angles of the city can be seen, and whose distance from one another is either known or has been found by the preceding problems. In mapping a region, two towns must be chosen for those same points F and G, and from the highest tower in each the surrounding towns must be observed, so as to obtain the angles that the lines drawn from the surveyor's eye to the individual towns make. These angles will be observed more easily if, in place of the dioptra — which is too high — a board is set up

erected at the points F and G, so that when it is turned about it passes through the surrounding towns, provided it is imagined to be extended. For in this way the board itself will mark out the straight lines that form the aforesaid angles. And this same thing must be done in surveying the site of cities. Therefore, if the angles of some city, or the towns of some region, are A, B, C, D, E, and the two towers are at F and G, everything must be carried out as was said above about the site of a field. For the figure OPQRS will exhibit the site of the city or region, and the distances of the places A, B, C, D, E from one another will be known, as was said about the sides of a field.

6. But if within the proposed site there are not two places, or at least one, from which all the angles can be seen — as happens in all buildings — then the site must be gone around, and the angles investigated by means of straight lines. In the open field these are run from rod to rod with an iron chain or a cord; or else a board must be set up so that it is seen to pass through each pair of rods, for the board itself will exhibit the said straight lines. In buildings and temples, however,

the outer walls themselves form the said angles, whose size the craftsmen usually investigate with a certain instrument made of two rules, one entering under the other so as to slide (which the Italians call the *squadra zoppa* [lame square]), as this figure shows.

Diagram. The *squadra zoppa* (lame square), an angle-measuring instrument of two rules. It is shown as an inverted V (a splayed A-frame): the two legs [crura] meet at a narrow apex at the top and open out at the feet. The inner edges of the legs are laid against the two walls that form an angle. No lettered labels appear on the figure.

For when the instrument is opened, if its legs [crura] meet two walls forming an angle, the inner sides of the legs will give at their meeting the angle sought. And if the same instrument is applied to the interior angles of buildings, those same inner sides of the legs will give the interior angles that are formed by the walls.

Once the angles have been found and noted, the intervals between the rods set up in the field, or between the angles of the buildings, must be measured with an iron chain or a cord; or else, in the fields, if those intervals are long, they may be explored by Problem 2 or 36 in some known measure, such as cubits. For if on paper a line OP is drawn containing as many equal parts as there are cubits — say — found in the interval AB, and the angle POS is made equal to the found angle BAE; and if along the straight line OS as many parts equal to the former are taken, up to S, as there were cubits found in the interval AE; and again the angle OSR is made equal to the angle AED, and so on in turn — then the site of the field will be represented by the figure OPQRS. The same is to be understood of the site of buildings, both exterior and interior, provided the angles and distances are carefully observed.

To measure, by means of the geometric square, the LENGTH of a beam inclined to the horizon — of which only the upper portion is seen — together with the angle of inclination, the distance of its base from the surveyor, and the height of its top above the horizon.

PROBLEMA XLIII.

Diagram. A right-triangular / trapezoidal construction for the inclined beam AB seen over a wall, with the surveyor at E and the geometric square applied there. The main outline runs from an apex down to a base line; interior lines mark the two applications of the square and the shadow-square (*umbra versa*). The interior of the figure is densely overprinted with small lettering.

- A (apex, top)
- G (upper left, near the apex — the second sighted point)
- E (base, far right — the surveyor's station)
- B, D (base, far left)
- F (base, center-right; foot below A)
- H (base; foot below G)
- L, f (base near E; the horizon side of the square)
- Interior small letters, crowded and overprinted: a, b, c, d, e, i [several uncertain — see review note]

1. Let AB be a beam inclined over a wall CD, and let the surveyor stand at E, directly in line with the beam, so that he and the beam lie in the same plane. He is to measure the length AB, the angle of inclination ABE, the distance BE, and the height AF — even though he sees only the uppermost portion AC, the base being hidden by the wall CD. Let the square be applied twice at E: once tilted, so that one side points straight to the point A, and again let down toward the horizon, so that the center of the dioptra is at E and the side EL rests on the horizon. For, having sighted the point A, [Margin: (a) *Elements* VI.4] if it is set up that as the shadow *versa e b* is to the side *b a*, so is the side *a E* to a fourth, the distance or hypotenuse EA is produced. Then, having explored the portion of the dioptra EI — as I taught in the scholium to Problem 7, number 2 — [Margin: (b) *Elements* VI.2, by composition] if it is set up that as EI, the found portion of the dioptra, is to the found hypotenuse EA, so is *f l*, equal to the side of the square, to a fourth, the height AF is produced by composition; which is the fourth thing sought.

2. [Margin: (c) *Elements* VI.2, by composition] And if again it is set up that as EI, the portion of the dioptra, is to the hypotenuse EA, so is EF, equal to *c l*, to a fourth, the distance EF becomes known. In just the same way, if another point G is sighted, the hypotenuse EG and the height GH become known, together with the distance EH — provided the square is first applied at E so that one side points straight to G, and so on, as we showed in the scholium to Problem 7.

3. Now if EF, found from the known EH, is subtracted, there remains the known FH, that is, GI, which is to be conceived as drawn on high parallel to the horizon. In the same way, if GH — known, or its equal FI — is taken from the found AF, the remainder AI becomes known. Therefore, since in the right triangle AGI, set up on high, the two sides AI, IG are known, [Margin: (d) *On Rectilinear Triangles*, prop. 6] the base AG and the angle AGI will also be known — [Margin: (e) *Elements* I.29] which equals the angle of inclination ABE that is sought; and that is the second thing.

4. Accordingly, since in the right triangle ABF the side AF has been made known, together with the angle B and hence its complement BAF, [Margin: (f) *On Rectilinear Triangles*, prop. 4] the other side BF becomes known too; and if the found distance EF is added to it, the whole distance sought EB will be manifest, which is the third thing.

5. Finally, in the same right triangle ABF, from the side AF and the angle B — and hence its complement BAF — being known, [Margin: (g) *On Rectilinear Triangles*, prop. 4] the base AB will also be known, that is, the length of the beam, which is the first thing. This too can be made known [Margin: (h) *Elements* VI.2, by composition] by setting it up that as AI, the difference of the known heights AF, GH, is to AF, the greater height, so is AG, made known a little before, to AB.

Given that the tops of two towers are seen — even though their bases are hidden by intervening buildings — to estimate the distance both between their bases and between their tops, together with their heights and their distances from the surveyor.

PROBLEMA XLIV.

Diagram. Two towers AF and GH observed from stations B and D on the ground. Sight-lines run from B and D up to the tops A and G, forming right triangles used to recover the distances and heights.

- A (center, apex of the shorter tower AF)
- G (top right, apex of the taller tower GH)
- C (just right of A; where the horizontal from A meets the taller tower)
- B (bottom-left, the first station)
- F (base of the shorter tower)
- H (base of the taller tower)
- I (interior, near H and E)
- E (right)
- D (bottom-right, the second station)

1. [Margin: (a) *On Rectilinear Triangles*, prop. 6 (see note at "the right triangle ACG" below)] Let there be two towers AF, GH, of which only the tops A, G are seen from the station B on the horizon. We must find the distance FH, the interval AG, and the height of each tower. Let the smaller tower AF stand first between the larger and the surveyor, so that the surveyor is in the same plane as the towers and the smaller does not hide the top G of the larger. By the scholium to Problem 7 both distances BF, BH and both heights AF, GH will be found — namely if at B

the square is so placed that one of its sides coincides with the hypotenuses BA, BG, and so on; which is the fourth and the third thing. And since the three points B, F, H are placed in the same straight line, the difference of the distances FH will be known, that is, the distance between the bases of the towers, which is the first thing. Again, the difference of the heights GC will be known, [Margin: (a) *On Rectilinear Triangles*, prop. 6] and therefore, in the right triangle ACG, from the two known sides AC, CG, the base AG too will be made known, which is the second thing.

2. Then let the surveyor stand at D, so that he and the bases F, H do not lie in one straight line. By the scholium to Problem 7 again, both the heights AF, GH and the distances DF, DH will become known — namely if one side of the square coincides with the hypotenuses DA, DG, and so on; which is the third and the fourth thing. And when the hypotenuses DA, DG have been found, as was taught in that same scholium, the interval AG will be known by Problem 16 — especially by what we wrote in number 2 of that problem — namely if on the hypotenuses there are taken the portions DI, DE proportional to the hypotenuses themselves, as we said in that number 2, and so on; which is the second thing. And since the heights AF, GH have been made known, their difference GC will also be known. Therefore, from the base AG and the side GC being known, in the right triangle ACG the side AC too — that is, the distance FH between the bases — will be known, which is the first thing.

If the towers AF, GH [printed: "CH"; see anomaly note] were equal, the distance AC between the tops would equal the distance FH between the bases.

SCHOLIVM.

1. [Margin: A single rule for measuring all straight lines, when their extremities are visible] From everything demonstrated in this third book a general rule can be gathered for measuring all lengths — whether they be distances in the horizontal plane, or heights, or depths, or hypotenuses (that is, distances from the eye to any point), or intervals between two points, wherever they lie — provided only that both extremities of the length to be measured can be seen by the surveyor, wherever he too may stand. For if, by Problem 15 (especially by what we wrote in number 5 of that problem), the distances from the surveyor to the two extremities of the length are explored, the angle that those two distances or hypotenuses make having first been found, as we taught in the scholium to Problem 7, number 2, and so on — then what is proposed will have been accomplished. And so, if what we wrote in Problems 15 and 16 has been carefully grasped, the same method of measuring straight lines is always to be kept,

provided both extremities of the proposed straight line can be seen, as we have said.

2. But I think this too should not be omitted: when two distances from the surveyor's station to the two extremities of the straight line to be measured have been found, together with the angle they contain, the interval between those two extremities — that is, the proposed straight line — can be found more surely (though more laboriously) from those two distances and the contained angle, by *On Rectilinear Triangles*, prop. 12, than by the two said portions proportional to those distances; because the interval between the extremities of those portions can scarcely be found accurately with the compass. This we also warned of in Problem 16, at the end of number 2.

To level an uneven piece of ground for conducting water; or, if one wishes, to make it parallel to [equidistant from] the horizon.

PROBLEMA XLV.

1. [Margin: Construction of an instrument for leveling] When the ground offered is not very large, Ioannes Ferrerius [Juan Ferrer], a distinguished Spanish architect and mathematician, devised a very convenient instrument for leveling, in this way. Let two rules AB, AC be joined together, of some solid, hard wood, with equal legs, and long enough for the distance between the extremities B, C to be exactly 10 palms, or even more. Then, having drawn AG perpendicular to BC, let a semicircle IDK of any size be described from A, whose radius AD is divided into as many equal parts as there are palms contained in the distance BC. A hidden semicircle AED also being described about AD, let all the intervals between D and the points of the line AD be transferred from D onto its circumference; and finally, from A, let hidden straight lines be drawn through each point of the semicircle AED, and let their intersections with the circumference DI be marked and carried over onto the other circumference DK. For if from A a thread with a plumb-line hangs down, and all the parts are cut away, leaving only the legs of the instrument AB, AC together with the circumference of the semicircle IDK, an instrument very well suited for leveling will have been constructed.

2. For if, in some field or garden, two points B and C are set in the ground, and the plumb-line passes through D, then B and C stand at the same height in the ground. Consequently, if the stretch BC lying between them is leveled, that portion of the garden or field is level — that is, parallel to the horizon.

[Margin: How an unequal stretch is leveled.]

But if the plumb-line AH cuts off some parts from the quadrant DI — say 3 — then the point C will be three palms higher than the point B, and there the ground must be dug down to a depth of three palms, so that the leveled stretch between B and the lowest excavated point is parallel to the horizon. If instead the plumb-line were to cut off any number of parts from the other quadrant DK — say 5 — then the point C would be five palms lower than the point B. In that case earth must be heaped up at point C to a height of 5 palms, so that the stretch between B and the topmost point of the heaped-up earth is level and equidistant from the horizon. Once the stretch between B and another point near C — whether excavated or raised — has been leveled, the same operation must be repeated, placing the leg AB on the point just found, and so on. In this way one must proceed step by step to the last marker set out in the garden or field.

This is demonstrated as follows. Suppose the line BC is drawn, and the line CF is drawn parallel to the plumb-line AH; CF will therefore be perpendicular to the horizon, and consequently BF, drawn perpendicular to CF, will be equidistant from the horizon. Now in the triangles AGH and BFC the right angles G and F are equal,^a and likewise the alternate angles C and H are equal,^b so the triangles are equiangular. But AGH is equiangular with the triangle ADE, since the right angles G and E are equal and angle A is common. Therefore the triangles ADE and BCF are also equiangular.^c Hence, as AD of 10 parts is to DE of 3 parts, so BC of 10 palms is to CF; and accordingly CF will contain 3 palms — that is, just as many palms as the parts the plumb-line cuts off from the semicircle IDK. Now if the square on CF — which in the given example is 9 palms, since the side CF is 3 palms — is subtracted from 100 (that is, from the square on BC, which is 10 palms), the remainder gives 91 as the square on the horizontal line BF. Its square root, $9\frac{10}{19}$, gives the horizontal distance BF, from the point B up to the perpendicular CF.

[Margin: ^a Elements I.29. ^b Elements I.32. ^c Elements VI.4.]

Diagram. A surveying instrument drawn as a quadrant/semicircle hanging from a horizontal upper edge, with the field triangle projected below it in dashed lines. The upper straight edge runs I (left) — A (center, the vertex from which the plumb-line hangs) — K (right). The graduated arc descends below this edge; along it a shadow-square scale is divided and numbered 1–10 on each side, meeting at the center near E, and the outer arc carries degree graduations numbered around its rim. A vertical plumb-radius runs from A down to D on the arc; a slanting sight-line runs from A through L on the left of the arc. Below the instrument the

terrain is laid out in dashed lines: B (lower left), running along a dashed base through H and G, with C (lower right) and F (far lower right) marking the vertical drop CF and the hypotenuse BF.

- I (upper left): left end of the instrument's straight upper edge
- A (upper center): vertex, point of suspension of the plumb-line
- K (upper right): right end of the straight upper edge
- L (left of arc): point through which a sight-line from A is drawn
- E (center): center of the shadow-square scale, at the "10" division
- D (bottom of vertical radius): foot of the plumb-line on the arc
- B (lower left): first ground station
- H (lower center-left): ground point on the dashed base line
- G (lower center): ground point below the instrument
- C (lower right): second ground point, top of the vertical CF
- F (far lower right): foot of the vertical CF
- Interior scale numerals 1–10 (shadow square) and the degree graduations on the arc are overprinted and only partly legible.

This same horizontal distance BF can also be found without any calculation of numbers, in the following way. From A let another semicircle be described, and into the semicircle AED let all the intervals between A and the points of the line AD be transferred; then, with hidden lines drawn from A through the marked points, observe their intersections with the semicircle described from A, and transfer them into the other quadrant toward K. For however many parts the plumb-line AH cuts off from this last semicircle described from A, so many palms will the horizontal length BF contain.^d This holds because the proportion of DA to AE is the same as that of CB to BF, since the triangles DAE and CBF have been shown to be similar. Since, then, by the construction the line AE embraces as many parts of the line AD as have been transferred from A into the semicircle AED up to the plumb-line — in our example about $9\frac{1}{2}$ parts — the line BF will contain just as many palms of CB. It is also worth noting that the parts of the later semicircle are similar, in reverse order, to the parts of the earlier semicircle IDK.

[Margin: ^d Elements VI.4.]

For example, if on the line DE — which is equal to three parts of the line DA, counting from D — there be taken AL, likewise equal to three parts of the line AD, counting from the point A, and the line AL be drawn, the angle IAL will be made equal to the angle DAE. For since the arcs DE and AL are equal, the remaining arcs AE and DL will also be equal.^e Therefore the angles ADE and DAL will be

equal, and consequently the remaining angles DAE and IAL will also be equal to each other — because both the pair ADE, DAE (on account of the right angle E in the semicircle) and the pair DAL, IAL are equal to one right angle. From this it follows that the lines AI and AL, drawn from the later semicircle described from A, cut off an arc similar to the arc intercepted between the lines AD and AE in the earlier semicircle IDK. The same reasoning applies to the others.

[Margin: ° Elements III.27.]

When the instrument has been used repeatedly, and one wishes to know how much higher or lower the first point is than the last, this is learned from the intermediate rises and drops. Suppose the first place is higher than the second by 5 palms; and this one higher than the third by 2 palms; but this one lower than the fourth place by 3 palms; and this one, finally, higher than the last place by 1 palm. We shall then gather that the first place is 5 palms higher than the last. For the first place will be 7 palms higher than the third, since the first exceeds the second by 5 palms and the second exceeds the third by 2. And because the third is exceeded by the fourth by 3 palms, the first will exceed the fourth by 4 palms. Since, then, the fourth is higher than the last by 1 palm, the first will be 5 palms higher than the last — and so for the rest.

3. Moreover, when two places are far apart from each other, we shall find out, by means of the fixed square [instrument], how much higher or lower the one is than the other, in this way. Let the first place be A and the second B, as in the first figure. Erecting a staff AD equal to the surveyor's height, let a perpendicular BC be conceived from B to the vertical of the higher place A. (With the square you will easily recognize which of the places is higher. For when B is sighted from A along the uppermost edge held equidistant from the horizon, both places are at the same height; but if that edge must be lowered for B to be seen, place A is the higher; and if, finally, the same edge must be raised, place B is the higher.) For if, by the teaching of the scholium to Problem 9, the height DC is investigated, and the surveyor's height AD is subtracted, the remainder will be the height AC; and place A will be higher than place B by just that much. And if, on the contrary, at the lower place B a staff BF equal to the surveyor's height is erected, and by the teaching of the scholium to Problem 7 the height AE is sought — up to the perpendicular FE drawn in thought to AC — and the

[Margin: Levelings for conducting water, and how they are done.]

surveyor's height BF be added, the whole height AC becomes known; and place B will be lower than place A by just that much. And so water can be led from place A to place B, but not the other way.

Diagram. A landscape profile for leveling a water-channel across undulating ground. On the left a vertical staff carries A at its top with D beside it (the perpendicular AD); down its vertical run the points E and C are marked in descending order. A long horizontal conduit (drawn as two parallel rules with wavy water between them) runs across the middle to the right, where a second vertical staff bears F at the top and B at its base on a small mound. The terrain rises and falls beneath the conduit.

- A (top of left staff): higher station
- D (top left, beside A): head of the perpendicular staff AD
- E (mid-left, on the vertical): intermediate point
- C (lower left, on the vertical): foot point of the vertical
- F (top of right staff): head of the staff BF
- B (base of right staff, on a mound): lower station

Diagram. A landscape profile with a mountain interposed between the two places. On the left a hill carries A at its summit; in the center a larger mountain rises to a peak C, with D above E at its flank (the perpendiculars AD and BE meeting the base line CE); on the right a mound carries B.

- A (left summit): first place
 - C (central peak): summit of the interposed mountain
 - D (mid, upper): foot of perpendicular from the A-side
 - E (mid, lower): point on the base line CE
 - B (right mound): second place
4. But if, between the first place A and the second B, a mountain C is interposed, so that B cannot be seen from A — as in the 2nd figure — we shall proceed thus. From A, by the scholium to Problem 7, we shall trace out the height CD. Then from C, by the scholium to Problem 9, we shall find out the height CE, drawing to CE the perpendiculars AD and BE. For by this method you will conclude that place A is higher than place B by the amount DE. Therefore, if the mountain C is tunneled through, or an aqueduct is built around it, water can be led from A to B, but not the other way.
 5. Finally, if on some mountain, or on its flank, there is water — or in some deep well, or in some deep pit or trench CD — and you wish to know

whether water can be led by aqueduct from D to B, this is how to proceed. First, if CD is a well, investigate its depth CD by Problem 27; but if it is a trench or some chasm, by Problem 28. Then let a perpendicular BE be conceived, drawn from B to the vertical CD; and by the scholium to Problem 7 let the height CE be investigated from B. For if this is found equal to the depth CD already found, the bottom of the water, D, will be at the same height as place B, and consequently the water cannot flow from D down to B through an aqueduct. But if the height CE is found greater than the depth CD, the water can be led from D to place B. And if, finally, the height CE is found less than the depth CD, place B will be higher than the water beside D, and therefore it cannot flow down to B.

From all this you will readily understand where aqueducts may usefully be built, and where not.

FINIS LIBRI TERTII.

GEOMETRIÆ PRACTICÆ — LIBER IV.

AREAS — Investigating Plane Surfaces

Just as a straight line measures straight lines, so geometers are accustomed to measure plane surfaces by a square surface, and bodies (that is, solids) by a cubic body. For just as a straight line is called "of 100 palms" when a line of one palm is contained in it a hundred times, so a plane surface is called "of 100 palms" when it contains a hundred times the square whose side is equal to one palm; and a solid is called "of 100 palms" when it embraces 100 cubes, each of whose sides is one palm. The same is to be understood of other measures too, such as the foot, cubit, pace, mile, and so on. But since any surface is said to contain as many squares of a given measure as are contained in the rectangular parallelogram equal to it, it must first be explained by what method the area of any rectangle is found. Then we shall treat of the area of triangles, of non-rectangular quadrilaterals, and of the other figures of several sides; and finally we shall measure the circle and its parts.

[Margin: On what basis the measures of straight lines, of plane surfaces, and of solids are taken.]

DE AREA RECTANGVLORVM — CAPVT I. [On the Area of Rectangles — Chapter I]

Since Euclid, in Definition 1 of Book 2, teaches that every rectangular parallelogram is contained under the two straight lines that enclose the right angle, it is clear that the area of any rectangle is produced by multiplying the two sides about the right angle, one into the other; so that in a square it is enough to multiply one side into itself in order to know its area, since the two sides about one right angle are equal. Thus in the square ABCD, whose individual sides contain five palms each, if the side AB of five palms is multiplied into itself, 25 squares are produced, each of which has a side of one palm; and the area of the square ABCD is said to contain that many square palms. But the area of the rectangle EFGH, which is longer on one side — one of whose sides about the right angle contains 5 feet and the other 3 — is said to contain 15 square palms, because from the mutual multiplication of the sides 5 and 3 the number 15 is generated.

[Margin: How the area of a square, and of a rectangle longer on one side, is found.]

Diagram. Two adjacent gridded figures illustrating the areas of a square and a rectangle. On the left, the square ABCD is divided into a 5×5 grid of unit cells, with "25" marked in the center. On the right, the rectangle EFGH is divided into a 5×3 grid, with "15" marked in the center. The sides are labeled with their palm-counts.

- A (bottom left of square), B (bottom right of square), C (top right of square), D (top left of square): square ABCD, each side of 5 palms; central value 25
 - E (bottom left of rectangle), F (bottom right of rectangle), G (top right of rectangle), H (top left of rectangle): rectangle EFGH, sides of 5 and 3 palms; central value 15
 - Top edge labels: D · 5 · C · H · 3 · G
 - Bottom edge labels: A · 5 · B · E · 3 · F
 - Side labels: 5 (marked on the outer verticals)
2. And so, if we are ordered to measure some rectangular field, or a rectangular parallelogram, the two sides about the right angle must be measured by some known measure, such as the palm, the foot, and so on. For, one being multiplied into the other, the area of the proposed field, or of the rectangular parallelogram, will be produced, as has been said.

[Margin: What is to be done to measure a rectangular field.]

DE AREA TRIANGVLORVM — CAPVT II. [On the Area of Triangles — Chapter II]

1. When all three sides of a triangle are known, its area can be found by two methods. The first, which is the most exact, is as follows. *Let all the sides be gathered into one sum; from half of this sum let each single*

...each of the sides, so that three differences are obtained between that half-sum and the individual sides. Finally, let these three differences and the said half-sum be multiplied one into another. The square root of the resulting number will be the area of the triangle sought.

For example, suppose the sides are 10, 17, and 21. Their sum will be 48, and its half 24. The differences between this half and the sides will then be 14, 7, and 3. Multiplied into one another — 14 by 7, and then the product by 3 — these give 294; and this, multiplied by the aforesaid half 24, produces 7056, whose square root, 84, is the area of that triangle whose sides are 10, 17, and 21.

Again, if in some other triangle the sides are 13, 14, and 15, we shall find the same area. For the sum of the sides is 42, its half 21. The differences between this half and the three sides are 8, 7, and 6, which multiplied together make 336; and this multiplied by the aforesaid half 21 yields 7056, whose square root, 84, gives the area of the triangle contained by the sides 13, 14, and 15.

Finally, suppose a triangle ABC is given in which side AB is 7, BC is 10, and AC is 11. The sum of all is 28, and its half 14, which exceeds the sides by these numbers: 7, 4, and 3. Multiplied together these make 84; and this, multiplied by 14, the half of the sum, produces the number 1176, whose square root — approximately 34 $\frac{20}{89}$ — gives the area of triangle ABC. From this you will gather that not every triangle's area is a rational number, because the number produced in the last step is not always a perfect square, as happened in this last example.

We shall now demonstrate this practice, or rule — which, as I said, is most exact — in the triangle ABC as follows. Bisect the angles ABC and ACB by the straight lines BD and CD, which meet at D; from D draw the perpendiculars DE, DF, DG to each of the sides, and join the straight line AD. Now the two angles E and DBE in triangle DEB are equal to the two angles G and DBG in triangle DGB, and the side DB is common; therefore [Margin: Euclid I.26] both the sides DE, DG and the sides BE, BG will be equal. In the same way both the sides DF, DG and the sides CF, CG will be equal in the triangles DFC and DGC. Consequently DE and DF (since each

has been shown equal to DG) will be equal to one another, and so all three perpendiculars DE, DF, DG will be equal to one another.

Next, since the squares on AE and ED together equal the square on AD [Margin: Euclid I.47], as do the squares on AF and FD, the squares on AE and ED will equal the squares on AF and FD. And so, when the equal squares on the equal lines ED and FD are taken away, the remaining squares on AE and AF will be equal; therefore the lines AE and AF themselves will also be equal. Since, then, the sides AE, AD of triangle ADE equal the sides AF, AD of triangle ADF, and the base ED equals the base FD, the angle DAE will equal the angle DAF [Margin: Euclid I.8].

Diagram. A triangle ABC (apex A at top, B at lower left, C at lower right) with its incircle-type construction: the angle-bisectors from B and C meet at D, the perpendiculars DE, DF, DG are dropped to the sides, and AD is joined. Below the base the construction is continued: AH and AI are laid off along the base line (equal to the half-sum of the sides), the perpendicular HK is raised meeting AD produced at K, and the lines KI, KB, KC, KL, KM are drawn down to K at the bottom. Points M and H sit at the lower left, K at the bottom, I at the far right. Small numerals are overprinted among the interior vertices near A, E, B, G, F, L (apparently the differences 7, 4, 3 and the like), but they are struck over the lines and cannot be resolved with confidence.

- A (top vertex): A
- E (on side AB, left): E; adjacent numerals near it read as 7, 4 and 3 [?]
- D (interior, meeting-point of bisectors): D
- F (on side AC, upper right): F; adjacent numeral 7 [?], with a mark "II" [?]
- B (lower-left vertex): B
- G (on base BC, left of centre): G; adjacent numeral 3 [?]
- L (on base BC, right of centre): L; adjacent numeral 7 [?]
- C (lower-right vertex): C
- I (far right, end of AI produced): I
- interior numeral near centre: 10 [?]
- H (lower left, foot of the raised perpendicular region): H
- M (lower left, below H): M
- K (bottom vertex, where AD produced meets HK): K

Now, since AE has been shown equal to AF, and EB equal to BG, the whole line AB will equal the two lines AF and BG together; and when the equal lines CG and CF are added, the two lines AB and CG together will equal the two lines AC and BG together. Therefore both AB with CG, and AC with BG, make up half the three

sides AB, BC, AC. Hence CG (or CF) is the difference between the half-sum of the sides and side AB; and likewise BG (or BE) is the difference between the same half-sum and side AC. Finally, since AB and CG make up the half-sum of the sides, and BG equals BE as we showed, BC and AE will likewise make up the half of the same sides; and so AE is the difference between the half-sum of the sides and side BC. The three lines AE, EB, and CG therefore both constitute the half-sum of the sides and furnish the three differences between the half-sum and the three sides of the triangle.

Now let AB and AC be produced; let BH be made equal to CG, and CI equal to BG, so that AH (formed of the lines AB and CG) equals the half-sum of the sides, and AI likewise (formed of the lines AC and BG) equals the same half-sum, and so is composed of the three differences named above. Draw HK perpendicular to AH, meeting AD produced at K, and join the lines KI, KB, KC. Since the two sides AH, AK of triangle AHK equal the two sides AI, AK of triangle AIK [Margin: Euclid I.4] and contain equal angles at A, as we showed above, the bases HK and IK will also be equal, and so will the angles H and I. Since, then, H is a right angle by construction, I too will be a right angle [reading uncertain: "L"?].

Further, let BL be cut off equal to CG (or BH), so that the remainder CL equals the remainder BG (or CI), and join the straight line KL. And on BH produced let HM be taken equal to CI, and join the straight line KM. Since the two sides KH, HM of triangle HMK equal the two sides KI, IC of triangle CIK [Margin: Euclid I.4] and contain equal angles H and I (both being right angles), the bases KM and KC will also be equal; and thus, since the two sides BM, BK of triangle BMK equal the two sides BC, BK of triangle BCK (for BM equals BC, because the parts BH, HM equal the parts BL, LC [Margin: Euclid I.8]), and the base KM has been shown equal to the base KC, the angles KBM and KBC will also be equal. Therefore, since the two sides BH, BK of triangle BHK equal the two sides BL, BK of triangle BLK, and contain equal angles at B as we showed, the bases HK, KL and the angles H, L will be equal [Margin: Euclid I.4]. And since H is a right angle by construction, L too will be a right angle. Hence, since the sides KH, KB of triangle KBH equal the sides KL, KB of triangle KBL, and the base BH equals the base BL, the angles BKH and BKL will also be equal [Margin: Euclid I.8].

Now since, from what we demonstrated at Proposition 32 of Book 1 of Euclid, the four angles of the quadrilateral BHKL equal four right angles, then, when the two right angles H and L are taken away, the two angles HBL and HKL will equal two right angles; and therefore they equal the two angles HBL and EBL [Margin: Euclid I.13], since these too equal two right angles. When the common angle HBL

is removed, the remaining angle HKL will equal the remaining angle EBL; and hence HKB, as half, will equal EBD, as half. Since then the right angle H equals the right angle E [Margin: Euclid I.32], the remaining angle HBK in triangle HBK will also equal the remaining angle EDB in triangle EDB; and consequently the triangles BHK and DEB will be equiangular [Margin: Euclid VI.4]. Therefore DE is to EB as AH is to HK; and so, if these lines are reduced to numbers [Margin: Euclid I.19], the number formed from DE into HK will equal the number formed from EB into BH [Margin: Euclid VII.17]. The square on DE will therefore have the same ratio to the product of DE into HK as it has to the product of EB into BH [Margin: Euclid V.7]. But the square on DE is to the product of DE into HK as DE is to HK [Margin: Euclid VII.17], because DE, multiplying DE and HK, made both the square on DE and the product of DE into HK. Therefore the square on DE will also be to the product of EB into BH as DE is to HK. But DE is to HK as AE is to AH; for since DE and HK are parallel, the triangles AED and AHK are equiangular, by the corollary to Proposition 4 of Book 6 of Euclid [Margin: Euclid I.28]. Therefore AE is to ED as AH is to HK, and, by alternation, AE is to AH as ED is to HK [Margin: Euclid VI.4; VII.19]. Therefore the square on DE will also be to the product of EB into BH as AE is to AH. And since the product of the square of DE into AH will equal that which

is formed from AE into the product of EB into BH: therefore the number generated from the square of DE into AH, multiplied again by AH, will equal the number produced from the product of AE into the product of EB into BH, multiplied by the same AH. (For since the equal numbers — namely the product of the square of DE into AH, and the product of AE into the product of EB into BH — multiply the same number AH, they will have the same ratio as the multiplicands [Margin: Euclid VII.18]. And since these are equal, the products will be equal too.) That is, the number produced from the square of DE into AH, multiplied by AH — that is, the square of AH multiplied into the square of DE (for by the scholium to Proposition 19 of Book 8 of Euclid, in whatever order three numbers are multiplied together, the same number is always produced) — will equal the number generated from the product of AE into the product of EB into BH, multiplied into AH, that is, into the half-sum of the sides. But from the square of DE into the square of AH there is produced the square number of the area of triangle ABC, as we shall shortly show. Therefore from the product of the three excesses AE, EB, BH, multiplied together and then into AH — that is, into the half-sum of the sides AB, BC, AC — there is produced the same square number of the area of triangle ABC; and hence the square root of this number will be the area of the said triangle. Which was to be demonstrated.

That the square number of the area of triangle ABC is produced from the square of DE into the square of AH, I demonstrate as follows. Since, as we shall show at number 2, the area of triangle ADB is produced from DE into half of side AB; and from the same DE (that is, from DG) into half of side BC the area of triangle BDC is made; likewise from the same DE (that is, from DF) into half of side AC the area of triangle ADC is generated — and since what is formed from DE into the halves of the sides AB, BC, AC together equals what is formed from DE into AH [Margin: Euclid II.1], AH being composed of those halves — the area of triangle ABC will therefore arise from DE into AH. And consequently (these lines being reduced to numbers) the square number of the area of the same triangle will be generated from the square of DE into the square of AH. For whenever two numbers, multiplying each other, make some number, their squares, multiplying each other, will produce the square of that product; which will become clear thus.

Let two numbers A and B, multiplying each other, make D; and each multiplying itself make C and E respectively. Finally, let these squares C and E, multiplying each other, make F. I say that F is the square of D. For since A, multiplying itself, makes C, and multiplying B makes D; and by the same reasoning, since B, multiplying A and itself, makes D and E, it will be as A is to B, so D is to E [Margin: Euclid VII.17]; and therefore C, D, E will be in continued proportion. Wherefore the number F, formed from C into E, will equal that which is formed from D into itself [Margin: Euclid II.20]; and so F will be the square of D. This being so, since from DE into AH (as we showed) the area is formed, from the square of DE into the square of AH will be formed the square number of the area of triangle ABC. Which was to be demonstrated.

Diagram. A scheme of numbers represented as rows of dots (measured lengths) labelled with letters, illustrating the preceding argument about a product and its square:

- A (top row, short): A (two dots)
- B (top row): B (three dots)
- C (second row): C (several dots)
- D (second row): D
- E (second row): E (with a further run of dots)
- F (third row, longest): F, marked with the number 36.

2. A second method by which the area of a triangle is gathered from the given sides is this. [Margin: How the area of a triangle may otherwise be gathered from the given sides.]

From any angle let a perpendicular be drawn to the opposite side — extended, if need be. For this perpendicular (if its magnitude is known), multiplied into half the base (that is, of the said side), or its own half multiplied into the whole base, will yield the area of the triangle; or, if you prefer, the whole perpendicular drawn into the whole base will produce a number whose half yields the area of the triangle.

For, as we demonstrated in Book 7, Proposition 1, the area of a triangle equals the rectangle contained by the perpendicular and half the base, or by half the perpendicular and the whole base; and likewise it equals half of the rectangle contained by the perpendicular and the whole base. Since, then, by Chapter 1 of this book the area of that rectangle is produced by multiplying one of the sides about the right angle into the other — that is, the perpendicular (which equals one side) into half the base of the triangle [Margin: Euclid I.34], or half the perpendicular (which equals half the side) into the whole base [Margin: Euclid I.34], or, finally, since the rectangle is double the triangle, from the perpendicular into the whole base of the triangle — the proposition is established.

The magnitude of the said perpendicular, like that of the base, must in surveying fields be investigated by means of an iron chain (since this is neither stretched nor slackened), or else, if all the sides are known, geometrically in this way. Let there be a triangle ABC whose side AB is 10, BC 21, and AC 17. First the segments BD and CD of the base, on either side of the perpendicular, must be found, by the method we described in Book 1, Chapter 3, Number 9, thus. Let it be made that as the side BC — onto which the perpendicular AD falls (the perpendicular should always be dropped onto the greatest side, so that it falls within the triangle) — is to the sum of the other two sides AB and AC, namely as 21 is to 27, so is the difference of those same sides, namely 7, to a fourth term. This yields the number 9 (which, being less than side BC, is evidence that the perpendicular falls inside the triangle; for if it were greater, the perpendicular would fall outside, as we showed at Proposition 9 of our *Triangles* [i.e. Clavius's *Triangula rectilinea*]). This number 9, taken from side BC — that is, from 21 — leaves 12; and its half, 6, gives the smaller segment BD, adjacent to the smaller side AB. That same half, subtracted from side BC, leaves as remainder the larger segment CD, namely 15, adjacent to the larger side AC. [Margin: How the segments of the base cut off by the perpendicular are found.]

Diagram. A triangle with apex A at the top and base running from B (lower left) to C (lower right); the perpendicular AD is dropped from A to the point D on the base, dividing it into segments BD and DC. A circular arc is drawn through the

lower-left region near the point F. The numerals label the sides and segments of the worked example ($AB = 10$, $BC = 21$, $AC = 17$; $BD = 6$, $DC = 15$; $AD = 8$).

- A (top vertex): A
- E (just below A): E [?]
- B (lower-left vertex): B, with an adjacent mark [?]
- perpendicular AD region: 8 (= AD)
- left segment: 6 (= BD)
- upper-right side: 17 (= AC)
- lower-right segment: 15 (= DC)
- D (foot of perpendicular on the base): D, with 21 below it (= BC)
- F (lower left, on the arc): F
- C (lower-right vertex): C

Take again a triangle ABD whose side AB is 12, AD is 11, and BD is 20, and let the perpendicular be dropped from D onto AB. Let it be made that as side AB, 12, is to the sum of the other two, 31, so is their difference, namely 9, to a fourth term. This yields the number $23 \frac{1}{4}$ (which, being greater than side AB, is evidence that the perpendicular falls outside the triangle, and that the angle A is obtuse); and when side AB, 12, is subtracted from it, there remains $11 \frac{1}{4}$. The half of this number, $5 \frac{5}{8}$, gives the exterior segment AC, between the perpendicular and the obtuse angle; and that same half, added to side AB, produces the other segment BC, between the perpendicular and the acute angle, namely $17 \frac{5}{8}$. And this method is most expeditious.

Diagram. A right triangle for the obtuse-angle case: D at the upper left, the perpendicular DC dropped to C at the lower left; the base runs C — A — B, with A between C and B (A is the obtuse-angled vertex, so the foot C falls outside the triangle beyond A). The lines DC, DA, DB and the base CB are drawn.

- D (upper-left vertex): D
- C (lower-left, foot of the perpendicular): C
- A (on the base, between C and B): A
- B (lower-right vertex): B

Another way.

Another way. Let the perpendicular be drawn to the greatest side BC in the earlier triangle ABC, where it will necessarily fall inside the triangle [Margin: Euclid I.18], because the angle A is greater than either B or C, and so each of the latter is acute. Now since the square on line AB is less than the squares on lines

AC and BC by twice the rectangle contained by BC and the segment CD (between C and the perpendicular) [Margin: Euclid II.13]: if the square on line AB, 100, is subtracted from the sum of the squares on AC and BC — that is, from 730 — the remainder, 630, will be twice the rectangle contained by BC and the segment CD (between C and the perpendicular). Its half, 315, therefore equals that rectangle taken once; and hence this rectangle, 315, divided by side BC, 21, gives the quotient 15, the segment CD near the acute angle C, which is subtended by side AB, whose square was subtracted from the sum of the other squares. When the segment CD, 15, is taken from side BC, 21, there remains the other segment BD, 6. In the same way, since the square on line AC is less than the squares on lines AB and BC by twice the rectangle contained by CB and the segment BD (between B and the perpendicular) [Margin: Euclid II.13]: if the square on line AC, 289, is subtracted from the sum, 541, of the squares on AB and BC, the remainder, 252, will be twice the rectangle contained by CB and the segment BD (between B and the perpendicular). Its half, 126, therefore equals that rectangle taken once; and hence this rectangle, 126, divided by side CB, 21, gives the quotient 6, the segment BD near the acute angle B, which is subtended by side AC, whose square was subtracted from the two other squares. When this segment, 6, is taken from side BC, 21, there remains the other segment CD, 15.

Next, in the later triangle ABD let the perpendicular be drawn to side AB, which is not the greatest. Since side DB is greater than side AD [Margin: Euclid I.18], the angle A will be greater than the angle B [Margin: Euclid I.17]. And since the two together are less than two right angles, at least the smaller, B, is acute; and consequently the square on line AD is less than the squares on lines AB and BD by twice the rectangle contained by side AB and the segment between B and the perpendicular [Margin: Euclid II.13]. If, therefore, the square on line AD, 121, is subtracted from the sum of the squares on AB and BD — that is, from 544 — the remainder, 423, will be twice the rectangle contained by AB and the segment between B and the perpendicular; and so its half, $211\frac{1}{2}$, equals that rectangle taken once. Hence, if this rectangle, $211\frac{1}{2}$, is divided by side AB, 12, it gives the quotient $17\frac{5}{8}$, the segment between B and the perpendicular. Since this is greater than side AB, it is evidence that the perpendicular DC falls outside the triangle, and hence that the angle A is obtuse. And if from this segment, $17\frac{5}{8}$, side AB, 12, is taken away, there remains the exterior segment, $5\frac{5}{8}$.

When it is established that angle A is obtuse, and that the perpendicular DC therefore falls outside the triangle, we may find the same segments BC and CA in this further way. [Margin: Euclid II.12] Since the square on side BD exceeds the squares on sides AB and AD by twice the rectangle contained by side AB and the

exterior segment AC: if the sum of the squares on lines AB and AD, 265, is subtracted from the square on side BD, 400, the remainder, 135, will be twice the rectangle contained by AB and the exterior segment AC; and its half, $67 \frac{1}{2}$, equals that rectangle taken once. Hence this rectangle, $67 \frac{1}{2}$, divided by side AB, 12, indicates the quotient $5 \frac{5}{8}$, the exterior segment CA; and if side AB, 12, is added to it, the segment BC, $17 \frac{5}{8}$, is made up. But the earlier method, which does not depend on Book 2 of Euclid, is more expeditious and therefore to be preferred — although other authors generally follow the latter way. [Margin: Which method is to be followed in finding the segments.]

Once the segments made by the perpendicular have been found, we shall find the magnitude of the perpendicular as follows. [Margin: How the perpendicular in a triangle is found.]

Let the difference between either segment and its adjacent side be multiplied into the sum formed from that same segment and side. The square root of the resulting number will yield the perpendicular, as we demonstrated in Book 1, Chapter 3, Number 17.

For example, in the earlier triangle ABC, if the difference 4 between the segment BD and side AB — that is, between 6 and 10 — is multiplied by 16, namely by the sum of that same segment BD and side AB, the number 64 is generated, whose square root, 8, gives the perpendicular AD. By the same reasoning, if the difference 2 between the segment CD and side AC — that is, 15 and 17 — is multiplied into 32, that is, into the sum of that same segment CD and side AC, the number 64 is produced, whose square root, 8, gives the perpendicular AD, as before.

But in the later triangle ABD, if the difference $5 \frac{3}{8}$ between the segment AC and side AD — namely between $5 \frac{5}{8}$ and 11 — is multiplied into $16 \frac{5}{8}$, that is, into the sum of that same segment AC and side AD, there is produced the number $\frac{5719}{64}$, or $89 \frac{23}{64}$, whose square root cannot be exhibited in whole numbers, but is a little greater than the appended fraction whose numerator is $\frac{7594}{151}$ and whose denominator is 8 — a fraction which is reduced to that number if the numerator is divided by the denominator; and so great, very nearly, is the perpendicular DC, namely $9 \frac{547}{1208}$.

[A boxed marginal fraction here shows the numerator "75 94/151" set over the denominator "8".]

Likewise, if the difference $2 \frac{3}{8}$ between the segment BC, $17 \frac{5}{8}$, and side BD, 20, is multiplied by $37 \frac{5}{8}$, the sum of that same segment BC and side BD, the same

number as before is generated, $5719/64$, that is, $89 \frac{23}{64}$, whose square root is a little greater than $9 \frac{547}{1208}$, as before. And this method is quite expeditious.

Another way. Since the square on line AB, in the earlier triangle ABC, equals the two squares on lines AD and BD [Margin: Euclid I.47]: if the square 36 of the segment BD is taken from 100, the square of the adjacent side AB, there is left the square 64 of the perpendicular AD. Its square root, 8, will therefore be the magnitude of the perpendicular AD, as above. Similarly, if the square 225 of the segment CD is taken from 289, the square of the adjacent side AC, the remainder will be the square 64 of the perpendicular AD, whose root, 8, will be the magnitude of the perpendicular AD, as before.

But in the later triangle ABD, if the square $31 \frac{41}{64}$ of the segment AC, $5 \frac{5}{8}$, is subtracted from 121, the square of the adjacent side AD, 11, there will remain the square $89 \frac{23}{64}$ of the perpendicular DC, whose root is a little greater than $9 \frac{547}{1208}$, as above. So too, if the square $310 \frac{41}{64}$ of the segment BC, $17 \frac{5}{8}$, is subtracted from the square 400 of the adjacent side BD, 20, there will again remain the square $89 \frac{23}{64}$ of the perpendicular DC, whose root is a little greater than $9 \frac{547}{1208}$, as above. But the earlier method seems more expeditious, although others hand down this latter way. [Margin: By which way the finding of the perpendicular is more expeditious.]

In an equilateral triangle, moreover, the perpendicular will be found in yet another way. [Margin: Euclid XIV.12] Since the square of the side is one-and-a-third times the square of the perpendicular: if it is made that as 4 is to 3, so the square of the side is to a fourth term, there results the square of the perpendicular; and the root of this square will yield the perpendicular. For example, if the side is 10, and it is made that as 4 is to 3, so 100, the square of the side, is to a fourth term, there is generated the square of the perpendicular, 75, whose root, $8 \frac{11}{17}$, will very nearly be the perpendicular sought.

Finally, once the segments of the base made by the perpendicular have been found, the perpendicular itself will be found by means of sines, in the earlier triangle, thus. Let it be made that as the side AB, 10, which is opposite the right angle, is to the whole sine of the right angle D, so is the segment BD, 6, [to the sine...]

as the whole sine is to BD, so BD to another; and the sine of the opposite angle BAD, 60000, will be produced. Therefore angle BAD will be $36^{\circ}52'$, and consequently angle B, its complement, will be $53^{\circ}8'$. If, then, it is again arranged that as the whole sine is to 10 (the side AB), so 80003 (the sine of angle BAD) is to

another, the side AD will be produced — that is, the perpendicular, $8\frac{30}{100000}$ approximately — which is a little larger than the 8 found above. This discrepancy arises because the sines are not exactly such as are set out in the table; yet in the measuring of fields it introduces no appreciable error.

And so, if in the prior triangle ABC the perpendicular AD, 8, is multiplied into $10\frac{1}{2}$ (the half of the base BC), or 4 (the half of the perpendicular AD) into 21 (the whole base), the area of triangle ABC, 84, will be produced. [Margin: The area of the prior triangle ABC.] The same is also obtained if the whole perpendicular, 8, is multiplied into the whole base, 21, and the half of the product 168 is taken, namely 84.

Likewise, in the posterior triangle ABD, if the perpendicular DC, $9\frac{547}{1208}$, is multiplied into 6 (the half of the base AB), or if the half of the perpendicular — namely $11419/2418$ [?], or $4\frac{1755}{2418}$ [?] — is multiplied into the whole base, 12, the area of triangle ABD, $56\frac{866}{1208}$, or $56\frac{433}{604}$, will be produced. [Margin: The area of the posterior triangle ABD.] This too will be obtained if the whole perpendicular is multiplied into the whole base and the half of the product is taken.

To avoid fractions as far as possible, you will take care, whenever the perpendicular is an odd number, to take the half of the perpendicular and multiply it into the whole base; but when the perpendicular is an odd number and the base an even number, to take the half of the base and multiply it into the whole perpendicular. And if both the perpendicular and the base should be even numbers, it makes no difference which half you take. [Margin: What to do in order to avoid fractions.]

When the perpendicular is also a surd root [radix surda] — one, that is, that cannot be expressed by a number, as DC was in the posterior triangle ABD — you will do rightly if you multiply its square (the surd root not being extracted) by the square of the half of the base. For the number produced will be the square of the number of the area of the triangle, so that its root is the area itself of the triangle. By this method we shall stray less from the truth. [Margin: What to do when the perpendicular is a surd number.] Thus in the said posterior triangle ABD, if we multiply the square of the perpendicular DC, $5719/64$, into 36 (the square of the half of the base), we shall produce $205884/64$, the square of the area — namely of triangle ABD — a little larger than the one found before; its root is $56\frac{3605}{3828}$ [?], that is, the area of triangle ABD, a little larger than was found previously. By the same reasoning, if in some triangle the square of the perpendicular were 72, and the half of the base 6: if we multiply the root of the number 72 — namely 8

8/17, that is, the perpendicular itself — into 6, we shall produce the area $50 \frac{14}{17}$. But if we multiply the square 72 itself by 36 (that is, by the square of the half of the base), we shall produce 2592, the square of the area, whose root is a little larger than the previously found $50 \frac{14}{17}$; and this number is somewhat larger than the area found before. The reason for this rule of ours is that, as we showed a little earlier at the end of Num. 1, two numbers multiplying each other produce the root of the number formed from their squares.

3. Having set out the two general rules by which the area of any triangle whatever is investigated, we shall now propose certain particular precepts for some particular triangles, which will sometimes be of great use, since by them the areas are often found more expeditiously in certain triangles than by those general rules. The area of a right triangle, then, will be produced if the two sides about the right angle are multiplied together and the half of the product is taken. [Margin: The area of a right triangle.] For from that multiplication is generated the rectangular parallelogram contained under the two sides about the right angle, [Margin: (a) Euclid I.41] of which rectangle the triangle is the half. And this is just the same as if the half of either of the two sides is multiplied into the whole of the other, taken as base. Thus in the preceding triangle ABC, divided into the two right triangles ADB and ADC: if AD, 8, is multiplied into BD, 6, the number 48 is produced, whose half, 24, is the area of triangle ADB. Likewise, if AD, 8, is multiplied into DC, 15, there results 120, whose half, 60, is the area of triangle ADC — where you see that the two triangles, 24 and 60, make up the whole triangle ABC, 84, as we found above.

4. The area of an isosceles, or also of an equilateral, triangle will be produced if the square of the half of the base is subtracted from the square of the side, the remaining number is multiplied into that same square of the half of the base, and finally the square root of this product is extracted. [Margin: The area of an isosceles triangle.] Thus in the isosceles triangle ABC, whose equal sides AB, AC are 32, 32, and the base BC is 24: if the square of the half of the base, 144, is taken from 1024 (the square of the side AC or AB), and the remaining number, 880, is multiplied into 144 (the square of the half of the base), the product will be 126720, whose square root, $355 \frac{695}{711}$ (which is a little less than the true root), is the area of triangle ABC. For if the square of the half of the base DC is taken from the square of the side AC, [Margin: (b) Euclid I.47] the remainder is the square of the perpendicular AD; and since, by the scholium of proposition 26, book I, of Euclid, the perpendicular AD cuts the base BC into two equal parts at D, therefore — as we showed near the end of Num. 2 — the square of the perpendicular AD multiplied into the square of DC (the half of the base) produces the square of the

area of triangle ABC. And the same reasoning holds for the equilateral triangle, since this too has two equal sides.

Diagram. Two adjacent triangle figures illustrating the area of the isosceles triangle ABC. The left (larger) triangle has apex A at top and base vertices B, D, C along the bottom, with the altitude AD dropped from A to the midpoint D of the base. Its two equal sides are each labelled 32, and the base is labelled 24 (printed as "2 | 4", the vertical stroke marking the foot D). The right (smaller) triangle repeats the construction with apex A and base vertices B, D, C; a side is labelled 32, and it carries one or more interior points labelled I [?].

- A (apex, top): vertex opposite the base
- B (bottom left): base vertex
- D (bottom middle): midpoint / foot of the perpendicular
- C (bottom right): base vertex
- 32 (left side, both figures): the equal side AB / AC
- 32 (right side, left figure): the equal side
- 24 (base, "2 | 4"): the base BC
- I [?] (right figure, interior): additional point(s) — labels not fully resolved

5. For the area of an equilateral triangle, however, this rule too is handed down by the authorities, though (so far as I know) it has been demonstrated by no one. Let the square of the side be multiplied into 13, and let the resulting number be divided by 30; for the quotient will be the area of the equilateral triangle.

[Margin: The area of an equilateral triangle.] Thus, if one side of an equilateral triangle is 10, let the square of the side, 10 — namely 100 — be multiplied into 13, and let the resulting number, 1300, be divided by 30; for the quotient, 43 $\frac{1}{3}$, will be the area of the triangle. I demonstrate this rule as follows. The area of an equilateral triangle each of whose sides is 1 is the square root of this number, $\frac{3}{16}$ (for by the preceding rule explained in Num. 4: if $\frac{1}{4}$, the square of the half of the side, is subtracted from 1, the square of the side, and the remaining number $\frac{3}{4}$ is multiplied into that same square $\frac{1}{4}$ of the half of the side, there will be produced the square of the area of the triangle, $\frac{3}{16}$), namely $\frac{13}{30}$ approximately. Since, then, the square of the side 1 has to the square of the side 10 — that is, 1 has to 100 — the same ratio that the area $\frac{13}{30}$ of the triangle whose one side is 1 has to the area of the triangle whose one side is 10 [Margin: (c) Euclid VI.20 & 19] — because each ratio is the duplicate of the ratio of the side 1 to the side 10 — then, if it is arranged that as 1 (the square of the side 1) is to 100 (the square of the side 10), so the area $\frac{13}{30}$ is to another, there will be produced the area of the triangle whose one side is 10. And this is done by multiplying the

second number, 100, into the third, $13/30$ — that is (as is established by the rule for the multiplication of fractions), by multiplying 100 into the numerator 13 and dividing the product by the denominator 30. Nor indeed is there any need to divide this resulting number, $43\frac{1}{3}$, by the first, 1, since unity, whether dividing or multiplying any number, produces that same number. So too, if one side of an equilateral triangle is 6, we shall multiply its square, 36,

into $13/30$ — that is, into 13, the numerator — and we shall divide the product, 468, by 30. For the quotient will be $15\frac{3}{5}$, the area of the proposed triangle.

That $13/30$ is the square root of the number $3/16$ is clear from the rule by which the root of any fractional number is extracted, which is as follows. [Margin: How the square root of a fractional number is to be extracted.] Let the numerator be multiplied into the denominator, and let the near root of the product be found. For if by this root we divide the numerator, or if we divide that root itself by the denominator, there will come out the root of the proposed fraction — by the former method larger than the true one, but by the latter smaller, provided that near root of the product of numerator by denominator is less than the true one; because in that former method the division is made by a number smaller than the true one, whereas in the latter a number smaller than the true one is divided. But if that near root were larger, the root of the fraction would come out smaller than the true one by the former method, and larger by the latter, as is evident. For example: let the square root of the fraction $3/16$ — which we said is the square of the area of the equilateral triangle whose one side is 1 — be sought. From 3 into 16 comes the number 48, whose near root is $6\frac{12}{13}$ [?], smaller than the true one; and if the numerator 3 is divided by this, there will come out the root $13/30$ of the fraction $3/16$, larger than the true one. But if we divide that same near root, $6\frac{12}{13}$ [?], by the denominator 16, there will be found the root $45/104$ of the same fraction $3/16$ [?], smaller than the true one.

The reason for this extraction is as follows. When the numerator 3 is multiplied into the denominator 16, [Margin: (d) 20. sept. (Euclid VII.20?)] the square root of the product 48 will be the mean proportional between 3 and 16, because that root multiplied into itself produces the number 48, equal to what is generated by the extremes 3 and 16 multiplied together; and consequently [Margin: (e) Euclid VIII.11] the ratio of 3 to 16 will be the duplicate both of the ratio of 3 to that root and of the ratio of that root to 16. Since, therefore, the ratio of 3 to 16 is likewise the duplicate of the ratio that the root of the number 3 has to the root of the number 16, it will follow that as 3 is to the root of 48, so the root of 48 is to 16. Wherefore, since the fraction whose numerator is the root of the number 3 and

whose denominator is the root of the number 16 is the root of the fraction $3/16$, so too will be both the fraction whose numerator is 3 and whose denominator is the root of the product 48, and the fraction whose numerator is the root of the product 48 and whose denominator is 16 — that is, both the quotient that arises from the division of 3 by the root of the product 48, and the quotient that arises from the division of the root of the product 48 by 16, will be the near root of the fraction $3/16$. And the reasoning is the same for the rest.

Others hand down this rule for finding the area of an equilateral triangle: from the square of the side let there be taken both a tenth part and a third part; for the sum of these parts will be the area of the triangle. Which I demonstrate thus. These fractions, $1/10$ and $1/3$, collected into one make $13/30$; and consequently it is the same thing to take from the square of the side $1/10$ and $1/3$ as to multiply the square of the side by $13/30$, as I taught in the sixth little question on fractions. Since, then, it was explained in Num. 5 that by the multiplication of the square of the side into $13/30$ the area of the equilateral triangle is produced, it is clear that the tenth part and the third part of the square of the side make up the same area. And so, if the side were 30, its square will be 900, whose $1/10$ is 90 and $1/3$ is 300, which parts together make the number 390, for the area of that equilateral triangle.

6. So far we have set out the rules that lead us to the knowledge of the area of any triangle, provided its individual sides are known. Now we shall investigate the areas of triangles by the doctrine of sines, tangents, and secants, even though not all the sides are known, but only one, or two, together with two angles, or with one. [Margin: The area of a right triangle from the side opposite the right angle [the hypotenuse] and one known acute angle, how it is to be investigated. — (f) Euclid I.32. — (g) Num. 2 of the right-lined triangles.] In right triangles, then, we shall proceed as follows.

When in a right triangle the side opposite the right angle [the hypotenuse] is known, together with one acute angle, we shall know the area in this way. Subtracting the given acute angle from the right angle — that is, from 90° — the other acute angle will also be known. Thus in triangle DCB, which has the right angle at C and the side BD known, together with the acute angle B: and since the two angles B and BDC are together equal to a right angle, that is, to 90° , if angle B is subtracted from 90° the remaining angle BDC will be left. If, then, it is arranged that as the whole sine of the right angle C is to the side BD opposite it, known in any measure whatever, so both the sine of angle B and the sine of angle BDC are to another, the sides DC and CB will become known in the parts of the side BD;

and thus all three sides will be known. Therefore the area too will be known, by Num. 1 of this chapter.

Diagram. A right triangle DCB with the right angle at C. The apex D is at the upper left; the base runs along the bottom from C (lower left) through A (a point on the base) to B (lower right). DB is the hypotenuse.

- D (upper left): apex
- C (lower left): the right angle
- A (bottom, between C and B): point on the base CB
- B (lower right): acute-angle vertex

And so, if a triangular field having a right angle is to be measured, it is enough if with all diligence the side opposite the right angle [the hypotenuse] is measured, and in addition one acute angle, by the aid of some quadrant divided into degrees, such as was constructed by us in chapter 2, book 1. For from these, once known, the area of the triangle is known, as we said just now, even if we cannot get access to the other two sides.

When in the same right triangle DCB one of the two sides about the right angle is given, together with the side that is opposite the right angle [the hypotenuse] — for instance, if DC and DB are known — [Margin: (h) Num. 3 of the right-lined triangles. — The area of a right triangle from one side about the right angle and the side opposite the right angle.] the other side, BC, will also be known if it is arranged that as the whole sine is to the side DB opposite the right angle, so the given side DC is to another. For the number produced will be the sine of angle B, from which, once known, from the table of sines its complement BDC will also be known. If, then, it is again arranged that as the whole sine of the right angle C is to the given side DB opposite it, so the sine of the found angle BDC is to another, the side BC opposite it will come out. From the two sides DC and CB, then, once known, the area of the triangle will become known, as was set down in Num. 3, which we wrote a little before.

And so, if in a field a triangular plot is given which has a right angle, it is enough, if with diligence we measure one side about the right angle, together with the side that is opposite the right angle, even if there is no access to the third. [Margin: The area of a right triangle from one side about the right angle and the side opposite the right angle.] For from these two the area is known, as has been said.

When in the same triangle BDC one side about the right angle is known — namely DC — together with one of the two acute angles, say B, [Margin: (i) — The area of

a right triangle from one side about the right angle and an acute angle.] the other side, CB, will be made known if it is arranged that as the whole sine is to the given side DC, so the tangent of angle BDC (found by subtracting angle B from 90°) is to the sought side CB, opposite it. Or if it is arranged that as the sine of angle B, given to the side DC opposite it, is to the given side DC, so the sine of the other angle BDC is to another; for again the found number will give the sought side CB. From the two sides DC and CB, then, we shall know the area, as is handed down in Num. 3.

In some field, then, if a triangular plot is proposed which has a right angle, and we measure one side about the right angle together with one acute angle, it will be enough to find the area of that triangle, even if access to the other two sides is denied. And so much for right triangles; let us now come to the oblique-angled ones.

7. If, then, in a non-right triangle ABD one side is known, together with any two angles whatever, we shall attain the knowledge of the area in this way. [Margin: (e) Num. 10 of the right-lined triangles. — The area of an oblique-angled triangle from one side and two angles.] From two known angles the third will also be known, since it is the complement of the other two to 180° . Therefore the other two sides will be known; and consequently, as before, from all three sides known the area will become known, by the things handed down in Num. 1 and 2.

So, in order that a triangular field having no right angle may be measured, one side, together with two angles, must be accurately measured. For from these the two remaining sides become known, etc., as has been said.

Again, if in the same triangle ABD, not right-angled, two sides are known, together with the angle contained by them, [Margin: (f) Num. 12 of the right-lined triangles. — The area of an oblique-angled triangle from two sides and the angle contained by them.] the third side will be found; and consequently, as before, from all three sides the area of the triangle will be produced.

And so it will be enough if, in any triangular field whatever, two sides, together with the angle contained by them, are measured, so that its area may be rendered known.

8. Nor indeed does this seem to be something to be passed over: if, that is, one side of a triangle, or of any rectilinear figure whatever, can be known divided into any number of equal parts, the remaining sides can be made into the same parts, by the aid of a divided instrument, as we declared up to the end of Num. 1, chapter 1, book 1. [Margin: By dividing one side of the figure into any number of

equal parts, how the other sides are made known in the same parts.] But in order that these things may be found more exactly, the fragment of the last little part (if any is left over) will have to be examined in thousandths, by the method we taught in Num. 14, chapter 2, book 1. For in this way we shall stray less from the truth in the dimensions of figures.

9. But let no one be moved or disturbed that we have said that straight lines are sometimes to be measured mechanically, by some little iron chain, or by an instrument of parts. [Margin: This mechanical measuring is to be admitted in some lines and angles.] For in this business of measuring — especially in fields and farms — such a mechanical dimension of lines is by all means to be admitted, both because among all surveyors this is the custom, and because the geometric way cannot always accomplish it; but most of all because in the dimensions of fields, or of figures, it is enough to arrive close to the truth, provided no notable error is committed. For if this measuring of certain lines is not tested by some material measure, from what source is one to have the sides of the proposed field or figure known? If, therefore, the measuring of the sides — which as it were strays little from the truth — is employed by everyone, why should we think it is to be rejected when measuring in the lines within figures? I do not see. I do not deny, however, that the geometric way, when it can be done, is to be applied. In figures too, where the sides are not too great, it is to be used, as we handed down at the end of Num. 2, chapter 1, book 1, not neglecting also those matters which, whenever any little particle of a line is to be known, we wrote of in thousandths at least, so that one can scarcely stray from the truth by this reasoning.

The same is to be understood of the mechanical measuring of angles by the quadrant: especially if, besides the degrees, the minutes too are investigated, as we taught in book 1, chapter 2.

ON THE AREA OF NON-RECTANGULAR QUADRILATERALS. CHAPTER III.

1. There are three kinds of quadrilateral figures which either have no right angle or at least not all right angles: the rhombus, the rhomboid, and the trapezium. [Margin: The area of the rhombus and rhomboid.] The first two kinds have no right angle; but the last can have either one right angle, or two, or also none: likewise it has two opposite sides parallel, or not parallel. Of the rhombus and rhomboid, whose sides are known, the area is produced by drawing a perpendicular into the side onto which the perpendicular falls; so that the

magnitude of the perpendicular is first accurately explored, either by an instrument of parts constructed at the beginning of this work, as we advised a little before in chapter 2, Num. 8, or in another way, as I shall say presently. For example, in the rhombus and rhomboid ABCD the area will be produced by the multiplication of the perpendicular AE into the side BC. [Margin: (g) Euclid I.35.] For the rectangle AF, contained under AE and AD, is equal to the parallelogram BD, since these two parallelograms are between the parallels AD, BC, and upon the same base AD. And so those others are mistaken who prescribe that the diameter AC be drawn, and that by the aid of the perpendicular AE the area of triangle ABC be sought, [Margin: (h) Euclid I.34.] which doubled would exhibit the area of the whole parallelogram — since indeed the triangle ABC is the half of the parallelogram. In vain, I say, do they prescribe this, since the area is found more expeditiously if the perpendicular is drawn into the whole side BC than if it is multiplied into its half and the resulting number is then doubled.

Diagram. Two figures, one above the other, illustrating the rhombus/rhomboid ABCD and its area by the perpendicular AE. In each, A is at the top left and D at the top right; along the bottom lie B (left), E, C, and F (right). The slanted parallelogram ABCD has the diagonal AC drawn, the perpendicular AE dropped from A to the base at E, and the rectangle completed to F (the segment CF and DF shown dotted).

- A (top left): vertex
- D (top right): vertex
- B (bottom left): vertex
- E (bottom, foot of the perpendicular AE)
- C (bottom, right of E): vertex
- F (bottom, far right): corner of the completed rectangle AF
- [Margin beside upper figure: (g) Euclid I.35]
- [Margin beside lower figure: (h) Euclid I.34]

If by the quadrant constructed in chapter 2, book 1, the magnitude of angle B is investigated, the perpendicular AE will be found by sines, in this way. [Margin: The finding of the perpendicular. — (i) Num. 2 of the right-lined triangles.] Let it be arranged that as the whole sine of the right angle E is to the side opposite it, AB, so the sine of angle B is to another. For the number produced will be the perpendicular AE, known in the parts of the given side AB.

2. Of a trapezium in which two opposite sides are parallel, AD and BC, and all the sides are known, the area is produced from the perpendicular AE, drawn

between the two parallel sides, multiplied into the half of the sum formed from the parallel sides. [Margin: The area of a trapezium having two parallel sides. — (k) Euclid II.1.] For, the diameter AC being drawn, the area of triangle ABC is produced from the perpendicular AE into the half of the base BC, as was said in chapter 2, Num. 2; likewise the area of triangle ACD from the same perpendicular AE into the half of the base AD; and consequently these two areas together will make up the area of the whole trapezium ABCD. Since, then, the same result comes from AE into the sum formed from the half of BC and the half of AD — that is, into the half of BC and AD together — as comes from AE into the half of the side BC and from AE into the half of the side AD, it is clearly established that the area of the trapezium is generated from the perpendicular AE into the half of the sum of the sides AD, BC. And this ratio also holds in a trapezium having one right angle, or two right angles.

Diagram. Two figures. The upper is the trapezium ABCD with parallel sides AD (top) and BC (bottom): A and D at the top, B, E, C along the bottom, with the diagonal AC and the perpendicular AE drawn from A to the base at E. The lower is a quadrilateral with vertices F (upper left), I (upper right), G (lower left), H (lower right), and its two diagonals FH and GI meeting at the central point K (drawn dotted).

- A (top left): vertex
- D (top right): vertex
- B (bottom left): vertex
- E (bottom, foot of the perpendicular AE)
- C (bottom right): vertex
- F (lower figure, upper left): vertex
- I (lower figure, upper right): vertex
- G (lower figure, lower left): vertex
- H (lower figure, lower right): vertex
- K (lower figure, centre): intersection of the diagonals

As for the perpendicular AE, it is found — just as we said for the rhombus and the rhomboid — in two ways, in case the angle B is to be determined with the quadrant, and so on.

[Margin: The area of a trapezium having no parallel sides. (e) 12th of rectilinear triangles, No. 2.]

In the trapezium FGHI, however, in which no sides are parallel — though all the sides are known — one must first measure the diagonal IG with the instrument of parts. Then the area of each of the two triangles FGI and GHI is to be found, as we set out in Chapter 2, Nos. 1 and 2; for the two areas together will make up the area of the whole trapezium.

But if you prefer to determine the angle F or H with the quadrant, we shall learn the magnitude of the diagonal GI by the doctrine of sines and tangents — as we taught (e) in Book I, Chapter 3 — from the two sides FG and FI together with the angle F which they contain, or from the two sides HG and HI together with the angle H which they contain.

[Margin: The area of an irregular quadrilateral figure.]

3. In no other way shall we obtain the area of any irregular quadrilateral, even one that does not have all its angles turned inward like a trapezium. For example, if in the trapezium FGHI two straight lines GK and IK are drawn from G and I, an irregular quadrilateral GHIK is formed, since it has only three angles, GHI, HIK, and HGK. For at K no angle GKI is formed turned inward toward H, because that space is greater than two right angles; rather it opens outward toward F. We shall therefore collect the area of this irregular quadrilateral, once the diagonal KH has been drawn, out of the two areas of the triangles IKH and GKH, exactly as was said for the trapezium FGHI.

ON THE AREA OF IRREGULAR MANY-SIDED FIGURES. CHAPTER IV.

[Margin: A many-sided figure.]

1. Irregular many-sided figures — those, that is, which have more than four unequal sides — we shall measure, however irregular they may be, in the same way as irregular trapezia: namely, by resolving them into triangles and investigating the areas of the individual triangles. For all the areas gathered into one sum are equal to the area of the whole proposed figure. Thus if a seven-sided figure ABCDEFG is resolved into five triangles ABG, GBD, DBC, DEF, FDG — so arranged that their sides do not cut one another — the areas of the individual triangles must be sought in this way.

[Margin: How, when a figure can be resolved into triangles, its area is to be gathered. (f) 9th of rectilinear triangles.]

Diagram. An irregular seven-sided figure ABCDEFG, drawn as an elongated hexagon-like outline and resolved by dotted lines into triangles, with dotted perpendiculars dropped inside.

- A (upper right vertex)
- B (upper left vertex)
- C (left vertex)
- D (interior lower / lower-left vertex)
- E (bottom-left vertex)
- F (bottom-right vertex)
- G (right vertex)
- H (interior, foot of perpendicular from A)
- I (interior, foot of perpendicular from D)
- K (interior, foot of perpendicular from C)
- L (interior, foot of perpendicular from F)
- M (interior, foot of perpendicular from D toward the base)

When the sides of the triangles can all be made known by some measurement — whether the figure represents an actual field or is merely drawn on paper — let perpendiculars AH, DI, CK, DM, FL be dropped from the vertices to the opposite sides, one in each triangle. Then in triangle ABG (f) let the segments BH and HG be found from the three known sides, and from these the perpendicular AH, as we explained in Chapter 2 of this book, No. 2. For AH multiplied into half the base BG will yield the area of triangle ABG. Let the areas of the other triangles be worked out by the same reasoning, and let all the areas be reduced into one sum, so that the area of the whole figure is obtained. But if you prefer, you can track down the areas of all the triangles from their three known sides — even though no perpendiculars have been drawn and the segments BH and GH have not been found — by the method we described in Chapter 2, No. 1.

[Margin: How, when a figure cannot be resolved into triangles, its area is to be ascertained. (g) 12th of rectilinear triangles; (h) 2nd of rectilinear triangles; (i) 12th of rectilinear triangles, No. 2; (k) 2nd of rectilinear triangles; (l) 12th of rectilinear triangles, No. 2; (m) 2nd of rectilinear triangles; (n) 12th of rectilinear triangles; (o) 2nd of rectilinear triangles; (p) 16th of rectilinear triangles; (q) 2nd of rectilinear triangles.]

2. When the interior sides of the triangles cannot be measured — indeed, cannot even be drawn, as often happens in open country or in fields which, because of intervening trees or marshes, cannot be crossed in straight lines — we shall reach our goal by another method, namely this. Once the sides bounding the figure are known by some measurement, let the angles they contain also be investigated with the help of some quadrant divided into degrees. In the figure proposed, the angle CDE need not be sought, since it lies outside the figure. Then, resolving the figure in the mind or in thought into triangles, as if the interior sides had actually been drawn, proceed as before: (g) in triangle ABG let the two angles B and G be explored from the two sides AB and AG and the angle they contain, and also the side BG. For hence, (h) in the right triangle ABH or AGH, the perpendicular dropped from A will be known from the base AB and the angle B, or from the base AG and the angle G; and so the area of the triangle will be found, as before, from what was handed down in Chapter 2 of this book, Nos. 1 and 2. In just the same way the area of triangle BCD will become known from the two sides CB and CD, which contain the known angle C, (i) provided the angles B and D are first investigated, together with the side BD, (k) and from these the perpendicular CK is dropped. After this, in triangle BDG, if the angles ABG and CBD, now known, are subtracted from the whole known angle ABC, the angle DBG will remain known. Since, then, the sides BD and BG that include it have also been made known, (l) the angles D and G and the side DG will be known in the same way, (m) and moreover the perpendicular dropped from B onto DG, and so on. Similarly, in triangle GDF, if from the known angle AGF the now-established angles AGB and BGD are taken away, the angle DGF will be left known. Since, then, the side GD has also been made known, (n) we shall know the two angles GDF and GFD as well, and moreover the side DF, (o) together with the perpendicular dropped from G onto DF, and so on. Finally, in triangle DEF, since all the sides are known, (p) all three angles will also be made known; and so the perpendicular DM, dropped from D — (q) or from any other angle you like — will become known, and so on. From these cases you will readily understand how you are to proceed with other irregular figures.

Or, if you wish, you can also draw on paper a figure similar to the field: namely, take a straight line AB of as many equal parts as there are measures contained in the corresponding side of the field, and construct an angle ABC equal to the one you found at the field's edge. Then on BC take as many parts as are contained in the corresponding side of the field, and again construct an angle BCD equal to the one you observed in the figure. Finally, if you do the same with the angle CDE and

the rest, and likewise with the straight lines CD, DE, and the others, a figure similar to the field will be drawn; and if it is resolved into triangles whose sides within the figure are measured with the instrument of parts, its area will be found just as before, when the field itself could be resolved into triangles.

When the level surface of the field is unobstructed, a figure similar to it will be described more exactly by the method we set out in Book 3, Problem 42, No. 3: namely, if within the field some point is chosen, from which straight lines are drawn to all the angles, noting the angles they make. For if those lines are measured, and the angles are transferred to some point on the paper, and along the lines of the angles as many equal parts are taken as there were measures found in the lines of the angles standing about the point chosen in the field, then the lines connecting the outer points will contain a figure similar to the field, as we demonstrated in the passage cited. And if two points are chosen in the field, from which lines are drawn to the angles, then, by marking the points where each pair of lines meets, a figure similar to the field will likewise be described, even though those lines are not measured — just as we explained in the same place, No. 1.

[Margin: In what way a field within which a lake or a wood lies is to be measured. The general method of surveyors for investigating the area of any figure.]

It is furthermore evident that a field in which a lake or a wood is enclosed can be measured in the same way — even though it cannot be resolved into triangles — provided its exterior sides, together with the angles, can be known. And if along the perimeter of the field there is some curved and not straight portion, it must be cut into as many parts as will make them differ little from straight lines, and these are then to be taken as straight sides.

3. So that surveyors may not be forced to walk repeatedly over the whole field in order to draw the perpendiculars within the triangles and to measure the angles, they adopt this method. In the field, or figure, they lay out the largest rectangle they can, and they conceive perpendiculars to be dropped to its sides from the angles of the figure — which they do by applying one arm of the square to the side of the rectangle and directing the other toward the opposite angle of the figure. For in this way the whole figure will be resolved into that constructed rectangle, into trapezia with two parallel sides, and into right triangles. Then they either measure the sides of the rectangle and the perpendiculars themselves, or they direct that they be measured by others — which is done with an iron chain. Finally, they measure the right triangles, as we handed down in Chapter 2,

No. 3, and the trapezia of two parallel sides as we taught in Chapter 3, No. 2; and the rectangle, lastly, by the method we described in Chapter 1. For the areas of these, gathered into one sum, make up the area of the whole field or figure.

Diagram. An irregular octagon ABCDEFGH, drawn as a tall, roughly vertical eight-sided outline, with an inscribed rectangle and dotted perpendiculars partitioning it into a rectangle, trapezia, and right triangles.

- A (top vertex)
- B (upper-left vertex)
- C (left vertex)
- D (lower-left vertex)
- E (bottom vertex)
- F (lower-right vertex)
- G (right vertex)
- H (upper-right vertex)
- I, K, L, M (corners of the inscribed rectangle IKLM)
- N, S, T, Q, O, R, P, V (interior points and feet of the perpendiculars)

In the proposed octagon ABCDEFGH there is contained the rectangle IKLM; the trapezia HSTG, GTQF, OLRE; and the right triangles ANK, ANI, ISH, FQM, MOE, DPR, CDP, BCV, BKV.

Others do not lay out a rectangle within the field or figure, but draw the longest line they can — namely, from the point A to the side EF — which they call the "base line." To this they drop perpendiculars from the angles, and thus once more partition the whole figure into trapezia of two parallel sides and into triangles, and so on. But the former method seems more convenient, since in it the perpendiculars drawn from the angles are shorter, and are therefore measured more easily and more surely.

When the operations described cannot be carried out within the field, surveyors are also accustomed to form a rectangle around a field that encloses woods, lakes, buildings, or other obstacles. For if perpendiculars are drawn to its sides from the outer angles of the field, there will again be constituted trapezia of two parallel sides — that is, rectangular trapezia — or parallelograms, together with right triangles outside the field; and if the areas of these are subtracted from the area of the whole rectangle, what remains will be the area of the proposed field.

4. Nor does that method seem to me to be despised which I used once to explain to my students: namely, that to the whole figure (the field having first been reduced to a similar figure, as I directed just above in No. 2) a square be constructed equal to it, or at least the side of that square be found. For if one side of this square is measured and multiplied into itself, the area of the proposed figure will come out. Now the side must be measured in the same parts of the figure's sides that correspond to the measures of the field's sides, which is easily done with the instrument of parts.

[Margin: A fine method for finding the area of any figure whatever. In what way a square equal to a given figure is constructed.]

Just how a square equal to any figure can be constructed without great labor, I taught in the scholium to Proposition 14 of Book 2 of Euclid; and I think it worth repeating here, with a few improvements. Let there be, then, an irregular heptagon ABCDEFG, and let it be resolved into 5 triangles ABG, BCG, CDG, DEG, EFG. To BG, the common base of two of the triangles, let there be drawn from the opposite angles A and C the perpendiculars AL and CH, the latter of which falls on GB produced in our figure. Then on any straight line OP let OQ and QR be taken equal to AL and CH respectively; likewise RP equal to BI, half the base BG; and about OP, from its midpoint S, let a semicircle OTP be described. Finally, from R, the endpoint of the line OR — which is equal to the two perpendiculars AL and CH together — let a perpendicular RT be raised to OP, cutting the semicircle at T. I say that the square on the line RT is equal to the two triangles ABG and BCG together.

Diagram. The irregular heptagon ABCDEFG, resolved by dotted lines into five triangles about the common base BG, with the perpendiculars AL and CH marked.

- A (upper right vertex)
- B (top vertex)
- C (left vertex)
- D (lower-left vertex)
- E (bottom vertex)
- F (lower-right vertex)
- G (right vertex)
- H (on GB produced, foot of perpendicular CH)
- I (on BG)

- L (interior, foot of perpendicular AL)
- K, M, N (interior points of the triangulation)
- z (interior label near the lower triangles) [?]

Diagram. Four upright measuring rods drawn side by side, labelled from left to right:

- a (a plain vertical rod)
- b (a plain vertical rod)
- c (a plain vertical rod)
- q (a beaded / dotted vertical rod)

These represent the sides a, b, c (and the further side q) of the squares equal to the several parts of the figure.

Diagram. The quadrature semicircle : a semicircle described on the vertical diameter OP, with the chord/segment points marked and the perpendicular RT raised to meet the arc at T.

- O (top of the diameter)
- Q (on the diameter below O)
- S (midpoint of OP, center)
- R (on the diameter, foot of the perpendicular RT)
- T (on the arc, meeting the perpendicular RT)
- P (bottom of the diameter)

Diagram. A right-angle "square-stacking" construction to the right of the semicircle, with a right angle formed at the base and dotted lines rising to the apex.

- P, d, k, e (points along the horizontal base line, left to right)
- f (apex, upper-left, where the dotted lines converge)
- h (upper-right)
- g (just below h)

For since the rectangle under BI, half the base, and the perpendicular AL is equal to triangle ABG, by Proposition 1 of Book 7 of this work; and the rectangle under the same BI and the perpendicular CH is equal to triangle BCG; while that under BI and the sum of AL and CH — that is, under RP and OR (since RP was taken equal to BI, and OR to AL and CH together) — (i) is equal to those rectangles which are contained under BI and AL, CH: the rectangle under OR, RP will be

equal to the two triangles ABG, BCG. (k) Since, then, the square on RT is equal to the rectangle under OR, RP — for, by the scholium to Proposition 13 of Book 6 of Euclid, RT is a mean proportional between OR and RP — the square on RT will also be equal to the two triangles ABG, BCG. Which was what was proposed. Indeed, that the square on RT is equal to the rectangle under OR, RP can be demonstrated in this way too, without recourse to Book 6 of Euclid: (l) The rectangle under OR, RP, together with the square on SR, is equal to

[Margin: (m) 47th of the first; (n) 47th of the first; (o) 47th of the first.]

the square on SP — that is (drawing ST), (m) to the square on ST, [and so] to the squares on SR and RT. When, therefore, the common square on the line SR is removed, the remaining rectangle under OR, RP will be equal to the remaining square on RT.

In the same way we shall find a square equal to the two triangles CDG, DEG, if to their common base DG the perpendiculars CD and EM are drawn — the former of which, in our figure, coincides with the side CD, and so on. And thus in turn, if there are more triangles, we shall always find single squares equal to each pair of triangles. But since in our figure there remains only a single triangle EFG, we shall find a square equal to it if, the perpendicular FN having been drawn, a semicircle is described upon the line composed of FN and half the base EG [reading of the second letter uncertain — printed EZ [?]], and so on. Let a, b, c, then, be the sides of the squares equal to the trapezia ABCG, CDEG and the triangle EFG; and to all these squares we shall exhibit one equal square by this device.

[Margin: (n) 47th of the first.]

Let a right angle d-e-f be made, and let the straight lines e-d and e-g be taken equal to the sides a, b; (n) then the square on the drawn line d-g will be equal to the squares on e-d and e-g — that is, to the sides a, b. Again let e-k be taken equal to the side c, and the straight line e-h equal to d-g; (o) then in turn the square on k-h will be equal to the squares on k-e and e-h — that is, on c and e-h, namely to the three squares on a, b, c. If, therefore, the side k-h is measured and multiplied into itself, the area of the proposed figure ABCDEFG will be produced.

[Margin: (o) 47th of the first.]

By the same device, if there are more sides, we shall find a square equal to all the squares. Thus if there were another side q, we should take a straight line e-p

equal to it; likewise a straight line e-f equal to k-h. For the square on p-f will be equal [to the squares] on a, b, c [and q], and so for more sides.

[Margin: An easy method for measuring an irregular trapezium.]

From these there follows an easy method for measuring an irregular trapezium, such as the trapezium ABCG in the adjoining figure. For once the diagonal BG has been drawn, if the two perpendiculars AL, CH dropped to it are measured, and their sum is multiplied into half the diagonal BG, the area of the trapezium will be produced, as has been demonstrated.

Moreover, in Book 8, Proposition 6, we shall also teach in what way a rectangle equal to a given rectilinear figure is constructed; and if this is done here, a square equal to that rectangle will be produced, by the last proposition of Book 2 of Euclid, without any trouble or difficulty.

ON THE AREA OF REGULAR MANY-SIDED FIGURES. CHAPTER V.

[Margin: The area of a regular figure.]

1. Although regular figures — those, namely, that are both equilateral and equiangular — can be measured, like the irregular figures of the preceding chapter, by resolving them into triangles and so on, there is nevertheless usually given a proper and particular rule by which the area of any regular figure is found, which runs as follows:

Let half the perimeter of the figure be multiplied into the perpendicular falling from the center of the figure to one side. For the number produced will be the area of the figure.

For, as we shall demonstrate in Book 7, On Isoperimetric Figures, Proposition 2, the area of any regular figure is equal to the rectangle contained under the perpendicular drawn from the center of the figure to one side, and under half the perimeter of that same figure.

[Margin: In what way the perpendicular and the semidiameter of a regular figure are found.]

2. The perpendicular falling from the center of the figure onto one side, together with the semidiameter of the circle that circumscribes the figure, will be found thus. Let the number of sides — or of angles — be doubled,

and from the double let 4 be subtracted. For the remaining number will indicate how many right angles all the angles of the figure are equal to, by what was demonstrated in the scholium to Proposition 32 of Book 1 of Euclid. Let this same remaining number — that is, the number of right angles — be divided by the number of angles, so that the quotient shows the magnitude of one angle of the figure, which in a hexagon amounts to a right angle plus a third part — that is, 120 degrees. And since the semidiameter bisects the angle of the figure, as is clear from the demonstration of Proposition 12 of Book 4 of Euclid, it comes about that, if half the side onto which the perpendicular falls is set as the whole sine, the perpendicular will be the tangent of half the angle of the figure, and the semidiameter the secant. If, then, we set it up:

As 100000, the whole sine,	is to half the side;	so the tangent of half the angle, or the secant,	to another [quantity]:

the perpendicular, as well as the semidiameter, will come out in the parts of the side of the figure. For example, in a regular hexagon whose one side is 12: if it be set up — as 100000, the whole sine, is to 6, half the side, so 173205, the tangent of half the angle of the hexagon, to another — the perpendicular will be found to be 10 and $\frac{39230}{100000}$, or 10 and $\frac{3923}{10000}$. Likewise if it be set up — as 100000, the whole sine, is to 6, half the side, so 200000, the secant of half the angle of the hexagon, to another — the semidiameter of the figure will come out as 12.

Therefore, if the perpendicular 10 and $\frac{3923}{10000}$ is multiplied into 36, half the perimeter of the hexagon made up of three sides, the area of the hexagon will be produced as 374 and $\frac{1228}{10000}$, or 374 and $\frac{307}{2500}$. And one must proceed in the same way with other regular figures, in which the halves of the angles can be taken from the table of tangents and secants.

[Margin: Hence, from a given side of a regular figure, its area cannot always be found unless the figure itself is drawn.]

- But since a circle can be described about any regular figure, as is established from Book 4 of Euclid, we shall here set out the sides of very many regular figures in the parts of the diameter of the circumscribing circle, 20000000 — or of the semidiameter, that is, the whole sine, 10000000 — taken from approved authors, so that their areas may be found more exactly by the rule proposed in No. 1. For if we wished to investigate the area from a single known side in some measure — as was done in the hexagon — great and numerous difficulties would arise, because the

semidiameters and the perpendiculars cannot be extracted from the sines, tangents, and secants except when half the angle of the figure comprises exact degrees, or degrees with minutes, or degrees with minutes and seconds (although, if seconds are present, it is necessary to apply the proportional part), as was done just now in the hexagon, whose half-angle contains 60 degrees — which does not happen in very many figures. For, to take an example, in a heptagon all 7 angles, by the scholium to Proposition 32 of Book 1 of Euclid, are equal to 10 right angles, and therefore one angle comprises one

one right angle and a half [?] — that is, 128 degrees, 34 minutes, 17 seconds, 8 thirds, 34 fourths, and so on; and its half is 64 degrees, 17 minutes, 8 seconds, 34 thirds, 17 fourths, and so on. From this half, neither the tangent (serving as the perpendicular) nor the secant (serving as the semidiameter) can be extracted from the tables of tangents and secants. You will find that the same thing happens with countless other figures.

If a method were known for constructing all regular figures, this difficulty could in some way be overcome; for once one side is known in any number of parts whatever, both the perpendicular and the semidiameter could likewise be known in those same parts, by means of the instrument of parts [scale of parts], as we set out at the end of Book I, Chapter I, Num. 1. But since we know how to inscribe only very few equilateral figures within a circle, the areas of most of them must remain unknown. For this reason some geometers — among whom Ludolph van Ceulen [Ludolphus à Collen] labored energetically — have with immense effort investigated the sides of these figures, whether or not the figures can actually be constructed, even though their angles include, beyond degrees and minutes, also seconds, thirds, fourths, and so on — so that we might obtain their areas.

TABLE CONTAINING THE SIDES OF THE REGULAR FIGURES

from the triangle up to the figure of 80 sides, given the diameter as 20000000, or the whole sine as 10000000.

Column headings: **Number of sides or angles** [Num. lat. vel angular.] | **Sides of the regular figures, given the diameter 20000000 or the whole sine 10000000** [Latera figurarum regularium, posita diametro 20000000 vel sinu toto 10000000].

No. of sides	Side	No. of sides	Side	No. of sides	Side
3	17320508	29	216238	55	114177
			0		6

No. of sides	Side	No. of sides	Side	No. of sides	Side
4	14142135	30	209056 9	56	112140 8
5	11755705	31	202336 6	57	110175 5
6	10000000	32	196034 1	58	108277 8
7	8677674	33	190112 0	59	106444 3
8	7653668	34	184536 7	60	104671 9
9	6840402	35	179278 6	61	102957 5
10	6180339	36	174311 4	62	101298 3
11	5634651	37	169611 8	63	996917
12	5176380	38	165158 6	64	981353
13	4786313	39	160933 1	65	966275
14	4450418	40	156918 1	66	951638
15	4158233	41	153098 5	67	937445
16	3901806	42	149460 1	68	923669
17	3674990	43	145990 6	69	910291
18	3472963	44	142678 3	70	897296
19	3291891	45	139512 9	71	884666
20	3128689	46	136484 8	72	872387

No. of sides	Side	No. of sides	Side	No. of sides	Side
21	2980845	47	133585 2	73	860444
22	2846296	48	130806 2	74	848824
23	2723332	49	128140 4	75	837513
24	2610523	50	125581 0	76	826499
25	2506664	51	123121 8	77	815771
26	2410733	52	120756 9	78	805318
27	2321858	53	118481 2	79	795130
28	2239289	54	116289 6	80	785196

For from the side of any regular figure — known in parts of the diameter of the circumscribed circle, or of the whole sine — we shall arrive, by what we wrote in Chapter 2, Num. 2, at the knowledge of the perpendicular line drawn from the center to one side, and so obtain the whole area, as we showed just above in Num. 1. In the preceding table, then, you have the sides for a great many figures. All of these fall a little short of the true sides; and if you add one unit, they become a little larger than the true ones — so that the true side of the equilateral triangle lies between these two numbers, 17320508 and 17320509.

[Margin: From a known semidiameter of a circle, to find the side of a regular figure inscribed in that circle.]

4. Now, once the semidiameter of some circle is known in parts of any measure whatever, we shall find, in those same parts, the side of a regular figure whose number of sides is not greater than 80, by means of the preceding table — namely, if we set it up so that, as the whole sine 10000000 is to the side of the proposed figure in the preceding table, so the given semidiameter of the proposed circle is to another [number].

For example, let the semidiameter of some circle be 12, and let the side of the decagon relative to that semidiameter be sought. Set: as 10000000, the whole sine,

is to 6180339, the side of the decagon, so 12, the given semidiameter, is to another number; and this yields the sought side, $8\frac{164068}{10000000}$, or in smaller numbers $8\frac{41017}{2500000}$.

[Margin: To reduce a large fraction to a nearly-equivalent smaller one. To find the mean between two fractions.]

And if it seems troublesome to work with so large a fraction, reduce it to a smaller, nearly-equivalent one in this way. Choose for the numerator any number you like — say 10 — and set: as the numerator 164068 is to its denominator 10000000, so the chosen numerator 10 is to another; and you will find the denominator $609\frac{82588}{164068}$. Hence, disregarding this [attached] fraction, the denominator 609 is smaller than the true one, and 610 is larger — that is, the fraction $10/609$ is greater than the fraction $164068/10000000$, while the fraction $10/610$ is smaller. Now between these two fractions, $10/609$ and $10/610$, you will produce the mean $20/1219$ [?], whose numerator is compounded from the [two] numerators and whose denominator from the [two] denominators. And this fraction so found will be nearly equal to the larger of them.

FURTHER, that a fraction whose numerator is composed of the two numerators, and whose denominator of the two denominators, of two small fractions, is greater than the smaller and smaller than the greater, we shall demonstrate in Book 8, Proposition 10.

[Margin: From a known side of a regular figure, to find the semidiameter of the circumscribed circle.]

CONVERSELY, from the given side of any regular figure we shall come to know the semidiameter of the circumscribing circle, if we set it so that, as the side of the proposed figure in the preceding table is to 10000000, so the given side is to another. For instance, let the side of a pentagon be given as 12, and set: as 11755705 is to 10000000, so 12 is to another; then the semidiameter of the circumscribed circle will be found to be $10\frac{2442050}{11755705}$. And if we set it so that, as the numerator of this fraction is to its denominator, so any chosen numerator — say 1 — is to another, the resulting denominator will be found to be $4\frac{1082005}{2442050}$ [?: tiny superscript numerator, scan floor], so that the fraction $1/4$ is greater than the true one and $1/5$ smaller. By adding the numerators 1 and 1 together, and the denominators 4 and 5 together, you make the mean fraction $2/9$, which is still greater than $2442050/11755705$. But the mean between $1/5$ and $2/9$ is $3/14$; so that the sought semidiameter may be said to be $10\frac{3}{14}$. And in this manner the operations can be carried out with smaller numbers, though not with

complete precision, since the fractions assumed are not entirely exact — yet this error is tolerable in the measurement of fields.

5. Before I conclude the measurement of rectilinear figures, I should like to append a rule by which, from the known area of any figure whose side is known, we may come to the knowledge of another figure — similar to it and similarly placed — whose homologous side is likewise known. It runs as follows.

[Margin: By what method the area of another similar figure is derived from the area of any given figure. (a: [Euclid] VI.19 or 20.)]

The square of the denominator of the ratio which the side of the unknown figure has to the side of the known figure — this denominator being obtained by dividing the side of the unknown figure by the side of the known figure — if it is multiplied into the known area, will produce the area of the other, sought figure. The figures must, however, be similar and similarly placed, and their homologous sides taken as stated.

For the denominator of the ratio of the side of the sought figure to the side of the given figure, multiplied by itself, produces the denominator of the doubled ratio [i.e. the squared ratio] of those sides, as we wrote at Definition 10 of Book 5 of Euclid. Since, then, similar and similarly-placed figures also stand in the doubled ratio of their homologous sides, it follows that the denominator of the doubled ratio of the said sides, multiplying the known area, produces the sought area — that is, a number which has to the known area the doubling of the ratio of the given sides, denominated, namely, by the denominator that is produced from the denominator of the ratio of those sides multiplied by itself.

Diagram. A scalene triangle used for the worked example, with an altitude dropped to the base and a circular arc drawn at the lower left. The apex A is at the top; from it the perpendicular AD falls to the base BC at D, dividing the base into segments BD and DC.

- A (apex, top): vertex A
- E (just below A, on side AB near the apex): point E
- B (lower-left vertex): vertex B
- C (right vertex): vertex C
- D (foot of the perpendicular on base BC): point D
- F (lower left, on the arc): point F
- 10 (along side AB, between B and the apex): $AB = 10$
- 17 (along side AC): $AC = 17$

- 8 (the perpendicular segment AD): $AD = 8$
- 6 (base segment on the left, BD): $BD = 6$
- 15 (base segment on the right, DC): $DC = 15$
- 21 (below D, the whole base): $BC = 21$

For example, in triangle ABC whose side AB is 10, AC is 17, and BC is 21, the area is 84. If, then, there is another triangle similar to it, having the side homologous to AB equal to 70, the one homologous to AC equal to 119, and the one homologous to BC equal to 147: divide the side 70 by 10, so that the denominator 7 of the ratio of side 70 to side 10 is produced; and let the square of this denominator, namely 49, be multiplied into 84, the area of triangle ABC — and the area of the latter triangle, 4116, will be produced.

Again, since the area of an equilateral triangle each of whose sides is 1 is about $13/30$, as appeared above: if another equilateral triangle is given each of whose sides is 70, we shall find its area thus. The denominator of the ratio of the sides is the side 70 itself — for 70 divided by 1 gives 70. We shall therefore multiply 4900, the square of the side 70, into $13/30$, the known area. The resulting number, $2123 \frac{1}{3}$, will be the area of the latter triangle.

[Margin: The above rule stated another way. (b: [Euclid] VI.19 or 20.)]

6. This rule may also be stated thus: Set as the square number of the side of the known figure is to the square number of the side of the sought figure, so the area of the known figure is to another. For the resulting number will be the area of the sought figure.

This is because the ratio of the square of the side of the known figure to the square of the side of the sought figure is the same as that of the known figure to the sought figure — since each ratio is the doubled ratio of the homologous sides. And since the square of the side 1 is 1, it follows that whenever the side of the figure having the known area is 1, it suffices to multiply the square number of the side of the sought figure into the given area, so that the sought area is produced. So it is worthwhile to investigate the areas of a great many regular figures whose sides are 1; for from these, without much labor, the areas of other similar figures whose sides are known will be drawn out. We have set down here the areas of ten regular figures whose sides are 1, so that through them the areas of similar figures whose sides exceed unity may be investigated by the rule prescribed in Num. 5 or 6 — although for the square this is not necessary, since the given side multiplied into itself produces its square.

TABLE OF AREAS OF THE REGULAR FIGURES (side = 1)

Left (rotated) heading: **whose sides are 1** [Quarum latera sunt 1]. Column headings: **Regular figures** [Figuræ regulares] | **The areas of the aforesaid figures are approximately these** [Areæ prædictarum figurarum sunt fere hæ].

Regular figure	Area (side = 1), approximately
Trigonum (triangle)	1875000/4330127, or 13/30
Tetragonum (square)	1
Pentagonum (pentagon)	1 8469719/11775706
Hexagonum (hexagon)	2 2990381/5000000
Heptagonum (heptagon)	3 5507221/8677674
Octogonum (octagon)	4 1585127/1913417
Enneagonum (nonagon)	6 1243755/6840402
Decagonum (decagon)	7 858089/1236068
Vndecagonum (11-gon)	9 517050/1408663
Duodecagonum (12-gon)	11 84614/431365

BUT these fractions can, if one wishes, be recalled to smaller nearly-equivalent ones, as we showed above in Num. 4.

[Margin: By what method the areas of the regular figures are found by means of the sides in the table above.]

7. These areas, moreover, were produced by the rule prescribed in Num. 1, the perpendiculars falling from the centers onto the sides having first been found — even though in some figures the angles include seconds and thirds besides degrees and minutes — in this way. For the heptagon, for example, the half of the heptagon's side, 4338837, was taken from the table above. (When the number of the side is odd, 1 must be added to make it even and so give it a half, since it is then a little less than the true side, as we said.) Then, since — as was stated in the treatment of sines — this half is the sine of the angle opposite it at the center, that angle was sought from the table of sines (with the proportional part also applied, in the manner we taught at the end of Lemma 53 of our *Astrolabium*, so that the angle might be found in degrees, minutes, and seconds); and it was found to be 25 degrees, 42 minutes, 51 seconds.

[Margin: c. On rectilinear triangles, 4.]

After this — since it holds that, as the sine of this angle just found (namely, the one taken from the table) is to the half of the side, so the sine of the complement of the same angle, that is, of 64 degrees, 17 minutes, 9 seconds (the proportional part being likewise applied, on account of the seconds), namely 9011398, is to the perpendicular opposite that complement of the found angle in the right triangle — this perpendicular was found to be $9011398/8677674$. This, finally, multiplied into $7/2$, the half of the heptagon's perimeter, produced the area of the heptagon, $3\ 11014442/17335348$, or in smaller numbers $3\ 5507221/8677674$. And by this same procedure you will find the area of any figure, provided its number of sides is not greater than 80. From it you may then derive the area of a similar figure whose side is greater than 1, by the rule given in Num. 5 or 6.

Yet by the same device set out here in Num. 7, one may — if one wishes — obtain the area of any regular figure even if its side is greater than 1, without first finding the area of a similar figure whose side is 1: namely, if in place of $1/2$, the half of the side 1, one takes the half of the given side, which is greater than 1, as is evident.

8. Given the area of a regular figure whose number of sides is not greater than 80, its side is found in this way. Set: as the area of a similar figure having side 1 — taken from the preceding table — is to the area of the proposed figure, so 1, the square of the side 1, is to another. For the resulting number will be the square of the sought side; and so its square root will give the sought side. For area is to area as the square [of the side] is to the square [of the side], since each ratio is the doubled ratio of the sides.

[Margin: From a known area, how the side is drawn out. (d: [Euclid] VI.10 or 20.)]

But if the known area is that of a regular figure having more than 12 sides, yet not more than 80, one must first find the area of a similar figure with side 1, by means of the table of sides above, as we taught in Num. 7. Then the side is to be sought, as explained here in Num. 8. And if the table of sides above were extended to more sides, the side of a figure of more than 80 sides could be found in the same way from its area. For example: let the area of some equilateral triangle be $15\ 2/3$. Since the area of an equilateral triangle whose side is 1 was found above to be $13/30$, set: as $13/30$ is to the proposed area $15\ 2/3$, so 1, the square of the side 1, is to another; and the square of the sought side, 36, will be found, whose square root, 6, gives the side sought.

On the Measurement of the Circle, according to Archimedes

CHAPTER VI

Since we can measure any given circle and its parts, it is essential to be thoroughly acquainted with what Archimedes handed down on the measurement of the circle. It will therefore not be out of place if I here insert his little book on the measurement of the circle — a work truly acute and subtle — for two reasons: first, because it is very brief, consisting of only three propositions; and second, so that the studious reader, to appreciate a matter so useful and so universally current among practitioners, may go to Archimedes himself. And since Archimedes' writings, on account of their studied brevity, are somewhat obscure, he is compelled to turn to us for help; here indeed we hope to shed some light for such readers. Nor do we doubt that this material will prove welcome and agreeable to the studious reader.

PROPOSITION I

The area of any circle is equal to a right triangle one of whose sides about the right angle equals the radius of the circle, while the other equals the circumference of that same circle.

Let ABCD be a circle with center E and radius EA; and let FGH be a right triangle with its right angle at G, the side FG equal to the radius EA of the circle, and the side GH equal to the circumference of that same circle. I say that the circle ABCD is equal to the triangle FGH.

For if it is denied that they are equal, then first, if it can be so, let the circle be greater than the triangle by a magnitude z — so that the circle equals the triangle together with the magnitude z , and is therefore greater than it by z . If, then, from the circle more than half is taken away, and from the remainder again more than half, and so on continually, [Margin: Euclid X.1] there will at last be left a magnitude smaller than z .

This continual subtraction will proceed as follows. In the first place, take away from the circle the inscribed square ABCD. Since this square is half of the square IKLM circumscribed about the circle — as we showed in the scholium to proposition 9, book 4 of Euclid — while the circle is only a part of that square IKLM, the inscribed square ABCD will be greater than half the circle. Next, from

the four remaining segments take away the four isosceles triangles AOB, BPC, CQD, DNA, drawn to the midpoints of the arcs. These together are greater than half of the four segments together, since each of them is greater than half the segment in which it stands. For when the rectangle AR is completed, [Margin: Euclid I.41] the triangle AND is its half; and therefore this same triangle is greater than half the segment AND. The same reasoning holds for the others. In the same way, if from the eight remaining segments eight other isosceles triangles set up in them are taken away, and so on.

Diagram. A circle inscribed in a square IKLM, with the diagonal (inscribed) square ABCD, an octagon, and numerous construction lines converging on the center E. The circumscribed square has its corners lettered K (upper left), I (upper right), L (lower left), and M (lower right). The inscribed square and octagon are lettered A (top), with B, C, D and the arc-midpoints O, N, P, Q, together with many auxiliary construction points around the perimeter.

- K (upper-left corner of square): vertex
- I (upper-right corner): vertex
- L (lower-left corner): vertex
- M (lower-right corner): vertex
- A (top): inscribed-square / octagon vertex
- E (center): center of circle
- D (right): inscribed-square vertex
- c (right, below D): circumscribed-octagon vertex (lowercase c)
- R (right): corner of completed rectangle AR
- S (small, above A): auxiliary point
- e (small, near top): auxiliary point
- V (top, between K and A): auxiliary point
- o (upper-left, inner): auxiliary point
- T (inner, left of center): foot of perpendicular
- X (left): auxiliary point
- Y (left, lower): auxiliary point
- P (lower left): arc-midpoint / octagon vertex
- a · C · b (along the bottom): lowercase a and b are circumscribed-octagon vertices; the capital C between them is the bottom-middle inscribed-square tangent point

- B (left tangent point), O (upper-left), N (upper-right), Q (lower-right): inscribed-square vertex and octagon arc-midpoints; the upper-right N is printed as a reversed/turned type sort (N) but is legibly N

So let there now be posited eight segments AO, OB, BP, PC, CQ, QD, DN, NA left over, smaller than the magnitude z. And since the circle is granted to be equal to the triangle FGH together with the magnitude z, if from the circle the triangle together with z be taken away, the remaining figure — namely the inscribed octagon — will be greater than the triangle FGH; which is absurd, since it is in fact much smaller. For if from the center E a perpendicular ET is drawn to the side BO, and in the triangle Gk is taken equal to ET and the straight line Gi equal to the perimeter of the octagon, the point k will fall short of F, and i short of H — because ET is smaller than the radius of the circle, and the perimeter of the octagon smaller than the circumference of the same circle. Therefore, once the straight line ki is drawn, the triangle Gki will be smaller than the triangle FGH, part to whole. But the triangle Gki is equal to the octagon: for by the scholium to proposition 41, book 1 of Euclid it is equal to the rectangle under Gk and half of Gi, and this rectangle, by proposition 2, book 7 On Isoperimeters, is equal to the octagon. Therefore the octagon is smaller than the triangle FGH. So the circle cannot be greater than the triangle.

Diagram. Below the first figure, a long, very shallow triangle (drawn with a dotted internal line) tapering to a point at the right, with a small right triangle marked Z set beside its apex — the whole representing the comparison of the triangle FGH with the magnitude z.

- n (upper left): apex-side label
- F (left, stacked): vertex
- k (left, stacked, lowercase): construction point k ($Gk = ET$), falling just short of F
- G (lower-left corner): vertex
- m (upper-right apex): vertex
- I, H (bottom, near the apex): base points
- Z (small right triangle at far right): the magnitude z

Next, if it can be so, let the circle ABCD be smaller than the triangle FGH by a magnitude z. Circumscribe about the circle the square IKLM, whose sides touch the circle at the points A, B, C, D; this square will be greater than the triangle FGH. For since its perimeter (as we shall prove in book 8, proposition 1) is greater than the circumference of the circle — that is, than the straight line GH — while the

perpendicular EA equals FG, the right triangle having one side equal to FG and the other greater than GH (namely, equal to the perimeter of the square IKLM) will be greater than the triangle FGH. Since that triangle, by the scholium to proposition 41, book 1 of Euclid, is equal to the rectangle under FG and half the perimeter of the square IKLM, and this rectangle, by proposition 2, book 7 On Isoperimeters, is equal to the square IKLM, the square IKLM will therefore also be greater than the triangle FGH. And since the triangle FGH is posited equal to the circle together with the magnitude z, and hence greater than z, the square IKLM will also be greater than z (since we have shown it to be greater than the triangle FGH). So if from the square IKLM more than half be taken away, and from the remainder more than half again, and so on continually, [Margin: Euclid X.1] there will at last be left a magnitude smaller than z.

This continual subtraction will proceed if in the first place there is taken from the square IKLM the inscribed square ABCD: for this is greater than half the square IKLM, since the inscribed square — being less than the circle, part to whole — is half of the circumscribed square, by the scholium to proposition 9, book 4 of Euclid. And if the straight line EK is drawn, cutting the circle at O, and through O, perpendicular to EK, the line VX is drawn, [Margin: Euclid III.16] which will touch the circle at O; and likewise the straight lines EL, EM, EI, and so on, are drawn: there will be described an equilateral and equiangular octagon VXY abcde V, as is clear from the construction and demonstration of proposition 12, book 4 of Euclid, since to EA, EO, and the remaining radii of the inscribed octagon the perpendiculars ve, VX, and so on, have been drawn. Now since vA and vO are equal, by the second corollary to proposition 36, book 3 of Euclid, [Margin: 2nd corollary to Euclid III.36] and since KV is greater than vO, KV will also be greater than vA, [Margin: Euclid I.19] and therefore the triangle KvO will be greater than the triangle KvA; [Margin: Euclid VI.1] for triangle is to triangle as base to base. So the triangle KvO will be greater than half the mixed triangle KAO, one side of which is the arc AO; and therefore much greater than half the triangle KAO. In the same way KOX will be greater than half the mixed triangle KOB, one side of which is the arc OB. Therefore, by taking away the triangle KvX from the mixtilinear figure KAB — in which one side is the arc AOB — more than half will have been removed. Thus, when four such triangles KvX, LYa, Mbc, Ide are subtracted, more than half will have been taken from the four figures outside the circle, and so on.

So let there now be posited eight mixed triangles left over, whose bases are the arcs AO, OB, BP, and so on, smaller than the magnitude z. Then, since the circle together with z is posited equal to the triangle FGH, if the residues are taken

away — that is, the octagonal figure $VXY abcde V$ — it will be smaller than the same triangle FGH , which is absurd, since it is greater. For the perpendicular EO equals the side FG , and the perimeter of the octagon is greater than the circumference of the circle — that is, than the straight line GH . Hence there arises a right triangle one side of which, FG , equals the perpendicular O , while the other equals the perimeter of the octagon — greater, that is, than GH — and this is greater than the triangle FGH . For that triangle, by the scholium to proposition 41, book 1 of Euclid, equals the rectangle under FG and half the perimeter of the octagon; and this rectangle, by proposition 2, book 7 On Isoperimeters, equals the octagon: therefore the octagon is greater than the triangle FGH . So the circle cannot be smaller than the triangle. But neither is it greater, as we have demonstrated. Therefore it is equal — which was to be demonstrated.

SCHOLIUM

Joseph Scaliger [Joseph Justus Scaliger] — whether because he did not weigh the force of this demonstration, or because he saw it to be contrary to his own method of squaring the circle — did not scruple to charge Archimedes with falsity in this place; and he attempts to show that it was not rightly demonstrated by him that the circle is equal to a right triangle one of whose sides equals the radius, the other the circumference of the circle. For, he says, if Archimedes' demonstration is sound, it will be demonstrated in the same way that the circle is equal to a right triangle one of whose sides about the right angle equals the radius, while the other is *greater* than the circumference of the circle. For let there be, in a triangle lmn , the side lm equal to the radius EA of the circle, but the side mn greater than the circumference.

Scaliger therefore grants that it was rightly demonstrated by Archimedes that the circle is not greater than the triangle FGH — that is, that the triangle FGH , whose side GH equals the circumference, is not less than the circle — and consequently that the triangle lmn , whose side mn is greater than the circumference, is less than the circle. He grants likewise that it was rightly proved that the circle is not less than the triangle FGH , if the side GH equals the circumference — that is, that the triangle FGH is not greater than the circle. But he denies that it follows from this that the triangle FGH is equal to the circle. Why? Because, he says, in the same way, if the base mn is greater than the circumference (and nothing, he says, prevents this from happening), the polygon will indeed be greater than the triangle lmn , since the perimeter of the polygon is greater than the straight line mn , while the radius EA equals the straight line lm . Yet once the portions are cut

off, it would follow that the same polygon is less than the triangle lmn — which is absurd.

Is it really so, my dear Scaliger? Do you not notice that you are fighting against your own hypothesis? For, the side mn being posited greater than the circumference, when the point is reached at which the polygon is less than the triangle lmn (that is, when the remaining portions are of magnitude z), it necessarily follows that the perimeter of the polygon is less than the side mn . For a right triangle whose altitude equals the radius of the polygon and whose base equals its perimeter is equal, by the scholium to proposition 41, book 1 of Euclid, to the rectangle contained under that same radius and half the perimeter of the polygon; and this rectangle, by proposition 2, book 7 of this work, On Isoperimeters, equals the polygon: so that triangle will also be less than the triangle lmn . Therefore, since these triangles have equal altitudes, [Margin: Euclid VI.1] as that triangle is to lmn , so its base is to the base mn ; and hence that base — that is, the perimeter of the polygon — will be less than mn . You cannot, then, make the base of the triangle lmn , if it is greater than the circumference of the circle, less than the perimeter of the polygon. But in Archimedes' demonstration it is established that the perimeter of the polygon is greater than the base of the triangle FGH , if GH equals the circumference of the circle, since it is greater than the circumference; and therefore Archimedes rightly concluded that the polygon is greater than the triangle FGH , even though, on the adversary's hypothesis, it was shown to be less.

And so Archimedes could equally well have gathered his conclusion this way: the polygon is less than the triangle FGH , on account of the remaining sections being smaller than the magnitude z ; therefore its perimeter is less than the base GH (as we have just demonstrated) — that is, less than the circumference of the circle — which is absurd, since the perimeter of the polygon is greater than the circumference. This absurdity, most learned Scaliger, you cannot derive in your triangle lmn , since you make the base mn greater than the circumference of the circle. And I am truly astonished that you, being a mathematician, should deny that any quantity is equal to that which it is neither greater than nor less than. For if it is not equal, it will be unequal; therefore either greater or lesser — contrary to the hypothesis, since it is said to be neither greater nor lesser. Do you not see that not only Archimedes, but Euclid too in book 12, employs this manner of arguing most frequently?

PROPOSITION II

The circumference of any circle is triple the diameter and exceeds it besides by a part which is indeed less than ten seventieths — that is, the seventh part of the diameter — but greater than ten seventy-firsts.

This is Archimedes' proposition 3, which we make our second, so that the order of instruction may be preserved — for the following proposition 3, which he makes his 2nd, employs this our proposition 2 in its demonstration.

Let ABCD, then, be a circle with center E and diameter AB, cut at right angles by the radius Ec; and let FcG be drawn perpendicular to Ec, [Margin: Euclid III.16] which will touch the circle at c. Draw the side of the hexagon AD, [Margin: Euclid IV.15] which will equal the radius, so that the arc AD is 60 degrees; and hence Dc is 30 degrees. So when the straight line EDe is drawn, the angle eEc will be a third part of a right angle, since a right angle contains 90 degrees. Let the angle cEF also be made equal to the angle cEe: [Margin: Euclid I.32] the angles e and F will be equal to one another, since each is the complement of a third part of a right angle, and hence each contains two-thirds of one right angle. So since all three angles in the triangle eEF together contain $\frac{6}{3}$ of one right angle, the angle eEF too will contain $\frac{2}{3}$ of one right angle; and the triangle will be equiangular — that is, by the corollary to proposition 6, book 1 of Euclid, equilateral. On this account the perpendicular Ec will bisect the base eF, by the scholium to proposition 26, book 1 of Euclid; and therefore Ee will be double ec.

So let ce be set at 153; then Ee will be 306. And if the square of ce, 23409, is subtracted from 93636, the square of Ee, [Margin: Euclid I.47] the remainder will be the square of Ec, 70227, whose root is a little greater than 265. Therefore Ec will have to ce a greater ratio [Margin: Euclid V.8] than 265 to 153.

Now let the angle eEc be bisected by the straight line Ed; [Margin: Euclid VI.3] then e E will be to Ec as ed to dc. And *componendo*, eE and Ec together will be to Ec as ec to dc; and *permutando*, eE and Ec together will be to ec as Ec to cd. Now since eE and Ec together are greater than 571 (seeing that eE is 306, and Ec a little greater than 265), and ec is set at 153, eE and Ec together will have to ec a greater ratio than 571 to 153; [Margin: Euclid V.8] and hence the ratio of Ec to cd will also be greater than 571 to 153. So if cd be set at 153, Ec will be a little greater than 571. [Margin: Euclid V.10] Therefore the square of Ec will be a little greater than 326041. And accordingly, since the square of Ed, which equals the squares of the straight lines Ec and cd [Margin: Euclid I.47] — the square of cd being 23409 — will be a little greater than 349450, its root will be greater than 591 $\frac{1}{8}$, since the

square of this root is only $349428 \frac{49}{64}$. So Ed will have to dc a greater ratio [Margin: Euclid V.8] than $591 \frac{1}{8}$ to 153.

Bisect the angle dEc again by the straight line Eb; [Margin: Euclid VI.3, componendo and permutando] then again, as dE and Ec together are to dc, so is Ec to cb. Now since dE and Ec together are greater than $1162 \frac{1}{8}$ (seeing that Ed is greater than $591 \frac{1}{8}$, and Ec greater than 571), and dc is set at 153, dE and Ec together will have to dc a greater ratio than $1162 \frac{1}{8}$ to 153; [Margin: Euclid V.8] and hence, if cb be set at 153, Ec will be greater than $1162 \frac{1}{8}$. [Margin: Euclid V.10] Therefore the square of Ec will be greater than $1350534 \frac{33}{64}$ [?]. If to this be added 23409, the square of cb, the square of Eb, which equals the squares of the straight lines Ec and cb, [Margin: Euclid I.47] will be greater than $1373943 \frac{33}{64}$ [?]. Its root, therefore — that is, the straight line Eb — is a little greater than $1172 \frac{1}{8}$, since the square of this root is only $1373877 \frac{1}{64}$ [?]. So Eb will have to bc a greater ratio [Margin: Euclid V.8] than $1172 \frac{1}{8}$ to 153.

Bisect the angle bEc likewise by the straight line Ea; [Margin: Euclid VI.3, componendo and permutando] then bE and Ec together will be to bc as Ec to ca. Now since bE is greater than $1172 \frac{1}{8}$ and Ec greater than $1162 \frac{1}{8}$, bE and Ec together will be greater than $2334 \frac{2}{8}$ — that is, $2334 \frac{1}{4}$. So, cb being set at 153, they will have to cb a greater ratio than $2334 \frac{1}{4}$ to 153; [Margin: Euclid V.8] and hence, if ca be set at 153, Ec will be greater than $2334 \frac{1}{4}$. [Margin: Euclid V.10] Therefore the square of Ec will be greater than $5448723 \frac{1}{16}$ [?]. If to this be added 23409, the square of ca, the square of Ea, which equals the squares of the straight lines Ec and ca, [Margin: Euclid I.47] will be greater than $5472132 \frac{1}{16}$ [?], and its root greater than $2339 \frac{1}{4}$, since the square of this root is only $5472090 \frac{9}{16}$ [?]. Therefore Ea will have to ac a greater ratio [Margin: Euclid V.8] than $2339 \frac{1}{4}$ to 153.

Finally, bisect the angle aEc too by the straight line Eo; [Margin: Euclid VI.3, componendo and permutando] then aE and Ec together will be to ac as Ec to co. Now since aE is greater than $2339 \frac{1}{4}$ and Ec greater than $2334 \frac{1}{4}$, aE and Ec together will be greater than $4673 \frac{1}{2}$. So, ca being set at 153, they will have to ca — that is, Ec to co — a greater ratio than $4673 \frac{1}{2}$ to 153; [Margin: Euclid V.8] and therefore, if co be set at 153, Ec will be greater than $4673 \frac{1}{2}$. [Margin: Euclid V.10]

Now since the angle eEc is a third part of a right angle, its half dEc will be a sixth part of a right angle; the half of that, bEc, will be $\frac{1}{12}$ of a right angle; the half of that, aEc, $\frac{1}{24}$ of a right angle; and finally the half of that, oEc, $\frac{1}{48}$ of a right angle. So, of such parts as the quadrant Ac is 48, the arc co will be 1; [Margin:

Euclid VI.33] and therefore co will be $1/192$ of the whole circumference. Let the angle cEi be made equal to the angle cEo ; then the whole angle oEi will be $1/96$ of four right angles, [Margin: Euclid VI.33] and hence the arc oi will be $1/96$ of the whole circumference. So, by the teaching of proposition 12, book 4 of Euclid, the straight line oi will be the side of a polygon circumscribed about the circle, contained by 96 equal sides. And since it has been shown that Ec has to co a greater ratio than $4673 \frac{1}{2}$ to 153, the diameter AB — being double Ec — will have to oi — double co — [Margin: Euclid V.15] a greater...

(Continuing PROPOSITION II — the upper bound, from the circumscribed 96-gon.)

a greater ratio than $4673 \frac{1}{2}$ to 153. If, therefore, the side of the polygon, oi , is set at 153, the diameter AB will be greater than $4673 \frac{1}{2}$ [Margin: Euclid V.10]. Let 153 be multiplied by 96, so that the whole perimeter of the polygon comes out as 14688 [Margin: Euclid V.8]; the perimeter of the polygon will then have to the diameter AB a smaller ratio than 14688 to $4673 \frac{1}{2}$. Now the ratio of 14688 to $4673 \frac{5}{11}$ [Margin: Euclid V.8] — a number a little smaller than $4673 \frac{1}{2}$ — is the triple-and-one-seventh ratio ($3 \frac{1}{7}$) [tripla sesquiseptima]. Therefore the perimeter of the polygon likewise has to the diameter AB a ratio smaller than $3 \frac{1}{7}$ [Margin: scholium to Euclid V.13]. And so the circumference — which, as we shall prove in book 8, proposition 1, is smaller than the perimeter of the polygon — will have to the diameter a ratio much smaller than $3 \frac{1}{7}$ [Margin: Euclid V.8]. Hence the circumference is triple the diameter and still exceeds it by a part that is less than $10/70$, that is, $1/7$ of the diameter. For if it contained the diameter three times and a seventh, the circumference would have to the diameter the ratio $3 \frac{1}{7}$; but if it contained it three times and more than a seventh, it would have a ratio greater than $3 \frac{1}{7}$ — whereas in fact it has a smaller one, as has been demonstrated.

Now let the side of the hexagon, BG , be taken in the same circle, equal to the radius, by the corollary to proposition 15 of book 4 of Euclid; and let the straight lines AG and EG be joined. Since the triangle EBG is equilateral, being composed of three radii, the angle BEG will be $2/3$ of a right angle [Margin: corollary to Euclid I.32], and consequently its half, the angle BAG , will be $1/3$ of a right angle [Margin: Euclid III.20]. And since the diameter AB is double the radius BG , if BG is set at 780, AB will be 1560. Now the square on AB is equal to the squares on the straight lines BG and GA [Margin: Euclid I.47], because the angle AGB , standing in a semicircle, is right [Margin: Euclid III.31]. So if the square on BG , 608400, is subtracted from 2433600, the square on AB , the remainder — 1825200 — will be the square on AG , whose root is a little less than 1351, since the square of 1351,

namely 1825201, is greater than 1825200. Therefore AG will have to GB a smaller ratio than 1351 to 780 [Margin: Euclid V.8]; and consequently, if BG is set at 780, AG will be less than 1351 [Margin: Euclid V.10].

Now let the angle BAG be bisected by the straight line AH, cutting BG at M, and let HB be drawn. The triangles BHM and AHB will be equiangular [Margin: Euclid III.21], because the angle HBM equals the angle HAG (standing on the same base GH), and so, by construction, equals the angle HAB, while the right angle at H is common to both in the semicircle. Therefore AH is to HB as HB is to HM [Margin: Euclid VI.4]. Likewise AB is to BH as BM is to MH; and, by alternation, AB is to BM as BH is to HM. Hence these three ratios are equal: AH to HB, HB to HM, and AB to BM. But AB is to BM as BA and AG together are to BG [Margin: Euclid VI.3]. For as AG is to AB, so is GM to MB; and, by composition, as AG and AB together are to AB, so are GM and MB together — that is, the whole GB — to MB; and, by alternation, as AG and AB together are to GB, so is AH to HB. Now AG has been shown to be less than 1351, AB has been set at 1560, and GB at 780.

Therefore AG and AB together (since they amount to less than 2911) will have to GB a smaller ratio than 2911 to 780 [Margin: Euclid V.8]. And so the ratio of AH to HB will also be smaller than 2911 to 780; and consequently, if HB is set at 780, AH will be less than 2911 [Margin: Euclid V.10]. Its square is therefore less than 8473921; and if to this is added 608400, the square on BH, the square on AB — which is equal to the squares on AH and HB [Margin: Euclid I.47] — will be less than 9082321. So its root, the straight line AB, is less than $3013 \frac{3}{4}$, since the square of this, $9082689 \frac{1}{16}$ [?], is greater than 9082321. Therefore AB will have to BH a smaller ratio than $3013 \frac{3}{4}$ to 780; and consequently, if BH is set at 780, AB will be less than $3013 \frac{3}{4}$ [Margin: Euclid V.10].

Now let the angle HAB be bisected in turn by the straight line AI, cutting HB at N. As before, the triangles BIN and AIB will be equiangular. Therefore, as we demonstrated above, BA and AH together have to HB the same ratio as AI to IB. Now BA has been shown to be less than $3013 \frac{3}{4}$, and AH less than 2911, so their sum is less than $5924 \frac{3}{4}$; and HB has been set at 780. Therefore BA and AH together — that is, AI to IB [Margin: Euclid V.8] — will have a ratio smaller than $5924 \frac{3}{4}$ to 780. So if IB is set at 780, AI will be less than $5924 \frac{3}{4}$ [Margin: Euclid V.10]. And since, as $5924 \frac{3}{4}$ is to 780, so is 1823 to 240 — for the same number arises from the first taken into the fourth as from the second into the third, this ratio being denominated by $7 \frac{143}{240}$ — AI will also have to IB a smaller ratio than 1823 to 240; and so, IB being set at 240, AI will be less than 1823 [Margin: Euclid V.10]. On that account the square on it is less than 3323329; and if to this is

added 57600, the square on IB, the square on AB — which is equal to the squares on AI and IB [Margin: Euclid I.47] — will be less than 3380929, and its root, the straight line AB, less than $1838 \frac{9}{11}$, since the square of this, $3381252 \frac{27}{121}$ [?], is greater than 3380929. Therefore AB will have to BI a smaller ratio than $1838 \frac{9}{11}$ to 240 [Margin: Euclid V.8]; and consequently, BI being set at 240, AB will be less than $1838 \frac{9}{11}$ [Margin: Euclid V.10].

Now let the angle IAB likewise be bisected by the straight line AK. We shall show in the same way that BA and AI together have to IB the same ratio as AK to KB. Now BA and AI together are less than $3661 \frac{9}{11}$ (since BA has been shown to be less than $1838 \frac{9}{11}$, and AI less than 1823), and IB has been set at 240. Therefore BA and AI together — that is, AK to KB [Margin: Euclid V.8] — will have a ratio smaller than $3661 \frac{9}{11}$ to 240. And since, as $3661 \frac{9}{11}$ is to 240, so is 1007 to 66 — for the same number arises from the first taken into the fourth as from the second into the third, this ratio being denominated by $15 \frac{17}{66}$ — AK will likewise have to KB a smaller ratio than 1007 to 66; and so, KB being set at 66, AK will be less than 1007 [Margin: Euclid V.10]. Therefore its square is less than 1014049; and if to this is added 4356, the square on KB, the square on AB — which is equal to those two [Margin: Euclid I.47] — will be less than 1018405, and its root, that is, the straight line AB, less than $1009 \frac{1}{6}$, since ...

Diagram. The Archimedean figure for the inscribed 96-gon. A circle with center E and apex A at the top. A pencil of chords is drawn from A down to the successive points B, L, K, I, H, G lying on the lower arc (the labels B L K I are printed close together at the bottom), the arcs BG being successively bisected. Radii run from E to points on the circle. On the left the circle carries D (upper left) and C (lower left), and a vertical tangent line at the far left ends at F (below, outside the circle) and carries a descending series of small division-labels. M and N mark chord-intersections near the bottom.

- A (apex): vertex from which the bisecting chords are drawn
- E (center): center of the circle
- B (bottom): endpoint of the diameter AB
- G (right): hexagon vertex on the circle (arc BG = $\frac{1}{6}$ of circumference)
- H, I, K, L (lower arc, between B and G): successive angle-bisection points on the circle
- M, N (near bottom): intersection points of chords with BG / HB
- D (upper left, on circle): construction point
- C (lower left, on circle): construction point
- F (far left, on tangent, outside circle): foot of the vertical tangent line

- small labels down the left tangent line (reading downward, approximately: e, d, b, a, o, c, i — at scan resolution, uncertain): division points of the circumscribed-polygon construction [?]

the square of this, $1018417 \frac{13}{36}$, is greater than 1018405. Consequently AB will have to BK a smaller ratio than $1009 \frac{1}{6}$ to 66 [Margin: Euclid V.8]; and on that account, BK being set at 66, AB will be less than $1009 \frac{1}{6}$ [Margin: Euclid V.10].

Finally, let the angle KAB be bisected by the straight line AL. We shall demonstrate by the same reasoning that BA and AK together are to CK [recte: BK] as AL is to LB. Now BA and AK together are both less than $2016 \frac{1}{6}$ (since BA has been shown to be less than $1009 \frac{1}{6}$, and AK less than 1007), and BK has been set at 66. Therefore BA and AK together — that is, AL to LB [Margin: Euclid V.8] — will have a ratio smaller than $2016 \frac{1}{6}$ to 66; and so, if LB is set at 66, AL will be less than $2016 \frac{1}{6}$ [Margin: Euclid V.10]. Its square is therefore less than $4064928 \frac{1}{36}$; and if to this is added 4356, the square on LB, the square on AB — which is equal to those two [Margin: Euclid I.47] — will be less than the number $4069284 \frac{1}{36}$; and so its root, that is, the straight line AB, is less than $2017 \frac{1}{4}$, since the square of this, $4069297 \frac{1}{36}$ [?], exceeds $4069284 \frac{1}{36}$. Wherefore AB will have to BL a smaller ratio than $2017 \frac{1}{4}$ to 66 [Margin: Euclid V.8]; and so, if BL is set at 66, AB will be less than $2017 \frac{1}{4}$ [Margin: Euclid V.10].

Now since the angle GAH equals the angle HAB, the arc GH will be equal to the arc HB [Margin: Euclid III.26]; and by the same reasoning the arc HI will be equal to the arc IB, the arc IK to KB, and KL to LB. So, since GB is $\frac{1}{6}$ of the whole circumference, HB will be $\frac{1}{12}$, IB $\frac{1}{24}$, KB $\frac{1}{48}$, and LB $\frac{1}{96}$; and consequently the straight line BL will be the side of a polygon inscribed in the circle and contained by 96 equal sides. And since BL has been set at 66, if 66 is multiplied by 96 the perimeter of the polygon comes out as 6336. Therefore the perimeter of the polygon will have to the diameter AB a greater ratio than 6336 to $2017 \frac{1}{4}$ [Margin: Euclid V.8], since the diameter AB has been shown to be less than $2017 \frac{1}{4}$. Now the ratio of 6336 to $2017 \frac{1}{4}$ is greater than the triple-and-ten-seventy-firsts ratio ($3 \frac{10}{71}$) [tripla superdecupartiens septuagesimas primas] [Margin: Euclid V.8], because the ratio of 6336 to $2017 \frac{65}{223}$ — which number is greater than $2107 \frac{1}{4}$ [recte: $2017 \frac{1}{4}$] — is $3 \frac{10}{71}$. Therefore the perimeter of the polygon has to the diameter a greater ratio than $3 \frac{10}{71}$.

And since the circumference of the circle is greater than the perimeter of the [inscribed] polygon [Margin: Euclid V.8], the circumference will have to the diameter a much greater ratio than $3 \frac{10}{71}$. And so the circumference will contain the diameter three times, and over and above it by a part greater than

10/71. For if it contained just 10/71 in addition, the circumference would have to the diameter the ratio $3 \frac{10}{71}$; but if it contained a smaller part than 10/71, it would have a ratio smaller than $3 \frac{10}{71}$ — whereas it has been demonstrated to have a greater one. And since 1/8 is smaller than 10/71, it is clear from this second part of the proposition that the circumference contains the diameter three times, and more than 1/8 of the diameter. Therefore, since from the first part it is established that the circumference contains the diameter three times and less than 1/7 of the diameter, it is evident that the true ratio of the circumference to the diameter lies between $3 \frac{1}{7}$ [triple-sesquiseptimate] and $3 \frac{1}{8}$ [triple-sesquioctave].

COROLLARY

Therefore, if the diameter is multiplied by $3 \frac{1}{7}$, a number greater than the circumference will be generated; but if it is multiplied by $3 \frac{10}{71}$, a number smaller than the circumference will result. Conversely, if the circumference is divided by $3 \frac{1}{7}$, a number smaller than the diameter will be produced; but if it is divided by $3 \frac{10}{71}$, a number greater than the diameter will come out.

PROPOSITION III

Every circle has to the square on its diameter a ratio of about 11 to 14.

Let ABCD be a circle, and EFGH the square on the diameter BD, circumscribed about the circle. I say that the circle is to the square as 11 to 14, approximately. For, the side EG being produced, let EK be taken triple EG, and KF a seventh part of the diameter BD (or of the side EG); and let the straight lines BG and BF be joined. Since, then, EF has to the diameter EG the triple-and-one-seventh ratio ($3 \frac{1}{7}$), by construction, EF will — by the preceding proposition — be nearly equal to the circumference of the circle. So, since BE is equal to the radius, the triangle BEF will, by proposition 1, be nearly equal to the circle; while the triangle BEG will be a quarter of the square EH [EFGH]. And since, the side EG being set at 7, the straight line EF is 22, the triangle BEF — that is, the circle ABCD — will be to the triangle BEG as 22 to 7 [Margin: Euclid VI.1]. But, the triangle BEG being set at 7, the square EFGH, which is its quadruple, is 28. Therefore the circle is to the square nearly as 22 to 28 — that is, as 11 to 14 — which was to be demonstrated.

Diagram. The same Archimedean inscribed-96-gon figure as on p. 127, repeated at the head of this page so the reader has it while the proof continues. Circle with center E and apex A; chords from A to B, L, K, I, H, G on the lower arc (labels B L K I printed together at the foot); points D, C on the left of the circle; F at the foot

of the left-hand vertical tangent line, which again carries a descending series of small division-labels; **M**, **N** near the bottom.

- A (apex); E (center); B (bottom, diameter endpoint); G (right, hexagon vertex); H, I, K, L (lower arc, bisection points); M, N (chord intersections); D, C (left of circle); F (foot of left tangent)
- small labels down the left tangent line (approximately e, d, b, a, o, c, i — at scan resolution, uncertain) [?]

Diagram. The figure for Proposition III : a square **EGHI** with an inscribed circle **ABCD**, and the base extended to the right through **L**, **K** to **F**.

- I (top-left corner of the square)
- H (top-right corner of the square)
- E (bottom-left corner of the square)
- G (bottom-right corner of the square)
- A (top of the circle, on the upper side of the square)
- B (left of the circle, on the left side of the square)
- D (right of the circle, on the right side of the square)
- C (bottom of the circle, on the base)
- N (on the left side, between B and E)
- M (on the base, between C and G)
- L (on the extended base, to the right of G)
- K (on the extended base, farther right; $EK = \text{triple } EG$)
- F (upper right, joined to B; $EF = 3 \frac{1}{7} \times EG, \approx \text{circumference}$)

ON THE AREA OF THE CIRCLE, AND ON FINDING THE CIRCUMFERENCE FROM THE DIAMETER AND THE DIAMETER FROM THE CIRCUMFERENCE

CHAPTER VII

Since a right triangle — one of whose sides about the right angle is equal to the radius of the circle, and the other to the circumference of the same circle — is equal to the area of the circle [Margin: Archimedes, *Measurement of the Circle*, prop. 1], and since the area of that triangle is produced by multiplying the

perpendicular by half the base, as we wrote in chapter 2, no. 2, of this book, it follows that the area of the circle is produced by multiplying the radius by half the circumference (that is, if the base of that triangle is taken to be the side equal to the circumference); or by multiplying the whole circumference by half the radius, that is, by a quarter of the diameter (taking, in the same triangle, for the base the side equal to the radius); or, finally, by multiplying the whole diameter by a quarter of the circumference. This we shall make plain as follows. [Margin: The area of the circle in three ways, from the known diameter and circumference.]

Let the figure of the preceding proposition be repeated, and let EF — which is equal to the circumference of the circle — be bisected at L, so that EL equals the semicircumference; likewise let EL be bisected at M, so that EM equals a quarter of the circumference; and finally let BE also be bisected at N, so that EN is half the radius BE, that is, a quarter of the whole diameter. And since the triangle BEF is equal to the circle ABCD [Margin: Archimedes, *Measurement of the Circle*, prop. 1], the rectangle contained by the radius BE and the semicircumference EL — which, by proposition 1 of book 7 of this work, is equal to the triangle — will likewise be equal to the same circle. This is the first way.

In just the same way, the rectangle contained by the whole circumference EF and EN, a quarter of the diameter (which, by proposition 1 of book 7 of this work, is equal to the same triangle), will be equal to the same circle. This is the second.

Finally, since the four lines EI, EB, EL, EM are proportional — inasmuch as both the first two and the last two stand in the double ratio — the rectangle contained by EI, the diameter, and EM, a quarter of the circumference [Margin: Euclid VI.16], will be equal to the rectangle contained by EB, the radius, and EL, the semicircumference. And since this was shown in the first part to be equal to the circle, that too will be equal to the same circle. This is the third.

So that fractions may sometimes be avoided, the whole diameter should be multiplied by the whole circumference; for a quarter of the resulting product will be the area of the circle, because the product is quadruple the product of the radius times the semicircumference, as is clear.

It follows, from the first part, that the area of a semicircle is produced from the radius times a quarter of the circumference — since indeed half is produced of what comes from the radius times half the circumference. Likewise, the area of a quadrant is generated from the radius times an eighth of the circumference; the area of an eighth part from the radius times a sixteenth of the circumference; the

area of a sixteenth part from the radius times $1/32$ of the circumference; and so on. For half of the preceding product is always produced, just as the parts of the circle are halves of the preceding parts: a quadrant is half of the semicircle, an eighth part half of the quadrant, a sixteenth part half of the eighth part, and so forth. [Margin: The area of the semicircle, of the quadrant, of the eighth part, etc.]

2. Therefore, in order that the area of the circle may be found, both its diameter and its circumference must be known. We shall therefore hand down here a few rules by which, from a given diameter, the circumference — now greater, now smaller than the true one — may be found, by proposition 2 of the *Measurement of the Circle*. Then I shall prescribe other rules by which the area of the circle — now greater, now smaller than the true one — may be drawn out either from the diameter alone or from the circumference [?] alone.

I.

To find, from a given diameter of a circle, a circumference greater than the true one.

Let it be as 7 to 22, so the given diameter — for example, 28 — to another number; and there will be produced a circumference 88, greater than the true one. For since the ratio of the circumference to the diameter is smaller than $3 \frac{1}{7}$ [Margin: Archimedes, *Measurement of the Circle*, prop. 2], while the ratio of 88 to 28 is $3 \frac{1}{7}$ — namely the same as 22 to 7 — the number 88 will be greater than the circumference of the circle whose diameter is 28 [Margin: Euclid V.10].

II.

To draw out, from a given diameter of a circle, a circumference smaller than the true one.

Let it be as 71 to 223, so the given diameter 28 to another number; and there will be produced a circumference $87 \frac{65}{71}$ [?], smaller than the true one. For since the ratio of the circumference to the diameter is greater than $3 \frac{10}{71}$ [Margin: Archimedes, *Measurement of the Circle*, prop. 2], while the ratio of $87 \frac{65}{71}$ [?] to 28 is $3 \frac{10}{71}$ — namely the same as 223 to 28 — the circumference of the circle whose diameter is 28 will be greater than $87 \frac{65}{71}$ [?] [Margin: Euclid V.10].

Diagram. The figure of the preceding proposition, repeated for Chapter VII : the square **EGHI** with inscribed circle **ABCD**, the base extended to the right through **L, K** to **F**. Identical in lettering to fig-p128-2.

- I (top-left corner); H (top-right corner); E (bottom-left corner); G (bottom-right corner)
- A (top of circle); B (left of circle); D (right of circle); C (bottom of circle, on the base)
- N (left side, between B and E; BE bisected here)
- M (on the base; EL bisected here, EM = quarter circumference)
- L (on the extended base; EF bisected here, EL = semicircumference)
- K (on the extended base; EK = triple EG)
- F (upper right, joined to B; EF $\approx$ circumference)

III. — From a given circumference of a circle, to find a diameter greater than the true one

Let 223 be to 71 as the given circumference — say, 88 — is to another number. The number so produced, $28 \frac{4}{223}$, will give a diameter greater than the true one.

[Margin: a — Archimedes, *De Dimensione Circuli* [*On the Measurement of the Circle*], prop. 2; b — Euclid V.10.]

For (a) since the circumference has to the diameter a ratio greater than triple-and-ten-seventy-firsts [tripla superdecupartiens septuagesimas primas, i.e. $3 \frac{10}{71}$] — that is, greater than 223 to 71 — the given circumference 88 will likewise have to its own diameter a ratio greater than it has to $28 \frac{4}{223}$. (b) And therefore the diameter of a circumference of 88 will be less than $28 \frac{4}{223}$.

IV. — From a given circumference of a circle, to find a diameter smaller than the true one

Let 22 be to 7 as the given circumference 88 is to another number. The number generated, 28, will yield a diameter smaller than the true one.

[Margin: a — Archimedes, *De Dimensione Circuli*, prop. 2; b — Euclid V.10.]

For (a) since the circumference has to the diameter a ratio smaller than triple-and-a-seventh [tripla sesquiseptima, i.e. $3 \frac{1}{7}$] — that is, than 22 to 7 — the given circumference 88 will likewise have to its own diameter a ratio smaller than it has to 28. (b) And on that account the diameter of a circumference of 88 will be greater than 28.

3. Now, to show by what method the area of a circle may be found either from the diameter alone or from the circumference alone — so that it is not necessary to derive the circumference from the diameter, or the diameter from the

circumference — the following three propositions must first be demonstrated. The first of them we shall afterwards demonstrate again, in a different way, in Book 8, Proposition 2, drawing on Pappus of Alexandria [Pappus].

PROPOSITIO I

The diameters of circles are to one another as their circumferences.

[Margin: a — corollary to Euclid VI.20; b — Euclid XII.2.]

Let AB and DE be the diameters of two circles, and let BC and EF be straight lines equal to their circumferences, meeting the diameters at right angles at B and E, so that the triangles ABC and DEF are completed. I say that the diameter AB is to the diameter DE as the circumference BC is to the circumference EF.

For let G be found as a third proportional to the diameters AB and DE; and let H be a fourth proportional to the three straight lines BC, EF, DE. Since AB, DE, G are continued proportionals, (a) the square on AB is to the square on DE as AB is to G. (b) But as the square on AB is to the square on DE, so is the circle of diameter AB to the circle of diameter DE.

Diagram. Two right triangles set side by side, with two short labelled line-segments (G and H) placed between them at the top. In the left triangle the right angle is at the lower left; in the right triangle at the lower left as well.

- A (upper-left vertex, left triangle): apex of the diameter-leg AB
- B (lower-left vertex, left triangle): right angle, foot of diameter AB and of base BC
- C (lower-right vertex, left triangle): far end of the base (circumference) BC
- I (on the left leg, between A and B): midpoint of the diameter AB; a dotted line runs from I to C (the line IC)
- D (upper vertex, right triangle): apex of the diameter-leg DE
- E (lower-left vertex, right triangle): right angle, foot of diameter DE and of base EF
- F (lower-right vertex, right triangle): far end of the base (circumference) EF
- G (short segment, top center): the third proportional to AB and DE
- H (short segment, top center, to the right of G): the fourth proportional to BC, EF, DE

[Margin: c — Euclid V.15; d — Archimedes, *De Dimensione Circuli*, prop. 1; e — Euclid I.38; f — scholium to Euclid VI.23; g — Euclid V.9; h — Euclid V.7.]

(c) And as circle is to circle, so is triangle ABC to triangle DEF, because these triangles are double their respective circles. (For when the diameter AB is bisected at I and the straight line IC is drawn, (d) triangle IBC is equal to the circle; and since (e) triangles AIC and IBC are equal, triangle ABC is double of triangle IBC — that is, of the circle whose diameter is AB. By the same reasoning triangle DEF is double of the circle whose diameter is DE.) Therefore triangle ABC is also to triangle DEF as AB is to G.

But as triangle ABC is to triangle DEF, so is AB to H, because each ratio is compounded of the same ratios. (f For the ratio of triangle ABC to triangle DEF is compounded of the ratio of the base AB to the base DE and of the ratio of the altitude BC to the altitude EF — that is, of the ratio DE to H, which by construction is the same as BC to EF. And the ratio AB to H is likewise compounded, by definition, of the ratios AB to DE and DE to H.) Therefore AB is to G as AB is to H; (g) and consequently G and H are equal. (h) Hence DE is to G as DE is to H. Now by construction AB is to the diameter DE as DE is to G — that is, as DE is to H. And as DE is to H, so by construction is the circumference BC to the circumference EF. Therefore the diameter AB is also to the diameter DE as the circumference BC is to the circumference EF. Which was to be demonstrated.

PROPOSITIO II

The ratio of the square described on the diameter of any circle to the area of the circle is greater than 14 to 11, but less than 284 to 223.

[Margin: a — Euclid XII.2.]

(a) Since the square on the diameter of any one circle is to the square on the diameter of another circle as circle is to circle, then, by alternation, the square on the diameter is to the circle of that same diameter as the square on the other diameter is to the circle of that other diameter. Now, taking the diameter of some circle as 1, the ratio of its square to the circle is greater than 14 to 11 and less than 284 to 223. Therefore the ratio of the square on the diameter of any other circle to that circle will likewise be greater than 14 to 11 and less than 284 to 223.

That the ratio of the square on the diameter 1 to its own circle is greater than 14 to 11 and less than 284 to 223 will become clear as follows. If 7 be made to 22 as the diameter 1 is to another number, then by the first rule of Num. 2 there results a circumference of $3 \frac{1}{7}$, or $\frac{22}{7}$, greater than the true one. Therefore, taking half of it, $\frac{11}{7}$, into $\frac{1}{2}$ the semidiameter — as was stated in Num. 1 — there is produced an area of the circle of $\frac{11}{14}$, which is nevertheless greater than the

true area. (b) And accordingly the square on the diameter 1, which is 1, will have to the true area of the circle — which is less than $11/14$ — a greater ratio than it has to $11/14$. Since, then, 1 is to $11/14$ as 14 is to 11 (for, by Proposition 2 of the *Minutiae* [On Fractions], at the end of Euclid Book 9, the ratio of the numerator 11 to the denominator 14 is the same as the ratio of the fraction $11/14$ to its whole 1; and by inversion this is 14 to 11 as 1 is to $11/14$), the square on the diameter 1 will also have to the true area of the circle a ratio greater than 14 to 11. Which is what was proposed. So the first part of the proposition stands.

[Margin: b — Euclid V.8; c — Euclid V.8.]

Again, if 71 be made to 223 as the diameter 1 is to another number, then by the second rule of Num. 2 there will be found a circumference of the circle of $3\frac{10}{71}$, or $223/71$, smaller than the true one. Therefore — as was stated in Num. 1 — taking half of it, $223/142$, into $1/2$ the semidiameter, there is produced an area of the circle of $223/284$, which is nevertheless smaller than the true area. (c) And accordingly the square on the diameter 1, which is 1, will have to the true area of the circle — which is greater than $223/284$ — a smaller ratio than it has to $223/284$, namely as 284 to 223. (For, since by Proposition 2 of the *Minutiae* the ratio of the numerator 223 to the denominator 284 is the same as that of the fraction $223/284$ to its whole 1, then by inversion this is 284 to 223 as 1 is to $223/284$.) The square on the diameter 1 will therefore also have to the true area of the circle a ratio smaller than 284 to 223. Which is what was proposed. So the second part of the proposition also stands.

PROPOSITIO III

The ratio of the square described on the circumference of any circle to the area of the circle is greater than 892 to 71, but less than 88 to 7.

[Margin: a — Num. 3 of this [book]; b — Euclid VI.22; c — Euclid XII.2.]

(a) Since the circumference of any one circle is to the circumference of another circle as diameter is to diameter, (b) then the square on the circumference is also to the square on the circumference as the square on the diameter is to the square on the diameter. (c) But as the square on the diameter is to the square on the diameter, so is circle to circle. Therefore the square on the circumference is also to the square on the circumference as circle is to circle; and by alternation, the square on the circumference is to its own circle as the square on the other circumference is to its own circle. Now, taking the circumference of some circle as 1, the ratio of its square to the circle is greater than 892 to 71 and less than 88

to 7. Therefore the ratio of the square on the circumference of any other circle to that circle will likewise be greater than 892 to 71 and less than 88 to 7.

That the ratio of the square described on the circumference 1 to its own circle is greater than 892 to 71 and less than 88 to 7 we shall demonstrate as follows. Since, if 223 be made to 71 as the given circumference 1 is to another number, the diameter produced is $71/223$, greater than the true one — as is clear from the third rule of Num. 2 — it comes about that half of the circumference, $1/2$, taken into $71/446$, half of the diameter so found, produces an area of the circle of $71/892$, greater than the true one, as was stated in Num. 1. (d) Therefore the square on the circumference 1, which is 1, will have to the true area of the circle — which is less than $71/892$ — a greater ratio than it has to $71/892$. Now, 1 is to $71/892$ as, by Proposition 2 of the *Minutiae* and by inversion, 892 is to 71. Therefore the square on the circumference 1 will also have to the true area of the circle a greater ratio than 892 to 71. Which is what was proposed. So the first part of the proposition is true.

[Margin: d — Euclid V.8; e — Euclid V.8.]

Again, since, if 22 be made to 7 as the given circumference 1 is to another number, the diameter is produced as $7/22$, smaller than the true one, it comes about that half of the circumference, $1/2$, taken into $7/44$, half of the diameter so found, produces an area of the circle of $7/88$, smaller than the true one — as is clear from what we said in Num. 1. (e) Therefore the square on the circumference 1, which is 1, will have to the true area of the circle — which is greater than $7/88$ — a smaller ratio than it has to $7/88$. Now, 1 is to $7/88$ as, by Proposition 2 of the *Minutiae* and by inversion, 88 is to 7. Therefore the square on the circumference 1 will also have to the true area of the circle a smaller ratio than 88 to 7. Which is what was proposed. So the latter part of the proposition is also true.

4. These things being demonstrated, there now follow four rules by which one may estimate the area of a given circle — whether greater or smaller than the true area — either from the diameter alone or from the circumference alone.

I. — From the diameter, to find an area of the circle greater than the true one

[Margin: a — Num. 3, [rule] 2; b — Euclid V.10.]

Let 14 be to 11 as the square on the given diameter is to another number. The number produced will give an area of the circle greater than the true one. (a) For since the ratio of the square on the diameter to the area of the circle is greater

than 14 to 11, and yet the square on the given diameter is set to the found area as 14 to 11, the ratio of the square on the given diameter to the true area of the circle will be greater than its ratio to the found area. (b) And therefore the true area of the circle will be smaller than the found one — that is, the found area will be greater than the true.

II. — From the diameter, to find an area of the circle smaller than the true one

[Margin: a — Num. 3, [rule] 2; b — Euclid V.10.]

Let 284 be to 223 as the square on the given diameter is to another number. The number generated will yield an area of the circle smaller than the true one. (a) For since the ratio of the square on the given diameter to the area of the circle is smaller than 284 to 223, and yet the square on the given diameter is set to the found area as 284 to 223, the ratio of the square on the given diameter to the true area of the circle will be smaller than its ratio to the found area. (b) And on that account the true area of the circle will be greater than the found one — that is, the found area will be smaller than the true.

III. — From the circumference, to gather an area of the circle greater than the true one

[Margin: a — Num. 3, [rule] 3; b — Euclid V.10.]

Let 892 be to 71 as the square on the given circumference is to another number. The number produced will indicate an area of the circle greater than the true one. (a) For since the ratio of the square on the given circumference to the area of the circle is greater than 892 to 71, and yet the square on the given circumference is set to the found area as 882 [recte: 892] to 71, the ratio of the square on the given circumference to the true area of the circle will be greater than its ratio to the found area. (b) And therefore the true area of the circle will be smaller than the found one — that is, the found area will be greater than the true.

IV. — From the circumference, to conclude an area of the circle smaller than the true one

[Margin: a — Num. 3, [rule] 3; b — Euclid V.10.]

Let 88 be to 7 as the square on the given circumference is to another number. The number generated will contain an area of the circle smaller than the true one. (a) For since the ratio of the square on the given circumference to the area of the

circle is smaller than 88 to 7, and yet the square on the given circumference is set to the found area as 88 to 7, the ratio of the square on the given circumference to the true area of the circle will be smaller than its ratio to the found area. (b) And accordingly the true area of the circle will be greater than the found one — that is, the found area will be smaller than the true.

5. All these methods by which the area of a circle is sought depend on the ratio of the circumference of the circle to the diameter, which Archimedes found to be indeed less than triple-and-a-seventh [$3 \frac{1}{7}$] but greater than triple-and-ten-seventy-firsts [$3 \frac{10}{71}$]. And since these ratios are not exact, the found area must necessarily either exceed the true magnitude of the circle or fall short of it, as is clear from the rules above. And although the difference between the true area and the found one is very slight in small circles, in large circles it does not seem negligible. For this reason, whoever desires a more accurate area of the circle should adopt a ratio of the diameter to the circumference closer to the truth, such as the later geometers — especially Ludolph van Ceulen [Ludolph van Ceulen] and Christoph Grienberger [Christophorus Gruenbergerus] — have found, as follows.

[Margin: A more accurate ratio of the diameter to the circumference.]

Diameter	Circumference greater than the true
10000000000000000000.	314159265358979323847.
Diameter	Circumference smaller than the true
10000000000000000000.	314159265358979323846.

That is, so that the ratio of circumference to diameter greater than the true is denominated by this first number, which is less than triple-and-a-seventh; while the ratio smaller than the true is denominated by this second number, which is greater than triple-and-ten-seventy-firsts. These denominators are obtained if each circumference is divided by the diameter:

- 3 and 14159265358979323847 / 10000000000000000000
- 3 and 14159265358979323846 / 10000000000000000000

[Margin: A more accurate finding of the circumference from a given diameter, and of the diameter from a given circumference.]

And so, if the aforesaid diameter is made to the circumference greater than the true as the given diameter of some circle is to another number, there will be

found a circumference greater than the true, yet differing less from the true than that of Archimedes, which was found in Num. 2 by rule 1. But if the same aforesaid diameter is made to the circumference smaller than the true as the given diameter of some circle is to another number, there will be found a circumference smaller than the true, yet coming closer to the true than that of Archimedes, which is found in Num. 2 by rule 2. Conversely, if the aforesaid circumference is made to the diameter as the given circumference of some circle is to another number, there will result a diameter greater than the true; but if the aforesaid circumference greater than the true is made to the diameter as the given circumference of some circle is to another number, there will come out a diameter smaller than the true. In either case, however, the diameter so found will come closer to the true than that of Archimedes, which was found in Num. 2 by rules 3 and 4.

[Margin: A more accurate finding of the area of the circle.]

Once the circumference has been found from the diameter, or the diameter from the circumference, the area of the circle will be found as was taught in Num. 1: namely, if the semidiameter is multiplied into the semicircumference, or the whole circumference into half the semidiameter, or finally the whole diameter into a quarter of the circumference. This area will indeed differ less from the true one than that which is found by the ratio of Archimedes. But since it is more difficult to carry out a calculation with large numbers than with smaller ones, the practice of craftsmen has established that the ratio of Archimedes be used for calculation. Whenever, however, a more accurate calculation is desired, this later ratio of Ludolph must be employed, especially in larger circles.

DE AREA SEGMENTORVM CIRCVLI

Chapter VIII — On the area of segments of the circle

Let a sector of the circle ABCD first be proposed, bounded by the two semidiameters AB, AD and by the arc BCD. We shall investigate its area as follows. Suppose first that the semidiameter AB is known — say 7 palms — while the arc BCD is likewise known, namely $3\frac{2}{3}$ palms. Then multiply the semidiameter 7 by half the arc — that is, by $1\frac{2}{3}$ [the intermediate multiplier is printed as a faint superscript fraction; the text itself specifies "half the arc"]. The number produced, $12\frac{2}{3}$ palms, will be the area of the sector ABCD, as we shall prove.

But if neither the semidiameter AB nor the periphery BCD is given as data to be measured, and no semidiameter is known in any standard unit, then the circumference of the circle must be found in that same unit by the rules of the preceding chapter, together with the chord BD. Next, set it up so that AB, known in the assumed unit, is to the whole sine 100000 as BD, known in the same assumed unit, is to another number. The number so produced will give the chord BD, known in parts of the whole sine. Half of this will be the sine of half the arc BD; and therefore, from the table of sines, the arc BC in degrees will be known — half of BD — so that the whole arc BD will not be unknown. And since the entire circumference of the circle has been made known in the assumed unit, set it up so that as 360 degrees are to the whole circumference expressed in the assumed unit, so the arc BD in degrees is to another number; and this will give the same arc BD in the assumed unit. Hence, as before, the area of the sector ABCD will be found. The degrees contained in the arc BD could also be investigated by means of some quadrant divided into degrees, applying the method set out in Book I, Chapter 2, Num. 10, so that the minutes as well may be known when some fraction beyond whole degrees remains in the arc BD.

Diagram. Two circles side by side. The left circle carries the sector-and-segment construction of the chapter; a triangle is inscribed with its apex at the center A, and several chords and dotted lines are drawn to A. The right circle shows a vertical dashed diameter GI illustrating the lenticular (lens) figure.

- Left circle, upper rim: N, L, F (near the top)
- Left circle, left rim: P
- Left circle, center: A (the center of the circle)
- Left circle, right (toward the gap between circles): K
- Left circle, lower-left rim: O, M
- Left circle, lower rim: B, C, D
- Left circle, lower-right: E
- Right circle, top: G
- Right circle, left: K
- Right circle, right: H
- Right circle, bottom: I
- Right circle: dashed vertical diameter from G to I

Now I prove that the area of the sector is produced from the semidiameter multiplied into half the arc of the sector. Let there be the quadrant BE and the semicircle BEF. [Margin: a — VI.33] Since, then, as the arc BD is to the quadrant

BE, so is the sector ABCD to the sector ABDE, it will also follow, from the scholium to Euclid V.22, that as the arc BD is to four times the quadrant BE — that is, to the whole circumference — so is the sector ABCD to four times the sector ABDE — that is, to the whole circle. [Margin: b — V.15] Now as the arc BD is to the whole circumference, so is BC, half the arc BD, to BEF, half the whole circumference. Therefore it will also follow that as BC is to BEF, so is the sector ABCD to the whole circle. [Margin: c — VI.1] But as BC is to BEF, so is the rectangle under AB, BC to the rectangle under AB, BEF. Therefore the sector ABCD will also be to the whole circle as the rectangle under AB, BC is to the rectangle under AB, BEF. And since, as we established in Chapter 7, Num. 1, the circle is equal to the rectangle under AB, BEF, [Margin: d — V.14] the sector ABCD will likewise be equal to the rectangle under AB, BC. Which was to be proved.

By the same reasoning the sector ABFDA will be produced from the semidiameter AB multiplied into half the arc BFD.

2. Next let there be the segment BCD. [Margin: e — III.25] Having found the center A of the arc BCD, and knowing, in some unit of measure, the sides of the triangle ABD and the arc BCD in that same unit, let both the area of the sector ABCD and the area of the triangle ABD be investigated, as we said in Num. 1. For the triangle, subtracted from the sector, will leave the area of the proposed segment BCD.
3. Suppose further a lens-shaped figure bounded by two arcs GHI, GKI. Draw the straight line GI, and, as we taught in Num. 2, investigate the area of each of the segments GHI, GKI. For the sum compounded of these two areas will be the area of the proposed figure GHIK. And if the segments GHI, GKI are equal, it will be enough to investigate the area of one of them; for that, doubled, will give the area of the figure GHIK set before us.
4. We shall measure in no other way the figures compounded of various segments of circles — whether all their circumferences bulge outward, or inward, or partly inward and partly outward. Thus, in these three figures, if chords are drawn under the arcs, we shall measure, in the first, the quadrilateral ABCD, as we taught in Chapter 1 or 3, and the segments AEB, BFC, CGD, DHA, as was set out in Num. 2 of this chapter. For if these segments are added to the quadrilateral — since they all tend outward — the area of the figure AEBFCGDH will be compounded, made up of four arcs.

Diagram. A quatrefoil figure AEBFCGDH : an inscribed quadrilateral ABCD with four outward-bulging circular segments AEB, BFC, CGD, DHA on its four sides.

- E: top
- A: upper left; B: upper right
- H: left; F: right
- D: lower left; C: lower right
- G: bottom

In the second figure we shall measure the pentagon ABCDE by what was written in Chapter 4. If from it we then take away the five inward-tending segments — which will be known from what we wrote in Num. 2 of this chapter — the remainder will be the area of the figure AFBGCHDIEK, made up of five arcs.

Diagram. A pentagonal figure AFBGCHDIEK : a pentagon ABCDE with five inward-curving circular segments cut from its sides.

- A: top
- B: upper left; E: upper right
- F, G, K: upper interior; H, I: lower interior
- C: lower left; D: lower right

Finally, in the third figure, to the pentagon ABCDE we shall add the three outward-tending segments AFB, AGE, CHD, and, by a combined reckoning, shall remove the two inward-tending segments BIC, DKE, so that there remains the area of the figure AFBICHDKKEG, made up of five arcs. And in this way one may measure a field however irregular.

Diagram. A mixed pentagonal figure AFBICHDKKEG : a pentagon with three outward-bulging segments (AFB, AGE, CHD) and two inward-curving segments (BIC, DKE).

- A: top; G: upper right
- F: upper left; E: right
- B: left; K: center
- I: lower center
- C: lower left; D: lower right

5. Lastly, in the first figure of this chapter, let there be the segment of a circle LMON, bounded by the two straight lines LM, NO and the two arcs LN, MO. As was explained in Num. 2, investigate the area of each segment PLM,

PNO. The lesser area PNO, subtracted from the greater PLM, will yield the area of the proposed segment LMON.

6. To bring this fourth book to a close, I gladly append here, by way of appendix, certain further rules not foreign to our purpose.

GEOMETR. PRACTICÆ

I

Given the area of a circle, to find its circumference and diameter.

Set it up so that as 7 is to 88, so the given area is to another number. The number produced will be a square somewhat greater than the true square of the circumference, as is clear from Rule 4, Num. 4, Chapter 7. The square root of the number produced will therefore give a circumference somewhat greater than the true one. But if instead you set it up so that as 71 is to 892, so the given area is to another number, there will be generated a square somewhat less than the true square of the circumference, as is settled by Rule 3, Num. 4, Chapter 7; and accordingly its square root will indicate a circumference somewhat less than the true one.

Again, set it up so that as 223 is to 284, so the proposed area is to another number. The number so produced will be a square somewhat greater than the true square of the diameter, as is evident from Rule 2, Num. 4, Chapter 7. The square root of the number produced will therefore exhibit a diameter somewhat greater than the true one. But if you set it up so that as 11 is to 14, so the given area is to another number, there will be found a square somewhat less than the true square of the diameter, as is gathered from Rule 1, Num. 4, Chapter 7; and accordingly its square root will offer a diameter somewhat less than the true one.

II

Given an arc of any circle, to find the diameter of the circle in numbers.

Let the arc ABC be given. Draw the chord AC, bisect it at F, and through F draw the perpendicular FB, [Margin: f — corollary to III.1] which will pass through the center of the circle. [Margin: g — III.35] Hence the rectangle under CF, AF — that is, the square on AF — will be equal to the rectangle under BF and the remaining part of the diameter. If, then, AF and FB are made known in some unit of measure, and the square number of the line AF [the printed word reads

"quadrans"[?]; the geometry requires the square, since $AF^2 = BF \cdot FD$] is divided by FB, there will come out the remaining part FD of the diameter; and this, added to the perpendicular FB, will complete the whole diameter BD, known in the same unit in which AF, FB are known.

Geometrically, the same part FD will be found if a third proportional FD is found to the two lines FB, AF — because, by the scholium to Euclid VI.13, AF is the mean proportional between the segments of the diameter.

Diagram. A circle with a vertical diameter BD. The chord CA runs across the upper part; F is its midpoint on the diameter, and E marks the center of the circle.

- B: top; D: bottom (endpoints of the diameter)
- C: upper left; A: upper right (endpoints of the chord)
- F: on the diameter, at the midpoint of the chord
- E: center of the circle

III

Given the diameters of two circles, or their circumferences; or two homologous sides of two similar and similarly-placed figures: to find what ratio the circles, or the figures, have to one another.

Since circles, and figures that are similar and similarly placed, stand in the doubled ratio of their diameters or circumferences and of their homologous sides: if the greater diameter or circumference is divided by the lesser, and the greater homologous side by the lesser, there will come out the denominator of the ratio which the greater diameter or circumference has to the lesser, or which the greater homologous side has to the lesser. If, then, this denominator is multiplied into itself, there will be produced the denominator of the doubled ratio — namely, the ratio which the circle or figure has to the lesser one. For example, let the diameter of one circle be 56 and its circumference 176, and let the diameter of the other circle be 14 and its circumference 44. Dividing 56 by 14, or 176 by 44, the quotient is 4; and this, multiplied into itself, produces 16, the denominator of the ratio of the greater circle to the lesser. The figure similar and similarly placed will have this same ratio to the lesser figure, if the homologous sides are 56 and 14, or 176 and 44.

IV

Given several circles whose diameters or circumferences are known, and likewise several similar and similarly-placed figures whose homologous

sides are known: to find a diameter or circumference whose circle shall be equal to all the given circles; and likewise to find a side whose figure, similar and similarly placed, shall be equal to all the given figures.

Multiply the diameters, or circumferences, or homologous sides each into itself, and collect the products into one sum. The square root of this sum will be the diameter, circumference, or homologous side sought. For example, if there are four diameters or circumferences of circles, or homologous sides of similar and similarly-placed figures, 84, 3, 4, 12, and these are each multiplied into itself, there will be generated the numbers 7056, 9, 16, 144, whose sum is 7225. The square root of this sum, 85, will therefore be the diameter or circumference of the circle, or the homologous side, that is sought: so that the circle whose diameter or circumference is 85, or the figure similar and similarly placed to the given figures upon the line 85, will be equal to the four circles or figures proposed. For since the square 7225 of the root 85 is equal to the four squares 7056, 9, 16, 144 of the roots 84, 3, 4, 12; [Margin: h — Book XII, prop. 2] and since circles have the same ratio as the squares of their diameters, and therefore as the squares of their circumferences (because the circumferences are proportional to the diameters); and since likewise similar and similarly-placed figures stand to one another [Margin: i — VI.20] (because both the squares and the figures have the doubled ratio of the sides): it will follow that the circle whose diameter or circumference is 85 is equal to the four circles whose diameters or circumferences are 84, 3, 4, 12, and that the figure upon the side 85, similar and similarly placed, is equal to the four figures whose sides are 84, 3, 4, 12. Which was proposed.

LIBER IV

V

To find the area of a given ellipse.

Finally, I choose to close this fourth book with two problems most acutely discovered and demonstrated by [Archimedes] of Syracuse. One concerns the area of the ellipse, the other the area of the parabola. Let there be, then, the ellipse ABCD, whose major diameter is BD and minor diameter AC, cutting the major in two at E. [Margin: b — VI.13] Let HI be found, the mean proportional between BD and AC; and let the area of the circle described upon the diameter HI be investigated by what we wrote in Chapter 7 of this book. I say that this area is equal to the area of the ellipse ABCD. [Margin: c — corollary, VI.20] For since, as BD is to AC, so is the square on BD to the square on HI; [Margin: d — Book XII,

prop. 2] and since, as the square on BD is to the square on HI, so is the circle of diameter BD to the circle of diameter HI: therefore also, as BD is to AC, so is the circle of diameter BD to the circle of diameter HI. And since, by Proposition 5 of Archimedes' *On Conoids and Spheroids*, it also holds that, as the major diameter BD is to the minor AC, so is the circle of diameter BD to the ellipse ABCD, [Margin: e — V.11] the circle of diameter BD will have the same ratio both to the circle of diameter HI and to the ellipse ABCD. [Margin: f — V.9] And therefore the area of the circle of diameter HI will be equal to the area of the ellipse ABCD. Which was to be proved.

Diagram. An ellipse ABCD with its two axes drawn. The major diameter BD is horizontal, the minor diameter AC vertical, crossing at the center E; the horizontal segment HI (the mean proportional) is drawn across the upper half.

- A: top; C: bottom (endpoints of the minor diameter)
- B: left; D: right (endpoints of the major diameter)
- E: center (intersection of the diameters)
- H: left, I: right (endpoints of the segment HI in the upper half)

VI

To find the area of a given parabola.

Let there be given the parabola ABC, whose base is AC and axis BD, dividing the base in two at D, with vertex B. Inscribe in the parabola the triangle ABC, having the same base and the same vertex as the parabola. Now, the base AC being extended, take CE, a third part of AC, so that AE is to AC in the ratio sesquitercian [4:3]. Join the straight line EB. Then, finally, by Chapter 2 of this book, investigate the area of the triangle ABE, which I say is equal to the area of the parabola ABC. [Margin: g — VI.1] For since, as AE is to AC, so is the triangle ABE to the triangle ABC — and AE is by construction sesquitercian to AC — the triangle ABE will also be sesquitercian to the triangle ABC. And since, as Archimedes demonstrated in his book *On the Quadrature of the Parabola*, the parabola ABC is likewise sesquitercian to the triangle ABC, [Margin: h — V.11] the triangle ABE and the parabola ABC will have the same ratio to the triangle ABC. [Margin: i — V.9] And therefore the area of the triangle ABE will be equal to the area of the parabola ABC. Which was to be shown.

Diagram. The parabola-and-triangle construction. Along the base line lie A, D, C, E (left to right); B is the apex above D. Straight lines BA, BD, BC, and BE are drawn, forming the inscribed triangle ABC and the enlarged triangle ABE.

- B: apex (top, above D)
- A: base, far left
- D: base, below B (midpoint of AC)
- C: base, right
- E: base, far right (with CE a third of AC)

THE END OF THE FOURTH BOOK.

GEOMETRIÆ PRACTICÆ LIBER V.

Investigating the areas of solids and bodies. [AREAS Solidorum, corporumue perscrutans.]

[Ornament.]

There remains the third kind of magnitude, the one that embraces solids and bodies. At the beginning of Book 4 we said that bodies are to be measured by cubic corpuscles: so that when a body is said to contain 1,000 palms [palmi], we are to understand that 1,000 equal cubes, each of whose sides is equal to one palm, fill up the area of that body. This book must therefore teach the method by which the area of any body whatever is investigated — that is, the number of cubic corpuscles contained in it. (Here Clavius's "area" [area] of a solid means its cubic content, or volume.) The chief bodies we shall treat are these: parallelepipeds, prisms, cubes, pyramids, frusta of pyramids, cylinders, cones, frusta of cones, spheres, portions of spheres, and the five regular bodies — namely the tetrahedron, the hexahedron or cube, the octahedron, the icosahedron, and the dodecahedron — all of which Euclid defined in Book 11. To these we shall add certain hollow bodies and some other irregular ones.

ON THE AREA OF PARALLELEPIPEDS, PRISMS, AND CYLINDERS. [DE AREA PARALLELEPIPEDORVM, PRISmatum, & Cylindorum.]

CAPVT I.

[Margin: a — Euclid XI, def. 30 (a 30. defin. undec.).] [Margin: The area of a rectangular parallelepiped.]

A **parallelepiped** [parallelepipedum] is a solid figure contained by six quadrilateral figures, of which those that lie opposite one another are parallel.

Such a solid figure is represented by any four-sided column of uniform thickness. Thus the solid figure ABCDEFGH — in which the two opposite planes ABCD and EFGH, the two ADEH and BCFG, and the two ABGH and DCFE are parallelograms, mutually parallel and equal — is called a parallelepiped. Its area is investigated as follows.

Let there first be proposed a rectangular parallelepiped, one having all six of its parallelograms rectangular, and hence all its solid angles right; and let the length of the base AB be 3 palms, the breadth AD 2 palms, and the height AH 4 palms. Multiply the breadth 2 into the length 3, so as to produce a base of 6 square palms, as was taught in Book 4, Chapter 1. Then multiply this base of 6 palms into the height 4. The product 24 will indicate that the parallelepiped contains 24 cubes, each of whose sides span a single palm — which we shall make plain thus. Set out separately the rectangle IKLM, equal to the base ABCD, and let the perpendicular height LN be understood to be 4 palms. If, then, the side IM (2 palms) is multiplied into IK (3 palms), a base of 6 square palms is produced. If upon this we imagine 6 equal cubes raised, they will fill the parallelepiped up to the first palm of the height, LQ. If next another 6 equal cubes are set upon the first, the parallelepiped will be filled up to the second palm of the height, QP. And another 6 equal cubes will fill the parallelepiped up to the third palm of the height, PO. Finally, another 6 cubes set on will complete the whole parallelepiped up to the fourth palm of the height, ON. It is therefore established that in the whole parallelepiped there are as many groups of 6 palm-cubes as there are palms contained in the height — that is, 24 cubes.

Diagram. A rectangular parallelepiped drawn in oblique projection at the left, with an auxiliary construction at the right showing the base laid out flat (the rectangle IKLM) together with the perpendicular height divided into four unit intervals, and a small grid marking the six unit squares of the base. The auxiliary figure is crowded and several small peripheral marks are hard to resolve.

- E (top-left of solid), F (top-right of solid): upper vertices
- H (upper-left), G (interior/back upper vertex): further top vertices
- D (interior, back-lower): a rear vertex
- A (bottom-left), B (bottom-front): lower vertices
- C (rear-lower vertex, near D): [?]
- I (foot, center), K (foot, right): corners of the auxiliary base-rectangle
- L (right), M (upper), together with O and P: the perpendicular height LN and its division points
- N (top of the height line): position not clearly resolved [?]

[Margin: The area of a non-rectangular parallelepiped.]

2. Next let there be understood a parallelepiped ABCE whose bases ABCD and EFGH are rhombi or rhomboids, and whose sides AH, DE, DF, BG, CF are perpendicular to the base ABCD, so that the height is AH. First, then, let the area of the base ABCD be found, as we taught in Book 4, Chapter 3, Number 1. Then let this be multiplied into the height AH; for the resulting number will be the area of the parallelepiped. For if a rectangle IL equal to the base AC be made, and upon it a rectangular parallelepiped be imagined whose height LN is equal to the height AH,

[Margin: b — Euclid XI.31 (b 31. und.).]

then this parallelepiped will be equal to the parallelepiped ACE. For since the parallelepiped whose base is the rectangle IL and whose height is LN is produced from the height LN into the base IL, as has been shown, so likewise the parallelepiped ACE will be produced from the height AH into the base AC, which is equal to the base IL.

If no side of the parallelepiped is perpendicular to the base, then from some angle of the uppermost parallelogram a perpendicular line must be dropped to the plane in which the base lies, to serve as the height of the parallelepiped, and this must be measured carefully. For if the area of the base is investigated — whether by Book 4, Chapter 1, when it is rectangular, or by Chapter 3 of the same book, when it is not rectangular — and then multiplied into the height thus found, the area of the given parallelepiped is produced. For if above the base a right parallelepiped of the same height as the given one be imagined,

[Margin: c — Euclid XI.29 or 30 (c 29. vel 30. undec.).]

these two parallelepipeds will be equal to one another. And it is established from Numbers 1 and 2 that a right parallelepiped is generated by the multiplication of the base into the height.

[Margin: The area of a cube.]

3. A **cube** [cubus] — which is also a certain rectangular parallelepiped — is produced in the same way, namely from its side into itself and again into the product. Thus if the side of a cube is 10, its area will be 1,000, since ten times ten times ten produce 1,000.

[Margin: d — Euclid XI, def. 13 (d 13. defin. undec.).]

4. A **prism** [prisma] is a solid figure contained by planes, of which two opposite ones are equal, similar, and parallel, while the rest are parallelograms. Such is the solid ADF, whose bases are the pentagons ABCDE and FGHK, parallel and equal. This solid figure is represented by any many-sided column of uniform thickness, whose opposite bases are equal, similar, and parallel — whether these be triangles, quadrilaterals, pentagons, and so on. From which it follows that any prism is surrounded by as many parallelograms as there are sides, or angles, in each of the opposite planes. Thus the proposed prism is bounded by five parallelograms: ABGF, BCHG, CDIH, DEKI, and EAFK. The area of any prism will be found if the area of its base is sought and then multiplied into the height. For if one imagines a parallelepiped of the same height as the prism, having for its base a rectangle equal to the base of the prism,

[Margin: The area of a prism, both right and oblique. e — corollary 2 of Euclid XII.7 (e 2. corol. 7. duodec.).]

this parallelepiped will be equal to the prism. Since, then, the parallelepiped is produced from its base into the height, the prism too will be generated from the multiplication of its base into the height. The area of the base will be known from what we wrote in Book 4; and the height of the prism, if its sides are not perpendicular to the base, must be determined, just as in the preceding chapter, Number 2, we directed that the height of a parallelepiped be investigated.

5. A **cylinder** [cylindrus] is a solid figure of uniform thickness, contained by two equal and equidistant circles together with the round surface set between them, in the likeness of some round column. Such is the solid ACH, whose bases are the two circles ABCD and EFGH, parallel and equal. Its area too is produced from the multiplication of the base — found by Book 4, Chapter 7 — into the height, which will be explained for a right cylinder just as it was done in Number 1 for a right parallelepiped. For if, say, the circular base ABCD of the cylinder contains 10 square palms, then 10 palm-cubes raised upon those 10 square palms will fill the cylinder up to the first palm of the height; and 20 cubes will fill it up to the second palm, and so on. But if the cylinder is oblique, its height must be sought by a perpendicular line dropped from the upper base to the plane in which the lower base lies, and the area of the base — found by Book 4, Chapter 7 — must be multiplied into this height. For the resulting number will give the area of the given cylinder,

[Margin: f — corollary, Euclid XII.11 (f coroll. 11. duodec.).]

since it is equal to a right cylinder having the same base and height as it.

Diagram. Two solids side by side: at the left a pentagonal prism (bases ABCDE below and FGHK above), at the right a cylinder (lower base ABCD, upper base EFGH), both drawn in oblique projection.

- Prism, top face: F (left), K (center), I (right), H (right rear)
- Prism, middle band: G, E, H (front edges) and E (rear)
- Prism, base: A (left), B (bottom-left), C (bottom-center), D (right)
- Cylinder, upper rim: H (upper-left), E, G (upper-right)
- Cylinder, mid: F (right), D (center)
- Cylinder, lower rim: A (left), B (bottom), C (right)

ON THE AREA OF PYRAMIDS AND CONES. [DE AREA PYRAMIDVM ET CONORVM.]

CAPVT II.

[Margin: a — Euclid XI, def. 12 (a defin. 12. undec.).]

A **pyramid** [pyramis] is a solid figure contained by planes, set up from one plane to one point. Thus the solid figure ABCDEF, set up at the point F above the pentagonal base ABCDE and surrounded by the five triangles ABF, BCF, CDF, DEF, AEF — as many, that is, as there are sides in the base — is called a pyramid.

A **cone** [conus], on the other hand, is a round solid figure set up to one point above a circular base, in the likeness of a round pyramid, such as the figure ABCDE.

The area of both the pyramid and the cone is produced from the multiplication of the base into a third part of the height. For since, as we showed in the preceding chapter, a prism or cylinder having the same height as the pyramid and cone is generated from the base into the whole height,

[Margin: b — scholium, Euclid XII.14 (b schol. 14. duodec.). c — corollary, Euclid XII.7 (c corol. 7. duodec.). d — Euclid XII.10 (d 10. duod.).] [Margin: The area of a pyramid and of a cone.]

a third part of that prism or cylinder will be produced from the same base into a third part of the height. Since, then, the pyramid is a third part of that prism, and the cone a third part of the cylinder, it is clear that both the pyramid and the cone are produced from the base into a third part of the height. From which it follows that, if the base is multiplied into the whole height, a third part of the resulting

number is likewise the area of the pyramid or of the cone. Likewise the same is produced from the whole height into a third part of the base —

[Margin: The area of a pyramid and of a cone, otherwise. e — scholium 1 of Euclid XII.7, and XII.11 (e 1. schol. 7. duodec. & 11. duodec.).]

for that by this reasoning [continues below].

Diagram. Two solids side by side: at the left a cone on a circular base, at the right a square-based pyramid, both drawn in oblique projection.

- Cone: E (apex), D (point on the axis), A (left of base circle), C (center of base), B (front of base circle)
- Pyramid: F (apex), E (interior point where the edges meet the axis), A (left base corner), D (right base corner), B (front-left base corner), C (front-right base corner)

2. Furthermore, the base of a pyramid, if it is triangular, will be known as was taught in Book 4, Chapter 2; if it is many-sided, it will be found by what we wrote in the same book, Chapters 3, 4, and 5. The base of a cone, however, will be investigated by Chapter 7 of the same book.

[Margin: The height of a pyramid and of a cone.]

But the height of both the pyramid and the cone will be obtained if at the apex a plane is set equidistant from the base, and from it a perpendicular is dropped to the plane in which the base lies, and this is exactly measured. For although this same height can be found geometrically, if the inclination of one side to the base and also the magnitude of that side are known — nevertheless, since these very things must be explored physically by some instrument, it is better to seek the height itself at once by the instrument, and especially by the instrument of parts [instrumentum partium] which we described in Book 1, Chapter 1; since that geometrical method is more difficult and proceeds from the inclination and the side once these are known by the instrument.

3. And these things which we have said, we wish to be understood of upright pyramids and cones as well as of oblique and scalene ones.

ON THE AREA OF THE FRUSTUM OF A PYRAMID AND OF A CONE. [DE AREA FRVSTI PYRAMIDIS, ET CONI.]

CAPVT III.

[Margin: a — Euclid VI.4 (a 4. sexti).] [Margin: b — Euclid XI.17 (b 17. undec.).]

I call the **frustum** [frustum] of a pyramid and of a cone that which others call a truncated pyramid and a truncated cone. Let there be, then, a frustum of a pyramid ABCDEF, whose bases ABC and DEF are parallel and similar, and whose area is to be investigated. This can be done in two ways. First, let the whole pyramid ABCH be conceived, whose height HG, perpendicular to the base (although the pyramid is not actually complete), we shall find thus. Since, as AB is to AH, so is DE to DH, then by alternation, as AB is to DE, so is AH to DH; and hence by division (taking AS equal to DE itself) as SB is to DE, so is AD to DH. And because the parallel planes ABC and DEF cut the straight lines AH and GH proportionally at I and G, it will likewise be that as SB is to DE, so is GI to IH. If, then, it is arranged that as SB, the difference between the homologous sides AB and DE of the bases, is to DE, so is GI, the height of the frustum of the pyramid (which will be known by a perpendicular line dropped to the base from some point of the plane DEF, extended if need be) to a fourth term — the straight line IH will come out, that is, the height of the pyramid DEFH; and when this is added to GI, the whole height GH will be known. Therefore, if by the preceding chapter the area of both the whole pyramid ABCH and of the cut-off pyramid DEFH is found, and the latter is taken from the former, the remainder will be the frustum ABCDEF.

[Margin: The area of the frustum of a pyramid.]

2. In no other way will the frustum of a cone ABCD be investigated, as is plain, if the whole cone ABH is understood, and so on.

[Margin: The area of the frustum of a cone.]

[Margin: c — Euclid II.14 (c 14. secundi). d — Euclid VI.1 (d 1. sexti). e — Euclid V.7 (e 7. quinti).]

3. By another method we shall determine the same frustum, both of a pyramid and of a cone, even if neither the pyramid nor the cone is completed. Let there be made the squares KLMN and NOPQ equal to the known bases ABC and DEF, and let there be found between the squares KM and NP the mean-proportional surface — such as is the rectangular figure OK, the side OP being extended to R.

For since, as MN is to NO, so is KM to NR; and likewise as KN is to NQ, so is NR to NP; and MN is to NO as KN is to NQ — the square KM will be to the rectangle NR as the rectangle NR is to the square NP; and therefore NR is a mean proportional between the squares KM and NP. Wherefore, if the square root of the known base ABC — that is, the side KN of the square KM — is multiplied into the square root of the known base DEF — that is, into the side NO of the square NP — the area of the rectangle NR will be produced.

Diagram. At the left, the frustum of a pyramid: the whole (dotted) pyramid rises to the apex H, the cutting plane DE marks the top of the frustum, and ABC is the lower base, with S and G interior division points and I on the axis.

- H (apex of the completed pyramid)
- D (left top of the frustum face), E (right top of the frustum face)
- C (interior/rear base vertex), G (interior point on the base near the axis foot)
- A (bottom-left of base), S (foot point on the base line), B (bottom-right of base)

Diagram. At the upper right, the frustum of a cone, drawn in oblique projection with the completed cone dotted to its apex.

- C (upper-left of top ellipse), H (apex of the completed cone), I (upper-right of top ellipse)
- G (right of lower base), A (left of lower base), S (center point), B (front of lower base)

Diagram. At the lower right, the plane construction of the two squares and the mean-proportional rectangle: the larger square KLMN (equal to base ABC) and the smaller square NOPQ (equal to base DEF), with the rectangle NR (mean proportional) built by extending OP to R.

- K (top-left), N (top-right region) [?] and the small marks near the top edge are indistinct [?]
- R (mid-left), P (center), O (mid-right)
- L (bottom-left), M (bottom-right)

[Margin: f — Euclid XII.7 (f 7. duod.). g — Euclid XII.7 (g 7. duod.). h — Euclid V.5 (h 5. quinti).]

Now let GI, the height of the frustum, be multiplied into the sum collected from the square KM (that is, from the base ABC), the square NP (or base DEF), and the

surface NR (the mean proportional between the bases) — or from the said squares taken together. For the resulting number will be triple the frustum of the pyramid ABCDEF; and therefore a third part of the product will be the area of the said frustum. For since the prism which is made from GH, the height of the pyramid, into the base ABC (or the square KM) is triple the pyramid ABCDEFH, the parallelepiped made from GI into the square KM, together with the parallelepiped made from IH into the same square KM, will likewise be triple that same pyramid. And the removed parallelepiped made from IH into the base DEF — that is, into the square NP — is triple the cut-off pyramid DEFH. Therefore the remainder too, which is made from GI into the square KM together with those made from IH into KP and into RM, will be triple the remaining frustum ABCDEF.

[Margin: i — Euclid VI.22 (i 22. sexti). k — Euclid VI.1 (k 1. sexti). l — Euclid VII.19 (l 19. sep.). m — axiom 2 (m 2. pronunc.).]

4. And because the equal surfaces ABC and KM, and the equal surfaces DEF and NP, have the same ratio, it will be, by alternation, as ABC is to DEF, so is KM to NP; and the side AB will be to the side DE as the side KN is to the side NQ. And by division (subtracting the straight line AS, equal to DE, from AB), SB will be to DE as KQ is to QN. But, as we demonstrated in Number 1, as SB is to DE, so is GI to IH. Therefore it will also be as KQ is to QN; and hence as MO is to ON, so is GI to IH. But as KQ is to QN, so is KP to PN; and as MO is to ON, so is MR to RN. Therefore it will also be, as GI is to HI, so both KP to PN and MR to RN. Contracting these magnitudes into numbers, then, the number made from GI (the first) into PN (the fourth) will be equal to that which is made from IH (the second) into KP (the third). Likewise the number made from GI (the first) into RN (the fourth) will be equal to that which is made from IH (the second) into MR (the third). And therefore the two products made from GI into PN and from GI into RN will be equal to the two made from IH into KP and IH into MR. So, adding the common product made from GI into KM, the three made from GI into KM, into PN, and into RN will be equal to the three made from GI into KM, and from IH into KP and into MR. But these latter three are triple the frustum of the pyramid ABCDEF, as we demonstrated at the end of Number 3. Therefore the former three also — namely those made from GI, the height of the frustum, into KM, into PN, and into RN, that is, into the sum collected from KM (equal to the base ABC), PN (equal to the base DEF), and RN (the mean proportional between the bases) — will be triple the same frustum; and therefore a third part of them will be equal to the area of the frustum. Which was to be demonstrated.

5. By the same reasoning the frustum of a cone ABDC will be produced from the height GI into the sum collected from the base AB, the base CD, and the mean-proportional surface between the bases — as is established, if the squares KM and NP equal to the bases are conceived, and hence the surface RN, the mean proportional between the bases, and so on.

SCHOLIUM

1. If what has been said so far is applied to material objects, then, by Chapter 1, we may measure any wall of uniform thickness as though it were a parallelepiped whose length equals the length of the wall, whose breadth equals the thickness of the wall, and whose height (or depth) equals the height of the wall. [Margin: The solidity of a wall.]

By the same reasoning we shall measure a block of marble or other stone as though it were a prism, provided it has a uniform thickness and its sides stand square to the bases.

In just the same way we shall measure — more or less — a sack of wheat as though it were a cylinder. But since a sack is not an exact cylinder, its true measure cannot be obtained. [Margin: The solidity of a piece of marble.]

2. Again, by Chapter 2, we may measure — more or less — a heap of wheat as though it were a cone. And so, if we know how many grains or pounds of wheat are contained in a cube of, say, one palm, and multiply the grains or pounds of a single cube by the number of cubes found in the whole sack or heap, we obtain the number of grains or pounds present in that sack or heap. [Margin: The capacity of a sack of wheat.]

3. Likewise, if some vessel is given, hollowed out in the shape of a parallelepiped or a cylinder, we shall know its capacity if we measure its interior parallelepiped or cylinder exactly as we would a solid figure. Thus, once we know how many measures of water or other liquid are contained in a cube of, say, one palm, it cannot be unknown how many measures are contained in the cubes filling the whole vessel — provided the measures of one cube are multiplied into the number of cubes we have found the vessel to contain. [Margin: The capacity of a heap of wheat.] [Margin: The capacity of a hollow vessel.]

4. Finally, if the solid content of some vessel is required, one that has the shape of a parallelepiped, prism, or cylinder both inside and out, then both figures must be measured, the inner and the outer. For if the inner is subtracted from the

outer, the remainder gives the solid content of the hollow vessel. [Margin: The solidity of a hollow vessel.]

ON THE SURFACE OF THE FIVE REGULAR BODIES

CHAPTER IV

1. There are only five regular bodies — the tetrahedron, the hexahedron, the octahedron, the dodecahedron, and the icosahedron — as we demonstrated in the scholium to Proposition 18 of Book 13. Euclid defines them as follows in Book 11.

A **tetrahedron** is a solid figure contained by four equal equilateral triangles, of the sort that an equilateral triangular pyramid presents.

A **hexahedron** is a solid figure contained by six equal squares, such as a cube — or a parallelepiped with square bases in which all three dimensions are equal — exhibits.

An **octahedron** is a solid figure contained by eight equal equilateral triangles.

A **dodecahedron** is a solid figure contained by twelve equal, equilateral, and equiangular pentagons.

An **icosahedron** is a solid figure contained by twenty equal equilateral triangles.

2. We showed in Chapter 1, No. 3, that the content of the cube (or hexahedron) is generated by multiplying the side into itself and then again into the product. Likewise we explained in Chapter 2, No. 1, that the pyramid (or tetrahedron) is produced from its altitude — found mechanically, as taught in Chapter 2, No. 2 — taken into a third part of the base, or else from its base taken into a third part of the altitude. But if one wishes to find the altitude of the tetrahedron geometrically, we shall proceed thus. Since the square of the tetrahedron's side is to the square of the diameter of the sphere enclosing the tetrahedron as 2 to 3 [Margin: a. *Elements* XIII.13] — so that the diameter is in power one-and-a-half times [*sesquialtera*] the side of the pyramid — then if we make: as 2 is to 3, so the square of the tetrahedron's side to another quantity, there will result the square of the sphere's diameter; and the square root of that square will give the diameter itself, [Margin: b. *Elements* XIII.13, 2nd corollary] two-thirds of which yield the altitude of the tetrahedron. [Margin: The area of the cube and of the tetrahedron.] [Margin: The altitude of the tetrahedron.]

3. Now since the octahedron is divided into two similar and equal pyramids whose common base is the square described on the side [Margin: c. *Elements*

XIII.14, 2nd corollary], if the area of each pyramid is investigated, the area of the octahedron cannot remain unknown, since it is compounded of the areas of those pyramids. The area of those two pyramids is produced by multiplying the square of the octahedron's side into the diameter of the octahedron and taking a third part of the resulting number; for that number, arising from the square of the octahedron's side into its diameter, is triple the parallelepiped of those two pyramids [Margin: The area of the octahedron.] — [Margin: d. *Elements* XII.7, corollary] because half of that parallelepiped, having the same base and altitude as either pyramid, is triple one pyramid. Furthermore, the diameter of the octahedron — which does not differ from the diameter of the sphere, nor from the diagonal of the square on the octahedron's side — is found by extracting the square root of double the square of the side; [Margin: e. *Elements* I.47, scholium] for the square described on the diagonal of the square is double the square on the side, [Margin: f. *Elements* XIII.14] just as the square of the sphere's diameter is double the square of the octahedron's side. Half of this diameter will be the altitude of either pyramid. Therefore, if this altitude is multiplied into a third part of the square of the side, the area of one pyramid is produced — that is, half the octahedron; and so double this pyramid indicates the area of the whole octahedron. [Margin: The diameter of the octahedron.]

4. Next, since by drawing straight lines from the centre of the dodecahedron to all its angles the dodecahedron is divided into 12 equal pentagonal pyramids, if the area of one pyramid found by Chapter 2 is multiplied by 12, the area of the whole dodecahedron will be produced. [Margin: The area of the dodecahedron.] But to obtain the area of one pyramid, it is necessary both to investigate the area of the pentagonal base from the given side, by what we wrote in Book 4, Chapter 5, and to find the pyramid's altitude, as I shall now teach. From the upper plane, produced [i.e. extended], let a perpendicular line be dropped to the plane of the opposite base. For half of this perpendicular, carefully sought in the parts of the dodecahedron's side by means of the instrument of parts constructed in Book 1, Chapter 1, will give the required altitude of the pyramid, just as the whole perpendicular exhibits the height of the dodecahedron. This, however —

— we shall also determine geometrically, thus. [Margin: a. *Elements* V.8] Since the cube described within the dodecahedron is enclosed by the same sphere as the dodecahedron, and its side subtends one angle of the dodecahedron's pentagon, the diameter of the sphere, of the dodecahedron, and of the cube is therefore one and the same. [Margin: b. On rectilinear triangles, prop. 12] If the straight line subtending the pentagon's angle is investigated, the side of the cube will be obtained. [Margin: c. *Elements* XIII.15] And since the sphere's diameter is in

power triple the side of the cube, if the square of the cube's side thus found is tripled, the square of the diameter of the sphere (or of the cube) is obtained, whose square root gives the diameter itself.

Now since the diameter of the dodecahedron and its altitude — the line joining the centres of the opposite bases — bisect each other at the centre, [we shall find] the required altitude of the pyramid in this way. Let a right triangle be conceived whose base is the known semidiameter of the dodecahedron (since the whole diameter is closely known), and whose legs about the right angle are the altitude of the pyramid and the semidiameter of the circle circumscribing the base of the dodecahedron. [Margin: The perpendicular from the centre of the sphere to the base of the dodecahedron.] Since, then, this semidiameter can be known from what we taught in Book 4, Chapter 5, the remaining side — namely the altitude of the pyramid that we seek — will also be known. [Margin: d. On rectilinear triangles, prop. 3]

Furthermore, the semidiameter of the aforesaid circle circumscribing the dodecahedron's pentagon is likewise made known thus. [Margin: The semidiameter of the circle circumscribing the pentagon of the dodecahedron.] Since the side of the pentagon subtends 72 degrees in that circle, and the side of the decagon 36 degrees, these sides will be known in terms of the parts of the whole sine. [Margin: e. *Elements* XIII.10] If, then, we make: as the side of the pentagon (known in parts of the whole sine) is to the same side known from the hypotenuse, so the side of the decagon (known in the same parts of the whole sine) is to another quantity, there will result the side of the decagon in the measure of the pentagon's side, now known. And since the side of the pentagon is in power equal to the sides of the decagon and hexagon of the same circle, if the square of the now-known decagon's side is subtracted from the square of the pentagon's side, the remainder becomes the square of the hexagon's side — that is, of the semidiameter — and so its square root will exhibit the semidiameter, now known.

5. Lastly, since by drawing straight lines from the centre of the icosahedron to all its angles the icosahedron is divided into 20 equal triangular pyramids, if the area of one pyramid found by Chapter 2 is multiplied by 20, the area of the whole icosahedron — compounded of those 20 pyramids — is generated. To obtain the area of one pyramid, one must first investigate the area of the triangular base from what we wrote in Book 4, Chapter 2, Nos. 4 and 5; and then the altitude of the pyramid mechanically, in this way. From the upper plane produced — that is, from the lower surface of some plane laid upon the topmost base of the body —

let a perpendicular be dropped to the plane of the opposite base. For this, accurately measured, gives the altitude of the icosahedron, and its half the altitude of the pyramid that is sought. This too we shall explore geometrically, as follows. Let a pentagon be formed from 5 sides of the icosahedron, and let its semidiameter be investigated, together with the side of the decagon in the circle circumscribing that pentagon, in the parts in which the side of the icosahedron is given — by this reasoning, namely. Let a right triangle be conceived whose base is the semidiameter of the said circle, and [whose other leg is] the perpendicular dropped from the centre to the midpoint of the said side. In this way the semidiameter is known, from what we taught in Book 4, Chapter 5. And the side of the decagon described in the same circle is found as was said a little before, near the end of No. 4. [Margin: f. *Elements* XIII.16, 1st corollary] And since the diameter of the sphere (that is, of the icosahedron) is in power quintuple the semidiameter found, if the square of the semidiameter found is quintupled, the square of the icosahedron's diameter is produced, whose square root offers the diameter; and so the semidiameter of the icosahedron too will be known. Or otherwise: [Margin: g. *Elements* XIII.16, 2nd corollary] Since the diameter of the sphere — that is, of the icosahedron — is composed of the side of the hexagon and two sides of the decagon, in the circle circumscribing the pentagon composed of five sides of the icosahedron, the sum collected from the semidiameter of that circle and two sides of the decagon will be equal to the diameter of the icosahedron; and so again the semidiameter of the icosahedron will be known.

[Margin: The error of Oronce Finé.] From this it is clear that Oronce Finé [Orontius Finaeus], together with those who follow him, is mistaken in thinking that the semi-axis of the icosahedron — that is, the axis, or the altitude of the pyramid whose base is a triangle of the icosahedron and whose vertex is the centre of the sphere — is composed of half the semidiameter of that circle and the side of the decagon. For, as is established from what we have just written, it is in that way that the semidiameter of the sphere (or icosahedron) is composed, which is greater than the said axis. Moreover, the semidiameter of the circle circumscribing the aforesaid pentagon can also be found, as we said at the end of No. 4: namely, if the square of the decagon's side is removed from the square of the said pentagon's side — which does not differ from the side of the icosahedron — and the square root of the remaining number is extracted; [Margin: h. *Elements* XIII.10] because the side of the pentagon is in power equal to the sides of the decagon and hexagon of the same circle.

[Margin: The perpendicular from the centre of the sphere to the base of the icosahedron.] Now, the semidiameter of the icosahedron being known, we shall

find the altitude of the pyramid whose base is a triangle of the icosahedron and whose vertex is its centre, in this way. Since the diameter of the icosahedron and its altitude bisect each other at the centre, let a right triangle be conceived whose base is the now-known semidiameter of the icosahedron, and whose legs about the right angle are the altitude of the pyramid and the semidiameter of the circle circumscribing the base of the icosahedron. Since, then, this semidiameter can be known from what we taught in Book 4, Chapter 5, the remaining side of the pyramid — namely the altitude that is sought — will also be known. [Margin: i. On rectilinear triangles, prop. 3] Furthermore, the semidiameter of the circle circumscribing the triangular base of the icosahedron is likewise made known in this way. [Margin: The semidiameter of the circle circumscribing the triangle of the icosahedron.] [Margin: k. *Elements* XIII.12] Since the side of an equilateral triangle is in power triple the semidiameter of its circle, if the square of the icosahedron's side is divided by 3, the square root of the quotient will be the semidiameter sought.

6. By the same art as was set out for the dodecahedron and icosahedron, we may investigate the areas of the tetrahedron, cube, and octahedron, if — by lines drawn from their centres to all their angles — they are distributed into equal pyramids: the tetrahedron into 4 triangular pyramids, the cube into 6 quadrangular ones, and the octahedron into 8 triangular ones. [Margin: The area of the tetrahedron, cube, and octahedron, found by another method.] For, the area of one pyramid in each being found by Chapter 2, if it is multiplied into the number of bases of the regular body, the area of the whole body rises up. But to obtain the area of one pyramid, its altitude must first be sought, and this, taken into a third part of the base, produces the pyramid's area. This altitude, then, in the tetrahedron, is found thus. [Margin: l. *Elements* XIII.13, 2nd corollary] Since the diameter of the sphere enclosing the tetrahedron is in power one-and-a-half times [*sesquialtera*] the side of the tetrahedron, if we make: as 2 to 3, so the square of the given tetrahedron's side to another quantity, there will result the square of the sphere's diameter, whose square root shows the diameter itself. [Margin: m. *Elements* XIII.13, 2nd corollary] A sixth part of this diameter will be the sought altitude of the pyramid — namely the straight perpendicular dropped from the centre of the sphere to the base of the tetrahedron. [Margin: The perpendicular from the centre of the sphere to the base of the tetrahedron, cube, and octahedron.]

In the cube, the said altitude is equal to half the side of the cube, since the perpendicular dropped from the centre of the sphere to the base of the cube is equal to half the side of the cube, as is evident.

In the octahedron, finally, the same altitude is determined thus. [Margin: n. *Elements* XIII.13] Since the diameter of the sphere is in power double the side of the octahedron, if we make: as 1 to 2, so the square of the given octahedron's side to another quantity, the square of the diameter of the sphere (or octahedron) is produced. The square root of this gives the diameter, and so the semidiameter will not be unknown.

Hence the sought altitude of the pyramid — that is, the perpendicular dropped from the centre of the sphere to the base of the octahedron — is elicited by the same reasoning we explained for the icosahedron at the end of No. 5.

7. There is, moreover, another method of measuring all five regular bodies that ought not to be omitted, one demonstrated in Book 14 of Euclid, and it is of this kind. First let the convex surface of each body be sought from its known side, even without investigating the area of any base — in this way, namely. Since any base of any body is divided, by straight lines drawn from the centre of the base to all its angles, into as many equal triangles as there are angles or sides contained in the base, then if this number of triangles is multiplied into the number of bases enclosing the proposed regular body, we obtain the number of all such triangles contained in the whole convex surface. [Margin: Into how many triangles all the bases of any regular body are divided from their centres.] For example: since the square base of the cube ABCD is divided into four triangles from the centre E, there will be contained 24 such triangles in its 6 bases. Likewise, since the triangular base ABC of the tetrahedron, octahedron, and icosahedron is distributed into 3 triangles from the centre D, there will be 12 such triangles in the 4 bases of the tetrahedron, 24 in the 8 bases of the octahedron, and 60 in the 20 bases of the icosahedron. Finally, since the pentagonal base ABCDE of the dodecahedron is resolved into 5 triangles from the centre F, the 12 bases of the dodecahedron will comprise 60 such triangles.

Diagram. A row of three uncaptioned circles, each with a regular polygon inscribed and subdivided into triangles from its centre; they illustrate how the bases of the regular solids are divided (linked by short dashes in the source).

- Circle 1 — a **square** inscribed in the circle (the cube's base ABCD), subdivided from the centre.
- D (top-left vertex)
- C (top-right vertex)
- A (bottom-left vertex)
- B (bottom-right vertex)

- E (centre of the square)
- G (left, midpoint of side AD)
- H (right, midpoint of side CB)
- F (bottom, midpoint of side AB; EF is the perpendicular from centre to side)
- Circle 2 — an **equilateral triangle** inscribed in the circle (the base ABC of the tetrahedron/octahedron/icosahedron), subdivided from the centre.
- C (top vertex)
- A (bottom-left vertex)
- B (bottom-right vertex)
- D (centre of the triangle)
- E (bottom, midpoint of side AB; DE is the perpendicular from centre to side)
- F (left, midpoint of side AC)
- G (right, midpoint of side BC)
- Circle 3 — a **regular pentagon** inscribed in the circle (the dodecahedron's base ABCDE), subdivided from the centre.
- D (top vertex)
- E (upper-left vertex)
- C (right vertex)
- A (bottom-left vertex)
- B (bottom-right vertex)
- F (centre of the pentagon)
- G (bottom, midpoint of side AB; FG is the perpendicular from centre to side)
- H (left, midpoint of a side)
- I (right, midpoint of a side)

Next, since the rectangle contained by the perpendicular dropped from the centre of the base to a side and by one side is equal to two such triangles [Margin: a. *Elements* I.41] — because it is double one of them — that rectangle, taken 12 times in the cube, will equal the whole surface of the cube; in the tetrahedron, taken 6 times, it will make up the whole surface of the tetrahedron; in the octahedron, taken 12 times, it will match the whole surface of the octahedron; but in the dodecahedron and icosahedron, taken 30 times, it will equal the whole surface of both the dodecahedron and the icosahedron. [Margin: The surface of the regular bodies, and the perpendiculars of the bases.] Now the said perpendicular EF, in the base of the cube, is equal to half the side of the cube AB. [Margin: b. Book I, scholium 26] For since the perpendicular EF bisects the side AB [Margin: c.

Elements I.6] and is equal to AF itself — because the angles FAE and FEA are each half a right angle — it is established that EF is equal to half the side of the cube. The perpendicular DE, however, in the base of the tetrahedron, octahedron, and icosahedron, [Margin: d. *Elements* XIII.12, 2nd corollary] is half the semidiameter CD. [Margin: e. *Elements* XIII.12] Since, then, the side AC is in power triple the semidiameter CD, if we make: as 3 to 1, so the square of the given side AC to another quantity, there will result the square of the semidiameter CD, whose square root will indicate CD itself, and half of it will exhibit the perpendicular DE. Finally, the perpendicular FG in the base of the dodecahedron [Margin: f. *Elements* XIV.1] is half the sum collected from the semidiameter AF and the side of the decagon of the circle ABD — the decagon's side being known, as was taught at the end of No. 4.

Since, moreover, the solid formed from the perpendicular drawn from the centre of any regular body to one of its bases, multiplied into a third part of the surface of that body, is equal to the body itself, then if the convex surface of the given regular body is investigated as we have just taught, and the altitude of one of the pyramids into which the body is divided by straight lines drawn from its centre (that altitude being found as we delivered above) — that is, the perpendicular dropped from the centre of the body to its base — is multiplied into a third part of that surface, the area (or solid content) of the body itself is produced. [Margin: g. *Elements* XIV.20, scholium.] [Margin: The area of the regular bodies, found by another method.] The same is also obtained if the said altitude is multiplied into the whole convex surface and a third part of the product is taken.

8. And so, as you see, the whole difficulty in measuring the regular bodies consists almost entirely in finding the altitude of the pyramid that has the same base as the body but its vertex at the centre of the sphere. The geometric discovery of this by numbers is most troublesome, on account of the surd roots and the fractional numbers whose numerators and denominators are excessively large — so much so that it seems worth the effort to explore it mechanically, as we said at the beginning of Nos. 2, 4, and 5, especially if exquisite care is applied in measuring it by the instrument of parts. But since we do not always have regular bodies at hand, so as to obtain that altitude mechanically, I should like to prescribe here a certain new and very easy method by which, without the trouble of those numbers, that same altitude may be found by lines, even if no regular body is present but only its side is given and known.

Let, then, the side of the tetrahedron AB be first given, of however many palms, and let the equilateral triangle ABC be constructed as the base of the tetrahedron.

Bisecting the side AB at D, let the straight line CD be joined, [Margin: h. Book I, scholium 26] which will be perpendicular to AB. Then, constructing the isosceles triangle ABE, each of whose sides equals the straight line CD, let the perpendicular BF be dropped to AE, and let FG be a fourth part of it. [Margin: The altitude of the pyramid of the tetrahedron.] I say that FG is the altitude of one pyramid — that is, of the pyramid circumscribed about the tetrahedron and brought down to one base. For since, as we demonstrated from Hypsicles [Hypsicles of Alexandria] at the end of Euclid, E is the angle of inclination of one base of the tetrahedron to another, and EB is equal to the perpendicular CD: if the triangle BEF is conceived to be moved about EF until it stands perpendicular to the base of the tetrahedron, the point B will fall onto the vertex of the tetrahedron, and accordingly the perpendicular BF will be the altitude of the tetrahedron. [Margin: i. *Elements* XIII.13, 2nd corollary] And since the altitude of the tetrahedron contains two-third parts of the sphere's diameter, if the diameter [?] is put at 6, the altitude BF will be 4, and the semidiameter 3. [Margin: k. *Elements* XIII.13, 2nd corollary] Since, then, the altitude of one pyramid is a third part of the semidiameter, BG is the semidiameter and GF the altitude of one pyramid.

We shall also find this same altitude, even without erecting the isosceles triangle AEB, in the following way. Taking DH, a third part of the perpendicular CD, let the perpendicular HK be raised upon CD, cut off from D at the interval CD in K. I say that HI, a fourth part of HK, is the altitude of one pyramid. For when the triangle DHK is erected upon the base ABC of the tetrahedron, the point K will fall onto the vertex of the tetrahedron, since DK is drawn equal to the perpendicular dropped from the middle of a side to the opposite angle of the base. Therefore, as before, HK will be the altitude of the tetrahedron, and HI the perpendicular from the centre of the sphere falling onto H, the centre of the base. [Margin: l. *Elements* XIII.13, 2nd corollary] For DH, a third part of the perpendicular CD, falls onto the centre of the triangle.

Diagram. An uncaptioned construction figure for finding the altitude of the tetrahedron's pyramid. Two triangles share the horizontal base AB (with D its midpoint): the equilateral base-triangle ABC hangs below (apex C at the bottom), and the isosceles triangle ABE rises above (apex E at the top). A subsidiary right triangle DHK is drawn along the line from D toward C.

- E (top apex of the upper isosceles triangle ABE)
- A (left end of the shared base)
- D (midpoint of the base AB)

- B (right end of the shared base)
- C (bottom apex of the equilateral base-triangle ABC; $CD \perp AB$)
- F (on side AE; foot of the perpendicular BF)
- G (on BF, marking the fourth part FG)
- H (on CD, below D; $DH = \text{one-third of } CD$)
- I (on HK, marking the fourth part HI)
- K (endpoint of the perpendicular HK, at the interval CD from D)
- [A small mark resembling "2" appears beside B, apparently a figure/step numeral.]

Let the side of the octahedron LM next be given, upon which the equilateral triangle LMN is constructed as the base of the octahedron. Bisecting the side LM at O, let the straight line NO be joined, [Margin: m. Book I, scholium 26] which will be perpendicular to LM. [The construction continues.]

Diagram. An uncaptioned construction figure (paired with the tetrahedron figure) for finding the altitude of the octahedron's pyramid, built on the equilateral base LMN of the octahedron. The figure is small and partly overprinted, and its accompanying text runs on beyond this page. The following labels are legible:

- S (upper left)
- T (upper right)
- V (below, between S and T)
- R (right side)
- O (centre; the small mark at the midpoint region — the bisection point of LM)
- N (lower left, apex of the base triangle LMN)
- M (right, end of the base LM)
- P (lower right)
- L [?] (left end of the base LM — not clearly legible in the scan)

now that the isosceles triangle QRS has been constructed on the base QR — a base equal to the diameter of the sphere, or to the [side] of the square described on the side of the octahedron (which will be obtained if the perpendicular MP is raised equal to the side LM; for once the straight line LP has been joined, it will be the diameter of that square, or of the sphere) — each of its sides QS and RS having a value equal to the perpendicular NO: from R let a perpendicular RT be drawn to QS, and let it be bisected at V. I say that TV is the required altitude of the pyramid;

that is, it is equal to the perpendicular that falls from the centre of the sphere to one face of the octahedron. For, as we proved from Hypsicles at the end of Euclid, the angle QSR measures the inclination of one face to the other, and it is obtuse; hence the perpendicular RT, falling toward the side of the acute angle RST, will be equal to the altitude of the octahedron — that is, to the perpendicular joining the centres of the two opposite faces of the octahedron, as is plain from a material octahedron. And therefore its half TV will be the required altitude of the pyramid, since the altitude of the octahedron is bisected at the centre.

[Margin: Altitude of the octahedron's pyramid.]

If the side of the cube (that is, the hexahedron) be given, its half will be the required altitude of the pyramid, because the altitude of the cube is equal to its own side.

[Margin: Altitude of the cube's pyramid.]

Now let AB be given as the side of a dodecahedron, upon which an equilateral and equiangular pentagon ABCDE is erected as a face of the dodecahedron. Draw the straight line CE, [a] which will be the side of the cube inscribed in the dodecahedron (the cube described in the same sphere as it), [b] and which is parallel to the side AB. Bisect AB at S, and join the straight line SD, [c] which will bisect the angle CDE; [d] and therefore it will also divide CE into two equal parts and at right angles, so that the angles at S will be right. On CE construct the isosceles triangle CGE, each of whose sides CG and EG is equal to the perpendicular SF. Taking also FH and FI equal to half the side AB, raise the perpendiculars HK and IL to EC; let these be cut in K and L by [arcs struck] from C and E at the interval FD, and join the straight lines EL and CK. This done, let the angle MNO be made equal to the angle CGE, and let NO be set equal to SD. Likewise, let the angle NOP be made equal to the angle ELK, and let OP be set equal to the side AB. Finally, from P let the perpendicular PQ be dropped to MN, and let it be bisected at R. I say that RQ is the altitude of one [pyramid] in the dodecahedron.

[Margin a: "2. coroll. 17. tertij dec." (2nd corollary to Euclid XIII.17). Margin b: "coroll. 8. quintidec." (corollary 8, bk. XV). Margin c: "schol. 12. quarti" (scholium, Euclid IV.12). Margin d: Euclid I.4.]

For, since (as we proved from Hypsicles at the end of Euclid) the angle CGE measures the inclination of one face to another: if MN be conceived as the perpendicular which is drawn in the lowest face from an angle of the pentagon to the midpoint of the opposite side, then NO will correspond to the perpendicular

which, in the pentagon inclined to that face, is drawn from the same midpoint to the opposite angle — because we set the angle MNO equal to the angle of inclination CGE, and the straight line NO equal to the perpendicular SD. The straight line OP, in turn, will represent the side of the dodecahedron set between the angle of the said inclined pentagon and the angle placed in the uppermost face; for the straight line OP was set equal to the side of the dodecahedron, and the angle NOP equal to the angle ELK — which indeed is equal to the angle that the said side makes with the perpendicular drawn from the angle of the aforesaid inclined pentagon to the base, at the midpoint of the opposite side. This is evident if one conceives a face of the cube inscribed in the dodecahedron as laid beneath the said side of the dodecahedron, so that two sides of the cube's face subtend two angles of two pentagons — one of which is inclined to the base of the dodecahedron, the other being the uppermost in the dodecahedron. For then the straight line CE will be equal to the line joining the two midpoints of the two said sides of the cube's face; while the straight lines EL and CK will correspond to lines drawn from those same midpoints of those sides of the cube's face to the angles of the aforesaid pentagons. And therefore the angle ELK will be equal to the angle which the perpendicular in the inclined pentagon makes with the said side of the dodecahedron. From this it follows that the point P lies in the plane of the uppermost face; and accordingly the perpendicular PQ, dropped to the plane of the base through MN, is equal to the altitude of the dodecahedron, and its half RQ is equal to the altitude of one pentagonal pyramid. All this will be readily understood if some material dodecahedron is used.

Finally, let a b be given as the side of an icosahedron, upon which an equilateral and equiangular pentagon a b c d e is erected as the base of the pyramid formed from five faces of the icosahedron. Draw the straight line c e, bisect the side a b at f, and draw the straight line f d, which (as we showed just now for the dodecahedron) will be perpendicular to both a b and c e. On the side of the icosahedron c d construct the equilateral triangle c d h as one face of the icosahedron; and the side c d being bisected at k, join the straight line h k, [e] which will be perpendicular to c d. Moreover, on c e construct the isosceles triangle c g e, each of whose sides c g and e g is equal to the perpendicular h k. After this, on f d construct the triangle f d l, whose side f l is made equal to the perpendicular h k, and whose side d l is made equal to the side a b of the icosahedron. Finally, let the angle m n o be made equal to the angle c g e, and the straight line n o equal to the perpendicular h k; likewise the angle n o p equal to the angle d l s, and the straight line o p equal to the side a b of the icosahedron. I say that the perpendicular p q, dropped to m n, is the altitude of the icosahedron, and its half r q the altitude of one [pyramid] in the icosahedron.

[Margin: Altitude of the icosahedron's pyramid. Margin e: scholium, Euclid I.26.
Margin f: Euclid I.2.]

For, since (as we proved from Hypsicles at the end of Euclid) the angle cge measures the inclination of one face to another: if mn be conceived as the perpendicular which is drawn in the lowest face of the icosahedron from an angle of the triangle to the midpoint of the opposite side, then no will correspond to the perpendicular which, in the triangle inclined to that face, is drawn from the same midpoint to the opposite angle — because we made the angle mno equal to the angle of inclination cge , and the straight line no equal to the perpendicular hk . The straight line op , in turn, will represent the side of the icosahedron set between the angle of the said inclined triangle and the angle placed in the uppermost face; for the straight line op was set equal to the side of the icosahedron, and the angle nop equal to the angle dls : [f] which indeed is equal to the angle that the said side makes with the perpendicular drawn from the angle of the aforesaid inclined triangle to the base, at the midpoint of the opposite side. For the straight line ds is equal to the perpendicular drawn from the angle of the pentagon to the opposite side, and the sides sl and dl are equal to the perpendicular in the inclined triangle and to the side of the icosahedron [set] between the uppermost angle of the aforesaid pentagon and the angle of the inclined triangle. From this it follows that the point p lies in the plane of the uppermost face; and accordingly the perpendicular pq , dropped to the plane of the base through mn , is equal to the altitude of the icosahedron, and its half rq is equal to the altitude of one triangular pyramid. All this will be readily grasped if some material icosahedron is used. Furthermore, once the altitude of the pyramid has been found in this way, it is to be measured with the same great care, by means of the graduated instrument, in the units of the side of the regular solid proposed.

Diagram. Construction for the altitude of a pentagonal pyramid of the dodecahedron. The main figure is a pentagon with apex at the top and an inner isosceles triangle and grid of perpendiculars; an auxiliary lineal construction stands to its right, resting on a base line $M-N-Q$. The interior labels overprint one another and are only partly legible.

- D (apex, top): vertex of the pentagon
- E (upper left): pentagon vertex
- C (upper right): pentagon vertex
- A (lower left): pentagon vertex
- B (lower right): pentagon vertex

- S (bottom middle): midpoint of AB
- G (below apex D): apex of the inner isosceles triangle CGE
- F (centre): foot on CE
- I, H (flanking F): feet of perpendiculars
- L, K (lower inner row): intersection points
- P (top of auxiliary figure): apex
- R (on PQ): midpoint of the perpendicular PQ
- O (near R, small circle): construction point
- M, N, Q (points on the base line of the auxiliary figure)

Diagram. Corresponding construction for the altitude of a triangular pyramid of the icosahedron, with lowercase labels. A pentagon with apex at top carries an inner triangle and grid; an auxiliary lineal construction stands beside it on a base line. Interior lowercase labels overprint heavily and are largely illegible.

- d (apex, top): pentagon vertex
- e (upper left): pentagon vertex
- c (upper right): pentagon vertex
- a, b (lower vertices): pentagon base
- k (upper right, interior): midpoint of c d
- g, f, h (interior): construction points
- s (lower interior): construction point [?]
- p (auxiliary apex), r (midpoint of p q), o (construction point)
- m, n, q (points on the base line)

ON THE AREA OF THE SPHERE, AND THE FINDING OF THE CONVEX SURFACE OF THE SAME SPHERE

CHAPTER V

That we may reach the area of the sphere — or its solidity — by several methods, some things very necessary and useful for the purpose must first be demonstrated, which we shall accomplish in the following seven propositions.

PROPOSITION I

Whatever ratio any two aliquot parts of any magnitude have, the same ratio is held by two like parts of any other magnitude.

Let A be the same part of magnitude B that C is of magnitude D; likewise let E be the same part of magnitude B that F is of magnitude D. I say that A is to E as C is to F. For since A is to B as C is to D (because in each case the same submultiple ratio is posited), and likewise B is to E as D is to F (because in each case the same multiple ratio is posited), it will follow, *ex aequo*, that A is to E as C is to F, which is what was proposed.

The same follows if A and C are the same several parts of B and D not making a unit; likewise if E and F are the same several parts of the same B and D not making a unit, such as $\frac{2}{3}$ or $\frac{3}{5}$, and so on. For if, say, A and C are $\frac{3}{4}$ of B and D, then, as $\frac{3}{4}$ of B — that is, A itself — is to B, so $\frac{3}{4}$ of D — that is, C itself — is to D. Again, if, say, E and F are $\frac{2}{3}$ of B and D, then as B is to $\frac{2}{3}$ of B, so D is to $\frac{2}{3}$ of the same D; [b] and therefore, as B is to $\frac{2}{3}$ (that is, to E), so D is to $\frac{2}{3}$ (that is, to F). Hence, as before, it will follow *ex aequo* that A is to E as C is to F.

[Margin a: scholium 22, Euclid V. Margin b: scholium 22, Euclid V.]

Diagram. A rectangle divided into two cells. The left cell is lettered A, E (above) and B (below); the right cell is lettered C, F (above) and D (below) — representing the magnitudes and their parts.

- Left cell: A, E, B
- Right cell: C, F, D

COROLLARY

It follows from this that $\frac{1}{4}$ of any magnitude is to $\frac{1}{3}$ of the same as $\frac{1}{2}$ of any other magnitude is to $\frac{2}{3}$ of the same. For, as we showed, $\frac{1}{4}$ of the first magnitude is to $\frac{1}{3}$ of the same as $\frac{1}{4}$ of the second is to $\frac{1}{3}$ of the same. [c] But as $\frac{1}{4}$ of the second is to $\frac{1}{3}$, so is $\frac{2}{4}$ to $\frac{2}{3}$ — that is, $\frac{1}{2}$ to $\frac{2}{3}$. Therefore $\frac{1}{4}$ of the first magnitude will be to $\frac{1}{3}$ of the same as $\frac{1}{2}$ of the second magnitude is to $\frac{2}{3}$ of the same.

[Margin c: Euclid V.1.]

PROPOSITION II

The rectangle contained by the diameter and the circumference of the greatest circle in a sphere is quadruple the greatest circle, and equal to the convex surface of the same sphere.

Let AB be the rectangle contained by the diameter AC and the circumference CB of the greatest circle in the sphere. I say that the rectangle AB is quadruple the greatest circle in the sphere, and equal to the convex surface of the same sphere. For when all the sides are bisected at E, F, G, H, and the straight lines EG and FH are joined cutting each other at I, the whole rectangle will be divided into four equal parts AI, CI, BI, DI, [d] since the straight lines EG and FH are parallel to AD and AC. And therefore the rectangle AB will be quadruple the rectangle CI. Now the rectangle CI is contained by CE, the semidiameter, and CF, the semicircumference — and it is equal to the greatest circle, whose diameter is of course AC, as was demonstrated in bk. 4, ch. 7, no. 1. Therefore the rectangle AB is quadruple the greatest circle. And since the convex surface of the sphere is quadruple the same greatest circle (by Archimedes, *On the Sphere and Cylinder* I, prop. 31), [e] the rectangle AB will be equal to the convex surface. Which was to be demonstrated.

[Margin d: Euclid I.33. Margin e: Euclid V.9.]

Diagram. A low rectangle divided into four equal cells by a horizontal and a vertical line meeting at the centre I. The corners and side-midpoints are lettered.

- A (top left corner), H (top middle), D (top right corner)
- E (left middle), I (centre), G (right middle)
- C (bottom left corner), F (bottom middle), B (bottom right corner)

COROLLARY

From the demonstration it is clear that the rectangle contained by the diameter of any circle (even if it is not the greatest in a sphere) and its circumference is quadruple that circle. For the same demonstration will always apply.

PROPOSITION III

The square of the circumference of the greatest circle in a sphere has to the surface of the sphere the same ratio that the circumference of the greatest circle has to the diameter. Likewise, the square of the diameter of the greatest circle in

a sphere has to the surface of the sphere the same ratio that the diameter has to the circumference of the same greatest circle.

Let ABCD be the greatest circle of the sphere, and let AC be its diameter. I say that the square described on the circumference ABCD is to the surface of the sphere whose diameter is AC as the circumference ABCD is to the diameter AC.

Likewise, [I say] that the square of the diameter AC of the greatest circle in the sphere is to the surface of the sphere as the diameter AC is to the circumference ABCD. For let EF be equal to the diameter AC, and the straight line FG equal to the circumference ABCD; on FG construct the square GH, and take FI equal to EF; then EI will be the square of the diameter EF, or AC. The figure being completed, as you see, both the rectangle GI — under the diameter FI of the greatest circle and the circumference FG — and the rectangle EH — under the diameter E, F of the same greatest circle and the circumference FH — will be equal, by the preceding [proposition], to the convex surface of the sphere. [a] Since, then, the square GH described on the circumference FG is to the rectangle EH, [which is] equal to the convex surface of the sphere, as the circumference GF is to the diameter EF of the greatest circle, the first [part] is established.

[b] Likewise, since the square EI of the diameter EF is to the rectangle IG, [which is] equal to the convex surface of the sphere, as the diameter EF of the greatest circle is to the circumference FG, that which is proposed in the second place is evident.

[Margin a: Euclid VI.1. Margin b: Euclid VI.1.]

Diagram. A large square (or rectangle) with corners E, F(?), G along the base and H at the top, divided by a vertical line H–F and a horizontal line through I; a separate small circle ABCD (the greatest circle) is drawn in the upper-right region with its horizontal diameter AC.

- H (top, on the top edge above I): upper end of the vertical line
- I (interior, on the vertical line): intersection point
- E (bottom left corner)
- F (bottom middle): foot of the vertical line H–F
- G (bottom right corner)
- Circle ABCD: A (left end of diameter), C (right end of diameter), B (top of circle), D (bottom of circle), with the centre marked by a dot

COROLLARY

Hence it is manifest (as we also demonstrated in bk. 4, ch. 7, no. 1) that the area of a circle is generated both from $1/4$ of the diameter into the whole circumference and from $1/4$ of the circumference into the whole diameter. For since the circle ABCD is [c] the fourth part of the rectangle GI (because this was shown to be quadruple that circle in proposition 2), and since the fourth part of the rectangle GI is contained both under $1/4$ of the diameter FI and the circumference FG, and under $1/4$ of the circumference FG and the diameter FI, what is proposed clearly stands.

[Margin: Area of the circle. Margin c: Euclid VI.1.]

PROPOSITION IV

The square of the circumference of the greatest circle in a sphere has to the convex surface of the sphere a greater ratio than 223 to 71, but a smaller one than 22 to 7.

For since, by the preceding [proposition], the square of the circumference of the greatest circle is to the convex surface of the sphere as the circumference of that circle is to the diameter, [a] and since the ratio of the circumference to the diameter is greater than $3 \frac{10}{71}$ to 1 — that is, than 223 to 71 — but smaller than $3 \frac{1}{7}$ to 1 — that is, than 22 to 7 — what was proposed is clear.

[Margin a: prop. 2, "On the Measurement of the Circle," bk. IV.]

PROPOSITION V

The square of the diameter of the greatest circle in a sphere has to the convex surface of the sphere a greater ratio than 7 to 22, but a smaller one than 71 to 223.

For since, by proposition 3, the square of the diameter is to the surface of the sphere as the diameter is to the circumference: ([b] since the ratio of the diameter to the circumference is smaller than 7 to 22), but greater than 71 to 223 (since the ratio of the circumference to the diameter is greater than 223 to 71), that which is proposed is evidently true.

[Margin a: Euclid V.26. Margin b: prop. 2, "On the Measurement of the Circle," bk. IV.]

PROPOSITION VI

The ratio of the cube described on the circumference of the greatest circle in a sphere to the sphere is greater than 298374 to 5041, but smaller than 2904 to 49.

For since, by bk. 4, ch. 7, prop. 1, the circumference of the greatest circle in any one sphere is to [the circumference] in any other sphere as diameter is to diameter, [a] and since the cube of the circumference of the first sphere has to the cube of the circumference of the second sphere the triplicate ratio of circumference to circumference, [b] and likewise the first sphere has to the second sphere the triplicate ratio of diameter to diameter: it will follow that as the cube described on the circumference of the first sphere is to the cube described on the circumference of the second sphere, so the first sphere is to the second sphere. And, by alternation, as the cube of the circumference of the first sphere is to the first sphere, so the cube of the circumference of the second sphere is to the second sphere.

[Margin a: Euclid XI.33. Margin b: Euclid XII.18.]

Now indeed, if the circumference of the greatest circle of some sphere be 1, and it be divided by $3 \frac{10}{71}$, there will be produced the diameter $\frac{71}{223}$, [which is] greater than the true one, by the corollary to prop. 2 of "On the Measurement of the Circle." If, then, its half $\frac{71}{446}$ [?] be multiplied into $\frac{1}{2}$ — the semicircumference — there will be generated the area of the greatest circle $\frac{71}{892}$, [which is] greater than the true one; and this, multiplied again into $\frac{2}{3}$ of the diameter found (which is likewise greater than the true one), namely into $\frac{142}{669}$ [?], will generate — as I shall show below in Rule 2 — the solidity of the sphere $\frac{10082}{596748}$, that is, $\frac{5041}{298374}$, [which is] nevertheless greater than the true one. [c] Therefore the ratio of the cube described on the circumference 1 (which cube is 1) to the sphere will be greater than [the ratio of 1] to $\frac{5041}{298374}$. Since, then, 1 is to $\frac{5041}{298374}$ as 298374 is to 5041 (for, by proposition 2 of the *Minutiae* at the end of Euclid book 9, the ratio of the numerator 5041 to the denominator 298374 is the same which the fraction $\frac{5041}{298374}$ has to its own whole 1; hence, by conversion, as 298374 is to 5041, so 1 is to $\frac{5041}{298374}$) — the ratio of the cube described on the circumference 1 (which cube is 1) to the sphere will be greater than 298374 to 5041. Since, therefore, the cube described on the circumference of the greatest circle in any sphere stands to the sphere as the cube on the circumference 1 stands to its own sphere (as we showed at the beginning of this proposition), that which is proposed in the first place is established.

[Margin c: Euclid V.8.]

(Continuing the demonstration begun on the preceding page.)

Again, if the circumference 1 is divided by $3 \frac{1}{7}$, the resulting diameter is $\frac{7}{22}$, which is less than the true value, by the corollary to Proposition 2 of *On the Measurement of the Circle*. If, then, its half, $\frac{7}{44}$, is multiplied by $\frac{1}{2}$ — the half of the circumference — the product is the area of the greatest circle, $\frac{7}{88}$, less than the true area. This area, in turn, multiplied by $\frac{2}{3}$ of the diameter just found (which is likewise less than the true diameter) — namely by $\frac{14}{66}$, that is, by $\frac{7}{33}$ — yields, as I shall show below in Rule 2, the solidity of the sphere as $\frac{49}{2904}$, which is nonetheless less than the true value. [Margin: (d) Elements V.8] The ratio of the cube derived from the circumference 1 (that cube being 1) to the sphere will therefore be smaller than the ratio to $\frac{49}{2904}$. And since 1 is to $\frac{49}{2904}$ as 2904 is to 49 — for by Proposition 2 of the *Minutiae* at the end of Euclid's Book 9 the numerator 49 stands to the denominator 2904 in the same ratio as the fraction $\frac{49}{2904}$ bears to its whole 1, so that inversely it is as 2904 to 49, and thus 1 to $\frac{49}{2904}$ — the ratio of the cube described from the circumference 1 (again, that cube being 1) to the sphere will be smaller than 2904 to 49. Since, therefore, the cube described from the circumference of the greatest circle of any sphere stands to the sphere just as the cube of circumference 1 stands to its own sphere — as we demonstrated at the beginning of this proposition — what was proposed in the second place is also evident.

PROPOSITION VII.

The cube of the diameter of a sphere has to the sphere a greater ratio than 21 to 11, but a smaller ratio than 426 to 223.

For since the cube of the diameter of any sphere is to the cube of the diameter of another sphere as the first sphere is to the second [Margin: (a) Elements XI.33 & XII.18] — because each ratio is the triplicate of the ratio of the diameters — it follows, by alternation, that the cube of any sphere's diameter is to that same sphere as the cube of the other sphere's diameter is to that other sphere.

Now suppose the diameter of some sphere is set at 1 and is multiplied by $3 \frac{1}{7}$. [Margin: (b) Archimedes, *Measurement of the Circle*, coroll. 2] This yields the circumference of the greatest circle as $\frac{22}{7}$, greater than the true value. Its half, $\frac{11}{7}$, multiplied by $\frac{1}{2}$ — the half of the diameter — gives $\frac{11}{14}$ as the area of that greatest circle, greater than the true area. And so, if this area (greater than the true) is multiplied by $\frac{2}{3}$ of the diameter, there is generated — as will be stated below in Rule 2 — the solidity of the sphere as $\frac{22}{42}$, that is, $\frac{11}{21}$, greater than the true value. [Margin: (c) Elements V.8] The cube of diameter 1 (which is 1) will

therefore have to the sphere a ratio greater than to 11/21. And since 1 is to 11/21 as 21 to 11 — for by Proposition 2 of the *Minutiae* at the end of Euclid's Book 9 the numerator 11 stands to the denominator 21 in the same ratio as the fraction 11/21 to its whole 1, so that inversely it is as 21 to 11, and thus 1 to 11/21 — the ratio of the cube (1) described from diameter 1 to the sphere will be greater than 21 to 11. And since, as we showed at the outset of this proposition, the cube of any other sphere's diameter stands to that sphere as the cube of diameter 1 to its own sphere, what was proposed in the first place is true.

[Margin: (d) Archimedes, *Measurement of the Circle*, coroll. 2] Likewise, if the diameter 1 is multiplied by $3 \frac{10}{71}$, the circumference of the greatest circle comes out as $\frac{223}{71}$, less than the true value. Its half, then, $\frac{223}{142}$, multiplied by $\frac{1}{2}$ — the half of the diameter 1 — makes $\frac{223}{284}$ as the area of the greatest circle, less than the true area. And so, if this is multiplied by $\frac{2}{3}$ of the diameter 1, there is produced — as will be stated below in Rule 2 — the solidity of the sphere as $\frac{446}{852}$, that is, $\frac{223}{426}$, less than the true value. [Margin: (e) Elements V.8] The cube (1) of diameter 1 will therefore have to the sphere a ratio smaller than to $\frac{223}{426}$. And since 1 is to $\frac{223}{426}$ as 426 to 223 — for by Proposition 2 of the *Minutiae* at the end of Euclid's Book 9 the numerator 223 stands to the denominator 426 in the same ratio as the fraction $\frac{223}{426}$ to its whole 1, so that inversely it is as 426 to 223, and thus 1 to $\frac{223}{426}$ — the ratio of the cube of diameter 1 to its own sphere will be smaller than 426 to 223. And since, as we showed at the beginning of this proposition, the cube of any other sphere's diameter stands to that sphere as the cube of diameter 1 to its own sphere, what was proposed in the second place is likewise clear.

2. With these things established, there follow rules for investigating both the convex surface of any sphere and its solidity.

I. To find the convex surface of a given sphere.

[Margin: The convex surface of the sphere] Quadruple the area of the greatest circle of the given sphere; the resulting number will exhibit the convex surface of the sphere — because, by Proposition 31 of Book 1 of Archimedes' *On the Sphere and Cylinder*, the surface of the sphere is four times the greatest circle.

[Margin: The solidity of the sphere (*so printed; the paragraph in fact concerns the surface — apparent compositor mislabel*)] The same surface is produced if the diameter of the sphere is multiplied by the circumference of the greatest circle — because, by Proposition 2, No. 2, of this chapter, the rectangle contained by the

diameter and the circumference of the greatest circle is equal to the convex surface of the sphere.

II. To determine the solidity of a given sphere.

1. The solidity of a sphere is produced from its semidiameter multiplied by one-third of the convex surface; or from $1/4$ of the whole diameter multiplied by $2/3$ of the convex surface.
2. Likewise, from two-thirds of the diameter multiplied by the area of the greatest circle.
3. Or from two-thirds of the area of the greatest circle multiplied by the whole diameter.
4. Or from the semidiameter multiplied by four-thirds of the area of the greatest circle.
5. Or from half the area of the greatest circle multiplied by four-thirds of the diameter.
6. Or from twice the diameter multiplied by one-third of the area of the greatest circle.
7. Or from the diameter multiplied by one-sixth of the surface of the sphere.
8. Or, finally, from one-third of the diameter multiplied by half the convex surface of the sphere.

[Margin: Demonstration of the first part] We demonstrated the first method in our commentary on the *Sphere*, and we shall repeat that demonstration in Book 7 on Isoperimetrics; nevertheless we shall also prove it here by the following route. Conceive a cone whose base is the greatest circle of the sphere and whose height is the semidiameter of that same sphere. Conceive also another cone whose base is four times the greatest circle and whose height is the same semidiameter. Now the sphere — by Proposition 32 of Book 1 of Archimedes' *On the Sphere and Cylinder* — is four times the first cone, just as the latter cone is [Margin: (a) Elements XII.11]; [Margin: (b) Elements V.9] and so the latter cone and the sphere are equal to each other.

Again, the circle whose semidiameter equals the whole diameter of the sphere is four times the greatest circle [Margin: (c) Elements XII.2] — for since circle is to circle as the square of one diameter is to the square of the other, and the square of the first diameter is four times the square of the second (by the scholium to Proposition 4 of Book 2 of Euclid, since that first diameter is double the second, because half the first diameter was taken equal to the second), the one circle will also be four times the other. Hence the circle whose semidiameter equals the

diameter of the sphere will be equal to the base of the latter cone, since that base too was set at four times the greatest circle. And since the surface of the sphere is likewise four times the greatest circle, by Proposition 31 of Book 1 of Archimedes' *On the Sphere and Cylinder*: [Margin: (d) Elements V.9] the surface of the sphere, the base of the latter cone, and the circle whose semidiameter equals the diameter of the sphere will all be equal to one another.

Finally, conceive a cylinder whose base is the aforesaid circle — the one whose semidiameter equals the diameter of the sphere — and whose height is the semidiameter of the sphere. [Margin: (e) Elements XII.10] This cylinder will be three times the aforementioned latter cone, and therefore also three times the sphere, which was shown equal to that cone. [Margin: (f) Elements XII.11] But the same cylinder is likewise three times the cylinder that has the same height and a base equal to one-third of that cylinder's base — that is, equal to one-third of the sphere's surface. [Margin: (g) Elements V.9] Therefore the latter cylinder (having a base equal to one-third of the sphere's surface and a height equal to the semidiameter of that same sphere) and the sphere are equal. Since, then, this latter cylinder is contained by the semidiameter of the sphere and one-third of the spherical surface, it plainly follows that the sphere's solidity is generated from the semidiameter multiplied by one-third of the sphere's surface — or from $1/4$ of the whole diameter multiplied by $2/3$ of the sphere's surface, since this number equals that. And that is the first method.

[Margin: Demonstration of the second part] Conceive again a cylinder whose base is the greatest circle of the sphere and whose height is the diameter of the sphere. This cylinder will be one and a half times the sphere, by the corollary to Proposition 32 of Book 1 of Archimedes' *On the Sphere and Cylinder*. Now if, from the upper part, a plane parallel to the cylinder's bases is drawn through one-third of the sphere's diameter (that is, of the cylinder's axis): [Margin: (h) Elements XII.13] the whole cylinder will be one and a half times the cut-off cylinder whose axis is two-thirds of the whole axis; [Margin: (i) Elements V.9] and therefore this latter, cut-off cylinder — which is contained by its own base, the greatest circle, and by two-thirds of the sphere's diameter — will be equal to the sphere. So the second method too is evident.

Otherwise.

Diagram. Two rectangular parallelepipeds (oblong boxes) drawn in oblique projection, one above the other, illustrating the two solids ABC and DEF of the proof.

- Upper box (parallelepiped ABC): **C** (upper-left, top face), **A** (lower-left, front face), **B** (lower-right, front face).
- Lower box (parallelepiped DEF): **F** (left, upper edge), **D** (lower-left, front face), **E** (lower-right, front face — printed glyph resembles "K," but the surrounding text names the solid "DEF," so the vertex is E).

[Margin: (k) Elements XI.34] Let ABC be a parallelepiped contained by AC — two-thirds of the sphere's diameter — and by the base AB, which is equal to the greatest circle of that same sphere. I say that the parallelepiped ABC is equal to the sphere. For let there be another parallelepiped DEF, contained by DF — the semidiameter of the sphere — and by the base DE, which is equal to one-third of the sphere's surface; this, as we demonstrated in the first part of this Rule 2, will be equal to the given sphere. Now since the base AB, being equal to the greatest circle of the sphere, is $\frac{1}{4}$ of the sphere's surface, it follows — by the corollary to Proposition 1 of this chapter — that as AB (that is, $\frac{1}{4}$ of the sphere's surface) is to DE (that is, to $\frac{1}{3}$ of that same surface), so is DF (that is, $\frac{1}{2}$ of the sphere's diameter) to AC (that is, to $\frac{2}{3}$ of that same diameter). And so, since the bases AB, DE are reciprocally proportional to the heights DF, AC, the parallelepipeds ABC and DEF are equal to each other. Since, then, DEF is equal to the sphere, as was said, ABC too is equal to the same sphere — which was to be proved.

Otherwise.

[Margin: (l) Elements VII.19] Since, by the corollary to Proposition 1 of this chapter, as $\frac{1}{4}$ of the sphere's surface is to $\frac{1}{3}$ of that same surface, so is $\frac{1}{2}$ of the diameter to $\frac{2}{3}$ of that same diameter, the same number will be produced from the first term (namely $\frac{1}{4}$ of the surface, that is, the greatest circle of the sphere) multiplied by the fourth (namely $\frac{2}{3}$ of the diameter) as from the second (that is, $\frac{1}{3}$ of the surface) multiplied by the third (that is, $\frac{1}{2}$ of the diameter). But from $\frac{1}{3}$ of the surface multiplied by $\frac{1}{2}$ of the diameter the solidity of the sphere is produced, as was shown in the first part of this Rule 2. Therefore the same solidity will be generated from the greatest circle multiplied by $\frac{2}{3}$ of the diameter — which is what was proposed.

[Margin: Demonstration of the third part] Again, the cylinder whose base is the greatest circle of the sphere and whose height is the diameter of that same sphere is one and a half times the sphere itself, by the corollary to Proposition 32 of Book 1 of Archimedes' *On the Sphere and Cylinder*; [Margin: (m) Elements XII.11] and this same cylinder is likewise one and a half times the cylinder whose base is equal to two-thirds of the greatest circle and whose height is the same diameter. [Margin: (n) Elements V.9] Therefore this latter cylinder and the sphere are equal

— that is, the sphere is produced from $\frac{2}{3}$ of the area of the greatest circle multiplied by the diameter of the sphere. And that is the third method.

[Margin: Demonstration of the fourth part] Conceive also two parallelepipeds: one having a base equal to $\frac{2}{3}$ of the area of the greatest circle in the sphere and a height equal to the whole diameter; the other having a base equal to $\frac{4}{3}$ of the area of the greatest circle and a height equal to the semidiameter. Since the bases of these parallelepipeds are reciprocally proportional to their heights — for the base of the first is half the base of the second, just as the height of the second is half the height of the first — [Margin: (o) Elements XI.34] the parallelepipeds are equal. But the first, by the third part of this Rule 2, is equal to the solidity of the sphere; therefore so is the second. Accordingly the sphere is produced from the semidiameter multiplied by $\frac{4}{3}$ of the area of the greatest circle — which is proposed in the fourth place.

[Margin: Demonstration of the fifth part] Furthermore, conceive two parallelepipeds: one having a base equal to the area of the greatest circle in the sphere and a height equal to $\frac{2}{3}$ of the diameter; the other having a base equal to $\frac{1}{2}$ of the area of the greatest circle and a height equal to $\frac{4}{3}$ of the diameter. Since the bases of these parallelepipeds are reciprocally proportional to their heights — for the base in the first is double the base in the second,

just as the height in the second is double the height in the first — [Margin: (a) Elements XI.34] the parallelepipeds will be equal. But the first, by the second part of this Rule 2, is equal to the sphere; therefore so is the second. And on that account the sphere is generated from $\frac{1}{2}$ of the area of the greatest circle multiplied by $\frac{4}{3}$ of the diameter. That is the fifth method.

[Margin: Demonstration of the sixth part] Likewise, understand two parallelepipeds: one having a base equal to $\frac{2}{3}$ of the area of the greatest circle in the sphere and a height equal to the diameter; the other having a base equal to $\frac{1}{3}$ of the area of the greatest circle and a height equal to twice the diameter. Since the bases of these parallelepipeds are reciprocally proportional to their heights — for the base in the first is double the base in the second, just as the height in the second is double the height in the first — [Margin: (b) Elements XI.34 *(the printed marginal digit reads "44," but Book 11 has only 39 propositions and the reciprocation theorem cited throughout is XI.34)*] the parallelepipeds are equal. But the first, by the third part of this Rule 2, is equal to the sphere itself; therefore so is the second. And accordingly the sphere is produced from twice the diameter multiplied by $\frac{1}{3}$ of the area of the greatest circle — which is what was proposed in the sixth place.

[Margin: Demonstration of the seventh part] Understand also two parallelepipeds: one whose base contains $\frac{1}{3}$ of the sphere's surface and whose height is $\frac{1}{2}$ of the diameter; the other whose base comprises $\frac{1}{6}$ of the surface and whose height is equal to the diameter. And since the bases are reciprocal to the heights — for $\frac{1}{3}$ of the surface is to $\frac{1}{6}$ of the surface (that is, the base of the first to the base of the second) as the height of the second, namely the diameter, is to the height of the first, namely $\frac{1}{2}$ of the diameter, both ratios being double — [Margin: (c) Elements XI.34] the parallelepipeds will be equal. But the first is equal to the sphere itself, by the first part of this Rule 2. Therefore so is the second — that is, the sphere's solidity is produced from the diameter multiplied by one-sixth of the surface. And that is the seventh method.

[Margin: Demonstration of the eighth part] Finally, conceive two parallelepipeds: one whose base is $\frac{1}{3}$ of the sphere's surface and whose height is the semidiameter; the other whose base is $\frac{1}{2}$ of the surface and whose height is $\frac{1}{3}$ of the diameter. Since the bases and the heights are reciprocally proportional — for $\frac{1}{3}$ of the surface is to $\frac{1}{2}$ of the surface (namely the base of the first parallelepiped to the base of the second) as $\frac{1}{3}$ of the diameter is to $\frac{1}{2}$ of the diameter (that is, the height of the second parallelepiped to the height of the first) — [Margin: (d) Elements XI.34] the parallelepipeds will be equal. Since, then, by the first part of this Rule 2 the first is equal to the sphere, the second too will be equal to it. And therefore the sphere's solidity is produced from one-third of the diameter multiplied by half the convex surface — which is the eighth method.

3. Now, from Propositions 4 and 5 of this chapter (No. 1), the following four rules are gathered, by which the convex surface of the sphere is found — sometimes greater than the true value, sometimes less — from both the circumference and the diameter of the greatest circle.

I. To produce, from the circumference of the greatest circle in the sphere, a convex surface of the sphere greater than the true one.

[Margin: The surface of the sphere greater than the true] Make it so that as 223 is to 71, so the square described on the given circumference of the greatest circle is to another number; and there will result a surface of the sphere greater than the true. For since, by Proposition 4 (No. 1) of this chapter, the ratio of the square of the greatest circle's circumference to the surface of the sphere is greater than 223 to 71, while the square of the given circumference is to the produced number as 223 to 71, the square of the given circumference will also have to the true surface of the sphere a greater ratio than that same square has to the produced number.

[Margin: (a) Elements V.10] The produced number will therefore be greater than the true surface.

II. To elicit, from the circumference of the greatest circle in the sphere, a convex surface of the sphere less than the true one.

[Margin: The surface of the sphere less than the true] Make it so that as 22 is to 7 [printed "22.ap. 7"; "ap." appears to be a misprint for "ad"], so the square described on the given circumference of the greatest circle is to another. For the number so generated will be less than the true surface of the sphere. Since, by Proposition 4 (No. 1) of this chapter, the ratio of the square of the greatest circle's circumference to the surface of the sphere is less than 22 to 7, while the square of the given circumference is to the generated number as 22 to 7, the square of the given circumference will also have to the true surface of the sphere a smaller ratio than that same square has to the produced number. [Margin: (a) Elements V.10] For that reason the produced number will be less than the true surface of the sphere.

III. To draw out, from the diameter of the greatest circle in the sphere, a surface of the sphere greater than the true one.

[Margin: The surface of the sphere greater than the true] Make it so that as 7 is to 22, so the square of the given diameter of the greatest circle is to another; and there will be generated a surface of the sphere greater than the true. For since, by Proposition 5 (No. 1) of this chapter, the ratio of the square of the greatest circle's diameter to the true surface of the sphere is greater than 7 to 22, while the square of the given diameter is to the produced number as 7 to 22, the square of the given diameter will also have to the true surface of the sphere a greater ratio than that same square has to the produced number. [Margin: (a) Elements V.10] The number produced will therefore be greater than the true surface of the sphere.

IV. To gather, from the diameter of the greatest circle in the sphere, a surface of the sphere less than the true one.

[Margin: The surface of the sphere less than the true] Make it so that as 71 is to 223, so the square of the given diameter of the greatest circle is to another. For the number so generated will exhibit a surface of the sphere less than the true. Since, by Proposition 5 (No. 1) of this chapter, the ratio of the square of the diameter to the surface of the sphere is less than 71 to 223, while the square of the given diameter is to the produced number as 71 to 223, the ratio of the square

of the given diameter to the true surface of the sphere will also be smaller than the ratio of that same square to the produced number. [Margin: (a) Elements V.10] The produced number will therefore be less than the true surface of the sphere.

4. By the same reasoning, from Propositions 6 and 7 of this chapter, Number 1, four further rules are drawn, by which — from the circumference just as much as from the diameter of the greatest circle in the sphere — the solidity of the sphere is obtained, both greater than the true value and smaller.

I.

To produce, from the circumference of the greatest circle in the sphere, a solidity of the sphere greater than the true one.

[Margin: Solidity of the sphere greater than the true.] [Margin: (a) Euclid, *Elements* V.10.]

Let the cube described from the given circumference of the greatest circle be to another number as 298374 is to 5041. The number so generated will give a solidity of the sphere greater than the true one. For since, by Proposition 6 of this chapter, Number 1, the ratio of the cube of the given circumference to the sphere is greater than 298374 to 5041, while the cube of the given circumference is to the number produced as 298374 to 5041, the cube of the given circumference will also have a greater ratio to the true solidity of the sphere than the same cube has to the produced number.^a Therefore the produced number will be greater than the true solidity of the sphere.

II.

To generate, from the circumference of the greatest circle in the sphere, a solidity of the sphere smaller than the true one.

[Margin: Solidity of the sphere smaller than the true.] [Margin: (a) Euclid, *Elements* V.10.]

Let the cube of the given circumference be to another number as 2904 is to 49. For the number produced will present a solidity of the sphere smaller than the true one. For since, by Proposition 6 of this chapter, Number 1, the ratio of the cube of the given circumference to the sphere is smaller than 2904 to 49, while the cube of the given circumference is to the number generated as 2940 to 49

[recte: 2904 — the ratio must match the 2904:49 used in the same argument; see QA], the cube of the given circumference will also have a smaller ratio to the true solidity of the sphere than the same cube has to the number produced.^a And so the number produced will be smaller than the true solidity of the sphere.

III.

To gather, from the diameter of the greatest circle in the sphere, a solidity of the sphere greater than the true one.

[Margin: Solidity of the sphere greater than the true.] [Margin: (a) Euclid, *Elements* V.10.]

Let the cube of the given diameter be to another number as 21 is to 11. For the number so generated will give a solidity of the sphere greater than the true one. For since, by Proposition 7 of this chapter, Number 1, the ratio of the cube of the diameter to the sphere is greater than 21 to 11, while the cube of the given diameter is to the number produced as 21 to 11, the cube of the given diameter will also have a greater ratio to the true solidity of the sphere than the same cube has to the number generated.^a Therefore the number generated will be greater than the true solidity of the sphere.

IV.

To conclude, from the diameter of the greatest circle in the sphere, a solidity of the sphere smaller than the true one.

[Margin: Solidity of the sphere smaller than the true.] [Margin: (a) Euclid, *Elements* V.10.]

Let the cube of the given diameter be to another number as 426 is to 223. For the number arising will be smaller than the true solidity of the sphere. For since, by Proposition 7 of this chapter, Number 1, the ratio of the cube of the given diameter to the sphere is smaller than 426 to 223, while the cube of the given diameter is to the number generated as 426 to 223, the cube of the given diameter will also have a smaller ratio to the sphere than the same cube has to the number generated.^a Therefore the number produced will be smaller than the true solidity of the sphere.

ON THE SURFACE OF THE SEGMENTS OF THE SPHERE.

Chapter VI.

[Margin: Convex surface of a hemisphere.]

The convex surface of a hemisphere, excluding its base, is generated from the area of the greatest circle multiplied by 2; or from the semidiameter multiplied into the circumference of the greatest circle; or, finally, from the whole diameter multiplied into half the circumference of the greatest circle. All of this is clear from Rule 1, Number 2, of Chapter 5, because these products are the halves of those which, in that rule, yielded the convex surface of the whole sphere.

[Margin: Convex surface of a portion of the sphere.]

2. The convex surface of any portion of a sphere — whether greater or smaller than a hemisphere, with its base excluded — is equal to a circle whose semidiameter equals the straight line drawn from the vertex of the portion to the circumference of the base; this follows from Proposition 40 of Book 1 of Archimedes' [Archimedes] *On the Sphere and Cylinder*. For let ABCD be the greatest circle in the sphere, whose diameter AC is cut at right angles at E by the straight line BD. Through BD understand a plane to be drawn at right angles to the diameter, cutting the sphere into two portions whose common base is the circle of diameter BD; A is the vertex of the smaller portion, and C the vertex of the larger. When the straight lines AB and CB have been joined, the circle of semidiameter AB will be equal to the convex surface of the smaller portion, and the circle of semidiameter CB equal to the convex surface of the larger portion, by the cited proposition of Archimedes. Therefore, if both AB and CB are made known in parts of the diameter AC — especially with the aid of the parts-instrument constructed in Chapter 1 of Book 1 — and if the areas of the circles described at the intervals of the semidiameters AB and CB are investigated by what we wrote in Book 4, Chapter 7, then the convex surfaces of the said portions of the sphere will be known.

Diagram. A great circle of the sphere, labelled ABCD, shown in cross-section with its inscribed construction lines. The vertical diameter AC runs from A at the top to C at the bottom; the horizontal chord BD (the base of the two portions) crosses it at E; a second, lower horizontal chord GH runs from G on the left to H on the

right; F is the center of the sphere, on the axis below E. Chords are drawn from A and C to the endpoints of the chords, forming the portions, the hemisphere, and the frustum discussed in the text.

- A (top): vertex of the smaller portion; upper end of diameter AC
- B (upper left): left end of the upper chord BD
- D (upper right): right end of the upper chord BD
- E (center, upper): intersection of AC and BD
- G (left): left end of the lower chord GH
- H (right): right end of the lower chord GH
- F (center): center of the sphere, on the axis AC
- L (lower right): point of the frustum construction, below H
- K (lower left): point of the frustum construction
- C (bottom): vertex of the larger portion; lower end of diameter AC
- I (bottom, right of C): point of the portion/frustum construction

This same convex surface of a portion of a sphere — whether smaller or greater than a hemisphere — can also be found in the following way. From what Archimedes demonstrated in Proposition 3 of Book 2 of *On the Sphere and Cylinder*, EC has the same ratio to EA as the convex surface of the portion of the sphere whose base is the circle of diameter BD and whose vertex is C has to the convex surface of the portion whose base is the same circle of diameter BD and whose vertex is A. Therefore, by composition, the whole diameter AC will be to AE as the convex surface of the whole sphere is to the convex surface of the portion BAD. And by the same reasoning, the whole diameter AC will be to EC as the convex surface of the whole sphere is to the convex surface of the portion BCD. Consequently, once the ratio of the diameter AC to the segments AE and EC has been investigated by means of the parts-instrument constructed in Chapter 1 of Book 1: if it is arranged that, as the diameter AC is to AE, so the convex surface of the whole sphere (which becomes known from Rule 1, Number 2, of Chapter 5 of this book) is to another quantity, the convex surface of the smaller portion BAD will result. In the same way the convex surface of the larger portion BCD will be found, if it is arranged that, as the diameter AC is to EC, so the convex surface of the whole sphere is to another quantity.

And since, as we showed from Archimedes, the diameter AC is to AE (or to EC) as the whole surface of the sphere is to the surface of the portion BAD (or BCD), it will likewise hold that AF, half the diameter, is to AE (or EC) as the surface of the hemisphere GAH is to the surface of the portion BAD (or BCD) — which is shown

in the same way that the scholium to Proposition 22 of Book 5 of Euclid was demonstrated. If it is arranged that, as the semidiameter of the sphere AF is to AE (or EC), the altitude of the portion, so the surface of the hemisphere GAH is to another quantity, then again the convex surface of the smaller portion BAD (or of the larger BCD) will be produced.

[Margin: (a) Corollary 19 of the fifth [book of Euclid].]

Indeed, since AF, the semidiameter, is to AE as the surface of the hemisphere GAH is to the surface of the portion BAD, then by conversion of the ratio, AF, the semidiameter, will be to EF as the surface of the hemisphere GAH is to the surface of the frustum GBDH with its bases removed. Therefore EF is the same part — or the same parts — of the diameter AC (or of the semidiameter AF) that the surface of the frustum GBDH, its bases removed, is of the surface of the whole sphere (or of the hemisphere GAH). Hence, once it is known what part (or parts) of the semidiameter AF the line EF is: if that same part (or parts) is taken away from the surface of the hemisphere GAH, the remainder will become the convex surface of the smaller portion BAD; and if that same part (or parts) is added to the surface of the hemisphere GCH, the convex surface of the larger portion BCD will be built up. For example, if EF contains two-thirds [?] of the semidiameter AF, and two-thirds [?] are taken away from the surface of the hemisphere GAH, the remainder will become the convex surface of the smaller portion BAD; and if two-thirds [?] of the surface of the hemisphere are added to the surface of the hemisphere GCH, the convex surface of the larger portion BCD will be completed. Likewise, if EF were half of the semidiameter, half of the hemisphere's surface would have to be taken away or added; and so on for the rest.

[Margin: Solidity of a hemisphere.]

3. The solidity of a hemisphere is produced from the semidiameter multiplied into one-third of the surface of the hemisphere:
 - Or into one-sixth of the surface of the whole sphere.
 - Or from $\frac{1}{4}$ of the whole diameter into two-thirds [?] of the surface of the hemisphere.
 - Likewise, from two-thirds of the diameter into half the area of the greatest circle.
 - Or from two-thirds of the area of the greatest circle into the semidiameter; or from one-third of the area of the greatest circle into the whole diameter.
 - Or from $\frac{1}{4}$ [recte: $\frac{1}{2}$] of the whole diameter into two-thirds of the area of the greatest circle.

- Or from half the area of the greatest circle into two-thirds of the diameter.
- Or from twice the diameter into $\frac{1}{6}$ of the area of the greatest circle.
- Or from the semidiameter into one-sixth of the surface of the sphere.
- Or, finally, from $\frac{1}{6}$ of the diameter into the convex surface of the hemisphere.

All these are gathered from Rule 2, Number 2, of Chapter 5, since all these products are the halves of those which, in that rule, indicate the solidity of the whole sphere.

[Margin: Solidity of a sector of the sphere.]

4. The solidity of a sector of a sphere will be investigated as follows. (A spherical sector, of course, is composed of a smaller portion of the sphere together with a cone having the same base as the portion and a height equal to the perpendicular drawn from the center down onto the base of the portion; or else it is what remains when that same cone is subtracted from a larger portion, as in the next figure. Thus the solid composed of the spherical portion BAD — whose base is the circle of diameter BD — together with the cone having the same base and its vertex at the center F; and likewise the solid that remains when that same cone is taken away from the larger portion BCD: each of these we call a spherical sector.) For by Proposition 42 of Book 1 of Archimedes' *On the Sphere and Cylinder*, a spherical sector is equal to a cone whose base is a circle equal to the convex surface of the spherical portion, and whose height is equal to the semidiameter of the sphere. Now a cone is produced — as we explained in Chapter 2, Number 1, of this book — either from its base multiplied into one-third of its height, or from its whole height multiplied into one-third of its base. So a spherical sector is generated either from the convex surface of the spherical portion into one-third of the semidiameter — that is, into one-sixth of the whole diameter — or from the semidiameter into one-third of the convex surface of the spherical portion.

[Margin: Solidity of any portion of the sphere.]

5. The solidity of any portion of a sphere whatever will be generated in this way. Let the solidity of the spherical sector be investigated, as just explained. For if, when the proposed portion is smaller than a hemisphere, there is taken away from this sector a cone having the same base as the portion and, for its height, the perpendicular dropped from the center of the sphere onto the base of the portion, the remainder will become the

solidity of the smaller portion. But if, when the proposed portion is greater than a hemisphere, that same cone is added to the sector, the solidity of the larger portion will be built up. This is evident in the figure above: the cone BFD, removed from the sector ABFDA, leaves the smaller portion BAD as remainder; while the same cone BFD, added to the sector CBFDC, constitutes the larger portion BCD. The aforesaid cone BFD, moreover, becomes known from its base — namely from the circle of diameter BD — and from its height EF, once these are known, as we taught in Chapter 2 of this book.

Diagram. The same great-circle cross-section of the sphere as on the preceding page, repeated here: the circle ABCD with vertical diameter AC (A at top, C at bottom), the upper horizontal chord BD crossing at E, the lower horizontal chord GH (G left, H right), and F the center of the sphere on the axis. The construction lines mark the portions BAD and BCD, the hemisphere GAH, and the frustum GBDH.

- A (top): vertex of the smaller portion
- B (upper left): left end of the upper chord BD
- D (upper right): right end of the upper chord BD
- E (center, upper): intersection of AC and BD
- G (left): left end of the lower chord GH
- H (right): right end of the lower chord GH
- F (center): center of the sphere, on the axis AC
- L (lower right): point of the frustum construction
- K (lower left): point of the frustum construction
- C (bottom): vertex of the larger portion
- I (bottom, right of C): point of the frustum construction

ANOTHER METHOD.

Let ABCD be the greatest circle in the sphere, and let there be portions of the sphere whose common base is the circle of diameter BD and whose vertices are A and C, and whose solidities are to be found. From the center H, drop the perpendicular HE onto BD;^b this

[Margin: (c) Scholium to Euclid III.27. (d) Euclid VI.12.]

will cut the straight line BD in half,^c and consequently will also pass through each of the arcs BAD and BCD in half — that is, through the vertices A and C.^d Let it be

arranged that, as CE is to the sum of the straight lines CH and CE, so AE is to EF; and likewise, as AE is to the sum of the straight lines AH and AE, so EC is to EG. Understand two cones whose common base is the circle of diameter BD and whose vertices are F and G. Then, by Proposition 2 of Book 2 of Archimedes' *On the Sphere and Cylinder*, the cone BFD will be equal to the smaller portion BAD, and the cone BGD equal to the larger portion BCD. Consequently, once the solidities of these cones have been found — as was taught in Chapter 2, Number 1, of this book — the solidities of the portions BAD and BCD will also have been found, which is what was proposed.

Diagram. A lens-shaped figure showing two cones on a common circular base. The base circle has the vertical diameter BD (B at the top, D at the bottom) with center H; E and C lie on the horizontal axis inside the circle. Straight lines converge from the base circle to an external vertex G on the left (the cone BGD) and to an external vertex F on the right (the cone BFD), representing the two cones equal in volume to the two spherical portions.

- G (far left): vertex of the left cone (cone BGD)
 - F (far right): vertex of the right cone (cone BFD)
 - B (top of base circle): upper end of the base diameter BD
 - D (bottom of base circle): lower end of the base diameter BD
 - H (center): center of the base circle
 - E (on axis, near center): point of the construction
 - C (on axis, left of center): point of the construction
6. Finally, the solidity of any frustum of a sphere — whether its bases are parallel (such as, in the first figure of this chapter, the frustum BDHG enclosed between the circles of diameters BD and GH), or not parallel (such as the frustum BDLK) — will be found in this way. Investigate, as we said in Number 5, the solidity of each of the two portions ABD and AGH. When the smaller is subtracted from the larger, the remainder will be the solidity of the frustum BDHG. Likewise, once the vertex I of the portion KIL has been found, if the solidity of each of the two portions BCD and KLI is found, and the smaller is taken from the larger, the solidity of the frustum BDLK will remain known.

[Margin: Solidity of any frustum of the sphere.]

ON THE SURFACE OF THE SPHEROID, AND OF ITS PORTIONS.

Chapter VII.

[Margin: Solidity of the spheroid.]

Let ABCD be an ellipse whose major axis is AC and minor axis BD, the latter cutting the former at right angles. We shall obtain the solidity of the spheroid — that is, of the solid produced by revolving the ellipse about its axis — as follows. A plane drawn through BD at right angles to the axis AC makes a circle (as Federico Commandino [Federico Commandino] demonstrates on Proposition 12 of Archimedes' book *On Conoids and Spheroids*), whose diameter is BD and center E. Therefore, by Proposition 29 of Archimedes' *On Conoids and Spheroids*, the half-spheroid ABD is double a cone having the same base as that half — namely the circle of diameter BD — and the same height EA. Therefore, if the solidity of this cone is investigated by Chapter 2 of this book and doubled, the solidity of the half-spheroid will arise; and this, doubled, will yield the solidity of the whole spheroid.

[Margin: Solidity of the portions of the spheroid.]

2. Draw FG parallel to the minor axis BD, cutting the major axis at H at right angles. If, then, a plane is drawn through FG at right angles to the axis, a circle will be formed in the spheroid having FG as diameter and H as center, as Federico Commandino demonstrated on Proposition 12 of Archimedes' *On Conoids and Spheroids*; and the portions of the spheroid FAG (the smaller) and FCG (the larger) will be cut off. The solidity of each will be found as follows. By Proposition 31 of Archimedes' *On Conoids and Spheroids*, the cone whose base is the circle of diameter FG and whose axis is HA has to the smaller portion of the spheroid FAG the ratio that the axis HC of the larger portion has to the sum of the straight lines EC and HC. So if it is arranged that, as HC, the axis of the larger portion, is to the sum of the straight lines EC and HC, so the aforesaid cone is to another quantity (which cone will be known from Chapter 2 of this book), the solidity of the smaller portion of the spheroid FAG will result.

Again, since by Proposition 33 of Archimedes' *On Conoids and Spheroids* the cone whose base is the circle of diameter FG and whose axis is HC has to the larger portion of the spheroid FCG the ratio that the axis HA of the smaller portion has

to the sum of the straight lines EA and HA: if it is arranged that, as HA, the axis of the smaller portion, is to the sum of the straight lines EA and HA, so the aforesaid cone (which you will measure by Chapter 2 of this book) is to another quantity, the solidity of the larger portion of the spheroid FCG will be produced.

Diagram. An upright ellipse labelled ABCD, the cross-section of the spheroid. The major axis AC is vertical (A at top, C at bottom); the minor axis BD is horizontal through the center E (B at left, D at right). Above the center, the horizontal chord FG (F at left, G at right) is drawn parallel to BD, cutting the major axis at H; FG is the diameter of the circle that divides the spheroid into the portions FAG and FCG.

- A (top): upper vertex of the major axis
- C (bottom): lower vertex of the major axis
- B (left): left end of the minor axis BD
- D (right): right end of the minor axis BD
- E (center): center of the ellipse
- F (left, above center): left end of the chord FG
- G (right, above center): right end of the chord FG
- H (on the major axis, above center): center of the chord FG

ON THE SURFACE OF THE PARABOLIC CONOID.

Chapter VIII.

[Margin: Solidity of the parabolic conoid.]

Let ABC be a parabola whose axis BD is perpendicular to the base AC. We shall measure the solidity of the parabolic conoid — which the parabola produces when revolved about its axis — as follows. By what Federico Commandino demonstrated on Proposition 12 of Archimedes' *On Conoids and Spheroids*, a plane drawn through AC at right angles to the axis BD makes a circle whose diameter is AC and center D. Therefore, by Proposition 23 of Archimedes' *On Conoids and Spheroids*, the parabolic conoid ABC is one-and-a-half times [sesquialterate to] a cone whose base is the circle of diameter AC and whose axis is BD. Therefore, if it is arranged that, as 2 is to 3, so the aforesaid cone (which we shall measure by Chapter 2 of this book) is to another quantity, the solidity of the parabolic conoid ABC will spring forth.

Diagram. A parabolic dome, the cross-section of the parabolic conoid, labelled ABC. The base AC is horizontal (A at the lower left, C at the lower right) with its midpoint D; the apex B is at the top, and the axis BD is the vertical line from B down to D. E is a point on the axis inside the figure, marked with a short vertical segment (an ordinate).

- B (top): apex of the parabola
- A (lower left): left end of the base AC
- C (lower right): right end of the base AC
- D (center of base): foot of the axis BD
- E (on the axis, interior): point of the construction

On the Area of the Hyperbolic Conoid

Chapter IX.

Let the upper figure be taken to be a hyperbola, and let the line E be equal to half the transverse diameter that lies between two opposed hyperbolas — that is, equal to the line drawn from the center of the hyperbolas to the vertex B. Let a circle be formed again, whose diameter is AC, by a plane passed through AC, drawn at right angles to the axis, in the manner Federico Commandino [Federico Commandino] demonstrated on proposition 12 of the book of Archimedes [Archimedes] *On Conoids and Spheroids*. This, then, is how we shall hunt down the solidity of the hyperbolic conoid produced when the hyperbola ABC is revolved about the axis BD.

[Margin: The solidity of the hyperbolic conoid.]

By proposition 27 of Archimedes's *On Conoids and Spheroids*, the hyperbolic conoid ABC has to the cone whose base is the same as the base of the conoid — namely the circle of diameter AC — and whose axis is the same BD, the same ratio that the line compounded of the axis BD and three times E has to the line compounded of the axis BD and twice E. So proceed thus: as the line compounded of the axis BD and twice E is to the line compounded of the axis BD and three times E, so let the aforesaid cone (which we shall measure by chapter 2 of this book) be to a fourth magnitude; that fourth magnitude will be the solidity of the hyperbolic conoid ABC.

On the Area of Casks

Chapter X.

Since casks do not everywhere keep the same shape, one can scarcely prescribe a rule by which the capacity of a given cask may be found exactly. The proof of this is that writers have described its measurement in various ways. I too, then, will state what seems to me plausible.

Let ABCDEF be a cask, having at its ends the circles AF and CD, and the bung-hole B. Through B imagine a plane passed at right angles to the line KL that joins the centers of the circles AF and CD, cutting the cask into two halves. If, then, the staves of the cask are curved at B and at E, and are afterwards extended along lines that are nearly straight — the sort of cask I have often seen at Rome — the halves of the cask, ABEF and CDEB, will resemble truncated cones. If you measure these by what was written in chapter 3 of this book, their sum will give the capacity of the given cask. Bear in mind, however, that the depth of the cask BE and the diameter of the circle AF must be measured inside the staves, so that their thickness is excluded, and so that one has a truncated cone whose bases are the circles BE, AF, and so on.

[Margin: The capacity of a cask.]

2. But if the staves of the cask really are in some way circular — as some maintain — the cask must be conceived as a kind of frustum of a spheroid. For the curvature of the staves does not perceptibly differ from the curvature of a spheroid whose axes are the straight lines KL and BE. Yet since only the minor axis — that is, the depth of the cask — is given, we shall find the major axis KL from a point A on the given ellipse in this way.

Diagram. An ellipse (the cross-section of the spheroid), drawn with its horizontal major axis KL and its vertical minor axis BE crossing at the center G. Each end of the major axis carries a small lens-shaped circle seen edge-on: the circle AF at the left end and the circle CD at the right end. Construction lines run from A across the interior.

- B (top): top endpoint of the vertical minor axis (the bung-hole)
- A (upper left): top of the left end-circle AF
- C (upper right): top of the right end-circle CD
- G (center): midpoint of the figure, intersection of the two axes
- K (left): left endpoint of the major axis

- L (right): right endpoint of the major axis
- H (just below and left of center): intersection of the arc struck from A with the perpendicular GH
- I (below center, on the minor axis): where the line from A through H cuts the minor axis
- F (lower left): bottom of the left end-circle AF
- D (lower right): bottom of the right end-circle CD
- E (bottom): bottom endpoint of the vertical minor axis

With a radius equal to half the minor axis, BG, describe from A an arc cutting, at H, the line GH — which is drawn at right angles to the minor axis through its midpoint G. Then from A, through H, let a straight line be drawn cutting the minor axis at I. For the line AI will give the halves GK and GL of the major axis, as we demonstrated in Book 1 of our *Astrolabe*, in the scholium to Lemma 50. And so, if both the half-spheroid BKE and the smaller portion AKF are measured, as was taught in chapter 7, and the solidity of the portion AKF is subtracted from the solidity of the half BKE, there will remain the capacity of the half-cask ABEF; and this, doubled, will yield the capacity of the whole cask. I judge that neither of these two ways of measuring a cask falls far short of the true capacity of the cask. Still, in the meantime I stand ready — should anyone find a more accurate method — to accept it gladly and with thanks.

[Margin: The capacity of a cask, another way.]

On the Area of Wholly Irregular Bodies

Chapter XI.

1. Some writers hand down a certain mechanical rule for measuring bodies that are altogether irregular — so much so that they cannot fall under the geometric rules explained up to now. Of this kind are statues, urns, amphorae, and fragments of stone, which are neither of uniform thickness nor have sides that are perfectly straight or perpendicular to their bases, and so on. This rule, then, which seems in no way to be despised, is as follows.

[Margin: The solidity of any irregular body.]

Let a wooden box be prepared from smoothed planks, like a kind of parallelepiped, and coated with pitch so that it can hold water. This box must be of such length, breadth, and height that the body to be measured, placed inside it,

can be entirely covered by water. Set the box level with the horizon by means of a level or a plumb-line, and pour into it just enough water to cover the body once it has been put in; then carefully mark the top surface of the water on the planks of the box, so that one has the height

of the water up to the bottom of the box. Then take out the body — taking care, however, that no water spills outside the box — and, after the water has come to rest, mark its sides again. If, by chapter 1 of this book, we then measure two parallelepipeds whose common base is the bottom, or base, of the box, but whose heights are the perpendiculars from the marked water-levels down to the base, and subtract the smaller from the larger, there will be left a parallelepiped exactly equal to the solidity of the proposed body. You obtain the same parallelepiped, too, if you multiply the height between the two marked water-levels into the base of the box. There are those who, once the water has been poured into the box, first mark its sides on the planks; then, once the body has been put in, mark the sides of the same water. For if the height between the later sides and the earlier is multiplied into the base of the box, the solidity of the inserted body will be produced.

2. For urns and amphorae, whether they are of stone or of terracotta, we shall proceed thus. Let the vessel be filled with sand, and its mouth so stopped that water can in no way get in. Then place the vessel in the water contained within the box, as though it were some irregular body, and investigate its solidity, as we said in number 1. Next pour out the sand and mark the water-levels before the empty vessel is put in. Finally, with the empty vessel inserted, mark the water-levels once more. For if the height between the later and the earlier levels is multiplied into the base of the box, the solidity of the vessel alone will be produced; and this, subtracted from the earlier solidity, will leave the known capacity of the vessel.

On the Convex Surface of the Cone and Right Cylinder

Chapter XII.

Since we have demonstrated from Archimedes the method by which the convex surface of a sphere and of its portions is to be investigated, there will perhaps be no lack of someone who desires the same for the cone and the right cylinder. This will be obtained from what Archimedes himself demonstrated in Book 1 of *On the*

Sphere and Cylinder, in the following way. Given any right cone whatever, its convex conic surface, excluding the base, will be equal to a circle whose radius is the mean proportional between the slant side of the cone and the radius of that cone's base — by proposition 14 of Book 1 of Archimedes's *On the Sphere and Cylinder*.

[Margin: The conic surface, with its base removed, and to what circle it is equal.]

2. And if a right cone is cut by a plane parallel to the base, the convex surface of the frustum of the cone, with its bases removed, will be equal to a circle whose radius is the mean proportional between the slant side of the frustum and the straight line compounded of the radii of the two bases — by proposition 16 of Book 1 of Archimedes's *On the Sphere and Cylinder*.

[Margin: The surface of the frustum of a cone, with its bases removed, and to what circle it is equal.]

Likewise the conic surface of a right cone has to its own base the same ratio that the slant side of the cone has to the radius of that same cone's base — by proposition 15 of Book 1 of Archimedes's *On the Sphere and Cylinder*.

[Margin: The ratio of the conic surface to its own base.]

4. Finally, the convex surface of a right cylinder, with its bases removed, is equal to a circle whose radius is the mean proportional between the slant side of the cylinder and the diameter of that same cylinder's base — by proposition 13 of Book 1 of Archimedes's *On the Sphere and Cylinder*.

[Margin: The cylindrical surface, with its bases removed, and to what circle it is equal.]

End of the Fifth Book.

Practical Geometry, Book VI.

In which are treated geodesy; the increasing and decreasing of figures in a given ratio; likewise the finding of two mean proportionals between two given straight lines; and finally the extraction of roots.

Now that we have finished with what we set out at the beginning concerning the measurement of magnitudes, it remains for us to treat also of the division of rectilinear surfaces. This part of practical geometry is called by its proper name *geodesy* [Geodaesia]. For δαίω [Greek: δαίω (daiō, "I divide" or "I apportion")]

means the same as "I apportion" or "I divide." I know that most people call that part which measures magnitudes and land "geodesy" as well; but these people, on the authority of John Pediasimus [Pediasimus] concerning the measuring and partitioning of land, stray far from the truth. For, he says, the measurement of land is divided into two parts, namely geometry and geodesy. The measurement of an area according to the art is indeed a measuring of land, and is rightly called geometry; but the division of one and the same area, or place, among different persons is a certain partitioning of land, and is most justly called geodesy. So says Pediasimus.

[Margin: What Geometry and Geodesy are.]

Federico Commandino did indeed publish, in the year 1570, a treatise on the division of surfaces ascribed to a certain Muḥammad al-Baghdādī [Machometus Bagdedinus], an Arab; and he himself wrote another, shorter and more general work on the same subject. Both little books are quite penetrating and richly learned. Later, Simon Stevin [Simon Stevin] of Bruges took up the same subject by another route, and — certainly in my judgment — an easier and more general one; yet in it something seems to be wanting if it is to fit all rectilinear surfaces (which he himself appears to intend), as those who attentively read through his geometric problems will readily judge. For the proposed matter can in no way be carried out unless two propositions are first demonstrated. The first of these he assumes without demonstration as a principle, while of the second he makes no mention at all, even though it is thoroughly necessary — and Muḥammad al-Baghdādī demonstrated it a little differently than we do. These two propositions, then, we shall demonstrate at the beginning of this book, the latter indeed far more generally than Muḥammad did — which I leave to the kind reader to judge. Then we shall take up the division of rectilinear surfaces, following in the footsteps of the same Stevin, except where the matter must be demonstrated more generally. Of the method of Muḥammad and of Federico Commandino, however, we shall say nothing: partly because their treatise is in everyone's hands, so that anyone who wishes may read it; partly because, when a figure of many angles is proposed, the student will divide it only with difficulty and labor unless he keeps in mind the division of all the preceding figures — which does not happen with our method; and finally because their method suits only ordinary figures, namely those which have all their angles turned inward, as many angles, that is, as the figure itself has sides, whereas our approach embraces also those figures which have their angles turned partly inward and partly outward.

Geometria Practica — Liber VI

THEOREM 1. PROPOSITION 1.

If a magnitude is cut in any way whatever into any number of parts, and some other magnitude into the same number of parts, taken in order proportional to them, then any number of parts of the first magnitude taken together will have to all the remaining parts taken together the same ratio that the corresponding number of parts of the second magnitude taken together have to all their remaining parts taken together. And if any part of the first magnitude is cut in any way into two parts, while the part of the second magnitude answering to it is cut into two other parts proportional to those two, then in that case too the whole magnitudes will be cut proportionally.

Let the magnitude AB be cut in any way into any number of parts AC, CD, DE, EF, FB; and let another magnitude GH of any kind whatever — even if it be of a different kind — be cut into the same number of parts GI, IK, KL, LM, MH, taken in order proportional to them. I say, for example, that the two parts AC, CD together are to the remaining three DE, EF, FB together as the two GI, IK together are to the remaining three KL, LM, MH together, and so on. For since AC is to CD as GI is to IK, then, by composition [*componendo*], AD is to CD as GK is to IK. But CD is to DE as IK is to KL. Therefore, by equality of ratios [*ex aequali*], AD is to DE as GK is to KL.

Diagram. Two parallel horizontal line-segments, one above the other. The upper segment is divided into five parts, its division points labelled from left to right: A, C, D, E, F, B. The lower segment is likewise divided into five parts, labelled from left to right: G, I, K, L, M, H. Two further subdivision points appear, corresponding to the text's later division of DE at N and of KL at O; in the woodcut these marks are set toward the right of the two lines (a reversed N printed above H, and O printed below the lower line near L–M), and their exact placement is obscured by the coarse cut.

- Upper segment: A – C – D – E – F – B
- Lower segment: G – I – K – L – M – H
- N (subdivision of DE, upper line) [position uncertain in the cut] [?]
- O (subdivision of KL, lower line) [position uncertain in the cut] [?]

Again, since by conversion [*convertendo*] BF is to FE as HM is to ML, then also by composition BE is to FE as HL is to ML. But FE is to ED as ML is to LK. Therefore, *ex aequali*, BE is to ED as HL is to LK; and by composition, BD is to ED as HK is to

LK; and by conversion, DE is to DB as KL is to KH. And so, since it has been shown that AD is to DE as GK is to KL, and that DE is to DB as KL is to KH, then, ex aequali, AD is to DB as GK is to KH.

We shall show in the very same way that AC is to CB as GI is to IH. For again, by conversion, composition, and equality of ratios, BC is to DC as HI is to KI; and by conversion, CD is to CB as IK is to IH. Since then AC is to CD as GI is to IK, and CD is to CB as IK is to IH, it follows, ex aequali, that AC is to CB as GI is to IH.

By like reasoning, AF is to FB as GM is to MH. For again, by composition and equality of ratios, AF is to EF as GM is to LM. And since EF is to FB as LM is to MH, it follows, ex aequali, that AF is to FB as GM is to MH; and so for the rest. Thus the first part of the theorem is established.

Next, let the third part — say DE — be cut in any way into two parts DN, NE; and let the third part KL likewise be cut into two parts KO, OL proportional to them. I say that also AN is to NB as GO is to OH. For by conversion, EN is to ND as LO is to OK; and by composition, ED is to DN as LK is to KO. Hence, since CD is to DE as IK is to KL, and DE is to DN as KL is to KO, it follows, ex aequali, that CD is to DN as IK is to KO. And thus the parts AC, CD, DN are proportional to the parts GI, IK, KO.

Again, since by conversion FE is to ED as ML is to LK, and by composition DE is to NE as KL is to OL, it follows, ex aequali, that FE is to EN as ML is to LO; and by conversion, NE is to EF as OL is to LM. And so all the parts AC, CD, DN, NE, EF, FB are proportional to all the parts GI, IK, KO, OL, LM, MH. Therefore, as was demonstrated in the first part, AN is to NB as GO is to OH. Thus the second part too is established.

PROBLEM 1. PROPOSITION 2.

Given a rectilinear figure, upon a given straight line intercepted between two other straight lines, to construct an equal quadrilateral whose opposite side, intercepted between those same two lines, is parallel to the given straight line. And, given any two unequal rectilinear figures, to cut away from the greater, by a line parallel to one side, a rectilinear figure equal to the smaller — whenever this can be done, which will be recognized from the solution of the problem itself.

1. Let the rectilinear figure A be given, and the given straight line BC, intercepted between two straight lines BD, CE. It is required first to construct, upon the given straight line BC, a quadrilateral equal to the rectilinear figure A, whose opposite side, intercepted between those same straight lines BD, CE, is parallel to the given straight line BC. Now if the two straight lines BD, CE are parallel (which happens

only when the two angles B, C together equal two right angles), the problem is accomplished [Margin: a — Euclid, *Elements* I.45] if upon the straight line BC a parallelogram BE is constructed, equal to the rectilinear figure A, whether at the angle BCE or at the angle CBD.

Diagram. At the left, a rhombus (tilted square) enclosing the letter A — the given rectilinear figure. At the right, a parallelogram lying between two parallel oblique lines; its four vertices are labelled D (upper left), E (upper right), B (lower left), C (lower right).

- A: the given rectilinear figure (in the rhombus)
- D, E: upper side of the parallelogram (parallel to BC)
- B, C: the given base line BC

2. When the angles B, C are right angles, we shall accomplish the problem more easily in this way. For the given rectilinear figure A, construct — by the scholium to Proposition 14 of Euclid's Book 2, or rather by what we set out in No. 4, Chapter 4, Book 4 of this Practical Geometry — a square FGH equal to it: namely, by resolving the rectilinear figure into triangles or trapezia, constructing a square equal to each triangle or trapezium, and at last reducing all those squares to one, as we explained at length in the places cited. Then, for the two straight lines BC, FG, let a third proportional BD be found, and through D let DE be drawn parallel to BC. The rectangle BE will then be equal to the given rectilinear figure A, [Margin: b — Euclid, *Elements* VI.17] since it is equal to the square FGH, because the three straight lines BC, FG, BD are continued proportionals.

Diagram. At the left, a kite-shaped quadrilateral enclosing the letter A, with a dotted diagonal running from A across to the vertex D. To its right, two vertical parallel lines carry the construction: near the top, points I and E (with a dotted line D–E and the marginal point K just beyond E); lower down, H above B, and C to the right at the foot of the parallels. At the lower left stands a square labelled F (lower left corner), G (lower right corner), H (upper right corner). Several construction lines are dotted, and the interior is crowded.

- A: given rectilinear figure (in the kite), with dotted diagonal to D
- F, G, H: the square equal to A
- B, C: base line; D, E: the constructed parallel side
- I, K: points on the parallels

[Continued on p. 149:] C. For, once the rectangle CD has been constructed equal to the square FH — that is, to the rectilinear figure A, as has been said — then, if

the side DE is produced until it cuts the straight lines BI, CK at IK, [Margin: a — Euclid, *Elements* I.35] the parallelogram BK will be equal to the rectangle CD, that is, to the rectilinear figure A.

3. But if the two straight lines BD, CE are not parallel, let them meet toward the parts D, E at F (which happens when the two angles DBC, ECB are less than two right angles). Suppose the first task is, upon BC toward F, to construct a trapezium equal to the rectilinear figure A, having its side opposite BC parallel to BC. For this to be possible, the rectilinear figure A must be less than the triangle BCF, as is evident.

So, for the rectilinear figure A, let there be constructed — by the scholium to Proposition 14 of Euclid's Book 2, or rather by what we set out in No. 4, Chapter 4, Book 4 of this work, as we just said — a square G equal to it, whose side is HI. [Margin: b — Euclid, *Elements* II.14] Likewise let a square K be constructed equal to the triangle BCF, whose side is LM. Then, for the two straight lines LM, HI, let a third proportional LN be found; and when a straight line O has been found that is the mean proportional between LM and MN, let it be arranged that as LM is to O, so BF is to FD; and finally, through D let DE be drawn parallel to BC. I say that the trapezium BE is equal to the rectilinear figure A.

[Margin: c — Euclid, *Elements* VI.19] For the triangle BCF has to the triangle DEF the duplicate ratio of the side BF to the side DF — that is, of the straight line LM to the straight line O. But LM also has to MN the duplicate of the ratio it has to O, since LM, O, MN are continued proportionals. Therefore the triangle BCF is to the triangle DEF as LM is to MN; and, by conversion of the ratio, the triangle BCF is to the trapezium BE as LM is to LN. [Margin: d — Euclid, *Elements* VI.20, corollary] Since then LM is to LN as the square K is to the square G — because LM, HI, LN are continued proportionals — it also holds that the triangle BCF is to the trapezium BE as the square K is to the square G; that is, as the triangle BCF (which is equal to K) is to the rectilinear figure A (which is equal to G). And so, since the triangle BCF has the same ratio to the trapezium BE as to the rectilinear figure A, [Margin: e — Euclid, *Elements* V.9] the trapezium BE and the rectilinear figure A are equal — which was proposed.

Diagram. Upper right cluster of the composite figure for this problem. At the left, an irregular quadrilateral enclosing the letter A (the given rectilinear figure); at its right, a square enclosing the letter K (the square equal to the triangle BCF). Below the square runs a horizontal construction-line of proportional segments, its points labelled from left to right: P (joined by a dotted stroke) — M — N — L,

with O and Q set just below at the right end. These carry the segments LM, MN, LN (and the mean O, the whole LP, and Q).

- A: given rectilinear figure
- K: square equal to triangle BCF
- P, M, N, L: proportional segments (LM, MN, LN, LP)
- O: mean proportional; Q: mean between LP and LM

Diagram. Lower cluster: a large right triangle with apex F at the right, its base descending to C at the lower left; across the top run the points R, B, D toward F, and the interior carries the constructed parallels and their intersection points V, a, E, T (with S and X toward the lower-left corner near C). At the lower right stands a square enclosing the letter **G** (the square equal to A), its corners labelled Z (upper right), H (lower left), Y/I (lower right). Many construction lines are dotted and overprint the interior labels.

- Triangle BCF: vertices B, C, F (with base line and the points R, S marking the far-side construction)
- D, E: the parallel DE (trapezium BE toward F); R, S: the parallel RS (trapezium BS toward R,S)
- V, a, T: interior construction points [?]
- G: square equal to A, with corners Z, Y, I resolved; corner H [?] (cropped in the scan)

4. Next, suppose that upon BC — toward R, S, where the angles RBC, SCB are greater than two right angles, and not toward the meeting-point F — a trapezium is to be constructed equal to the rectilinear figure A (of whatever magnitude it may be), having its side opposite BC parallel to it. [Margin: f — Euclid, *Elements* II.14] Again let a square K be made equal to the triangle BCF, with side LM; and another square G equal to the rectilinear figure A, with side HI, by the methods we taught for Proposition 14 of Euclid's Book 2, or rather in No. 4, Chapter 4, Book 4 of this work. Then, for the sides LM, HI, let a third proportional MP be found, set in a straight continuation of LM; and between the whole LP and LM let the mean proportional Q be found; and finally let it be made that as Q is to LP, so FB is to FR, and let RS be drawn parallel to BC. I say that the trapezium BS is equal to the rectilinear figure A.

[Margin: g — Euclid, *Elements* VI.19] For the triangle BCF has to the triangle RSF the duplicate ratio of the side FB to the side FR — that is, of the ratio of Q to LP. But the ratio of LM to LP is likewise the duplicate of the ratio of LM to Q, or of Q to LP, since the three straight lines LM, Q, LP are continued proportionals.

Therefore the triangle BCF is to the triangle RSF as LM is to LP; and hence also, by the converse separation of the ratio [*divisione rationis contraria*] that we demonstrated in the scholium to Proposition 17 of Euclid's Book 5, the triangle BCF is to the trapezium BS as LM is to MP. [Margin: h — Euclid, *Elements* VI.20, corollary] But LM is to MP as the square K is to the square G, since the three lines LM, HI, MP are continued proportionals. Therefore the triangle BCF is likewise to the trapezium BS as the square K is to the square G. Since then a square K has been constructed equal to the triangle BCF, [Margin: i — Euclid, *Elements* V.14] the trapezium BS will also be equal to the square G — that is, to the rectilinear figure A, to which the square G was constructed equal — which was proposed.

5. But if it should happen that the two straight lines BF, CF meet at so remote a point that it can scarcely be obtained (which happens when the lines are nearly parallel), we shall accomplish the problem, even without having the meeting-point F, in this way. Taking any point T on one of them — namely on CF — let TV be drawn parallel to the other, BF; and for the two BC, CV let a third proportional X be found. Then, having constructed (by the scholium to Proposition 14 of Euclid's Book 2, or rather as we taught in No. 4, Chapter 4, Book 4 of this work) a square G equal to the rectilinear figure A, let a fourth proportional IY be found to the three BC, X, HI, and let YZ be drawn parallel to the sides of the square. [Margin: k — Euclid, *Elements* VI.19, corollary] And since the triangle BCF (were it completed) is to the triangle VCT as the straight line BC is to the straight line X — [Margin: l — Euclid, *Elements* VI.1] that is, as HI is to IY, that is, as the square G is to the rectangle IZ — while the triangle BCF is greater than the square G, or the rectilinear figure A (for whenever a trapezium equal to a given rectilinear figure is to be constructed toward the side of angles less than two right angles, the triangle must be greater than the rectilinear figure), [Margin: m — Euclid, *Elements* VI.14] the triangle VCT will likewise be greater than the rectangle IZ. Therefore, as was taught in No. 3, let a trapezium Vb be constructed equal to the rectangle IZ; and for the three straight lines CV, Va, CB let a fourth proportional BD be found [Margin: n — Euclid, *Elements* VI.4, scholium] (the straight line drawn Ca will pass through D, if the fourth BD has been rightly found; [Margin: o — Euclid, *Elements* VI.4] just as, conversely, the line Ca, if it is drawn exactly, will yield the fourth proportional sought BD); and let DE be dropped parallel to BC. I say that the trapezium BE is equal to the given rectilinear figure A.

For since the trapezium BE is similar to the trapezium Vb, by what we demonstrated for Proposition 18 of Euclid's Book 6, [Margin: p — Euclid, *Elements* VI.20, corollary] the trapezium BE will be to the trapezium Vb as the straight line BC is to the straight line X — that is, as HI is to IY, [Margin: q — Euclid, *Elements*

VI.1] that is, as the square G is to the rectangle IZ. Since then the trapezium Vb is equal to the rectangle IZ, [Margin: r — Euclid, *Elements* V.14] the trapezium BE will likewise be equal to the square G — that is, to the rectilinear figure A — which was proposed.

6. In no other way, on the other side of the angles RBC, SCB — which are greater than two right angles — even without having the meeting-point F, we shall construct a trapezium equal to the rectilinear figure A, of whatever magnitude it is set. For in this case it is not necessary that A be less than the triangle BCF, were it completed. For, drawing from any point T of the straight line CF a line TV parallel to BF, and producing it, let a third proportional

[Continued on p. 150:]...to the two straight lines BC, CV be found: X. Then, having constructed a square G equal to the rectilinear figure A, let a fourth proportional IY be found to the three straight lines BC, X, and HI, and let YZ be drawn parallel to the sides of the square, so that again the triangle BCF is to the triangle VCT as the square G is to the rectangle IZ. After this, as we directed in No. 4, let a trapezium Ve be constructed equal to the rectangle IZ; and for the three straight lines CV, Vd, CB let a fourth proportional BR be found [Margin: a — Euclid, *Elements* VI.4, scholium] (the straight line Cd, drawn through R, will pass through it if the fourth BR has been rightly found; [Margin: b — Euclid, *Elements* VI.4] just as, conversely, the line Cd, if drawn accurately, will cut off the fourth proportional sought BR); and let RS be dropped parallel to BC. I say that the trapezium BS is equal to the rectilinear figure A.

For since the trapezium BS is similar to the trapezium Ve, by what we showed in the scholium to Proposition 18 of Euclid's Book 6, [Margin: c — Euclid, *Elements* VI.20, corollary] the trapezium BS will be to the trapezium Ve as the straight line BC is to the straight line X — that is, as the straight line HI is to IY, [Margin: d — Euclid, *Elements* VI.1] that is, as the square G is to the rectangle IZ. Since then the trapezium Ve is equal to the rectangle IZ, [Margin: e — Euclid, *Elements* V.14] the trapezium BS will likewise be equal to the square G — that is, to the rectilinear figure A — which was proposed.

7. Now, given any two rectilinear figures A and BCDEFGHI, let the latter be taken as the greater, and let it be required to cut away from it a rectilinear figure having a side parallel to the side BI and equal to the former A, which is set as the smaller — if indeed this can be done. It can always be done in figures having all their angles turned inward, but in others not always. By what we set out in the scholium to Proposition 14 of Euclid's Book 2, or rather in No. 4, Chapter 4, Book 4 of this work, let a square KM be constructed equal to the smaller rectilinear

figure A. Then, from the angle C, which is nearest the side BI, draw CO parallel to BI, and by the same method construct a square PQR equal to the rectilinear figure BO; and for the two straight lines KN, PQ, having found the third proportional KS, draw ST parallel to KL. [Margin: f — Euclid, *Elements* VI.17] Then the rectangle KT, contained by the first line KL and the third KS, will be equal to the square of the mean PQ — that is, to the rectilinear figure BO. And since in this example KS is found to be less than the side KN, and hence the rectangle KT less than the square KM (that is, than the rectilinear figure A), the rectilinear figure BO will also be less than the rectilinear figure A. Therefore, from the nearer angle H, drawing again HV parallel to CO or BI, let a square be made equal to the rectilinear figure CH, whose side is X. And for the two straight lines KN and X, having found the third proportional SN — which in this example terminates at the end of the side KN — [Margin: g — Euclid, *Elements* VI.17] the rectangle SM, contained under the first line ST and the third SN, will again be equal to the square of the mean X, that is, to the rectilinear figure CH. Since then KT has also been shown equal to BO, the whole square KM — that is, the rectilinear figure A — will be equal to the whole rectilinear figure BCVHOI; and thus from the greater rectilinear figure, by the straight line HV parallel to the side BI, we have cut off a rectilinear figure equal to the smaller rectilinear figure A, which was to be done.

Diagram. The figure for numbers 7–8. At the upper left an irregular polygon with its left vertices E, F, G and along its top the points R, V, C, B; toward the right H, O, I; the parallels CO and HV cut across its interior (drawn dotted). Below stands a triangle enclosing the letter A (its right angle at right), and beneath that a long rectangle carrying the proportional-segment construction, with points labelled Q, P (upper left), then N, V, S (with a small – mark) across the top, R at lower left, X at far lower left, and M, Z, T along the foot toward the square K at the right, with L at the lower-right corner and a small **d** near M–Z. Construction lines are dotted and heavily overprint the labels.

- BCDEFGHI: the given greater (irregular) rectilinear figure
- A: the smaller figure (in the triangle); KM, PQR: the squares
- K, L, M, N, S, T, Z: points of the rectangle/segment construction
- O, V, H: points of the parallels CO, HV (V resolved; O, H [?] — cropped in the scan)

8. But if the third proportional found to the two straight lines KN and X had been less than SN — namely equal to SY, so that the rectangle SZ were equal to the square of the straight line X, or to the rectilinear figure CH — then from the nearest angle D another parallel Da would have to be drawn, and a square

constructed equal to the rectilinear figure DH; and to the straight line KN and the side of this last-found square a third proportional would be adjoined, and a length Yb equal to it cut off. And if Yb were less than YN, one would still have to draw from the nearest angle G a parallel to the side BI, make a square equal to the rectilinear figure comprehended between this parallel and Da, and adjoin to the straight line KN and the side of this square a third proportional, and so on. And so one must proceed successively until a third proportional SN is found that terminates at N (such as was the third proportional SN found for the two KN and X), or whose end falls beyond N. For when it terminates at N, the rectilinear figure comprehended between BI and the last parallel will be equal to the square KM — that is, to the rectilinear figure A — as was shown for the rectilinear figure BH a little before, in this No. 8. But when the end of the third proportional falls beyond N — namely at b — so that the rectangle Yd is equal to the last square found (that is, to the rectilinear figure DH), and thus the whole rectilinear figure between BI and the last parallel (namely the rectangle Kd) is greater than the square KM or the rectilinear figure A, then the last rectilinear figure DH will be greater than the rectangle Nd, because KZ is equal to BH, and Yd to DH. Therefore, if upon the straight line Da, between the straight lines DV, aH, a trapezium Df is constructed by the parallel ef equal to the rectangle Nd, as was taught above in No. 3, then the rectilinear figure contained between the parallels BI and ef will be equal to the square KM — that is, to the rectilinear figure A.

For the rest, it is not necessary that the parallel always be drawn from the nearest angle in the figure BF. For when one's judgment estimates roughly that a parallel drawn from some angle that is not the nearest would cut off a rectilinear figure smaller than the square KM, or not much greater, then the parallel should be drawn from that angle, and a square equal to the cut-off rectilinear figure constructed, and the rest completed as before.

Finally, if it should ever be found that the cut-off rectilinear figure is not much less than the square, then, as we taught in No. 3, upon that parallel a trapezium should be constructed equal to the rectangle by which the square KM exceeds the cut-off rectilinear figure.

9. From all this I think the diligent reader sufficiently understands how he ought to proceed in other examples. For if, say, from this proposed most irregular rectilinear figure a portion equal to some other smaller rectilinear figure is to be cut off by a line parallel to the side AM, we shall produce MA as far as N. And if the triangle ABN is found to be equal to the given smaller rectilinear figure (which will be known if the square constructed equal to the triangle is equal to

the square that is constructed equal to the given smaller figure), then the straight line AN will accomplish the problem. But if it is greater, we shall construct upon AN, toward B, a trapezium equal to the excess by a parallel to AN. And if it is less, we shall draw the parallel LO. For if the rectilinear figure NL is found equal to the deficiency, the parallel LO will accomplish the problem; but if it is greater, we shall construct upon LO, toward MN, by a parallel to MN, a trapezium equal to the excess — for that parallel will solve the problem. But if it is less, we shall produce OL to P; and if the triangle KLP is equal to the deficiency, the whole parallel OP will satisfy the question; but if greater, [Margin: h — Euclid, *Elements* VI.25] we shall construct at the angle K a triangle similar to the triangle KLP and equal to the excess; thus

Diagram. The "most irregular" rectilinear figure of No. 9. An irregular many-sided polygon; along its top run the points C, R, O, N, B (with F at the upper left); down the left side E, T, S, and the interior points D, L; at the right A, M, K; along the foot G, X, V, H, I, Q, P. Interior parallels (to AM, MN, etc.) are drawn dotted and cross the figure, and the point-labels are crowded and overprinted.

- F, C, R, O, N, B: upper vertices/points
- E, T, S, D, L: left and interior points
- A, M, K: right side (with AM the side of reference)
- G, X, V, H, I, Q, P: lower points
- Interior points R, O, N, B, A, M, K, T, S, D, L, H, I, Q, P resolve; the far-left vertices E, C, G, X, V remain [?] (off-tile). Vertex N is engraved mirror-reversed (N̄) in the woodcut — a compositor artifact; the label is plain N.

LIBER VI

so that this triangle, together with the figure cut off by the parallel LO, equals the given smaller figure. From this you will conclude that in this case the problem cannot be solved by a single parallel, since two parallels — namely LO, and the one that cuts off a triangle similar to KLP — together cut from the whole figure BG a part equal to the given smaller figure.

But if the triangle KLP is less than the aforesaid deficiency — so that this triangle, together with the figure cut off by the parallel LO, is less than the given smaller figure — we shall draw the parallel QR through D. Now if the figure PR equals the deficiency by which the figure KPLMABNO falls short of the given smaller figure, then what was required has been done by the parallel QR. But if PR is greater, a parallel drawn alongside QR toward OP, cutting off a figure equal to this excess,

will satisfy the problem. On the other hand, if the figure PR is less than the aforesaid deficiency, and the triangle CDR is found equal to this last deficiency (by which the figure PR falls short of the aforesaid deficiency), the parallel DQ will resolve the question. But if the triangle CDR is greater than this last deficiency, then — if at C a triangle be constructed equal to the excess and similar to triangle CDR — two parallels will satisfy the question, namely DQ and the base of the said constructed triangle; and in this case the problem cannot be satisfied by a single parallel. [Margin: Elements VI.25] Finally, if the triangle CDR turns out to be less than that same last deficiency, we shall draw the parallel IS. And if the figure DI equals the amount by which the triangle CDR is less than that last deficiency, the whole figure ISDCBAMLKI will equal the given smaller figure. But if the figure DI is unequal, we shall go on further, as has now often been said, until we find a figure equal to the given smaller figure; and one equal figure will certainly be found, since the whole figure BG is assumed to be the greater.

You see, then, that one can easily judge when the problem can be solved by a single parallel and when it cannot, but only by two. For whenever, in the course of the construction, we meet with a triangle of the sort that KLP and CDR were — one from which a triangle similar to it and equal to some excess must be cut off — the problem cannot be solved except by two parallels.

PROBL. 2. PROPOS. 3.

Given any rectilinear figure divided into triangles from some one point, to find straight lines proportional, in order, to those triangles.

Let ABCDEF be any rectilinear figure, divided into the triangles ABC, ACD, ADE, AEF by straight lines drawn from the angle A (or from some assigned point on one of the sides) to all the opposite angles; and let there be found, for these triangles, as many straight lines proportional in order. From every angle except A, drop perpendiculars BI, CL, DK, DN, EM, FO onto the straight lines issuing from A, to serve as the altitudes of the triangles. (It makes no difference if some of these perpendiculars fall onto those lines produced beyond the triangles — as here DK and DN do.) Thus each triangle has two altitudes, except the two outermost, which have only one apiece.

Next, on any straight line GH mark off GP equal to the altitude BI of the first triangle ABC, and PQ equal to the altitude DK of the second triangle ACD taken with respect to the same base AC. After this, [Margin: a. Elements VI.12] contrive that as CL, the altitude of the second triangle with respect to the base AD, is to EM, the altitude of the third triangle ADE with respect to the same base AD, so is PQ to

QR. And as DN, the altitude of the third triangle ADE with respect to the base AE, is to FO, the altitude of the fourth triangle AEF with respect to the same base AE, so is QR to RH — and so on, if there be more triangles, always taking the two altitudes dropped onto the common base, and so forth. I say that the four straight lines GP, PQ, QR, RH are proportional, in order, to the four triangles.

For, as was demonstrated by us in the scholium to Elements VI.1, triangle ABC is to triangle ADC as the altitude BI is to the altitude DK, on account of the common base AC — that is, as GP is to PQ, since these were taken equal to those altitudes. For the same reason, triangle ACD is to triangle ADE as the altitude CL is to the altitude EM — that is, as PQ is to QR, since by construction CL is to EM as PQ is to QR. And finally, by like reasoning, triangle ADE is to triangle AEF as the altitude DN is to the altitude FO — that is, as QR is to RH, since by construction DN is to FO as QR is to RH. What was proposed is therefore established.

Diagram. A vertical straight line GH at the left, marked off (top to bottom) at G, P, Q, R, H, giving the four segments GP, PQ, QR, RH proportional to the four triangles. At the right is the polygon ABCDEF, with apex A at the top, divided by lines from A into the triangles ABC, ACD, ADE, AEF; the perpendicular altitudes fall to the feet marked BI (foot I), CL (foot L), DK and DN (feet K, N, on the produced lines at the base), EM (foot M) and FO (foot O).

- A (top apex): vertex from which the dividing lines are drawn
- B (upper left): vertex; perpendicular BI
- I (interior): foot of perpendicular from B
- F (right): vertex; perpendicular FO
- O (below F): foot of perpendicular from F
- C (lower left): vertex; perpendicular CL
- L (lower center): foot of perpendicular from C
- E (lower right): vertex; perpendicular EM
- M (bottom center): foot of perpendicular from E
- D (bottom): vertex; perpendiculars DK, DN
- K (bottom left): foot on the produced line
- N (bottom right): foot on the produced line
- G, P, Q, R, H (left vertical line): the proportional segments GP, PQ, QR, RH

ALITER

Let ABCDEF again be a rectilinear figure divided from the point A into the triangles ABC, ACD, ADE, AEF. Since each pair of adjacent triangles makes up a

quadrilateral whose diagonal is the side common to the two triangles — as ABCD is — we shall draw through D, parallel to the diagonal AC, the line DO, which cuts the side BC produced in O. Likewise in the quadrilateral ACDE we shall draw, parallel to the diagonal AD, the line EP, which cuts the side CD produced in P. And in the quadrilateral ADEF we shall draw, parallel to the diagonal AE, the line FQ, which cuts the side DE produced in Q. Then on any straight line GN take GH and HK equal to BC and CO respectively. To the three lines CD, DP, HK find the fourth proportional KL. And finally, to the three lines DE, EQ, KL let the fourth proportional LN be found. I say that the four straight lines found — GH, HK, KL, LN — are proportional to the four triangles.

For when the lines AO, AP, AQ are drawn from A to the points of intersection O, P, Q, [Margin: b. Elements I.37] triangle ACD will equal triangle ACO, triangle ADE will equal triangle ADP, and triangle AEF will equal triangle AEQ. [Margin: c. Elements VI.1] Since, then, as BC is to CO — that is, as GH is to HK — so is triangle ABC to triangle ACO, that is, to triangle ACD; and likewise as CD is to DP — that is, as HK is to KL — so is triangle ACD to triangle ADP, that is, to triangle ADE; and as DE is to EQ — that is, as KL is to LN — so is triangle ADE to triangle AEQ, that is, to triangle AEF: what is proposed is evident.

Diagram. A vertical line GN at the left, marked at G, H, K, L, N, giving the segments GH, HK, KL, LN. At the right is the figure ABCDEF: apex A at the center-left, with B, C at the left, D at the bottom, E at the right, F at the top; the diagonals AC, AD, AE are drawn, and the parallels DO, EP, FQ (dotted) meet the produced sides in O (bottom), P (right), Q (upper right).

- A (center left): vertex from which the diagonals issue
- B (left): vertex
- C (lower left): vertex
- D (bottom center): vertex; parallel DO
- E (right center): vertex; parallel EP
- F (top): vertex; parallel FQ
- O (bottom): intersection of DO with BC produced
- P (right): intersection of EP with CD produced
- Q (upper right): intersection of FQ with DE produced
- G, H, K, L, N (left vertical line): the proportional segments GH, HK, KL, LN

Diagram. A horizontal strip (illustrating the third method described on the following page): a baseline I–R–T–X–M divided into four cells by verticals at R, T, X, with the top vertices S (over R), V (over T), Y (over X). The cells are the equal-

altitude parallelograms whose bases IR, RT, TX, XM are proportional to the four triangles.

- I (left end of baseline): start of the strip
- R, T, X (baseline divisions): feet of the verticals RS, TV, XY
- M (right end of baseline): end of the strip
- S, V, Y (top): the upper vertices of the parallelograms

ALITER

The two methods set out are the most expeditious, but they are proper to the triangles into which the figure is divided by straight lines drawn from some one given point on any side, or issuing from some angle. The same problem can, however, be carried out in another way, one that suits figures of any kind, though it is not so expeditious. We shall proceed thus. Let the same figure as before be divided into triangles, or even into figures of several sides. [Margin: a. Elements I.44 or I.45] On the first triangle ABC (or the first figure), construct an equal rectangle — or any other, non-rectangular, parallelogram — IS. On the straight line RS construct another parallelogram ST, equal to the second triangle ACD (or the second figure), having the angle SRT equal to the angle I. Again, on the straight line TV construct another, VX, equal to the third triangle ADE (or the third figure), having the angle VTX equal to the same angle I. And finally, on the straight line XY construct another, YM, equal to the fourth triangle AEF (or the fourth figure), having the angle YXM equal to the same angle I; and so on, if there be more triangles or figures. I say that the straight lines IR, RT, TX, XM are proportional to the triangles, or figures. For out of the four rectangles, or parallelograms, one whole is constituted, as is clear from the demonstration of Elements I.45, because the angles I, SRT, VTX, YXM are equal, and consequently all four have the same altitude. [Margin: b. Elements VI.1] Therefore the straight lines IR, RT, TX, XM are proportional to the parallelograms, and hence also to the triangles or figures — which is what was proposed.

PROBL. 3. PROPOS. 4.

To divide a given rectilinear figure, by a straight line drawn from any angle or from a point on one of the sides, in a given ratio — so that the antecedent of the ratio inclines toward whichever part you prefer.

Let ABC first be any triangle, to be divided into two parts by a straight line from the angle A, so that the part toward B has to the remaining part the given ratio D to E. Cut the side BC, which lies opposite the given angle, in F — by the method we

taught in the scholium to Elements VI.10 — so that BF is to FC as D is to E; and draw the straight line AF. I say that triangle ABF is to triangle AFC as D is to E. [Margin: a. Elements VI.1] For triangle ABF is to triangle AFC as BF is to FC, that is, as D is to E.

Next, let the same triangle ABC be divided into two parts by a straight line from the point F, given on the side BC, so that the part toward B has to the remaining part the given ratio D to E. Draw from the given point F to the opposite angle A the straight line FA, so that the whole triangle is cut into two triangles. [Margin: b. Prop. 3 of this book] Find two straight lines GH, HI having the same ratio that triangle ABF has to triangle AFC; and let the whole GI be cut at H so that GH is to HI as D is to E. Now since the point H falls at the end of the first line GH, and since triangle ABF is to triangle AFC as GH is to HI — that is, as D is to E — the straight line FA, drawn from the given point F to the opposite angle A, will divide triangle ABC into two parts in the given ratio D to E.

Let there again be given a ratio K to L, and let triangle ABC be divided from the point F into two parts of this same ratio. Let the whole GI be cut at M, so that GM is to MI as K is to L. And since the division-point M falls within the first part GH of the whole line GI, we shall cut BA, the base of the first triangle, opposite the given point F, in N, so that BN is to NA as GM is to MH. I say that the drawn straight line FN accomplishes the problem — that is, that triangle BFN is to the trapezium FNAC as K is to L. For since the straight lines GH, HI were found proportional to the triangles ABF, AFC, and since we cut both the first part GH at M and the first triangle ABF (by the straight line FN) proportionally, [Margin: c. Elements VI.1] then, triangle BFN being to triangle NFA as BN is to NA — that is, as GM is to MH — [Margin: d. Prop. 1 of this book] it will follow that as GM is to MI, that is, as K is to L, so is triangle BFN to the trapezium FNAC, which is what was proposed.

Finally, let a ratio O to P be given, and let triangle ABC be cut into two parts of this same ratio. Let the whole GI be cut at Q, so that GQ is to QI as O is to P. Since the division-point Q falls within the second part HI of the whole line GI, we shall divide AC, the base of the second triangle opposite the given point F, in R, so that AR is to RC as HQ is to QI. I say that the drawn straight line FR accomplishes the problem — that is, that the trapezium ABFR is to the triangle RFC as O is to P. For since the straight lines GH, HI were found proportional to the triangles ABF, AFC, and since we cut both the second part HI at Q and the second triangle AFC (by the straight line FR) proportionally, [Margin: e. Elements VI.1] then, triangle AFR being to triangle CFR as AR is to RC — that is, as HQ is to QI — [Margin: f. Prop. 1

of this book] it will follow that as GQ is to QI, that is, as O is to P, so is the trapezium ABFR to the triangle RFC, which is what was proposed.

But now, if the antecedent of the ratio ought to incline toward C, and the given ratio is O to P, this will be done most conveniently if the triangle be divided from F according to the ratio P to O, so that the antecedent inclines toward B, as we have taught. For then the part toward C will have to the remaining part the ratio that O has to P, by inverse proportionality. And this I wish to be understood in other figures as well.

Diagram. The triangle ABC (apex A at top, B at lower left, C at lower right), with the point F on the base and the cevians AF, FN, FR drawn to the division-points N (on AB) and R (on AC). Alongside, at the right, is the auxiliary line GI (with H and the division-points M, Q, P) and the short segments representing the given ratios (O, and the stacked ratio-lines D–E, K–L). Readable labels: A, O, N, P, R, K, B, H, F, G, D, C, L, Q, M, E, I — all resolved at native 4×4 (); the left-flank vertex on side BA, the foot of cevian FN, is N (not M).

Let ABCDEF next be any many-sided figure, to be cut into two parts by a straight line drawn from the angle A, so that the part toward B has to the remaining part the given ratio M to N. When straight lines have been drawn from the given angle A to all the opposite angles, dividing the figure into four triangles, [Margin: g. Prop. 3 of this book] let straight lines GH, HI, IK, KL be found proportional to them. Then let the whole GL be cut in O, so that GO is to OL as M is to N. And since the division-point O falls within the third line IK, we shall cut DE, the base of the third triangle opposite the given angle A, in P — just as IK is cut in O — and we shall draw the straight line AP.

Diagram. The many-sided figure ABCDEF (apex A at top, B at upper left, C at left, D at the bottom, E at lower right, F at right), divided by lines from A into four triangles; the cevian AP (dotted) is drawn to the point P on side DE. At the right are the short segments of the given ratio M, N; below is the auxiliary line GL, marked at G, H, I, K, L, with the division-point O. Readable labels: B, A, F, C, D, P, E, M, N, G, H, I, K, L, O — the tiled (left) portion verified at native 4×4 (); the right-hand labels (E, F, K, L) are corroborated by the surrounding text.

I say that the figure ABCDIA [recte: ABCDPA] is to the figure APEFA as GO is to OL, that is, as M is to N. For since the triangles of the given figure are proportional, in order, to the straight lines GH, HI, IK, KL by construction, and since we divided both the third triangle ADE and the third line IK proportionally, by the straight line AP and at the point O — [Margin: a. Elements VI.1] since DP is to PE (that is,

IO is to OK) as triangle ADP is to triangle APE — [Margin: b. Prop. 1 of this book] the whole magnitudes too will have been cut proportionally; that is, ABCDPA will be to APEFA as GO is to OL, that is, as M is to N, which is what was proposed.

But if the point O, dividing the line GL into two parts of the given ratio M to N, were to fall on one of the points H, I, K — say on I, the end of the second line HI — then the straight line AD, which terminates the second triangle ACD, would divide the whole figure in the given ratio; for then, by the first part of Proposition 1 of this book, ABCDA would be to ADEFA as M is to N, as is evident.

Furthermore, let there be a rectilinear figure ABCD, to be divided from the point E, given on the side AB, into two parts, of which the first — toward B — has to the latter part the given ratio K to L. When straight lines have been drawn from the given point E to all the opposite angles, dividing the figure into as many triangles less one as the figure has sides, [Margin: c. Prop. 3 of this book] let straight lines FH, HI, IG be found proportional to the triangles EBC, ECD, EDA. Then let the whole FG be cut in M according to the given ratio K to L. Since the division-point M falls within the first line FH, we shall divide the base BC of the first triangle EBC, opposite the given point E, in X, just as FH is cut in M. For when the straight line EX is joined, [Margin: d. Prop. 1 of this book] triangle EBX will be to the figure EXCDAE as FM is to MG, that is, as K is to L — because the triangles EBC, ECD, EDA are proportional to the straight lines FH, HI, IG by construction, and the first parts, triangle EBC and line FH, are cut proportionally by the straight line EX and at M; [Margin: e. Elements VI.1] since EBX is to EXC as BX is to XC, that is, as FM is to MH. The proposition is therefore established.

If the given ratio is N to O, then when FG is cut in P according to the ratio N to O, the division-point P will fall within the second line HI. Therefore, if the base CD of the second triangle ECD, opposite the given point E, be cut in Q — just as HI is cut in P — and the straight line EQ be joined, [Margin: f. Prop. 1 of this book] then again the figure EBCQE will be to the figure EQDA as FP is to PG, that is, as N is to O.

If, finally, the given ratio is R to S, then when FG is cut in V according to the ratio R to S, the division-point V will fall within the third line IG. For which reason, if the base DA of the third triangle EDA, opposite the given point E, be cut in T — just as IG is cut in V — and the straight line ET be joined, [Margin: g. Prop. 1 of this book] then again the figure EBCDTE will be to the triangle ETA as FV is to VG, that is, as R is to S.

And by this method one must proceed in all other figures that have as many sides as angles — that is, in those whose angles all turn inward.

Diagram. The quadrilateral ABCD (A at top, B at left, D at right, C at the bottom), with the point E on the side AB, from which the cevians EC, ED (and the auxiliary lines EX, EQ, ET) are drawn to the division-points X, Q, T on the bases BC, CD, DA. At the right is the auxiliary line FG, marked F, M, H, P, I, V, G, with division-points M, P, V; and the short segments of the given ratios N–O, K–L, R–S. Readable labels: A, E, T, B, D, X, Q, C; F, M, H, P, I, V, G; N, O, K, L, R, S [some interior letters overprinted].

We shall accomplish this same problem in a rectilinear figure whose angles turn partly outward and partly inward — provided that straight lines can be drawn, from an angle or from a point given on a side, dividing the figure into triangles that cut none of its sides. As in this eight-sided figure ABCDEFGH, whose five angles B, C, D, F, H turn inward and whose remaining three — BAH, DEF, FGH — turn outward: straight lines are drawn from the angle A to all the angles except the two adjacent ones B, H, cutting no side of the figure. [Margin: h. Prop. 3 of this book] If, then, six straight lines IK, KL, LM, MN, NO, OP be found proportional to the six triangles ABC, ACD, ADE, AEF, AFG, AGH, and the whole line LP be cut in S according to the given ratio Q to R; and the base DE of the third triangle (for the division-point S falls within the third line LM), opposite the given point E, be divided in T, just as the line LM is divided in S, and the straight line AT be drawn: [Margin: i. Prop. 1 of this book] the figure ABCDTA will be to the figure ATEFGHA as IS is to SP, that is, as Q is to R.

Diagram. The eight-sided figure ABCDEFGH (B at the top, A below it, C and D at the left, E at the bottom center, F and G at the lower right, H at the right), with straight lines drawn from A to the non-adjacent angles, dividing it into six triangles; the cevian AT (dotted) is drawn to the point T on side DE. Below is the auxiliary line IP, marked I, K, L, M, N, O, Q, R, P, with the division-point S. Readable labels: B, A, V, C, D, T, E, F, G, H; I, K, L, M, N, O, Q, R, P, S [interior letters partly overprinted].

From the angle B or H, however, the proposed figure could not be divided in just any ratio, because the lines issuing from either of them to the opposite angles partly cut the sides and partly fall outside the figure. But if the given portion were less than the figure BCDEB (understanding, that is, the straight line BE to be drawn) compared with the figure BEFGHAB, then indeed the whole figure could be divided from B in the given ratio: for two triangles BCD (drawing, namely, the line BD) and BDE would be formed at the point B, whose bases are the sides CD,

DE of the figure; while four others — ABE, AEF, AFG, AGH — would be formed at the point A, whose bases likewise are the sides AB, EF, FG, GH of the figure, and so on.

In the same way we shall divide any rectilinear figure whatever, even the most irregular, in a given ratio — not indeed from any arbitrary angle or given point (unless from it straight lines can be drawn to all the opposite angles, except the two adjacent ones, cutting no side of the figure, such as the point V would be in the preceding figure), but from some particular point; provided the figure is first divided into triangles from several points, so that each triangle has at least one side which is also a side of the figure. Thus if this figure ABCDEFHHIKA be divided into eight triangles, [Margin: k. Prop. 3 of this book] and for them as many proportional lines be found on the straight line LT, and the whole line LT be cut in Y according to the given ratio V to X; and the base IK of the sixth triangle (since the point Y falls within the sixth line QR) be cut in Z, just as QR is cut in Y; [Margin: l. Elements VI.1] so that, the straight line FZ being drawn, the triangle FIK is cut just as its base IK — that is, the line QR — is cut: [Margin: m. Prop. 1 of this book] the figure ABCDEFZKA will be to the figure FZIHGF as LY is to YT, that is, as V is to X. And so with the rest.

Diagram. A highly irregular figure (labelled ABCDEFHHIKA in the text) drawn as two interlocking star-shaped clusters and divided into eight triangles numbered 1–8. Vertices read (left cluster) B, A, C, D, E with the cevians and triangles 1, 2, 3, 4, 5 about A and K; (right cluster) I, H, G, F with triangles 6, 7, 8 and the point Z. Below is the long auxiliary line LT, marked L, M, N, O, P, Q, R, S, T, with the division-points (a near P, and Y near Q/R); at the left the short segments of the given ratio V, X. Readable labels: A, B, C, D, E, F, G, H, I, K; the region-numbers 1–8; Z; V, X; L, M, N, O, P, Q, R, S, T; Y [several interior labels (?) under overprinting].

SCHOLIUM

[Margin: How a given figure may be divided — from a given angle, or from a point on one of its sides — into any number of equal parts.]

Once these things are properly understood, we shall be able to cut any figure whatever into any number of equal parts from a given angle — or from a point on one of its sides — from which straight lines can be drawn to all the angles except the two nearest, in such a way that they cut none of the figure's sides. For suppose the proposed figure is to be cut, say, into 7 equal parts. We divide it in the ratio 1 to 6 — that is, according to the submultiple named from the denominator of the parts, less one, into which the figure is to be divided. In this

way the first part will be one seventh of the whole figure, since the second comprises 6 such parts. Next we cut this latter part in the ratio 1 to 5, so that the first part contains one sixth of it — that is, one seventh of the whole figure. After this we divide the latter part of this second division in the ratio 1 to 4. Again we divide the latter part of this third division in the ratio 1 to 3. Then we cut the latter part of this fourth division in the ratio 1 to 2. And finally we divide the latter part of this fifth division into two equal parts — that is, in the ratio 1 to 1.

In just the same way we may divide an irregular figure — one in which straight lines cannot be drawn from any angle, or from a point on a side, to the opposite angles without some of the figure's sides being cut — into any number of equal parts, using different angles or points. For if, say, the last figure of this proposition is to be divided into 5 equal parts, we cut off from it, at some angle or point, a fifth part. Then from the larger part comprising four fifths of the whole figure we take away, at some angle or point of it, a fourth part; likewise a third part from the larger part of this division; and finally a half from the last part of this division's remainder. By this procedure the whole figure will be divided into 5 parts, exactly as happens in the straight line AB. For if we are directed to divide AB into 7 equal parts, we accomplish it by first taking away the seventh part AC; then one sixth [$\frac{1}{6}$, printed as a small superscript fraction — glyph faint] (CD) from the remaining line CB; then from the remaining DB the fifth part DE; from the remaining EB the fourth part EF; from the remaining FB the third part FG; and finally the remaining line GB we cut in two at H — that is, we cut off from it the half GH.

We shall carry out the same thing more easily when, from a given angle or point, straight lines can be drawn to all the angles (the two nearest excepted) without cutting any side of the figure, in this way. We cut a line composed of straight segments proportional to the triangles of the figure into as many equal parts as we are directed to divide the figure into. For if we divide the bases of the triangles — those opposite the given angle or point, which correspond to the lines onto which the division points fall — in the same way that the corresponding lines are cut, and then draw straight lines from the given angle or point to the division points, what is proposed will be done. For example, if the second figure of this proposition is to be cut into 5 equal parts, we divide the line GL into 5 equal parts GH, Hb, ba, aK, KL. And since the first point H falls at H, the triangle ABC will be the fifth part of the figure [Margin a: Prop. 1 of this book], since GH is to HL as the triangle ABC is to the rest of the figure. Next we cut the base of the second triangle at d, just as the second line HI is cut at b; and the base of the third triangle at e, just as the third line IK is cut at a; and we draw the straight lines Ad,

Ae. And since the fourth point K falls at K, the end of the fourth line IK, the figure will be divided into 5 equal parts ABC, ACd, AdDe, AeE, AEF [Margin b: Prop. 1 of this book], because these parts are proportional to the parts GH, Hb, ba, aK, KL.

Nor will it be difficult to adapt this same method to irregular figures, such as the last one of this proposition. For if it is to be divided, say, into three equal parts, the line LT must be cut into three equal parts La, ab, bT. And the base DE of the fourth triangle must be divided at d, just as the fourth line OP is divided at a; likewise the base KI of the sixth triangle must be cut at f, just as the sixth line QR is cut at b. For if from K, the angle opposite the base DE, the straight line Kd is drawn, and likewise from the angle F, opposite the base KI, the straight line Ff, then the three parts of the figure — ABCDdKA, KdEFfK, fFGHIf — will be equal to one another [Margin c: Prop. 1 of this book], since they are proportional to the straight lines La, ab, bT. And the same holds for the rest.

Diagram. The upper figure of two on this page. At the left, an irregular polygon divided from its apex into triangles, with interior lines (some dotted) running from the apex A to the division points on the far side; below it a straight reference line running left to right (G...L) marked with several division ticks; and at the right a set of short horizontal segment-marks (with end-serifs) representing the given magnitudes. The interior point-letters are small and overprinted on the woodcut lines.

- A (apex, top center): vertex from which the dividing lines radiate
- B (upper left): polygon vertex
- C (left): polygon vertex
- D (lower left): polygon vertex / division point
- E (right): polygon vertex
- F (upper right): polygon vertex
- f, P (interior, lower center): division points
- M (right, on a segment-mark): magnitude label
- N (right, on a segment-mark; printed as a reversed "И" = mirror-reversed N, a compositor artifact — the magnitude-mark pair is M, N)
- G — H — I — K — L (lower straight reference line, left to right): division points, with small lowercase b, a resolved beneath; a faint base mark ("e"-like) and a trailing tick ("o") remain [?]

PROBLEM 4. PROPOSITION 5.

To divide a given rectilinear figure by a straight line parallel to a given straight line in a given ratio, so that the antecedent of the ratio inclines toward whichever part you choose.

Let there first be a triangle ABC, to be divided into two parts by a line parallel to the side BC, so that the part toward A has to the remainder the given ratio D to E. One of the two sides from which the dividing line need not be equidistant — namely AC — is divided at F so that the ratio AF to FC is the same as D to E, the start being made from the angle A toward which the antecedent of the ratio must incline. Between the whole side AC and its part AF (which ends at the angle A that is opposite the side to which the parallel is to be drawn) let the mean proportional AG be found, and through G let GH be drawn parallel to BC. I say that this parallel accomplishes the problem — that is, that the ratio of the triangle AGH to the trapezium BCGH is the same as D to E. [Margin a: Elements VI, cor. 19] For since the triangle ABC is to the triangle AGH as the side AC is to the straight line AF — because the three straight lines AC, AG, AF are in continued proportion, and the triangles ABC, AGH on AC, AG are similar and similarly placed — then by conversion of the ratio (convertendo) the triangle ABC will be to the trapezium BCGH as AC is to FC. Therefore, by separation (dividendo), the triangle AGH will be to the trapezium BCGH as AF is to FC — that is, as D is to E, which was proposed. We may also gather it this way. [Margin b: Elements VI, cor. 19] Since the triangle ABC is to the triangle AGH as the straight line AC is to AF, then dividendo the trapezium BG will be to the triangle AGH as FC to AF; and by inversion (convertendo) the triangle AGH will be to the trapezium BG as AF to FC — that is, as D to E.

Diagram. The lower figure. A triangle ABC with the apex A at the top right, B at the bottom right, and C at the bottom left. A line GH is drawn parallel to the base BC; a dotted line runs from A across the figure. To the left stand two vertical segment-marks representing the given ratio, labeled D and E.

- A (top right): apex
- B (bottom right): base vertex
- C (bottom left): base vertex
- K (on side AB, upper): point on the slant side
- L (interior, left): point on the parallel
- F (near C, above G): division point
- G (on side AC near C): foot of the parallel GH

- M (on side AB, lower right): point
- H (on the base region): foot of the parallel
- I (on the base): division point
- D, E (left, on the two vertical segment-marks): the given ratio $D : E$

LIBER VI.

ANOTHER WAY

When any side whatever — say BC — has been divided at I according to the given ratio D to E, and the straight line AI has been joined, [Margin a: Elements VI.1] the triangle ABI will be to the triangle AIC as BI to IC — that is, as D to E. [Margin b: Prop. 1 of this book] If, then, upon BC, between the straight lines BA and CA, a trapezium BG is set up by the parallel GH equal to the triangle AIC (which is the consequent), the remaining triangle AGH will be equal to the remaining triangle ABI. [Margin c: Elements V.7] Hence the triangle AGH will be to the trapezium BG as the triangle ABI to the triangle AIC — that is, as BI to IC, or as D to E. But the former way is more expeditious; I have chosen nonetheless to propose this other one as well, because it is almost entirely general and fits all many-sided figures, as will appear below.

We shall carry out what is proposed in no other way if the antecedent of the ratio should incline toward the side BC, to which the straight line is to be drawn parallel. For, the side AC being again divided at K according to the given ratio D to E — so that the antecedent of the ratio begins from the side BC — let the mean proportional AL be found between the whole AC and its part AK (which is the consequent of the ratio), and through L let LM be drawn parallel to the side BC. I say that this parallel accomplishes the problem — that is, that the ratio of the trapezium BL to the triangle ALM is the same as D to E. [Margin d: Elements VI, cor. 19] For since the triangle ABC is to the triangle ALM as CA to AK — because the three straight lines AC, AL, AK are in continued proportion, and the triangles ABC, ALM on AC, AL are similar and similarly placed — then dividendo the trapezium BL will be to the triangle ALM as CK to AK, that is, as D to E, which was proposed. This same thing can also be done by that other, general method, though not so expeditiously. For, any side whatever — say BC — being divided at I according to the given ratio D to E, and the straight line AI being joined, [Margin e: Elements VI.1] the triangle ABI will be to the triangle AIC as BI to IC, that is, as D to E. [Margin f: Prop. 2 of this book] If, then, upon BC, between the straight lines BA and CA, a trapezium BL is set up by the parallel LM equal to the triangle ABI (which is the antecedent), the remaining triangle ALM will be equal to the

remaining triangle AIC. [Margin g: Elements V.7] Hence the trapezium BL will be to the triangle ALM as the triangle ABI to the triangle AIC — that is, as BI to IC, or as D to E.

But now let the triangle ABC be divided in the given ratio D to E by a line parallel to any given line F. If F is parallel to one of the sides, we shall partition the triangle in the given ratio by a straight line parallel to that side, just as we have already taught — whether the antecedent should incline toward the angle opposite that side, or toward the side itself — and it will be done as directed, [Margin h: Elements I.30] since that parallel is also parallel to the given straight line.

But if the given straight line F is parallel to none of the sides, let a parallel to it be drawn from some angle, falling inside the triangle — such as BG. If, then, the antecedent of the ratio is to be set toward the parts A, we cut the side AC at H in the given ratio D to E. Let the point H first fall between G and C; and between GC and its part CH (which ends at the angle C, opposite the parallel BG) let the mean proportional CI be found; and through I let IK be drawn parallel to BG, or to F. I say that this accomplishes the problem — that is, that the trapezium ABKI is to the triangle IKC as D to E. [Margin i: Elements VI, cor. 19] For since (the straight line BH being drawn) the triangle GBC is to the triangle IKC as GC to CH — because the three lines GC, CI, CH are in continued proportion, and the triangles are similar and similarly placed — [Margin k: Elements VI.1] and since also GC is to CH as the triangle GBC is to the triangle HBC, [Margin l: Elements V.9] the triangles IKC, HBC will be equal; and consequently the remaining trapezium ABKI will be equal to the remaining triangle ABH. [Margin m: Elements V.7] Therefore the trapezium ABKI will be to the triangle ICK as the triangle ABH to the triangle HBC — that is, as AH to HC, or as D to E, which was proposed.

Diagram. An elongated triangle for the constructions above. The apex is at the upper left, where the vertex A and the point F lie close together (F being where the parallel BG, produced, meets the extension); B is at the lower left and C at the far lower right. Short segment-marks at the top and right give the ratio (D, E and o, P). The interior division points L, M, G form a small cluster near the middle, with dotted construction lines; I and H lie on the hypotenuse AC descending toward C; N and K lie on the base BC.

- F, A (upper left): apex cluster
- D, E (top, on segment-marks): a given ratio
- o, P (right, on segment-marks): the given ratio O : P
- L, M, G (interior, middle cluster): division / mean-proportional points

- I (on hypotenuse AC): division point
- H (on hypotenuse AC, lower): division point
- N (on base BC, left): foot of a parallel
- B (lower left): base vertex
- K (on base BC): division point
- C (lower right): base vertex

Next let the point L (we now suppose the given ratio to be E to D, and AC to be divided at L according to the given ratio) fall between A and G. Then, the mean proportional AM being found between GA and its part AL (which ends at the angle A, opposite the parallel BG), let MN be drawn through M parallel to GB, and therefore also to F. I say that this accomplishes the problem — that is, that the triangle AMN is to the trapezium MNBC as E to D. For, the straight line BL being joined, [Margin n: Elements VI, cor. 19] since the triangle ABG is to the triangle AMN as AG to AL — because the three lines AG, AM, AL are in continued proportion, and the triangles are similar and similarly placed — [Margin o: Elements VI.1] and since also AG is to AL as the same triangle ABG is to the triangle ABL, [Margin p: Elements V.9] the triangles AMN, ABL will be equal; and consequently the remaining trapezium MNBC will be equal to the remaining triangle LBC. [Margin q: Elements V.7] Therefore the triangle AMN will be to the trapezium MNBC as the triangle ABL to the triangle LBC — [Margin r: Elements VI.1] that is, as AL to LC, or E to D, which was proposed.

But if the point of division should fall at G — which would happen if the given ratio were O to P — the parallel BG itself would accomplish the problem, [Margin s: Elements VI.1] since the triangle ABG is to the triangle GBC as AG to GC, that is, as O to P. And if the antecedent must be set toward C, and the given ratio were P to O, [Margin t: Elements VI.1] then likewise the triangle CBG will be to the triangle ABG as CG to GA, or as P to O.

ANOTHER WAY

When any side whatever — namely AC — has been divided at H according to the given ratio D to E, and the straight line BH joined, [Margin u: Prop. 2 of this book] since the point H falls between G and C, let a trapezium BI be made upon BG, between the straight lines BC and GC, by the parallel IK, equal to the triangle BGH. Consequently the whole trapezium ABKI will be equal to the whole triangle ABH, and therefore also the remaining triangle IKC to the remaining triangle BCH. [Margin x: Elements V.7] Wherefore the trapezium ABKI will be to the

triangle IKC as the triangle ABH to the triangle BCH, [Margin y: Elements VI.1] that is, as AH to HC, or as D to E.

Again, when the side AC has been divided at L according to the given ratio E to D, and the straight line BL joined — since the point L falls between A and G — [Margin z: Prop. 2 of this book] let a trapezium BM be made upon BG, between the straight lines BA and GA, by the parallel MN, equal to the triangle BGL. Consequently both the whole trapezium MNBC will be equal to the whole triangle LBC, and the remaining triangle AMN to the remaining triangle ABL. [Margin a: Elements V.7] For this reason the triangle AMN will be to the trapezium MNBC as the triangle ABL to the triangle LBC, [Margin b: Elements VI.1] that is, as AL to LC, or as E to D, which was to be done.

GEOMETR. PRACTICÆ

We shall complete the problem in no other way if the antecedent of the ratio should incline toward the parts C. For, the side CA being divided at L in the given ratio D to E — since the point L falls between A and G — we shall find the mean proportional AM between GA and AL, and through M we shall draw MN parallel to BG. I say that this accomplishes the problem — that is, that the trapezium MNBC is to the triangle AMN as D to E. For, the straight line BL being joined, [Margin a: Elements VI, cor. 19] since the triangle ABG is to the triangle AMN as AG to AL — because the three lines AG, AM, AL are in continued proportion, and the triangles are similar and similarly placed — [Margin b: Elements VI.1] and since also AG is to AL as the same triangle ABG is to the triangle ABL, [Margin c: Elements V.9] the triangles AMN, ABL will be equal; and consequently the remaining trapezium MNBC will be equal to the remaining triangle LBC. Therefore the trapezium MNBC will be to the triangle AMN as the triangle LBC to the triangle ABL, [Margin d: Elements VI.1] that is, as CL to LA, or as D to E.

But if the given ratio is E to D, then, the same side CA being divided at H in the given ratio E to D, and the straight line BH drawn — since the point H falls between G and C — we shall find the mean proportional CI between GC and CH, and through I we shall draw IK parallel to BG. I say that this accomplishes the problem — that is, that the triangle CIK is to the trapezium IKBA as CH to HA, or as E to D. For, the straight line BH being joined, [Margin e: Elements VI, cor. 19] since the triangle CBG is to the triangle CIK as CG to CH — because the three lines CG, CI, CH are in continued proportion, and the triangles are similar and similarly placed — [Margin f: Elements VI.1] and since also CG is to CH as the same triangle CBG is to the triangle CBH, [Margin g: Elements V.9] the triangles CIK, CBH will be equal; and therefore the remaining trapezium IKBA will be equal to the

remaining triangle HBA. [Margin h: Elements V.7] Wherefore the triangle CIK will be to the trapezium IKBA as the triangle CBH to the triangle HBA, [Margin i: Elements VI.1] that is, as CH to HA, or as E to D, which was to be done.

The same may also be done another way: if, when the point of division L falls between A and G, [Margin k: Prop. 2 of this book] a trapezium GN is constructed by the parallel MN equal to the triangle BGL; and if, when the point of division H falls between G and C, the trapezium BI is set up equal to the triangle BGH by the parallel IK; and so on.

Diagram. The upper figure — the same elongated triangle repeated for the constructions of this page. The apex cluster F, A is at the upper left, B at the lower left, and C at the far lower right. Segment-marks at top and right carry the ratios (D, E and o, P). The interior points L, M, G cluster near the middle with dotted lines; I and H lie on the hypotenuse AC toward C; N and K on the base BC.

- F, A (upper left): apex cluster
- D, E (top, on segment-marks): a given ratio
- o, P (right, on segment-marks): the given ratio O : P
- L, M, G (interior, middle cluster): division / mean-proportional points
- I, H (on hypotenuse AC): division points
- N, B, K (on / at the base BC): foot of parallel, vertex, division point
- C (lower right): base vertex

Moreover, let the quadrilateral ABCD be divided by a line parallel to the side CD into two parts, so that the part toward A has to the remainder the given ratio E to F. By what we set forth in the scholium to Proposition 14 of Book 2 of Euclid — or better, by what we handed down at Number 4, Chapter 4, Book 4 of this work — let a square GHIK be constructed equal to the quadrilateral ABCD; and let the side GK be cut at L in the given ratio E to F, and through L let LM be drawn parallel to GH. Then upon CD, between the straight lines CB and DA, [Margin l: Prop. 2 of this book] let a quadrilateral DO be made equal to the rectangle LI, having the side NO parallel to the side CD. And if indeed the point O falls on the side CB, or on the point B itself, the straight line NO will accomplish the problem. For since the square GI is equal to the whole quadrilateral AC, and the rectangle taken away, LI, is equal to the quadrilateral taken away, DO, the remaining rectangle GM will also be equal to the remaining rectilinear figure ANO. [Margin m: Elements V.7] Wherefore ANO will be to DO as GM to MK, [Margin n: Elements VI.1] that is, as GL to LK, or as E to F, which was proposed.

But if the point O falls on CB produced — which will happen if the given ratio is Q to R (for, the side GK being divided at S in the given ratio Q to R, and ST drawn parallel to the side GH, if upon CD, between the straight lines DA and CB, [Margin o: Prop. 2 of this book] a quadrilateral DO is made equal to the rectangle KT, then O will fall beyond B, and the parallel NO will cut the side AB at P) — [Margin p: Prop. 2 of this book] then we shall construct upon NP, between the straight lines NA and PA, by the parallel VX, a trapezium NV equal to the triangle BOP, and it will be done as directed. For, the common rectilinear figure CDNPB being added, the whole rectilinear figure CDXVB will be equal to the whole rectilinear figure DO — that is, to the rectangle KT; and therefore the remaining triangle AVX will be equal to the remaining rectangle GT. [Margin q: Elements V.7] Hence the triangle AVX will be to the rectilinear figure CDXVB as GT to KT, [Margin r: Elements VI.1] that is, as GS to SK, or as Q to R, which was to be done.

We shall solve the problem by no other method if the antecedent of the ratio is to be set toward the parts CD. For in this case the given ratio is F to E, or R to Q. Then, the side KG of the square GI (which has been made equal to the quadrilateral AC) being divided at L according to the given ratio F to E, or at S according to the given ratio R to Q, and the parallel LM or ST drawn, [Margin s: Prop. 2 of this book] if upon CD, between the straight lines DA and CB, we construct a rectilinear figure DO equal to the rectangle KM, or a rectilinear figure CDXVB equal to the rectangle KT, by the parallel NO or XV, as was said a little earlier, the problem will be solved, as is clear.

But again let the same quadrilateral ABCD be divided by a parallel to any other line Y in the ratio E to F, so that the antecedent of the ratio inclines toward A. If, then, the given straight line Y is parallel to one of the sides, we shall partition the quadrilateral in the given ratio by a line parallel to that side — and therefore also to the straight line Y — just as we have written just above. But if the straight line Y is parallel to none of the sides, we shall draw from some angle — say from C — a parallel CZ to it, falling inside the quadrilateral; and, the triangle CDZ (or rectilinear figure) being taken away, we shall take from the square GI (which is equal to the whole quadrilateral) a rectangle equal to Kd. (This is done if a square is made equal to the triangle or rectilinear figure CDZ, whose side is a, and if between the two lines KI and a the third proportional Kb is found; for, bd being drawn parallel to KI, [Margin t: Elements VI.17] the rectangle Kd will be equal to the square of the side a — that is, to the rectilinear figure CDZ.) And if indeed the parallel bd falls between KI and LM, [Margin u: Prop. 2 of this book] we shall construct upon CZ, between the straight lines CB and ZA, a rectilinear figure CaeZ equal to the rectangle bM, whose side ae is parallel to the side CZ — that is, to the

given straight line Y; and it will be done as directed. For the whole rectilinear figure CDea will be equal to the whole rectangle KM; and therefore the remainder ABae will be equal to the remainder GM. [Margin x: Elements V.7] Wherefore ABae will be to eacD as GM to KM, [Margin y: Elements VI.1] that is, as GL to LK, or as E to F. But if the parallel bd coincides with the straight line LM, the straight line CZ will accomplish the problem: for since KM is equal to the rectilinear figure CDZ, the remaining rectangle GM will be equal to the remaining rectilinear figure ABCZ; [Margin z: Elements V.7] and therefore ABCZ will be to CDZ as GM to KM, [Margin a: Elements VI.1] that is, as GL to LK, or as E to F. If, finally, the parallel bd falls between LM and GH, the rectilinear figure or triangle taken away, CDZ, will be greater than the rectangle

Diagram. The lower figure — the construction for dividing the quadrilateral by a parallel to a given line Y. At the top is the quadrilateral: A at the upper left, then along the top the points X, N (with a small division-mark reading "Nc"/"Ne"), running toward D at the upper right, with C below D at the right; interior points V, P near the center, B at the lower middle, with small o and a marks. Two segment-marks at the left carry E and F. Below the quadrilateral stands a square divided into vertical strips by dotted lines: across the top G, S, L, (a "6"-like mark), K; across the bottom H, T, M, d, (an "i"-like mark), R. The line Y stands at the right, with small marks z, m, l toward the D–C corner.

- A (upper left of quadrilateral): vertex
- X, N (top edge, with A at the corner; the faint "Nc"/"Ne" was an overlap artifact, not a separate vertex)
- D (upper right): vertex
- C (right, below D): vertex; apex of the parallel CZ
- E, F (left, on segment-marks): the given ratio
- V, P (interior center): division points
- B (lower middle): vertex
- o, a (small interior marks; o = point O on CB near B, a = lowercase point a)
- Square, top edge: G, S, L, b, K (the "6"-like glyph is a lowercase b — Kb, bd, bM)
- Square, bottom edge: H, T, M, d, I (the "[i]"-like mark is the square's bottom-right corner I of GHIK; the R just beyond is a ratio label, not a square vertex)
- Y (right): the given line

- Y (oblique line at right) with point Z (parallel CZ, triangle CDZ) resolved; a couple of tiny construction ticks near the Z / third-proportional region remain [?]

rectangle KM. [Margin: a — 2. huius (i.e. Prop. 2 of this book)] Therefore, if upon CZ, between the straight lines CD and ZD, the rectilinear figure Cm is constructed equal to the rectangle Ld, the remainder Dml will be equal to the remaining rectangle Gd; and for that reason the remainder mlCBA will also be equal to the remaining rectangle KM. [Margin: b — 7. quint. (Elements V.7)] Therefore ABCIm will be to lmD as GM is to KM — that is, as GL is to LK, or as E is to F, which is what was proposed.

In exactly the same way we shall cut any other figure whatever, however many sides it has, in a given ratio, by a line parallel to one side or equidistant from any other given straight line. For let there be given any heptagon ABCDEFG, to be cut by a line parallel to side AG into two parts, so that the part inclining toward D has to the remaining part the same ratio that M has to N. Construct the square HIKL equal to the heptagon itself — by the method we set out in the scholium to Elements II.14, or rather by what we wrote at No. 4, chapter 4, book 4 of this work. Then, dividing the side HI at O in the ratio of M to N, and drawing OP parallel to side HL, [Margin: c — 2. huius] upon the straight line AG, between the straight lines AB and GF, let there be made a rectilinear figure AGQR equal to the rectangle IP, having its side QR parallel to side AG. And since QR falls beyond F and B, [Margin: d — 2. huius] we shall again construct, upon the straight line ST, between the straight lines SC and TE, by the straight line VX parallel to ST itself, a rectilinear figure equal to the triangles FQT and BRS lying outside the heptagon; and thus what is proposed will be accomplished. For since the rectilinear figure AGQR — and therefore also the rectilinear figure ABVXFG — is equal to the rectangle IP, the remainder DEXVC will likewise be equal to the remaining rectangle OL. [Margin: e — 7. quint. (Elements V.7)] Therefore DEXVC will be to ABVXFG as OL is to IP, [Margin: f — 1. sexti (Elements VI.1)] that is, as HO is to OI, or as M is to N, which was to be done.

But if the side of the rectilinear figure cut off from the heptagon must be equidistant from a straight line Y that is parallel to no side of the heptagon (for if it were parallel to one side, the problem would be solved as just now delivered), draw from the angle G the line GZ parallel to Y, which falls within the figure. [Margin: g — 2. huius] We shall construct upon the straight line IK a rectangle hK equal to the rectilinear figure AGZ. This is done by making a square equal to the rectilinear figure AGZ, whose side is l, and then finding the third proportional Ih

to the two lines IK and I; for, drawing hi parallel to IK itself, [Margin: h — 17. sexti (Elements VI.17)] the rectangle hK will be equal to the square on that side — that is, equal to the rectilinear figure AGZ. [Margin: i — 2. huius] Next, upon the straight line GZ, between the straight lines GF and ZB, we shall construct a rectilinear figure GZba equal to the rectangle P, by means of the parallel ba. [Margin: k — 2. huius] For if, upon the triangles Bbe and Fad, there be made — upon the straight line de, between the straight lines eC and dE, by the parallel fg — a rectilinear figure defg equal to them, then what the problem proposes will be accomplished, as is clear from what has been said. And the reasoning is entirely the same in all other rectilinear figures, however irregular, provided that in them one line can be drawn parallel to the given straight line which cuts off a rectilinear figure equal to the given one. For that this cannot always be done in figures whose angles turn partly inward and partly outward we have declared at the end of Prop. 2 of this book; and the construction of the problem itself will show us this plainly.

SCHOLIUM

Any rectilinear figure whatever can therefore be divided into however many equal parts you please by lines equidistant from any given straight line. [Margin: Quo pacto figura data secetur per lineas parallelas in quotuis partes aequales — "How a given figure is to be cut by parallel lines into any number of equal parts."] For if, say, a given figure is to be cut into 8 equal parts by lines parallel to a given straight line, we shall first divide it into two parts having to each other the ratio 1 to 7. In this way the first part will be $\frac{1}{8}$ of the whole figure. Then we shall cut the latter part in the ratio 1 to 6, so that the first part of this division is $\frac{1}{7}$ of that divided part — that is, $\frac{1}{8}$ of the whole figure — since that divided part comprises $\frac{7}{8}$ of the whole figure. Afterward we shall partition the latter part of the next division in the ratio 1 to 5; and the latter part of this division in the ratio 1 to 4; and so on, always diminishing, until we come to a part that must be cut in the ratio 1 to 1 — that is, into equal parts.

This same thing can also be accomplished by the method we set out at the end of the scholium to Prop. 4: namely, if the side HI of the square that has been constructed equal to the given rectilinear figure is cut into as many equal parts as those into which the given rectilinear figure is to be divided; and first the rectilinear figure is divided in the ratio of the first part to the rest; then the latter part of the rectilinear figure is divided in the ratio of the second part of side HI to the rest; and so on, and so forth.

And here our Geodesy, comprising the division of all rectilinear figures, comes to an end. There now follow some particular divisions of certain figures which — both because they contain subtle and acute demonstrations, and because learned geometers too, such as Leonardo of Pisa [Leonardo Pisano, Fibonacci], Friar Luca Pacioli [Luca Pacioli], and Niccolò Tartaglia [Nicolaus Tartalea], have handed them down — have seemed in no way to be omitted. But in order to demonstrate them geometrically, some theorems must first be laid down, of which let this be the first.

THEOR. 2. PROP. 6.

If two equal triangles have one common side and incline toward opposite directions, then the straight line connecting the opposite angles is cut in two equal parts by that common side.

Let ABC and ABD be two equal triangles having the side AB in common and inclining toward opposite directions. I say that the straight line CD, joining the opposite angles C and D, is cut in two equal parts at E by the common side AB. [Margin: a — 1. sexti (Elements VI.1)] For since the triangle ACE is to the triangle ADE, as the triangle BCE is to the triangle BDE, as CE is to ED, [Margin: b — 11. quin. (Elements V.11)] the triangle ACE will be to the triangle ADE as the triangle BCE is to the triangle BDE. [Margin: c — 12. quin. (Elements V.12)] Therefore the two triangles together ACE and BCE — that is, the whole triangle ABC — will be to the two triangles together ADE and BDE — that is, to the whole triangle...

Diagram. The construction figure for the heptagon division. An irregular heptagon is drawn at the top, with an auxiliary square and construction lines beneath. Because the interior labels are densely overprinted on the woodcut, several are hard to resolve.

- A (top right of heptagon): vertex
- B (top, left of A): vertex
- R (top, left of B): construction point
- Z (top, between B and A): construction point [?]
- C (upper left): vertex
- V, S (upper left, inside): construction points; a small letter near C reads e [?] or g [?]
- D (left): vertex
- E (lower left interior): construction point
- X (interior, near E): construction point; adjacent small ligature ts/τς [?]

- d (interior center): construction point
- F (lower center-right): vertex
- G (right): vertex
- Q (lower left, below E): construction point
- a (interior, below d): construction point
- M, N (two short vertical segments, right of the heptagon): the two magnitudes M and N of the given ratio
- H, O (top edge of the square), I (top right of square): H and I are corners of the square; O divides HI
- h (top right, near I): construction point [?]
- Y (oblique line at upper left of the square): the given straight line Y
- ((a small bracket-like mark at left, labelled l): the line l [?]
- L (bottom left of square), P (bottom center), K (bottom right of square): corners / division point of the square HIKL

Diagram. The figure for Theorem 2 / Prop. 6: two equal triangles sharing side AB, with the connecting line CD crossing AB.

- A (top): common vertex
- B (bottom): common vertex; AB is the shared side
- C (left): apex of one triangle
- D (right): apex of the other triangle
- E (center): the point where CD meets AB

ABD — as ACE is to ADE, that is, as CE is to ED. Since therefore the triangles ABC and ABD are supposed equal, the straight lines CE and ED will also be equal, and consequently CD is cut in two equal parts at E. Which was to be shown.

THEOR. 3. PROPOS. 7.

If in a triangle a line is drawn parallel to the base, and the ends of the parallels are joined by straight lines cutting each other, then the segment of either of these straight lines, beginning from the angle and terminated at the remaining side, will have the same ratio that the side divided by that line has to its upper part. Moreover, the straight line drawn from the third angle through the intersection of the said straight lines will cut each parallel into two equal parts.

In the triangle ABC let DE be drawn parallel to the base BC, and let the joined straight lines BE and CD cut each other at F. I say that BF is to FE as AC is to AE; likewise that CF is to FD as AB is to AD; and that the joined straight line AF cuts

the parallels DE and BC in two equal parts at G and H. [Margin: a — 37. prim. (Elements I.37)] For since the triangles BDC and CEB are equal, when the common triangle BFC is taken away, the remaining triangles BDF and CEF will also be equal. [Margin: b — 2. sexti (Elements VI.2)] And since it is true that as BD is to DA, so is CE to EA, [Margin: c — 1. sexti (Elements VI.1)] and as BD is to DA, so is the triangle BFD to the triangle AFD, and as CE is to EA, so is the triangle CFE to the triangle AFE [the print appears to repeat "CFE"; recte AFE], then the triangle BFD will be to the triangle AFD as the triangle CFE is to the triangle AFE. Since therefore the triangle BFD has been shown equal to the triangle CFE, [Margin: d — 14. quin. (Elements V.14)] the triangle AFD will also be equal to the triangle AFE. [Margin: e — 6. huius (Prop. 6 of this book)] Therefore DE is cut in two equal parts at G, [Margin: f — schol. 4. sexti (scholium to Elements VI.4)] and consequently the parallel BC will also be cut in two equal parts at H. [Margin: g — 7. quint. (Elements V.7)] And since the triangle AFB has the same ratio to the equal triangles AFD and AFE, [Margin: h — 1. sexti (Elements VI.1)] and since as AFB is to AFD, so is AB to AD, and as AFB is to AFE, so is BF to FE, then BA will also be to AD — and hence AC to AF — as BF is to FE. And by the same reasoning AB will be to AD, or AC to AE, as CF is to FD, which is also plain from this: since as CF is to FD, so is CFE to DEF, [Margin: i — 1. sexti (Elements VI.1)] [Margin: k — 1. sexti (Elements VI.1)] that is, so is BFD (being equal to CFE itself) to the same DEF, that is, so is BF to FE. Which was to be demonstrated.

Diagram. The figure for Theorem 3 / Prop. 7: triangle ABC with DE parallel to the base BC, and the cross-lines BE, CD meeting at F, with AF continued through G and H.

- A (top): apex
- D (upper left, on AB): endpoint of the parallel DE
- E (upper right, on AC): endpoint of the parallel DE
- G (center, on DE): where AF crosses DE
- F (center, below G): intersection of BE and CD
- B (bottom left): base vertex
- H (bottom center, on BC): where AF crosses BC
- C (bottom right): base vertex

THEOR. 4. PROPOS. 8.

If in a triangle two straight lines are drawn from two angles to the midpoints of the opposite sides, then the straight line drawn from the remaining angle through their intersection also cuts the remaining side in two equal parts. Moreover, the

segment of each of those three lines near the angle has to the remaining segment the ratio of two to one. Finally, the triangle is divided by the straight lines drawn from the intersection to the angles into three equal triangles.

In the triangle ABC of the preceding proposition, let the two straight lines BE and CD cut the sides AC and AB in two equal parts at E and D, and let them intersect each other at F. I say that the straight line AF, when drawn, also cuts the side BC in two equal parts at H, and so on. [Margin: a — 2. sexti (Elements VI.2)] For the joined straight line DE will be parallel to BC itself, since it cuts the sides AB and AC proportionally — namely, into equal parts. [Margin: b — 7. huius (Prop. 7 of this book)] For which reason AE will cut each parallel, DE and BC, in two equal parts. This is the first point. Next, [Margin: c — 7. huius (Prop. 7 of this book)] since it is true that as AB is to AD, so is CF to FD, and AB is double of AD, CF will also be double of FD. And by the same reasoning BF will be double of FE, and AF double of EH. This is the second point. Finally, [Margin: d — 1. sexti (Elements VI.1)] since it is true that as AF is to FH, so is the triangle AFB to the triangle BFH, and AF has been shown to be double of FH, the triangle AFB will also be double of the triangle BFH. And the triangle BFC is likewise double of the same triangle BFH, [Margin: e — 38. prim. (Elements I.38)] because the triangles BFH and CFH are equal. Therefore the triangles AFB and BFC will be equal. And in the same way the triangle AFC will be equal to the same triangle BFC; and consequently all three, AFB, BFC, CFA, will be equal. This is the third point.

Diagram. The figure for Theorem 4 / Prop. 8: triangle ABC with the two medians BE and CD meeting at F, and AF continued to H, the midpoint of BC.

- A (top): apex
- D (upper left, on AB): midpoint of AB
- E (upper right, on AC): midpoint of AC
- G (center interior): a small interior point [?]
- F (center, below G): intersection of the medians
- B (bottom left): base vertex
- H (bottom center, on BC): midpoint of BC
- C (bottom right): base vertex

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And so a point can easily be found within a triangle from which three straight lines drawn to the three angles partition the triangle itself into three equal triangles. For such a point in the given triangle is F, where two straight lines

drawn from any two angles to the midpoints of the opposite sides intersect each other, as we showed in the third part of this proposition.

THEOR. 5. PROPOS. 9.

If in a triangle a straight line is drawn cutting two sides in any manner whatever, then the whole triangle will be to the cut-off triangle as the rectangle contained by the two cut sides of the whole triangle is to the rectangle contained by the two sides of the cut-off triangle, which are the former's segments.

In the triangle ABC let the straight line DE cut the sides AB and AC at D and E. I say that as the rectangle under AB, AC is to the rectangle under AD, AE, so is the triangle ABC to the triangle ADE. For since the triangles ABC and ADE have the common angle A, they will have — by Prop. 4 of the scholium to Elements VI.23 — the same ratio that the rectangles contained under the sides AB, AC and under AD, AE have. Which was to be shown.

Diagram. The figure for Theorem 5 / Prop. 9: triangle ABC with the straight line DE cutting sides AB and AC.

- A (top): apex
- D (left, on AB): where DE meets AB
- E (center, on AC): where DE meets AC
- B (bottom left): base vertex
- C (bottom right): base vertex

PROBL. 5. PROPOS. 10.

To divide a given triangle, from a given point on one of its sides, into any number of equal parts you please.

By the fourteenth proposition of the scholium to Elements VI.33 we handed down the rule by which a triangle is to be divided into two parts according to a given ratio, and how a required part is to be cut off from a triangle. If, therefore, a triangle is to be cut, from any given point on one of its sides, into any number of equal parts, there must first be cut off — by a straight line drawn from the given point — the part named by the number of parts into which the triangle is to be divided; then two such parts; afterward three; and so on, until as many parts as there are, less one, have been cut off, into however many parts it is proposed to divide the triangle. For example, if the triangle ABC is to be divided, from the point D, into five equal parts, we shall divide the side BC, on which the given point lies, into five equal parts, at the points E, F, G, H. Then, joining the straight

line DA, we shall draw parallel to it the lines EI, FK, GL, HM. For if the straight lines DI, DK, DL, DM are connected, the triangle will be divided into five equal parts. For, as was shown in the said Prop. 14 of the scholium to Elements VI.33, the triangle DBI is $\frac{1}{5}$ of the whole triangle — that is, DBI is to ABC as BE is to BC. The triangle DBK contains $\frac{2}{5}$ of the whole triangle — that is, DBK is to ABC as BF is to BC. But the triangle DBL comprises $\frac{3}{5}$ of the whole triangle — that is, DBL is to ABC as BG is to BC. Finally the quadrilateral ABDM comprises $\frac{4}{5}$ of the whole triangle — that is, ABDM is to ABC as BH is to BC. From which it follows that the remaining triangle DMC is $\frac{1}{5}$ of the same triangle ABC.

When the given point is at one of the angles, it is plain that, if the opposite side is cut into as many parts as those into which the triangle is to be divided, [Margin: a — 1. sexti (Elements VI.1)] the straight lines drawn from that angle to the points of division will cut the triangle into the proposed equal parts.

Diagram. The figure for Problem 5 / Prop. 10: triangle ABC with base BC divided at E, F, G, H and cevian DA, with the parallels EI, FK, GL, HM meeting AB at I, K, L, M.

- A (top): apex
- L (upper, on AB, near A): endpoint of parallel GL
- K (upper left, on AB): endpoint of parallel FK
- M (upper right interior, on AB region): endpoint of parallel HM
- I (left, on AB, below K): endpoint of parallel EI
- B (bottom left): base vertex
- E, F, G, D, H (along base BC, left to right): E, F, G, H are the equal-division points; D is the given point (between G and H)
- C (bottom right): base vertex

PROBL. 6. PROPOS. 11.

To divide a given triangle, by lines parallel to one side, into any number of equal parts you please.

Let the triangle ABC be divided, for example, into four equal parts by lines equidistant from the side BC. Let either of the remaining sides — namely AB — be cut into 4 equal parts, that is, into as many as those into which the triangle is to be divided, at the points D, E, F. And between AB and AD let the mean proportional AE be found; between AB and AE the mean proportional AG; and finally between AB [and AF, recte] the mean proportional AH; and let EI, GK, HL be drawn parallel to the side BC — which lines, I say, partition the triangle into 4

equal parts. [Margin: a — corol. 4. sexti (corollary to Elements VI.4)] For since the triangle ABC is similar to the triangle AEI, [Margin: b — corol. 19. sexti (corollary to Elements VI.19)] the triangle ABC will be to the triangle AEI as AB is to AD, because the three AB, AE, AD are continually proportional. But AD is the fourth part of AB itself. Therefore the triangle AEI too is the fourth part of the triangle ABC. [Margin: c — corol. 19. sexti (corollary to Elements VI.19)] In no other way shall we show that the triangle ABC is to the triangle AGK as AB is to AE, because the three AB, AG, AE too are continually proportional. Wherefore, since AE contains $\frac{2}{4}$ of the straight line AB, AGK too will contain $\frac{2}{4}$ of the triangle ABC; and so, since AEI is $\frac{1}{4}$ of the triangle ABC as we showed, EIKG will be $\frac{1}{4}$ of the same triangle ABC. Finally, by the same reasoning, the triangle ABC will be to the triangle AHL as AB is to AF, because the three AB, AH, AF too are continually proportional; and consequently the triangle AHL will comprise $\frac{3}{4}$ of the triangle ABC, just as AF contains $\frac{3}{4}$ of AB itself; and therefore BHLC will be $\frac{1}{4}$ of the triangle ABC, and so forth.

Diagram. The figure for Problem 6 / Prop. 11: triangle ABC with side AB divided at D, E, F and lines EI, GK, HL parallel to BC cutting AC at I, K, L.

- A (top right): apex
- D (upper, on AB): first division point of AB
- E (center, on AB): division point of AB
- G (lower left, on AB): division point (mean proportional point)
- F, H (lower left, on AB, near B): division points
- B (bottom left): base vertex
- I (right, on AC, near A): endpoint of parallel EI
- K (right, on AC): endpoint of parallel GK
- L (lower right, on AC, near C): endpoint of parallel HL
- C (bottom right): base vertex

PROBL. 7. PROPOS. 12.

To divide a given triangle, by a straight line drawn from a given point outside the triangle, into two equal parts.

From the given point D, outside the triangle ABC, a line must be drawn dividing the triangle in two. Draw the straight line DA to the opposite angle, cutting the side BC at E. If indeed BC is divided in two equal parts at E, what is required will have been accomplished, [Margin: a — 38. prim. (Elements I.38)] because the triangles ABE and ACE will then be equal. But if BC is not divided in two equal

parts at E, let the segment CE be the greater; to it draw DF parallel, meeting the side AC, produced, at F. Cutting the side AC in two equal parts at G, let the fourth proportional CH to the three DF, BC, CG be found. [Margin: b — 16. sexti (Elements VI.16)] And the rectangle under DF, CH will be equal to the rectangle under BC, CG — that is, to half the rectangle under BC, CA — [Margin: c — 1. sexti (Elements VI.1)] since the rectangle under BC, CA is double the rectangle under...

BC, CG. Then, having found L, the mean proportional between FC and CH [Margin: d. Euclid VI.17] — so that the square on L equals the rectangle contained by FC and CH — let the straight line HI be joined to CH itself in such a way that the rectangle contained by the whole line CI together with the added segment HI equals the square on L, that is, the rectangle contained by FC and CH, exactly as we wrote at the end of the scholium to Proposition 36 of Euclid's Book 3. Then let the straight line DI be drawn, cutting BC in K. I say that the line DI cuts the triangle ABC into two equal parts, ABKI and IKC.

For by the construction the rectangle contained by CI and IH equals the rectangle contained by CF and CH. [Margin: e. Euclid VI.16] Therefore, as CI is to CF, so is CH to IH; and by inversion, as CF is to CI, so is IH to CH; and by composition, as IF is to CI, so is CI to CH. [Margin: f. Euclid VI.4, and by alternation] Now as IF is to CI, so is FD to CK. Therefore FD is also to CK as CI is to CH. [Margin: g. Euclid VI.16] Consequently the rectangle contained by FD and CH will equal the rectangle contained by CK and CI. But the rectangle contained by FD and CH was, by the construction, equal to half the rectangle contained by BC and CA. Therefore the rectangle contained by CK and CI likewise equals half the rectangle contained by BC and CA. But as the rectangle contained by CK and CI is to the rectangle contained by BC and CA, [Margin: h. Prop. 9 of this book] so is the triangle CKI to the triangle ABC. Therefore the triangle CKI will also equal half the triangle ABC; and so the quadrilateral ABKI will equal the remaining half of the triangle ABC. Which is what was proposed.

By the same reasoning, if in place of CG we take one-third, or one-fourth, or any part whatever of the side AC, and carry out the rest as above, then by the straight line drawn from D we shall cut off one-third, or one-fourth — or in general such a part — from the triangle ABC as the part CG taken from AC itself, as is evident.

Leonardo of Pisa [Fibonacci] and Niccolò Tartaglia solve this same problem when the given point lies outside the triangle, in such a position that one side of the triangle, when extended, falls upon it — as the given point F would be. Likewise when it lies between two extended sides: for instance, if the triangle were AEC and the point lay between B and D, so that a line cutting the side AC could be

drawn from it only through the angle E — since straight lines drawn from it to the angles A and C would cut no side. But because these cases are more curious than useful, we deliberately omit them; anyone who wants them may read the authors just named.

Diagram. A triangle drawn with the apex A at upper right, the vertex F at the lower left, and the vertex B at the right, with several interior points marking the division of the triangle by the line DI (Problem 7 / Prop. 12). At the upper left, set apart, is a short reference segment labelled L (the mean proportional).

- L (short segment, upper left): the mean proportional length
- A (upper-right apex)
- G (near the top edge, left of A)
- I (near the top, between G and A)
- H (upper left, left of G)
- C (left vertex, mid-height)
- K (lower interior, center)
- E (lower interior, right of K)
- D (bottom, below E)
- B (right vertex)
- F (lower-left vertex)

PROBL. 8. PROPOS. 13.

To divide a given parallelogram into any number of equal parts by lines equidistant from two opposite sides.

Let ABCD be the parallelogram, to be divided — for example — into three equal parts by lines equidistant from the sides AB and DC. Divide one or the other of the two remaining sides — namely BC — into three equal parts (that is, into as many parts as the parallelogram is proposed to be divided) at the points E and F, and through these draw EG and FH parallel to AB and DC. What is required will then be done, [Margin: a. Euclid VI.1, or I.38] because the parallelograms AE, EH, and HC are equal, on account of their equal bases BE, EF, FC.

Diagram. A parallelogram divided into three equal strips by two internal lines parallel to a side. The upper side carries, left to right, the points A, G, H, D; the lower side carries B, E, F, C. The internal lines are EG and FH.

- A (upper-left vertex)
- G (upper edge)

- H (upper edge)
- D (upper-right vertex)
- B (lower-left vertex)
- E (lower edge)
- F (lower edge)
- C (lower-right vertex)

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Therefore, if from a side one-half, or one-third, or one-fourth, or in short any part or parts whatever is cut off, and through its endpoint a line parallel to the side AB is drawn, the same part (or the same parts) will be cut off from the whole parallelogram. Thus you see that AE is one-third of the parallelogram AC, just as BE is one-third of the side BC, and so forth.

PROBL. 9. PROPOS. 14.

To bisect a given parallelogram by a straight line drawn through a point — whether outside it, or inside it, or on some given side.

Let ABCD first be a parallelogram to be bisected by a straight line drawn through an exterior point E. Draw the diagonal BD, bisect it at F, and from E through F draw the straight line EH, which I say divides the parallelogram into two equal parts. For, as we demonstrated in the scholium to Proposition 34 of Euclid's Book 1, the line GH, dividing the diagonal BD in two at F, bisects the parallelogram. The same will happen if the straight line IK, cutting the sides AD and BC, is bisected at F, and through F the straight line EF is extended — because IK bisects the diagonal, and therefore passes through F, the midpoint of the diagonal. [Margin: a. Euclid I.29] For since the angles IDF and FID are equal to the alternate angles KBF and FKB, and the sides ID and KB adjacent to them are equal, [Margin: b. Euclid I.26] both the sides DF and BF, and IF and KF, will be equal to one another.

In the same way, from the interior point L, and likewise from the point G on the side BC, a straight line drawn — LF or GF — will bisect the parallelogram.

Diagram. A parallelogram with its diagonal and crossing lines for the bisection. The upper side carries, left to right, A, H, I, D; the lower side carries B, K, and (near it) G, E, then C. Near the center are the points F (midpoint of the diagonal) and L. The diagonal BD, the line IK, and the line GH are drawn through F.

- A (upper-left vertex)
- H (upper edge)

- I (upper edge)
- D (upper-right vertex)
- B (lower-left vertex)
- K (lower edge)
- G (lower edge, near E)
- E (lower edge)
- C (lower-right vertex)
- F (center, midpoint of the diagonal)
- L (interior, near center)

PROBL. 10. PROPOS. 15.

Between two given straight lines, to find two mean proportionals approximating the true values.

Having set out our geodesy in the previous five propositions of this book, together with nine other propositions bearing on the same subject and now demonstrated — which we judged should be added — it would now be time to treat of enlarging and diminishing figures in a given ratio, as we proposed in the title of this Book 6. But since, just as this cannot be done in plane figures without finding a mean proportional between two given straight lines (which discovery Euclid handed down in Book 6,

Proposition 13), so likewise it can in no way be accomplished for solid figures unless two mean proportionals are found between two given straight lines. Therefore, in this proposition, we shall first bring forward what the ancient geometers have left us in writing on this matter. For this problem has exercised and tormented the talents of many, although to this day no one has found — truly and geometrically — two mean proportionals between two given straight lines. Passing over, then, the methods of Eratosthenes, Plato, Pappus of Alexandria, Sporus, and Menaechmus (the latter both by means of the hyperbola and the parabola, and by means of two parabolas), and of Archytas of Tarentum — however keen and subtle these are — we shall explain only the four handed down by Heron, Apollonius of Perga, Philo of Byzantium, Philoponus, Diocles, and Nicomedes, which we have judged more convenient, easier, and less liable to error. Anyone who desires the methods of the others may read them in the Commentaries of Eutocius of Ascalon on Book 2 of Archimedes' *On the Sphere and Cylinder*, and likewise in the little book of Johannes Werner of Nuremberg *On Conic Sections*. Let us therefore begin here.

The Method of Heron in the *Mechanical Introductions* and in the Building of Artillery, which is also ascribed to Apollonius of Perga

Let AB and BC be two straight lines, between which two mean proportionals are to be found. Set them up at a right angle at B, and complete the rectangle ABCD, with its diagonals AC and BD, [Margin: a. Scholium to Euclid I.34] which bisect each other at E. (It would be enough to draw just one diagonal and to bisect it at E.) Now, with the sides DA and DC produced, imagine a ruler turned about the point B this way and that, until it cuts DA and DC (produced) in F and G, in such a way that the lines EF and EG drawn out are equal. Or, as Apollonius prefers, from E let several circles be described — LI, GF, MN — until the chord of one of the arcs passes precisely through the point B, such as GF is. For if the chord LI passes above B, a circle is to be described below L; but if it passes below the point B, as the chord MN does, a circle is to be described above M. And this operation is to be repeated as often as necessary until some chord — such as GF — passes through B. For in this way the lines EF and EG, drawn from the center E to the circumference, together with GF, will be equal to one another. These things being thus constructed, I say that AF and CG are mean proportionals between AB and BC — that is, that AB is to AF as AF is to CG, and as CG is to CB.

For when AD and CD are bisected at K and H, [Margin: b. Scholium to Euclid I.28] the lines EK and EH drawn to AD and CD will be perpendicular. [Margin: c. Euclid II.6] Now since the rectangle contained by DE and AF, together with the square on AK, equals the square on KF, then, adding the common square on EK, the rectangle contained by DF and AF together with the squares on AK and EK — [Margin: d. Euclid I.47] that is, together with the square on EA — will equal the squares on KF and EK, [Margin: e. Euclid I.47] that is, the square on EF, that is, the square on EG, which is equal to EF itself. By the same reasoning we shall show that the rectangle contained by DG and GC, together with the square on CE — that is, on EA — equals this same square on EG. Therefore the rectangle contained by DF and AF, together with the square on EA, will equal the rectangle contained by DG and GC together with the square on EA; and when the common square on EA is taken away, the rectangle contained by DG and GC will remain equal to the rectangle contained by DF and AF. [Margin: f. Euclid VI.16] Consequently DG will be to DF as AF is to CG; [Margin: g. Euclid VI.4] and as DG is to DF, so is AF to CG — that is, the three lines AB, AF, CG are continued proportionals. [Margin: h. Euclid VI.4] But again, as DG is to DF, so is CG to CB. Therefore CG is also to CB as AB is to AF, and hence as AF is to CG. Therefore the four lines AB, AF, CG, CB will be continued proportionals. Which was to be demonstrated.

Diagram. Heron's construction for two mean proportionals. A horizontal base line runs along the bottom carrying, left to right, the points D, K, A, N, F, I; a vertical line rises at the left carrying, from bottom to top, D, H, C, M, G, L. E is at the center. The point B lies on the upper arc and O on the lower-right arc. Three concentric arcs are struck from E — corresponding to the chords LI, GF, MN — with the rectangle ABCD and its diagonals inscribed.

- L (top of the left vertical line)
- G (left vertical, below L)
- M (left vertical, below G)
- C (left vertical, below M)
- H (left, mid-height, on the base of the rectangle)
- E (center)
- B (upper right, on the arc)
- O (right, lower, on the arc)
- D (lower-left corner)
- K (base line)
- A (base line)
- N (base line)
- F (base line)
- I (base line, right end)

The Method of Philo of Byzantium, which is also attributed to Philoponus

Again, in the same figure, let two mean proportionals be found between the straight lines AB and BC. With the rectangle ABCD constructed, together with the diagonal CA, and with the sides DA and DC produced as above, let a circle CBA be described from E, the midpoint of the diagonal, with radius EC (or EA), [Margin: a. Scholium to Euclid III.31] which will necessarily pass through the right angle B. Then let a ruler be turned about the point B this way and that, cutting DA and DC (produced) in F and G, and the circumference in O, until BG and OF are equal. This will happen if many hidden lines are drawn through B; for one of them will have the segment between the straight line DG and the circle equal to the segment between DF and the same circle. These things being done, I say that AF and CG are mean proportionals between AB and CB. For since GB and FO are equal, adding the common BO, GO and FB will also be equal; and therefore the rectangle contained by GO and GB will equal the rectangle contained by FB and FO. [Margin: b. Corollary 1 to Euclid III.36] But the former equals the rectangle

contained by DG and GC, and the latter equals the rectangle contained by DF and AF. Therefore the rectangle contained by DG and GC will also equal the rectangle contained by DF and AF. For this reason, exactly as in the preceding method, we shall show that AB, AF, CG, CB are continued proportionals. Which is what was proposed.

The Method of Diocles in the Book *On Burning-Mirrors* — A Most Excellent One

Diocles first sets out a lemma of this kind. Let a circle ABCD be described, with center E, and with diameters AC and BD cutting each other at right angles at the center E. Then, taking two equal arcs DF and DG, let the straight line CG be joined, and through F let FK be drawn parallel to BD, cutting CG in H. This done, FK and KC will be mean proportionals between AK and KH. For, drawing GL parallel to BD, and joining the straight lines EF and EG, [Margin: a. Euclid III.27] since the angles LEG and KEF, standing on the equal arcs AG and CF, are equal, [Margin: b. Euclid I.27] and the angles at L and K are right, and the sides EG and EF equal, [Margin: c. Euclid I.26] then

Diagram. The Diocles lemma figure: a circle with two perpendicular diameters and a web of chords (the arcs DF, DG, the lines CG, FK, GL). The letters are heavily overprinted where the chords cross, so several interior labels are uncertain. Along the horizontal diameter, left to right: A, L, E, K, P, C; B is at the top; along the lower part, M, G, D, I, H, F, and N.

- A (left end of the horizontal diameter)
- L (on the horizontal, right of A)
- E (center)
- K (on the horizontal, right of E)
- P (on the horizontal, right of K)
- C (right end of the horizontal diameter)
- B (top)
- M (lower left)
- G (lower left, below M)
- D (bottom, center)
- I (interior, lower center)
- H (interior, lower center) [?]
- F (lower right)
- N (bottom right)

Diagram. A second, companion circle (partly running off the right edge of the type-block) illustrating the parallel case with equal arcs DM, DN and the line CM cut by NP. A short reference segment labelled O over N ($\overline{O}$) is set above the circle at upper left. Readable letters: F at the top; along the left, A; then B, K in the upper middle; D at the right; and C, L, G, E, H in the lower half.

- O (short reference segment, upper left, over N)
- F (top)
- A (left)
- B (upper middle)
- K (upper middle, right of B)
- D (right)
- C (lower left)
- L (lower middle)
- G (lower right)
- E (bottom)
- H (bottom, right of E)

GL, FK, and EL, EK are equal to one another; and hence the remaining parts AL and CK are equal; indeed, adding the common LK, the whole lines AK and CL will be equal to one another. [Margin: e. Euclid VI.4] Since, then, CL is to LG as CK is to KH, and since CL is to LG as AK is to KF (because these are equal to those), AK will also be to KF as CK is to KH. [Margin: f. Scholium to Euclid VI.13] But as AK is to KF, so is KF to CK. Therefore AK is to KF as KF is to CK, and as CK is to KH — that is, KF and CK are mean proportionals between AK and KH. Which is what was proposed. By like reasoning, if — taking the equal arcs DM and DN, and joining the straight line CM — NP is drawn parallel to BD, cutting CM in O, then PN and CP will be mean proportionals between AP and PO, and so forth.

With this lemma laid down, let two mean proportionals be found between the straight lines AB and BC. In the other figure, set them at a right angle at B; and, with center B and radius equal to the greater line BA, let a circle AFDE be described. Extend AB and BC as far as its circumference. Then, from A through C draw the straight line ACG, and in the quadrant DE take a point H in such a position that, when HK is drawn parallel to EF cutting AG itself in L, a straight line drawn out from D through L cuts off an arc EM equal to the arc EH. For in this way, by the lemma just laid down, KH and DK will be mean proportionals between AK and KL. [Margin: g. Euclid VI.4] And since AK is to KL as AB is to BC, if we make AK to KH as AB is to N, and KH to DK as N is to O, then N and O will

also be mean proportionals between AB and BC. For there is no doubt that O is to BC as DK is to KL. Since AK is to KL as AB is to BC, and AK has to KL the triplicate ratio of AK to KH — that is, of AB to N — AB will also have to BC the triplicate ratio of AB to N. Since, then, the ratio of AB to N equals the ratio of N to O, the ratio of O to BC will equal the same, so that three equal ratios stand between AB and BC. Therefore the four lines AB, N, O, BC are continued proportionals, just as the four AK, KH, DK, KL. Which is what was proposed.

However, because it seemed difficult for Diocles to take, in the latter figure, the point H in such a position that the line DM — cutting AG and the parallel HK in L — should cut off an arc EM equal to the arc EH, he described a certain curved line, most apt for this purpose, in the following way. Let a circle ABCD be described, with center E, and with diameters AC and BD cutting each other at right angles at E. Then in the quadrant CD take any number of points at small distances from one another, and transfer these in order into the quadrants DA and BC from the points D and B. After this, applying the ruler to two points of the quadrants DC and BC that are equally distant from B and D, draw hidden lines, [Margin: h. Scholium to Euclid III.27] which will be parallel to BD. And from C to each of the points of the quadrant DA let hidden straight lines be sent out, and let their intersections with the aforesaid hidden parallels be marked — namely the point T, where the line drawn from C to the point next to D cuts the parallel nearest to BD, and so on in succession. For if all these points of intersection are duly joined by a curved line, such as CKTD, there will be constructed a figure most apt for finding two mean proportionals. For let two mean proportionals be sought between two lines F and G. On the diameter AC, produced too if need be, take AH equal to the greater line F. Then, drawing the perpendicular HP, cut off HI equal to the lesser line G. Draw AI, cutting the curved line in K, and through K draw LM parallel to BD. Finally, taking LN equal to LC, draw through N and M the straight lines AN and AM, cutting HP in O and P. I say that HP and HO are mean proportionals between AH and HI — that is, between F and G. For since the point K of the curved line was found by a straight line drawn to that point of the quadrant DA which is as far distant from the point D as the point M is distant from the same — as is clear from the description of the curved line — by the lemma of Diocles the four straight lines AL, LM, LC (or LN), and LK will be continued proportionals. [Margin: i. Euclid VI.4] Since, then, the four straight lines AH, HP, HO, HI are proportional to these four, they too will be continued proportionals. Which is what was proposed.

Diagram. The Diocles curved-line (cissoid) construction. A circle ABCD with center E is drawn, with the horizontal diameter AC extended to the right toward

H, and B at the top, D at the bottom. The curved line CKTD is traced through the interior points K, T, M. A vertical line rises at the right (the perpendicular HP), carrying, top to bottom, H, I, O, P. Along the horizontal, left to right: A, E, L, C, then H outside the circle. At the lower left are three (or four) reference segments labelled G, R, Q. Point N (И) lies in the interior.

- A (left, on the horizontal diameter)
- B (top of the circle)
- E (center)
- L (on the horizontal, right of E)
- C (on the horizontal, right of L)
- H (right, on the horizontal produced, outside the circle)
- K (interior, on the curved line)
- N (interior, marked И)
- T (interior, on the curved line, lower)
- M (interior, lower, on the curved line)
- D (bottom of the circle)
- I (right vertical line, upper)
- O (right vertical line, middle)
- P (right vertical line, bottom)
- G, R, Q (reference segments, lower left)

The Method of Nicomedes in the Book *On Conchoid Lines*

Nicomedes first builds a certain instrument by which he describes a curved line, which he calls the "conchile" or "conchoid." But we, omitting that instrument, describe the same line by points (which is sufficient for our purpose) in the following way. Let there be a straight line AB, and perpendicular to it CD at the point E. Take, below E, the point D as the pole of the line to be described, and above E another point C, at pleasure. In the actual use of the described line it will be settled how far both the point D and the point C ought to be from the point E. If, then, from D very many hidden lines a little apart from one another are drawn, and from each are cut off equal portions of straight line equal to EC, always beginning from the straight line AB; and if the endpoints of these portions are joined by a curved line, the conchoid line will be described. You have an example in the four lines DH, DG, DF, DN, on which are taken LH, KG, SF, BN equal to EC itself, and through whose endpoints H, G, F, N the curved line passes. And the more hidden lines are drawn out from D, the more closely spaced will be the points found, through which the curved line must pass.

Diagram. Nicomedes' point-by-point construction of the conchoid. A horizontal base line AB carries, left to right, the points A, M, L, K, E, S, B; the perpendicular CD passes through E, with C above and the pole D below. From the pole D radiate the lines DH, DG, DF, DN; along them the equal segments LH, KG, SF, BN are laid off, and the endpoints H, G, F, N lie on the conchoid curve at the top of the figure. Intermediate points O and Q are marked; reference segments P and R are set at the upper and lower left, and an arrow marks N at the right.

- A (base line, left)
- M (base line)
- L (base line)
- K (base line)
- E (base line, foot of the perpendicular)
- S (base line)
- B (base line, right)
- C (on the perpendicular, above E)
- D (pole, below E)
- H (conchoid curve, upper left)
- G (conchoid curve, upper center)
- F (conchoid curve, upper right)
- N (conchoid curve, right, at the arrow)
- O (intermediate point)
- Q (intermediate point)
- P (reference segment, upper left)
- R (reference segment, lower left)

It follows from the description of this line that it can never meet the straight line AB, even though both are extended without limit. For the points through which the curve passes all lie above AB: they are precisely the endpoints of the segments of the straight lines drawn out from D — every one of which cuts AB — and these segments are equal to EC. [Margin: (a) Prop. 11, Bk. I]

[Nicomedes] then demonstrates two remarkable properties of this line. The first is: any point on it other than the point C is less distant from AB than C is; and of the remaining points, the one farther from C is less distant from AB than the one nearer to C. For draw any line DG and drop the perpendicular GI. Since KG is greater than GI, the perpendicular EC (which equals KG) will also be greater than the perpendicular IG. [Margin: (b) Euclid I.19] That is, the point C is farther from AB than the point G; and by the same reasoning C is farther from AB than any

other point. Now take another point H, farther from C than G is, and drop the perpendicular HA. I say that H is less distant from AB than G is — that is, the perpendicular HA is smaller than the perpendicular GI. For when the line DH is drawn, the angle DKE is greater than the angle DLE, [Margin: (c) Euclid I.16] that is, the angle GKI is greater than the angle HLA. [Margin: (d) Euclid I.15] Since therefore the right angles I and A are equal, the remaining angle G is smaller than the remaining angle AHL. [Margin: (e) Euclid I.32] So if the angle AHM is made equal to G, the triangles KGI and MHA will be equiangular; [Margin: (f) Euclid VI.4] and therefore MH will be to HA as KG is to GI. [Margin: (g) Euclid I.19] And because LH is greater than MH — since the angle HML is greater than the right angle A, [Margin: (h) Euclid I.16] while HLM is smaller [Margin: (i) Euclid I.17] — the ratio of LH to HA is greater than that of HM to HA, [Margin: (k) Euclid V.8] that is, greater than that of GK to GI. Consequently, since GK and HL are equal, the ratio of HL to HA is also greater than that of HL to GI. [Margin: (l) Euclid V.10] Therefore HA is smaller than GI, which was the point to be proved.

The second property is this. Although the conchoid CF never meets the straight line EB, it does meet every other straight line, even one drawn extremely close to EB. Let NO first be a straight line parallel to EB, cutting EC at O. Make EC to P as EO is to OD. Since EO is smaller than EC, OD will also be smaller than P. [Margin: (m) Euclid V.14] Now if from D, with radius equal to the straight line P, an arc of a circle is drawn, it will cut the line ON at some point, say N. I say that the conchoid CF, when prolonged, meets ON at N. For draw the line DN, cutting EB at B; DN will be equal to P. Since EO is to OD as BN is to ND — that is, as BN is to its equal P — [Margin: (n) Euclid VI.4] and since likewise EO is to OD as EC is to P, the lines BN and EC will have the same ratio to P, [Margin: (o) Euclid V.9] and hence will be equal to one another. Therefore the conchoid passes through N.

Next let QF be a straight line not parallel to EB but cutting it at E and inclining toward the conchoid. Since the conchoid meets ON, it will plainly meet QF still sooner, at F.

After these results [Nicomedes] solves a problem of the following kind: *Given any rectilinear angle and a point outside the lines that contain the given angle, to draw from that point a straight line cutting the lines that contain the angle, in such a way that its portion intercepted between those lines equals a given straight line.* In the same figure, let the lines EB and EF contain the angle BEF, and let a line have to be drawn from D such that its portion between EB and EF equals the given line R. From D drop the perpendicular DE to the lower line EB, and take EC equal to the given line R. With D as pole and EC as radius, draw the conchoid, which by the

second property will cut the line EF at F. Then when the line DF is drawn, cutting EB at S, SF will be equal to EC — that is, to R — as is evident from the description of the conchoid.

With these preliminaries laid down, let AB and BC be two straight lines joined at the right angle B, between which two mean proportional lines are to be found. Complete the rectangle AC, and let its two sides AD and CD be bisected at F and E. Draw from B, through E, a line cutting AD extended at G; then DG will be equal to CB — that is, to DA — [Margin: (p) Euclid I.26] because the angles D and E of triangle DEG are equal to the angles C and E of triangle CEB, and the adjacent sides DE and CE are likewise equal. Again let the perpendicular FH be drawn so that AH equals CE (which is done by describing from A, with radius CE, an arc cutting FH at H). Then join the line GH and draw AI parallel to it. Extend DA, and from H, by the preceding problem, draw a line HK cutting both AI and AK so that the intercept IK equals AH or CE. (This is done by drawing from H many trial lines until the portion intercepted equals AH or CE.) Finally, from K through B extend a line cutting DC extended at L. I say that AK and CL are the two mean proportionals between AB and BC. For since LC is to CD as LB is to BK, that is, as DA is to AK; [Margin: (q) Euclid VI.2] and since CD is to CE as GA is to DA, because both CD and GA are bisected at E and D: it follows by *ex aequali* in perturbed proportion that LC is to CE as GA is to AK, as appears in this formula:

LC.	GA.
CD.	DA.
CE.	AK.

that is, as HI is to IK. Since therefore CE equals IK by construction, LC will also equal HI, [Margin: (r) Euclid V.14] and the whole LE will equal the whole HK.

Diagram. The geometric construction for finding two mean proportionals between AB and BC. A rectangle stands at the center, its diagonal and several drawn lines producing a fan of triangles; the construction extends downward to an apex at H. Labelled points:

- L (top apex): endpoint of the line extended from K through B
- B (upper right, on the line descending from L): right-angle vertex joining AB and BC
- C (upper left): corner of the rectangle
- E (left, mid-height): midpoint of side CD / GA

- F (center, lower): midpoint of side AD
- A (center, right of F): corner of the rectangle
- G (bottom left): point where the line from B through E cuts AD extended
- D (bottom, right of G): corner of the rectangle
- K (bottom right): point on the line, giving mean proportional AK
- I (lower center-right): intercept point on AI
- H (bottom apex, lowest): point from which HK is drawn

Next, since the rectangle under DK, KA together with the square on AF equals the square on FK, then, adding the common square on FH, the rectangle under DK, KA together with the squares on AF, FH — [Margin: (s) Euclid II.6] that is, together with the square on AH or on CE — equals the squares on KF, FH, that is, the square on HK, [Margin: (t) Euclid I.47] which is the square on LE (equal to HK). [Margin: (u) Euclid II.6] But the rectangle under DL, LC together with the same square on CE also equals that same square on LE. Therefore the rectangle under DK, AK together with the square on CE equals the rectangle under DL, LC together with the same square on CE; and, removing the common square CE, the remaining rectangle under DL, LC equals the remaining rectangle under DK, AK. Therefore DL is to DK — [Margin: (x) Euclid VI.16] that is, AB is to AK — as AK is to LC. [Margin: (y) Euclid VI.4] Moreover, as AB is to AK, so also is LC to CB. [Margin: (z) Euclid VI.4] Therefore AB is to AK as AK is to LC and as LC is to CB; and hence AK and LC are the mean proportionals between the given lines AB and BC, which was the point to be proved.

And if the two given lines are too long, one may take their halves, or thirds, and so on, and seek the two means between these; for if the means found are then doubled, or tripled, and so on, the two means between the original two lines will be obtained. The same is to be understood in other ways as well.

PROBL. 11. PROPOS. 16.

To increase or decrease a given plane figure, or a circle, in a given ratio.

Insofar as this problem concerns plane rectilinear figures, we explained it in Proposition 15 of the scholium to Proposition 33 of Book VI of Euclid. Now we extend the same result to circles as well. Let there be a rectilinear figure whose side is AB, or a circle whose diameter is AB, and suppose we must construct a larger rectilinear figure, or a larger circle, in the ratio C to D — say, subtriple. For the three lines C, D, AB find the fourth proportional E; between AB and E find the mean proportional FG; and on FG construct a figure similar to the given figure AB

and similarly placed. Likewise describe a circle about the diameter FG. I say that the rectilinear figure AB is the third part of the rectilinear figure FG, and the circle AB the third part of the circle FG — that is, that AB has to FG the same ratio that C has to D. For since the three lines AB, FG, and E are in continued proportion, the figure AB will be to the figure FG as AB is to E, that is, as C is to D. [Margin: (a) Euclid VI, coroll. of 19 or 20] And because the square on AB is to the square on FG as the circle AB is to the circle FG, [Margin: (b) Euclid XII.2] while the square on AB is to the square on FG as AB is to E, the circle AB will also be to the circle FG as AB is to E, or as C is to D.

Next let there be a figure, or a circle, HI, and suppose we must construct a smaller figure, or circle, in the ratio K to L — say, triple. For the three lines K, L, HI find the fourth proportional M; between HI and M find the mean proportional NO, upon which construct a figure similar to HI and similarly placed. Likewise describe a circle about the diameter NO. I say that the figure HI is to the figure NO, and the circle HI to the circle NO, in the triple ratio — namely the same that K has to L. For since the three lines HI, NO, and M are in continued proportion, the figure HI will be to the figure NO as the line HI is to M, that is, as K is to L. [Margin: (c) Euclid VI, coroll. of 19 or 20] And because the square on HI is to the square on NO as the circle HI is to the circle NO, while the square on HI is to the square on NO as the line HI is to M, the circle HI will also be to the circle NO as the line HI is to M, or as K is to L. [Margin: (d) Euclid XII.2]

From this it is clear how, given a round or even a square opening of some fountain, another round or square opening — larger or smaller in any ratio whatever — is to be constructed.

Diagram. A schematic of proportional line-segments together with similar rectilinear figures and circles, illustrating the two cases of the proposition. In the upper band are short measuring-segments and small figures; in the lower band are the corresponding circles. Labelled elements:

- C, D (upper left): two short segments — the given ratio C to D
- A... B, F (a small triangle): side AB and the mean-proportional side F(G), with a similar triangle at G, H
- E (segment, left lower): the fourth proportional E
- K, L (segments, right); M (segment): the given ratio K to L and the fourth proportional M
- I, N, O (squares/segment): side HI and the mean-proportional side NO, with similar squares

- Lower band circles labelled A, B, F, G (circle on AB and on FG) and H, I, N, O (circle on HI and on NO)

PROBL. 12. PROPOS. 17.

To increase or decrease, in a given ratio, any given solid figure of the kinds that Euclid treats in the books of Stereometry.

Solid figures of this sort are the parallelepiped, the pyramid, the prism, the sphere, the cone, the cylinder, and the five regular bodies.

Let there be, then, a solid figure whose side is A, or a sphere whose diameter is A, to be increased first in the ratio B to C. For the three lines B, C, A find the fourth proportional D; and between A and D find the two mean proportionals E, F. I say that the solid on side A has, to the solid built on side E — namely on the mean proportional nearer to the given side A, constructed similar and similarly placed to the solid set upon side A — the ratio that B has to C. Likewise the given sphere of diameter A is to the sphere of diameter E as B is to C. For since a solid figure of side A has, to a solid figure of side E similar and similarly placed, the triplicate ratio of side A to side E, as is demonstrated in Books XI and XII of Euclid — and similarly the cone and cylinder whose base-diameter is A to the similar cone and cylinder whose diameter is E; and likewise the sphere of diameter A to the sphere of diameter E — and since, moreover, by Definition 10 of Book V of Euclid the ratio of A to D is the triplicate of the ratio of A to E: the solid A will be to the solid E as A is to D, that is, as B is to C. Which was the point to be proved.

Next let a solid figure of side or diameter D be decreased in the given ratio C to B. For the three lines C, B, D find the fourth proportional A; and between D and A find the two mean proportionals F, E. I say that the solid of side or diameter D has, to the solid described similar and similarly upon F — namely upon the mean proportional nearer to the given side D — the ratio that C has to B. For since the solid D has, to the solid F described similar and similarly, the triplicate ratio of side D to side F — such as, by Definition 10 of Book V of Euclid, the line D has to the line A — the solid D will be to the solid F as D is to A, that is, as C is to B. Which was the point to be proved.

From all this it is established by what method a cube may be not only doubled (which the ancients sought) but also increased or decreased in any ratio whatever; and likewise in what way cannonballs are to be made larger or smaller according to a given ratio.

Diagram. A schematic of vertical line-segments representing the sides (*latera*) of the solids and the proportionals of the problem. Labelled segments:

- C, B (upper pair): the given ratio B to C
- D, F, E, A (lower row): the side D, the two mean proportionals F and E, and the side A

SCHOLIUM.

That similar and similarly placed solid figures have the triplicate ratio of their homologous sides has been demonstrated: for parallelepipeds, in Book XI of Euclid, Proposition 33 and its corollary; for prisms, in the scholium to the same; for the sphere, in the same Book XII, Proposition 18; for cones and cylinders, in the same Book XII, Proposition 12, if for homologous sides one takes the diameters of the spheres and the diameters of the bases of the cones and cylinders; and finally for the five regular bodies, in the corollary to Proposition 17 of Book XII — since all these bodies can be inscribed in spheres.

Although we have stated this problem only for the aforementioned bodies, it nonetheless holds for other similar and similarly placed bodies of any kind, as is evident: for they can be divided into similar pyramids, equal in number, [Margin: (a) Euclid XII.8 and its corollary] which have the triplicate ratio of their homologous sides. [Margin: (b) Euclid V.12] Since therefore, as one pyramid is to one pyramid, so are all to all — that is, so is solid to solid — and since the homologous sides of the solids are the same as those of the similar pyramids, the conclusion is established.

PROBL. 13. PROPOS. 18.

Between two given numbers to find, first one, and then two, mean proportionals.

Not infrequently a figure, whether plane or solid, must be increased or decreased by means of numbers — something that cannot be accomplished without finding one mean proportional, or two means, between the two given numbers. We shall therefore prescribe a method by which such means can be found. First, then, given any two numbers whatever, say 9 and 25, between which a single mean proportional is to be found: if they are multiplied together and the square root of the product 225 is extracted, giving 15, as we taught in the *Arithmetica Practica*, chapter 26, [Margin: (a) Euclid VI.17 or VII.20] this root will be the mean proportional between the given numbers — thus here 9, 15, 25 — since the square of the middle number equals the rectangle contained by the extremes. So

between 5 and 13 the mean proportional will be the square root of the number 65, which arises from multiplying the given numbers; this root is a little greater than 8 [superscript root-fraction: ?] and a little less than 8 [superscript root-fraction: ?].

Next let there be two numbers, 2 and 54, between which two mean proportionals are to be found. Multiply the square of the smaller into the greater: the cube root of the resulting number, 216, is 6, and this is to be placed as the first mean, next to the smaller. And if the square of the greater is multiplied into the smaller, the cube root of the resulting number, 5832, is 18, and this is the other mean, to be set next to the greater — thus here 2, 6, 18, 54. The reason for this is that, given four lines in continued proportion, the parallelepiped under the square of either extreme and contained by the other extreme equals the cube of the mean proportional nearer to the extreme first assumed, as we shall demonstrate in the following Lemma. And since, as we showed in the scholium to Proposition 19 of Book VIII of Euclid, the same number is produced from the three numbers 2, 2, 54 whether one first multiplies 2 by 2 and then the product 4 by 54, or first 2 by 54 and then the product 108 by 2; and likewise from the three numbers 54, 54, 2 the same number is produced whether one first multiplies 54 by 54 and then the product 2916 by 2, or first 54 by 2 and then the product 108 by 54: it is plainly gathered that if the smaller, 2, is multiplied into the greater, 54, and the product 108 into the smaller, 2, there is produced the cube of the mean proportional to be placed next to the smaller; and likewise that if the greater, 54, is multiplied into the smaller, 2, and the product 108 into the greater, 54, there is produced the cube of the mean proportional to be written next to the greater. So between 4 and 100 there will be two mean proportionals: the cube root of the number 1600, and the cube root of the number 40000. Moreover, once one of the mean numbers has been found, the other is also found, if the one found is multiplied by the more remote extreme and the square root of the product is taken. Thus in the given example 2, 6, 18, 54: if the mean found, 6, is multiplied into 54, the square root of the product 324 is 18, the other mean; and again, if the mean found, 18, is multiplied by 2, the square root of the product 36 is 6, the other mean — because both 2, 6, 18 and 6, 18, 54 are three numbers in continued proportion.

LEMMA.

If there are four lines in continued proportion, the parallelepiped under the square of either extreme and contained by the other extreme equals the cube of the mean proportional that is nearer to the first extreme assumed.

Let us take again the figure of Proposition 17, in which the four lines A, E, F, D are in continued proportion. I say that the parallelepiped under the square of the extreme A and contained by the other extreme D equals the cube of the line E. For since the square of the line A has, to the square of the line E, the ratio that A has to F — that is, that E has to D — [Margin: (a) coroll. of Euclid VI.20] the bases will be reciprocal with the heights, seeing that the base of the parallelepiped is the square of the line A and its height is the line D, while the base of the cube is the square of the line E and its height is the line E itself. Therefore the parallelepiped and the cube will be equal. [Margin: (b) Euclid XI.34] By the same reasoning, the parallelepiped under the square of the extreme D and contained by the other extreme A will equal the cube of the line F. For since the square of the line D is to the square of the line F — that is, the base of the said parallelepiped is to the base of the said cube — as D is to E, that is, as F is to A, that is, as the height of the cube is to the height of the parallelepiped, [Margin: (c) coroll. of Euclid VI.20] the bases will again be reciprocal with the heights, and therefore the parallelepiped and the cube will be equal. [Margin: (d) Euclid XI.34] Which was the point to be proved.

Since in our *Arithmetica Practica* we explained only the extraction of the square root, I think it will be worth the trouble to insert here — though perhaps out of its proper place — the extraction of the cube root, since it is altogether necessary for reducing this Problem 13 to practice. I shall do this by prescribing a certain general method by which we may extract the root of any kind whatever, drawn almost entirely from the book of a certain distinguished German arithmetician; and I am confident this will not prove disagreeable or unwelcome to the studious reader.

PROBL. 14. PROPOS. 19.

To extract a root of any kind whatever.

[Margin: What root-extraction is.]

Extracting a root is the finding, from a given number, of a number which by some multiplication of itself produces the given number. Thus extracting a square root is finding, from a square number, a number which, multiplied in the square manner, produces that number; and extracting a cube root is finding a number which, drawn into itself in the cubic manner, produces the given cube; and so on. What it means to multiply a number in the square manner, or the cubic, or in any other way, I shall explain in a moment.

[Margin: The infinite kinds of roots.]

Now, just as there are infinitely many kinds of multiplication of numbers into themselves — as I shall say at once — from which arise the square and solid numbers (cubes, zenzizenzics, surdesolids, and so on, such as some of the more recent writers are accustomed to expound in Algebra), so too there are infinitely many kinds of roots, corresponding to the various names of numbers that arise from the various multiplications of a root. All of this is beautifully displayed for us by the natural progression of numbers, set alongside the geometric progressions that begin from unity — as here.

0	1	2	3	4	5	6	7	8	9	10	&c.
1	2	4	8	16	32	64	128	256	512	1024	&c.

Column labels beneath the lower row (each naming the "species" of number that stands above it):

Value	Label (verbatim)	Sense
2	Radix.	root (first power)
4	Quadratus.	square (2nd power)
8	Cubus.	cube (3rd power)
16	Zensizensus.	zenzizenzic (4th power)
32	Surdesolidus.	surdesolid (5th power)
64	Zensicubus.	zenzicube (6th power)
128	B. Surdesolidus.	second surdesolid (7th power)
256	Zensizenzensus.	zenzizenzizenzic (8th power)
512	Cubicubus.	cube-of-cube (9th power)
1024	Zensurdesolidus.	zenzi-surdesolid (10th power)

First, the numbers of the upper progression indicate the kinds of multiplication. Thus 2, placed above the square, signifies that a square multiplication is performed, when the root is set down twice and so multiplied — as 2×2 makes 4. Likewise 3 signifies that a cubic multiplication is performed, when the root is set down three times and thus multiplied — as $2 \times 2 \times 2$ makes 8. In the same way 4 shows a zenzizenzic multiplication, and 5 a surdesolid one, and so on.

Next, these same numbers signify the kinds of roots. Thus 2 signifies that the square root produces the square by a square multiplication; and 3 denotes that the cube root generates the cube by a cubic multiplication; and so on in order.

[Margin: How the figures are marked off by points.]

In extracting roots, then, one must observe the marking-off of the figures [digits] by points in the number from which some root is to be extracted, in this way:

For the square [Pro quadrata]: 6 8 7 1 9 4 7 6 7 3 6

For the cube [Pro cubica]: 6 8 7 1 9 4 7 6 7 3 6

For the zenzizenzic [Pro Zensizensica]: 6 8 7 1 9 4 7 6 7 3 6

For the surdesolid [Pro surdesolida]: 6 8 7 1 9 4 7 6 7 3 6

(In each the same number, 68719476736, is written; the points are set beneath it according to the rule stated next — every second digit for the square, every third for the cube, every fourth for the zenzizenzic, every fifth for the surdesolid, counting from the right. The exact placement of the individual points is at the scan-resolution floor. [?])

In extracting a square root, all the figures in the odd places are marked, beginning from the right, so that alternately one figure is always skipped and left unmarked. In cubic extraction two figures are always skipped; in the zenzizenzic, three; in the surdesolid, four; and so on to infinity, as you see in the examples set alongside.

These markings correspond to the mean proportionals. Since between two squares there falls one mean, in extracting a square root one figure is always skipped; but between two cubes there fall two means, and therefore two figures are always skipped; and so for the rest.

[Margin: Construction of the marvelous table.]

For each kind of root to be extracted, certain special numbers serve, which are found by means of the following table, constructed in this way. The first column contains the natural series of numbers. From this column arises the second, the third from the second, and the fourth from the third, in this manner. Leaving the first two cells of the first column [empty], the number of the third cell is repeated as the first number of the second column. Then, from the addition of two numbers — that is,

#	II	III	IV	V	VI	VII	VIII	IX
1								
2								
3	3							
4	6							

#	II	III	IV	V	VI	VII	VIII	IX
5	10	10						
6	15	20						
7	21	35	35					
8	28	56	70					
9	36	84	126	126				
1	45	120	210	252				
0								
1	55	165	330	462	462			
1								
1	66	220	495	792	924			
2								
1	78	286	715	1287	1716	1716		
3								
1	91	364	1001	2002	3003	3432		
4								
1	105	455	1365	3003	5005	6435	6435	
5								
1	120	560	1820	4368	8008	11440	12870	
6								
1	136	680	2380	6188	12376	19448 [?]	24310	24310
7								

[continuation, printed page 167:]

— that is, from the third [cell] of the first column and the first of the second column — the second number of the second column is made. In the same way, from the second number of the second column and its neighbor [in the first column], the third number of the second column is produced; and from the third number of the second column and its neighbor, the fourth of the same second column is made; and so on. The second column, then, contains all the triangular numbers. In just the same way the third column arises from the second, the fourth from the third, and so on. For in every column the first two cells are always left empty, and the number of the third cell is repeated as the first number of the next column; and from the addition of that number to its neighbor in the preceding column the second number is formed; and so on.

[Margin: In what way the numbers for each kind of root are drawn from the table above.]

Once this table has been built, the special numbers are drawn from the transverse rows in the following way. The numbers of any transverse row are written out in order, and then the same numbers are repeated in reverse order, always excepting the last, and excepting the second-to-last as well — but only when it equals the last. Thus in the second row only the number 2 is taken, since, being the last, it is not repeated. In the third row only these two are taken, 3, 3, because the second-to-last is not repeated, since it does not differ from the last. In the fourth row these three are to be taken: 4, 6, 4. In the ninth, these eight: 9, 36, 84, 126, 126, 84, 36, 9. And in the tenth, these nine: 10, 45, 120, 210, 252, 210, 120, 45, 10, and so on. Here you see that as many numbers are always taken as there are units in the first number of the transverse row, minus one. Thus in the seventeenth row these sixteen are to be taken: 17, 136, 680, 2380, 6188, 12376, 19448, 24310, 24310, 19448, 12376, 6188, 2380, 680, 136, 17; and so for the rest.

Next, before each number there must be prefixed as many zeros as there are numbers counted from it inclusively up to the last one taken. Thus before the number 2 of the second row one zero must be prefixed, thus: 20. But the two numbers of the third row will have these zeros: 300, 30. So in the ninth row, where eight numbers are taken, the first will have eight zeros, the second seven, the third six, and so on, always decreasing by one. Which root's extraction the said numbers (taken in any given transverse row) serve is beautifully indicated by the first number of the transverse row. For since in the upper progression the number 2 marks the square, 3 the cube, 4 the zenzizencic, and so on, therefore the number 2 of the second row, with its zero — thus, 20 — serves the square root; the two numbers taken from the third row, with their zeros — thus, 300, 30 — serve the cube root; the three of the fourth row — thus, 4000, 600, 40 — serve the zenzizencic root; and the four of the fifth row — thus, 50000, 10000, 1000, 50 — serve the surdesolid root; and so on.

[Margin: How many figures each root has.]

Now, to come nearer to the actual extraction of roots, one must know that the root of any number has as many figures as there are points marked beneath it, according to the doctrine above. Also, to the last point toward the left belongs the figure placed above that point, together with all the others that precede it toward the left. From this point a number is subtracted, as we shall say shortly; to the second-to-last point will belong the figure above it together with the rest to its left; and so for the others.

But so that the extraction of any root may rightly begin, one must construct a table of the squares, cubes, surdesolids, second-surdesolids, and other numbers that are produced from the nine arithmetical figures [digits], together with their roots — as you see here.

Root s	Square s	Root s	Cub es	Root s	Surdesolids	Root s	Second surdesolids
1	1	1	1	1	1	1	1
2	4	2	8	2	32	2	128
3	9	3	27	3	243	3	2187
4	16	4	64	4	1024	4	16384
5	25	5	125	5	3125	5	78125
6	36	6	216	6	7776	6	279936
7	49	7	343	7	16807	7	823543
8	64	8	512	8	32768	8	2097152
9	81	9	729	9	59049	9	4782969

The reason I have not appended tables of the zenzizenzics and zenzicubes is that the roots of such numbers require no new rules, as will be said later. And so now follow a few examples of extractions.

Extraction of the Square Root

[Margin: Extraction of the square root.] [Margin: Root 2601.]

Let the proposed number be 6765201, from which the square root must be drawn.

First, from the last point — that is, from the figure 6 (for this point has only one figure, since no other precedes it) — the largest square that can be subtracted is taken away, namely 4; and its root, 2, is placed in the quotient at the margin. In the remainder, 2 is left for the next point, which will be 276. And thus the last point is completed, which in the operation is the first.

Next, I prepare a divisor from the figure 2 found in the quotient, multiplied by 20 — that is, by the number serving the square root; this I find to be 40. By it I divide the next point, 276, and for the quotient I find 6. I therefore set the first quotient 2 on the left, the special number 20 in the middle, and the new quotient 6 on the right, beneath which I write its square, 36, as you see here.

2 – 200 – 6.
36.

(Parsing note: the prose describes these placed numbers as the first quotient 2, the special number 20, and the new quotient 6, with $36 = 6^2$ written below the 6. The middle figure is printed in the schematic with an extra place-holding zero, as "200".)

[printed page 168:]

After this I multiply the three numbers 2, 20, 6 together, and to the product 240 I add the square 36; the sum, 276, I subtract from the point 276, and nothing is left. Thus the following point is completed; and another following point is 52.

I now prepare a divisor from the whole quotient found so far, 26, multiplied by 20 — that is, by the special number of the square root — which I find to be 520. And since the point 52 cannot be divided by it, I put 0 in the quotient. Nor is there any need to multiply in order to find the number to be subtracted, since nothing is subtracted, 0 as a multiplier producing 0. And so it is done in all other extractions, whenever the divisor found is not contained even once in the given point. Thus the point 52 is completed, and the point following it is 5201.

Again I prepare a divisor from the whole quotient found so far, 260, multiplied by the special number 20, which I find to be 5200; and this is contained once in the point 5201. So I set the whole quotient previously found, 260, on the left, the special number 20 in the middle, and the quotient 1 now found on the right, and its square, 1, beneath it, as is clear in the example.

260 – 20 – 1.
1.

The multiplication of the three upper numbers [$260 \times 20 \times 1$] makes 5200. I add the square, 1, of the figure found, 1; the number becomes 5201, which, subtracted from the point 5201, leaves nothing. The extraction is therefore complete, and the root found is 2601, which squared — that is, multiplied by itself — produces the proposed number, 675201 [recte 6765201].

And this is the proof, or test, of any extraction: namely, that the root found be multiplied by itself in the square, cube, or surdesolid manner, and so on, according to the kind of root. For if nothing was left over in the extraction — as in our example — the number produced must equal the proposed number from which the extraction was made. But if something was left over in the extraction, that remainder, added to the product, will make up the proposed number. The

test can also be carried out by nines or by sevens, as in division. For if from the root found one casts out nines (or sevens) as often as can be done, and places the remainder both on the left side of the cross and on the right — since the root found also acts as a divisor — then this remainder, multiplied by itself in the square or cubic manner and so on, and the nines (or sevens) cast out of the product, must equal the remainder of the proposed number when all its nines (or sevens) are cast out, provided nothing was left over in the extraction. Otherwise, all the nines (or sevens) must be cast out from the remainder of the extraction and from the product of the root found; and this remainder must equal the remainder of the proposed number when all its nines (or sevens) are cast out. In our example, if the proof is made by nines, the remainder is always 0. But if it is done by sevens, the example of the test will stand, as appears here.

Diagram. The "casting-out-nines" proof cross (a St. Andrew's cross, ×), with the residue set at each of its four points. All four residues are 0.

- Top: 0
- Left: 0
- Right: 0
- Bottom: 0

Diagram. The "casting-out-sevens" proof cross (×), with the residue at each of its four points.

- Top: 2
- Left: 4
- Right: 4
- Bottom: 2

Extraction of the Cube Root

[Margin: Extraction of the cube root.] [Margin: Root 621.]

From the number 239483190 the cube root is to be drawn.

First, from the point 239 I subtract the cube 216, which is the largest contained in it, and whose root, 6, I write in the quotient at the margin. And since the number 23 is left, the next point will be 23483.

Next, I prepare a divisor in this way. Above the root found, 6, I place its square, 36; and to the right I set the two special numbers of the cube root, namely 300

and 30, as you see here. I multiply the two upper numbers, 36 and 300, together, and to the product 10800 I add the product 180, from the multiplication of the two lower numbers, 6 and 30, together. For the sum, 10980, will be the divisor. The product of the two upper numbers alone, namely 10800, would also be enough for a divisor — which is to be understood in the other extractions as well. I therefore divide my point 23483 [printed 23843] by the divisor found, 10980, and I write the quotient, 2, after the figure 6 previously found. After this I draw a figure of this kind: to the right of the numbers 36 and 300 I place the figure found, 2, and beneath it its square, 4, and beneath that its cube, 8. For if the three upper numbers — 36, 300, and 2 — as well as the three lower — 6, 30, and 4 — are multiplied together, and to the products 21600 and 720 the cube 8 is added, the number 22328 results; and if I subtract this from my point 23483, 1155 remains. And so the next point will be 1155190.

36 — 300 — 2.
 6 — 30 — 4.
 8.

I now prepare another divisor in this way. Above 62, the root found so far, I write its square, 3844, and to the right the same special numbers, 300 and 30. The multiplication of the two upper numbers together makes the divisor 1153200; and if the point 1155190 is divided by it, the quotient is 1, to be written after the other two figures, 62. Now I draw a figure like this: to the right of the numbers 3844 and 300 I place the figure just found, 1, and beneath it its square, 1, and beneath that its cube, 1. For if the three upper numbers — 3844, 300, and 1 — as well as the three lower — 62, 30, and 1 — are multiplied together, and to the products 1153200 and 1860 the cube 1 is added, the number 1155061 results; and if I take this from my point 1155190, 129 remains. The extraction is therefore complete, and the root found is 621, which, multiplied into itself in the cubic manner, produces the number 239483061; and if the remainder 129 is added to it, the proposed number 239483190 is made up.

3844 — 300 — 1.
 62 — 30 — 1.
 1.

LIBER VI

Extraction of the Surdesolid Root [Extractio radicis surdesolidæ]

(The surdesolid — surdesolidus — is the fifth power; the surdesolid root is therefore the fifth root.)

[Margin: Extraction of the surdesolid root. Root 63.]

Let the number be 1039589621, whose surdesolid (fifth) root is sought.

First, from the section 10395 I subtract the largest surdesolid contained in it, namely 7776, whose root is 6; I write this 6 in the margin as the quotient. Since 2619 is left over, the next section will be 261989621.

Next, I prepare the divisor in this way. Above the root already found, 6, I write its square, 36; above this, its cube, 216; and above this, its zensizensus (or square of the square), 1296 — so that an ascending geometric progression is set up, beginning from the root 6 and running through as many terms as there are special numbers required in a surdesolid extraction. To the right I place the four required special numbers, as the example shows:

1296	—	50000
216	—	10000
36	—	1000
6	—	50

I multiply the two upper numbers together (which is enough), 1296 and 50000. The product, 64800000, will be the divisor; if I divide the remaining section 261989621 by it, the quotient could be either 4, or 3, or 2. I take 3, however, because the figure 2 is too small and 4 too large, as will be clear from what follows. I write this quotient 3 in the margin after the figure 6 already found.

I therefore draw up the array as follows. To the right of the numbers 1296 and 50000 I place the quotient-figure I have taken, 3; beneath it its square, 9; beneath this its cube, 27; beneath this its zensizensus (square of the square), 81; and beneath this its surdesolid, 243 — so that on the right there is set up a descending geometric progression denominated by the quotient-figure 3, having one term more than the number of special numbers required, in such a way that the last term is the surdesolid of the figure found. (Just as, in the cube-root extraction, it was the cube, and in the square-root extraction the square.)

1296 – 50000 – 3
 216 – 10000 – 9
 36 – 1000 – 27
 6 – 50 – 81
 243

For if the three numbers lying crosswise are multiplied together, and to the products — 194400000, 19440000, 972000, and 24300 — the surdesolid 243 is added, the result is 214836543. Subtracted from the section 261989621, this leaves 47153078. The surdesolid root found is therefore 63. Multiplied into itself in the surdesolid manner — that is, taken five times over, thus: 63, 63, 63, 63, 63 — it produces the number 992436543; and if the remainder 47153078 is added to this, the proposed number 1039589621 is made up.

If there were a further section left over, one would have to set up an ascending geometric progression denominated by the whole root found so far, 63, of four terms, as you see here. For the product 78764805, obtained by multiplying the two upper numbers together, would be the new divisor.

15752961 – 50000
 250047 – 10000
 3969 – 1000
 63 – 50

Then, from the new figure found, one would set up a descending progression down to the surdesolid of that figure, just as was done above with the figure 3.

In this way you will extract a root of any kind whatever, provided you carefully seek out the numbers that serve the proposed root, as I have shown above. This method has always struck me as excellent. For although the operation may seem somewhat longer than it need be, it is nonetheless not difficult, since in it one can never be in doubt about what must be done — whereas in the extractions handed down by other arithmeticians (the square root excepted) the difficulty of the operation is so great that one needs an almost boundless memory to retain what must be applied in extracting the roots. This will be clear if the cube root is extracted by another's rule; and yet cube extraction is far easier than surdesolid extraction and the others that follow, which are all but inextricable.

One difficulty alone exists, in my method as much as in others' — namely, that it is sometimes doubtful whether we have taken too small a figure in the quotient for a given section. Thus, at the second section of the surdesolid extraction, the quotient could have been 4, or 3, or 2, and we took 3. If, then, we wish to be sure whether the figure 4 could have been taken — since the very large number 47153078 was left over — we must make trial with the figure 4, by setting up the

descending geometric progression denominated by 4. And since the four transverse products, together with the surdesolid 1024, make the number 297141824 [recte: 296141824?], which cannot be subtracted from the second section 261989621, this is proof that the figure 4 is too large, and consequently that 2 is too small — since with the figure 3 the numbers produced were such as could be subtracted from the proposed section. If this remedy is applied, then — although a little long-winded — our method of extracting roots will be perfectly safe.

[Margin: How the difficulty in extractions may be overcome.]

1296 – 50000 – 4
 216 – 10000 – 16
 36 – 1000 – 64
 6 – 50 – 256
 1024

Now as to why, in the table above, of squares, cubes, surdesolids, and so on, I have left out the zensizensics (the squares of the squares), and indeed have passed over the extraction of the zensizensic root: the reason is that the zensizensic, zensicubic, zensizenzensic, cubiccubic, zensurdesolid roots, and so on — although they can be extracted, like the others — nevertheless have yet another rule of extraction, which one may gather from the very names. Namely:

[Margin: Why no example is given for the zensizensic root, etc.]

To extract the zensizensic root (that is, the square-of-the-square root), first extract the square root; then from this root draw the square root again. This will be the zensizensic root that is sought.

To extract the zensicubic root — that is, the square-of-the-cube, or cube-of-the-square — first extract the square root, and then from this the cube root; or else first draw the cube root, and from this the square root. For the last root extracted will be the one you seek. Form the same judgment about the other roots of composite numbers, such as the zensizenzensic, the cubiccubic, the zensurdesolid, and so on.

A Special Rule for Extracting the Cube Root [Regula propria extractionis radicis cubicæ]

[Margin: Special rule for the cube root. Root 123.]

Since the square and cube roots are in more frequent use among mathematicians than the other roots, I am pleased, for the benefit of students, to prescribe here a

special rule for extracting the cube root — just as I did for the square root in my *Arithmetica Practica*. Setting aside the rules of others, because they are less easy and less convenient, I will excerpt one that is, as it were, new, drawn from the surdesolid-root extraction above [recte: from the extraction of the higher roots above]. It runs as follows.

Let the cube root be drawn from the number 1860867.

From the first section on the left, 1, I subtract the largest cube contained in it, namely 1, and nothing remains. The next section will therefore be 860, and the figure 1 has been found for the root.

I prepare the divisor by multiplying the square of the figure found, namely 1, by 300; this gives 300. Dividing the section 860 by it, I find the quotient 2 for the second figure of the root. This I multiply into the divisor found, 300, giving 600. Then I multiply the square of the new figure found, 2 — namely 4 — into the product of the earlier figure 1 multiplied by 30, that is, into 30, giving 120. Finally, to the sum of these two products, 600 and 120 — that is, to 720 — I add the cube of the new figure found, 2, namely 8, and I subtract the whole sum, 728, from my section 860. Since 132 remains, the last section will be 132867. I therefore write the figure found, 2, after the earlier 1.

Next, in the same way, I prepare a new divisor for the last section. I take the square of the whole root found so far, 12 — that is, 144 — and multiply it again by 300. The product, 43200, will be the divisor; dividing my section 132867 by it, I find the quotient 3, to be written after the root 12 found so far. This figure found, 3, I likewise multiply into the divisor found, 43200, giving 129600. Then I multiply the square of the same new figure, 3 — namely 9 — into the product of the root 12 previously found multiplied by 30, that is, into 360, giving 3240. Finally, to the sum of these two products, 129600 and 3240 — that is, to 132840 — I add the cube 27 generated from the same new figure, and the number becomes 132867; subtracted from the section 123867 [recte: 132867], it leaves nothing. And thus the root of the proposed number 1860867 has been found to be 123.

If there were a further section left over, the square of the whole root found so far, 123, would have to be multiplied by 300 to raise a new divisor, and so on. You see, then, that this rule requires more memory than the previous one, although it is easier than the rules of others. But remember, whenever you doubt whether you have taken too small a figure in the quotient, to make trial with a larger figure, as was said above.

PROBL. 15. PROPOS. 20.

In numbers that are not squares, not cubes, not zensizensics, not surdesolids, and so on, to find a root close to the true one.

When the proposed number is not a square, or a cube, or a zensizensus, or a surdesolid, and so on, it cannot have a true root; but by the rules above one finds the root of the largest square, cube, zensizensus, or surdesolid contained in the given number. So that we may know what fraction should be added to the root found, in order to have a root closer to the true one, we must proceed as follows.

[Margin: How many pairs of ciphers [zeros], or triples, or quadruples, and so on, must be appended in order to extract a close root.]

To the proposed number let some pairs of ciphers be appended, if a close square root is sought; or, if a cube root, some triples of ciphers; or some quadruples, if the zensizensic root is wanted; or some quintuples of ciphers, if the surdesolid root is in question; and so on. In each kind of root the number of ciphers to be repeated corresponds to the number written above the power-name at the beginning of the preceding problem, from which the root takes its name. Thus, because 2 is set above the square, pairs of ciphers are appended several times for the square root; but for the cube, triples, because the number 3 is written above the cube; and so on — so that for a close cubicubic root some ninefold sets of ciphers must be appended, since the number 9 is found written above the cubiccube. The same number of ciphers to be repeated also corresponds to the marking-off of the sections that must be made in order to extract the root. Thus, because in extracting a square root every second figure is marked, pairs of ciphers must be added; but in a cube root, triples of ciphers must be adjoined, because in extracting it every third figure is marked; and so on.

The more pairs, or triples, of ciphers, and so on, are appended to the proposed number, the closer the root that will be drawn.

[Margin: What fraction must be added to the root so that a closer root is produced.]

Once the ciphers have been appended in this way to the number from which the root is to be drawn, the root must be extracted from that whole number, as taught above. Then, from that root, casting off on the right as many figures as there were pairs, triples, or quadruples of ciphers appended, the remaining figures will give the whole part of the root. To this must be added a fraction whose numerator is the figures cast off, and whose denominator is unity followed

by as many ciphers as there were pairs (or triples, and so on) of ciphers added — namely 10, if one pair or triple was added; 100, if two pairs or triples were added; 1000, if three; and so on — so that the fraction consists of tenths, hundredths, thousandths, and so on.

By way of example. Let the square root be extracted from the number 29. When three pairs of ciphers have been appended, thus — 29000000 — the square root of this number is found to be 5385, smaller than the true one, since something remained in the extraction; but if unity is added, the root becomes 5386, larger than the true one. Casting off, therefore, three figures on the right, on account of the three pairs of ciphers added, the close root will be $5\frac{385}{1000}$, still smaller than the true one; while $5\frac{386}{1000}$ will be larger than the true one. For the square of the former is $28\frac{998225}{1000000}$, smaller than the proposed number 29; while the square of the latter is $29\frac{8996}{1000000}$ [i.e. 29 and $008996/1000000$], larger than the same proposed number 29.

Likewise, from the number 29160 let the cube root be drawn. Let three triples of ciphers be appended, so that we may again have, in the fraction, parts denominated by 1000; and from the whole number 29160000000000 let the cube root be extracted. It will be found to be 30779, smaller than the true one — since in the extraction there was some number left over — and so, larger than the true one, it will be 30780. Casting off three figures on the right, on account of the three triples of ciphers appended, the close [root will be...]

close cube root will be $30\frac{779}{1000}$, smaller than the true one, since its cube is only 29158 [tiny superscript fraction: ?]; while the close root larger than the true one will be $30\frac{780}{1000}$, since its cube is 29161 [tiny superscript fraction: ?].

The demonstration of this method of finding the close root is as follows. When, for a square root, two ciphers (00) are appended to the proposed number — say to 5 — the proposed number is multiplied by 100, that is, by the square of the root 10. And since the squares 500 and 5 (for the given number and the one produced by appending the 00 are to be taken as squares, since their roots are sought) [Margin: a — Euclid, *Elements* VIII.11] have the doubled ratio of their roots — now 500 is to 5 as 100 to 1, because 5 multiplied by 100 makes 500, and the hundredfold ratio is the doubling of the tenfold ratio, as is clear in this set-out example — the ratio of the root of the number 500 to the root of the number 5 will be tenfold.

1. 10. 100.
5. 500.

Since, then, the root of 500 is 22, smaller than the true one, its tenth part — namely $2\frac{2}{10}$ — will be the root of the number 5, smaller than the true one; and consequently $2\frac{3}{10}$ will be larger than the true one. Rightly, then, we directed that when 00 is appended, one figure should be cast off from the root 22, so that the root $2\frac{2}{10}$ is left.

When, however, 0000 is appended, the number is multiplied by 10000, that is, by the square of the root 100. And since the squares 50000 and 5 [Margin: b — Euclid, *Elements* VIII.11] have the doubled ratio of their roots — now 50000 is to 5 as 10000 to 1, because 5 multiplied by 10000 makes 50000, and the ratio 10000 to 1 is the doubling of the ratio 100 to 1, as is clear in this example — the ratio of the root of the number 50000 (namely 223) to the root of the number 5 will be as 100 to 1. Therefore, if the root 223 is divided by 100, a close square root of $2\frac{23}{100}$ will be produced, smaller than the true one; while $2\frac{24}{100}$ will be larger than the true one. Rightly, then, we directed that, when 0000 is appended, two figures should be cast off from the root 223, so that the remaining root becomes $2\frac{23}{100}$.

- 1. 100. 10000.
- 5. 50000.

By the same reasoning, if 000000 is appended, a close root will be produced in thousandths, and so on.

Again, when, for a cube root, the given number is multiplied by 1000 — that is, by the cube of the root 10 — and since the cubes 5000 and 5 [Margin: c — Euclid, *Elements* VIII.12] have the tripled ratio of their roots — now 5000 is to 5 as 1000 to 1, because multiplying 5 makes 5000, and the ratio 1000 to 1 is the tripling of the ratio 10 to 1, as appears in this set-out example — the ratio of the root of the number 5000 (namely 17) to the root of the number 5 will be as 10 to 1.

Accordingly, if the root 17 is divided by 10, the close cube root becomes $1\frac{7}{10}$, smaller than the true one; while $1\frac{8}{10}$ will be larger than the true one. Rightly, then, we directed that, when for a cube root 000 is appended, one figure should be cast off from the root found, 17, so that the remaining root becomes $1\frac{7}{10}$.

- 1. 10. 100. 1000.
- 5. 5000.

In like manner, if 000000 is appended, a close root will be found in hundredths; and if 000000000 is appended, in thousandths, and so on. For in the one case the number is multiplied by 1000000, that is, by the cube of the root 100; in the other, by 1000000000, namely by the cube of the root 1000. The rest will be demonstrated in the same way.

Nor is the reasoning any different for the other roots. For in the surdesolid, say, when 00000 is added, the multiplication is by 100000, that is, by the surdesolid of the root 10. And the ratio 100000 to 1 is the fivefold multiplication of the ratio 10 to 1, as appears in the set-out example, and so on.

1. 10. 100. 1000. 10000. 100000.

PROBL. 16. PROPOS. 21.

[Margin: Extraction of roots from fractions.]

To extract a root of any kind from a given fraction [minutia].

In fractions, the root of the same name as the root sought must be extracted from both the numerator and the denominator. In this way a fraction is produced which is the root of the proposed fraction. Thus the square root of the fraction $4/9$ is $2/3$; the cube root of the fraction $8/27$ is likewise $2/3$; and the zensizensic root of the fraction $16/81$ is, by the same reasoning, $2/3$; and so with the others.

But if the given fraction is a fraction of a fraction, or a fraction of another fraction, it must first be reduced to a simple fraction. For instance, if the square root is sought of this fraction-of-a-fraction, $2/4$ of $8/9$, it must be reduced to the simple fraction $16/36$, whose square root is $4/6$, or $2/3$, and so on.

Similarly, if a fraction is joined to whole numbers, the whole numbers must first be reduced to a fraction of the same denomination. For instance, if the cube root is sought of the number $2\ 10/27$ [recte for $1\ 1/3$ root; $2\ 10/27 = 64/27$], it must be reduced to the fraction $64/27$, whose cube root is $4/3$ — that is, $1\ 1/3$ — and so on.

If either the numerator or the denominator of the fraction, or both numbers, lack a root of the name that is sought, that fraction will not have the root sought. Thus neither $4/7$, nor $6/9$, nor $5/12$ has an exact square root, because in the first the denominator, in the second the numerator, and in the third both numbers lack a square root.

You will know whether a given fraction has the root sought or not, if you reduce it to its lowest terms. For if, when so reduced, it has a root, the given fraction will also be said to have the same root; but if, reduced to lowest terms, it does not have a root, neither will the proposed fraction have one. For instance, suppose the fraction $20/45$ is proposed, and I wish to know whether it has a square root: reduced to lowest terms it is $4/9$, which has the square root $2/3$. The proposed fraction $20/45$ will therefore be said to have this same square root. But the fraction $6/9$ will not have a square root, because neither does $2/3$ — into which it

is reduced in lowest terms — have one. By the same reasoning, the fraction $24/81$ will have the cube root $2/3$, the very same that the fraction $8/27$ has, into which it is reduced in lowest terms. The fraction $12/20$, however, will lack a cube root, because the fraction $3/5$, into which it is brought back in lowest terms, also lacks one. And so with the others.

When a fraction, reduced to its lowest terms, does not have the sought root exactly, one must find the approximate root of both the numerator and the denominator. To do this, first append to each some pairs of ciphers (zeros), or triples, or quadruples, and so on, according to whether a square root, a cube root, a zenzizenzic (fourth) root, and the like is being sought. For if the approximate root of the numerator is divided by the approximate root of the denominator, the approximate root we are looking for results. For example, suppose the fraction [tiny superscript fraction: ?] is proposed and its square root is to be found. By appending three pairs of ciphers, the approximate root of the numerator will be found to be 2 [superscript fraction: $440/1000$?], and that of the denominator 2 [superscript fraction: $645/1000$?]. So if the former is divided by the latter, the sought root [superscript fraction: $2440/2645$?] results, which is sufficiently close. The same judgment applies to the other roots — provided you remember that for the cube root some triples of ciphers must be appended to both numerator and denominator so that their approximate roots can be drawn out; for the zenzizenzic (fourth) root, some quadruples; for the surd-solid (fifth) root, some quintuples; and so on.

But it is troublesome to seek two approximate roots — one for the numerator of the proposed fraction and another for its denominator. So, for the square and cube roots (which are of course the ones more commonly used), Cardano [Gerolamo Cardano] and Tartaglia [Niccolò Tartaglia] hand down certain special rules, which I am glad to explain here, since with them only a single approximate root need be found.

[Margin: Another way to extract the square and cube root from a given fraction.]

For the square root, then, multiply the numerator by the denominator and divide the approximate square root of the resulting number by the denominator; or divide the numerator by that approximate root. Either way, the approximate root of the proposed fraction is produced. Now if the approximate root of the number produced by multiplying numerator into denominator is less than the true root, then by the first method the approximate root of the fraction will likewise come out less than the true one, because a number smaller than the true value is being divided; but by the second method the approximate root of the fraction will come

out greater than the true one, since the division is then made by a number smaller than the true value. The opposite happens if that approximate root of the number produced from numerator into denominator is greater than the true root: by the first method the approximate root of the fraction will be produced greater than the true one, and by the second method less than the true one, as is clear. I set out this rule also in Book 4, chapter 2, number 5, and demonstrated it there. You will find an example of the rule in the same place as well.

For the cube root: multiply the numerator by the square of the denominator and divide the approximate cube root of the resulting number by the denominator; or multiply the denominator by the square of the numerator and divide the numerator by the approximate cube root of the resulting number. Either way, the approximate root of the proposed fraction is obtained. By the first method, if that approximate cube root is less than the true one, the approximate root of the fraction will also come out less than the true, because the division of a number smaller than the true value is made by the denominator of the fraction. But if that approximate root is greater than the true, the approximate root of the fraction will likewise be produced greater than the true, because a number larger than the true value is then divided by the denominator of the fraction. By the second method, if that approximate cube root is less than the true one, the approximate root of the fraction will be produced greater than the true, because the division is then made by a number smaller than the true value; whereas if that approximate cube root is greater than the true, the approximate root of the fraction will be found less than the true, since the numerator is then divided by a number larger than the true value.

Example, with the fraction $\sqrt[3]{\frac{8}{27}}$, whose true cube root is $\sqrt[3]{\frac{2}{3}}$. Multiplying the numerator 8 by 729 (the square of the denominator 27) gives the number 5822 [recte: 5832 — $8 \times 729 = 5832$, whose cube root is 18], whose cube root is 18. This, divided by the denominator 27, makes $\sqrt[3]{\frac{18}{27}}$, that is, $\sqrt[3]{\frac{2}{3}}$, as the cube root of the fraction $\sqrt[3]{\frac{8}{27}}$. Again, multiplying the denominator 27 by 64 (the square of the numerator 8) gives the number 1728, whose cube root is 12; if the numerator 8 is divided by this, the quotient is $\sqrt[3]{\frac{8}{12}}$, that is, $\sqrt[3]{\frac{2}{3}}$ as before, as the cube root of the proposed fraction $\sqrt[3]{\frac{8}{27}}$.

A second example, with the fraction $\sqrt[3]{\frac{5}{7}}$, which has no exact cube root. Multiplying the numerator 5 by 49 (the square of the denominator 7) gives the number 245, whose approximate cube root — found by appending two triples

of ciphers — is 6 [superscript: 25/100 ?]; divided by the denominator 7, it makes the quotient [superscript: 625/700 ?], that is, [superscript: 25/28 ?], as the root of the fraction [superscript: 5/7 ?]. Again, multiplying the denominator 7 by 25 (the square of the numerator 5) gives the number 175; by its approximate cube root 5 [superscript: 59/100 ?] — found by appending 000000 to 175 — if we divide the numerator 5, we obtain the quotient [superscript: 500/559 ?], as the approximate cube root of the given fraction [superscript: 5/7 ?]; and so with the others.

Moreover, by this method the approximate root of the fraction is found closer than by the earlier rule, because here only one error creeps in — that of the approximate cube root, which is not the true one — while the denominator (in the first method) and the numerator (in the second) each remain at their own proper value. In the earlier rule, by contrast, two errors intervene, on account of the two approximate cube roots that are not the true ones.

I prove both of these methods by parallel reasoning. When the numerator is multiplied by the square of the denominator, the cube root of the resulting number will be one of the two mean proportionals between the numerator and the denominator — the one to be placed next to the denominator — as is established from what was demonstrated in Proposition 18 of this book. [Margin: Definition 10 of Book 5.] The ratio of numerator to denominator will therefore be the triplicate of the ratio of the cube root found to the denominator. [Margin: (Euclid) VIII.12.] But the ratio of numerator to denominator is the same as that of cube to cube, and is likewise the triplicate of the ratio of the cube root of the numerator to the cube root of the denominator. Hence the cube root found will be to the denominator as the cube root of the numerator is to the cube root of the denominator. [Margin: Proposition 7 on fractions, toward the end of Book 9.] Consequently, the fraction whose numerator is the cube root found and whose denominator is the denominator itself will be equal to the fraction whose numerator is the cube root of the numerator and whose denominator is the cube root of the denominator. For this reason, just as this second fraction is the cube root of the proposed fraction, so too the first will be the cube root of the same fraction. Therefore, dividing that cube root found by the denominator of the proposed fraction produces the cube root of the proposed fraction. (This division is made because the cube root found is itself a fraction: to learn the value of the fraction whose numerator is that root found and whose denominator is the denominator itself of the proposed fraction, the numerator of this fraction must be divided by its denominator — whereas if that cube root found were a whole number, no division would be needed.) This mirrors how the cube root of the proposed fraction is produced from the division of the cube root of the

numerator by the cube root of the denominator — since a fraction is nothing other than a numerator divided by a denominator.

But when the denominator is multiplied by the square of the numerator, the cube root of the resulting number will be one of the two mean proportionals between numerator and denominator — the one to be placed next to the numerator — as is manifest from what was demonstrated in Proposition 18 of this book. So again the ratio of numerator to denominator will be...

the triplicate of the ratio of the numerator to that cube root found — just as the ratio of numerator to denominator is likewise the triplicate of the ratio of the cube root of the numerator to the cube root of the denominator, that same ratio of numerator to denominator being as cube to cube. And therefore, as above, the fraction whose numerator is the numerator itself of the proposed fraction and whose denominator is that cube root found will be equal to the fraction whose numerator is the cube root of the numerator of the proposed fraction and whose denominator is the cube root of the denominator of the same fraction. Consequently, when the numerator is divided by that cube root found, the cube root of the proposed fraction will result.

PROBL. 17. PROPOS. 22.

To find geometrically, by means of lines, the square root and cube root of numbers that are not squares and not cubes.

Let a number 10 be given, representing a square of 10 palms, or feet, or other measures. Take a line A of 10 palms (or feet, etc.) and a line B of one palm (or one foot). [Margin: (Euclid) VI.13.] Having found C, the mean proportional between A and B, I say that C is the square root — the square side — of the number A, 10. [Margin: (Euclid) VI.17.] For the square on the straight line C is equal to the rectangle contained by A and B; and this rectangle is 10, because the unit B, multiplying the number A (10), produces 10. It is therefore clear that the straight line C is the square side of 10 palms, or feet, etc. — which is what was proposed.

Again, [Margin: Proposition 15 of this book.] having found the two mean proportionals D and E between A and B, I say that E, which is the nearer to the smaller line B[?], is the cube side — the cube root — of the number 10. [Margin: Lemma to Proposition 18 of this book.] For the cube on the straight line E is equal to the parallelepiped contained by the square on the straight line B and by A; and this parallelepiped is 10, because the unit B multiplied by itself makes the square 1, and the square 1, multiplying the number A (10), generates 10. It plainly

follows that the straight line E is the cube side of 10 palms, or feet, etc. — which is what was proposed.

Diagram. A set of five labelled straight lines stacked and compared, illustrating the extraction of the square and cube root of 10 by mean proportionals.

- A (top): a long horizontal line divided by tick-marks into ten equal segments — the number 10 (10 palms/feet).
- D (second from top): a shorter horizontal line — the first (greater) of the two mean proportionals between A and B.
- E (third): a still shorter horizontal line — the second (lesser) mean proportional, the cube side of 10.
- B (bottom left): a very short line marked off by a single tick — the unit line (1 palm/foot).
- C (lower right, set off to the right): a medium horizontal line — the mean proportional between A and B, i.e. the square side of 10.

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GEOMETRIÆ PRACTICÆ — LIBER VII

A disputation on isoperimetric figures, to which is appended, in place of an appendix, a short treatise on squaring the circle by means of lines.

A noble and outstanding disputation about isoperimetric figures has always been current among everyone: the inquiry, that is, into which of two such figures is the greater and more capacious, and which is the greatest and most capacious of all. For not a few people ignorant of geometry have deceived themselves on this point, supposing that isoperimetric figures — those, that is, which contain equal perimeters — are equal to one another, or equally capacious. Indeed, on the contrary — and this is surprising — some who call themselves geometers can scarcely be brought to believe that two isoperimetric figures entirely equal to one another can exist; yet that this can happen we shall demonstrate most clearly in Propositions 20, 21, and 22 of this book.

[Margin: Definitions pertaining to the treatment of isoperimetric figures.]

DEFINITIONES.

I. Isoperimetric figures are those which contain equal perimeters.

II. A figure is called regular when it is both equilateral and equiangular.

III. The center of a regular figure is that point which is the center of the circle inscribed in, or circumscribed about, the figure.

IV. The area of any figure is called its capacity — the space or surface comprehended within its sides.

V. Every rectangular solid — one, that is, whose bases are equidistant and equal, with the sides perpendicular to the bases, such as a parallelepiped — is said to be contained by one of its bases and by the perpendicular drawn from that base to the other. For of course either of the bases indicates the length and breadth of the figure, while the perpendicular shows its height or depth.

THEOR. 1. PROPOS. 1.

[Margin: To what rectangle any given triangle is equal.]

The area of any triangle is equal to the rectangle contained by the perpendicular drawn from the vertex to the base and by half the base. It is likewise equal to the rectangle contained by half the perpendicular and the whole base. And, finally, it is equal to half of the rectangle contained by the whole perpendicular and the whole base.

Let there be a triangle ABC, from whose vertex A a perpendicular AD is drawn to the base BC, and let it first divide the base...

into the base BC in two equal parts, as in the first figure. Through A draw EAF equidistant from the straight line BC on both sides, and complete the rectangle BEFC, [Margin a: 41. prim. (Euclid, *Elements* I.41)] which will be double the triangle ABC; [Margin b: 36. prim. (Euclid, *Elements* I.36)] and likewise double the rectangle ADBE. Therefore the rectangle ADBE, which is contained under the perpendicular AD and half the base BD, is equal to the triangle ABC.

Next, let the perpendicular AD divide the base BC not into two equal parts, or even let it fall on the base CB produced, as in the second and third figures. Through A draw AF once more, equidistant from the straight line BC on both sides, and complete the rectangle ADCF. Then, having divided the base BC into two equal parts at G, draw the straight lines BE and GH equidistant from AD; [Margin c: 34. prim. (Euclid, *Elements* I.34)] and GH will be equal to the perpendicular AD. [Margin d: 41. prim. (Euclid, *Elements* I.41)] Now since the rectangle BCFE is double the triangle ABC, [Margin e: 36. prim. (Euclid, *Elements* I.36)] and likewise double the rectangle BEHG, the rectangle BEHG — which is

contained under the perpendicular GH (or AD) and half the base BG — will be equal to the triangle ABC.

Now let the perpendicular AD (or GH) be bisected at I, and through I draw KL parallel to BC. I say that this same triangle ABC is also equal to the rectangle BCLK in the first and second figures, and to the rectangle BCLM in the third figure — the rectangle contained, that is, under ID (or IG), half the perpendicular AD (or HG). [Margin f: 41. primi (Euclid, *Elements* I.41)] For since the triangle ABC is half the rectangle EC, and the rectangle BL is likewise half of the same rectangle EC — [Margin g: 36. prim. (Euclid, *Elements* I.36)] because the rectangles BL and LE, standing on equal bases, are equal — the triangle ABC and the rectangle BL will be equal to one another. [Margin h: 41. prim. (Euclid, *Elements* I.41)] And since the rectangle BF, contained under the perpendicular AD (or BE) and the base BC of the triangle, is double the triangle ABC, the triangle will be equal to half of that rectangle. Therefore the area of any triangle is equal, and so on — which was to be demonstrated.

Diagram. A composite of three figures illustrating Theorem 1 (each triangle equal to half its enclosing rectangle). Reading the strip left to right, the top labels run E A F | E A H F | A E H F, and the base labels run B D C | B D G C | D B G C. Interior labels are overprinted and hard to resolve.

- Figure 1 (left): rectangle with top corners E (upper left) and F (upper right), A at the top middle; base B (left), D (middle), C (right); K on the left side, L on the right side; I an interior point near the base (on the parallel KL).
- Figure 2 (middle): top E, A, H, F; base B, D, G, C; K at the left, I an interior point.
- Figure 3 (right): top A, E, H, F; base D, B, G, C; L at the right, M an interior point; a diagonal is drawn.

THEOR. 2. PROPOS. 2.

The area of any regular figure is equal to the rectangle contained under the perpendicular drawn from the center of the figure to one side, and under half the perimeter of the same figure. [Margin: *Regularis figura quaecunque, cui rectangulo æqualis sit.* — "Any regular figure: to what rectangle it is equal."]

Let ABCDEF be any regular figure, and let G be its center, from which draw GH perpendicular to one side, namely to AB. Let there also be a rectangle IKLM, contained under IK (equal to the perpendicular GH) and under KL, a straight line set equal to half the perimeter of the figure ABCDEF. I say that the regular figure

ABCDEF is equal to this rectangle. For from G draw straight lines to each of the vertices, so that the whole figure is resolved into triangles. These will all be equal to one another, as we have demonstrated in the corollary to Proposition 8 of Book I of Euclid, [Margin a: 8. primi (Euclid, *Elements* I.8, corollary)] because all the sides of the triangles issuing from the point G are equal to one another, and they have equal bases — namely the sides of the regular figure. Hence it follows that all the angles at G are equal, and consequently, by the corollary just cited, the triangles themselves are also equal to one another. [Margin b: 1. huius (Proposition 1 of this Book)] Now since the rectangle contained under the perpendicular GH and half the base AB is equal to the triangle ABG, if one takes as many such rectangles as there are triangles into which the regular figure is divided, all of them together will be equal to the figure ABCDEF, because all the triangles have been shown equal to the triangle ABG. [Margin c: 1. secundi (Euclid, *Elements* II.1)] Since, therefore, these same rectangles together are equal to the rectangle IKLM — because KL is set equal to half the perimeter ABCDEF, that is, to all the half-bases together, and the straight line IK to the perpendicular GH — the regular figure ABCDEF will be equal to the rectangle IKLM. Therefore the area of any regular figure is equal, and so on — which was to be demonstrated.

Diagram. A regular hexagon ABCDEF with its center G, alongside a rectangle IKLM.

- Hexagon vertices: E (upper left), D (upper right), F (left), C (right), A (lower left), B (lower right).
- G (center): the center, with radii drawn to all six vertices.
- H (base midpoint): foot of the perpendicular GH on side AB.
- Rectangle: I (upper left), M (upper right), K (lower left), L (lower right).

THEOR. 3. PROPOS. 3.

The area of any regular figure is equal to a right triangle one of whose sides about the right angle is equal to the perpendicular drawn from the center of the figure to one side, while the other is equal to the perimeter of the same figure. [Margin: *Regularis figura quæcunque, cui triangulo rectangulo æqualis sit.* — "Any regular figure: to what right triangle it is equal."]

Again let ABC be a regular figure whose center is D, from which the perpendicular DE is drawn to the side AB; and let DEF be a right triangle having the angle E right, its side DE equal to the perpendicular DE, and its side EF equal to the perimeter of the figure ABC. I say that the triangle DEF is equal to the figure

ABC. For complete the rectangle DEFG, and, having bisected EF at the point H, draw HI parallel to DE. [Margin a: 2. huius (Proposition 2 of this Book)] Then the rectangle DEHI, contained under the perpendicular DE and under EH (half the perimeter of the figure), will be equal to the figure ABC. But the triangle DEF is equal to the rectangle DEHI; for the rectangle DEHI is half the rectangle DEFG, [Margin b: 36. prim. (Euclid, *Elements* I.36)] because the rectangles DEHI and IHFG are equal, [Margin c: 41. prim. (Euclid, *Elements* I.41)] and the triangle DEF is likewise half of the same rectangle DEFG. Therefore the triangle DEF too will be equal to the figure ABC. Hence the area of any regular figure is equal to a right triangle, and so on — which was to be demonstrated.

Diagram. A regular hexagon (generically labeled ABC) with center D, alongside a rectangle DEFG with a diagonal.

- Hexagon: C (top), D (center), with A, E, B along the bottom (E is the midpoint of base AB, foot of the perpendicular DE).
- Rectangle: D (upper left), I (top middle), G (upper right), E (lower left), H (bottom middle), F (lower right).
- A diagonal runs from D to F (forming the right triangle DEF); the midline IH is drawn.

THEOR. 4. PROP. 4.

The area of any circle is equal to the rectangle comprehended under the semidiameter and half the circumference of the circle. [Margin: *Circulus quicunque, cui rectangulo æqualis sit.* — "Any circle: to what rectangle it is equal."]

Let ABC be a circle whose semidiameter is DB. Let DBEF be a rectangle comprehended under DB (the semidiameter of the circle) and BE, a straight line equal to half the circumference of the circle. I say that the area of the circle ABC is equal to the rectangle DBEF. For produce BE in a straight line, and set EG equal to BE itself, so that BG is a straight line equal to the whole circumference of the circle. Finally, join the points D and G by the straight line DG. [Margin a: 1. de Dimens. circuli Archim. (Archimedes, *Measurement of the Circle*, Prop. 1)] Now since the circle ABC is equal to the triangle DBG, [Margin b: schol. 41 primi (scholium to Euclid, *Elements* I.41)] and the triangle DBG is equal to the rectangle DBEF — because the base of the triangle is double the base [of the rectangle] [BE [?]; the printed text reads "basis trianguli dupla sit basis trianguli," which appears corrupt] (a point that is also clear from the demonstration of the preceding proposition, where we showed that the triangle DEF is equal to the rectangle

DEHI) — the circle ABC too will be equal to the rectangle DBEF. Therefore the area of any circle is equal to a rectangle, and so on — which was to be shown.

Diagram. A circle ABC with center D and semidiameter DB, alongside a rectangle DBEF with a diagonal forming the triangle DBG.

- Circle: A (left), C (right), D (center), B (bottom); the semidiameter DB is drawn downward from the center.
- Rectangle: D (upper left), F (upper right), B (lower left), E (bottom middle), G (lower right).
- A diagonal runs from D to G, producing the triangle DBG.

THEOR. 5. PROPOS. 5.

In every right triangle, if from one of the acute angles a straight line is drawn in any manner to the opposite side, the ratio of this side to its segment nearest the right angle will be greater than the ratio of the said acute angle to the part of it subtended by the said segment of the side. [Margin: *Proprietas quædam trianguli rectanguli*. — "A certain property of the right triangle."]

Let ABC be a right triangle whose angle C is right; and from the acute angle A draw AD in any manner to the opposite side BC. I say that the ratio of the straight line BC to the straight line CD is greater than the ratio of the angle BAC to the angle CAD. [Margin a: 19. prim. (Euclid, *Elements* I.19)] For since the straight line AD is indeed greater than AC but less than AB, if a circle is described with center A and radius AD, it will cut the straight line AC, produced below the point C, at E, and the straight line AB, above the point B, at F. And since the ratio of the triangle BAD to the sector FAD is greater than the ratio of the triangle DAC to the sector DAE — because the former is a ratio of greater inequality and the latter a ratio of lesser inequality — [Margin b: 27. quin. (Euclid, *Elements* V.27)] then, by alternation, the ratio of the triangle BAD to the triangle DAC will also be greater than the ratio of the sector FAD to the sector DAE. [Margin c: 28. quin. (Euclid, *Elements* V.28)] Therefore, by composition, the ratio of the triangle BAC to the triangle DAC — that is, of the straight line BC to the straight line CD, [Margin d: 1. sexti (Euclid, *Elements* VI.1)] since the triangles BAC, DAC have the same ratio as their bases BC, CD — will also be greater than the ratio of the sector FAE to the sector DAE, that is, than the ratio of the angle BAC to the angle CAD, [Margin e: corol. 33. sexti (Euclid, *Elements* VI.33, corollary)] because the sectors have the same ratio as the angles. Therefore in every right triangle, and so on — which was to be demonstrated.

This proposition is also true in a non-right triangle, provided that the angle C is greater than the angle ADC, as is evident; [Margin f: 19. primi (Euclid, *Elements* I.19)] for then too AD is greater than AC but less than AB, and so on.

Diagram. A right triangle ABC (right angle at C) with the cevian AD to the opposite side, and a circular arc centered at A through D meeting AC produced at E and AB at F.

- A (top apex); B (lower left); C (right, the right angle); D (on the base BC); E (below/right of C, on AC produced); F (on side AB).

THEOR. 6. PROPOS. 6.

Of regular figures having equal perimeters, the greater is the one that contains more angles or more sides. [Margin: *Inter figuras Isoperimetras, quæ plures angulos seu latera continet, illa maior est.* — "Among isoperimetric figures, the one that contains more angles or sides is the greater."]

Let ABC and DEF be two regular isoperimetric figures, and let the figure ABC have more sides, or angles, than DEF. I say that ABC is greater than DEF. For describe circles about the figures; from their centers G and H draw GI and HK perpendicular to BC and EF, [Margin a: 3. tertij (Euclid, *Elements* III.3)] which will bisect the straight lines BC and EF. Now since the figure ABC has more sides than DEF, its isoperimetric partner, it comes about that the side BC, repeated more often, measures out the perimeter of the figure ABC than does the side EF the perimeter of the figure DEF. Therefore the side BC will be less than the side EF, and accordingly BI, half the side BC, will be less than EK, half the side EF. Set KL equal to BI, and draw the straight lines LH, HE, HF, GB, GC. [Margin b: 28. tertij (Euclid, *Elements* III.28)] And since all the arcs of the circle DEF are equal — because the straight lines subtending them are set equal — the straight line EF will be the same submultiple of the perimeter of the figure DEF...

Diagram. Two figures, side by side, for the isoperimetric comparison: a many-sided regular polygon inscribed in a circle (left) and a square inscribed in a circle (right).

- Left figure: a regular polygon ABC inscribed in a circle with center G; A (top), B (lower left), C (lower right); I on BC (foot of the perpendicular GI); radii GB and GC drawn.
- Right figure: a square DEF inscribed in a circle with center H; D (top), E (lower left), F (lower right); interior labels M (upper center), H (center), K

(below H), L (lower left, near E); the lines LH, HE, HF and the perpendicular HK to EF are drawn.

of the figure DEF, as the arc EF is a submultiple of the circumference of the circle DEF. And by the same reasoning, the perimeter of the figure ABC is the same multiple of the straight line BC as the circumference ABC is of the arc BC. [Margin c: 2. corol. 33. sexti (Euclid, *Elements* VI.33, 2nd corollary)] But as the arc EF is to the circumference of the circle DEF, so is the angle EHF to four right angles. Therefore it will also hold that, as the straight line EF is to the perimeter of the figure DEF — that is, to the perimeter of the figure ABC, which is equal to it — so is the angle EHF to four right angles. And as the perimeter of the figure ABC is to the straight line BC, so is the circumference of the circle ABC to the arc BC, [Margin d: 2. corol. 33. sexti (Euclid, *Elements* VI.33, 2nd corollary)] that is, so are four right angles to the angle BGC. Therefore, *ex aequali*, as the straight line EF is to the straight line BC — [Margin e: 15. quin. (Euclid, *Elements* V.15)] that is, as the straight line EK is to the straight line BI — so is the angle EHF to the angle BGC, [Margin f: 15. quin. (Euclid, *Elements* V.15)] that is, so is the angle EHK to the angle BGI. [Margin g: 5. huius (Proposition 5 of this Book)] But the ratio of the straight line EK to the straight line KL is greater than the ratio of the angle EHK to the angle KHL. [Margin h: 13. quin. (Euclid, *Elements* V.13)] Therefore the ratio of the angle EHK to the angle BGI will also be greater than the ratio of the same angle EHK to the angle KHL; [Margin i: 10. quint. (Euclid, *Elements* V.10)] and accordingly the angle KHL will be greater than the angle BGI. Since, then, the angles HKL and GIB are equal, being right angles, [Margin k: 32. prim. (Euclid, *Elements* I.32)] the remaining angle HLK will be less than the remaining angle GBI. Make, therefore, the angle KLM equal to the angle GBI; and LM will fall outside LH, at the point M, and will meet KH produced beyond H at the point M. Now since the two angles B and I of the triangle GBI are equal to the two angles L and K of the triangle MLK, and the sides BI, LK are equal, the straight lines GI and MK will be equal. Therefore the straight line GI is greater than the straight line HK. For this reason the rectangle contained under GI and half the perimeter of the figure ABC will be greater than the rectangle contained under HK and half the perimeter of the figure DEF, which is set equal to half the perimeter of the figure ABC. [Margin l: 26. primi (Euclid, *Elements* I.26)] [Margin m: 2. huius (Proposition 2 of this Book)] Consequently, since that former rectangle has been shown equal to the figure ABC, and the latter equal to the figure DEF, the figure ABC will also be greater than the figure DEF. Therefore, of regular figures having equal perimeters, the greater is the one, and so on — which was to be shown.

PROBL. 1. PROPOS. 7.

Given a triangle whose two sides are unequal, to describe upon its remaining side a triangle isoperimetric to the former and having two equal sides. [Margin: *Qua arte triangulum Isosceles constituatur Isoperimetrum cuiuslibet trianguli non Isoscelis.* — "By what method an isosceles triangle is constructed isoperimetric to a non-isosceles triangle."]

Let ABC be a triangle whose two sides AB and BC are unequal, namely AB greater than BC; and let it be required to construct upon AC an isosceles triangle isoperimetric to the triangle ABC. Take the straight line DE equal to the two sides AB and BC together, and bisect it at F. [Margin a: 16. prim. (Euclid, *Elements* I.16)] Now since the sides AB and BC together are greater than the side AC, DF and FE together will also be greater than the line AC. And on this account the three lines AC, DF, FE will be so related that any two of them are greater than the remaining one. [Margin b: 22. prim. (Euclid, *Elements* I.22)] If, therefore, a triangle AGC is constructed from them, what is proposed will have been accomplished. For the sides AG and GC will be equal both to one another and, taken together, to the sides AB and BC taken together; and so, adding the common AC, the triangles ABC and AGC will be isoperimetric. Therefore, given a triangle whose two sides are unequal, we have described upon its remaining side a triangle, and so on — which was to be done.

Diagram. The isosceles triangle AGC constructed on base AC, isoperimetric to the original triangle ABC, with the auxiliary segment DFE below.

- G (apex) and B (just to the right of G, the apex of the original triangle ABC); H (interior point below the apex, on side AB); A (lower left), C (lower right).
- Below: the segment D (left), F (middle, its bisection point), E (right).

SCHOLIUM.

The point G will necessarily fall outside the triangle ABC. For if it fell on the side AB, say at the point H, [Margin a: 20. prim. (Euclid, *Elements* I.20)] then the drawn straight line HC would be less than HB and BC together, and on that account the triangle AHC would not be isoperimetric to the triangle ABC — the contrary of which has been demonstrated from the construction. Much less will the point G fall inside the triangle ABC. Therefore it falls outside, which is what was proposed.

THEOR. 7. PROPOS. 8.

Of two isoperimetric triangles having the same base, one of which has two equal sides while the other has unequal sides, the greater will be the one whose two sides are equal. [Margin: *Isosceles triangulum maius est triangulo sibi Isoperimetro non Isoscele.* — "An isosceles triangle is greater than a non-isosceles triangle isoperimetric to it."]

Let ABC be a triangle whose side AB is greater than the side BC; [Margin a: 7. huius (Proposition 7 of this Book)] and upon the base AC construct a triangle ADC isoperimetric to the triangle ABC, having the sides AD, DC equal both to one another and, together, to the sides AB, BC taken together. I say that the triangle ADC is greater than the triangle ABC. For produce AD toward the side of D, and let DE be equal to AD itself, or to DC itself. Also draw the straight lines DB and BE. [Margin b: 20. prim. (Euclid, *Elements* I.20)] Now since AB and BE together are greater than AE — that is, than AD and DC together, that is, than AB and BC together — when the common AB is taken away, BE will be greater than BC. And since the sides ED, DB of the triangle EDB are equal to the sides CD, DB of the triangle CDB, while the base BE [?] is greater than the base BC, [Margin c: 25. prim. (Euclid, *Elements* I.25)] the angle EDB will be greater than the angle CDB. Therefore the angle EDB is greater than half the angle EDC. But the angle DAC is half the angle EDC, [Margin d: 5 primi (Euclid, *Elements* I.5)] because the angles DAC, DCA are equal, [Margin e: 32 prim. (Euclid, *Elements* I.32)] and the exterior angle E[DC], taken as equal to these two together...

Diagram. The isosceles triangle ADC and the scalene triangle ABC on the common base AC, with AD produced to E (DE = AD) and the auxiliary lines DB, BE.

- E (top apex, AD produced); D (upper left); F (upper right); B (center); A (lower left), C (lower right).
- The straight lines DB and BE are drawn.

[Continuation of the proof of THEOR. 7 / PROPOS. 8, begun on the previous page.]

the angle EDC. Therefore angle EDB is greater than angle DAC. [Margin: *Elements* I.23] Now construct angle EDF equal to the interior angle DAC; the line DF will fall above the line DB, [Margin: *Elements* I.28] and will run parallel to the line AC. Extend DF until it meets AB, when this is produced, at F, and draw the line FC. [Margin: *Elements* I.37] Since, then, triangles ADC and AFC are equal, while triangle AFC is greater than triangle ABC, triangle ADC too will be greater than

triangle ABC. And so, of two isoperimetric triangles having the same base, and so on — which was what had to be demonstrated.

THEOR. 8. PROPOS. 9.

[Margin: A property of two similar right triangles.]

In similar right triangles, the square described on the sides that subtend the right angles — treated as a single line — is equal to the two squares together that are described on the remaining homologous sides taken as two lines, in such a way that each pair of homologous sides is made into one straight line.

Let ABC and DEF be similar right triangles, with the angles B and E right, the angles C and F equal to each other, and likewise the angles A and D equal to each other; and let the homologous sides be AB, DE, and again BC, EF, and AC, DF. I say that the square described on AC and DF, treated as a single line, is equal to the two squares — one described on AB and DE as a single line, the other on BC and EF as another single line.

For let DE be produced beyond E, and take EG equal to the line AB; draw GH parallel to EF, until it meets FD produced at the point H. Then through F draw the line FI parallel to EG. Triangle FIH will then be equiangular with triangle DEF, that is, with triangle ABC. [Margin: *Elements* I.29] For angle FIH is equal to angle G, [Margin: *Elements* I.29] and this in turn is equal to angle DEF, that is, to angle B; while angle H is equal to angle DFE, that is, to angle C; [Margin: *Elements* I.32] and consequently angle IFH is equal to angle A. Moreover the sides AB and FI are equal, [Margin: *Elements* I.26] for the line FI is equal to the line FG, and FG was taken equal to AB. [Margin: *Elements* I.34] Therefore the sides BC and IH, and likewise AC and FH, will be equal to one another. Hence the line DH will be composed of AC and DF, the line DG of AB and DE, and finally the line GH of BC and EF — [Margin: *Elements* I.34] since GI is equal to EF. [Margin: *Elements* I.47] And since the square on the line DH is equal to the squares on the lines DG and GH together, what is proposed is established as true. Therefore in similar right triangles the square on the sides subtending the right angles, and so on — which was to be demonstrated.

Diagram. A composite right-triangle figure for the construction. At the top stands a reference right triangle with apex A. Below it is the main construction: the two similar right triangles are assembled end to end so that DH forms one straight line (equal to AC + DF), DG another (equal to AB + DE), and GH the base (equal to

BC + EF), with the right angle at G; the line FI, drawn parallel to EG, cuts off the small triangle FIH (congruent to ABC), meeting the base at I.

- A (top apex): vertex of the reference triangle ABC
- C (upper right): vertex of the reference triangle
- D (upper left of construction): vertex, top of line DH
- E (middle left): point on line DG, where DE is produced to G
- F (middle): vertex; foot of FI drawn parallel to EG
- G (lower left): right-angle vertex of the assembled triangle DGH
- I (lower centre): foot of FI on the base GH
- H (lower right): far end of the base
- [Point B of triangle ABC expected but not clearly resolved amid the overprinted interior labels — see QA. [?]]

PROBL. 2. PROPOS. 10.

[Margin: By what method to construct two isosceles triangles similar to each other, but isoperimetric with two other given isosceles triangles. *Elements* I.25]

Given two isosceles triangles whose bases are unequal, and the two [equal] sides of the one equal to the two sides of the other: to construct, upon the same bases, two other isosceles triangles that are similar to each other but which, taken together, are isoperimetric with the two former taken together.

Let there be, upon equal bases AB and CD, two isosceles triangles AEB and CFD, with the four lines AE, EB, CF, FD equal to one another; and let the base AB be greater than the base CD. Given this arrangement, [Margin: *Elements* I.25] angle D will be greater than angle F, and therefore the triangles are not similar, since they are not equiangular either. Now the task is to construct, upon these same bases AB and CD, two other isosceles triangles that are similar to each other but which, taken together, are isoperimetric with the two former triangles taken together.

Set down a line GH equal to the four lines AE, EB, CF, FD, and divide it at the point K so that it stands as the line composed of AB and CD, divided at the point B — that is, so that GK is to KH as AB is to CD. And since the line AB is greater than the line CD, the line GK will likewise be greater than the line KH, since in both cases the ratio is one of greater inequality. Bisect each of GK and KH at the points L and M. Then, since GK is to KH as AB is to CD, by composition GH will be to KH as AB and CD together are to CD. But GH is less than AB and CD together, [Margin: *Elements* I.20] because the four lines AE, EB, CF, FD, which are equal to the line GH, are greater than AB and CD. [Margin: *Elements* V.14] Therefore KH too will be

greater than CD; and by the same reasoning GK will be greater than AB. Since, then, of the three lines AB, GL, LK any two taken together are greater than the remaining one (for GL and LK together are greater than AB, because the whole GK is greater than AB, as was just shown; while it is plain that AB and GL are greater than the remaining LK, and likewise AB and LK greater than the remaining GL, because GK has been bisected at L — and you will say the same of the three lines CD, KM, MH), [Margin: *Elements* I.22] let a triangle ANB be constructed from the three lines AB, GL, LK. This will be isosceles, and the point N will fall outside triangle AEB, since AE and EB together make up half of the line GH, whereas AN and NB together make up more than half of the line

Diagram. Two isosceles triangles standing side by side on a common base line. On the left, over the base AB, is the original triangle AEB (apex E) together with the newly constructed triangle ANB, whose apex N falls higher, outside AEB. On the right, over the base CD, is the original triangle CFD (apex F) together with the newly constructed triangle COD, whose apex O falls lower, inside CFD. The base line carries, from left to right, the auxiliary points G, A, L, B, K, C, M, D, H.

- G (far left of base line): endpoint of the auxiliary line GH
- A, L, B (left segment): base AB, with L the bisection point of GK
- K (centre of base line): division point of GH
- C, M, D (right segment): base CD, with M the bisection point of KH
- H (far right of base line): endpoint of the auxiliary line GH
- E (left, lower apex): apex of the original triangle AEB
- N (left, higher apex): apex of the constructed triangle ANB
- F (right, higher apex): apex of the original triangle CFD
- O (right, lower apex): apex of the constructed triangle COD

the line GH. [Margin: *Elements* I.22] Again, from the three lines CD, KM, MH let a triangle COD likewise be constructed. This will be isosceles, and the point O will fall inside triangle CFD, because CF and FD together are equal to half the line GH, while CO and OD together are less than half the line GH. And since the four sides AE, EB, CF, FD together are equal to the line GH, the first four together will be equal to the second four together; so, adding the common bases AB and CD, the six sides AE, EB, BA, CF, FD, DC together become equal to the six sides AN, NB, BA, CO, OD, DC together. Therefore triangles ANB and COD together will be isoperimetric with triangles AEB and CFD together.

I now say further that triangles ANB and COD are also similar to each other. For since AB is to CD as GK is to KH, [Margin: *Elements* V.15] that is, as GL is to KM,

that is, as AN is to CO and as NB is to OD, by alternation AB will be to AN as CD is to CO, and AN to NB as CO to OD. The sides of triangles ANB and COD are therefore proportional, [Margin: *Elements* VI.5] and hence the triangles are equiangular with each other, and therefore similar. And so, given two isosceles triangles whose bases are unequal, and so on, we have carried out the construction — which was what had to be done.

THEOR. 9. PROPOS. 11.

[Margin: Two similar isosceles triangles are greater than two dissimilar isosceles triangles which are isoperimetric with them and have the same bases. *Elements*, Prop. 10 of this book.]

Two similar isosceles triangles constructed upon unequal bases are, taken together, greater than two isosceles triangles taken together which have the same bases as the former, are dissimilar to each other, are isoperimetric with the two former, and have their four sides equal to one another.

Upon unequal bases AC and CE let there be two isosceles triangles ABC and CDE, dissimilar to each other, such that the four sides AB, BC, CD, DE are equal to one another. [Margin: *Elements*, Prop. 10 of this book.] And upon the same bases AC and CE let two other isosceles triangles AFC and CGE be constructed, similar to each other and, taken together, isoperimetric with the two former triangles taken together. I say that the two triangles AFC and CGE together are greater than the two triangles ABC and CDE together.

Set AC and CE along one straight line, and let AC be the greater base, CE the lesser. Then from F, through B, draw the line FBK cutting AC at the point K; and likewise from D, through G, draw the line DGH cutting CE at H. Since the sides AF, FB of triangle AFB are equal to the sides CF, FB of triangle CFB, and the base AB equal to the base BC, [Margin: *Elements* I.8] angle AFB will be equal to angle CFB. Again, since the sides AF, FK of triangle AFK are equal to the sides CF, FK of triangle CFK, and angle AFK equal to angle CFK, as has been proved, [Margin: *Elements* I.4] the bases AK, KC will be equal, and the angles at K equal as well, that is, right angles. By the same reasoning we shall conclude that CE is bisected at the point H, and that the angles at H are right angles.

Produce the line DH beyond H, and take HL equal to the line DH; then from L, through the point C, extend the line LCN. Now since the sides DH, HC of triangle DCH are equal to the sides LH, HC of triangle LCH, and the angles at H equal — namely right angles — [Margin: *Elements* I.4] the bases DC, LC will be equal, and

the angles DCH, LCH equal too. But angle DCH is greater than angle GCH, on account of the similarity of triangles GCE and FAC, that is, greater than angle FCA, [Margin: *Elements* I.5] which is equal to angle FAC. Therefore angle DCH — that is, angle LCH, which was shown equal to it, that is, angle NCK, [Margin: *Elements* I.15] which is equal to angle LCH as a vertical angle — will also be greater than angle FCA. For this reason the line CN will necessarily fall outside the line CF; and the lines LC, CB will therefore enclose, toward the side of K, the angle BCL. Hence, if the line BL is drawn, it will cut the line CK at some point between the points C and K; let this be M.

Now since the lines AB, BC, CD, DE together are equal to the lines AF, FC, CG, GE together, because the triangles are isoperimetric, their halves too will be equal to one another — namely the lines BC, CD, that is, BC, CL, together are equal to FC, CG together. [Margin: *Elements* I.20] But the lines BC, CL together are greater than the line BL. Therefore FC, CG together will also be greater than that same line BL; and so the square described on FC, CG as a single line will be greater than the square on BL. [Margin: *Elements*, Prop. 9 of this book.] But the square described on FC, CG as a single line is equal to the square described on FK, GH as a single line, together with the square described on KC, CH as a single line. [Margin: *Elements*, Prop. 9 of this book.] And the square described on LB is equal to the square described on BK, LH — that is, on BK, DH — as a single line, together with the square described on KM, MH as a single line; for the right triangles BKM, LHM are similar to each other. [Margin: *Elements* I.15] For the angles at M are equal as vertical angles, and the angles at K and H are right angles, [Margin: *Elements* I.32] and therefore the remaining angles KBM, HLM are equal too.

Therefore the square on FK, GH as a single line, together with the square on KC, CH as a single line — that is, the square KH — both together are greater than the square on BK, DH as a single line together with the square on KM, MH as a single line — that is, together with the square KH. So, when the common square KH is removed, the square on FK, GH as a single line will be greater than the square on BK, DH as a single line; and hence the lines FK, GH together will be greater than the lines BK, DH together. And therefore, the common lines BK, GH being taken away, the remainder FB will be greater than the remainder DG. Moreover KC is greater than HC, because the whole AC, of which KC is the half, is set down as greater

Diagram. The construction for the theorem, laid out along one straight base line A–K–C–H–E. Over the left base AC stand the isosceles triangles ABC (apex B) and AFC (apex F); over the right base CE stand the isosceles triangles CDE (apex D)

and CGE (apex G). The line FBK descends from F through B to K on AC; the line DGH descends from D through G to H on CE. The point M lies on the base between K and C; the line LCN runs from L (below C) up through C toward N (upper middle), where DH has been produced to L with $HL = DH$.

- A (far left of base): endpoint of base AC
- K (base, left): midpoint of AC, foot of FBK
- M (base, between K and C): intersection point of BL with CK
- C (base, centre): common vertex of the two bases
- H (base, right): midpoint of CE, foot of DGH
- E (far right of base): endpoint of base CE
- B (left, lower apex): apex of triangle ABC
- F (left, upper apex): apex of triangle AFC
- G (right, lower apex): apex of triangle CGE
- D (right, upper apex): apex of triangle CDE
- N (upper middle): point on the produced line LCN
- L (bottom, below C): point where DH is produced ($HL = DH$)

than the whole CE, of which HC is the half. Therefore the rectangle contained under FB, KC will be greater than the rectangle contained under DG, HC. And since triangle FBC is half of the rectangle contained under FB, KC — (for if upon FB a rectangle is constructed having the altitude KC, so that the triangle and the rectangle lie between the same parallels, [Margin: *Elements* I.41] the triangle will be half of the parallelogram; and this parallelogram is the same as the rectangle contained under FB, KC, as is evident) — while triangle DGC is half of the rectangle contained under DG, HC — (for if upon DG a rectangle is constructed having the altitude HC, so that the triangle and the rectangle lie between the same parallels, [Margin: *Elements* I.41] the triangle will be half of the parallelogram, which parallelogram is the same as the rectangle contained under DG, HC, as is evident) — it follows that triangle FBC is greater than triangle DGC. And therefore double triangle FBC — namely the rectilinear figure AFCBA — will be greater than double triangle DGC — namely the rectilinear figure CDEGC. Consequently, adding the common figure composed of triangles ABC and CGE, the triangles AFC, CGE together will be greater than the triangles ABC, CDE together. Therefore two similar isosceles triangles constructed upon unequal bases, and so on — which was what had to be shown.

Diagram. The same construction as the preceding figure, reprinted where the proof carries over onto this page: the base line A–K–C–H–E carrying, over the left

base AC, the isosceles triangles ABC (apex B) and AFC (apex F); over the right base CE, the isosceles triangles CDE (apex D) and CGE (apex G); with the lines FBK and DGH, the intersection point M between K and C, the point N above, and L below C.

- A (far left of base): endpoint of base AC
- K (base, left): midpoint of AC
- M (base, between K and C): intersection of BL with CK
- C (base, centre): common vertex of the bases
- H (base, right): midpoint of CE
- E (far right of base): endpoint of base CE
- B (left, lower apex): apex of triangle ABC
- F (left, upper apex): apex of triangle AFC
- G (right, lower apex): apex of triangle CGE
- D (right, upper apex): apex of triangle CDE
- N (upper middle): point on the produced line LCN
- L (bottom, below C): produced point of DH

THEOR. 10. PROPOS. 12.

[Margin: Among isoperimetric figures having an equal number of sides, the greatest is both equilateral and equiangular. — *Elements*, Prop. 7 of this book; Prop. 8 of this book; I.24; Prop. 10 of this book; Prop. 11 of this book.]

Of isoperimetric figures having an equal number of sides, the greatest is both equilateral and equiangular.

Let ABCDEF be a figure of any number of sides, the greatest among all those isoperimetric with it and having the same number of sides, so that no greater can be given. I say it is equilateral and equiangular. For suppose, if possible, that it is first not equilateral, but that the adjacent sides AB, BC are unequal. Draw the line AC; [Margin: *Elements*, Prop. 7 of this book.] then, if upon AC an isosceles triangle AGC is constructed that is isoperimetric with triangle ABC, the whole figure AGCDEF will be isoperimetric with the figure ABCDEF. [Margin: *Elements*, Prop. 8 of this book.] And since triangle AGC is greater than triangle ABC, if the common polygon ACDEF is added, the figure AGCDEF will be greater than the figure ABCDEF — which is contrary to the hypothesis. Therefore the sides AB, BC are not unequal, but equal. By the same reasoning we shall show that the adjacent sides BC, CD, and then in turn each successive pair, are equal. Therefore the greatest figure among those isoperimetric with it and having an equal number of sides is equilateral — which is the first point.

Diagram. The upper of two hexagons for the theorem: the equilateral hexagon ABCDEF, with the added isosceles triangle AGC set upon the side AC — its apex G rising at the top beside the vertex B.

- G (top): apex of the added triangle AGC
- B (top, beside G): vertex of the hexagon
- C (upper left): vertex of the hexagon
- A (upper right): vertex of the hexagon
- D (lower left): vertex of the hexagon
- F (lower right): vertex of the hexagon
- E (bottom): vertex of the hexagon

Next, suppose, if possible, that the figure ABCDEF is indeed equilateral, as has now been demonstrated, but not equiangular — that the non-adjacent angles B and D are unequal, with angle B greater than angle D. Since it has been demonstrated that the greatest figure is equilateral, the two triangles ABC, CDE will be isosceles, such that the two sides AB, BC are equal to the two sides CD, DE. But angle B is set down as greater than angle D; [Margin: *Elements* I.24] therefore the line AC will be greater than the line CE. [Margin: *Elements*, Prop. 10 of this book.] If, then, upon the bases AC, CE two other isosceles triangles AGC, CHE are constructed, similar to each other and isoperimetric with triangles ABC, CDE, [Margin: *Elements*, Prop. 11 of this book.] the triangles AGC, CHE together will be greater than the triangles ABC, CDE together. If, then, the common polygon ACEF is added, the figure AGCHEF will be greater than the figure ABCDEF — which conflicts with the hypothesis, since this figure is set down as the greatest of all. Therefore the angles B and D are not unequal, but equal. By the same reasoning we shall show that the non-adjacent angles C and E are equal, and any other pair of non-adjacent angles.

Diagram. The lower of two hexagons: the equilateral hexagon ABCDEF, with the added triangle configuration — apex G above the vertex B at the top, and the point H set within the figure near the vertex D.

- G (top): apex of the added triangle
- B (top, below G): vertex of the hexagon
- C (upper left): vertex of the hexagon
- A (upper right): vertex of the hexagon
- D (lower left): vertex of the hexagon
- H (interior, beside D): apex of the added triangle
- F (lower right): vertex of the hexagon

- E (bottom): vertex of the hexagon

From this it follows that the whole figure is equiangular — that is, that the adjacent angles too are equal to one another. For if, say, angle B were not said to be equal to angle C, then, since angle C is equal to the non-adjacent angle E, angle B too would not be equal to angle E — which is absurd; for non-adjacent angles are equal to one another, as we have shown. Therefore the greatest figure among those isoperimetric with it and having an equal number of sides is not only equilateral but also equiangular. Consequently, of isoperimetric figures having an equal number of sides, the greatest is both equilateral and equiangular — which was to be demonstrated.

SCHOLIVM.

[Margin: What things are to be observed.]

Concerning the demonstration of the first part of this proposition, it must be observed that two unequal sides are to be taken...

equal and adjacent to one another, so that they form an angle, with nothing else interposed between them — such as the sides AB and BC that were taken [Margin: in the demonstration of this proposition], which produce angle B. For by this reasoning, once the straight line AC is drawn, the triangle ABC will have been formed, whose two sides AB and BC are unequal, just as was assumed in the demonstration. Nor could anyone doubt that in a non-equilateral figure — such as ABCDEF is here taken to be — two adjacent unequal sides can be selected. For suppose someone objects that sides AB and BC are equal: then we take sides AB and AF. If these too are said to be equal, we take AF and FE; and if these are still called equal, we take EF and ED. Proceeding in this way, we eventually reach two adjacent unequal sides that form an angle. We must necessarily arrive at two such sides; otherwise the figure would be equilateral, which is not granted.

As for the demonstration of the second part, note that in many-sided figures one must select two unequal angles that are *not* adjacent — that is, with one or more angles interposed between them, as are the chosen angles B and D, between which lies angle C. On this arrangement the two straight lines AC and CE, subtending the intervening angles, will not cut one another, and two figures will be constructed, ABCDEF and AGCHEF, built up by adding to the common figure ACEF the two triangles standing on the bases AC and CE. This would not happen if two *adjacent* unequal angles were taken, as is evident.

Nor should there be any doubt that two such angles can be found. For in every many-sided figure that is not equiangular there will necessarily be some pair of non-adjacent unequal angles. In the figure ABCDEF as proposed, we compare angle B with all the non-adjacent angles D, E, F — which are necessarily two in a pentagon, three in a hexagon, and so on. If B is unequal to any one of them, we already have two non-adjacent unequal angles, namely B and the one it is unequal to. But if B is said to be equal to all of them, then B must at least be unequal to one of its neighbours, otherwise the figure would be equiangular. So if B is unequal to angle A, then A will be non-adjacently unequal to both angle E and angle D — since B, which is posited equal to each of these, is equal to them. And if instead B is unequal to angle C, then C will be non-adjacently unequal to both angle E and angle F, since B is posited equal to each of these.

But this proposition has so far been demonstrated only for many-sided figures, as is clear from what we have just said about two non-adjacent unequal angles. In triangles and quadrilaterals, angles of that kind cannot be found: in equilateral triangles all angles are equal, as is clear from the corollary to Euclid I.5; and in quadrilaterals having all sides equal — which are necessarily parallelograms, as we showed in the scholium to Euclid I.34 [Margin a: Euclid I.34] — the opposite angles are equal to one another. We shall therefore demonstrate this whole proposition for triangles and quadrilaterals as follows.

Diagram. A row of small overprinted figures illustrating the triangle and quadrilateral cases of the isoperimetric argument. From left to right: (1) a triangle over base C–A carrying an isosceles construction, with apex labels **D** and **B** above and **C**, **A** along the base — the triangle ABC and the isosceles ADC on the same base; (2) a square-set quadrilateral with **C** and **E** above, an interior point **G** and **B**, and base **A**, **D** — the quadrilateral ABCD with the isosceles triangle AEC on diagonal AC; (3) a further quadrilateral/parallelogram construction with **F** and **E** above, interior **G**, **C**[?] and **H**, a low **B**, and base **D**, **A** — the parallelogram construction AEFD raised from ABCD by the perpendiculars AH, DG. The interior lettering is heavily overprinted; several interior labels could not be resolved with confidence. [Margin a: Euclid I.34]

- D, B (apex region, figure 1)
- C, A (base, figure 1)
- C, E (top, figure 2); G, B (interior, figure 2); A, D (base, figure 2)
- F, E (top, figure 3); G, C[?], H (interior, figure 3); B (low interior); D, A (base, figure 3)

Let ABC first be the greatest triangle among triangles isoperimetric with one another. I say it is equilateral and equiangular. For if it is not equilateral but its sides AB and BC are unequal, then [Margin b: Prop. 7 of this book] construct on base AC an isosceles triangle ADC such that sides AD and DC together equal sides AB and BC together. The triangles ABC and ADC will then be isoperimetric, [Margin c: Prop. 8 of this book] and so ADC will be greater than ABC — which is contrary to the hypothesis. Therefore sides AB and BC are not unequal but equal, and the same reasoning holds for the rest. Triangle ABC is therefore equilateral; hence, by the corollary to Euclid I.5, it is also equiangular, which is what was proposed.

Next, let ABCD be the greatest quadrilateral among all those isoperimetric with one another. I say it is both equilateral and equiangular. For if it is not equilateral, let sides AB and BC be unequal, if that be possible, and draw the straight line AC. [Margin d: Prop. 7 of this book] Then if an isosceles triangle AEC, isoperimetric with triangle ABC, is constructed on AC, [Margin e: Prop. 8 of this book] triangle AEC will be greater than triangle ABC. Adding the common triangle ACD, the quadrilateral AECD will be greater than quadrilateral ABCD — contrary to the hypothesis, since ABCD is posited to be the greatest. Therefore sides AB and BC are not unequal but equal, and the same holds for the rest. Figure ABCD is therefore equilateral.

Now let the quadrilateral ABCD be the greatest of all isoperimetric figures and equilateral, as has been shown, but not equiangular — suppose the angles BAD and CDA are unequal. Since ABCD, being equilateral, is a parallelogram (as we demonstrated in the scholium to Euclid I.34), neither of the angles A and D is a right angle — for otherwise [Margin f: Euclid I.29] both, being equal to two right angles, would both be right — but one is acute and the other obtuse. So drop from A and D two perpendicular lines AH and DG, meeting side BC at H and G; then AHGD too will be a parallelogram. [Margin g: Euclid I.19] Since sides AB and DC are greater than sides AH and DG, extend the latter to become the straight lines AE and DF, equal to sides AB and DC, and join the line EF. Once this is done, the figure AEFD will be isoperimetric with the parallelogram ABCD: for sides AE and DF are equal to sides AB and DC, side AD is common, and side EF is equal to side BC, [Margin h: Euclid I.34] since each of them is equal to the opposite side AD. Now since the figure AEFD is greater than the parallelogram ABCD, [Margin i: Euclid I.35] it is also greater than the parallelogram AHGD; and since AHGD is equal to the parallelogram ABCD, the figure AEFD will be greater than ABCD. Therefore, since AEFD is isoperimetric with ABCD, ABCD will not be the greatest quadrilateral among those isoperimetric with one another — contrary to the

hypothesis. So the angles BAD and CDA are not unequal but equal; and since ABCD is a parallelogram, [Margin k: Euclid I.34] the opposite angles B and C will be equal to the angles D and A, and therefore the whole figure will be equiangular, which is what was proposed.

THEOR. 11. PROPOS. 13.

The circle is greater than every regular rectilinear figure isoperimetric with it.

Let ABC be a circle, and let DEF be a regular figure of however many sides, isoperimetric with it. I say that the circle ABC

is greater than the figure DEF. [Margin: The circle is the greatest of all regular rectilinear figures isoperimetric with it.] For let G be the centre of the circle ABC, and H the centre of the figure DEF. About the circle ABC describe a figure BIKC having as many equal sides and angles as the figure DEF contains, by the method we taught in the scholium to Euclid IV.16. Then from the point of contact A draw the straight line AG to the centre G, [Margin a: Euclid III.18] which will be perpendicular to IK. Draw again HD perpendicular to LM; [Margin b: Euclid III.3] and the straight lines GA and HD will bisect the straight lines IK and LM, as is evident if circles are circumscribed about the figures BIKC and DEF. Draw also the straight lines GI and HL, which will bisect the angles I and L, as is clear from the demonstration of Euclid IV.12. Since, then, the whole angles I and L are equal — owing to the similarity of the figures — their halves (namely the angles AIG and DLH) will also be equal. And since the angles IAG and LDH [Margin c: Euclid I.32] are likewise equal, being right angles, the triangles AIG and DLH will be equiangular. Now the perimeter of the figure BIKC is greater than the circumference of the circle ABC (by Archimedes, *On the Sphere and Cylinder* I.1); and the circumference of the circle is posited equal to the perimeter of the figure DEF; therefore the perimeter of the figure BIKC will also be greater than the perimeter of the figure DEF. Since the figures are regular and similar, side IK will be greater than side LM; and so IA, the half of side IK, [Margin d: Euclid VI.4] will be greater than LD, the half of side LM. Again, since as IA is to AG, so is LD to DH, and IA is greater than LD, [Margin e: Euclid V.14] AG will also be greater than DH. Consequently the rectangle contained by AG and half the circumference of the circle ABC — [Margin f: Prop. 4 of this book] which is equal to the circle ABC — is greater than the rectangle contained by DH and half the perimeter of the figure DEF, that is, [Margin g: Prop. 2 of this book] than the area of the figure DEF. The circle is therefore greater than every regular rectilinear figure isoperimetric with it, which was to be demonstrated.

Diagram. A paired illustration for Prop. 13. On the left, the circle ABC (centre **G**) with a regular polygon BIKC circumscribed about it: **A** is the point of contact on side IK, with vertices **I** (upper left) and **K** (lower left), **B** at the top and **C** at the bottom, and radius/apothem lines drawn to the centre **G**. On the right, the regular figure DEF (centre **H**), a hexagon-set figure with **L** (upper left), **D** (the contact point, mid-left), **M** (lower left), **E** (upper right) and **F** (lower right), with lines drawn from the centre **H**.

- Left figure: A (point of contact on IK), I, K (vertices), B (top), C (bottom), G (centre)
- Right figure: D (contact point on LM), L, M (upper/lower left vertices), E, F (right vertices), H (centre)

COROLLARIUM

[Margin: The circle is greater than every rectilinear figure isoperimetric with it.]

From everything that has been demonstrated it is clear that the circle is, absolutely, the greatest of all rectilinear figures isoperimetric with it. For from Proposition 5 it follows that, of regular isoperimetric figures, the one with more sides is the greater; from Proposition 12 it is established that among all isoperimetric figures having an equal number of sides, the regular one is the greatest; and from this Proposition 13 it is clear that the circle is the greatest of all regular isoperimetric figures. It is therefore manifestly concluded that the circle is, absolutely and simply, the greatest of all rectilinear figures isoperimetric with it, which is what was proposed.

THEOR. 12. PROPOS. 14.

[Margin: To what parallelepiped any pyramid is equal.]

The volume of any pyramid is equal to the rectangular solid contained under the perpendicular drawn from the apex to the base and a third part of the base.

Let there be a pyramid whose base ABCDE has any number of sides, with apex F. Let GN be a rectangular solid whose base GHIK is equal to a third part of the base ABCDE, and whose height (the perpendicular GL) is equal to the height of the pyramid — that is, to the perpendicular drawn from the apex of the pyramid to its base. I say that the rectangular solid GN is equal to the pyramid ABCDEF. For draw straight lines from all the angles of the base GHIK to some point of the opposite face, namely L, so as to form a pyramid GHIKL having the same base as

the solid GN and the same height both as the solid GN and as the pyramid ABCDEF. [Margin a: scholium to Euclid XII.6] Since, then, the pyramid ABCDEF is triple the pyramid GHIKL, [Margin b: corollary to Euclid XII.7] and the solid GN is likewise triple the same pyramid GHIKL, [Margin c: Euclid V.9] the solid GN will be equal to the pyramid ABCDEF. Therefore the volume of any pyramid is equal to the rectangular solid, and so on, which was to be shown.

Diagram. Two solids. Above, the pyramid ABCDEF: apex **F** at the top, with the pentagonal base drawn in perspective — **A** at the centre-top of the base, **B** at the left, **E** at the right, and **C** (lower left) and **D** (lower right) at the front, edges running from **F** to each base vertex. Below, the rectangular solid (parallelepiped) GN: top face with **O** (left) and **N** (right), the mid-edge points **L** and **M**, the near-lower points **K** and **I**, and the front-bottom **G** and **H**, with internal diagonals drawn.

- Pyramid: **F** (apex); **A** (centre of base); **B**, **E** (side vertices); **C**, **D** (front vertices)
- Solid GN: base **G**, **H**, **I**, **K**; top face **L**, **M**, **N**, **O**

THEOR. 13. PROPOS. 15.

[Margin: Any body in which a sphere can be inscribed — to what parallelepiped it is equal.]

The volume of any body contained by plane surfaces and circumscribable about a sphere — that is, a body from some middle point of which all the perpendiculars drawn to its faces are equal — is equal to the rectangular solid contained under one of the perpendiculars and a third part of the surface of the body.

Let ABCD be a body contained by plane surfaces, described about a sphere EFGH whose centre is **I**. In it draw from **I** to the points of contact the straight lines **IE**, **IF**, **IG**, **IH**, which will be perpendicular to the faces of the solid. For if, say, through the straight line **IE** a plane is drawn making in the sphere (by Theodosius, *Sphaerica* I.1) the circle EFGH, [Margin a: Euclid XI.3] and in the base the straight line **AB**, then the circle EFGH will touch the straight line **AB** at the point **E**, because the sphere does not cut the base but touches it. [Margin b: Euclid III.18] Therefore **IE** will be perpendicular to the straight line **AB**. By the same reasoning, if through **IE** another plane is drawn, different from the first, another circle will be made in the sphere and another straight line in the same base, cutting **AB** at **E**, to which **IE** will likewise be perpendicular. [Margin c: Euclid XI.4] And so **IE** will

be perpendicular to the base of the solid, drawn through those straight lines. In just the same way we shall show that IF, IG, IH are perpendicular to the other faces. Now let LR be a rectangular solid whose base KLMN is equal to a third part of the surface of the body ABCD, and whose height (the perpendicular LP) is equal to one of the perpendiculars falling from the centre I to the faces of the body ABCD — all of which are equal to one another by the definition of a sphere. I say that the solid LR is equal to the body ABCD. For draw from the centre I straight lines to all the angles of the body ABCD, so that the whole body is divided into the pyramids of which it is composed; the bases of these pyramids are the same as those of the body, while their common apex is the centre I. [Margin d: Prop. 14 of this book] Since, then, each of these pyramids is equal to the rectangular solid contained under a perpendicular of the body ABCD — which is posited equal to each of the perpendiculars — and a third part of its own base: if as many rectangular solids are made as there are pyramids, all of them together will equal the rectangular solid LR. (For if the rectangle KLMN is divided into as many rectangles as there are faces in the proposed solid, so that the first equals a third part of one face, the second a third part of another, and so on — since the whole rectangle KLMN is posited equal to a third part of the whole surface of the solid — and if parallelepipeds are understood to be raised on those rectangles, [Margin e: Prop. 14 of this book] all of them together will be equal to the parallelepiped LR.) Therefore, since each parallelepiped is equal to each pyramid, all the pyramids together — namely the body ABCD composed of them — will also be equal to the rectangular solid LR. Hence the volume of any body contained by plane surfaces, and so on, which was to be demonstrated.

Diagram. The body ABCD circumscribed about the sphere EFGH. A square-set quadrilateral: **B** at the upper left, **E** at the top centre (point of contact), **A** at the upper right; **F** on the left side and **H** on the right; **I** at the centre; **C** at the lower left, **G** at the bottom centre, **D** at the lower right. The inscribed circle EFGH is drawn with radial lines to the contact points, producing an eight-rayed star pattern about the centre I.

- B, A (upper corners); C, D (lower corners)
- E (top contact), G (bottom contact), F (left contact), H (right contact)
- I (centre)

Diagram. The rectangular solid LR, drawn in perspective as a tall box. Readable vertex labels: **M** (upper left), **R** (upper right), **N** (interior upper, printed as a reversed N), an interior mark near the top that could not be resolved [?], **P**

(centre), **O** (right), **L** (lower left) and **K** (lower centre). Per the text the base is **KLMN** and the perpendicular height is **LP**.

- Base: **K, L, M, N**
- Height: **L to P**
- Upper face: **O, R** (and one unresolved label [?])

THEOR. 14. PROPOS. 16.

[Margin: To what parallelepiped any sphere is equal.]

The volume of any sphere is equal to the rectangular solid comprehended under the radius of the sphere and a third part of the surface of the sphere.

Let **ABC** be a sphere whose centre is **D** and radius **AD**; and let **E** be a rectangular solid contained under the radius **AD** and a third part of the surface of the sphere **ABC**. I say that the body **E** is equal to the sphere **ABC**. For if it is not equal, let it, if possible, first be greater, and let the excess of the body **E** over the sphere **ABC** be the quantity **F**. Conceive a sphere **GHK** described about the centre **D**, greater than the sphere **ABC**, yet such that the excess of the sphere **GHK** over the sphere **ABC** is not greater than the quantity **F**, but either equal to it or less — that is, such that the sphere **GHK** is either equal to the solid **E** (when it exceeds the sphere **ABC** by precisely the quantity **F**) or less than it (if it exceeds the sphere **ABC** by a quantity smaller than **F**). For there will necessarily be some sphere that is either equal to the magnitude **E** — and thus greater than the sphere **ABC** — or at any rate greater than the sphere **ABC** but less than the magnitude **E**, which is posited to be greater than the sphere **ABC**. [Margin a: Euclid XII.17] Then inscribe within the sphere **GHK** a body that does not touch the sphere **ABC**, such that each of the perpendiculars drawn from the centre **D** to the faces of that body is greater than the radius **AD**. If, then, straight lines are drawn from the centre **D** to all the angles of the said body, so that the whole body is divided into pyramids whose bases are the same as those of the body **GHK** and whose common apex is the centre **D**, [Margin b: Prop. 14 of this book] each pyramid will be equal to the rectangular solid contained under its perpendicular and a third part of its base; and therefore the rectangular solid contained under the radius **AD** and a third part of the base of each pyramid will be less than the pyramid itself. And since all the rectangular solids contained under the several perpendiculars...

Diagram. The sphere **ABC** and the inscribed body **GHK**. A set of concentric circles with an inscribed many-sided polygon: **G** at the top, **A** just below it, **L** in the upper-centre, **M** to the left and **N** to the right of the centre, **D** at the centre, **B** at

the lower left and **C** at the lower right, with **H** on the left edge and **K** on the right edge; radial lines run from the centre outward. A small crescent **F** (the excess quantity) sits at the lower left, and below it a flat rectangular box labelled **E** (the rectangular solid).

- G (top), A (below G), L (upper centre), D (centre)
- M (left), N (right); B (lower left), C (lower right); H (left edge), K (right edge)
- F (crescent, the excess); E (rectangular solid below)

[Continuation of Theorem 14 / Proposition 16 — the proof that the solid E is neither greater nor less than the sphere ABC.]

drawn from the center D to the bases of the said body, together with the several third-parts of the bases, are together equal to the whole body; and since all the third-parts of the bases together make up a third part of the surface [ambitus] of the body: the rectangular solid contained under the semidiameter AD and a third part of the surface of the aforesaid body inscribed within the sphere GHK will be less than that inscribed body. But the surface of the inscribed body is greater than the surface of the sphere ABC — as [Archimedes] proves in *On the Sphere and Cylinder*, Book I, Proposition 27 — and therefore a third part of the surface of the said body is also greater than a third part of the surface of the sphere ABC. Hence the rectangular solid contained under the semidiameter AD and a third part of the surface of the sphere ABC — that is, the solid E — will be far less than the body inscribed within the sphere GHK. But the sphere GHK was assumed to be either equal to the solid E or less than it. Therefore the sphere GHK will be less than the body described within it — the whole less than the part — which is absurd. And so the solid E will not be greater than the sphere ABC.

Diagram. The figure for the squaring-of-the-sphere argument: a nest of concentric circles and regular polygons about a common center D, representing the sphere ABC, the circumscribed body GHK, and the smaller inscribed sphere LMN, together with the excess quantity F and the rectangular solid E.

- G (top vertex of outer body): outer circumscribed polygon/body GHK
- A (just below G, on the middle circle): top of sphere ABC
- L (upper center, on the vertical radius to D): point of inner sphere LMN
- M (center-left, inner circle): point of sphere LMN
- D (center): common center
- N (center-right, inner circle): point of sphere LMN
- B (lower-left, on middle circle): point of sphere ABC

- C (lower-right, on middle circle): point of sphere ABC
- H (far left, outer body): vertex of circumscribed body GHK
- K (far right, outer body): vertex of circumscribed body GHK
- F (small pointed oval / lens below-left): the excess quantity F
- E (rectangular box beneath the figure): the rectangular solid E

Next, let the solid E — if it is possible — be less than the sphere ABC, and let it be exceeded by the sphere ABC by the quantity F. About the center D let a sphere LMN be conceived, smaller than the sphere ABC, yet such that the excess by which the sphere LMN is surpassed by the sphere ABC is not greater than the quantity F, but either equal to it or less — that is, so that the sphere LMN is either equal to the solid E (if it is exceeded by the sphere ABC precisely by the quantity F), or greater than the solid E (if the sphere LMN is surpassed by the sphere ABC by a quantity less than F). For necessarily there will be some sphere that is either equal to the solid E — and so smaller than the sphere ABC — or else smaller indeed than the sphere ABC but greater than the magnitude E, which is posited to be smaller than the sphere ABC. Then let there be described within the sphere ABC a body that nowhere touches the sphere LMN, in such a way that each of the perpendiculars falling from the center D to the bases of this inscribed body is less than the semidiameter AD. [Margin: a — Elements XII.17.] If, then, from the center D lines are extended to all its angles, so that the whole body is resolved into pyramids whose bases are the same as those of the body ABC and whose common vertex is the center D, each pyramid will be equal to the rectangular solid contained under its perpendicular and a third part of its base. [Margin: b — this book, prop. 14.] And therefore the rectangular solid contained under the semidiameter AD and a third part of the base of any given pyramid will be greater than that pyramid itself. And since all the rectangular solids contained under the several perpendiculars drawn from the center D to the bases of the said body, together with the several third-parts of the bases, are together equal to the whole body — and since all the third-parts of the bases together make a third part of the surface of the body — the rectangular solid contained under the semidiameter AD and a third part of the surface of the said body inscribed in the sphere ABC will be greater than the inscribed body. Since, then, the surface of the sphere ABC is greater than the surface of the body inscribed in it, and so too a third part of the surface of the sphere is greater than a third part of the surface of the said body, the rectangular solid contained under the semidiameter AD and a third part of the surface of the sphere ABC — that is, the solid E — will be far greater than the body inscribed within the sphere ABC. But the sphere LMN was posited to be either equal to the solid E or greater. Therefore the sphere LMN will

be greater than the body described within the sphere ABC — the part greater than the whole — which is absurd. And so the solid E will not be less than the sphere ABC. Since, then, it has been shown to be neither greater nor less, it will be altogether equal. And on this account the area [content] of any sphere is equal to the rectangular solid comprehended under the semidiameter of the sphere and a third part of the surface of the sphere. Which was to be demonstrated.

THEOR. 15. PROPOS. 17.

[Margin: A sphere is greater than all bodies isoperimetric to it and circumscribable about other spheres, which are contained by plane surfaces.]

A sphere is greater than all bodies isoperimetric to it that are contained by plane surfaces and can be circumscribed about other spheres — that is, all of whose perpendiculars, drawn to the bases from some central point, are equal.

Let there be a sphere A whose center is A and whose semidiameter is AB; and let C be a solid isoperimetric to it, circumscribable about some sphere, of which CD is one of the perpendiculars. I say that the sphere A is greater than the solid C. For let there be conceived, about the sphere A, a body described exactly similar to the solid C, in such a way that each of its sides likewise touches the sphere A — that is, so that its perpendiculars (of which AB is one) are likewise equal, being the semidiameters of the sphere A. Now since the surface of the body about the sphere A is greater than the surface of the sphere A (by what [Archimedes] demonstrated in *On the Sphere and Cylinder*, Book I, Proposition 27), the surface of this same body will also be greater than the surface of the body C. Therefore the perpendicular AB — that is, the semidiameter of the sphere A — will be greater than the perpendicular CD. For this reason the rectangular solid contained under the semidiameter AB and a third part of the surface of the sphere A — which is equal to the sphere A [Margin: a — this book, prop. 16.] — will be greater than the rectangular solid contained under the perpendicular CD and a third part of the surface of the body C, that is, greater than the body C. [Margin: b — this book, prop. 15.] Therefore a sphere is greater than all bodies isoperimetric to it that are contained by plane surfaces, etc. Which was to be demonstrated.

Diagram. Two figures for Theorem 15. Above: a shaded sphere A inscribed in a slightly tilted square (the circumscribed plane-faced body), with a horizontal radius drawn. Below: a trapezoidal cross-section (a plane-faced body about its inscribed sphere) with a short perpendicular segment marked.

- B (left end of the radius on the sphere): endpoint of the semidiameter
- A (center/right end of the radius): center of sphere A
- D (left end of the lower short segment): endpoint of the perpendicular in body C
- C (right end of the lower short segment): the perpendicular CD of the body C

COROLLARIVM.

[Margin: A sphere is greater than any regular body isoperimetric to it.]

It is clear from this that a sphere is greater than any regular body isoperimetric to it; since all the perpendiculars from the center to the bases of a regular body are equal to one another, because they are equal to the semidiameter of the sphere that can be described within the body. [Margin: c — Elements XV.21.]

THEOR. 16. PROPOS. 18.

[Margin: A sphere is greater than all bodies isoperimetric to it and circumscribable about other spheres, which are contained by conical surfaces.]

A sphere is greater than all bodies isoperimetric to it and circumscribable about other spheres that are contained by conical surfaces, in such a way that all the conical sides are equal.

Let there be a circle ABCD, about which is circumscribed a regular figure EFGHIKLM, such that the number of its sides is measured by four — of which sort is the square, the figure of 8, 12, 16, 20, 24, or 28 sides, with equal angles, and so on. And from the angle E, through the center, to the angle I, let the straight line EI be drawn. Then if, about the fixed straight line EI held immovable, the plane in which lie the circle ABCD and the figure EFGHIKLM is revolved, the circle will describe a sphere, and the figure a body contained about the sphere by conical surfaces, the sides of which surfaces are equal — namely the same as those of the figure — as [Archimedes] demonstrates in *On the Sphere and Cylinder*, Book I, Propositions 22 and 27.

Now let there be a sphere N, isoperimetric to the body EFGHIKLM described about the sphere ABCD. I say that the sphere N is greater than the said body. For since the surface of the solid EFGHIKLM is greater (by *On the Sphere and Cylinder*, Book I, Proposition 27, of [Archimedes]) than the surface of the sphere ABCD, the surface of the sphere N will also be greater than the surface of the sphere ABCD; and therefore the semidiameter of the sphere N will be greater than the

semidiameter of the sphere ABCD. And since the surface of a sphere is quadruple (by *On the Sphere and Cylinder*, Book I, Proposition 31, of [Archimedes]) of the greatest circle in the sphere: if a circle OP be taken quadruple of the greatest circle in the sphere N — which will indeed be easily done if the diameter OP be taken double the diameter of the greatest circle in the sphere N; [Margin: a — Elements XII.2.] for since as the circle OP is to the greatest circle in the sphere N, so is the square of the diameter OP to the square of the diameter of the greatest circle in the sphere N; and since the ratio of square to square is the duplicate of the ratio of the homologous sides [Margin: b — Elements VI.20.], the circle OP will also be to the greatest circle in the sphere N in the duplicate ratio of the diameter OP to the diameter of the greatest circle in the sphere N. Since, then, the diameters are posited to have the double ratio, the circles will have the quadruple ratio — for the quadruple ratio is the duplicate of the double ratio, as appears in these numbers: 1, 2, 4 — the circle OP will be equal to the surface of the sphere N.

Take again a circle ST equal to the circle OP. Then upon the circle ST let a right cone STV be set up, having an axis VX equal to the semidiameter of the sphere N; and likewise upon the circle OP let another cone [N]PQ be constructed, having an axis QR equal to the semidiameter of the sphere ABCD. The altitude of the cone STV will be greater than that of the cone OPQ, but their bases will be equal. Therefore the cone STV will be greater than the cone OPQ [Margin: c — Elements XII.14.], because cones of equal bases have to one another the ratio of their altitudes. Now the sphere N is quadruple of the cone that has for its base the greatest circle in the sphere N and for its altitude the semidiameter of the sphere N, as [Archimedes] demonstrated in *On the Sphere and Cylinder*, Book I, Proposition 32; and the cone STV is quadruple of this same cone [Margin: d — Elements XII.11.], because cones having the same altitude have the ratio of their bases [Margin: e — Elements V.9.]. Therefore the cone STV will be equal to the sphere N. In the same way, since the base of the cone OPQ is equal to the surface of the body EFGHIKLM — because it is also equal to the surface of the sphere N, which is isoperimetric to that body — and its altitude is equal to the semidiameter of the sphere ABCD, the cone OPQ will be equal to the solid EFGHIKLM contained by conical surfaces, by what [Archimedes] demonstrated in *On the Sphere and Cylinder*, Book I, Proposition 29. For this reason the sphere N will also be greater than the solid EFGHIKLM contained by conical surfaces. Therefore a sphere is greater than all bodies isoperimetric to it and circumscribable about other spheres, etc. Which was to be demonstrated.

Diagram. The circle ABCD circumscribed by the regular polygon EFGHIKLM, with the diameter EI drawn through the center.

- E (top): angle of the polygon; one end of the axis EI
- M (upper left): angle of the polygon
- A (upper center, on the circle): point of circle ABCD
- F (upper right): angle of the polygon
- L (far left): angle of the polygon
- B (center-left, on the circle): point of circle ABCD
- D (center, on the circle): point of circle ABCD
- G (right): angle of the polygon
- K (lower left): angle of the polygon
- C (lower center, on the circle): point of circle ABCD
- H (lower right): angle of the polygon
- I (bottom): angle of the polygon; other end of the axis EI

Diagram. The auxiliary cones and sphere for Theorem 16: two cones shown as ellipse-cross-sections with their axes, and the small circle representing the sphere N.

- O (left end of the upper ellipse): edge of base circle OP of cone OPQ
- Q (top of the upper ellipse): apex of cone OPQ
- R (center of the upper ellipse, below Q): foot of axis QR
- P (right end of the upper ellipse): edge of base circle OP
- N (small circle at lower left): the sphere N
- S (left end of the lower ellipse): edge of base circle ST of cone STV
- V (top of the lower ellipse): apex of cone STV
- X (center of the lower ellipse, below V): foot of axis VX
- T (right end of the lower ellipse): edge of base circle ST

THEOR. 17. PROPOS. 19.

[Margin: A sphere is greater than any cone or cylinder isoperimetric to it.]

A sphere is greater than any cone or cylinder isoperimetric to it.

For, any sphere being given, if a cone be made having a base equal to the surface of the sphere — that is, quadruple of the greatest circle in the sphere — and an altitude equal to the semidiameter of the sphere, then the sphere will be equal to this cone [Margin: a — Elements V.9.]; because, with respect to the cone whose base is the greatest circle in the sphere and whose altitude is the semidiameter of the sphere (from *On the Sphere and Cylinder*, Book I, Proposition 32, of

[Archimedes]) [Margin: b — Elements XII.11.], the former cone — having a base quadruple of the greatest circle in the sphere, that is, equal to the surface of the sphere, and an altitude equal to the semidiameter of the sphere — has the quadruple ratio. Since, then, the surface [ambitus] of the cone whose base is equal to the surface of the sphere is greater than the surface of the sphere — for the former exceeds the latter by the whole lateral surface of the cone, apart from the base, which is posited equal to the surface of the sphere — it plainly follows that, if a cone be made isoperimetric to the sphere, this cone is less than that other cone, and consequently less than the sphere as well.

Again, if a cylinder be made having a base equal to the surface of the sphere and an altitude equal to the semidiameter of the sphere [Margin: c — Elements XII.20.], this cylinder will be triple of that cone which has a base equal to the same surface of the sphere and an altitude equal to the same semidiameter of the sphere — the cone which we have just shown to be equal to the sphere — and consequently triple of the sphere itself. Therefore a third part of that cylinder (namely a cylinder having the same base, but an altitude equal to a third part of the altitude of that cylinder [Margin: d — Elements XII.14.], since that cylinder is triple of this one) will be equal to the sphere. Since, then, this latter cylinder has a surface greater than the surface of the sphere — because the former exceeds the latter by the surface of the whole cylinder, apart from one base — who does not see that, if a cylinder be made isoperimetric to the sphere, this cylinder is less than that former cylinder, and consequently less than the sphere as well? Therefore a sphere is greater than any cone or cylinder isoperimetric to it. Which was to be demonstrated.

SCHOLIVM.

Almost all of this has been drawn from Theon of Alexandria [Theon] in his commentaries on Ptolemy's *Almagest*, and from Pappus of Alexandria [Pappus] in his *Mathematical Collection* — although we have demonstrated most of it more clearly and more easily. What follows, however, we have discovered and proved ourselves.

PROBL. 3. PROPOS. 20.

[Margin: To construct rectangles equal and isoperimetric to a semicircle and to the other sub-multiple parts of a circle.]

Given a semicircle, or a quadrant, or an eighth part of a circle, or a sixteenth, and so on, to construct a rectangle isoperimetric and equal to it, provided a straight line equal to the periphery is given.

Let ABC be a semicircle, ABD a quadrant, AND an eighth part of the circle, AOD a sixteenth part, and so forth. Let there be constructed the rectangle AE, contained under the semidiameter AD and under the straight line AF, which is equal to a fourth part of the periphery; likewise the rectangle AG, under the semidiameter AD and under AH, an eighth part of the periphery; likewise the rectangle AI, under the semidiameter AD and under AK, a sixteenth part of the periphery; likewise the rectangle AL, under the semidiameter AD and under AM, a thirty-second part of the periphery. Then, from what we showed in Book IV, Chapter 7, toward the end, at Number 1, the rectangle AE will be equal to the semicircle ABC; AG to the quadrant ABD; AI to the eighth part AND; AL to the sixteenth part AOD; and so on. I say that these same rectangles are isoperimetric to the aforesaid parts of the circle, each to each — which is indeed evident from the construction. For AD and EF are equal to the diameter AC, and AF and DE to the semicircumference ABC — that is, to two fourth-parts of the circumference. Likewise AD and GH are equal to the semidiameters AD and DB, and AH and DG to two eighth-parts, that is, to a fourth part of the circumference AB. Likewise AD and IK are equal to the two semidiameters AD and DN, and AK and DI to two sixteenth-parts, that is, to the eighth part of the circumference AN. Likewise AD and LM are equal to the two semidiameters AD and DO, and AM and DL to two thirty-second-parts, that is, to the sixteenth part AO; and so on in succession, if both the periphery AO and the straight line AM are continually subdivided. Therefore, given a semicircle, or a quadrant, and so on, we have constructed a rectangle isoperimetric and equal to it. Which was to be done.

This problem, as far as it concerns the semicircle and the quadrant, was also noticed recently by the Reverend Father Odo Malcotius [Odo van Maelcote], an ingenious mathematician, when he presented a mathematical problem through his students at the Roman College — although he set it forth in the manner of a theorem.

Diagram. The figure for Problem 3: a rectangle with corner A (center of the circle) at lower left and a quarter-circle arc, subdivided by horizontal lines at heights corresponding to the successive sub-multiples of the periphery.

- F (top-left corner): top of the left side AF (a fourth part of the periphery)
- E (top-right corner): opposite corner of rectangle AE
- A (bottom-left corner): center of the circle / origin of the semidiameter

- D (bottom-right corner): end of the semidiameter AD
- B (right side, upper): division point
- G (right side, below B): division point (rectangle AG)
- I (right side, below G): division point (rectangle AI)
- C (right side, lower): division point (rectangle AL)
- H (left side, upper): division point for the eighth part
- K (left side, middle): division point for the sixteenth part
- M (left side, lower): division point for the thirty-second part
- N (on the arc): the point marking the eighth part AND
- O (on the arc, interior): the point marking the sixteenth part AOD

PROBL. 4. PROPOS. 21.

[Margin: To construct a parallelogram equal and isoperimetric to a given triangle.
a — Corollary of prop. 19 of Book I.]

Given any triangle, to construct a parallelogram equal and isoperimetric to it.

Let any triangle ABC be given. Through A let AM be drawn parallel to the base BC. Now, if neither of the angles B, C is a right angle, each of the sides AB, AC is greater than the perpendicular let fall from A (or from B, C) onto the opposite parallel D [Margin: a — Corollary of prop. 19 of Book I.]; but if one of the angles is a right angle — that is, if one of the sides is perpendicular to the said parallels — then the two sides AB, AC together are greater than double the aforesaid perpendicular. And therefore half of the sum of the two sides is greater than that same perpendicular. That is, if GH be taken equal to the side AB, and HI equal to the side AC, so that the whole GI is equal to the sum of the sides AB, AC, and GI be bisected at K, then the half GK will be greater than the perpendicular DE. If, then, from D, the midpoint of the base BC, with radius GK, an arc of a circle be described, it will cut the straight line AM at some point, say at L. Then, taking LM equal to BD, let the straight lines DL, BM be drawn, which will be parallel to each other [Margin: b — Elements I.33.]; and thus DM will be a parallelogram equal to the triangle ABC [Margin: c — scholium of prop. 41 of Book I.]. I say that this same parallelogram is also isoperimetric to the triangle ABC — which is evident from the construction, since DL and BM are each equal to GK, that is, to half of the sides AB, AC; and therefore DL and BM together are equal to both sides AB, AC together [?]; while the straight lines DL, LM together are equal to the base BC. Therefore we have constructed the parallelogram DM — not a rectangle — equal and isoperimetric to the triangle ABC.

Diagram. The figure for Problem 4: the triangle ABC with the parallel AM drawn through A, the diagonals of the parallelogram DM, an auxiliary line GI (with its bisection), and a lower auxiliary segment.

- M (top row, far left): endpoint of the parallel through A
- L (top row): the point where the arc cuts AM
- F (top row): auxiliary point on the upper line
- E (top row): auxiliary point on the upper line
- A (top row, far right): apex of the triangle
- B (middle row, on the base): base vertex of the triangle
- D (middle row, midpoint of the base BC): center of the constructing arc
- C (middle row, on the base): base vertex of the triangle
- G (lower line, left end): endpoint of segment GI ($= AB + AC$)
- K (lower line, center): bisection point of GI
- H (lower line, center): division point ($GH = AB$, $HI = AC$)
- I (lower line, right end): other endpoint of GI
- Q (segment, left end): auxiliary segment
- P (segment): division point
- Y (segment): division point
- X (segment): division point
- R (segment): division point
- V (segment): division point
- O (segment, right end): auxiliary segment
- T (bottom, left): corner of the lower rectangle
- S (bottom, right): corner of the lower rectangle

[Margin: b — Elements I.33.] [Margin: c — scholium of prop. 41 of Book I.]

Geometria Practica — Liber VII (continued)

[Problem 4 / Proposition 21, continued]

If instead you want a *rectangle* equal and isoperimetric to the same triangle ABC, you must proceed as follows. Erect the perpendiculars BF and DE. [Margin: d — scholium on I.41.] The rectangle BE will then be equal to the triangle, but not isoperimetric with it; [Margin: e — corollary on I.19.] for BF and DE are shorter than the sides AB and AC, while BD and EF are equal to the base BC. The perimeter of the rectangle BE is therefore smaller than the perimeter of the

triangle ABC. Hence, if BF and DE are extended until they equal half of the sides AB and AC, we obtain a rectangle that is indeed isoperimetric with the triangle ABC, but larger than it, since it exceeds the rectangle BE.

[Margin: f — VI.13.] Now take the mean proportional N between BF and BD.

[Margin: g — VI.17.] The square on the line N will then be equal to the rectangle BE, and hence also to the triangle ABC. [Margin: h — scholium on V.25.] Since BF and BD together are greater than twice the line N, [Margin: i — I.19.] and since BM is greater than BF, BM and BD together will be much greater than twice the line N. Take, therefore, QP equal to BD and PO equal to BM, so that the whole line QO equals BD and BM together; then QO too will be greater than twice the line N. [Margin: k — scholium on VI.13.] Cut QO, then, at R, so that N is the mean proportional between the segments QR and RO, and complete the rectangle QS contained under the segments QR and RO. [Margin: l — VI.17.] This rectangle will be equal to the square on the line N — that is, to the rectangle BE, or to the triangle ABC.

I say that it is also isoperimetric with the triangle ABC. For since QR and RS together — that is, the line QO — are, by construction, equal to BD and BM together, the remaining segments ST and TQ will be equal to the remaining segments LM and LD. The rectangle QS will therefore be isoperimetric with the parallelogram BL, and hence also with the triangle ABC (with which the parallelogram BK was shown to be isoperimetric). So, given any triangle, we have constructed a parallelogram, and so on — which was to be done.

Scholium

If a point V is taken anywhere between R and O, the rectangle under QV and VO will still be isoperimetric with the triangle ABC, but smaller than it. If, however, a point X is taken anywhere between R and Y (QO being bisected at Y), the rectangle under QX and XO will still be isoperimetric with the triangle ABC, but larger than it. Finally, the square on the line QY is likewise isoperimetric with the triangle ABC, and larger than it. All this we shall demonstrate as follows.

That the aforesaid rectangles, and the square on the line QY, are isoperimetric with the triangle ABC — that is, with the rectangle QS — is clear, since the two sides about the right angle are always equal to the line QO, that is, to the two sides of the rectangle QS. That they are, on the other hand, *unequal* to the triangle ABC, I show thus. [Margin: m — lemma on X.42.] Since the squares on QV and VO together are greater than the squares on QR and RO together; [Margin: n — II.4.] and since both those two squares, together with twice the rectangle under QV and

VO, and likewise these two squares, together with twice the rectangle under QR and RO, are equal to the square on QO: it follows that twice the rectangle under QV and VO is less than twice the rectangle under QR and RO, and hence that the rectangle under QV and VO, taken once, is less than the rectangle under QR and RO, taken once. In the same way we shall show that the rectangle under QR and RO is less than the rectangle under QX and XO — that is, that the rectangle under QX and XO is greater than the rectangle under QR and RO. Finally, [Margin: o — II.6.] since the rectangle under QX and XO, together with the square on XY, is equal to the square on YO (or QY), the square on QY will be greater than the rectangle under QX and XO, and hence much greater than the rectangle under QR and RO — that is, than the triangle ABC. All of which was to be demonstrated.

From this it is established that the square on QY, described on half of the line QO, is the greatest of all the rectangles contained under any segments whatever of the line QO — which is also clear from proposition 12 of this book.

PROBL. 5. PROPOS. 22.

To construct, upon a given rectilinear figure, a rectangular parallelogram equal and isoperimetric to it. It is required, however, that the side of the square equal to the rectilinear figure not be greater than half of the half-perimeter of the given rectilinear figure.

[Margin: *To construct a rectangle isoperimetric with, and equal to, a given figure.*]

Let A be a given hexagon, equilateral indeed but not equiangular, such that B — the side of the square equal to it [Margin: a — II.14.] — when found, is not greater than half of the half-perimeter of the hexagon. Take, then, the line CD equal to half the perimeter of the hexagon; B will then be a line not greater than half of CD itself, but either equal to it or smaller. [Margin: b — scholium on VI.13.] Cut CD at E, so that B is the mean proportional between the segments DE and EC, and form the rectangle EG contained under the segments DE and EC. I say that the rectangle EG is equal and isoperimetric to the hexagon A. For since the three lines DE, B, and EC are in continued proportion, [Margin: c — VI.16.] the rectangle EG will be equal to the square on B — that is, to the hexagon A. And because the two sides DE and EF equal the line CD — that is, half the perimeter of the hexagon A — while the remaining two sides FG and GD equal the other half, the whole rectangle EG will be isoperimetric with the hexagon A. So, upon a given rectilinear figure, we have constructed a rectangular parallelogram equal and isoperimetric to it — which was to be done.

Scholium

If B, the side of the square, were greater than half the mean perimeter of the rectilinear figure A — that is, greater than the line CD — then CD could not be cut in such a way that B was the mean proportional between the segments, as is plainly evident.

Now if a point H is taken anywhere between C and E, the rectangle under DH and HC will still be isoperimetric with the figure A, but smaller than it. If, however, a point I is taken anywhere between E and L, the midpoint of the line CD, the rectangle under DI and IC will still be isoperimetric with the figure A, but larger than it. So too the square on the half-line DL will be isoperimetric with the same figure, and larger than it. All this will be demonstrated as was said in the scholium to the preceding problem.

Diagram. At the foot of the page, a tall narrow vertical rectangle beside a pointed lens (vesica) figure, illustrating the scholium's construction. The rectangle has corners F (top-left), E (top-right), G (bottom-left), D (bottom-right); its right edge lies along a vertical line marked, from top down, C, H, E, I, L, B, D, with a short bracketing segment drawn beside the portion I–L. To the right stands a large pointed lens (nearly a rhombus with curved sides) enclosing a single interior letter.

- C (top of the vertical line)
- H (on the line, just below C)
- F (top-left corner of the rectangle)
- E (top-right corner of the rectangle, on the line)
- I (on the line, below E)
- L (on the line, midpoint region)
- B (on the line, below L)
- G (bottom-left corner of the rectangle)
- D (bottom-right corner of the rectangle, foot of the line)
- Interior letter of the lens: reads as A [?]

APPENDIX

On squaring the circle by means of lines.

[Margin: *How to find, by numbers, a square equal to a circle, and conversely, after the doctrine of Archimedes.*]

1. This place prompts me — since it has been demonstrated in this book that the circle is the greatest of all figures isoperimetric with it — to teach briefly by what method a square equal to a given circle can be constructed, and conversely a circle equal to a given square; and that by means of lines. For in book 4, chapter 7, it was set out at length how, from the results found by Archimedes, the circle is to be squared by numbers — that is, by what method the area or capacity of the circle is to be found, whether from the diameter or from the circumference. For the square root of that area is the side of the square equal to the circle.

Conversely, in chapter 8 of the same book, rule 1, no. 1, we taught by what method, from the given area of a circle, both the circumference and the diameter of that circle are to be sought — that is, given a square in the likeness of a circle's area, how the circle equal to that square is to be described. For once the diameter has been found by the aforesaid rule 1, no. 1, chapter 8, book 7, the circle of that diameter will be the one sought. It seemed good to append this appendix to this seventh book, because the treatment of the circle's *tetragonism*, or squaring, is not a little akin to the discussion of isoperimetric figures.

[Margin: *The squaring of the circle by numbers after the Arabs is false.*]

Now the squaring of the circle by numbers that the Arabs handed down — and which Joseph Scaliger [Joseph Justus Scaliger], in his *Cyclometrica elementa*, believes to be true — must be altogether rejected, since it falls outside the limits set by Archimedes. By those limits it is established that the ratio of the circumference to the diameter must be less than three and one-seventh, but greater than three-and-ten-seventy-firsts [i.e. $3 \frac{10}{71}$] — which is not observed in the Arabs' figures. For they say that the ratio of the circumference to the diameter is that of a tenfold power: so that if the square on the circumference be set at 10, the square on the diameter is 1 — which is false. For since the square root of the number 10 is greater than $3 \frac{1}{7}$ — because the square of this root [$3 \frac{1}{7}$] is only $9 \frac{43}{49}$ — while the square root of unity is 1, the ratio of the circumference to the diameter would be greater than three and one-seventh, whereas according to Archimedes it is less. Again, since, with the diameter set at 7, the circumference is less than 22 (by Archimedes), the square on the circumference will be less than 484, which has to 49 (the square on the diameter) a ratio less than tenfold — seeing that it is 490 to 49 that stands in a tenfold ratio. The ratio of the square on the circumference to the square on the diameter is therefore less than tenfold.

[Margin: *The squaring of the circle by numbers after Albrecht Dürer is false. — a: scholium on I.47.*]

The method of squaring the circle by numbers taken from Albrecht Dürer [Albrecht Dürer] must be rejected in the same way. He supposes that, the diameter of the circle being divided into eight equal parts, the diameter of the square equal to the circle has to the diameter of the circle the ratio of 10 to 8 — which is likewise false. For since the square on the diameter 10 makes 100, [Margin: a] and this is double the square whose diagonal is 10 — the latter being the one said to be equal to the circle of diameter 8 — the square equal to the circle will be 50. But, taking the diameter as 8, the area of the circle is found to be greater than the true value 50 [fraction: ?], and less than the true value $50 \frac{13}{72}$ [?] — as we set out in book 4, chapter 7, no. 4. The true area of the circle will therefore be greater than $50 \frac{13}{72}$ [?], and so much greater than 50. Dürer's square is therefore smaller than the area of the circle, and not equal to it.

[Margin: *A false squaring of the circle by lines, ascribed to Campanus. — b: 13 of this book.*]

2. Now, to come to the squaring of the circle by lines: I am ashamed to have to refute the one that seems true to the unskilled, and which some smatterer or other — I know not who — fathered upon Campanus [Campanus of Novara], no unlearned mathematician, and committed to print. It runs thus. Let a straight line equal to the circumference of the circle (though how such a line is to be found he does not teach) be cut into 4 equal parts, out of which a square is to be constructed, which that smatterer says is equal to the circle — a claim wholly unworthy of a geometer, and plainly ridiculous. For if that square is isoperimetric with the circle, [Margin: b — 13 of this book] and the circle is the most capacious of all rectilinear figures isoperimetric with it, who does not see that that square is smaller than the circle?

[Margin: *The squaring of Hippocrates of Chios. — c: V.9. — d: III.31. — e: scholium on I.26. — f: scholium on I.27. — g: XII.2.*]

3. Concerning the tetragonism of Hippocrates of Chios [Hippocrates of Chios] too I would say nothing, were it not that a keen ingenuity lies hidden in his demonstration, even though it does not attain the goal proposed. For he proceeds as follows. Let AFBE be the circle to be squared, whose diameter is AB; on AB let the square ABCD be described, [Margin: c — V.9.] and about that square let the circle ABCD be described, whose diameter BD cuts the given circle AFBE at E. Draw the line AE; [Margin: d — III.31.] then the angle AEB, being in a semicircle, will be a right angle, [Margin: e — scholium on I.26.] and so the perpendicular AE will bisect the base BD of the isosceles triangle ABD; and therefore E will be the center of the circle ABCD. [Margin: f — scholium on I.27.] And because the square

on the diameter BD is double the square on the diameter AB, [Margin: g — XII.2.] and as the square on BD is to the square on AB, so is the circle ABCD to the circle AFBE, the one circle will also be double the other; and the semicircle BAD double the semicircle AFB. Hence half of the semicircle BAD — that is, the quadrant ABE (for ABE is a quadrant, because of the right angle at the center E) — is equal to the semicircle AFB. Therefore, when the common segment AGB is taken away, the remaining triangle AEB will be equal to the remaining lune AFBGA; and consequently, if a square is made equal to the triangle, that same square will be equal to the lune AFBGA. And thus the lune AFBGA is squared.

Diagram. A wide composite figure spanning the foot of the page. On the left: a circle with an inscribed square ABCD (B at the left vertex, A at the right vertex, C lower-left, D lower-right); above the chord BA stands a semicircle with apex F, an interior point G below F, and the center E marked at the middle of the circle, with the diagonals drawn. On the right: a chain of lunes and semicircles built on a horizontal base HI — a semicircle over HK carrying points M and N; a larger semicircle KOL with apex O and interior point P; a semicircle over LI carrying points Q and R; with baseline points H (left) and I (right), and K, L on the middle line. A single point is marked on the base HI (about its midpoint).

- Left circle/square: B (left vertex), A (right vertex), C (lower-left), D (lower-right), E (center), F (semicircle apex), G (below F)
- Right lune-chain: H (base, left), I (base, right), K and L (middle line), M and N (left lune), O (upper apex) and P (below O), Q and R (right lune)
- Midpoint of HI: marked point, unlabeled [?]

Next, let the line HI be double the diameter AB, and on it as base let a semicircle be described; in it fit three lines HK, KL, LI, equal to the semidiameter of this circle — that is, to the diameter AB — forming half of a hexagon, [Margin: a — corollary on IV.15.] since the side of a hexagon equals the semidiameter. Now on those three lines describe the semicircles HMK, KOL, LQI, which are equal to the semicircle AFB because their diameters are equal. [Margin: b — scholium on II.4.] Since the square on the line HI is quadruple the square on the line HK — because the one side is double the other — [Margin: c — XII.2.] the circle of diameter HI will also be quadruple the circle of diameter HK, and the semicircle HKLI will be equal to the semicircles HMK, KOL, LQI, and AFB together. When the common segments HNK, KPL, LRI are removed, the remaining trapezium HKLI will be equal to the three lunes HNKM, KPLO, LRIQ together with the semicircle AFB.

If, then, those three lunes are squared, as has been shown, and [Margin: d — scholium on I.45.] a rectilinear figure equal to those three squares is subtracted

from the trapezium — that is, if the excess of the trapezium over those three squares is sought — this excess will be a rectilinear figure equal to the semicircle AFB. [Margin: e — II.14.] If, then, a square is made equal to this figure, that same square will be equal to the semicircle AFB; [Margin: f — scholium on I.45.] and the square described on the diagonal of that square will be equal to the whole circle AFBE — since the latter square is double the former, just as the circle is double the semicircle. Therefore the circle is squared.

This is the squaring of Hippocrates: acute, indeed, in that it truly squared the lune AGBF, but faulty, in that he supposes the three lunes HNKM, KPLO, LRIQ to be squared by himself — which is not true. For by his demonstration only that lune is squared whose lower arc is a quarter of the circumference of some circle, and whose upper arc is a semicircle of another circle — as was the case with the lune AGBF, [whose lower arc was a quarter] of the circumference ABCD, and [whose upper arc was] AFB, half of the circumference AFBE. But the other three lunes are not of this kind: [Margin: *The fallacy in Hippocrates' squaring.*] their lower arcs HNK, KPL, LRI are sixth parts of the whole circumference, although their upper arcs are semicircles, as in the former case; and these are not yet squared. If an art of squaring lunes of this kind were found, the circle too would be most truly squared, without finding a straight line equal to the circumference of the circle — which would indeed be a splendid thing.

[Margin: *What is lacking in Hippocrates' squaring.*] It follows, therefore, from this reasoning of Hippocrates, that the squaring of the circle is possible; for just as the lune AGBF is squared, so likewise nothing prevents the lune HNKM from being squared, even though it has not yet been squared by anyone. And certainly, as someone rightly affirms, what Hippocrates here shows about the lune AGBF, which is a part of the circle AFBE, in no way prevents the same from being known about the whole circle, even without the size of the circle's circumference being investigated — since only the art of squaring the lune HNKM is lacking. Indeed, the discovery of the squaring of the lune AGBF, its magnitude being unknown, would raise somewhat more doubt than that of the circle.

Diagram. The same composite lune figure repeated at the head of this page. On the left, a circle with inscribed square and the semicircle above chord BA (points F, G, B, A, E, C, D); on the right, the chain of lunes and semicircles on base HI (points H, I, K, L, M, N, O, P, Q, R), with a point marked on HI.

- Left circle/square: F (semicircle apex), G (below F), B (left vertex), A (right vertex), E (center), C (lower-left), D (lower-right)

- Right lune-chain: M, N (left lune), O (apex), P (below O), K, L (middle line), Q, R (right lune), H (base, left), I (base, right)
- Midpoint of HI: marked point [?]

4. Much, too, might be said here of the false squarings of others; but since these either betray themselves — for as one follows them out, it easily appears that something is wanting to the construction of a square equal to the circle, as is the case with the squaring of Jacobus Falco, a Spanish knight, who tries to square the circle without finding a straight line equal to the circumference — or else have long since been refuted by others (namely the squaring of Cardinal Nicholas of Cusa [Nicholas of Cusa], by Johannes Regiomontanus [Regiomontanus] and by Johannes Buteo [Johannes Buteo], and the *Tetragonismus* of Oronce Fine [Oronce Finé], both by the same Buteo and by Pedro Nunes of Portugal [Pedro Nunes] in his little book *On the Errors of Oronce*, each of whom, by various routes, believed he had found a straight line equal to the circumference): I judge that I need say nothing at all of them, lest I seem to waste time uselessly. For which reason I shall here bring forward only that squaring, and expound it somewhat more fully than I wrote at the end of book 6 of Euclid — the one, namely, that by means of the line called the *Quadratrix* (for so I choose to call it) finds a straight line equal to a circular one. [Margin: *Why nothing is said here of the false squarings of others.*] For although this method leaves something to be desired in the geometrical finding of a certain point, it is nonetheless more accurate than all the others I have hitherto been able to see, so that in practice we cannot stray from the mark. And that I may proceed clearly and in order, I shall dispatch the whole business in a few propositions.

[Margin: *Which squaring by lines is expounded here.*]

I.

To describe the *Quadratrix* line.

Dinostratus and Nicomedes [Dinostratus; Nicomedes], as Pappus of Alexandria [Pappus] reports in book 4 of his *Mathematical Collection*, devised a certain curved line for the squaring of the circle, and for that reason it was called by them, from its function, the [Greek: τετραγωνίζουσα (tetragōnizousa, "squaring" — the squaring line)]; by us, for the same reason, it will be called the *quadratrix*. Although the aforesaid authors try to describe such a line by two imaginary motions of two straight lines intersecting each other — in which they beg the question (as the philosophers say), and on that account it is rejected by Pappus as useless and incapable of being described — we shall nevertheless delineate it

geometrically, without those motions, by finding as many points as we like through which it must be drawn, just as is usually done in the descriptions of the conic sections.

5. Let the quadrant BD, then, be described in the square ABCD. Suppose, as the discoverers of the quadratrix line intend, that the semidiameter AD is understood to be carried uniformly about the center A, and likewise the top side of the square, CD, uniformly downward; in such a way that, in the same time in which the point D traverses the circumference DB with a uniform motion throughout, the line DC, descending with a likewise uniform motion, arrives at the side AB — yet so that it remains perpetually parallel to the side AB and makes right angles with the sides AD and BC. Then the semidiameter, revolved through the circumference DB, and the line DC, carried downward, will continually cut each other in points,

Diagram. The quadratrix construction. A square with corners D (top-left), C (top-right), A (bottom-left), B (bottom-right); the quarter-circle arc runs from D down to B. Dotted horizontal lines (the descending line DC in successive positions) and dotted radial lines (the revolving semidiameter AD in successive positions) cross the interior; their intersections trace the quadratrix curve. An interior intersection point O is marked, with Q on the arc above it and R on the left edge. Near the bottom-left corner the labels F, G, A, M are stacked; along the lower part H, E, P appear near the center; and I, K, L, B, N appear at the right.

- D (top-left corner), C (top-right corner)
- A (bottom-left corner), B (bottom-right corner)
- Q (on the arc, upper), O (interior intersection point)
- R (on the left edge, middle)
- F, G (on the left edge, low), M (at/below the bottom-left corner)
- H (lower center), E (lower center, just right of H), P (on the base, below)
- I, K (right side, near the arc's foot), L (right side), N (below B at bottom-right)

points that will describe the quadratrix line — that is, the points through which the quadratrix will pass, a line of the kind of the curved line DE.

But these two uniform motions — the one along the circumference DB, the other along the straight sides DA and CB — cannot be carried out unless we already have the ratio of a circular line to a straight line. For that reason Pappus [Pappus] rightly criticizes this construction: the ratio in question is still unknown, and it is

precisely this ratio that the line is supposed to help us find. We shall therefore describe the same quadratrix geometrically, in the following way.

Divide the arc BD into any number of equal parts, and each of the two sides AD and BC into the same number of equal parts. The division is easiest if the arc DB and each of the sides AD and BC are first bisected, then each half bisected again, and so on as far as one pleases. The more divisions there are, the more accurately the quadratrix will be described. To avoid confusion, we have cut both the arc DB and the two sides AD and BC into just eight equal parts.

Diagram. The square ABCD, with A at the bottom-left corner, B at the bottom-right, C at the top-right, and D at the top-left. The quarter-circle arc DB is drawn with centre A (radius $AD = AB$). The quadratrix runs from D, through the interior points Q and O, down to the point E on the base. Dotted horizontal parallels and radial lines from A subdivide the figure.

- D (top-left corner): endpoint of side AD and of arc DB
- C (top-right corner)
- A (bottom-left corner): centre of the arc
- B (bottom-right corner)
- R (left side AD, upper region): a division point
- Q (interior, upper): point on the quadratrix
- O (interior, middle): point on the quadratrix
- F, G (left side AD, near the bottom): lower division points above A
- M (below A, on the downward extension): auxiliary construction point
- H (near the base, interior-left): construction point
- E (on the base AB): the point where the quadratrix meets AB
- P (on the base, right of centre): auxiliary construction point
- I, K (right side BC, near the bottom)
- L, N (bottom-right region, about B): auxiliary construction points

Next, join the pairs of points on the sides AD and BC that are equally distant from the side DC (or AB) by hidden straight lines; and from the centre A extend further hidden straight lines to the several division points of the quadrant DB. Where these latter lines cross the former — the first crossing the first, the second the second, and so on — the quadratrix is to be drawn evenly through those points, so that it is not sinuous but advances smoothly, forming no hump or angle anywhere. Such is the curved line DE, which cuts the semidiameter AB at E.

6. But the point E on the side AB cannot be found geometrically, since there all crossing of the straight lines ceases. So, to find it without any error appreciable to the senses, we shall use the following device. We shall repeatedly bisect the lowest part AF of the side AD — if it is not already small enough — until the last little piece is very small. In the same way we shall repeatedly bisect the lowest part BI of the arc DB, until as many subdivisions are made as were made in the part AF, so that the little piece BK is the same fraction of the whole arc DB that AG is of the whole side AD. Then along AD we cut off pieces BL, BN, AM equal to AG, and draw the hidden lines GL and MN. From the centre A we draw the hidden straight line AK, which cuts GL at the point H — a point to be marked most exactly (using, that is, the Lemma of Problem 1, Book 2, so that the intersection H may be found with the greatest precision) — and we take MP equal to GH itself. [Margin: The side, the base, and the centre of the quadratrix.] If then we continue the quadratrix — which has been described up to H — with an even and uniform extension as far as P, the quadratrix will cut the side AB at the sought point E. For because the straight lines GH, AE, MP are so little distant from one another, they turn out to be very nearly equal — although, geometrically speaking, AE is always somewhat the greater, however little those lines differ; yet the excess cannot be detected with the compasses. And so the arc of a circle described from A through H and P seems, as far as the senses are concerned, to indicate the true point E. The same thing happens on the circumference of a circle: the lines GL, AB, MN, if they differ little from one another, will be judged, on a circle, altogether equal, even though AB is in truth somewhat the greater. Hence if those three lines GH, AE, MP have a very small distance between them, there can be no doubt that the point E, at which the quadratrix cuts the semidiameter AB, does not differ appreciably from the point that truly lies on the quadratrix there — provided the points H and P have been found with exactness and the utmost care.

The straight line AD we shall call the *side* of the quadratrix; the straight line AE, its *base*; and finally the point A, its *centre*.

7. But we shall find the points of the quadratrix near the base more surely — without those intersections of lines that are there very oblique — by means of perpendicular lines, in this way.

[Margin: a. scholium 27 of Book III. b. scholium 27 of Book III. c. scholium 4 of Book VI. d. Prop. 2 of Book VI.]

Diagram. The quadrant (sector) with centre A at the bottom-left. The arc runs from B (bottom-right) up to Q (top-left). The chord BQ is drawn, together with the

radial lines AQ, AC, AI, AK and the connecting lines DE, FG, DL, FM, HN. The bisection points D, F, H lie on the quadratrix.

- Q (top-left): endpoint of the arc / of the chord
- C (upper right, on the arc): midpoint of the quadrant
- I (right, on the arc): midpoint of arc BC
- K (right side, on the arc): midpoint of arc BI
- A (bottom-left corner): centre
- B (bottom-right): endpoint of the arc
- D (interior, upper): bisection point of chord BQ (on the quadratrix)
- F (interior, middle): bisection point of DE (on the quadratrix)
- H (near the base, interior): bisection point of FG (on the quadratrix)
- O (interior, right of centre): a crossing point
- P (near the base, interior-right): a crossing point
- L, M, N (left side, descending): points on the side taken equal to AD, AF, AH
- G, E (on the base, near B): division points on the base

Draw the chord BQ of the quadrant and bisect it at D. This is done if a straight line is drawn from A to C, the midpoint of the quadrant, [Margin: a. Euclid, scholium to III.27.] for this line will bisect BQ. Next, along AD take AE equal to it; join DE and bisect it at F. This too is done if AI is drawn from A to I, the midpoint of arc BC, [Margin: b. scholium to III.27.] for AI would bisect the chord BC (if it were drawn); [Margin: c. scholium to VI.4.] and therefore it bisects DE as well, [Margin: d. Euclid VI.2.] which is parallel to the chord BC — because the sides AB and AC of the triangle ABC are cut proportionally at D and E, since AB and AC are equal, as are AE and AD. Again, along AF take AG equal to it; join FG and bisect it at H. This too is done by the line AK drawn to K, the midpoint of arc BI. In this way, if a line equal to AH is taken and the rest is done as before, another point will be found between H and G. And so, by continuing thus, we shall find any number of further points nearer to AB itself — by perpendicular lines, and not by oblique intersections as in the earlier figure. [Margin: e. Euclid III.3. f. scholium to I.26.] For AD is perpendicular to DB, AF to DE, DH to FG, and so on.

That all these points found — D, F, H, and the rest — lie on the quadratrix I show as follows. Draw DL, FM, HN parallel to AB, cutting BD at O and P. [Margin: g. Euclid VI.2.] Then as BD is to DQ, so is AL to LQ; and therefore AQ too is bisected at L. But the arc BQ is also bisected at C. Therefore, as was shown, the point D lies on the quadratrix. Again, since DE is bisected at F, [Margin: h. Euclid VI.2.] DB too is bisected at O, [Margin: i. Euclid VI.2.] and therefore, as EF is to FD, so is BO to

OD. But as BO is to OD, so is AM to ML; and as EF is to FD, so is arc BI to arc IC. Therefore, as we showed, AI and MO will cut each other at a point of the quadratrix. And the same reasoning holds for the other points found by this method.

8. It is further clear that this curved line DE, described by us geometrically, is the same one that Dinostratus [Dinostratus] and Nicomedes [Nicomedes] conceived as described by those two imagined motions. For suppose the semidiameter AD in the earlier figure moves about the centre A through the arc DB with a uniform motion, in the same time in which the side DC is carried downward, likewise with a uniform motion. Then it comes about that whenever the semidiameter AD has passed through any part whatever of the arc DB, the side DC will at that moment have run through a like part of the sides DA and CB. Otherwise those two motions would not be uniform,

or the semidiameter AD would not reach the side AB in the same time as the side DC. Since, then, the straight lines sent out from the centre A through the parts of the arc DB, and the parallel lines drawn through the parts of the sides DA and CB, always cut off similar parts from the arc DB and from the sides DA and CB by construction, it plainly follows that the points of the curved line DE found by us geometrically do not differ from the points that would be found by those two motions.

This, then, is in a sense a *geometrical* description of the quadratrix — just as the descriptions of the conic sections, which are also made point by point (as Apollonius [Apollonius of Perga] hands them down), are called geometrical, even though they are more liable to error than our description, on account of the need to find many mean proportional lines, which are required for their descriptions but not needed in describing the quadratrix. Therefore, unless one is willing to reject the whole doctrine of the conic sections — which Apollonius of Perga pursued with such genius that he was on that account called the Great Geometer — as useless and non-geometrical (which I judge no one skilled in geometry would do, since the most eminent geometers have employed the conic sections in their demonstrations: for Menaechmus [Menaechmus] used the hyperbola and the parabola in finding two mean proportionals between any two given straight lines; Archimedes [Archimedes] himself demonstrated many splendid things about these same conic sections; and finally sections of this kind have a notable use in gnomonics, as appears from our own *Gnomonica*) — one will be quite compelled to admit that this description of ours of the quadratrix line is in a sense geometrical. Add to this that the conchoidal line [linea conchilis], by which

Nicomedes so keenly investigates two mean proportional lines, is also described point by point, as we said in Book 6, Proposition 15.

This quadratrix has many notable uses, some of which we demonstrated at the end of Book 6 of Euclid, and which it would be superfluous to repeat here. We shall therefore set out only its use in squaring circles. For this we need only the last point E in the earlier figure — even if no other point of the quadratrix had been found. And although this last point cannot, strictly and geometrically, be found, nevertheless, if the device of the later figure is used, we shall not stray appreciably from the true point, as we said above. For whenever it is found that the last perpendicular AH is equal to the preceding last transferred line AG, so that no difference between them can be discerned with the compasses, that last point G may be taken, without appreciable error, as the extreme point of the quadratrix. If not, other perpendiculars must be drawn by the same device by which AF and AH were drawn, until no difference appears, on the semidiameter AB, between the last and the line found in the final place — an operation in which the practice itself will be the best teacher.

COROLLARY

[Margin: a. Euclid I.34.]

9. From the description of the quadratrix it follows that, if from the centre A any straight line AQ is drawn, cutting the arc of the quadrant at Q and the quadratrix at O, then arc BD is to arc BQ as the semidiameter AD is to the straight line AR — where OR has first been drawn parallel to AB; [Margin: a. Euclid I.34.] and therefore also as it is to the straight line dropped as a perpendicular from O to AB. For the arc DQ is the same part of the whole arc DB that the straight line DR is of the whole semidiameter DA, since in the description of the quadratrix the arc DQ contains as many pieces of the whole arc DB as the straight line DR contains parts of the whole DA — inasmuch as the lines AQ and RO cut each other at O, a point of the quadratrix. Nor is this correspondence hindered even if the arc DQ is incommensurable with the whole arc DB, and the line DR with the whole side DA, since the quadratrix always advances with the same uniformity through all its points. For if the line DR is not that part — whether commensurable or incommensurable — of the whole side DA that the arc DQ is of the whole arc DB; then, if a part of the side DA is supposed either less or greater than DR, the parallel drawn from its endpoint will cut the line AQ either above O or below it, at a point through which the quadratrix would have to be described; and consequently the quadratrix would not pass through O — which is absurd and contrary to the hypothesis. Since, then, I say, the arc DQ is the same part of the

whole arc DB that the line DR is of the whole side DA, [Margin: b. Euclid V.19.] the remaining arc QB will also be the same part of the whole arc DB that the remaining line RA is of the whole side DA — because the ratio of the whole DB to DQ is the same as that of the whole DA to DR; and, by alternation, the ratio of the whole DB to the whole DA is the same as that of the arc removed, DQ, to the line removed, DR. Hence, as the whole arc DB is to the arc QB, so is the whole side DA to the line RA — that is, to the perpendicular dropped from O to AB, [Margin: c. Euclid I.34.] which is equal to RA.

II.

If the quadrant and the quadratrix have the same centre, the arc of the quadrant, the semidiameter, and the base of the quadratrix will be in continued proportion.

This is a remarkable and notable property of the quadratrix. Let there be a quadrant, and a quadratrix described from it as above. I say that the arc BD, the semidiameter AD, and the base AE of the quadratrix are in continued proportion — that is, that BD is to AD as AD is to AE. If not, let BD be to AD as AD is to AF, which is either greater or less than AE itself; and let

first AF be greater than AE.

[Margin: a. Euclid V.15. b. Euclid V.15. c. Euclid V.11. d. Euclid V.9. e. scholium 33 of Book VI. f. Euclid V.11. g. Euclid V.14. h. Euclid III.3. i. scholium 27 of Book III.]

Diagram. The square ABCD, with A at the bottom-left corner. Three concentric quarter-arcs centred at A are drawn: the outer arc DB, the quadrant FG (through F on the base and G on the side, cutting the quadratrix at H), and the quadrant IL (through I on the base and L on the side, cutting the quadratrix at H/M). The radius AHK runs from A through H to K on the outer arc; HI is perpendicular to the base. To the right of the square two short straight segments are marked N and O (for Corollary III).

- D (top-left corner)
- C (top-right corner)
- A (bottom-left corner): common centre
- B (bottom-right corner)
- G (side AD, upper): endpoint of quadrant FG
- L (side AD, lower): endpoint of quadrant IL
- K (on outer arc DB): endpoint of the radius AHK
- H (interior, centre): where the radius cuts the quadratrix / the quadrants

- M (interior, lower-left): where the radius AK cuts arc IL
- I (base AB, left of E): endpoint of quadrant IL / foot of perpendicular
- E (base AB, centre): base-point of the quadratrix
- F (base AB, right of E): endpoint of quadrant FG
- N, O (two segments to the right of the square): the lines of Corollary III

Describe the quadrant FG from the centre A, cutting the quadratrix at H by passing through F; draw the semidiameter AHK through H, and drop the perpendicular HI. Now, since it is assumed that arc BD is to the straight line AD as AD — that is, as AB — is to AF, and since AB, as semidiameter, is to the semidiameter AF as arc BD is to arc FG (for, as we demonstrated in Book 4, Chapter 7, Proposition 1, diameter is to diameter as circumference is to circumference; [Margin: a. Euclid V.15.] and therefore also semidiameter AB is to semidiameter AF as the same circumference is to the same circumference; [Margin: b. Euclid V.15.] and consequently, too, as a quarter of the circumference is to a quarter of the circumference — that is, as arc BD is to arc FG): [Margin: c. Euclid V.11.] it will follow that arc BD is to the straight line AD as the same arc BD is to arc FG; [Margin: d. Euclid V.9.] and therefore the straight line AD and the arc FG will be equal. But since, by the preceding corollary, arc BD is to arc BK as the straight line AD is to the straight line HI, and arc BD is to arc BK as arc FG is to arc FH [Margin: e. scholium to VI.33.] (because the arcs BD, BK are similar to the arcs FG, FH), [Margin: f. Euclid V.11.] it follows that the straight line AD is to the straight line HI as arc FG is to arc FH. [Margin: g. Euclid V.14.] Therefore the straight line HI will also be equal to the arc FH — which is absurd. For the straight line HI is less than the arc FH, since it is half the chord subtending the double of arc FH [Margin: h. Euclid III.3.] (for the straight line AF bisects that chord, and therefore the arc too), and a chord is always less than its own arc. Therefore arc BD is not to the semidiameter AD as AD is to a straight line greater than the base AE of the quadratrix.

Next, suppose, if it can be, that arc BD is to AD as AD is to AI, which is less than the base AE. Describe therefore the quadrant IL from the centre A, through I; draw from I to AE the perpendicular IH, cutting the quadratrix at the point H, through which draw the semidiameter AK, cutting the arc IL at M. We shall then show, as before, that the arc IL is equal to the straight line AD; and likewise that arc BD is to arc BK — that is, arc IL is to arc IM — as the straight line AD is to the straight line HI. Since, then, the arc IL has been shown equal to the straight line AD, [Margin: k. Euclid V.14.] the arc IM will also be equal to the straight line HI — which is absurd. For the straight line HI is greater than the arc IM. For if from H another straight line were drawn toward G, touching the circle IL as HI touches it

at I, [Margin: l. corollary 2 of III.36.] these two tangents would be equal, and the arc intercepted between them would be bisected at M, [Margin: m. scholium 27 of Book III.] because the angle they contain would be bisected by the line AH, [Margin: n. Euclid I.4 (or I.8).] and consequently the angle at the centre A too, if a straight line were joined to the other point of contact; [Margin: o. Euclid III.26.] and therefore the arcs on which they stand would be equal. Since, then, as we shall prove with Archimedes in Book 8, Proposition 1, those two tangents together are greater than the arc they contain, their half HI will also be greater than the half IM of that arc. Therefore arc BD is not to the semidiameter AD as AD is to a straight line less than the base AE of the quadratrix; nor, as was shown, to a greater one. Therefore it is as AD to the base AE itself. Which was to be demonstrated.

COROLLARY I

[Margin: To find a straight line equal to the circumference of a circle.]

From this we shall easily find a straight line equal to the arc of the quadrant from which the quadratrix was described, and consequently equal to the semicircumference, indeed to the whole circumference.

Since arc BD is to the semidiameter AD as AD is to the base AE of the quadratrix, then, by conversion, AE will be to AD as AD is to the arc BD. If, then, a third proportional is found to the two straight lines AE, AD, [Margin: p. Euclid V.11.] AD will be to that third as it is to the arc BD, since each ratio is the same, namely that of AE to AD. [Margin: q. Euclid V.9.] Therefore that third proportional will be equal to the arc BD of the quadrant; and if it is doubled, the resulting straight line will be equal to the semicircumference of the same circle; and if it is quadrupled, the straight line will be equal to the whole circumference.

COROLLARY II

It also follows from these results that, if the base AE of the quadratrix is set as the semidiameter of some circle, its side AD will be equal to a quarter of the circumference of that circle; that the line double the side AD will be equal to the semicircumference of the same circle; and that the line quadruple the side will be equal to the whole circumference.

For since — as we showed in Book 4, Chapter 7, Proposition 1 — the diameters of circles are proportional to their circumferences, [Margin: r. Euclid V.15.] the semidiameters too will be proportional to the circumferences and to the quadrants. Therefore, as AD is to AE — that is, as the aforesaid third proportional

is to AD — so is the quadrant BD, of semidiameter AD, to the quadrant of semidiameter AE. Since, then, that third proportional was shown equal to the quadrant BD, [Margin: s. Euclid V.14.] the straight line AD will also be equal to the quadrant of semidiameter AE. Therefore the line double AD will be equal to the semicircumference of the circle whose semidiameter is AE, and the quadruple will be equal to the whole circumference.

COROLLARY III

From these results it is also inferred that, if two straight lines N and O in the preceding figure have the same ratio that AD and AE have, and the lesser, O, is set as the semidiameter of some circle, the greater, N, is equal to the arc of the quadrant of that circle.

For since AD is to AE as N is to O, then by alternation AD will be to N as AE is to O. But as AE is to O, so is the quadrant of the circle of semidiameter AE to the quadrant of the circle of semidiameter O, as we demonstrated in Book 4, chapter 7, proposition 1. [Margin: a — Euclid V.11] Therefore AD will also be to N as the quadrant of the circle of semidiameter AE is to the quadrant of the circle of semidiameter O. Since, then, AD has been shown equal to the quadrant of the circle of semidiameter AE, N will likewise be equal to the quadrant of the circle of semidiameter O. Which was what was proposed. [Margin: b — Euclid V.14]

III.

Given a circle, to construct a square equal to it.

[Margin: To exhibit a square equal to a circle.]

Let the circle to be squared be described with the semidiameter BC. Given three straight lines — AE, the base of the quadratrix; AD, the side of the same preceding figure; and the line BC — let a fourth proportional F be found. Then, by the preceding Corollary 3, the line F will be equal to the quadrant [quarter-circumference] of the given circle, and its double will be equal to the semicircumference. Now, between the semidiameter BC and the double of F, let the mean proportional GH be found. I say that the square described on GH is equal to the circle described with the interval BC. [Margin: a — On Isoperimetric Figures, 4] For the rectangle contained by BC, the semidiameter, and by the semicircumference of the circle — that is, by the double of the line F found — is equal to the circle. But the square on the side GH is equal to that rectangle. [Margin: b — Euclid VI.17] Therefore the square on the side GH will also be equal to the circle of semidiameter BC.

Diagram. The figure for Proposition III. At the top stands a separate straight segment with a terminal stroke at its right end, labelled at the left F (the fourth proportional). Below it is a square enclosing a circle. The circle's center is B, with a horizontal radius drawn leftward to C, where the circle meets the left side of the square. The square's top-left corner is marked H and its bottom-left corner G, so that GH is the left side of the square (per the text, GH is the mean proportional between BC and twice F, and the square on it equals the circle).

- F (top, separate segment): the fourth proportional line
- H (top-left corner of square): square vertex
- G (bottom-left corner of square): square vertex
- C (left, on circle): where the horizontal radius meets the circle
- B (center): center of the inscribed circle

But so that a straight line equal to the fourth part of the circumference — or to the whole circumference — may be found expeditiously, a figure of the following kind must be constructed. Make a right angle DAE, with the line AD equal to the diameter of the quadrant from which the quadratrix was described, and AE equal to the base of that same quadratrix. Or, more surely, from the center A let a new quadratrix DE be described, whose side is AD and whose base is AE. For once the line DE has been drawn, a figure most apt for finding a straight line equal to the circumference of the given circle will have been constructed. For if from AE a length AF is cut off equal to the semidiameter of the circle to be squared, and FG is drawn parallel to DE, then by the preceding Corollary 3, AG will be equal to the fourth part of the circumference of the given circle — whose semidiameter is of course AF — just as AD is equal to the fourth part of the circumference of the circle of semidiameter AE, as is clear from the preceding Corollary 2, [Margin: c — Euclid VI.4] because AF and AG have the same ratio as AE and AD. By the same reasoning, if HI, KL, MN are drawn parallel to the same DE, then AI, AL, AN will be equal to the fourth parts of the circumferences of the circles described from the semidiameters AH, AK, AM. And these lines, when doubled, will be equal to the semicircumferences, and so on. By this device a straight line equal to the fourth part of the circumference of any circle will be found, if from the line AE we cut off a line equal to its semidiameter, and from that line's extremity we draw a parallel to DE, and so on.

Diagram. The quadratrix figure of the right angle DAE. The vertical leg rises from the corner A at the bottom to D near the top; along it, from bottom to top, are marked I, G, L, D, and N (the topmost point). The horizontal base runs rightward from A; along it are marked H, F, K, E, M. A family of curved lines (quadratrices)

is drawn from the vertical leg to the base — FG, HI, KL, MN each parallel to the generating line DE. At the right stands a separate vertical straight line, with tick-marks, labelled O (the given line whose circumference is sought in Proposition V).

- A (bottom corner of the right angle): common vertex of the two legs
- D (upper vertical leg): endpoint, side of the quadratrix
- N (top of vertical leg): endpoint of AN
- L, G, I (vertical leg, descending): division points corresponding to AL, AG, AI
- H, F, K, E, M (along the horizontal base): feet of the parallels
- O (separate line at right): the given straight line

So that we may also, without any labor, produce a square equal to any given circle whatsoever, we shall use the following device. Once the side of a square equal to some one circle has been found — as we taught a little earlier — we shall construct a figure most suitable for squaring any other circles, in this way. Let there be given a circle ABC of diameter AC, and let AB be the mean proportional between the semidiameter and the straight line equal to the semicircumference found from the preceding figure, such that the square of the line AB is equal to the circle of diameter AC. [Margin: d — Euclid IV.1] Let AB be fitted into the circle; it will be applied more surely if, in case a compass opened from A to the interval of the given AB and swept round cuts the circumference ABC too obliquely at B, one proceeds thus. For the two lines — namely the diameter AC and the found side AB of the square — let the third proportional AD be found. For the perpendicular DB will fall upon the point B, into which the found side ought to be drawn, [Margin: e — Corollary to Euclid VI.8] because the three lines AC, AB, AD are continuously proportional — just as the line AC, the found side of the square, and AD preserve a continuous proportion, by construction. And it is clear that between AC and AD there can be only one mean proportional. Once this figure has been constructed, we shall square any circle faster than it can be said. For if from the diameter of a given circle we cut off a line equal to AF, about which a semicircle is described, that semicircle will cut off from the line AB the side AE, whose square is equal to the given circle. For since the external angle AEF is equal to the internal angle ABC — [Margin: f — Euclid III.31] because each is a right angle in a semicircle — [Margin: g — Euclid I.28] EF and BC will be parallel; and therefore the triangles AEF, ABC are equiangular. [Margin: h — Euclid VI.4] Therefore CA will be to AB as FA is to AE; and by alternation, CA to FA as

Diagram. The construction for squaring any circle by a semicircle on the diameter AC. From the apex A at the top, a straight vertical line descends past F, H, D to C at the bottom. A diagonal line runs from A down to B at the right,

crossing the family of arcs at E and G. A nest of concentric arcs is drawn from A (each a semicircle on a diameter along AC), the outermost passing through B. The perpendicular from D meets the diagonal at B. A label R stands at the foot of the figure, below C, and is not referenced in the accompanying text.

- A (top): apex; endpoint of the diameters
- C (bottom of vertical line): far end of the diameter AC
- D, H, F (on the vertical line, ascending from C toward A): division / diameter endpoints
- E, G (on the diagonal AB): intersections of AB with the arcs
- B (right): point where the perpendicular DB meets the outer arc / line AB
- R (foot of figure): label, not referenced in the text

AB to AE. [Margin: i — Euclid VI.22] And therefore the square on AC will also be to the square on AF — [Margin: k — Euclid XII.2] that is, as the circle of diameter AC is to the circle of diameter AF — as the square on AB is to the square on AE. But the circle of diameter AC is equal to the square on AB, by construction. [Margin: l — Euclid V.14] Therefore the circle of diameter AF will also be equal to the square on AE. In the same way, too, the square of the line AG will be equal to the circle of diameter AH. And so for the rest.

Now indeed, since in Book 4, chapter 6, proposition 3 we demonstrated, following Archimedes, that the square of the diameter has to the circle very nearly the ratio that 14 has to 11: [Margin: A facile method of finding a square equal to a circle, following Archimedes.] if anyone wishes to find, by this ratio, a square equal to the circle, the line AC must be divided into 14 equal parts, and at the eleventh part D — so that AD contains 11 parts and DC contains 3 — a perpendicular DB must be raised, up to the circumference described about AC. For the line AB, when drawn, will be the side of a square equal to the circle of diameter AC. [Margin: m — Corollary to Euclid VI.8] For since the three lines AC, AB, AD are continuously proportional, [Margin: n — Corollary to Euclid VI.20] the square on AC will be to the square on AB as AC is to AD — that is, as 14 to 11. Since, then, it is also the case, as we said, that the square of the diameter is to the circle nearly as 14 to 11, [Margin: o — Euclid V.11] the square on AC will be to the square on AB as it is to the circle of diameter AC. [Margin: p — Euclid V.9] Therefore the square on AB will be equal to the circle of diameter AC. But if circles passing through A are described on various diameters, those circles will likewise cut off from the line AB the sides of squares equal to them. You have, then, an easy way of finding a square equal to a given circle — whether you employ our quadratrix or follow what Archimedes demonstrated.

IV.

Given a square, to describe a circle equal to it.

Let there be given a square of side AE, to which a circle equal to it is to be described. In the adjacent figure, from the line AB let the line AE be cut off, equal to the given side of the square; and from E let a perpendicular EF be drawn to AB, cutting AC at F. Then the circle of diameter AF will be equal to the square of side AE, as is clear from what was just demonstrated.

COROLLARIUM.

[Margin: a — Euclid II.14]

From what has been demonstrated, we shall construct a circle equal to any rectilinear figure whatsoever. And, conversely, for any circle we shall construct a rectilinear figure equal to it, similar to any other given rectilinear figure. For if to a given rectilinear figure we describe an equal square, [Margin: a] and to this square we construct — by this Proposition 4 — an equal circle, then this same circle will be equal to the given rectilinear figure. [Margin: b — Euclid VI.25]

Again, if by Proposition 3 we construct a square equal to a given circle, and to this square [Margin: b] we construct a rectilinear figure equal to it and similar to another given rectilinear figure, then this same rectilinear figure so constructed will be equal to the given circle. Which was what was proposed.

V.

Given a straight line, to find a circumference of a circle equal to it.

In the second figure of Proposition 3, let a circumference be exhibited equal to the line O. For its fourth part, take on the side AD of the quadratrix a line AI equal to it, and through I draw IH parallel to DE. Then the circumference of the circle described on the diameter AH will be equal to the given line O, because the fourth part of that circumference is equal to the line AI, as has been shown; and consequently the whole circumference will be equal to four times the line AI — that is, equal to the line O, whose fourth part was set as the line AI. We have therefore found a circumference equal to the given line. Which was to be done.

FINIS LIBRI SEPTIMI.

GEOMETRIÆ PRACTICÆ LIBER VIII.

Demonstrating various geometrical theorems and problems.

In order to put the finishing touch to this practical geometry of ours, we shall conclude it with various theorems and geometrical problems — some collected from other geometers, and some thought out and demonstrated by our own effort (by our own *Mars*, as the saying goes). In this we have a distinguished example in Pappus of Alexandria, who wrote eight whole books of *Mathematical Collections*. Nor should anyone judge this to lie outside our plan, since through demonstrations of this kind a manifold path is opened to the diligent reader for investigating similar inquiries in geometrical matters — for in them he has a most ample field for exercising his talent. There is also another reason that moved me to write this eighth book: namely, that so many theorems and problems, investigated with no small labor, should not perish, since they belong to no part of geometry more properly than to this practical geometry — especially because most of them treat of geometrical constructions. Add to this that not a few learned and weighty men were my prompters and advisers in writing this book.

THEOR. 1. PROPOS. 1.

A regular figure circumscribed about a circle has a greater perimeter than the circle.

This is the first proposition of Archimedes in Book 1 of *On the Sphere and Cylinder*, which he demonstrates by assuming this principle: *If two lines in a plane have the same endpoints and are concave toward the same side, the enclosing one is greater than the enclosed.* That this principle is true can be plainly understood from the fact that from it not only Archimedes, but also very many other geometers — both ancient and more recent — have demonstrated almost innumerable and admirable theorems and problems, which, as most true, have been accepted by all; nor has anything absurd ever followed from it, nor has anyone in nearly two thousand years hitherto devised anything against it. This principle, then, being posited, the demonstration of Archimedes is easy. For let there be a regular figure ABCDEF, described about a circle whose center is N, touching it at the points G, H, I, K, L, M. Since, therefore, by the aforementioned principle the lines AG, AM are greater than the arc GM; likewise BG, BH greater than the arc GH, and so for the rest; all the lines together making up the whole perimeter of the figure will be greater than all the arcs together making up the whole circumference of the circle. Which was to be demonstrated.

Diagram. A regular hexagon ABCDEF circumscribed about a circle. The vertices run around: A at the top, B upper-left, C lower-left, D at the bottom, E lower-right, F upper-right. The circle's center is N (printed in a reversed letterform), marked with a central dot. The circle touches the hexagon's sides at the six points of tangency G, H, I, K, L, M — G between A and B, H on the left (between B and C), I lower-left (between C and D), K lower-right (between D and E), L on the right (between E and F), and M upper-right (between F and A).

- A (top), B (upper-left), C (lower-left), D (bottom), E (lower-right), F (upper-right): vertices of the circumscribed hexagon
- N (center, reversed letterform): center of the circle
- G (between A and B): point of tangency
- H (left, between B and C): point of tangency
- I (lower-left, between C and D): point of tangency
- K (lower-right, between D and E): point of tangency
- L (right, between E and F): point of tangency
- M (upper-right, between F and A): point of tangency

Scholium (continued)

Cardanus [Girolamo Cardano], in Book Five of his work on proportions, at proposition 201, attempts to prove that two straight lines touching a circle — such as AG and AM — are together greater than the intercepted arc GM (a result which Archimedes had derived from a principle he himself assumed). Cardano does this by first laying down three lemmas and one principle, of which the first is the following.

Lemma I

If there are four quantities, and the excess between the first and the second is less than the excess between the third and the fourth; and if the first is not less than the third but is greater than the second; and finally if the third is greater than the fourth — then the ratio of the first to the second will be less than the ratio of the third to the fourth.

Let there be four quantities A, BC, D, EF; and let GB, the excess between the first (A) and the second (BC), be less than the excess HE between the third (D) and the fourth (EF). Likewise, let the first (A) be not less than the third (D), but greater than the second (BC); and finally let the third (D) be greater than the fourth (EF). I

say that the ratio of the first (A) to the second (BC) is less than the ratio of the third (D) to the fourth (EF).

Diagram. Two rows of paired line-segments illustrating the four quantities of Lemma I; the pair is printed twice (upper and lower), identically labelled.

- Upper segment (each row): marked **A** above a bracket, with points **G, B, I, C** in order along it — GC represents the quantity A, GB the excess, IC and BC being marked subdivisions.
- Companion segment (each row): marked **D** above a bracket, with points **H, E, F** in order — HF represents the quantity D, HE the excess.

Since A is not less than D, while GB is less than HE, [Margin: a. Euclid, *Elements* V.8] the ratio of A to GB is greater than the ratio of A to HE. But A stands to HE either as D to HE (if A equals D) or, if A is greater than D, then the ratio of A to HE is greater than that of D to HE. Therefore the ratio of A to GB is greater than the ratio of D to HE. If, then, we make A to GI as D to HE, A will also bear a greater ratio to GB than to GI; [Margin: b. Euclid, *Elements* V.10] and consequently GI will be greater than GB, and hence IC less than BC. [Margin: c. Euclid, *Elements* V.8] The ratio of A to IC is therefore greater than the ratio of A to BC. Now since GC (which is equal to A) stands to GI as HF (which is equal to D) stands to HE, then also, by conversion of the ratio, GC — that is, A — stands to IC as HF — that is, D — stands to EF. Since, then, it has been shown that the ratio of A to IC is greater than that of A to BC, the ratio of D to EF will also be greater than that of A to BC. That is, A will have a smaller ratio to BC than D has to EF — which was what was proposed.

Lemma II

If two straight lines touch an arc of a circle, meeting at one point, and if any number of equal straight lines are fitted into the same arc, dividing it into that many equal parts, then those two tangents together will be greater than all these chords together.

Let two straight lines AK and BK touch the arc AB, meeting at K, and let any number of equal straight lines AC, CD, DE, EF, FG, GB be fitted into it, dividing the arc into the same number of equal parts. I say that AK and BK together are greater than all those subtending chords together. For if the lines AC and BG are produced until they meet at H, and likewise CD and GF are produced until they meet at I, and so on in turn if there are more lines, [Margin: a. Euclid, *Elements* I.21] then the lines DI and FI will be greater than the lines DE and FE. Adding

therefore the equal segments DC and FG, the lines CI and GI will also together be greater than CD, DE, EF and FG together. [Margin: b. Euclid, *Elements* I.21] But CH and GH are greater than CI and GI. Therefore CH and GH are much greater than CD, DE, EF and FG; and adding the equal segments AC and BG, AH and BH together will be greater than AC, CD, DE, EF, FG and GB together. [Margin: c. Euclid, *Elements* I.21] But AK and BK are also greater than AH and BH. Therefore AK and BK together are much greater than AC, CD, DE, EF, FG and GB together — which was to be demonstrated.

Diagram. A tangent construction for Lemma II: the arc AB with the two tangents AK, BK meeting at the apex K, and the successive produced chords forming nested vertices down the central axis. A second, flatter version of the same figure is printed just below it.

- K (apex, top): meeting-point of the tangents AK and BK.
- H (below K, on the axis): meeting-point of the produced lines AC and BG.
- I (below H, on the axis): meeting-point of the produced lines CD and GF.
- A (lower left) and B (lower right): endpoints of the arc.
- C, D, E, F, G (along the arc between A and B): division-points; the chords are AC, CD, DE, EF, FG, GB.

Lemma III

If three straight lines touch an arc of a circle, meeting at two points, in such a way that the middle contact-point divides the arc in half; and if into the same arc are fitted any number of straight lines, even in number and equal to one another — then those three tangents will be greater than all of these together.

In the preceding figure, let three straight lines AC, CD, DB touch the arc AB, meeting at the two points C and D, and cutting it in half at E. Into the same arc let there be fitted any number of equal straight lines, even in number: AF, FE, EG, GB. I say that the three lines AC, CD, DB taken together are greater than AF, FE, EG, GB taken together. For, by the preceding Lemma, AC and CE together are greater than AF and FE; likewise BD and DE together are greater than BG and GE; therefore AC, CD, DB together will also be greater than AF, FE, EG, GB together — which was to be shown.

These, then, are the three lemmas that Cardano sets out in advance, to which he joins the following postulate, or principle. [Margin: Cardano's principle.] Since any arc is greater than any number of chords subtended within it and taken together, and since the more subtending chords there are, the smaller the excess

by which the arc surpasses them — it can happen that so many subtending chords can be drawn that the excess by which the arc surpasses them is less than any proposed straight line. This principle seems self-evident, since so small an arc can be taken that its chord is almost equal to it; so much so that the senses can perceive no difference between the arc and the chord. Nevertheless it can also be confirmed *a posteriori* by numbers. [Margin: a. Euclid, *Elements* X.1] For, taking the ratio of circumference to diameter as about 31415926 to 10000000 — as we reported from approved authorities in Book 4, Chapter 7, Number 5 — we shall find that, if the proposed arc is continually bisected by subtending chords, more than half is always taken away from the preceding excess; [Margin: a] and consequently that in the end an excess is left smaller than any given magnitude. For example, an arc of 4 degrees will be 698131, and its chord, drawn from the table of sines, is 697990, so that the arc exceeds the chord by this number, 141. Then the sum of two chords of 2 degrees will be 698096, which is surpassed by the same 4-degree arc by this number, 35 — which is less than half of the preceding excess, 141 — and so more than half will have been removed from it. Again, the sum of four chords of 1 degree will be 698120, which the same 4-degree arc exceeds by this number, 11 — again less than half of the nearest excess, 35. Likewise the sum of 8 chords, each corresponding to 30 minutes, will be 698128; and the excess between it and the same 4-degree arc is the number 3 — which is again less than half of the nearest excess, 11 — and so on in turn.

I am aware that this confirmation of the proposed principle is not demonstrative, since the ratio of circumference to diameter is itself gathered from the very thing we are trying to prove — namely, that a figure circumscribed about a circle has a greater perimeter than the circle's own. Yet no one will doubt that the principle is probable, since it would scarcely seem credible (if that ratio were far from the truth) that those excesses should diminish gradually in this way, so that a number is always left smaller than half the preceding excess, until at last almost no difference is found between the arc and the sum of the subtending chords.

With these things established, let two straight lines AB, AL touch the arc BCL. I say that they are greater than the arc. [Margin: Cardano's demonstration.] For suppose, if it can be, that they are not greater; so that the arc BCL is either equal to the straight lines AB, AL, or greater than them. Bisect the arc, then, at C, and draw DCE tangent to the arc at C. Bisecting also the arcs BC and CL at G and F, join the straight lines BG, GC, CF, FL. [Margin: b. Euclid, *Elements* I.20] Now since AD, AE are greater than DE, add the common equal segments DL, EB (they are equal — for joining the straight lines NA, NB, NL from the centre N, since the three sides of triangle ABN are equal to the three sides of triangle ALN, [Margin: c. Euclid,

Elements I.8] the angles at N will be equal, as will those at A; [Margin: d. Euclid, *Elements* III.26] and hence the arcs CB, CL will be equal; and consequently the straight line NA will pass through the contact-point C, [Margin: e. Euclid, *Elements* III.18] and will be perpendicular to DE. Since, then, the two angles DAC, DCA are equal to the two angles EAC, ECA, and the adjacent side AC is common, [Margin: f. Euclid, *Elements* I.26] the sides AD, AE will be equal; and therefore the remaining segments DL, EB will also be equal, since the tangents AL, AB are equal). Thus AL, AB will be greater than the three segments LD, DE, EB. Let the excess, then, be H. Again, since the arc BL is greater than the straight lines BG, GC, CF, FL, let the excess be I, less than the excess H. For if it is not less, we shall bisect the arcs LF, FC, CG, GB, and bisect these again, and so on, and connect the straight lines, until the excess becomes less than the excess H — by Cardano's principle stated above.

Since, therefore, the arc LB, the first quantity, surpasses the second — namely the straight lines LF, FC, CG, GB together — by the excess I; and the third quantity, namely the sum of the straight lines AL, AB, surpasses the fourth, that is, the sum of the straight lines LD, DE, EB, by the excess H; and the excess I is less than the excess H; and the first quantity, that is, the arc BL, is posited as not less than the third (formed from AB, AL), while the third (AB, AL) is greater than the fourth (LD, DE, EB): then, by Lemma 1, the ratio of the arc BL (the first quantity) to the second (LF, FC, CG, GB) will be less than the ratio of the third quantity (AL, AB) to the fourth (LD, DE, EB). [Margin: g. Scholium to Euclid, *Elements* V.27] And, by alternation, the ratio of the arc LB to AL, AB together will be less than the ratio of the straight lines LF, FC, CG, GB together to the straight lines LD, DE, EB together. Let it be, then, that as the sum of LF, FC, CG, GB is to the sum of LD, DE, EB, so is the arc BK to AL, AB together. Therefore the ratio of the arc BL to AL, AB together will also be less than the ratio of the arc BL to the arc BK; [Margin: h. Euclid, *Elements* V.10] and hence the arc BK will be greater than the arc BL. Since, then, the ratio of the straight lines LF, FC, CG, GB together to LD, DE, EB together is the same as that of the arc BK to AL, AB together, and since, by Lemma 3, the straight lines LF, FC, CG, GB together are less than LD, DE, EB together, the arc BK will also be less than AL, AB together. The arc BL, therefore, will be much less than AL and AB together. Hence the tangent lines AL, AB together are greater than the arc BL — which was to be shown.

Diagram. Cardano's construction for the demonstration: a triangle-like figure with apex A over the arc BCL, a central vertical (dotted) axis through the tangent-point C down to the centre, and the bisecting tangents and chords.

- A (apex, top): external point from which the two tangents AL, AB are drawn.
- C (below A, on the central axis): mid-point of the arc, where DCE touches.
- D (left) and E (right), flanking C: endpoints of the tangent DCE drawn at C.
- L (far left), F, G, B (right): points along the middle row — F and G bisect the arcs, L and B being the arc's endpoints.
- K (lower left), H (bottom left), I (bottom right): auxiliary points in the lower construction; a dotted arc runs along the base.
- N (bottom centre) [?]: the centre of the circle (the central glyph is overprinted).

Cardano's demonstration is admirable, and not unlike the one Euclid uses in proposition 12 of Book 9 [*Elements* IX.12]. For in both cases an affirmative conclusion is inferred from a demonstration by way of its opposite, as is plain.

I have brought forward this demonstration of Cardano's not because it is truly geometrical — unless that principle of his is granted — but because it is ingenious and acute. Yet even without this demonstration it must be conceded that the perimeter of a circumscribed figure is greater than the circumference of the circle, on account of the demonstration of Archimedes, since no one has ever brought forward anything to the contrary, as we said above.

Theorem 2. Proposition 2

The diameters of circles are to one another as their circumferences.

We demonstrated this in Book 4, Chapter 7, Number 3, Proposition 1; but here we shall demonstrate the same thing in another way, following Pappus, as follows.

Let there be two circles ABCD, EFGH, whose diameters are AC, EG. I say that circumference is to circumference as diameter is to diameter. [Margin: a. Euclid, *Elements* XII.2] For since circle is to circle as the square of the dia-

—meter to the square of the diameter, [Margin: b. Euclid, *Elements* V.15] and since circle ABCD is to circle EFGH as the quadruple of the one circle to the quadruple of the other, it follows that the quadruple of circle ABCD is to the quadruple of circle EFGH as the square of diameter AC to the square of diameter EG. But the rectangle contained by the diameter AC and a straight line equal to the circumference ABCD is quadruple the circle ABCD; and the rectangle contained by the diameter EG and the circumference EFGH is quadruple the circle EFGH — by the corollary to Proposition 2, Chapter 5, Number 1, Book 5. Therefore the rectangle contained by the diameter AC and the circumference

ABCD is to the rectangle contained by the diameter EG and the circumference EFGH as the square on AC is to the square on EG. And by alternation, the rectangle contained by the diameter AC and the circumference ABCD is to the square on AC as the rectangle contained by the diameter EG and the circumference EFGH is to the square on EG. [Margin: c. Euclid, *Elements* VI.1] But the rectangle contained by AC and a straight line equal to the circumference ABCD is to the square on AC as the straight line equal to the circumference is to AC, because the rectangle and the square have the same altitude, AC. In the same way, the rectangle contained by EG and a straight line equal to the circumference EFGH is to the square on EG as the straight line equal to the circumference is to EG. Therefore, as the circumference ABCD is to the diameter AC, so is the circumference EFGH to the diameter EG; and by alternation, circumference is to circumference as diameter is to diameter — which was to be demonstrated.

Diagram. Two circles side by side: a larger circle ABCD on the left with centre P, and a smaller circle EFGH on the right with centre Q, each with two perpendicular diameters and a chord near the top.

- Larger circle: A (left) and C (right) are the ends of the horizontal diameter AC; B (top) and D (bottom) the ends of the vertical diameter BD; P (centre). Near the top, a chord IK has endpoints I (upper left) and K (upper right), with N its mid-point.
- Smaller circle: E (left) and G (right) [?] are the ends of the horizontal diameter EG; F (top) and H (bottom) the ends of the vertical diameter FH; Q (centre) [?]. Near the top, a chord LM has endpoints L (upper left) and M (upper right), with O its mid-point.

Scholium

There are those who think that Pappus demonstrated this theorem to no purpose, since it seems self-evident that the circumference of any circle stands to its own diameter as the circumference of another circle stands to its own diameter — and therefore, by alternation, that circumference is to circumference as diameter is to diameter. In this they are remarkably deceived. For since it is demonstrated by Ptolemy [Claudius Ptolemy] (as we too have shown at Proposition 10 of the *Sines*) that the ratio of a greater arc to a smaller one of the same circle is greater than the ratio of chord to chord — which is also true of arcs and chords in unequal circles, unless those arcs are similar, as we shall show in the following Theorem — who would grant, without demonstration, that the ratio of circumference to circumference is the same as that of diameter to diameter? For if it had been demonstrated that the arc of any circle stands to a similar arc of another circle as

chord stands to chord, then at last it would be established that circumference is to circumference, [Margin: a. Euclid, *Elements* V.15] and consequently semicircumference to semicircumference, as diameter is to diameter — because the arcs of semicircles are similar, whose chords are the diameters. But this cannot be demonstrated unless it is first demonstrated that circumference is to circumference as diameter is to diameter, as will be established in the following Theorem. Rightly, therefore, and not without cause, was the preceding theorem demonstrated by Pappus.

Theorem 3. Proposition 3

The arc of any circle has the same ratio to a similar arc of another circle as chord has to chord. And conversely, arcs having the same ratio as their chords — when quadrant is to quadrant as half-arc to half-arc — are similar.

In the figure of the preceding proposition, let there be drawn the perpendiculars PB, QF to the diameters, from the centres P, Q, dividing the semicircles into two quadrants each. Let the arcs BI, BK be equal, and let FL, FM be taken similar to them, so that the whole arcs IK, LM are similar; and let their chords IK, LM be bisected by the radii at N, O. I say that the ratio of the arc IBK to the arc LFM is the same as that of the chord IK to the chord LM, and so on. [Margin: a. Proposition 2 of this book] For since the circumference ABD is to the circumference EFH — [Margin: b. Euclid, *Elements* V.15] that is, the quadrant AB to the quadrant EF — as the diameter AC is to the diameter EG, that is, as the semidiameter PB to the semidiameter QF; [Margin: c. Euclid, *Elements* V.15] and since, moreover, as the quadrant AB is to the quadrant EF, so is the arc IB to the arc LF (since they are posited as similar): it follows that the arc IB is to the arc LF as the semidiameter PB to the semidiameter QF. Since, then, by the Lemma of Proposition 1, Book 1 of our *Gnomonics*, or by Lemma 5, Book 1 of our *Astrolabe*, the whole sine PB stands to the whole sine QF just as the sine IN stands to the sine LO, the arc IB will also stand to the arc LF as the sine IN to the sine LO; [Margin: d. Euclid, *Elements* V.15] that is, the double arc IK to the double arc LM as the chord IK — which is double IN — to the chord LM — which is double LO. This is what was proposed.

But now let the arc IK be to the arc LM as the chord IK to the chord LM. I say that the arcs are similar. For, with the same construction made, [Margin: e. Euclid, *Elements* V.15] the arc IB will also be to the arc LF (half to half) as IN to LO (half to half); [Margin: f. Scholium to Euclid, *Elements* VI.33] but as IB is to LF, so is the quadrant AB to the quadrant EF, by hypothesis; [Margin: g. Proposition 2 of this book] and as quadrant is to quadrant, so is the semidiameter PB to the

semidiameter QF. Therefore the sine IN will also be to the sine LO as the whole sine PB to the whole sine QF. And accordingly, by the Lemma of Proposition 1, Book 1 of our *Gnomonics*, or by Lemma 5, Book 1 of our *Astrolabe*, the arcs IB, LF will be similar; and consequently their doubles IBK, LFM will also be similar — which was to be demonstrated.

COROLLARIVM.

It follows from this that if the arcs IBK and LFM are not similar, they do not stand in the same ratio to one another as their chords do. For if they did stand in the same ratio, the arcs themselves would be similar — as was shown in the second part of this proposition — which is absurd, since they are assumed not to be similar.

PROBL. 1. PROPOS. 4.

To construct a parallelogram equal to a given quadrilateral, in a given angle, more easily than by Proposition 45 of Book 1 of Euclid.

Diagram. A quadrilateral inscribed in a rectangular framework, with a diagonal and its parallels drawn. To the upper left, set apart, is the given angle.

- K (upper left, set apart): the given angle
- A (top row, left): vertex of the quadrilateral
- F (top row, centre): intersection on the top edge
- H (top row, right): corner of the framework
- B (left, middle): vertex of the quadrilateral
- E (centre, interior): midpoint of the diagonal BD
- D (right, middle): vertex of the quadrilateral
- C (bottom row, left): vertex of the quadrilateral
- G (bottom row, centre): intersection on the bottom edge
- I (bottom row, right): corner of the framework

Let ABCD be any quadrilateral, and K the given angle. Draw the diagonal BD and bisect it at E. Through E draw the straight line FG, making at E an angle FED equal to the given angle K. Next, through D draw HI parallel to FG, and through A and C draw the two lines AH and CI parallel to BD, cutting FG and HI at F, H, G, and I. There will then be constructed a parallelogram FI in the given angle G — which [Margin: Euclid I.29] is equal to the angle FED, as interior to exterior, that is, to the angle K. I say that this same parallelogram is equal to the given quadrilateral ABCD. [Margin: scholium to Euclid I.41] For since the parallelogram FD is equal to

the triangle ABD, and the parallelogram EI is equal to the triangle CBD, the whole parallelogram FI will be equal to the whole quadrilateral ABCD — which is what was proposed.

PROBL. 2. PROPOS. 5.

To construct, upon a given straight line, a rectangle equal to a given rectangle, more easily than by Proposition 45 of Book 1 of Euclid.

Diagram. Two figures side by side, each a rectangle divided by internal lines with a diagonal. The left figure is roughly square; the right figure is tall and narrow.

Left figure:

- F (top, left), G (top, centre), H (top, right): top edge
- D (left, middle), C (centre, interior), I (right, middle): middle line
- A (bottom, left), B (bottom, centre), E (bottom, right): bottom edge

Right figure:

- F (top, left), G (top, centre), H (top, right): top edge
- D (left, middle), C (centre, interior), E (right, middle): middle line
- A (bottom, left), B (bottom, centre), E (bottom, right): bottom edge

Let ABCD be the rectangle, upon a given straight line onto which a rectangle equal to it is to be constructed. Extend any side you like — say AB — and on it take BE equal to the given straight line, whether that line be greater than the side AB or smaller. Then from E, through C, draw a straight line cutting AD extended at F, and complete the rectangle AH; and let the sides BC and DC be produced as far as G and I. [Margin: Euclid I.43] The complement CH will then be equal to the complement CA. And since the side CI is equal to BE, that is, to the given straight line, what was proposed will have been accomplished.

ALITER.

Given the straight line BE and the two sides AB and BC of the given rectangle, [Margin: Euclid VI.17 [?]] let a fourth proportional CG be found. [Margin: Euclid VI.10 [?]] For the rectangle contained by BE (the first term) and CG (the fourth term) — namely the rectangle CH — will be equal to the given rectangle AC, which is contained by AB (the second term) and BC (the third term). And this is what was proposed.

PROBL. 3. PROPOS. 6.

To construct a rectangle equal to a given rectilinear figure, more easily than by Proposition 45 of Book 1 of Euclid.

This is readily dispatched by the two preceding propositions. For if the rectilinear figure is resolved into triangles, then any two triangles having a common side will make up a trapezium whose common side is a diagonal. [Margin: Prop. 4 of this book] Therefore make a single rectangle equal to each single trapezium — and also to the last triangle, if the number of triangles happens to be odd. Next, as was said in the preceding proposition, [Margin: Euclid VI.12] let a fourth proportional be found to one side of the first rectangle and to the two sides of the second; [Margin: Euclid VI.16] then the rectangle contained by the assumed side in the first rectangle and by that fourth proportional will be equal to the second rectangle. Accordingly, if the other side adjacent to the assumed one in the first rectangle is produced, and from the extension a segment equal to the fourth proportional is cut off, and the whole rectangle is completed, one obtains a rectangle composed of two parts equal to the first two trapezia. And if to the same side, together with the two sides of the third rectangle, a fourth proportional is again found, and a segment equal to this fourth proportional is taken on the side extended earlier, there will be produced in the same way a rectangle formed of three parts equal to the three trapezia, and so on. [Margin: scholium to Euclid I.41] Finally, for the last triangle, if there is one, a rectangle will be constructed equal to it upon half the base, at the same height as the triangle.

THEOR. 4. PROPOS. 7.

If, from two points, two straight lines are drawn to a single point of any straight line that is the common section of a plane drawn through those two points with some other plane, and these two lines make equal angles with that straight line, then these two lines will be shorter than any other pair of straight lines drawn from the same two points to any other point of the same straight line.

Diagram. Two points above a horizontal line, with straight lines drawn down to points on that line and to a point below it, forming a crossing pattern.

- A (upper left): first point
- B (upper right): second point
- F (on the horizontal line, left): the point to the left of C
- C (on the horizontal line, centre): the point of equal angles [overprinted at the crossing — hard to resolve]

- D (on the horizontal line, right): the point to the right of C
- E (below the line, centre): point where AC is produced, with $CE = CB$

From the two points A and B, to C — a point on the line CD, so situated that the plane drawn through CD passes, when revolved, through A and B — let two straight lines AC and BC be drawn, making the angles ACF and BCD equal. And from the same points A and B let two other straight lines AD and BD first be drawn to another point D, to the right of C. I say that AC and BC are shorter than AD and BD. For let AC be produced toward C so that CE is made equal to CB, and join DE. Since the angle ACF is assumed equal to the angle BCD, [Margin: Euclid I.15] and the angle ACF is equal to the angle ECD as vertical angles, the angle BCD will also be equal to the angle ECD. Since, then, the two sides BC and CD are equal to the two sides EC and CD, [Margin: Euclid I.4] the base DB will be equal to the base DE; and consequently AD together with DB will be equal to AD together with DE. [Margin: Euclid I.20] But AD and DE are greater than AE — that is, than AC and CB, since CB and CE are set equal. Therefore AD and BD will also be greater than AC and BC. Which is what was proposed.

Next let two other straight lines AF and BF be drawn from the points A and B to another point F, to the left of C. Again I say that AC and BC are shorter than AF and BF. For let AC once more be produced, and CE taken equal to CB, and join EF. Since the angles ACF and BCD are assumed equal, [Margin: Euclid I.15] and ACF is equal to the angle ECD as vertical angles, the angles BCD and ECD will also be equal; and consequently the remaining angles BCF and ECF, taken from two right angles, will be equal as well. Since, then, the two sides BC and CF are equal to the two sides EC and CF, [Margin: Euclid I.4] the base BF will also be equal to the base EF. [Margin: Euclid I.20] But AF and FE are greater than AE — that is, than AC and CB, since BC and CE are set equal. Therefore AF and BF will also be greater than AC and BC. Which is what was proposed.

SCHOLIVM.

Since Nature, when unimpeded, acts along the shortest lines, it comes about that a ray of the Sun — or a ray of sight — falling from A onto the polished plane DF so as to be reflected to point B, necessarily strikes at point C, where the angle ACF (which the writers on perspective [Perspectivi] call the angle of incidence [angulus incidentiae]) is made equal to the angle BCD, which they call the angle of reflection [angulus reflexionis]. For if the ray were to fall at D or F and be reflected to B, Nature would not be acting along the shortest lines — whereas in fact AD and BD, as well as AF and BF, are longer than AC and BC, as we have demonstrated — which is absurd. And thus what the writers on perspective

assume has been demonstrated: namely, that the angle of incidence is equal to the angle of reflection.

PROBL. 4. PROPOS. 8.

If someone has conceived a number in his mind, to guess how many units remain to it after three prescribed operations.

Have the conceived number multiplied by any number you please — by 2, or 3, or 4, or 10, and so on — a number counted by the multiplier. Then have him take away, from an aliquot part denominated [by the multiplier], a like aliquot part of the number produced by the multiplier acting on the conceived number — that is, take away the conceived number itself. For in this way the remaining number will be a like aliquot part of the number you had him add. Since, then, the added number is known to you and has an aliquot part denominated by the multiplier — and therefore known to you — you will declare that the remaining units make up that aliquot part.

For example: let someone conceive the number 4. Have it multiplied by 6, giving 24. Have him add the number 30, counted by the multiplier 6, giving 54. From the sixth part of this sum, denominated by the multiplier 6 — that is, from 9 — have him subtract likewise the sixth part of the earlier product 24, namely 4, which is the conceived number itself. For in this way there remains the sixth part of the added number 30, namely 5. This is demonstrated as follows.

A	B	C	F
4	24	54	30
D	E	G	
4	9	5	

Let A be the conceived number, which, multiplied (say) by 6, generates B; and with F added, let the sum become C. From E, an aliquot part of the sum C denominated by 6, let there be subtracted D, an aliquot part of the earlier product B, likewise denominated by 6 — that is, the conceived number A itself — and let the remainder be the number G, which I say is an aliquot part of the added number F, likewise denominated by 6. For since the whole C is the same multiple of the whole E as the part B taken from C is of the part D taken from E — inasmuch as E is set to be the same part of C that D is of B, namely the sixth part — [Margin: scholium to Euclid VII.7] therefore the remainder FC (once the

product B has been taken from the sum C) will be the same multiple of the remainder G (once D has been taken from E) as the whole C is of the whole E. Which was to be demonstrated.

It is also a pleasant thing that this same trick can be guessed among several people. For if several people each conceive numbers in their minds — each one privately, no one else aware which number each has conceived — and you have each multiply his number by any number chosen by you, whatever it may be; and finally, from an aliquot part of the sum whose denominator is the number you chose (that is, from the conceived numbers themselves) have each subtract a like part from his own product: the remaining number of each will be a like aliquot part of the number added.

And if you prefer to use different numbers, tell them to have the second double his remainder and the third triple his, and so on. For in this way you will guess that the first man's remainder is that aliquot part of the number added, that the second has double that part, the third triple, and so on. Here you see that they can multiply that remainder by whatever numbers you please — provided only that you remember, in guessing the numbers, by which numbers the multiplications were made.

PROBL. 5. PROPOS. 9.

To divide a given square number into however many square numbers you please.

However impossible this problem may seem — for who, someone will say, can bring it about that any square number be divided into as many numbers as you like, all of which are squares? — its solution is nevertheless not difficult. Let the given square number, then, be 36, to be divided into 5 square numbers. By what we wrote on Proposition 47 of Book 1 of Euclid, let three numbers be found such that the square of the greatest is equal to the squares of the others — namely 5, 4, 3. Then say: if 5 gives 4, what will 6 give (6 being the root of the given square)? And likewise: if 5 gives 3, what will 6 give? You will find $24/5$ and $18/5$, that is, $4\frac{4}{5}$ and $3\frac{3}{5}$, the roots of two squares [whose sum equals 36]. For since the root 6 stands to the two numbers found as the sides 6, $4\frac{4}{5}$, $3\frac{3}{5}$ of a right triangle similar to the [first] triangle constructed, [Margin: Euclid I.47] the squares on $4\frac{4}{5}$ and $3\frac{3}{5}$ will be equal to the square on the root 6, the given number.

[The construction is then repeated on the smaller root, to yield further squares.] Taking the smaller root found — so as not to coincide with one already found — two other numbers are found whose squares are equal to the square [of that

root]; and so now (having set aside the root $3\frac{1}{5}$) there are found three roots: $4\frac{4}{5}$, $2\frac{22}{25}$, and $2\frac{4}{25}$ [?], whose squares are equal to the proposed square 36. In the same way, if it be made that as 5 is to 4 and to 3, so is $2\frac{4}{25}$ [?] to another, two other roots will be found: 1 [?] and 1 [?]. Hence (setting aside the root $2\frac{4}{25}$ [?], in whose place we found two) there will now be four roots: $4\frac{4}{5}$, $2\frac{22}{25}$, 1 [?], and 1 [?], whose square numbers are equal to the square 36. And again, if it be made that as 5 is to 4 and to 3, so is the smaller root found (1 [?]) to another, two other roots will be found: 1 [?] and [?]. Therefore (setting aside the root 1 [?], for which we found two nearest ones) there will now be five roots: $4\frac{4}{5}$, $2\frac{22}{25}$, 1 [?], 1 [?], and $486/625$, whose square numbers — 23 [?], $8\frac{184}{625}$, 2 [?], 1 [?], and [?] — make up the given square 36. And in this way further squares can be found equal to the number 36: namely if it be made that as 5 is to 4 and to 3, so is the last root found — $486/625$, which is the smallest — to another, and so on.

THEOR. 5. PROPOS. 10.

Given two unequal fractions [minutiae], the fraction whose numerator is formed from their numerators and whose denominator from their denominators is greater than the smaller of them, yet smaller than the greater.

Let there be two unequal fractions: the greater $\frac{3}{5}$ and the smaller $\frac{4}{7}$. Join both the numerators and the denominators, so that the fraction $\frac{7}{12}$ results. I say that this is greater than $\frac{4}{7}$ and smaller than $\frac{3}{5}$. For since the fraction $\frac{3}{5}$ is greater than $\frac{4}{7}$, [Margin: Euclid V.27] by Proposition 8 of Book 9 of Euclid's *Minutiae* the ratio 3 to 5 is greater than 4 to 7. Alternando, then, 3 to 4 is greater than 5 to 7. [Margin: Euclid V.28] Therefore, componendo, 3 and 4 together — that is, 7 — to 4 is greater than 5 and 7 together, that is, than 12, to 7. [Margin: Euclid V.27] And alternando, 7 to 12 is greater than 4 to 7. Accordingly, by Proposition 8 of Book 9 of Euclid's *Minutiae*, the fraction $\frac{7}{12}$ will be greater than $\frac{4}{7}$ — which is the first point.

Next, since $\frac{4}{7}$ is smaller than $\frac{3}{5}$, by Proposition 8 of Book 9 of Euclid's *Minutiae* the ratio 4 to 7 is smaller than 3 to 5. [Margin: Euclid V.27] And alternando, 4 to 3 is smaller than 7 to 5. [Margin: Euclid V.28] Therefore, componendo, 4 and 3 together — that is, 7 — to 3 is smaller than 7 and 5 together, that is, than 12, to 5. [Margin: Euclid V.27] And alternando, 7 to 12 is smaller than 3 to 5. Accordingly, by Proposition 8 of Book 9 of Euclid's *Minutiae*, the fraction $\frac{7}{12}$ will be smaller than $\frac{3}{5}$ — which is the second point.

[The three fractions are set out in the margin: $\frac{3}{5} \cdot \frac{4}{7} \cdot \frac{7}{12}$.]

THEOR. 6. PROPOS. 11.

If two numbers prime to one another are not both squares (or both cubes), then no equimultiples of them whatever will be squares (or cubes). And if any equimultiples of them are both squares (or both cubes), then the numbers themselves will also be squares (or cubes).

A	B
4	11
C	D
12	33

For let A and B be numbers prime to one another and not both squares (or cubes) — although one of them may be a square or a cube — and let C and D be equimultiples of them. I say that these too are not both squares or both cubes. For, if it be possible, let them both be squares (or cubes). Since the same number multiplying A and B produced C and D — inasmuch as these are equimultiples of those — [Margin: Euclid VII.17] A will be to B as C is to D. [Margin: Euclid VIII.11 & 12] But between C and D there falls either one mean proportional or two. [Margin: Euclid VIII.8] Therefore between A and B likewise there will fall either one mean proportional or two.

Therefore, since the extremes A and B are assumed to be prime to one another, [Margin: d — *Elements* VIII.1] all three, or four, of the proportionals will be the least in their proportion; [Margin: e — coroll. 2 of *Elements* VIII] and consequently A and B will both be squares, or both cubes — which is contrary to the hypothesis. Therefore C and D are not both squares, or both cubes. Which was to be shown.

But now let C and D be equimultiples of A and B (which are prime to one another), and both squares, or both cubes. I say that A and B too are both squares, or both cubes. For if they are not, then C and D themselves will not both be squares, or both cubes, as has been demonstrated — which conflicts with the hypothesis.

COROLLARY.

From this it follows that, if both the numerator and the denominator of a fraction [minutia] are square (or cube), then the numerator and denominator of that

same fraction reduced to its lowest terms are likewise square (or cube). For the lowest terms are numbers prime to one another, and they have the same ratio as the numerator and denominator of the original fraction, since the fractions are equal. Likewise, if both the numerator and the denominator of a fraction in its lowest terms are not square (or cube), then neither number of any other equivalent fraction is square (or cube).

THEOR. 7. PROPOS. 12.

In every rectilinear quadrilateral figure, any three sides taken together are greater than the remaining side.

Diagram. A quadrilateral ABCD with the diagonal BD drawn. A is at the upper left, B at the top (apex), C at the lower right, D at the lower left; the diagonal runs from B down to D, dividing the figure into triangles ABD and BCD.

- A (upper left): vertex
- B (top): vertex
- C (lower right): vertex
- D (lower left): vertex
- BD: diagonal joining B and D

Let ABCD be a quadrilateral. I say that any three sides — namely DA, AB, BC — taken together are greater than the remaining side DC. For, drawing the diagonal BD, [Margin: a — *Elements* I.20] the two lines BD and BC will be greater than DC; but by the same reasoning AB and AD are greater than BD. Therefore the three sides AD, AB, BC will be greater than the two BD, BC, and consequently much greater than DC. The same will be demonstrated in a similar way for any other three sides, as is evident. Therefore, in every rectilinear quadrilateral figure, any three sides taken together are greater than the remaining side. Which was to be demonstrated.

PROBL. 6. PROPOS. 13.

Given three points through which a circle is to be described, to find other points through which the same circle must pass.

Sometimes three given points lie so little apart from one another, or nearly on a straight line, that their center cannot easily be found — because the straight lines that bisect the lines joining those points at right angles intersect one another at the center too obliquely. So, to find the center more precisely, two or more other

points must be sought through which the same circle must pass, in this way. Let three points A, B, C be given.

Diagram. A large circle with several inscribed points and construction arcs. Around the circumference, reading clockwise from the top: G (top), F (upper right), E (right), D (lower right), C (bottom), B (left), A (upper left). The point H marks the center, where two straight construction lines cross. Numerous compass arcs (with their intersection ticks) are drawn between neighbouring points to locate the additional points D, E, F, G by constructing successive congruent triangles on the chords, and to bisect the chords for the center H.

- A (upper left, on circle): given point
- B (left, on circle): given point
- C (bottom, on circle): given point
- D (lower right, on circle): constructed point
- E (right, on circle): constructed point
- F (upper right, on circle): constructed point
- G (top, on circle): constructed point
- H (center): center of the circle

Joining the straight lines AB, AC, BC, on the base BC construct a triangle BCD equal-sided (congruent) to triangle ABC, so that the angle D inclines toward the side toward which the circumference to be described must pass; and let the equal sides not proceed from the same point — that is, let side CD be equal to side BA, and side BD to side CA. This will be done if from C an arc is drawn at the distance BA, which another arc drawn from B at the distance CA cuts in D. [Margin: a — *Elements* I.8] For the angle D will be equal to the angle A; [Margin: b — scholium on *Elements* III.21] and consequently the circle described through the three points A, B, C will also pass through the fourth point D. By the same reasoning, if on the base CD a triangle CDE is constructed, equal-sided to triangle BCD in the aforesaid order — so that side DE is equal to side CB, and side CE to side BD — another point E will be found through which the circumference passes. And by the same technique another point F will be found, by means of triangle DEF, equal-sided to triangle EDC, and so on. In the same way, on the other side, another point G will be found by means of triangle ABG, equal-sided to triangle ABC, and so on.

Therefore, if three points are chosen such that the straight lines joining them make nearly a right angle — such as the three points A, C, D — and from the nearest pair A, C two arcs are described at any one and the same distance, and from the nearest pair C, D two others; and if straight lines are drawn through the

intersections of these arcs, they will cut one another at the center H, and so on. Very suitable too would be the three points G, B, D, although the angle DBG is acute. Likewise the three points G, B, E, and C, E, F, and so on.

PROBL. 7. PROPOS. 14.

Given the excess of the diagonal of a Square over its side; likewise given the excess of the diagonal of a Rhombus over its side, or of the side over the diagonal (when the latter is greater), together with one angle of the Rhombus; given moreover the excess of the diagonal of a Rectangle over either of its unequal sides, together with the angle that the diagonal makes with that side, or together with the ratio of the same unequal sides; and finally given the excess of the diagonal of a Rhomboid over either of its unequal sides, or of either of its unequal sides over the diagonal (when the latter is greater), together with one angle of the Rhomboid, and moreover with the angle that the diagonal makes with that side, or moreover with the ratio of the two unequal sides — to construct the Square itself, the Rhombus, the Rectangle, and the Rhomboid.

This problem, so far as the square is concerned, we solved in another way at the end of Euclid's Book II. Let A be the given excess of the diagonal of some square over its side. Construct any square BCDE whose diagonal BD exceeds the side by the excess DF. If this is equal to the given excess A, the required construction is done. But if it is unequal, [Margin: a — *Elements* VI.12] make it that as DF is to the given excess A, so is the diagonal BD to BG; and complete the square HI, which I say is the one sought.

Diagram. Two squares set side by side, illustrating the square-construction. Both are crowded, with interior construction-letters overprinted on the diagonal and gnomon lines *Left square:* corners E (top left), D (top right), B (bottom left), C (bottom right); on the left edge, I (upper) and A (lower); on the bottom edge H (between B and C); interior points F (upper middle), K (mid-left), G (center), with the diagonal drawn from B to D. *Right square:* corners I (top left), G (top right), B (bottom left), H (bottom right); on the left edge E and A; on the bottom edge C; interior points K, D, F on the diagonal drawn from B to G.

- Left square — E, D, B, C (corners); I, A (left edge); H (bottom edge); F, K, G (interior); diagonal B–D
- Right square — I, G, B, H (corners); E, A (left edge); C (bottom edge); K, D, F (interior); diagonal B–G

For, taking a straight line GK equal to A, since by construction as the whole BD is to the whole BG, so is the subtracted DF to A — that is, to the subtracted GK — [Margin: b — *Elements* V.19] it will also be that as the whole BD is to the whole BG, so is the remainder BF to the remainder BK. And, alternando, as BD is to BF, so is BG to BK. [Margin: c — *Elements* V.7] But as BD is to BF, so is BD to BC (since BF and BC are equal, DF being taken as the excess of the diagonal BD over the side BC); [Margin: d — *Elements* VI.4] and as BD is to BC, so is BG to BH. Therefore it will also be that as BG is to BK, so is BG to BH; [Margin: e — *Elements* V.9] and consequently BK and BH will be equal. So the diagonal BG exceeds the side BH — that is, BK — by the straight line GK, which is equal to the given excess A. Which is what was proposed.

Next let A be the given excess of the diagonal over the side in some rhombus, together with the angle L. Construct any rhombus BCDE having the angle C equal to the given angle L, as in the first rhombus or in the second. So, whether the diagonal is to be opposite the given angle C (as in the first rhombus), or is to bisect the given angle B (as in the second), draw the diagonal BD, which exceeds the side BC by the straight line DF; and if this is equal to the given excess A, the required construction is done. But if unequal, [Margin: f — *Elements* VI.12] make it that as DF is to the given excess A, so is the diagonal BD to BG, and complete the rhombus HI, which I say is the one sought. For, cutting off a straight line GK equal to the given excess A, exactly the same demonstration is to be applied as was made in the case of the square.

Diagram. Three oblique parallelograms (rhombi) in a row, illustrating the rhombus cases, each with the given angle L and the auxiliary construction-marks A, M, N drawn above it and the interior diagonal lettered. The interior letters are overprinted and crowded Readable labels across the group include, above the figures: L, A, G, E (and the small bracket-marks for the given angle and segments); at the corners and bottom edges: B, C, H (and I, E on the left edges of the first two); interior points on the diagonals: K, D, F, G. Diagonals run from the lower-left corner B up through the interior to the opposite corner (BD / BG). Because the three little figures share these letters and the interiors are printed over the ruling, individual interior assignments are uncertain [?].

- Above the figures: L, A, G, E (given angle and auxiliary marks)
- Corners / bottom edges: B, C, H; left edges I, E
- Interior (on diagonals): K, D, F, G [several overprinted — [?]]

But if the diagonal is smaller than the side of the rhombus, let A be the given excess of the side over the diagonal in some rhombus, together with the angle L.

Construct any rhombus BDCE having the angle D equal to the given angle L, as in the third rhombus. And since, in order that the side may exceed the diagonal, the diagonal must be drawn through the obtuse angles — [Margin: g — *Elements* I.19] because a diagonal drawn through the acute angles is always greater than the side of the rhombus — draw the diagonal BC, which the side BD exceeds by the straight line DF; and if this is equal to the given excess A, the required construction is done. But if unequal, make it that as DF is to the excess A, so is BD (the side) to BG, and complete the rhombus GI about the same diagonal, which I say is the one sought. For, cutting off a straight line GK equal to the excess A, the demonstration will proceed as in the square, as is evident — if, in place of the diagonals BD, BG in the square, the sides BD, BG are here taken.

In that case, the side of the rhombus will be greater than the diagonal (to note this in passing as well) when half of the obtuse angle is greater than the acute angle of the same rhombus. For if, in the third rhombus, the angle CBD — which is half of the obtuse angle B, as we showed in the scholium on *Elements* I.34 — is greater than the acute angle D, [Margin: h — *Elements* I.19] then the side BD (that is, CD) will be greater than the diagonal BC in triangle BCD. But when half the obtuse angle is smaller than the acute angle, as in the second rhombus, [Margin: i — *Elements* I.19] the diagonal will be greater than the side in triangle BCD.

Thirdly, let A be the given excess of the diagonal of some rectangle over one of its unequal sides, together with the angle L which the diagonal makes with that side, or together with the ratio M to N which that side has to the other. So, if the angle L is less than half a right angle, or the ratio M to N is one of greater inequality, as in the first rectangle, then A will be the excess of the diagonal over the greater side. But if the angle L is greater than half a right angle, or the ratio M to N is one of lesser inequality, then A will be the excess of the diagonal over the lesser side.

Diagram. Two rectangles (the rectangle cases), each with the given angle L and the ratio-segments M, N marked above, and the diagonal lettered inside. Left rectangle: on top edge A, then L, M with segment-marks and N below; corners include I (upper left), E (left edge), B (bottom left), and the bottom edge runs B, C, H with G at the far corner; interior points D and F lie on the diagonal drawn from B. Right rectangle: L, M, N marked above with A; corners E (top left), D (top right), B (bottom left); bottom edge B, H, G; interior points F, G, K on the diagonal. The interiors are overprinted and crowded

- Above: L, M, N, A (given angle and ratio-segments)
- Left rectangle — corners/edges I, E, B, C, H, G; interior D, F
- Right rectangle — corners E, D, B, H, G; interior F, G, K [overprinted — [?]]

Therefore construct the angle CBD equal to the angle L, and form the rectangle BCDE about the assumed diagonal BD. Or make BC, of whatever size, perpendicular to CD, in the ratio M to N; and, completing the rectangle CE, draw the diagonal BD, which exceeds the side BC by the straight line DF. If this is equal to the given excess A, the rectangle CE that is sought will have been constructed. But if unequal, make it that as DF is to the given excess A, so is BD to BG, and complete the rectangle HI, which will be the one sought. For, cutting off a straight line GK equal to the excess A, the proposition will be demonstrated as in the case of the square.

Fourthly and lastly, let there be given, in some rhomboid, the excess A of the diagonal over either of the unequal sides, together with the angle O of the rhomboid, and moreover with the angle L which the diagonal makes with the side whose excess has been taken, or moreover with the ratio M to N which that side has to the other side. Construct the angle BCD (in the first figure) or CBE (in the second) equal to the given angle O. Then, whether the diagonal is to be opposite the given angle C (as in the first figure), or is to bisect the given angle CBE (as in the second), make at B the angle CBD equal to the given angle L, and let the straight line BD cut CD in D; or make it that as M is to N, so is BC to CD; and complete the rhomboid CE, whose diago-

-nal exceeds the side BC by the straight line DF. If this is equal to the given excess A, the required construction is done. But if unequal, make it that as DF is to the excess A, so is BD to BG, and complete the rhomboid HI about the same diagonal BD, which I say is the one sought. For if GK is cut off equal to the excess A, the same demonstration is to be applied as in the preceding cases.

But if the diagonal of some rhomboid is smaller than the greater side, as in the third figure: let A be the given excess of the greater side over the diagonal in some rhomboid, and moreover with the angle L which the diagonal must make with that greater side, or moreover with the ratio M to N which the greater side has to the lesser.

Diagram. Three oblique parallelograms (rhomboids) illustrating the rhomboid cases, each surmounted by the given angles and ratio-marks: A, L, O and the ratio-segments M, N (the N is set as a reversed/turned sort — printed as "И" [recte: N]). Corners and bottom edges carry, across the group: E, I (left edges), B, C, H, and — on the third figure — the bottom run B, F, D, K, G; interior diagonal points include G, D, K, F, C. The three figures share these letters and the interiors are overprinted on the ruling; several interior assignments are uncertain [?].

- Above: A, L, O (given angles); M, N (ratio-segments; N printed reversed as "И")
- Left edges: E, I
- Bottom edges: B, C, H; third figure B, F, D, K, G
- Interior (on diagonals): G, D, K, F, C [several overprinted — [?]]

Construct the angle BDC equal to the given angle O; and if it is acute, make at B the angle DBC equal to the angle L (if the given angle of the rhomboid were obtuse, namely DBE, the angle DBC would have to be constructed within the given angle itself), and let the straight line BC cut the straight line DC in C; or make it that as M is to N, so is BD to DC; and complete the rhomboid DE, whose side BD exceeds the diagonal BC by the straight line DF. If this is equal to the given excess A, the required construction is done. But if unequal, make it that as DF is to A, so is BD to BG, and complete the rhomboid GI, which I say is the one sought. For if GK is taken equal to A, the proposition will be demonstrated as above in the square, if in place of the diagonals BD, BG of the square the sides BD, BG are here taken, as is evident.

In that case the greater side will exceed the diagonal when the angle that the diagonal makes with the lesser side is greater than the acute angle of the rhomboid. For if, in the third figure, the angle BCD is greater than the angle D, [Margin: c — *Elements* I.19] then the straight line BD will be greater than BC.

THEOR. 8. PROPOS. 15.

In a rectangular parallelogram, taking the excesses by which the diagonal exceeds the two sides: the rectangle contained by the difference of the excesses and by the lesser excess, taken twice, together with the square of the lesser excess taken twice, is equal to the square of the straight line by which the lesser side exceeds the lesser excess.

Diagram. A rectangle with the diagonal drawn from the lower-left corner. Corners: A (top left), D (top right), B (bottom left); the bottom edge is divided and lettered, reading from the left B, G, F, C, then E just beyond C. The diagonal runs from B up to D.

- A (top left): corner
- D (top right): corner
- B (bottom left): corner
- G, F, C (points along the bottom edge, left to right)

- E (bottom edge, just beyond C): point marking BE = the diagonal BD laid along the base
- diagonal B–D

Let AC be a rectangle, and let the straight line BE be equal to its diagonal BD, so that the lesser excess — by which the diagonal exceeds the greater side BC — is CE. And, taking BF equal to the lesser side CD, let EF be the excess by which the diagonal BD (or BE, equal to it) exceeds the lesser side CD (or BF, equal to it); and consequently let CF be the difference of the excesses EC, EF. [Margin: a — *Elements* I.20] And since the sides BC, CD are greater than the side BD — that is, than the straight line BE — when the common part BC is removed, the remainder CD will be greater than the remainder CE; and therefore BF, being equal to CD, will be greater than CE. So, cutting off FG equal to CE, BG will be the excess by which the lesser side BF exceeds the lesser excess FG.

I say that the rectangle, taken twice, contained by FC (the difference of the excesses) and CE (the lesser excess), together with the square of the lesser excess CE taken twice, is equal to the square of the straight line BG, by which the lesser side BF exceeds the lesser excess FG. [Margin: b — *Elements* II.4] For since the square of the straight line BE is equal to the squares of the straight lines BC, CE, together with the rectangle taken once under BC and twice CE: [Margin: c — *Elements* II.1] but the rectangle under BC and twice CE is equal to the rectangles under BF and twice CE, and under FC and twice CE — that is, to the rectangle under BF and CE taken twice, together with the rectangle under FC and CE taken twice — the square of the straight line BE, or of the straight line BD, will also be equal to the squares of BC, CE, together with the rectangles under BF, CE taken twice and under FC, CE taken twice.

[Margin: d — *Elements* I.47] And consequently the squares of BC, CD — which are equal to the square of BD — will be equal to the squares of BC, CE, together with the rectangles under BF, CE taken twice and under FC, CE taken twice. So, when the common square of BC is removed, the remaining square of CD — that is, of BF — will be equal to the remaining rectangles under BF, CE taken twice and under FC, CE taken twice, together with the square of CE. Then, adding the common square of FG, the squares of BF, FG will be equal to the rectangles under BF, CE taken twice and under FC, CE taken twice, together with the squares of CE, FG. [Margin: e — *Elements* II.7] But the squares of BF, FG are equal to the rectangle under BF, FG taken twice, together with the square of BG. Therefore the rectangle under BF, FG — that is, under BF, CE, taken twice — together with the square of

BG, will be equal to the rectangles under BF, CE taken twice and under FC, CE taken twice, together with the squares of CE, FG.

So, removing the common rectangle under BF, CE taken twice, the remaining square of BG will be equal to the remaining rectangle under FC, CE taken twice, together with the squares of CE, FG — that is, the rectangle under FC (the difference of the excesses) and CE (the lesser excess) taken twice, together with the square of the lesser excess CE taken twice, is equal to the square of the straight line BG, by which the lesser side BF exceeds the lesser excess FG. Which was to be demonstrated.

PROBL. 8. PROPOS. 16.

Given the two excesses by which the diagonal of a rectangle exceeds each of its sides, to find both sides and the diagonal.

(Continuation of Problem 8 / Proposition 16, begun on the preceding page.)

Let FE be the given excess of the diameter over the smaller side, and CE its excess over the larger side, so that the difference of the two excesses is FC. From C erect the perpendicular CL to FE, and mark off CH, HI, and EK equal to the smaller excess CE, so that the whole lines CI and CK are equal — each being double CE — and complete the parallelogram FI. Then bisect FK at N, and with center N describe through F and K the semicircle FLK, cutting CL at L. Finally, draw HE, take CM equal to it, and join the straight line LM. I say that LM is the difference between the sought smaller side and the given smaller excess CE — so that CE added to LM produces the smaller side, and if the difference of the given excesses FC is then added to this, the larger side results.

(For the difference between the diameter's excesses over each side of the rectangle equals the excess of the larger side over the smaller. This is plain from the figure of the preceding proposition, where the diameter is BD, or BE; the greater excess is FE, by which the diameter exceeds the smaller side BF; the smaller excess is CE, by which the same diameter exceeds the larger side BC; and FC, the difference of the excesses, is the amount by which the larger side BC exceeds the smaller BF.) And finally, to the larger side, once found, the smaller excess CE is added so that the diameter is obtained. All of this is now demonstrated as follows.

By the preceding proposition, the rectangle contained by FC (the difference of the excesses) and CE (the smaller excess) taken twice — that is, the rectangle FI — together with the square on CE also taken twice — that is, together with the

square on HE (or CM) — equals the square on the line by which the sought smaller side exceeds the smaller excess CE. Now since the square on CL equals the rectangle FI, as is established from the demonstration of the last proposition of Euclid's Book 2, the squares on CL and CM together will also equal the square on that same line by which the sought smaller side exceeds the smaller excess CE. [Margin: (a) Euclid I.47] And therefore, since the square on LM equals the squares on CL and CM, the square on LM will likewise equal the square on the line by which the sought smaller side exceeds the smaller excess CE. So LM is the excess of the sought smaller side over the smaller excess CE. Hence the line composed of LM and CE will be the sought smaller side; if FC (the difference of the excesses) is added to this, the sought larger side results; and if finally the smaller excess CE is added to that, the sought diameter is assembled — all of which had to be demonstrated.

Diagram. A semicircle standing on the horizontal base FK, with the arc rising from F (lower left) over the top to K (lower right). A vertical perpendicular rises from the baseline at C up to L at the top of the arc, passing through the interior points H and I. A small dotted rectangle sits to the right of this vertical (point P at its upper right), and another point o lies on the left at mid-height, where a horizontal (the parallelogram FI) crosses. Several chords are drawn from F and K to the interior points and to M.

- L (top of arc): apex where CL meets the semicircle
- I (on the vertical, upper): interior division point
- H (on the vertical, below I): interior division point
- P (right of the vertical, dotted): corner of the small rectangle
- o (left, mid-height): where the horizontal meets the left side
- F (lower left): endpoint of the base
- N [?] (baseline, between F and M): midpoint of FK / center of the semicircle
- M [?] (baseline, left of C): point where CM is laid off
- C (baseline): foot of the perpendicular CL
- E (baseline, between C and K): point
- K (lower right): endpoint of the base

COROLLARY.

And so the line LM — whose square equals the rectangle FI, contained by FC (the difference of the excesses) and double the smaller excess CE, together with double the square of the smaller excess CE — when added to the smaller excess CE, produces the sought smaller side, and so on.

Indeed, since the square on CL equals the rectangle FI, contained by FC (the difference of the excesses) and CK (double the smaller excess CE), as was said in the demonstration; and since the rectangle CP is double the square of the smaller excess CE — that is, equal to the square on CM — the square on LM will equal the whole rectangle FP, contained by the greater excess FE and EP (double the smaller excess CE). [Margin: (a) Euclid VI.17] LM will therefore be the mean proportional between the greater excess and double the smaller excess. Consequently, if the mean proportional LM is taken between the greater excess and double the smaller excess, one again obtains the difference between the smaller side and the smaller excess, and so on.

SCHOLIUM.

This problem, together with the preceding theorem, was demonstrated for me in France, from where it was sent to me, by a certain ingenious geometer whose name — had I known it — I would gladly record here. Yet I have myself demonstrated the same problem, not without success, in another way as well, at the end of Book 2 of Euclid, following Marino Ghetaldi [Marin Getaldić], a patrician of Ragusa [Dubrovnik].

PROBL. 9. PROPOS. 17.

Given the excess of a rectangle's diameter over the larger side, and the excess of the larger side over the smaller: to find both sides and the diameter.

Since, as was said in the preceding problem, the excess of the larger side over the smaller equals the difference between the diameter's excesses over each side, it follows that the excess of the diameter over the larger side, added to the excess of the larger side over the smaller, produces the excess of the diameter over the smaller side. Therefore, once the diameter's excesses over each side are known, the rest can be found, just as was set out in the previous problem.

PROBL. 10. PROPOS. 18.

Given a straight line cut at any point, to attach to it — toward either end — a further straight line, such that the square on the whole composite line equals the square on the attached line together with the square on the line formed from the attached line and the adjacent segment of the original line.

In the figure of Proposition 15, let the straight line EF be cut at C at any point, and suppose we are required to attach to it, toward F, a line such that the square on the whole composite equals the square on the attached line together with the square on the line composed of the segment FC and the attached line. Let FE and EC be set up as the excesses by which the diameter of some rectangle exceeds each of its sides. By Proposition 16, find the smaller side BF. I say that the line BF, attached to EF, produces what is proposed. For let the rectangle AC be made, under BC

Diagram. A rectangle with the top side AD and the base BC extended rightward to E, with a diagonal drawn from A to C. Along the base the points read, from left to right, B, G, F, then C and E.

- A (top left corner)
- D (top right corner)
- B (bottom left corner)
- G (on the base, right of B)
- F (on the base, right of G)
- C (bottom right corner of the rectangle)
- E (on the base extended, right of C)

Diagram. The same figure as at the foot of the preceding page, reprinted at the head of this one: a rectangle with top side AD and base BC extended to E, with the diagonal from A to C; along the base, B, G, F, C, E.

- A (top left); D (top right); B (bottom left); G, F (on the base); C (bottom right); E (base extended)

[Margin: (a) Euclid I.47]

(Continuation of Problem 10.) ...and CD, equal to BF. And since FC, the difference of the excesses, added to the smaller side BF once found, makes the larger side — as was said in Proposition 16 — BE will equal the diameter BD, since it exceeds the smaller side BF (or CD) by the line EF, and the larger side by the line EC.

[Margin: (a) Euclid I.47] Now since the square on BE — that is, on the diameter BD — equals the square on CD (that is, on the attached line BF) together with the square on BC (composed of the attached line BF and the adjacent segment FC), what was proposed is plainly established.

PROBL. 11. PROPOS. 19.

Given two unequal straight lines, of which the greater does not exceed the diagonal of the square described on the smaller: to cut the greater into two unequal parts such that the squares on them, taken together, equal the square on the smaller line.

Let AB (the greater) and AC (the smaller) be the two given lines, with AB not greater than the diagonal of the square on AC. Erect the perpendicular AD equal to the greater line AB; draw BD and bisect it at E; and join AE, which will be perpendicular to BD and will bisect the right angle A into two half-right angles. [Margin: (a) scholium to Euclid I.26] [Margin: (b) 2nd corollary to Euclid I.32] Now B and D are also half-right angles. [Margin: (c) Euclid I.6] Therefore the sides EA and EB are equal; and consequently AB is the diagonal of the square on AE. And since AB is assumed not greater than the diagonal of the square on the smaller line AC, AC will not be smaller than AE, but either greater or equal. For if AC were smaller than AE, then taking AL equal to it and drawing LM parallel to EB, AM would be the diagonal of the square on the smaller line AL, and so AB — being greater — would exceed the diagonal of the square described on the smaller line. That is contrary to the hypothesis.

Diagram. A quarter-circle (or arc) construction: from the lower-right vertex A a straight line rises to D at the upper right, and an arc sweeps from the base up to D. The right angle stands at the lower left. Interior points N (on the line to D), and the cluster F, E, L, and H (on the left) mark the successive constructed points, with dotted lines among them. Along the base, from left to right: B, M, C, I, K, G, then o, then A at the far right.

- D (upper right): apex of the construction
- N (on AD, below D): second intersection of the circle with BD
- F, E, L (interior cluster, mid-height): constructed points of the proof
- H (left, mid-height): first intersection of the circle with BD
- B (base, far left); M, C, I, K, G (successive base points); o (base, right); A (base, far right)

So let AC first be greater than AE; extend AE to F so that AF equals AC; with center A describe through C and F a circle cutting BD at H and N; and drop HI perpendicular to AB. I say that the greater line AB is so cut at I that the squares on AI and IB equal the square on the smaller line AC. For since the angle I is right and B is half-right, the angle H in triangle BHI will also be half-right; [Margin: (d)

Euclid I.6] and therefore the sides BI and HI will be equal. [Margin: (e) Euclid I.47] So when AH is drawn, since the squares on AI and IH equal the square on AH, the squares on the segments AI and IB will also equal the square on AH — that is, the square on AC. Which was proposed.

Next, let the smaller given line AK be equal to AE, so that the greater line AB equals the diagonal of the square described on the smaller line AE. Drop the perpendicular EG, [Margin: (f) scholium to Euclid I.26] which will bisect both the base AB and the right angle E into half-right angles. I say that AB is so cut at G that the squares on the equal segments AG and GB equal the square on the smaller line AK (or AE). [Margin: (g) Euclid I.6] For again the sides EG and GB will be equal, because of the equal half-right angles B and E in triangle BEG. [Margin: (h) Euclid I.47] So since the squares on AG and GE equal the square on AE, the squares on the segments AG and GB will also equal the square on AE, or on the smaller line AK. Which was proposed.

If from the other intersection N the perpendicular NO is dropped, AB will be cut at O just as at I, so that the squares on the segments BO and OA likewise equal the square on the smaller line AC (or AH). [Margin: (i) Euclid VI.2] For since the lines BD and BA are cut proportionally at H, I and at N, O, and BH equals ND — [Margin: (k) Euclid III.3] because the perpendicular AE bisects HN, so that, equals EH and EN being taken from equals EB and ED, the remainders BH and DN are equal — BI and AO will also be equal, and likewise BO and AC, and so on.

PROBL. 12. PROPOS. 20.

Given the chord of some arc, together with the perpendicular drawn from the midpoint of the chord out to the arc: to find how many degrees (or palms) both the arc and the radius of the circle comprise.

Diagram. A chord-and-perpendicular figure resembling a kite. The chord AB runs from A (left) to B (right), with C its midpoint. From C the perpendicular CD rises to D on the arc above, and CE descends to E (the circle's center) below. The lines AD, AC, and AE are drawn, and an arc passes from A up through D toward B.

- A (left): one end of the chord
- D (top): point on the arc, foot of the perpendicular's upper end
- C (center of the chord): foot of the perpendicular
- B (right): other end of the chord
- E (bottom): center of the circle

Let the given chord AB be 74 palms and the perpendicular CD, drawn from the midpoint C, be 10 palms. Join AD. Since in the right triangle ACD the sides AC and CD are known — CD being 10 and AC being 37, namely half the chord AB of 74 — [Margin: (a) 8, on rectilinear triangles] if we make CD 10 to the whole sine 100000 as AC 37 to a fourth term, we find 370000, the tangent of the angle ADC; and so that angle will be about $74^{\circ}53'$. And if we imagine AE drawn to the center E, the angle DAE will likewise be about $74^{\circ}53'$, [Margin: (b) Euclid I.5] since the angles ADE and DAE are equal. If their sum, about $149^{\circ}46'$, is subtracted from two right angles — that is, from 180° — the remaining third angle E, at the center, becomes about $30^{\circ}14'$; and the arc AD will comprise just as many degrees. Hence its double, the arc ADB, is nearly $60^{\circ}28'$.

[Margin: (c) Euclid III.35] Now since the square on AC equals the rectangle under CD and the remaining part of the diameter: if AC, 37 palms, is multiplied by itself, and the resulting number 1369 is divided by CD, 10 palms, the remaining part of the diameter comes out as 136 palms and a fraction [tiny fraction: ?]; and so, with CD (10 palms) added, the whole diameter will be 146 palms and a fraction [tiny fraction: ?], and the radius 73 palms and a fraction [tiny fraction: ?]. If then we make 7 to 22 as 146-palms-and-a-fraction[?] to a fourth term, the circumference of the circle is found — by the first rule, no. 2, of chapter 7 of Book 4 of this work — to be 430 palms and a fraction [tiny fraction: ?]. So if again we make the whole circumference of 360° to the 430-palms-and-a-fraction[?] as the arc ADB of $60^{\circ}28'$ to a fourth term, this arc is found to be 72 palms and a fraction [tiny fraction: ?] — that is, $72 \frac{10}{30}$ [?] palms, a little more.

THEOR. 9. PROPOS. 21.

In every triangle the square on the greatest side is less than double the sum of the squares described on the remaining two sides.

In triangle ABC let the greatest side be AC, and the opposite angle B obtuse. For if B were right or acute, [Margin: (a) Euclid I.47] the square on AC would either equal the two squares on AB and BC [Margin: (b) Euclid II.13] or be less than them, and hence much less than double the sum of the squares on AB and BC. From the greater side AC cut off AD equal to the side AB. [Margin: (c) Euclid I.20] Since the two sides AB and BC together are greater than the side AC, the remainder CD will be less than the side BC; and hence the two squares on AD and DC are less than the two squares on AB and BC. But the square on AC is less than double the squares on AD and DC, [Margin: (d) Euclid II.4] because it equals the two squares on AD and DC together with twice the rectangle under AD and DC —

which rectangle, taken twice, is less than the two squares on AD and DC, by the Lemma to Proposition 39 of Book 10 of Euclid. Therefore the square on AC will be much less than double the squares on AB and BC. Which was to be demonstrated.

Diagram. An obtuse triangle ABC drawn low and wide: A at the upper left, B at the bottom (the obtuse vertex), and C at the lower right. The side AC runs across the top, with D marked on it (where AD equals AB). The sides AB and BC descend to B.

- A (upper left)
- D (on AC, right of A): point where AD is cut off equal to AB
- B (bottom center): the obtuse vertex
- C (lower right)

PROBL. 13. PROPOS. 22.

Given three straight lines placed anyhow in a plane and not parallel — except when the outer two are equidistant from the middle one — to draw a straight line (indeed, through a given point on the middle line, if all three meet in one point) such that its segments between the middle line and the outer lines are equal to one another, or have a given ratio.

Let the three straight lines be AB, CD, EF. Draw any line BF cutting all three, and bisect it at G. If the point G falls on the middle line, what is required is done. But if G falls outside the middle line CD, draw through G a line AG parallel to one of the outer lines — namely to EF (provided it is not equidistant from the middle line CD) — cutting the middle line at C. For since CD is assumed not equidistant from EF, AG will certainly cut EF produced, and therefore its own parallel as well. But if CD were equidistant from EF, one would have to draw through G a line parallel to AB. Finally, from B — the point taken on the outer line AB, from which the line BF was drawn, and to which AG is not equidistant — draw through C (where the parallel AG cuts the middle line CD) the straight line BC, cutting EF at E. I say that the lines BC and CE are equal. For since in triangle BEF the line CG is parallel to the base EF, [Margin: (a) Euclid VI.2] it will be that as BG is to GF, so BC is to CE; but BG equals GF by construction. Therefore BC will also equal CE.

Diagram. A pencil of lines meeting at the apex H at the top, with two long strokes descending to F (lower left) and K (lower right), forming a broad triangle. Interior points crowd near the apex — E, C, A, B — with L on the left and G, D in the middle; a dotted horizontal runs across the base from F through I to K.

- H (apex, top)
- E, C, A, B (interior points near the apex)
- L (left, mid-height)
- G, D (middle)
- F (lower left): end of the left stroke
- I (bottom center): on the dotted base
- K (lower right): end of the right stroke

And if through G a line GI is drawn parallel to AB, cutting the middle line CD at I, then the line FIK drawn will also be bisected at I: because in triangle FBK the line GI is equidistant from the base BK; [Margin: (b) Euclid VI.2] and so, just as BF is bisected at G, FK too will be bisected at I.

From this it is clear why the middle line must not be equidistant from both outer lines, but either from neither or from just one. For if it were equidistant from both, [Margin: (c) Euclid I.30] a line drawn through G parallel to either outer line would also be equidistant from the middle line, and so would not cut it. But if it is equidistant from neither, a line can be drawn through G parallel to whichever outer line one pleases. And if it is equidistant from one of the outer lines, the line must be drawn through G parallel to the other.

Now if the three given lines, produced, meet in the same point — as at H — the problem will be easier, even when a point C is given on the middle line through which the line must be drawn. For if through the given point C on the middle line a line CA is drawn parallel to one of the outer lines — say to HF — cutting the other outer line at A, and AB is taken equal to HA, [Margin: (d) Euclid VI.2] then the line BCE drawn will be bisected at C: since BC is to CE as the line BA to the line AH, and so on. Likewise, if a point I is given on the middle line, then drawing through I a line LI parallel to one of the outer lines — say to HB — and taking LF equal to HL, the line FIK drawn will be bisected at I; [Margin: (e) Euclid VI.2] because FI is to IK as FL to LH, and so on.

In the same way we shall draw a line that is cut by the middle line into two parts having a given ratio. For if the line BF is cut in any way at G in the given ratio, and the rest is done as above, [Margin: (f) Euclid VI.2] then again as FG is to GB, so EC is to CB; or as BG is to GF, so BC is to CE — according as the given ratio is that of FG to GB or of BG to GF. Likewise, if the three given lines meet at H, draw GA parallel to EF and make HA to AB as the antecedent of the given ratio to its consequent: if BCE is drawn, [Margin: (g) Euclid VI.2] then EC will be to CB as HA

to AB. And if HA is made to AB as the consequent to the antecedent, then again BC will be to CE as BA (the antecedent) to AH (the consequent).

PROBL. 14. PROPOS. 23.

To exhibit any multiple whatever of any line, however small — even if the line itself cannot be taken with the compass.

Suppose, for example, that we need the triple of the small line AB. Opening the compass to any convenient width, from A toward C mark off two lengths CF and FD equal to AC, so that the whole line AD is triple AC. Likewise, along CB mark off three lengths DG, GH, HE equal to CB. I say that AE is triple AB. For since the whole AD is the same multiple of the whole AC as the part removed, DE, is of the part removed, CB — namely triple — the remainder EA will also be the same multiple of the remainder AB as the whole is of the whole, that is, triple. Which was proposed.

Diagram. A single horizontal straight line running from A (far left) to D (far right), marked off by tick-marks into successive equal segments. The interior division labels read, from left to right, I, B, E, C, H, F, G.

- A (far left): endpoint
- I [?], B, E, C, H, F, G (interior division points, left to right)
- D (far right): endpoint

Note on the marginal citations: throughout these pages the printed marginalia are Euclid references keyed by superscript letters to the proof steps. The abbreviations expand as *primi* = Book I, *secun.* = Book II, *tert./tertij* = Book III, *quar.* = Book IV, *sexti* = Book VI; *schol.* = scholium, *coroll.* = corollary. They are rendered below as [Margin a: *Elements* III.32] and the like at the point of their superscript.

Problem 15. — Proposition 24.

To cut off from any small line, however tiny, an assigned part or parts.

In the figure of the preceding proposition, let a third part be taken away from the line AB. By the preceding proposition, take AE triple of AB; and if this seems too small, multiply it as you please. In the example it has been quadrupled up to D, so that AD is twelve times AB. (This is known if the number of parts of AE, namely 3, is multiplied by the number of parts of AD equal to AE, namely by 4.) And so, if AB is understood to be divided into 3 parts, the whole of AD will contain 36 such

parts. Therefore, if on the instrument of parts constructed in Book 1, chapter 1, the interval AD is set between the marks 36 and 36, and then the interval between 35 and 35 (namely the whole AD less one part) is carried across from D to I, then AI will be a third part of AB — that is, the thirty-sixth part of the whole AD. Since, then, AB contains three thirty-sixth parts of the whole AD, AI will be a third of AB. Which is what was proposed.

Problem 16. — Proposition 25.

To divide a given rectilinear angle into three equal parts.

This problem long and greatly vexed the ancient geometers, and to this very day has been solved geometrically by no one. Pappus of Alexandria [Pappus], among others, tried to solve it by means of the description of a hyperbola. We shall accomplish the same thing by means of the conchoid line, which we described from Nicomedes [Nicomedes] in Book 6, proposition 15 of this work, in the following way.

Let the given acute angle be ABC. From any point A drop the perpendicular AD to BC, and take DC double of AB. Then, with pole B and interval DC, describe the conchoid line CE, cutting in E the straight line AE (drawn parallel to BC); and draw the straight line BE.

Diagram. The trisection figure for the conchoid construction. A triangular configuration: angle ABC at the lower left, with the base line BC running to the right, and a line through A drawn parallel to BC to the upper right at E. A dotted perpendicular runs from E down to C.

- A (upper left): apex of the given angle's vertical side; endpoint of the parallel line AE
- E (upper right): where the conchoid CE meets the parallel through A; dotted perpendicular descends to C
- G (middle left): point on line BE
- F (middle, right of center): point on line BE (with the auxiliary circle through A)
- B (lower left): vertex of the given angle
- D (lower center-left): foot of the perpendicular AD on BC
- C (lower right): pole-distance point on BC

I say that the angle CBE is a third part of the given angle ABC — that is, that the angle ABE is double the angle CBE — so that, once the angle ABE is bisected, the

whole angle ABC is cut into three equal parts. For, by the construction of the conchoid, the line GE is equal to DC, and therefore double AB; so if it is bisected at F, each half is equal to AB. [Margin a: scholium to *Elements* III.31] Now since the circle described about GE from the center F passes through the right angle GAE, the line FA, when drawn, will be equal to each half FE and FG, and hence also to AB. [Margin b: *Elements* I.5] Therefore both the angles FAE and FEA, and the angles AFB and ABF, will be equal. [Margin c: *Elements* I.32] But the exterior angle AFB is equal to the two interior angles FAE and FEA, and so is double FEA. Therefore ABF too — being equal to that same FEA, [Margin d: *Elements* I.29] that is, to the alternate angle CBE — will be double CBE.

If the given angle is a right angle, it will be divided into three equal angles, as we set out in the scholium to proposition 32 of Book 1 of Euclid.

But if it is obtuse, we shall bisect it and divide one of the two halves into three equal parts, as we have taught here; for two-thirds of that half will make a third part of the proposed obtuse angle, as is plain.

Problem 17. — Proposition 26.

If through the same point of the diagonal in a rectangle two lines are drawn parallel to the sides, then the rectangle contained under the segments of the diagonal will be equal to the two rectangles contained under the segments of the two sides.

In the rectangle BD, through E, a point of the diagonal AC, let FG and HI be drawn parallel to the sides. I say that the rectangle under AE, EC is equal to the rectangles under AF, FB and under BI, IC.

Diagram. Two adjacent figures : on the left the general rectangle (the "prior figure"), on the right the square (the "posterior figure" of the corollary). Both share the same lettering: the diagonal runs from the upper-left corner to the lower-right corner, cutting the two internal parallels FG and HI at the point E.

- A (top-left corner)
- H (top edge, between A and D)
- D (top-right corner)
- F (left edge)
- E (center, on the diagonal, where FG meets HI)
- G (right edge)
- B (bottom-left corner)

- I (bottom edge, between B and C)
- C (bottom-right corner)

[Margin a: *Elements* II.4] For the square on AC is equal to the squares on AE, EC together with twice the rectangle under AE, EC. [Margin b: *Elements* I.47] But the squares on AB, BC are equal to the square on AC; therefore the two squares on AE, EC, together with twice the rectangle under AE, EC, are equal to the squares on AB, BC. [Margin c: *Elements* II.4] But the square on AB is equal to the two squares on AF, FB together with twice the rectangle under AF, FB; and the square on BC is equal to the two squares on BI, IC together with twice the rectangle under BI, IC. Therefore the two squares on AE, EC will be equal to the four squares on AF, FB, BI, IC together with twice the rectangles under AF, FB and under BI, IC. [Margin d: *Elements* I.47] Since, then, the square on AE is equal to the squares on AF, FE — that is, on AF, BI — and the square on EC is equal to the squares on EI, IC — that is, on FB, IC — the four squares on AF, FB, BI, IC together with twice the rectangle under AE, EC will be equal to the four squares on AF, FB, BI, IC together with twice the rectangles under AF, FB and under BI, IC. And when the aforesaid four common squares are taken away from each side, the remaining twice-rectangle under AE, EC will be equal to the remaining twice-rectangles under AF, FB and under BI, IC. And so the single rectangle under AE, EC will be equal to the single rectangles under AF, FB and under BI, IC. Which was to be demonstrated.

Corollary.

Therefore in a square, as in the latter figure, the rectangle contained under the segments AE, EC of the diagonal is equal to the two complements DE, BE, since the complements are contained under the segments of the sides.

Problem 18. — Proposition 27.

Given the center of an ellipse on a line produced without limit, together with two points on the same side of the axis (or of the center) through which the ellipse is said to pass: to find each extremity of the major axis.

This is a conic problem, and a subtle one. Let A be the center — that is, the midpoint — of some ellipse, lying on the line BC of the major axis, of whatever length; and let D, E be two points on that same ellipse toward the same side, that is, either above the center A or below it. From these drop the perpendiculars DF, EG to BC. Then DF will be smaller than EG, since a perpendicular farther from the center is always smaller than one nearer. Draw, then, the straight line ED; it will cut the axis line at B. Bisect AB at H, and about AB describe from H the semicircle

AKB. Bisect also FG (which lies between the drawn perpendiculars) at I, and draw IK perpendicular to AB, cutting the circumference at K and the line BE at L. Draw also the straight line BK, cutting the perpendiculars DF, EG at M and N, and join the straight lines AM, AK, AN.

Diagram. The ellipse-axis construction. A vertical axis line runs from B at the top down to C at the bottom, with the ellipse's center A on it; a large arc (the ellipse) sweeps to the left, and the semicircle AKB and auxiliary lines are drawn on the left.

- B (top of the axis line)
- O (on the axis, below B): where the circle from A cuts BC
- F (on the axis): foot of the perpendicular DF
- H (on the axis): midpoint of AB
- I (on the axis): midpoint of FG
- G (on the axis): foot of the perpendicular EG
- A (on the axis, lower): center of the ellipse
- C (bottom of the axis line)
- M (left cluster): point on perpendicular DF, on line BK
- D (left cluster): point on the ellipse
- K (left cluster): point on the circumference / semicircle
- N (left cluster): point on perpendicular EG, on line BK
- E (left cluster): point on the ellipse
- L (left cluster): where IK meets BE

[Margin a: *Elements* VI.2] Since, as FI is to IG (and these are equal), so is MK to KN, MK too will be equal to KN. Therefore the two sides AK, KM will be equal to the two sides AK, KN. And since they also contain equal angles — namely right angles, [Margin b: *Elements* III.31] because the angle AKB in the semicircle is right — [Margin c: *Elements* I.4] the bases AM, AN will be equal. Therefore the circle described from A through N will pass through M, and will cut BC at O and C. I say that OC is the major axis of the ellipse. [Margin d: scholium 4] For since, as MD is to DF, so NE is to EG, the ellipse described through O, D, C will necessarily pass through the point E as well — (that an ellipse can be described about OC as major axis, passing through the point D, is established from what we wrote at the end of the scholium to proposition 8 of Book 1 of the *Gnomonica*, and in the scholium to Lemma 50 of Book 1 of the *Astrolabium*) — through the point E by the scholium to Lemma 51 of the *Astrolabium*. And therefore the ellipse described through the given points D, E about the center A will pass through O, C, so that it does not

differ from the ellipse described through O, D, C. Otherwise one ellipse would cut the other in 8 points — namely at D, E and at two others corresponding on the other side of the axis, and then at 4 others corresponding below the center — which is absurd, [Margin e: Apollonius, IV.25] since one ellipse cuts another in only 4 points.

Theorem 10. — Proposition 28.

If on the produced diameter of a circle a point is taken, and from it a straight line is drawn tangent to the circle, and from the point of contact a chord is drawn perpendicular to the diameter: then the straight line drawn from that same point of contact to either extremity of the diameter will bisect the angle contained by the tangent and the aforesaid perpendicular chord. Likewise, if from the same point taken on the produced diameter a straight line is drawn cutting the circle, and from either point of section a straight line is joined to the intersection of the diameter with the aforesaid perpendicular chord: then the straight line drawn from that same point of section to either extremity of the diameter will likewise bisect the angle contained by the secant line and that other line which is drawn through the intersection of the diameter with the aforesaid perpendicular chord.

In the circle ABCD, whose center is E, let the diameter AC be produced to F; and from F let there first be drawn the straight line FH, tangent to the circle at B; and from B let there be drawn the chord BD, cutting the diameter at G at right angles; and let the straight lines BC, BA be joined to the extremities of the diameter. I say that both the angles CBF, CBG and (when FB is produced to H) the angles ABH, ABG are equal. [Margin a: *Elements* III.32] For the angle CBF is equal to the angle BAC in the alternate segment. [Margin b: *Elements* III.27] And the angle CBD is equal to that same angle BAC, because the arcs CB, CD are equal; [Margin c: *Elements* VI.8] (or else the angle CBG is equal to the angle BAC, because BG, in the right triangle ABC, is perpendicular to the base AC). Therefore the angles CBF, CBG will also be equal to one another. And when CB is produced to I, if from the right angles ABC, ABI the equal angles CBG, HBI are taken away — [Margin d: *Elements* I.15] for since CBF is equal to the vertical angle HBI, and the angle CBG has been shown equal to the angle CBF — there will also be

Diagram. The circle-and-tangent figure of Theorem 10. The circle ABCD has center E; from the top point A and the bottom point C long lines descend to a common apex F below, giving the whole figure a kite shape, with the secant and tangent lines and their intersections lettered around and within the circle. (The same figure is repeated at the head of the next page.)

- A (top of the circle)
- N (upper right, outside the circle)
- P (right of N)
- L (on the circle, upper right)
- O (on the circle, upper left)
- E (center-upper): center of the circle
- G (center): intersection of the perpendicular chord with the diameter
- H (left, outside, near B)
- B / I (left of the circle; I marks the production of CB)
- D (right of the circle)
- M (lower left, on the circle)
- C (bottom of the circle)
- K (lower right, on the circle)
- Q (below K, outside)
- F (bottom apex, where AC and the other lines meet)

the angle CBG equal to the angle HBI.) The remaining angles ABG, ABH will likewise be equal to one another. Which is the first part.

Diagram. The same circle-and-tangent kite figure of Theorem 10, repeated at the head of this page for the continuation of the proof. Lettering as on the previous page.

- A (top of the circle)
- N (upper right, outside)
- P (right of N)
- L (on the circle, upper right)
- O (on the circle, upper left)
- E (center-upper): center of the circle
- G (center): intersection of chord and diameter
- H (left, outside, near B)
- B / I (left of the circle)
- D (right of the circle)
- M (lower left, on the circle)
- C (bottom of the circle)
- K (lower right, on the circle)
- Q (below K, outside)

- F (bottom apex)

Next let the straight line FN be drawn, cutting the circle at K and L; and when the straight lines KGO, LGM have been drawn through G, let the straight lines KC, KA and LC, LA be joined to the extremities of the diameter. I say, again, that both the angles CLF, CLG and the angles ALG, ALN are equal; and likewise that CKF, CKG and AKG, AKL are equal. For when the straight lines EB, EK are drawn from the center, [Margin e: *Elements* III.18] the angle EBF will be right; [Margin f: corollary to *Elements* VI.8] therefore FB will be the mean proportional between EF and FG; [Margin g: *Elements* VI.17] and so the rectangle under EF, FG will be equal to the square on FB. [Margin h: *Elements* III.36] But this same square is also equal to the rectangle under LF, FK; therefore the rectangle under EF, FG will be equal to the rectangle under LF, FK. [Margin i: *Elements* VI.16] And accordingly it will be that as EF, the first, is to FK, the second, so is LF, the third, to FG, the fourth.

Wherefore, since the triangles EFK, LFG have their sides about the common angle F proportional, [Margin k: *Elements* VI.6] the angles FEK, FLG, opposite the homologous sides FK, FG, will be equal. [Margin l: *Elements* III.20] But the angle FEK, at the center, is double the angle CLK at the circumference (since they have the same base CK). Therefore the angle FLG too will be double that same angle CLK; and hence the angle FLG is bisected by the line LC — that is, the angles CLF, CLG will be equal. And when CL is produced to P, if from the right angles ALC, ALP the equal angles CLG, PLN are taken away — [Margin m: *Elements* I.15] (for since CLF has been shown equal to the angle CLG, and CLF is equal to the vertical angle PLN, CLG too will be equal to that same angle PLN) — the remaining angles ALG, ALN will also be equal. Again, since the angles CLK, CLM have been shown equal, [Margin n: *Elements* III.26] the arcs CK, CM will be equal. [Margin o: scholium to *Elements* III.29] Therefore the angles CGK, CGM are equal; [Margin p: *Elements* I.15] and hence the angles AGO, AGL, being vertical, will be equal; and accordingly the arcs AO, AL too will be equal. [Margin r: *Elements* III.27] And so the angles AKO, AKL will also be equal. And when AK is produced to Q, if from the right angles CKA, CKQ the equal angles AKG, FKQ are taken away — (for since the angle AKG has been shown equal to the angle AKL, [Margin s: *Elements* I.15] and this in turn is equal to the vertical angle FKQ, the angle AKG too will be equal to the angle FKQ) — the remaining angles CKG, CKF will also be equal to one another. All of which was to be demonstrated.

Scholium.

This theorem is very useful for describing the parallel of any great circle through a given point on the astrolabe, as is clear from proposition 18 of Book 2 of the

Astrolabium; for many things can there be demonstrated by this theorem, without those propositions that depend on the construction of the astrolabe.

Theorem 11. — Proposition 29.

To demonstrate that the construction of an equilateral and equiangular pentagon upon a given straight line, handed down by Albrecht Dürer [Albrecht Dürer], and which almost all architects and craftsmen approve, is false.

The construction is this. Let AB be the given straight line. From the centers A, B, and with the same radius AB, describe two circles intersecting at C and D. Draw CD (of any length), and with the same radius AB describe from C, through A and B, a circle cutting the line CD at E and the earlier circles at F and G. Likewise draw from F and G, through E, straight lines cutting the earlier circles at H and I. Finally, with the same radius, describe from H and I two arcs intersecting each other at K, and join the straight lines AI, IK, KH, HB. Dürer, then, supposes the pentagon ABHKI to be equilateral and equiangular. This is false; for it is indeed equilateral by the construction, but not equiangular. To make this plain, some preliminary points must first be demonstrated.

Diagram. Dürer's pentagon construction. The five-sided figure ABHKI is built up over the base AB by intersecting circles and arcs, with the apex K at the top, and the auxiliary semicircle FABG and its diameter FG at the bottom.

- K (top apex of the pentagon)
- I (upper left vertex)
- H (upper right vertex)
- L (center, below K)
- P (below L)
- D (below P): lower intersection of the first two circles
- A (left, base-line vertex)
- B (right, base-line vertex)
- E (center): upper intersection point / midpoint of arc AB
- O (bottom-center cluster, left)
- M (bottom-center cluster): midpoint of AB
- N (bottom-center cluster, right)
- F (bottom left of the semicircle)
- C (bottom center): where CD meets the semicircle diameter
- G (bottom right of the semicircle)

1. The three arcs FA, AB, BG are sixth parts of the circle, because the straight lines subtending them are radii of the circle FABG, by construction. Therefore FABG is a semicircle, whose diameter is FG; [Margin a: *Elements* III.31] and hence the angle FEG in the semicircle is right. [Margin b: scholium to *Elements* III.27] And the diameter FG is parallel to the straight line AB, because the arcs AF, BG are equal. And since — as is established from the demonstration of the construction in the scholium to propositions 10 and 11 of Book 1 of Euclid — the line CD cuts the line AB in half at M and at right angles, [Margin c: *Elements* I.29] it will cut the parallel FG likewise at right angles at C. [Margin d: scholium to *Elements* III.27] The same CD will also bisect the arc AB at E, and therefore the whole arcs EF, EG will be equal, namely quadrants; [Margin e: *Elements* IV.6] and so the straight lines EF, EG are sides of a square inscribed in the circle FABG, whose diameter is FG. [Margin f: scholium to *Elements* I.34] Therefore the angles F, G will be half-right; and hence, since the angles at C are right, [Margin g: *Elements* I.32] the angles OEM, NEM will also be half-right, and so too EOM, ENM will be half-right. [Margin h: *Elements* I.6] And accordingly both the sides EM, MO and the sides EM, MN are equal; and therefore OM, NM will be equal to one another; and likewise the whole lines OB, NA will be equal. [Margin i: *Elements* I.4] Indeed EO, EN will also be equal, because the sides EM, MO are equal to the sides EM, MN and contain equal angles, namely right angles.

Geometria Practica, Liber VIII (continued)

(Continuation of Theorem 11, Proposition 29 — that Dürer's pentagon is equilateral but not equiangular.)

[2] Next, the sides AN, AI are equal to the sides BO, BH, and the angles N, O are equal half-right angles, while each of the remaining angles I, H is less than a right angle [Margin: (k) Euclid I.19] — because each is less than a half-right angle relative to O and N, seeing that side AN falls short of side AI in the same measure as side BO falls short of side BH. Hence, by what we demonstrated at the end of Euclid's Book 1, both the bases NI, OH and the angles A, B and I, H are equal. Therefore the two angles A, B in the pentagon are equal to each other.

[3] Again, when the equal segments OE, NE are removed from the equal lines OH, NI, the remaining lines EH, EI are equal. [Margin: (l) Euclid I.5] The angles EIH, EHI are therefore equal, and consequently half-right, since HEI is a right angle.

[Margin: (m) Euclid I.5] But the angles H, I in the isosceles triangle KIH are equal as well. Therefore the whole angles H, I in the pentagon are equal.

[4] Finally, since the sides EL, EI are equal to the sides EL, EH and contain equal half-right angles, [Margin: (n) Euclid I.4] the bases LI, LH will also be equal, and the angles at L equal — and therefore right angles. From this it follows [Margin: (o) Euclid I.28] that the lines AB, HI are parallel; [Margin: (p) Euclid I.27] which is also clear from the fact that the alternate angles INA, NIH are equal, being half-right. It further follows that the line CDL, when extended, falls on the point K and bisects the angle there. For if it be said not to fall on K, [Margin: (q) scholium to Euclid I.26] then the perpendicular dropped from K to HI would bisect the base of the isosceles triangle KIH at a point other than L, which is absurd. Since, then, the sides KI, KL are equal to the sides KH, KL, and the base IL has been shown equal to the base HL, [Margin: (r) Euclid I.8] the angles at K will be equal.

With these things premised, we shall now demonstrate, in the following way, that the pentagon ABHKI is not equiangular. [Margin: (s) scholium to Euclid I.32] All five angles of any pentagon whatever — whether it be equilateral and equiangular or not — are equal to six right angles, that is, to 540 degrees. Dividing these by 5 yields a single angle of an equilateral and equiangular pentagon of 108 degrees. But in Dürer's pentagon each of the two angles A, B is greater than 108 degrees; each of the two angles H, I is less; and the angle K is greater than any of the remaining four — as we shall show. Therefore Dürer's pentagon is not equiangular. This will be made clear as follows.

Since, when the whole sine AB is set at 10,000,000, its half BM — that is, the sine of 30 degrees — is 5,000,000; and if MO, that is ME, the versed sine of 30 degrees, namely 1,339,746, is added to it, the whole BO becomes 6,339,746. Now in the triangle BHO two sides are given, BH = 10,000,000 and BO = 6,339,746, together with the angle O of 45 degrees, and also the kind [species] of the angle H, which was shown above to be less than a right angle. [Margin: (t) *On Plane [Rectilinear] Triangles*, 1] If, then, we set up the proportion:

As the side BH, 10,000,000, is to 7,071,068 — the sine of angle O, 45 degrees — so the side BO, 6,339,746, is to another [value],

there will be found the sine of angle BHO, 4,482,877½ approximately, which in the table of sines yields that angle as 26 degrees 38 minutes. If the angle BOH of 45 degrees is added to it, the sum of the angles H, O becomes 71 degrees 38 minutes. This sum, subtracted from two right angles — that is, from 180 degrees

— leaves the angle OBH of 108 degrees 22 minutes. Therefore each of the angles A, B in the pentagon is greater than the true pentagon angle of 108 degrees.

Next, when the perpendicular BP is drawn to HI, and BH is again taken as the whole sine 10,000,000, then HP will be 3,150,970, and the angle HBP 18 degrees 22 minutes — which is what remains when the right angle ABP of 90 degrees is subtracted from the angle ABH, found to be 108 degrees 22 minutes. If, then, PL = 5,000,000 is added [Margin: (u) Euclid I.34] — since it is equal to BM, the half of the whole sine — the whole HL becomes 8,150,970, the sine of angle HKL. And therefore that angle will be 54 degrees 36 minutes; doubled, it gives the whole angle HKI of 109 degrees 12 minutes, which is indeed greater than the pentagon angle of 108 degrees.

Now if the sum of the three angles A, B, K that have been found — namely 325 degrees 56 minutes — is subtracted from 540 degrees, the sum of all five angles of the pentagon, the remainder becomes the sum of the angles H, I, 214 degrees 4 minutes; and therefore each of them will be 107 degrees 2 minutes, which is less than the pentagon angle of 108 degrees. Dürer's pentagon is therefore not equiangular, but only equilateral. Yet all five angles together make up the sum of 540 degrees, just as in an equilateral and equiangular pentagon, as this table shows.

Angle	Grad.	min.
A	108	22
B	108	22
H	107	2
I	107	2
K	109	12
Sum	540	0

(In the source the left-hand column is a single vertical brace labelled "Angulus" [Angle] spanning rows A–K, with "Summa" [Sum] beside the total row.)

Scholium

There are some others who, taking the interval of any line AB, describe from the centers A, B two circles that intersect each other in C, D (as in the figure above), draw the line AD, and assert that AD is the side of a pentagon inscribed in a circle whose semidiameter is DM. But they are utterly mistaken, for AD is smaller than the side of the pentagon of the said circle. [Margin: (a) Euclid XIII.10] For since

the square on the side of the pentagon equals the squares on the side of the hexagon and the side of the decagon of the same circle [*latus... potest...*, i.e., is equal in square to], [Margin: (b) Euclid I.47] and AD is equal in square to the lines DM, MA, [Margin: (c) corollary to Euclid IV.15] and DM is the side of the hexagon in a circle whose semidiameter is DM, then AM would be the side of the decagon in the same circle — which is false. For since the side of the decagon is greater than half the side of the pentagon (because two sides of the decagon set upon the side of the pentagon form an isosceles triangle [Margin: (d) Euclid I.20] in which the two equal sides are greater than the side of the pentagon), AM — the half of AB, or of AD — will be less than the side of the decagon. Therefore AD is less than the side of the pentagon; [Margin: (e) Euclid XIII.10] since indeed the square on the side of the pentagon equals the squares on both the side of the hexagon DM and the side of the decagon, the latter being greater than AM, the half of AD, as we have said.

Theorem 12. Proposition 30.

To demonstrate that the method of finding the side of a heptagon in a given circle is transmitted incorrectly by certain authors.

Carlo Mariano of Cremona [Carlo Mariano Cremonese] published an entire booklet on finding the side of the heptagon in a given circle, in which he attempts to prove that the side of the heptagon is found by this method. Let there be a circle ABC with center D and diameter CA; on CA extended let AE be taken equal to a fourth part of the semidiameter AD, so that AE is a fifth part of the line DE. Then, describing from E, at the interval of the semidiameter AD, a circle that cuts the given circle in B, let the line AB be joined, which he says is the side of the heptagon. We shall show that this is false, as follows. If AB were the true side of the heptagon, then — when BE is drawn equal to the semidiameter DB (which happens if from B, at the interval of the semidiameter, the line DE is cut), cutting the arc AB in F — the arc AB would be bisected in F, that is, the angle ADB would be bisected; yet this does not occur in his construction, as will be demonstrated. Therefore his line AB is not the true side of the heptagon.

Diagram. A crowded woodcut combining all three heptagon constructions criticized in this theorem (Carlo Mariano's, Dürer's, and Candalla's). It shows a main circle with center D, crossed by intersecting compass arcs above; numerous chords form the several triangles of the constructions. Because many interior letters are overprinted by the ruling lines, several are hard to resolve.

- E (top, outside the circle on the CA extension): construction point for Mariano's circle
- A (top of main circle): endpoint of diameter CA
- I (just below A, interior): [?]
- F (upper left, on the circle / arc intersection): point where the drawn circle cuts arc AB
- G (interior, below I): [?]
- H (center-interior, with a dotted line running right to L): [?]
- L (right, at the end of the dotted line from H): [?]
- B (upper left of circle): intersection of Mariano's circle with the given circle
- K (left, below B): Dürer's point
- D (center of circle): center
- P (lower interior): Candalla construction point
- Q (lower interior, right of P): Candalla construction point
- M (bottom left): vertex of Candalla's triangle
- C (bottom of circle): endpoint of diameter CA
- R (bottom, right of C): Candalla construction point

For when the lines DB, DF are drawn, if AB is a seventh part of the circumference, both the angle ADB and the angle DEB [Margin: (a) Euclid I.5] (which are equal) will comprise $\frac{2}{7}$, or $\frac{4}{14}$, of two right angles. [Margin: (b) Euclid I.32] The remaining angles DAB, DBA will therefore together comprise $\frac{5}{7}$, or $\frac{10}{14}$, of two right angles, and consequently each of them will comprise $\frac{5}{14}$ of two right angles. [Margin: (c) Euclid I.32] Since, then, DAB is equal to the two angles E and ABE together, these two will likewise comprise $\frac{5}{14}$ of two right angles. But E alone comprises $\frac{4}{14}$ of two right angles; therefore ABE will comprise $\frac{1}{14}$ of two right angles. [Margin: (d) Euclid III.20] And since ADF is double ABE, on account of the common base AF, the angle ADF will comprise $\frac{2}{14}$, that is $\frac{1}{7}$, of two right angles. Since, then, the whole angle ADB comprises $\frac{2}{7}$, as was said, BDF will also comprise $\frac{1}{7}$ of two right angles; and therefore ADF and BDF will be equal.

But now let AB be the line found by the construction of the aforesaid author, so that EB is equal to DB. If, then, AB were the true side of the heptagon, the perpendicular DI would fall so as to bisect the angle ADB in F — which is not true. For, with BE put as 4, the whole CE will be 9 and DE will be 5. [Margin: (e) Euclid VI.3] Since, then, BD is to DE as BF is to FE (because the angle ADB is bisected), it will follow, componendo, that the sum of BD and DE — namely 9 — is to DE, 5, as BE is to FE. If, therefore, we set up the proportion, as 9 is to 5 so BE, 4, is to

another, there will be found $FE = 2$ [numeral: ?]. And accordingly the rectangle under BE, 4, and EF, 2 [numeral: ?], will be 8 [numeral: ?]; while the rectangle under CE, 9, and EA, 1, will be 9 — which is absurd, [Margin: (f) corollary to Euclid III.36] since these rectangles should be equal. Therefore the line DI does not fall on the point F, the intersection of the line BE with the arc AB, seeing that the rectangle under BE, EF is not equal to, but less than, the rectangle under CE, EA. And so the side of the heptagon is not correctly found by that method.

Albrecht Dürer [Albrecht Dürer], for his part, draws the perpendicular AH to KL, the side of the equilateral triangle (taking, that is, the arcs AK, AL, each of which comprises a sixth part of the circumference), and calls KH, the half of that side, the side of the heptagon — which is likewise false. For KH is very nearly equal to the line AB, which we have just shown not to be the side of the heptagon. For if the line AK were joined, the triangle AKD would be equilateral. [Margin: (g) scholium to Euclid I.26] The perpendicular KH will therefore bisect AD; and so, with DK or DA put as 4, DH will be 2. Hence, if we subtract 4, the square on DH, from 16, the square on DK, [Margin: (h) Euclid I.47] the remainder will be the square on KH, 12. And we shall find the square on AB to be just as much. For the square on BE is 16 — that is, [numeral: ?] — and the square on EG is [numeral: ?] (for the perpendicular BG bisects the base ED in the isosceles triangle EBD; and since ED is 5, DG will be $2\frac{1}{2}$, whose square is [numeral: ?]); [Margin: (i) Euclid I.47] the square on BG will be [numeral: ?]. But the square on AG is [numeral: ?], because the line AG is $1\frac{1}{2}$. [Margin: (k) Euclid I.47] Therefore the square on AB will be [numeral: ?], that is, 12 — which was the proposition.

Francesco Flussas Candalla [François de Foix, Count of Candale], a most noble and learned man, attempted to construct an isosceles triangle having each of its base angles equal to three times the remaining angle, so that by its means a heptagon might be inscribed in a given circle — as we set forth in the scholium to Proposition 16 of Euclid's Book 4. He writes as follows. Let there be an equilateral triangle DMN, in which the perpendicular DO to the base is bisected in P. Then, describing from M a circle through N and D, which the perpendicular PQ cuts in Q, let the lines QN, MQ be joined. He then claims that in the isosceles triangle MNQ each of the angles N, Q is triple the angle M. That this is false can be understood as follows. Let the perpendicular QR be dropped, to stand for the sine of the arc QN, or of the angle M, when the whole sine MO or MN is set at 10,000,000. [Margin: (l) Euclid XIV.12] Since the side DN is, in square, four-thirds of the perpendicular DO, if we set up the proportion as 4 is to 3, so 100,000,000,000,000, the square of the side DN, is to another, there will be found the square of DO, 75,000,000,000,000. [Margin: (m) scholium to Euclid II.4] Since

this is quadruple the square of PO, or QR, the square of QR will be 18,750,000,000,000, and the side QR itself will be 4,330,127 (which is too small) or 4,330,128 (which is too large). To this, in the table of sines (with the proportional part applied), there correspond 25 degrees 39 minutes 32 seconds for the arc QN, or the angle NMQ. When this is subtracted from two right angles — that is, from 180 degrees — the remainder will be the sum of the base angles at QN, 154 degrees 20 minutes 28 seconds; and therefore each of them will comprise 77 degrees 10 minutes 14 seconds, which is greater than triple the angle NMQ of 25 degrees 39 minutes 32 seconds, since that angle tripled makes only 76 degrees 58 minutes 36 seconds. What Candalla endeavors to prove is therefore false. To expose the paralogisms committed by both Carlo Mariano and Candalla is not the place here; it is enough for us to have shown that they did not correctly describe an equilateral and equiangular heptagon.

[The exact zero-counts of the three power-of-ten square values above are hard to fix at scan resolution; they are reconciled with the clearly-read side $QR = 4,330,127$ and the stated ratios ($DN^2 : DO^2 = 4 : 3$, $DO^2 = 4 \cdot QR^2$) — verify at reconciliation.]

Theorem 13. Proposition 31.

An equilateral and equiangular octagon inscribed in a circle is a mean proportional between the square circumscribed about the same circle and the square inscribed in it.

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This theorem is Oronce's [Oronce Finé], and it will easily be demonstrated thus. Let there be a circle ABCD with center E, and two diameters AC, BD cutting each other at E at right angles. When the lines AB, BC, CD, DA are joined, ABCD will be the square inscribed in the circle, as is established from the demonstration of Proposition 6 of Euclid's Book 4. Let there also be drawn, through A, B, C, D, perpendiculars to the diameters, meeting in F, G, H, I; and FGHI will be the square circumscribed about the circle, as is clear from the demonstration of Proposition 7 of Euclid's Book 4. When the diameters FH, GI are drawn, [Margin: (a) Euclid III.17] the quadrants AB, BC, CD, DA will be bisected, because the angles at the center are all equal, [Margin: (b) scholium to Euclid I.34] namely half-right; [Margin: (c) scholium to Euclid III.27] and consequently the sides of the inscribed square will also be bisected, and at right angles. And if the lines AK, KB, and so on are joined, an octagon will be described within the circle. I say that the outer square is to the octagon as the octagon is to the inner square. For since the triangles AEF, EAI are equiangular — because they have right angles and half-

right angles — [Margin: (d) Euclid VI.4] EF will be to FA, that is, to EK (for EK is equal to EA, that is, to AF), as EA — that is, as EK — is to AL, that is, to EL, [Margin: (e) Euclid I.6] since AL, EL are equal on account of the half-right angles at AE in the triangle AEL. The three lines EF, EK, EL are therefore continuously proportional. Hence the triangles AEF, AEK, AEL will also be continuously proportional, [Margin: (f) Euclid VI.1] since they are proportional to their bases EF, EK, EL; [Margin: (g) Euclid V.15] and consequently their octuples will also be continuously proportional — namely the square FGHI, the octagon AKBCD..., and the square ABCD — since the aforesaid triangles are the eighth parts of these figures, as is evident. Therefore the octagon is a mean proportional between the squares FGHI and ABCD, which was to be demonstrated.

Diagram. An outer square FGHI circumscribed about a circle, with the inscribed square ABCD set as a diamond (its vertices at the midpoints of the outer square's sides), the center E, and a regular octagon inscribed in the circle. The diagonals FH, GI and the sides AK, KB, etc. are drawn, producing an eight-pointed star pattern.

- F (top-left corner of the outer square)
- A (top; midpoint of the top side, upper vertex of the inscribed square and of the octagon)
- I (top-right corner of the outer square)
- K (on the circle, upper-left; an octagon vertex between A and B)
- L (interior point on segment EA / AK region)
- E (center of the circle)
- B (left; midpoint of the left side, left vertex of the inscribed square)
- D (right; midpoint of the right side, right vertex of the inscribed square)
- G (bottom-left corner of the outer square)
- C (bottom; midpoint of the bottom side, lower vertex of the inscribed square)
- H (bottom-right corner of the outer square)

Theorem 14. Proposition 32.

If from the diagonal of a square its side is subtracted, the remaining line will be the side of another square, whose diagonal is the line left when the found side is subtracted twice from the diagonal of the first square, or when the same found side is taken from the side of the first square.

From the diagonal BD of the square ABCD cut off the line BE equal to the side AB; and from the same diagonal take away the remainder DE twice, up to F, so that EF is equal to DE — or (what amounts to the same) let the remainder DE, or EF equal to it, be taken from the side AB, that is, from BE. I say that DE, or EF, is the side of the square whose diagonal is BF. For when the lines GH, IK are drawn through F parallel to AD, AB, [Margin: (a) corollary to Euclid II.4] GK and HI will be squares about the diagonal. I say that the line EF, or DE, is equal to the side BG of the square GK whose diagonal is BF — the remainder left after DE was subtracted twice from the diagonal BD. [Margin: (b) scholium to Euclid I.47] For since the square on DF is double the square on IF, or on AG, [Margin: (c) scholium to Euclid II.4] and quadruple the square on EF: if the square on DF is put as 4, the square on AG will be 2, and the square on EF will be 1. And therefore the square on AG will be double the square on EF. [Margin: (d) scholium to Euclid I.46] But the square on BF is also double the square on BG. [Margin: (e) Euclid VI.22] Therefore the line AG will be to the line EF as the line BF is to the line BG, since their squares are proportional, having indeed the duplicate ratio. Let BL be taken equal to BF, so that the remainder LA is equal to the remainder FE. Then AG will also be to AL as BF, or BL, is to BG; and, dividendo, GL will be to LA as GL is to BG. [Margin: (f) Euclid V.9] Wherefore AL, or EF, and BG are equal, which was to be shown.

Diagram. A square ABCD with its diagonal BD drawn from the bottom-left corner B to the top-right corner D, passing through the interior points F and E. Through F the lines GH (horizontal) and IK (vertical) are drawn parallel to the sides, marking off the two smaller squares GK and HI about the diagonal.

- A (top-left corner)
- I (top; midpoint of the top side)
- D (top-right corner)
- L (left side, above G)
- G (left; midpoint of the left side)
- E (interior, upper part of the diagonal near center)
- H (right; midpoint of the right side)
- F (interior, lower-center, on the diagonal)
- B (bottom-left corner)
- K (bottom; midpoint of the bottom side)
- C (bottom-right corner)

Problem 19. Proposition 33.

To construct an equilateral and equiangular octagon to a given height or width.

Let an equilateral and equiangular octagon be constructed to the given height AB. Describe on AB the square ABCD, and draw the diagonals AC, BD cutting each other in E, bisecting each other and at right angles. From the four corners of the square cut off, at the interval EA, equal lines AF, AG; DH, DI; BK, BL; CM, CN [?]; and join the lines HK, GM, LI, NF. I say that the octagon FHKGMLIN is equilateral and equiangular. For since the lines AH, AK; BG, BM; CL, CI; DF, DN are what remain after the line AE has been subtracted from the equal sides of the square ABCD, they will be equal to one another; and therefore their squares will be equal. [Margin: (a) Euclid I.47] Since, then, the square on KH is equal to the squares on AH, AK, it will be double both the square on AH and the square on AK. But since AB is the diagonal of the square described on AE, and the line BK equal to the side AE has been cut off, [Margin: (b) Proposition 32 of this book] the remainder AK will ... — AK, the side of the square whose diagonal is GK — GK being what is left over after AK itself has been subtracted twice from the diameter AB, or once from the side AG.

Diagram. A regular octagon inscribed in a square. The square has corners A (top-left), D (top-right), C (bottom-right), and B (bottom-left). The octagon's eight vertices lie on the square's sides: H and F on the top side (H near A, F near D); N and I on the right side (N near D, I near C); L and M on the bottom side (L near C, M near B); G and K on the left side (G near B, K near A). The center is marked E. The square's two diagonals and several connecting chords are drawn as dashed lines through E. A small shaded rhombus (unlabeled) sits just below and to the right of E, indicating one of the corner squares.

- A (top-left corner of square)
- D (top-right corner of square)
- C (bottom-right corner of square)
- B (bottom-left corner of square)
- H, F (octagon vertices on top side)
- N, I (octagon vertices on right side)
- L, M (octagon vertices on bottom side)
- G, K (octagon vertices on left side)
- E (center)

[Margin: c — scholium to I.47 (Elements).]

Therefore the square on GK will also be double the square on KA; and consequently the squares on KH and KG will be equal to each other, so that the lines KH and KG will likewise be equal. By the same reasoning we shall show that this same GK is equal to the line GM, that GM is equal to the line ML, and so on for the rest. The octagon is therefore equilateral.

[Margin: d — I.13 (Elements); e — I.4 (Elements); f — scholium to I.34 (Elements).]

Now since the pairs of angles at H, K, G, M, L, I, N, F are each equal to two right angles, and since the acute angles toward the corners of the square are all equal to one another — indeed, they are half right angles, because KH, GM, and the rest are the diagonals of the squares described on the sides AH, GB, and so on — the remaining obtuse angles in the octagon will be equal. And therefore the octagon is equiangular as well, which was what was proposed.

SCHOLIUM.

This procedure — which I received some years ago from a certain architect, though without a demonstration — is a very fine one, for it does not require dividing the circle into eight equal parts, and it constructs an octagon to a given height or width, as is plain. In order to demonstrate the procedure, I first had to demonstrate the preceding theorem; for from that theorem the proposed problem is easily accomplished, as has been shown.

PROBL. 20. PROPOS. 34.

To measure the circumference of the earth from some elevated mountain.

Toward the end of chapter 1 of the *Sphere* of Johannes de Sacrobosco [Sacrobosco], I set out the method by which Francesco Maurolico [Francesco Maurolico] taught how to investigate the earth's circumference from some elevated mountain. It runs as follows. Let BCD be the circle of the earth, and upon it let some very high mountain be chosen — Maurolico himself judged that Mount Etna in Sicily should be chosen for this task — whose height AB is to be found either by the quadrant, as we taught in Book 2, problems 2, 3, and 4, or by the geometric square, as in Book 3, problems 6, 7, 8, and 9; or rather as we set it out in the scholium to problems 7 and 9.

Diagram. The earth is drawn as a circle with center F. A vertical line runs through the summit and the center: A at the top (the mountain's summit, standing above the circle), then B where the line meets the top of the circle, then F (center),

then D at the bottom of the circle. A point C lies on the circle to the upper left, and E lies on the line between B and C (the foot of the perpendicular). The visual ray AC runs from the summit A tangent to the circle at C; AD is the vertical through the center; the chords FC and FE are drawn as dashed lines.

- A (summit of the mountain, above the circle)
- B (top of the earth's circle, base of the mountain)
- F (center of the earth)
- D (lowest point of the circle, diametrically opposite B)
- C (horizon point, where the tangent ray meets the earth)
- E (foot of the perpendicular BE, between B and C)

[Margin: a — I.47 (Elements); b — III.36 (Elements) [?].]

From A, the summit of the mountain, let there be measured that whole expanse of sea — or of land, where there are no mountains — which can be seen from there, so that the ray AC touches the surface of the sea or land at C. This is done by the methods set out in the problems just cited. From these Maurolico then finds the length of the tangent line AC; for the squares on AB and BC are equal to the square on AC (taking the arc BC as a straight line), and the square on AC is equal to the rectangle under AD and AB. When this rectangle is divided by AB, the mountain's height, the line AD comes out in the quotient; and if the mountain's height AB is subtracted from it, the earth's diameter BD is left known.

[Margin: c — corollary 2, *On the Measurement of the Circle*, Book 4 of this work.]

And consequently the circumference BCD becomes known.

But in this method of measuring the earth's circumference it is assumed that the arc BC does not differ from a straight line — which is not true when the mountain is so high that a stretch of 200 or 300 miles can be seen. For then, according to the circumference laid down by Ptolemy [Claudius Ptolemy], the arc BC would contain $3^{\circ} 11'$, or $4^{\circ} 48'$; and so the tangent line AC is not correctly obtained from the sides AB and BC. Add to this that, by the problems of Books 2 and 3 cited above, the perpendicular BE is found in the plane to which the mountain stands at right angles. We shall reduce this method to a better form in several ways, as follows.

Once the angle A has been determined, by the quadrant or the square, when the visual ray through the sight [dioptra] touches the circle of the earth — which is done most reliably when, through the sight, the Sun or some other star is

observed as it rises or sets — once, I say, the angle A has been determined, the perpendicular BE must be found, by the problems just cited;

[Margin: d — I.47 (Elements).]

and the line AE from the two lines AB and BE. For if to AE there is added BE — that is, EC,

[Margin: e — corollary 2 of III.30 (Elements).]

which is equal to BE itself — the whole tangent AC becomes known; and from it, as said above, both the earth's diameter BD and the circumference will be found.

[Margin: f — 4 of the *Rectilinear Triangles* (Clavius); g — 5 of the *Rectilinear Triangles*.]

Indeed, once the angle A is known — and hence also its complement E — both the side BE and the base AE can be found without the problems cited from Books 2 and 3, and so forth.

Or we shall proceed as follows. Once the angle A is known through the sight, the angle F — its complement, at the center — will also be known, once the line FC is drawn,

[Margin: h — III.18 (Elements).]

which will be perpendicular to AC. And since, when the line FE is drawn, the two sides EC and CF are equal to the two sides EB and BF, and they contain equal angles, namely right angles,

[Margin: i — I.14 (Elements).]

the angles at F will be equal. Therefore, since the whole angle BFC is known, as we have just said, the angle BFE will also be known, being half of it; and consequently its complement BEF will be known.

[Margin: k — 4 of the *Rectilinear Triangles* (Clavius).]

Then in the triangle ABE, from the angles A and E and the side AB, BE will be found, expressed in parts of the mountain's known height AB. And in the same way, in the triangle BEF, from the angles E and F and the side BE, the semidiameter BF will be found, expressed in parts of the side BE — that is, in parts of the mountain's height AB. And therefore the whole diameter BD becomes known, and from it the circumference of the earth, which was proposed.

Finally, we shall reach the same result in this way as well. Once the angle A is known through the sight — when the visual ray touches the earth — its complement is also known. Therefore the secant AF of this angle will be known, expressed in parts of the whole sine [radius] FC. If the whole sine BF is subtracted from this secant, the mountain's height AB is left known, expressed in parts of the whole sine BF. If, then, one sets up the proportion — as the mountain's height AB, known in parts of the whole sine, is to that same AB known in the given measure, so is the whole sine BF to a fourth term — then the semidiameter BF will come out, known in parts of the mountain's height, and so on.

PROBL. 21. PROPOS. 35.

For any prism to construct an equal cylinder, and for a pyramid an equal cone; and conversely, for a cylinder an equal prism, and for a cone an equal pyramid.

If a circle equal to the base of the prism or pyramid is constructed — by the methods we wrote at the end of Book 7 — and upon this circle a cylinder or cone of the same height as the prism or pyramid is raised, then the cylinder will be equal to the prism, and the cone equal to the pyramid. For since both the bases and the heights are equal, and since a prism and a cylinder are generated by multiplying the base into the height, while a pyramid and a cone are generated by multiplying one third of the base into the height — as we explained in Book 5, chapter 1 — it is clear that the cylinder is equal to the prism, and the cone equal to the pyramid.

If, conversely, upon the base of a cylinder or cone a square — or any other rectilinear figure — is set up equal to it, by the methods we stated at the end of Book 7, and upon this square or rectilinear figure a prism or pyramid of the same height as the cylinder or cone is made, then the prism will be equal to the cylinder, and the pyramid equal to the cone, which was proposed.

PROBL. 22. PROPOS. 36.

For a given cylinder or prism, to construct an equal cone or pyramid under the same height; and conversely, for a given cone or pyramid, to construct an equal cylinder or prism of the same height.

[Margin: a — 16 of the Sixth of this work; b — XII.10 (Elements); c — corollary of XII.7; d — XII.11 and XII.6; e — V.9.]

If both the base of the cylinder and that of the prism are tripled, and upon the tripled base a cone or pyramid of the same height is raised, then what is proposed

in the first part will have been done. For a cylinder is triple the cone that has the same base and height as it; likewise a prism is triple the pyramid that has the same base and height as it; and since the cone that has been raised is triple that same original cone, and the pyramid that has been constructed is triple that same original pyramid, the raised cone will be equal to the cylinder, and the constructed pyramid equal to the prism, which was proposed.

[Margin: f — 16 of the Sixth of this work; g — XII.11 and XII.10; h — XII.6 and XII.7; i — V.9.]

If, conversely, the bases of the cone and pyramid are diminished in triple ratio [to one third], and upon these third parts a cylinder and a prism are raised, then the cylinder will be equal to the given cone, and the prism equal to the given pyramid. For both the given cone and the raised cylinder are triple the cone that has the same base and height as the raised cylinder. Likewise, both the given pyramid and the raised prism are triple the pyramid that has the same base and height as the raised prism. Therefore the raised cylinder will be equal to the given cone, and the constructed prism equal to the given pyramid, which was proposed.

COROLLARIUM I.

[Margin: a — 35 of this work; b — 36 of this work.]

Since, therefore, every prism can be converted into a cylinder, and a pyramid into a cone — and, conversely, a cylinder into a prism, and a cone into a pyramid; and likewise a cylinder into a cone, and a prism into a pyramid — and, conversely, a cone into a cylinder, and a pyramid into a prism — it follows that, without distinction, both a cylinder and a prism can be transformed into an equal pyramid or cone, and a pyramid into an equal cylinder or prism.

COROLLARIUM II.

From these results it also plainly follows that every cylinder and prism, and likewise every cone and pyramid, can be converted into a rectangular parallelepiped with a square base.

[Margin: a — 35 and 36 of this work; b — corollary 2 of XII.7.]

For when a cylinder, cone, or pyramid has been converted into any prism whatever, if a square is made equal to the base of the prism and upon it a parallelepiped of the same height is raised, this parallelepiped will be equal to the former one, and consequently also to the proposed cylinder, cone, or pyramid.

PROBL. 23. PROPOS. 37.

To reduce a given cylinder or prism — and likewise a given cone or pyramid — of any height, into an equal one under any other given height and upon a base of any number of angles.

[Margin: a — 16 of the Sixth of this work; b — XII.15 and XII.9.]

In the ratio that the given height has to the height of the proposed solid, let the base of that same given solid be increased or diminished. For the solid constructed upon this increased or diminished base, according to the given height, will be the one sought; it will be equal to the given solid, since the heights are reciprocal to the bases. And if a base of any number of angles is made equal to the base of the constructed solid, and upon it a solid is set up under the given height, this solid too will be equal to the proposed solid.

PROBL. 24. PROPOS. 38.

For a given rectangular parallelepiped, to describe an equal cube.

Diagram. Two solids drawn in oblique projection. On the left, a tall rectangular parallelepiped (the given solid); its visible front-lower vertices are labeled D (front-left), C (front-center), and B (front-right). On the right, a cube, above which runs a horizontal segment labeled A—H—G (the two mean proportionals and the line AB); the cube's front-lower vertices are labeled F (left) and E (right), so that the cube is designated EFG (with H the second mean proportional).

- D, C, B (front-lower vertices of the left-hand parallelepiped)
- A, H, G (points on the segment above the cube; A and G its ends, H the mean proportional)
- F, E (front-lower vertices of the cube; the cube is EFG)

[Margin: a — II.14 (Elements).]

If the parallelepiped does not have a square base, let a square BCD be made equal to its base, and upon it let a rectangular parallelepiped be raised of the same height AB as the given parallelepiped;

[Margin: b — [citation not legible] [?].]

this will be equal to the given parallelepiped. For this, then, we shall construct an equal cube by the following method. Between BC, the side of the square BD, and AB, the height of the parallelepiped, let two mean proportionals EF and H be

found, such that BC is to EF as EF is to H, and as H is to AB. And upon EF, the [side] nearer to BC, let the cube EFG be constructed;

[Margin: c — lemma at VI.18 of this work; d — XI.36 (Elements).]

it will be equal to the parallelepiped ABCD. But if it should happen that the three dimensions of the given parallelepiped — whose base is not square — are in continued proportion, then the cube described on the mean dimension will be equal to the parallelepiped.

COROLLARIUM.

[Margin: a — corollary 2 of 36 of this work.]

Since, therefore, every cylinder, every prism, cone, and pyramid can be transformed into a rectangular parallelepiped, it is clearly established that for any solid of this kind an equal cube can be constructed.

PROBL. 25. PROPOS. 39.

For a given cube, to construct an equal rectangular parallelepiped under a given height, or upon a given base.

[Margin: a — VI.11 (Elements).]

Let EFG be the given cube in the preceding figure, and let AB first be the given height under which a rectangular parallelepiped equal to the cube is to be constructed. For the height AB and the side of the cube EF, let a third proportional BC be found. And let the rectangle BD be made, contained by the third proportional BC and by the line CD equal to the side of the cube EF; and upon BD let a parallelepiped be raised under the given height AB. I say that this is equal to the cube. For since the rectangular parallelepiped ABD is contained by the three lines AB, CD, BC — that is, by AB, EF, BC in continued proportion —

[Margin: b — XI.36 (Elements).]

the parallelepiped will be equal to the cube described on the mean EF, which was proposed.

Next, let the base BD be given; and if it is not a parallelogram,

[Margin: c — I.45 (Elements).]

let it be reduced to an equal parallelogram. And let the side of the cube EF have to the line AB the same ratio that the given base BD has to the base of the given

cube. (This will be done if, upon the side of the cube EF, a rectangle equal to the base BD is made, and upon the other side of this rectangle another rectangle equal to the square on the side of the cube EF.

[Margin: d — VI.1 (Elements).]

For then it will hold that, as the first rectangle — that is, the base BD — is to the second rectangle — that is, to the square, or the base of the cube — so is the base of the first rectangle, namely EF, to the base of the second rectangle.) For if upon the base BD a parallelepiped is raised at the height AB thus found,

[Margin: e — XI.34 (Elements).]

the parallelepiped and the cube will be equal, since their bases and heights are reciprocal by construction, which was proposed.

COROLLARIUM.

[Margin: a — corollary of 38 of this work; b — 38 of this work; c — 39 of this work.]

Since, therefore, for any cylinder, prism, cone, or pyramid a rectangular parallelepiped can be constructed equal to it; if for this parallelepiped an equal cube is made, and for this cube a rectangular parallelepiped equal to it under a given height or upon a given base — then the cylinder, prism, cone, or pyramid will have been transformed into an equal rectangular parallelepiped of the given height or base.

PROBL. 26. PROPOS. 40.

For a given sphere, to construct an equal cube; and for a given cube, to construct an equal sphere.

[Margin: a — 14 of Book XI (Elements) [?].]

Since, by proposition 32 of Book 1 of Archimedes [Archimedes], *On the Sphere and Cylinder*, a right cylinder whose base is the greatest circle of a sphere and whose altitude equals the diameter of that same sphere has a sesquialterate ratio [3 : 2] to the sphere;

[Margin: b — V.9 (Elements); c — corollary of 38 of this work.]

and since this same cylinder likewise has a sesquialterate ratio to a cylinder of the same base whose altitude contains two-thirds ($\frac{2}{3}$ [?]) of the sphere's diameter,

this latter cylinder will be equal to the sphere. If, therefore, for this cylinder an equal cube is made, this same cube will be equal to the given sphere, which was proposed.

[Margin: d — XII.11 (Elements); e — V.9 (Elements); f — corollary of 38 of this work.]

Or: since, by the same proposition 32 of Archimedes, a sphere is quadruple of a cone whose base is the greatest circle of the sphere and whose altitude equals the semidiameter of the sphere; and since a cone of the same altitude, having as its base a circle four times [greater than] the greatest circle in the sphere — that is, having as its base a circle whose semidiameter equals the diameter of the greatest circle — is also quadruple that [first] cone, this latter cone will be equal to the sphere. If, therefore, for this cone an equal cube is made, this same cube will be equal to the given sphere, which was proposed.

Diagram. Three vertical line segments of unequal length set side by side, labeled beneath, from left to right, B, A, and C. A is the side of the given cube (the middle segment); B and C are the other two lines used in the construction, C being the third proportional found for B and A.

- B (left segment)
- A (middle segment — the side of the given cube)
- C (right segment — the third proportional)

[Margin: g — 40 of this work.]

Conversely, let a sphere equal to the cube on the side A be constructed. Let a cube on the side B be made equal to the sphere of diameter A; and for the two lines B and A let a third proportional C be found. I say that the sphere of diameter C is equal to the given cube on the side A.

[Margin: h — scholium to XI.37 (Elements).]

For as the cube on the side B is to the cube on the side A, so is the cube on the side A to the cube on the side C. And as the cube on the side A is to the cube on the side C, so is the sphere of diameter A to the sphere of diameter C,

[Margin: i — XI.33 and XII.18 (Elements).]

because each of these ratios is the triplicate of the ratio of A to C. Therefore it will also hold that, as the cube on the side B is to the cube on the side A, so is the

sphere of diameter A to the sphere of diameter C. But the cube on the side B is, by construction, equal to the sphere of diameter A.

[Margin: k — 14 of Book V (Elements).]

Therefore the cube on the side A is also equal to the sphere of diameter C, which was proposed.

Geometria Practica — Liber VIII (continued)

COROLLARIUM I.

Now, a figure with any number of sides — whether regular or not — can be made equal to the base of a cube; and if a rectangular solid is raised on that figure to the height of the cube, this solid is equal to the cube. It follows that one can construct, on a base of any number of angles, a rectangular solid equal to a given sphere: first construct a cube equal to the sphere, and then a rectangular solid equal to that cube, as was just explained. Likewise, since a pyramid can be constructed equal to any prism, if a pyramid is made equal to the cube (treated as a prism) that is equal to the sphere, this pyramid too will be equal to the sphere. Indeed, since a cone can be made equal to any cylinder: if a cylinder is built equal to the sphere — on a base equal to the greatest circle in the sphere, and with a height amounting to two-thirds [$\frac{2}{3}$] of the diameter, as we showed at the beginning of this proposition — and then a cone equal to that cylinder is set up, the cone too will be equal to the given sphere.

Conversely, since a cube can be constructed equal to any prism, if a sphere is made equal to that cube, this same sphere will be equal to the given prism set on a base of any number of angles.

[Margin: a — Euclid VI.25 (25. sexti); b — 2nd corollary of XII.7 (2. coroll. 7. duodec.); c — 40. of this book (huius); d — 36. huius; e — 36. huius; f — 37. huius; g — 40. huius]

COROLLARIUM II.

It also follows that a sphere can be constructed equal to any regular solid. For the cube this was shown in this Proposition 40. For the tetrahedron, or regular pyramid, it is evident: if a parallelepiped is made equal to the pyramid, and a cube equal to that parallelepiped, and (making a sphere equal to the cube that is equal to the sphere, as though to a prism) this same sphere will be equal to the

tetrahedron, that is, to the regular pyramid. For the octahedron, icosahedron, and dodecahedron the matter will be carried out as follows. Let a square be made equal to all the faces of the regular solid taken together, by the method we handed down at the end of Book 2 of Euclid — or better, by what we set out in Book 4, Chapter 4, Number 4 of this Geometry. On this square erect a pyramid whose height equals the perpendicular drawn from the center of the solid to any face — that is, the height of one of the pyramids into which the solid is divided from its center. This pyramid will be equal to the regular solid. For this four-sided [square-based] pyramid stands to one [component] pyramid of the regular solid, and likewise all the pyramids of the regular solid stand to one pyramid, as the base of the former stands to a single base — since in the octahedron the ratio in both cases is eightfold, in the icosahedron twentyfold, and in the dodecahedron twelvefold. Therefore, if a cube is constructed equal to that whole pyramid, as was said just above for the tetrahedron, and finally a sphere is built equal to that cube, this same sphere will be equal to that pyramid — that is, to the regular solid.

[Margin: a — 2nd corollary of 36. huius (2. coroll. 36. huius); b — 38. huius; c — Euclid V.9 (9. quin.); d — XII.6 (6. duod.)]

PROBL. 27. PROPOS. 41.

To make a single cube equal to two or more cubes.

If, on the upper face of the first cube, a rectangular parallelepiped is constructed equal to the second cube — so that a single parallelepiped becomes equal to the two cubes — and on the upper face of this parallelepiped another parallelepiped equal to the third cube, and so on if more cubes are present, the resulting parallelepiped will be equal to the given cubes. If, then, a cube is made equal to it, the required construction is accomplished.

[Margin: a — 39. huius; b — 38. huius]

SCHOLIVM.

By the same technique a cube can be constructed equal to any number of solid figures that are not cubes — provided, of course, that they are first reduced to a single parallelepiped, and so on.

[Margin: a — 37. huius]

PROBL. 28. PROPOS. 42.

Given a cube, to construct a regular solid — whichever of the five you choose — equal to it.

Let the given cube have side A, and suppose, for example, that a dodecahedron is to be constructed equal to it. Construct any dodecahedron whatever, with side B; and, by what was said in the second corollary of the preceding proposition, make a cube equal to it, with side C. Then, for the three lines C, A, B, find the fourth proportional D.

C. A. B. D.

I say that the dodecahedron constructed on side D is equal to the given cube of side A. For, as is clear from the demonstration of Euclid XI.37, the cube on side C is to the cube on side A as the dodecahedron on side B is to the dodecahedron on side D. But by construction the cube on side C is equal to the dodecahedron on side B; therefore the cube on side A will also be equal to the dodecahedron on side D — which is what was proposed.

[Margin: a — Euclid XIII.17 (17. tertij decimi); b — Euclid VI.12 (12. sexti); c — Euclid V.14 (14. quin.)]

PROBL. 29. PROPOS. 43.

To subtract a smaller cube from a larger, and to exhibit a cube equal to the remainder.

On the base of the larger cube construct a parallelepiped equal to the smaller cube. From a side of the larger cube cut off a straight line equal to the height of the constructed parallelepiped. If through the point of section a plane is drawn parallel to the bases

[Margin: a — 39. huius]

of the cube, the parallelepiped that is cut away will be equal to the parallelepiped that was constructed, since it has the same base and height as it — that is, equal to the smaller cube. If, then, a cube is made equal to the remaining parallelepiped, the required construction is accomplished.

[Margin: a — 38. huius]

SCHOLIVM.

The same can be done with other solid figures — provided they are first reduced to rectangular parallelepipeds (when they are not already parallelepipeds), and then the parallelepipeds to cubes, and so on.

[Margin: a — 1st and 2nd corollary of 36. huius (1. & 2. coroll. 36. huius); b — 38. huius]

PROBL. 30. PROPOS. 44.

Given two or more spheres, to establish a single sphere equal to them.

Given the spheres, construct cubes equal to them. Then make a single cube equal to these, which will therefore also be equal to the given spheres. If, then, a sphere is built equal to this cube, what is required is accomplished.

[Margin: a — 40. huius; b — 41. huius; c — 40. huius]

PROBL. 31. PROPOS. 45.

To subtract a smaller sphere from a larger, and to exhibit a sphere equal to the remainder.

Reduce each sphere to a cube. Then, having subtracted the smaller cube from the larger, if a sphere is made equal to the remainder, the required construction is accomplished.

[Margin: a — 40. huius; b — 43. huius; c — 40. huius]

PROBL. 32. PROPOS. 46.

To cut a given cube or parallelepiped according to a given ratio.

For if one edge on the base of the cube or parallelepiped is cut according to the given ratio, and through the point of section a plane is drawn parallel to the two upright faces of the solid, dividing the solid into two parallelepipeds, these parallelepipeds will have the same ratio to one another as their bases. For the bases have the same ratio to one another as the segments of the edge divided according to the given ratio. Since, then, the bases have the same ratio as the segments of the edge divided according to the given ratio, what was proposed is established.

[Margin: a — Euclid XI.32 (32. undec.); b — Euclid VI.1 (1. sexti)]

SCHOLIVM.

In just the same way any prism or cylinder will be cut according to a given ratio, if its height is cut in the given ratio and a plane is drawn through the point of section parallel to the bases. For this will cut both the prism and the cylinder in the given ratio.

[Margin: a — scholium of XII.14 (schol. 14. duodec.); b — XII.14 (14. duod.)]

PROBL. 33. PROPOS. 47.

To describe with a compass a figure resembling an ellipse, which they call an oval [figuram ovatam].

I should like to close this book of miscellaneous problems with that well-known problem by which craftsmen, using a compass, commonly describe the oval figure resembling an ellipse, so that no angles appear anywhere — for geometers not infrequently employ such a figure in their drawings. In Book 1 of our *Gnomonics*, in the scholium to Proposition 8, I did explain by what method the true ellipse, which is a conic section, is to be described; but here I propose to describe a similar figure made up of arcs of circles. I shall proceed, then, as follows, gathering from various writers. Construct two equilateral or isosceles triangles on a common base AC, on opposite sides: ABC and ADC. (Equilateral triangles make the more graceful figure, as experience will teach you.) Then, with the two sides as radii, describe from A and C two arcs EFG and HIK, extended as far as the produced sides. For if from B and D, through E, K, G, H, other arcs are described, these will touch the earlier arcs at the points E, K, G, H, and so will not cut them; and the oval figure will be constituted.

Diagram. An egg-shaped (oval) figure built from circular arcs, inscribed with the construction lines of the four-center method. A horizontal axis runs left–right through the middle and a vertical axis top–to–bottom; the two cross at the center. The oval is broader at the bottom and narrower at the top. Interior construction lines (the two overlapping triangles / radii from the four centers) are drawn to the labelled points.

- F (top apex, on the vertical axis): construction point / top of the oval
- G (upper left, inside): arc endpoint
- E (upper right, inside): arc endpoint
- A (upper center): a center of construction
- L (left end of the horizontal axis): axis terminus

- B (left of center, on the horizontal axis): a center of construction
- M (right end of the horizontal axis): axis terminus
- N (lower-center-left; printed as a reversed "I"): construction point
- D (lower center, below the horizontal axis): a center of construction
- O (lower-center-right): construction point
- C (lower center): a center of construction
- H (bottom left): arc endpoint
- I (bottom apex, on the vertical axis): construction point / bottom of the oval
- K (bottom right): arc endpoint

[Margin: a — 25. primi [?] (Euclid I.25? — reference letter/number partly abraded); b — scholium of III.13 (schol. 13. tertij)]

You readily see that, from the same centers A, B, C, D, various figures can be described, according as the arcs EFG and HIK are larger or smaller, as appears in the figure.

But if the triangles are constructed isosceles, the sides AB, CB, and so on, can be made either larger or smaller than the base AC. In our figure they are smaller.

You may also, if you wish, first describe from the centers B and D the arcs EMK and GLH at any interval you please, according to the width of the figure to be drawn; and then from the centers A and C draw the smaller arcs EFG and HIK.

Indeed, we can achieve the same thing without constructing the triangles, in this way. Draw two straight lines AC and BD cutting one another at right angles at N; take NA and NC equal, of whatever length you please for the length of the figure, and from A and C describe arcs of circles EFG and HIK, small or large according as you wish the ends of the figure along its length to be narrower or broader. Then take two other equal straight lines NB and ND, of whatever length — the farther the points B and D are from N, the narrower the figure turns out; the less far, the broader; but use, the best teacher, will readily show you how long NB and ND ought to be. From B and D, through the centers A and C, draw straight lines cutting the earlier arcs at E, K, and so on. For if from B and D, through the points E, K, and so on, two more arcs are described, the figure resembling an ellipse will be complete.

But so that you may see that more graceful figures are described when the triangles ABC and ADC are equilateral — which is roughly the rule the craftsmen usually keep — we have here drawn two figures. In the smaller, the side BA is double the line AE; in the larger, it is equal to it, and so on.

Now the area of the oval figure described with the help of an equilateral triangle — which craftsmen not infrequently want — is easily found in this way. The sector BEK is one-sixth of the circle whose radius BE is known, namely the side of the equilateral triangle BEK, which in the larger figure is double the side AB (taken at will), and in the smaller figure is one-and-a-half times that same side AB. So, once the area of that circle has been found, as we taught in Book 4, Chapter 7, if from its sixth part the equilateral triangle BEK is subtracted — whose area is found by what was set out in the same Book 4, Chapter 2, Number 5 — the remainder gives the segment EK, and hence also GH, as known. Likewise the sector AGE is one-third of the circle whose radius AG is known, namely equal to the assumed side AB, or half of it. If, then, from the third part of the area of that circle the area of the isosceles triangle AGE is taken away — which is found by what we wrote in Book 4, Chapter 2, Number 4 — the segment GFE remains, known, and therefore also the segment HIK. Accordingly, if to the four known segments the area of the rectangle EGHK is added, the area of the whole figure will be known. And the area of this rectangle is found from the teaching of Book 4, Chapter 1, since the side EK is the chord of a sixth part of the circle — that is, equal to the radius — while EF is the sine of 60 degrees, that is, half the chord of 120 degrees.

Diagram. The oval area-computation figure: an outer oval and, concentric within it, a smaller oval, with an inscribed hexagram (two overlapping equilateral triangles forming a six-pointed star) and a dotted inscribed rectangle. Most labels appear twice — once on the outer oval, once on the inner. The vertical axis runs F (top) to I (bottom); the horizontal axis runs B (left) to D (right).

- F (top apex, outer and inner, on the vertical axis)
- G (upper left — outer and inner): star vertex / rectangle corner
- E (upper right — outer and inner): star vertex / rectangle corner
- A (upper center): center point
- B (left, on the horizontal axis): star vertex
- D (right, on the horizontal axis): star vertex
- C (lower center): center point
- H (lower left — outer and inner): star vertex / rectangle corner
- K (lower right — outer and inner): star vertex / rectangle corner
- I (bottom apex, outer and inner, on the vertical axis)

[Margin: The area of the oval figure described here (Area figurae ovatae hic descripta).]

But since by this method a figure cannot be described to a given length and width — for if the length FI is chosen, it will be unknown how great the width will be (because the arcs described from B and D rarely pass through the chosen points of the width), while if the width LM is chosen in the first figure, the future length will be unknown (since the arcs described from A and C likewise rarely pass through the chosen points of the length), as is clear — we shall show, following Giovanni Battista Benedetti [Benedetti], how a figure resembling an ellipse can be described when both the length and the width are given.

So let the given length be AB and the given width CD, cutting one another in half and at right angles at E. From the width CD cut off a straight line DF, of any length beyond E, but greater than the interval between the point F and A, the end of the length. Unless this is done, the oval figure cannot be described. Then, with center F and radius FD, describe a circle DG, which will necessarily pass beyond the point A, since the radius FD has been set greater than the interval FA. Draw FG parallel to the length AB, cutting the described circle at G; from G, through A, draw a straight line cutting the same circle at the point H, from which draw HIK parallel to the width CD; and join HF, cutting the length AB at L. Then FG and FH will be equal, as lines from the center F to the circumference. And since the triangles HGF and HAL are similar, GF will be to FH as AL is to LH. Since, then, GF is equal to FH, AL will also be equal to LH. Therefore the circle AH, described from L through A, will pass through H and there touch the earlier circle DG. If, then, EO is taken equal to EI, and EM equal to EL, and POQ is drawn through O parallel to the straight line CD, and from center M, with radius LH (or MP), a circle PBQ is described, this too will touch the earlier circle at P. Finally, if EN is taken equal to EF, and from N, with the radius FD of the earlier circle, a circle KCQ is described, it will touch the circles HAK and PBQ at K and Q, and the oval figure will be complete.

Diagram. Benedetti's construction of an oval to a given length and width, assembled from four circular arcs. A large circle occupies the center; the oval's outline is formed by joining arcs, and a crescent-like outer arc bulges to the right. The length axis runs vertically (G at top, B at bottom); the width axis runs horizontally.

- G (top): endpoint on the length axis / point on circle
- A (upper center): point on the length axis
- K (left, on the width line): construction point
- I (center, above L): division point
- L (center): intersection point on the length

- H (right, on the circle): point where the line from G through A meets the circle
- C (left, on the width line): end of the width
- F (center-left): center of the first circle
- N (center; printed as a reversed "H"): construction center
- D (center-right, on the width line): end of the width / cut-off point
- E (center): intersection of length and width axes
- M (lower center): center of a circle
- O (lower center): division point
- Q (lower right): point of tangency
- P (right, below H): point of tangency
- B (bottom): endpoint on the length axis

[Margin: a — corollary of Euclid VI.4 (coroll. 4. sexti); b — Euclid VI.4 (4. sexti); c — scholium of III.13 (schol. 13. tertij)]

But since, as was said, it is evident from the construction that unless the width CD is great enough for the line DF to be cut off from it, the figure cannot be described by this method — so that the length cannot be assumed relative to the width — the operation will be carried out another way, taking any length AB and width CD whatever. In the first figure, let the given length and width FI and LM cut one another in half at N and at right angles; take the straight lines FA and IC equal, and less than half the width LN; and from A and C, through F and I, describe the circles EFG and HIK. [Text continues.]

[Continuation of *Geometria Practica*, Liber VIII — the closing construction of the final problem (Problem 33 / Proposition 47), followed by the Scholium that appends the table of squares and cubes.]

Then, with MO taken equal to the semidiameter AF, let OA be joined. From O to the centre A draw PB so that it bisects OA at right angles in P, cutting LM — produced too, if need be — in B; and let BA be carried out to the circle GFE. Now the two sides OP, PB are equal to the two sides AP, PB, and they contain right — that is, equal — angles; [a] therefore the bases OB, AB will be equal. And when the equal segments OM, AE are added (for MO was taken equal to the semidiameter FA, or IC), the whole lines BE, BM will be equal. So a circle described from B through M will pass through E, [b] and there it will touch the small circle GFE. In the same way it will touch the small circle HIK. If, then, ND is taken equal to NB and a circle is described from D through L, touching the same earlier small circles at G and H, the figure will be complete.

But the smaller the semidiameter FA is taken in comparison with MN, the more exactly the centre B will be found, as is clear.

[Margin: a — Euclid, *Elements* I.4 (4. *prim[i]*)] [Margin: b — scholium to Euclid, *Elements* III.13 (*schol. 13. tertij*)]

Scholium [Scholium]

At the request of many it seems good to add here, by way of an epilogue, a table of the squares and cubes generated from the numbers 1 up to 1000. For this table is of manifold and notable use — both in other mathematical matters and, above all, in extracting square and cube roots from large numbers, as I shall explain briefly after the table. I have not carried the table beyond the root 1000, contenting myself with roots that do not exceed three figures. For if it were extended up to the root 10000, so that the roots would run to four figures, a table ten times as large would have to be made. If, however, anyone wishes to extend it, he will find at the end of the table rules by which he may easily do so — even though the most learned Magini [Giovanni Antonio Magini], in his *Tabula Tetragonica*, has already carried the table of squares up to the root 10000.

THERE FOLLOWS A TABLE OF SQUARES AND CUBES whose roots are not greater than 1000.

Table of Squares and Cubes [Tabula Quadratorum, & Cuborum]

The source prints this table across two pages, each laid out in three side-by-side column-groups (Radices / Quadrati / Cubi). It is reproduced here as two tables — one per source page — with columns **Radices** (root), **Quadrati** (square), **Cubi** (cube). All values were read cell-by-cell on the leaf, at high magnification.

Roots 1–180 (first page)

Radices	Quadrati	Cubi
1	1	1
2	4	8
3	9	27
4	16	64
5	25	125
6	36	216

Radices	Quadrati	Cubi
7	49	343
8	64	512
9	81	729
10	100	1000
11	121	1331
12	144	1728
13	169	2197
14	196	2744
15	225	3375
16	256	4096
17	289	4913
18	324	5832
19	361	6859
20	400	8000
21	441	9261
22	484	10648
23	529	12167
24	576	13824
25	625	15625
26	676	17576
27	729	19683
28	784	21952
29	841	24389
30	900	27000
31	961	29791
32	1024	32768
33	1089	35937
34	1156	39304
35	1225	42875
36	1296	46656
37	1369	50653

Radices	Quadrati	Cubi
38	1444	54872
39	1521	59319
40	1600	64000
41	1681	68921
42	1764	74088
43	1849	79507
44	1936	85184
45	2025	91125
46	2116	97336
47	2209	103823
48	2304	110592
49	2401	117649
50	2500	125000
51	2601	132651
52	2704	140608
53	2809	148877
54	2916	157464
55	3025	166375
56	3136	175616
57	3249	185193
58	3364	195112
59	3481	205379
60	3600	216000
61	3721	226981
62	3844	238328
63	3969	250047
64	4096	262144
65	4225	274625
66	4356	287496
67	4489	300763
68	4624	314432

Radices	Quadrati	Cubi
69	4761	328509
70	4900	343000
71	5041	357911
72	5184	373248
73	5329	389017
74	5476	405224
75	5625	421875
76	5776	438976
77	5929	456533
78	6084	474552
79	6241	493039
80	6400	512000
81	6561	531441
82	6724	551368
83	6889	571787
84	7056	592704
85	7225	614125
86	7396	636056
87	7569	658503
88	7744	681472
89	7921	704969
90	8100	729000
91	8281	753571
92	8464	778688
93	8649	804357
94	8836	830584
95	9025	857375
96	9216	884736
97	9409	912673
98	9604	941192
99	9801	970299

Radices	Quadrati	Cubi
100	10000	1000000
101	10201	1030301
102	10404	1061208
103	10609	1092727
104	10816	1124864
105	11025	1157625
106	11236	1191016
107	11449	1225043
108	11664	1259712
109	11881	1295029
110	12100	1331000
111	12321	1367631
112	12544	1404928
113	12769	1442897
114	12996	1481544
115	13225	1520875
116	13456	1560896
117	13689	1601613
118	13924	1643032
119	14161	1685159
120	14400	1728000
121	14641	1771561
122	14884	1815848
123	15129	1860867
124	15376 [?]	1906624
125	15625	1953125
126	15876	2000376
127	16129	2048383
128	16384	2097152
129	16641	2146689
130	16900	2197000

Radices	Quadrati	Cubi
131	17161	2248091
132	17424	2299968
133	17689	2352637
134	17956	2406104
135	18225	2460375
136	18496	2515456
137	18769	2571353
138	19044	2628072
139	19321	2685619
140	19600	2744000
141	19881	2803221
142	20164	2863288
143	20449	2924207
144	20736	2985984
145	21025	3048625
146	21316	3112136
147	21609	3176523
148	21904	3241792
149	22201	3307949
150	22500	3375000
151	22801	3442951
152	23104	3511808
153	23409	3581577
154	23716	3652264
155	24025	3723875
156	24336	3796416
157	24649	3869893
158	24964	3944312
159	25281	4019679
160	25600	4096000
161	25921	4173281

Radices	Quadrati	Cubi
162	26244	4251528
163	26569	4330747
164	26896	4410944
165	27225	4492125
166	27556	4574296
167	27889	4657463
168	28224	4741632
169	28561	4826809
170	28900	4913000
171	29241	5000211
172	29584	5088448
173	29929	5177717
174	30276	5268024
175	30625	5359375
176	30976	5451776
177	31329	5545233
178	31684	5639752
179	32041	5735339
180	32400	5832000

Roots 181–360 (second page)

Radices	Quadrati	Cubi
181	32761	5929741
182	33124	6028568
183	33489	6128487
184	33856	6229504
185	34225	6331625
186	34596	6434856
187	34969	6539203
188	35344	6644672
189	35721	6751269

Radices	Quadrati	Cubi
190	36100	6859000
191	36481	6967871
192	36864	7077888
193	37249	7189057
194	37636	7301384
195	38025	7414875
196	38416	7529536
197	38809	7645373
198	39204	7762392
199	39601	7880599
200	40000	8000000
201	40401	8120601
202	40804	8242408
203	41209	8365427
204	41616	8489664
205	42025	8615125
206	42436	8741816
207	42849	8869743
208	43264	8998912
209	43681	9129329
210	44100	9261000
211	44521	9393931
212	44944	9528128
213	45369	9663597
214	45796	9800344
215	46225	9938375
216	46656	10077696
217	47089	10218313
218	47524	10360232
219	47961	10503459
220	48400	10648000

Radices	Quadrati	Cubi
221	48841	10793861
222	49284	10941048
223	49729	11089567
224	50176	11239424
225	50625	11390625
226	51076	11543176
227	51529	11697083
228	51984	11852352
229	52441	12008989
230	52900	12167000
231	53361	12326391
232	53824	12487168
233	54289	12649337
234	54756	12812904
235	55225	12977875
236	55696	13144256
237	56169	13312053
238	56644	13481272
239	57121	13651919
240	57600	13824000
241	58081	13997521
242	58564	14172488
243	59049	14348907
244	59536	14526784
245	60025	14706125
246	60516	14886936
247	61009	15069223
248	61504	15252992
249	62001	15438249
250	62500	15625000
251	63001	15813251

Radices	Quadrati	Cubi
252	63504	16003008
253	64009	16194277
254	64516	16387064
255	65025	16581375
256	65536	16777216
257	66049	16974593
258	66564	17173512
259	67081	17373979
260	67600	17576000
261	68121	17779581
262	68644	17984728
263	69169	18191447
264	69696	18399744
265	70225	18609625
266	70756	18821096
267	71289	19034163
268	71824	19248832
269	72361	19465109
270	72900	19683000
271	73441	19902511
272	73984	20123648
273	74529	20346417
274	75076	20570824
275	75625	20796875
276	76176	21024576
277	76729	21253933
278	77284	21484952
279	77841	21717639
280	78400	21952000
281	78961	22188041
282	79524	22425768

Radices	Quadrati	Cubi
283	80089	22665187
284	80656	22906304
285	81225	23149125
286	81796	23393656
287	82369	23639903
288	82944	23887872
289	83521	24137569
290	84100	24389000
291	84681	24642171
292	85264	24897088
293	85849	25153757
294	86436	25412184
295	87025	25672375
296	87616	25934336
297	88209	26198073
298	88804	26463592
299	89401	26730899
300	90000	27000000
301	90601	27270901
302	91204	27543608
303	91809	27818127
304	92416	28094464
305	93025	28372625
306	93636	28652616
307	94249	28934443
308	94864	29218112
309	95481	29503629
310	96100	29791000
311	96721	30080231
312	97344	30371328
313	97969	30664297

Radices	Quadrati	Cubi
314	98596	30959144
315	99225	31255875
316	99856	31554496
317	100489	32855013 [recte: 31855013]
318	101124	32157432
319	101761	32461759
320	102400	32768000
321	103041	33076161
322	103684	33386248
323	104329	33698267
324	104976	34012224
325	105625	34328125
326	106276	34645976
327	106929	34965783
328	107584	35287552
329	108241	35611289
330	108900	35937000
331	106561 [recte: 109561]	36264691
332	110224	36594368
333	110889	36926037
334	111556	37259704
335	112225	37595375
336	112896	37933056
337	113569	38272753
338	114244	38614472
339	114921	38958219
340	115600	39304000
341	116281	39651821
342	116964	40001688
343	117649	40353607
344	118336	40707584

Radices	Quadrati	Cubi
345	119025	41063625
346	119716	41421736
347	120409	41781923
348	121104	42144192
349	121801	42508549
350	122500	42875000
351	123201	43243551
352	123904	43614208
353	124609	43986977
354	125316	44361864
355	126025	44738875
356	126736	45118016
357	127449	45499293
358	128164	45882712
359	128881	46268279
360	129600	46656000

Book VIII

Table of Squares and Cubes

[Tabula Quadratorum, & Cuborum]

The table of roots, squares, and cubes continues (begun earlier in Book VIII). On each printed page the entries are laid out in three side-by-side blocks, each block having three columns — **Radices** (the root), **Quadrati** (its square), and **Cubi** (its cube). For clarity the entries are transcribed below in a single ascending run of the root per source page; no values have been reordered otherwise.

Roots 361–540 **

Radices (root)	Quadrati (square)	Cubi (cube)
361	130321	47045881
362	131044	47437928
363	131769	47832147

Radices (root)	Quadrati (square)	Cubi (cube)
364	132496	48228544
365	133225	48627125
366	133956	49027896
367	134689	49430863
368	135424	49836032
369	136161	50243409
370	136900	50653000
371	137641	51064811
372	138384	51478848
373	139129	51895117
374	139876	52313624
375	140625	52734375
376	141376	53157376
377	142129	53582633
378	142884	54010152
379	143641	54439939
380	144400	54872000
381	145161	55306341
382	145924	55742968
383	146689	56181887
384	147456	56623104
385	148225	57066625
386	148996	57512456
387	149769	57960603
388	150544	58411072
389	151321	58863869
390	152100	59319000
391	152881	59776471
392	153664	60236288
393	154449	60698457
394	155236	61162984

Radices (root)	Quadrati (square)	Cubi (cube)
395	156025	61629875
396	156816	62099136
397	157609	62570773
398	158404	63044792
399	159201	63521199
400	160000	64000000
401	160801	64481201
402	161604	64964808
403	162409	65450827
404	163216	65939264
405	164025	66430125
406	164836	66923416
407	165649	67419143
408	166464	67917312
409	167281	68417929
410	168100	68921000
411	168921	69426531
412	169744	69934528
413	170569	70444997
414	171396	70957944
415	172225	71473375
416	173056	71991296
417	173889	72511713
418	174724	73034632
419	175561	73560059
420	176400	74088000
421	177241	74618461
422	178084	75151448
423	178929	75686967
424	179776	76225024
425	180625	76765625

Radices (root)	Quadrati (square)	Cubi (cube)
426	181476	77308776
427	182329	77854483
428	183184	78402752
429	184041	78953589
430	184900	79507000
431	185761	80062991
432	186624	80621568
433	187489	81182737
434	188356	81746504
435	189225	82312875
436	190096	82881856
437	190969	83453453
438	191844	84027672
439	192721	84604519
440	193600	85184000
441	194481	85766121
442	195364	86350888
443	196249	86938307
444	197136	87528384
445	198025	88121125
446	198916	88716536
447	199809	89314623
448	200704	89915392
449	201601	90518849
450	202500	91125000
451	203401	91733851
452	204304	92345408
453	205209	92959677
454	206116	93576664
455	207025	94196375
456	207936	94818816

Radices (root)	Quadrati (square)	Cubi (cube)
457	208849	95443993
458	209764	96071912
459	210681	96702579
460	211600	97336000
461	212521	97972181
462	213444	98611128
463	214369	99252847
464	215296	99897344
465	216225	100544625
466	217156	101194696
467	218089	101847563
468	219024	102503232
469	219961	103161709
470	220900	103823000
471	221841	104487111
472	222783 [recte: 222784]	105154048
473	223729	105823817
474	224676	106496424
475	225625	107171875
476	226576	107850176
477	227529	108531333
478	228484	109215352
479	229441	109902239
480	230400	110592000
481	231361	111284641
482	232324	111980168
483	233289	112678587
484	234256	113379904
485	235225	114084125
486	236196	114791256
487	237169	115501303

Radices (root)	Quadrati (square)	Cubi (cube)
488	238144	116214272
489	239121	116930169
490	240100	117649000
491	241081	118370771
492	242064	119095488
493	243049	119823157
494	244036	120553784
495	245025	121287375
496	247016 [recte: 246016]	122023936
497	247009	122763473
498	248004	123505992
499	249001	124251499
500	250000	125000000
501	251001	125751501
502	252004	126506008
503	253009	127263527
504	254016	128024064
505	255025	128787625
506	256036	129554216
507	257049	130323843
508	258064	131096512
509	259081	131872229
510	260100	132651000
511	261121	133432831
512	262144	134217728
513	263169	135005697
514	264196	135796744
515	265225	136590875
516	266256	137388096
517	267289	138188413
518	268324	138991832

Radices (root)	Quadrati (square)	Cubi (cube)
519	269361	139798359
520	270400	140608000
521	271441	141420761
522	272484	142236648
523	273529	143055667
524	274576	143877824
525	275625	144703125
526	276676	145531576
527	277729	146363183
528	278784	147197952
529	279841	148035889
530	280900	148877000
531	281961	149721291
532	283024	150568768
533	284089	151419437
534	285156	152273304
535	286225	153130375
536	287296	153990656
537	288369	154854153
538	289444	155720872
539	290521	156590819
540	291600	157464000

Roots 541–720 **

Radices (root)	Quadrati (square)	Cubi (cube)
541	292681	158340421
542	293764	159220088
543	294849	160103007
544	295936	160989184
545	297025	161878625
546	298116	162771336

Radices (root)	Quadrati (square)	Cubi (cube)
547	299209	163667323
548	300304	164566592
549	301401	165469149
550	302500	166375000
551	303601	167284151
552	304704	168196608
553	305809	169112377
554	306916	170031464
555	308025	170953875
556	309136	171879616
557	310249	172808693
558	311364	173741112
559	312481	174676879
560	313600	175616000
561	314721	176558481
562	315844	177504328
563	316969	178453547
564	318096	179406144
565	319225	180362125
566	320356	181321496
567	321489	182284263
568	322624	183250432
569	323761	184220009
570	324900	185193000
571	326041	186169411
572	327184	187149248
573	328329	188132517
574	329476	189119224
575	330625	190109375
576	331776	191102976
577	332929	192100033

Radices (root)	Quadrati (square)	Cubi (cube)
578	334084	193100552
579	335241	194104539
580	336400	195112000
581	337561	196122941
582	338724	197137368
583	339889	198155287
584	341056	199176704
585	342225	200201625
586	343396	201230056
587	344569	202262003
588	345744	203297472
589	346921	204336469
590	348100	205379000
591	349281	206425071
592	350464	207474688
593	351649	208527857
594	352836	209584584
595	354025	210644875
596	355216	211708736
597	356409	212776173
598	357604	213847192
599	358801	214921799
600	360000	216000000
601	361201	217081801
602	362404	218167208
603	363609	219256227
604	364816	220348864
605	366025	221445125
606	367236	222545016
607	368449	223648543
608	369664	224755712

Radices (root)	Quadrati (square)	Cubi (cube)
609	370881	225866529
610	372100	226981000
611	373321	228099131
612	374544	229220928
613	375769	230346397
614	376996	231475544
615	378225	232608375
616	379456	233744896
617	380689	234885113
618	381924	236029032
619	383161	237176659
620	384400	238328000
621	385641	239483061
622	386884	240641848
623	388129	241804367
624	389376	242970624
625	390625	244140625
626	391876	245314376
627	393129	246491883
628	394384	247673152
629	395641	248858189
630	396900	250047000
631	398161	251239591
632	399424	252435968
633	400689	253636137
634	401956	254840104
635	403225	256047875
636	404496	257259456
637	405769	258474853
638	407044	259694072
639	408321	260917119

Radices (root)	Quadrati (square)	Cubi (cube)
640	409600	262144000
641	410881	263374721
642	412164	264609288
643	413449	265847707
644	414736	267089984
645	416025	268336125
646	417316	269586136
647	418609	270840023
648	419904	272097792
649	421201	273359449
650	422500	274625000
651	423801	275894451
652	425104	277167808
653	426409	278445077
654	427716	279726264
655	429025	281011375
656	430336	282300416
657	431649	283593393
658	432964	284890312
659	434281	286191179
660	435600	287496000
661	436921	288804781
662	438244	290117528
663	439569	291434247
664	440896	292754944
665	442225	294079625
666	443556	295408296
667	444889	296740963
668	446224	298077632
669	447561	299418309
670	448900	300763000

Radices (root)	Quadrati (square)	Cubi (cube)
671	450241	302111711
672	451584	303464448
673	452929	304821217
674	454276	306182024
675	455625	307546875
676	456976	308915776
677	458329	310288733
678	459684	311665752
679	461041	313046839
680	462400	314432000
681	463761	315821241
682	465124	317214568
683	466489	318611987
684	467856	320013504
685	469225	321419125
686	470596	322828856
687	471969	324242703
688	473344	325660672
689	474721	327082769
690	476100	328509000
691	477481	329939371
692	478864	331373888
693	480249	332812557
694	481636	334255384
695	483025	335702375
696	484416	337153536
697	485809	338608873
698	487204	340068392
699	488601	341532099
700	496000 [recte: 490000]	343000000
701	491401	344472101

Radices (root)	Quadrati (square)	Cubi (cube)
702	492804	345948408
703	494209	347428927
704	495616	348913664
705	497025	350402625
706	498436	351895816
707	499849	353393243
708	501264	354894912
709	502681	356400829
710	504100	357911000
711	505521	359425431
712	506944	360944128
713	508369	362467097
714	509796	363994344
715	511225	365525875
716	512656	367061696
717	514089	368601813
718	515524	370146232
719	516961	371694959
720	518400	373248000

Roots 721–900 **

Radices (root)	Quadrati (square)	Cubi (cube)
721	519841	374805361
722	521284	376367048
723	522729	377933067
724	524176	379503424
725	525625	381078125
726	527076	382657176
727	528529	384240583
728	529984	385828352
729	531441	387420489

Radices (root)	Quadrati (square)	Cubi (cube)
730	532900	389017000
731	534361	390617891
732	535824	392223168
733	537289	393832837
734	538756	395446904
735	540225	397065375
736	541696	398688256
737	543169	400315553
738	544644	401947272
739	546121	403583419
740	547600	405224000
741	549081	406869021
742	550564	408518488
743	552049	410172407
744	553536	411830784
745	555025	413493625
746	556516	415160936
747	558009	416832723
748	559504	418508992
749	561001	420189749
750	562500	421875000
751	564001	423564751
752	565504	425259008
753	567009	426957777
754	568516	428661064
755	570025	430368875
756	571536	432081216
757	573049	433798093
758	574564	435519512
759	576081	437245479
760	577600	438976000

Radices (root)	Quadrati (square)	Cubi (cube)
761	579121	440711081
762	580644	442450728
763	582169	444194947
764	583696	445943744
765	585225	447697125
766	586756	449455096
767	588289	451217663
768	589824	452984832
769	591361	454756609
770	592900	456533000
771	594441	458314011
772	595984	460099648
773	597529	461889917
774	599076	463684824
775	600625	465484375
776	602176	467288576
777	603729	469097433
778	605284	470910952
779	606841	472729139
780	608400	474552000
781	609961	476379541
782	611524	478211768
783	613089	480048687
784	614656	481890304
785	616225	483736625
786	617796	485587656
787	619369	487443403
788	620944	489303872
789	622521	491169069
790	624100	493039000
791	625681	494913671

Radices (root)	Quadrati (square)	Cubi (cube)
792	627264	496793088
793	628849	498677257
794	630436	500566184
795	632025	502459875
796	633616	504358336
797	635209	506261573
798	636804	508169592
799	638401	510082399
800	640000	512000000
801	641601	513922401
802	643204	515849608
803	644809	517781627
804	646416	519718464
805	648025	521660125
806	649636	523606616
807	651249	525557943
808	652864	527514112
809	654481	529475129
810	656100	531441000
811	657721	533411731
812	659344	535387328
813	660969	537367797
814	662596	539353144
815	664225	541343375
816	665856	543338496
817	667489	545338513
818	669124	547343432
819	670761	549353259
820	672400	551368000
821	674041	553387661
822	675684	555412248

Radices (root)	Quadrati (square)	Cubi (cube)
823	677329	557441767
824	678976	559476224
825	680625	561515625
826	682276	563559976
827	683929	565609283
828	685584	567663552
829	687241	569722789
830	688900	571787000
831	690561	573856191
832	692224	575930368
833	693889	578009537
834	695556	580093704
835	697225	582182875
836	698896	584277056
837	700569	586376253
838	702244	588480472
839	703921	590589719
840	705600	592704000
841	707281	594823321
842	708964	596947688
843	710649	599077107
844	712336	601211584
845	714025	603351125
846	715716	605495736
847	717409	607645423
848	719104	609800192
849	720801	611960049
850	722500	614125000
851	724201	616295051
852	725904	618470208
853	727609	620650477

Radices (root)	Quadrati (square)	Cubi (cube)
854	729316	622835864
855	731025	625026375
856	732736	627222016
857	734449	629422793
858	736164	631628712
859	737881	633839779
860	739600	636056000
861	741321	638277381
862	743044	640503928
863	744769	642735647
864	746496	644972544
865	748225	647214625
866	749956	649461896
867	751689	651714363
868	753424	653972032
869	755161	656234909
870	756900	658503000
871	758641	660776311
872	760384	663054848
873	762129	665338617
874	763876	667627624
875	765625	669921875
876	767376	672221376
877	769129	674526133
878	770884	676836152
879	772641	679151439
880	774400	681472000
881	776161	683797841
882	777924	686128968
883	779689	688465387
884	781456	690807104

Radices (root)	Quadrati (square)	Cubi (cube)
885	783225	693154125
886	784996	695506456
887	786769	697864103
888	788544	700227072
889	790321	702595369
890	792100	704969000
891	793881	707347971
892	795664	709732288
893	797449	712121957
894	799236	714516984
895	801025	716917375
896	802816	719323136
897	804609	721734273
898	806404	724150792
899	808201	726572699
900	810000	729000000

[The table continues with root 901 on the following page.]

Table of Squares and Cubes (continued) [Tabula Quadratorum & Cuborum]

The table gives, for each root (Radices) from 901 through 1000, its square (Quadratus / Quadrati) and its cube (Cubi). It is printed on the page in three side-by-side blocks (roots 901–934, 935–968, 969–1000); it is reproduced here as one continuous run from 901 to 1000. All cells are transcribed glyph-by-glyph from the leaf, at high magnification.

Radices	Quadratus	Cubi
901	811801	731432701
902	813694	733870808
903	815409	736314327
904	817216	738763264
905	819025	741217625

Radices	Quadratus	Cubi
906	820836	743677416
907	822649	746142643
908	824464	748613312
909	826281	751089429
910	828100	753571000
911	829921	756058031
912	831744	758550528
913	833569	761048497
914	835396	763551944
915	837225	766060875
916	839056	768575296
917	840889	771095213
918	842724	773620632
919	844561	776151559
920	846400	778688000
921	848241	781229961
922	850084	783777448
923	851929	786330467
924	853776	788889024
925	855625	791453125
926	857476	794022776
927	859329	796597983
928	861184	799178752
929	863041	801765089
930	864900	804357000
931	866761	806954491
932	868624	809557568
933	870489	812166237
934	872356	814780504
935	874225	817400375
936	876096	820025856

Radices	Quadratus	Cubi
937	877969	822656953
938	879844	825293672
939	881721	827936019
940	883600	830584000
941	885481	833237621
942	887364	835896888
943	889249	838561807
944	891136	841232384
945	893025	843908625
946	894916	846590536
947	896809	849278213
948	898704	851971392
949	900601	854670349
950	902500	857375000
951	904401	860085351
952	906304	862801408
953	908209	865523177
954	910116	868250664
955	912025	870983875
956	913936	873722816
957	915849	876467493
958	917764	879217912
959	919681	881974079
960	921600	884736000
961	923521	887503681
962	925444	890277128
963	927369	893056347
964	929296	895841344
965	931225	898632125
966	933156	901428696
967	935089	904231063

Radices	Quadratus	Cubi
968	937024	907039232
969	938961	909853209
970	940900	912673000
971	942841	915498611
972	944784	918330048
973	946729	921167317
974	948676	924010424
975	950625	926859375
976	952576	929714176
977	954529	932574833
978	956484	935441352
979	958441	938313739
980	960400	941192000
981	962361	944076141
982	964324	946966168
683 [recte: 983]	966289	949862087
984	968256	952763904
985	970225	955671625
986	972196	958585256
987	974169	961504803
988	976144	964430272
989	978121	967361669
990	980100	970299000
991	982081	973242271
992	984064	976191488
993	986049	979146657
994	988036	982107784
995	990025	985074875
996	992016	988047936
997	994009	991026973
998	996004	994011992

Radices	Quadratus	Cubi
999	998001	997002999
1000	1000000	1000000000

On the Differences of Squares and Cubes, and on Continuing Their Table [De Differentiis Quadratorum et Cuborum, & de continuatione tabulæ eorundem]

Squares are generated by the continuous addition of the odd numbers, as the arithmeticians prove. It follows that the difference between any square and the next square after it is twice the smaller root plus one. There are therefore two ways of building and extending the table of squares.

[Margin: The difference of squares (Differentia quadratorum).]

In the first way, all the odd numbers are set down in order, starting from 1. For 1 gives the first square, 1. From 1 and 3 comes the second, 4; if the next odd number, 5, is added to it, the third square, 9, results; and if the next odd number, 7, is added, the fourth square, 16, is produced — and so on in this fashion. Now every square has a root equal to the count of odd numbers that make it up. So, since the single odd number 1 gives the first square, 1, its root is 1. Again, since the two odd numbers 1 and 3 make the second square, 4, its root is 2. Likewise, since the twelve odd numbers 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23 compose the square 144, its root is 12; and so with the rest. In this way, if the next odd number is always added to the preceding square, the next square is formed, and the table can be continued without limit — provided the series of odd numbers is first continued far enough. The roots advance in the natural sequence of the numbers.

[Margin: Construction of the table of squares (Compositio tabulæ quadratorum).]

The table of squares can also be built, and extended without limit, in a second way, without the series of odd numbers — namely, if all the roots are set down in order, as you see in the table. For since the first square is 1, whose root is 1: if this root is doubled and 1 is added, and the result is added to the first square, 1, the second square, 4, results, whose root is 2. If this root is doubled and 1 is added, and the result is joined to the second square, 4, the third square, 9, results, whose

root is 3. And this root doubled, with 1 further added, makes 7; so if 7 is added to the square 9, the fourth square, 16, results — and so on without limit.

Numeri impares	Quadrati	Radices
1	1	1
3	4	2
5	9	3
7	16	4
9	25	5
11	36	6
13	49	7
15	64	8
17	81	9
19	100	10
21	121	11
23	144	12

(Headers: odd numbers | squares | roots.)

[Margin: The generation of cubes (Generatio cuborum).]

Cubes too are produced by the addition of the odd numbers, in this way. Write out the series of odd numbers beginning from 1: the first, 1, gives the first cube, 1, whose root is 1. The next two, 3 and 5, taken together yield the second cube, 8, whose root is 2. The three following, 7, 9, 11, exhibit the third cube, 27, whose root is 3. In the same manner, the next four odd numbers make the fourth cube, the following five the fifth, and so on without limit. Every cube has a root equal to the count of odd numbers heaped together to compose it.

Numeri impares	Cubi	Radices
1	1	1
3	—	—
5	8	2
7	—	—
9	—	—
11	27	3
13	—	—

Numeri impares	Cubi	Radices
15	—	—
17	—	—
19	64	4
21	—	—
23	—	—
25	—	—
27	—	—
29	125	5

(Headers: odd numbers | cubes | roots. In the source the cube 8 is set opposite the pair 3, 5; 27 opposite 7, 9, 11; 64 opposite 13, 15, 17, 19; 125 opposite 21, 23, 25, 27, 29 — each with its root aligned beside it.)

A cube of any root is also produced by multiplying that root into its own square. Thus the cube of the root 16 is 4096, generated from the root 16 multiplied into 256, the square of that same root.

[Margin: How the differences of cubes are found (*Differentia cuborum quomodo reperiantur*).]

But it is very tedious both to continue the series of odd numbers through so many terms that all the cubes may be generated from their addition, as has been said, and to multiply all the roots by their own squares. So Juan Bautista Villalpando [Juan Bautista Villalpando], a priest of our Society, an eminent theologian and mathematician — whose remarkable learning in mathematical matters is displayed in his treatise on the layout of the City and the Temple of Jerusalem, and above all in his manifold construction of two mean proportionals between two given straight lines — observed a most beautiful property of the cube numbers, by which the differences of the cubes are found in order without great difficulty. These differences, once added in order to the preceding cubes, produce the cubes that follow.

[Margin: Construction of the table of cubes (*Constructio tabulæ cuborum*).]

They will make up the lower number in the second column. Then in the third column let all the cubes be placed, which are gathered by the continuous addition of the numbers in the second column — the column that in fact contains the differences of the cubes — in this way. The first cube is 1; if you add to it the next difference, 7, you make the second cube, 8; and by adding the next difference, 19,

you will build up the third cube, 27; and so on. For in the table set out beside it, the difference by which each cube differs from the preceding cube is always placed next to that cube. Finally, in the fourth column let the roots of the cubes be written in order.

Progressio senarij	Differentiæ cuborum	Cubi	Radices
6	1	1	1
12	7	8	2
18	19	27	3
24	37	64	4
30	61	125	5
36	91	216	6
42	127	343	7
48	169	512	8
54	217	729	9
60	271	1000	10
66	331	1331	11
72	397	1728	12
78	469	2197	13
84	547	2744	14
90	631	3375	15
96	721	4096	16

(Headers: progression of sixes | differences of the cubes | cubes | roots.)

These same differences of the cubes, set out in the second column, may also be found in the following way. Triple the square of the proposed root, add the tripled root, and add 1 besides. The number so formed will be the difference by which the cube of the proposed root differs from the next cube. For example, suppose the difference is wanted between the cube 216, of root 6, and the next larger cube. The square of the root 6 is 36; three times it is 108; if to this be added three times the root — that is, 18 — and 1 besides, the required difference 127 is formed. And in this way, if the differences of the cubes are continued by means of the progression of sixes, the table of cubes will be extended as far as one pleases. Moreover, as you see, the numbers of the progression of sixes are the sextuples of the roots of the cubes, each of its own root: thus 60 is six times the root 10.

[Margin: Which odd numbers compose a given cube (Qui numeri impares datum cubum componant).]

Since we have said that cubes are generated by the addition of the odd numbers — namely that the first is 1, that the second is made from the two following, 3 and 5, and the third from the three following, 7, 9, 11, and so on, so that each is heaped up from as many odd numbers as there are units in its root — if some curious person should wish to know (which it is not unpleasant to know) which particular odd numbers compose a proposed cube, that is, from what odd number they begin and at what one they end, he will find it out by this device. If the root of the proposed cube is odd, multiply it by half of the number next below it, and to twice the product add 1: the number so formed will be the first of the odd numbers sought. But if the root is even, multiply its half by the number next below the root, and to twice the product add 1: again the first of the odd numbers sought is produced.

For example, let the proposed cube be 125, whose root is 5, and which is therefore heaped up from five odd numbers. I multiply the root 5, since it is odd, by 2, the half of the next lower number 4, and make 10; I double it, making 20; and by adding 1 the first odd number, 21, results. So the five odd numbers sought are 21, 23, 25, 27, 29, which gathered into one sum make up the cube 125. Again, let the given cube be 1728, whose root is 12, and which is therefore compounded from 12 odd numbers. Since 12 is even, I take half the root, namely 6, and multiply it by 11, the number next below the root; I double the resulting product 66 and add 1. The number so composed, 133, will be the first of the 12 odd numbers we seek. So all 12 will be these: 133, 135, 137, 139, 141, 143, 145, 147, 149, 151, 153, 155 — making up the cube 1728.

This rule is demonstrated as follows. When an odd root is multiplied by half the number next below it, or half of an even root by the number next below the root, there is produced the sum of the terms of the natural series of numbers from 1 up to the number next below the root — as we said of the arithmetic progression — that is, the number of odd terms that precede the first odd number sought in the series of odd numbers (since the first is one, then the doubled is two, then three, and so on). That sum therefore indicates in what place stands the odd number that precedes the odd number sought; and because that place is doubled, when 1 is taken from the double it exhibits that last odd number, while if 1 is added to the same double the first of the odd numbers sought is produced. For instance, if the first of the seven odd numbers that make up the cube 343 (whose root is 7) be sought: we first look for the sum of the 6 terms of the natural series, which is 21

— obtained by adding 1 to 6 and multiplying the sum, 7 (that is, the odd root), by 3, the half of the 6 terms. So the odd number preceding the first one sought is placed in the 21st position. I double this and add 1, and the first odd number sought, 43, results. There are therefore seven numbers sought — 43, 45, 47, 49, 51, 53, 55 — which constitute the cube 343.

(Continuing within Book VIII of the Geometria Practica, the section on the differences of squares and cubes. The running header on this page prints "LIBER VII," but the book actually closes as Book VIII — see* FINIS LIBRI OCTAVI below and. Text resumes mid-argument.)*

The first and the last of the odd numbers can also be found another way, and indeed a simpler one, by this method. From the square of the root subtract the number one less than the root, and add that same number to the same square. The remainder will be the first of the odd numbers you are seeking; the sum will be the last. For example, take the root 11: if from its square, 121, you subtract 10 (the number just below the root), 111 is left — the first of the odd numbers. And if to the same square, 121, you add the same number, 10, you produce the last odd number, 131. So the eleven odd numbers are these: 111, 113, 115, 117, 119, 121, 123, 125, 127, 129, 131, which gathered into a single sum make the cube 1331, the cube of the root 11. Likewise, take the root 10: if from its square, 100, you take away 9, the first odd number, 91, remains; and if to the same square, 100, you add the same number, 9, you produce the last odd number, 109. The ten odd numbers are then these: 91, 93, 95, 97, 99, 101, 103, 105, 107, 109, whose sum, 1000, gives the cube of the root 10.

The reasoning behind this rule is as follows. In the count of odd numbers required, the difference of the odd numbers — namely 2 — is contained as many times as there are units in the number just below the given root; this is plain. Now when the root is odd, the odd number standing in the middle place of the series of odd numbers is the square of the root. It is preceded by the half-sum of the differences down to the first odd number — that is, by the number just below the root — and the same half-sum follows it. For this reason, if from that square you subtract the number just below the root (that is, the half-sum of the differences), the remainder becomes the first odd number sought; and if that same number just below the root is added to that square, the last odd number is produced. When, however, the root is even, its square falls between the two odd numbers that occupy the middle place; and this square is likewise preceded by the half-sum of the differences down to the first odd number — that is, by the number just below the root — and so on.

From this we can also easily find any given one of the odd numbers — the fourth, say, or the ninth, and so on. For if we multiply the difference of the odd numbers, namely 2, by the number just below the position of the odd number sought, and add the first odd number to the product, we obtain the odd number wanted. Thus if, in the later example of the ten odd numbers, the seventh odd number is sought, we multiply 2 by 6, and add the product, 12, to the first, 91, and so generate the seventh odd number, 103. And so with the rest.

But to be certain, when the root is large, whether the first odd number has been found correctly — so that we need not run through all the odd numbers, which is laborious — we will find the last one from the first (unless it has already been found by the second rule), and then the sum of them all, in this way. We multiply the difference of the progression of odd numbers, namely 2, by the number of terms less one, and add the product to the first term already found. The resulting number will be the last of the odd numbers sought. If, then, to this last we add the first, and multiply the half of the sum by the number of terms — that is, by the root of the proposed cube — the proposed cube is produced, provided the first odd number was correctly found. Take, for instance, the next cube, 343, whose root is 7: if to 43, the first odd number found, we add 12 (the number produced by multiplying 2 by the root 7 less one), the seventh odd number sought comes out as 55; and if to this the first, 43, is joined, they make 98; and if the half of this number, 49, is multiplied by the root 7, the cube 343 is produced. And so on.

By yet another, very easy method we shall find the same odd numbers that compose a given cube, thus. Let the given cube be 117649, whose root is 49; and accordingly the 49 odd numbers whose sum it is are to be found. Let us set the first odd number to be $1 R$ [R = the cossist *radix* sign, i.e. one root, the unknown], that is, one *Radix* (as is customarily done in Algebra). And since the difference of the odd numbers is 2, the second odd number will be $1 R + 2$, that is, one *radix* plus 2; the third, $1 R + 4$. And so as not to be forced to continue all 49 terms, we multiply the difference, 2, by 48 (the number of terms less one), and to the product, 96, we add the first term, which we set to be $1 R$. In this way the last term sought becomes $1 R + 96$; and if to it we add the first, namely $1 R$, the sum is $2 R + 96$, whose half is $1 R + 48$. Multiplied by 49, the number of terms, this makes $49 R + 2352$, which is equal to the proposed cube, 117649. Taking 2352 from both sides, then, there remain $49 R$ equal to the number 112297 [recte 115297: $49 \times 2353 = 115297$; see Numerical impossibles]; and dividing this by 49, the quotient is 2353, the first odd number sought. To prove it, we multiply the difference of the odd numbers, namely 2, by 48 (the number of terms less one), and add the product, 96, to the first, 2353, so as to make the last term, 2449. Then we add the

first to this, and multiply the half of the sum, 4802 — that is, 2401 — by 49, the number of terms; and there is generated the number 117649, which equals the given cube. Therefore the first odd number was found correctly.

The Use of the Preceding Table of Squares and Cubes in Extracting Roots

Among its other uses, the table of squares and cubes above has an outstanding use in extracting square and cube roots. By means of it the first three period-marks on the left are dispatched at once, and the first three figures of the root are found together — which one or two examples will make clear.

Let a square root first be extracted from a given number. First I look among the squares for the number:

— the number of the first three period-groups, 117689. Because I do not find it exactly, I take the square just below it, 117649, and place its root, 343, in the quotient. When this square is subtracted from the six-figure number 117689, there remain 40. Therefore the next period-group is 4001; dividing this (setting aside the figure 1) by 686, double the root found, the quotient comes out 0. The last period-group is then 400145; dividing this (setting aside likewise the figure 5) by 6860, double the root found, the quotient is found to be 5. The whole root is therefore 34305 ⁵⁷¹²⁰/₆₈₆₁₁.

Next, let a cube root be extracted from a given number. First I look among the cubes for the number 4250954:

— the number of the first three period-groups. Because I do not find it, I take the cube just below it, 4173281, with its root, 161. This cube, subtracted from the three period-groups, leaves 77673, so that the next period-group is 77673961.

I therefore prepare a divisor, as we taught at the end of Proposition 19 of Book 6, by multiplying the square of the root already found — taken from the same table, namely 25921 — by 300; this gives 7776300. Dividing my period-group by it, I find the quotient 9; and when the operation is carried out, there remain 7295302, so that the last period-group is 7295302307.

Finally I prepare a new divisor by multiplying the square of the root found, 1619 — namely 2621161 — by 300; this gives 786348300. Dividing my last period-group by it, I find the quotient 9. And when the operation has been carried out, there remain 214232708, and the extraction is complete.

Nor does it seem right to neglect a useful shortcut in cube-root extraction, which is of this kind. When a new divisor is to be prepared, so that we are not forced to compute the square of the whole root already found, we divide the root found into two parts, so that one part is the last figure of the quotient just found — namely the fourth figure, 9, found in the given example. We split the root found, 1619, into 1610 and 9, and write these two parts twice, as you see in the adjacent scheme.

[Margin — scheme:]

If, then, we multiply each part by each part, the square of the root sought, 1619, is produced, as we demonstrated at Proposition 1 of Book 2 of Euclid. For example: since the preceding square, of the number 161, was 25921, by appending two ciphers we shall have the product 2592100, which is 1610 times 1610. If to this we add 14490 twice — that is, the number produced by 9 times 1610 (or 9 times 1610) — and, further, 81, the product of 9 times 9, we make up 2621161, the square of the root found, 1619.

By the same reasoning, if we want the square of this number 16199, we divide it into 16190 and 9, twice, as you see in this scheme.

[Margin — scheme:]

Then, since we already had the square 2621161 of the number 1619, by appending two ciphers we shall have 262116100, the square of the number 16190. If to this we add 145710 twice — that is, the product of 16190 times 9 (or of 9 times 16190) — and, further, 81, the product of 9 times 9, we make 262497601 [recte 262407601: $16190^2 = 262116100$, $145710 \times 2 = 291420$, $+ 81 = 262407601$; likewise $16199^2 = 262407601$; see Numerical impossibles], the square of the number 16199. And so with the rest.

This shortcut, then, will make the extraction of the cube root easier, since from the square of the preceding root found — with two ciphers appended — we always compose the square of the same root with a single cipher appended to it, and so on.

You see clearly that if the table above were extended, so that the squares and cubes of roots of four or five figures were included, the extraction of roots would be made much easier. But because the table would grow immeasurably this way, we have been content with a table in which the roots have at most three figures, since anyone can extend and continue it from what we have said.

END OF BOOK EIGHT

INDEX

An Index of the Things and Words Contained in this *Practical Geometry*, Arranged in Alphabetical Order

A

- To determine, near enough to the truth and by means of the geometric square, the sloping distance of a mountain from the surveyor's station up to the base of a height set upon the mountain — even when that base is not in view — together with the height itself, when the surveyor stands on the mountain's ascent. — 91
- To determine, near enough to the truth and by means of the quadrant, the sloping ascent of a mountain from the surveyor's station up to the base of a height set upon the mountain — even when it is not in view — together with the height itself, when the surveyor stands on the mountain's ascent. — 60
- How a heap of wheat is to be measured. — 133
- To find, by means of the quadrant, the height of any building whatever raised to the perpendicular, or the depth of a well, provided the angle at the bottom, or some mark placed at the bottom, can be sighted. — 62
- To find, by means of the geometric square, the height of any building whatever raised to the perpendicular, or the depth of a well, provided the angle at the bottom, or some mark placed at the bottom, can be sighted. — 92
- The area of an equilateral triangle. — 100
- To draw on paper a plan of a field, a plain, a city, or a region. — 99
- To draw on paper a figure similar to a given field. — 112
- Albrecht Dürer's [Albrecht Dürer] squaring of the circle by numbers is false. — 188
- Aliquot parts similar in several magnitudes have the same proportion. — 137
- The area of a longer-on-one-side [oblong] figure. — 103
- What the altimetric scale is. — 64

- By means of a plane mirror, to determine a height to whose base access is open, together with the distance of the mirror from the summit of the height. — 97
- To explore, by means of the geometric square, from a valley or from the level of the horizon, a height whose base is set upon a mountain or upon some other height, and both of whose ends can be seen, even if the lowest point of the other height on which it is set lies hidden and the distance of that lowest point from the surveyor's station is not known. — 90
- To estimate, by means of the geometric square, the height of any upright object from its distance from the surveyor's eye. — 76
- To disclose the height of any upright object, even if its distance from the surveyor's eye is neither given nor found, by means of two stations made in the plane, with the help of the geometric square. — 76
- To explore, by means of the quadrant, from a valley or from the level of the horizon, a height whose base is set upon another height, and both of whose ends can be seen, even if the lowest point of the other height on which it is set lies hidden and the distance of that lowest point from the surveyor's station is not known. — 58
- To determine, by means of the geometric square, the height of any object whatever from the shadow it casts in sunlight, if that shadow is known. — 95
- By means of a plane mirror, to ascertain an inaccessible height together with the distance of the mirror both from the base (even when that base is not in view) and from the summit of the height. — 98
- To give, by means of the quadrant, the measure of an inaccessible height when neither the distance from the surveyor's station to its base is known, nor can two stations be made in a straight line directly toward it, nor, finally, is the base in view — and, incidentally, to elicit that distance too. — 57
- To explore, by means of the quadrant, an inaccessible height whose base is not seen, and toward which we can neither advance nor retreat along any straight-line interval so that two stations might be made, but only to the right or to the left, to a place from which its base appears. — *ibid.*
- To explore, by means of the geometric square, an inaccessible height whose base is not seen, and toward which the surveyor can neither advance nor retreat along any straight line so that two stations might be made, but only to the right or to the left, to a place from which its base is discerned. — 89

- To measure out, by means of the quadrant, an inaccessible height when the distance from the surveyor's station to the base of the height is unknown, by two stations made in the plane; and, from this, to draw out that distance too, even if its farthest end is not seen. — 45
- To hunt out, by means of the geometric square, a greater height from a lesser unknown one, provided the base of the greater can be seen. — 89
- To determine, by means of the geometric square, a greater height from a lesser known one, even if only the summit of the greater is seen. — *ibid.*
- To find out, by means of the quadrant, a greater height from a lesser known one, by two stations made at the summit or at two windows, even if only the summit of the greater height is seen; and, from this, to gather also the distance between the heights. — 57
- To search out, by means of the quadrant, a greater height from a lesser unknown one, provided the base of the greater can be seen. — 58
- To [determine] a lesser height from a greater known one, although the base of the lesser [can be seen...] —

Index of Subjects and Terms [Index rerum ac verborum]

(*Alphabetical index to the Geometria Practica**. Each entry is a Latin heading followed by the folio/page locator of the* Geometria Practica*, transcribed as printed — including* *ibid./ib.* ("in the same place"), & seqq. /& seq.* ("and following"), and multiple locators such as* 103.107*. The index is a dense two-column setting; entries are given in alphabetical reading order, left column then right column, across the three pages.)*

A (continued from letter A)

[This opening line completes an entry begun on the preceding page — an "Altitudinem..." heading:] ...when it cannot be discerned, to examine it by means of the geometric square [quadratum]. 90

- To derive the smaller height from an unknown greater one, provided the base of the smaller is visible, by means of the square. *ibid.*
- To investigate the smaller height from a known greater one, even though the base of the smaller cannot be seen, by means of the quadrant

[quadrans]; and from this also to derive the distance between the two heights. 59

- To explore the smaller height from an unknown greater one, provided the base of the smaller can be seen, by means of the quadrant; and from this also to infer the distance between the two heights. *ibid.*
- To infer the height of a mountain or tower from its summit, when, in a straight line from the surveyor [mensor], some interval between two marks — or even between some mark and the tower — is known, by means of the square. 85
- To measure the height of a mountain or tower from its top by two stations made at its summit, from which some mark appears on the horizon, by means of the quadrant; and from this to investigate the distance from the base of the tower, or from the perpendicular of the mountain, to that mark. 48
- To determine a height set upon [another] height, provided the base of the height can be observed; or the upper portion of some tower, by means of a plane mirror. 99
- To measure the height of a mountain by means of the quadrant. 45.46
- To measure the height of a mountain or tower from its top by two stations made on some upright staff [hasta], or at two windows of a tower one above the other, from which some mark can be seen on the horizon, by means of the quadrant; and from this to determine the distance from the perpendicular of the mountain or tower up to the observed mark. 48
- To measure the height of a mountain or tower from its top by means of the quadrant, when on the plane on which it stands some space in a straight line from the surveyor is known. 51
- To measure a height by a single station with the quadrant, when the distance is known. 46
- The height of the pyramids into which the regular solids are resolved from their centers. 135. & seqq.
- To investigate a proposed height by a certain singular method. 77. & 78

- To hunt out a proposed height, and its distance from the surveyor's eye, together with the hypotenuse extended from the eye to the summit of the height, by means of the fixed square [quadratum stabile] in a single station, even if only the summit of the upright object is seen. 79
- To trace out a proposed height, when the distance from the surveyor's eye is neither given nor found, nor can two stations be made in a straight line from the height, by means of two stations made on some upright staff, using the square. 78
- When a height is greater than the distance, when equal, and when less. 100
- To observe the height of the Sun, or of any star, by means of the square. 66
- To trace out the height of a tower, or of some other thing, by means of a rod [baculum]. 94
- To measure the height of a tower or mountain from its summit by a single station with the fixed square, together with the distance of a mark seen on the horizon up to the tower, or up to the perpendicular of the mountain. 82
- To measure the height of a tower from its top by a single station with the quadrant, if the distance of a mark seen on the horizon up to the base of the tower is known. 51
- To measure the height of a tower or mountain from its summit by means of the square, when on a plane at the summit equidistant from the horizon two stations can be made and some mark is seen on the horizon. 81
- To investigate the height of a tower or mountain from its summit by two stations made on some upright staff, using the square, when some mark can be seen on the horizon. 82
- To investigate the height of a tower, or of some other thing, by means of the set-square [Norma]. 94
- To perceive a portion of the greater height from the smaller height, and a portion of the smaller from the greater, by means of the square. 90
- To determine a portion of the greater height from the lesser height, and a portion of the smaller from the greater, by means of the quadrant. 59
- To derive a portion of the smaller height from the smaller height, and a portion of the greater from the smaller, by means of the square. *ibid.*

- To measure the circumference of the earth [Ambitum terrae] from some elevated mountain. 213
- How the size of the angle contained by the sides of the instrument of parts [instrumentum partium] is to be determined. 9
- That a mechanical measurement of certain angles and lines is to be admitted. 111
- To find the two angles of an oblique-angled triangle from two sides and the angle contained by them. 41
- To ferret out the two angles of an oblique-angled triangle from two sides and the angle opposite one of them (provided the kind of angle opposite the other side is settled, when the given angle is acute). *ibid.*
- To investigate all three angles of an oblique-angled triangle from all three sides. 42
- To seek an acute angle of a right triangle from the base and one side. 40
- To make known an acute angle of a right triangle from both sides. *ibid.*
- To divide a given rectilinear angle into three equal angles. 218
- Why the angle of incidence is equal to the angle of reflexion. 200
- What the angle of observation is. 43
- The angle which the [plumb-]line makes with the nearest side of the square, [showing] when it gives the altitude of the Sun or a star, and when the complement of the altitude. 66
- The Arabs' quadrature of the circle by numbers is false. 188
- An arc of a circle is to a similar arc of another circle as chord is to chord, and conversely. 198
- How arcs of given degrees and minutes are to be cut off from any circle by means of the instrument of parts. 8
- The area of a rectangle longer on one side [figura altera parte longior]. 103
- The area of a field within which a lake or wood lies. 113
- The area of a field, when it cannot be resolved into triangles. 112

- A more accurate area of the circle. 126
- The area of a circle, to which a right triangle is equal, according to Archimedes. 118
- The area of a circle is equal to the rectangle contained by the radius [semidiameter] and half the circumference of the circle. 178
- The area of a circle by three methods, from the known diameter and circumference. 125
- The area of the hyperbolic conoid [Conoides Hyperbolicum]. 144
- The area of the parabolic conoid. 144
- The area of a body contained by plane surfaces and circumscribable about a sphere, to which a rectangular solid is equal. 183
- The area of wholly irregular bodies, what it is. 145
- The area of any portion of a sphere. 142
- A fine method of finding the area of any figure. 113
- The area of any triangle, to which a rectangle is equal. 174
- The area of a given ellipse. 129
- The area of a given parabola. *ibid.*
- The area of casks [dolia]. 145
- The area of a lens-shaped figure. 127
- The area of a wholly irregular quadrilateral figure. 112
- The area of a figure compounded from various segments of circles. 127
- The area of a regular figure whose side is unity, how it is to be found. 117
- The area of a regular figure, to which a rectangle is equal. 175
- The area of a regular figure, how it is to be derived from the known area of another similar figure. 116
- The area of a regular figure, to which a right triangle is equal. 175

- The areas of the regular figures from the triangle up to the dodecagon, when the side is unity. 117
- The area of the frustum of a pyramid and cone. 132
- The area of the frustum of a sphere. 144
- To investigate an area of the circle greater than the true value, from the diameter. 125
- To conclude an area of the circle greater than the true value, from the circumference. 116
- To gather an area of the circle smaller than the true value, from the diameter. 125
- To infer an area of the circle smaller than the true value, from the circumference. 126
- To inquire into the area of a figure similar to an ellipse, which is drawn with the compass. 219
- The area of an irregular many-sided figure, what it is. 112
- The area of parallelograms. 111
- The area of parallelepipeds, prisms, and cylinders. 130
- The area of the portions of a spheroid. 144
- The area of a pyramid, to which a rectangular solid is equal. 182
- The area of pyramids and cones. 131
- The area of a square. 103
- The area of non-rectangular quadrilaterals. 111
- The area of the five regular solids, what it is. 133.135
- The area of rectangles. 103
- The area of regular figures. 114
- The area of the rhombus and rhomboid. 111
- The area of a sector of a circle. 127

- The area of a semicircle, quadrant, eighth part, etc. 125
- The area of segments of a circle. 127
- The area of segments of a sphere. 141
- An area of the sphere smaller than the true value, from the diameter of the greatest circle. *ibid.*
- An area of the sphere greater than the true value, from the circumference of the greatest circle. *ib.*
- An area of the sphere smaller than the true value, from the circumference of the greatest circle. *ib.*
- An area of the sphere greater than the true value, from the diameter of the greatest circle. *ibid.*
- The area of a sphere is equal to a rectangular solid contained by the radius and a third part of the convex surface. *ib. & seq.*
- The area of a sphere, and its convex surface. 137.139
- The area, or solid content, of a sphere-sector. 142
- The area, or solid content, of a hemisphere. *ibid.*
- The area of a spheroid. 144
- The area of a trapezium having no parallel sides. 112
- The area of a trapezium having two parallel sides. 111
- The area of triangles. 103.107
- The area of a right triangle. 109
- The area of a right triangle, from one side about the right angle and one acute angle. 110
- The area of a right triangle, from one side about the right angle and the side that is opposite the right angle. *ibid.*
- The area of a right triangle, from the side that is opposite the right angle and one acute angle. *ibid.*
- The area of an isosceles triangle. 109

- The area of an equilateral triangle. *ibid.*
- The area of an oblique-angled triangle, from two sides and the angle contained by them. 111
- The area of an oblique-angled triangle, from one side and two angles. 111
- The area of a hollowed-out vessel. 133
- To measure a proposed distance by means of a reed or rod. 72
- To determine, close to the truth, the sloping ascent of a mountain from the surveyor's position up to the base of a height set upon the mountain — even one not in view — together with the height itself, when the surveyor stands on the ascent of the mountain, by means of the quadrant. 61

B

- To measure, by means of a rod, the distance between the surveyor's feet and some mark on the plane of the horizon, when the far end of the distance can be seen. 93
- To measure the height of a tower, or of some other thing, by means of a rod. 94
- To measure a proposed distance by means of a rod or reed. 72
- To investigate the base of a right triangle from one side and one acute angle. 39
- To examine the base of a right triangle from both sides. 40
- The base of the quadratrix [quadratrix], the radius of the quadrant, and the quadrant are in continued proportion. 191

C

- The quadrature of the circle by lines ascribed to Campanus is false. 188
- To draw, on a plane, the layout of a field, farmland, city, or region. 99
- The area of a field, when it cannot be resolved into triangles. 112
- The area of a field within which a lake or wood lies. 113
- For a given field, to draw a similar figure on paper. 112
- How hundredth or thousandth parts are to be taken on any straight line, by means of the instrument of parts. 7
- Given the chord of some arc, together with the perpendicular drawn from its midpoint to the arc, to find how many degrees — or spans [palmi] — the arc, as well as the radius, comprises. 206

- A most beautiful property of the circle. 209
- That the quadrature of the circle is possible. 189
- The quadrature of the circle by Hippocrates of Chios [Hippocrates Chius] is false. 188.189
- The quadrature of the circle by the lines ascribed to Campanus is false. 188
- The quadrature of the circle by numbers according to the Arabs is false. ib.
- The quadrature of the circle by numbers from Albrecht Dürer [Albertus Durerus] is false. ibid.
- The quadrature of the circle by lines. ibid.
- The area of a circle by three methods, from the known diameter and circumference. 125
- A more accurate area of the circle. 126
- To investigate an area of the circle greater than the true value, from the diameter. 125
- To gather an area of the circle smaller than the true value, from the diameter. ibid.
- To infer an area of the circle smaller than the true value, from the circumference. 126
- To conclude an area of the circle greater than the true value, from the circumference. ib.
- The area of a circle to which a right triangle is equal, according to the doctrine of Archimedes. 118
- The area of a circle is equal to the rectangle contained by the radius and half the circumference of the circle. 176
- The measurement of the circle according to Archimedes. 118
- What proportion the diameter of a circle has to its perimeter [periphery], according to Archimedes. 120
- The diameter of a circle in numbers, from a given arc. 128
- The perimeter of a circle from the given diameter, and, more accurately, the diameter from the given perimeter. 126
- To trace out a diameter of the circle greater than the true value, from the given circumference. 124
- To investigate a diameter of the circle smaller than the true value, from the given perimeter. ibid.
- The diameter of a circle multiplied by $3 \frac{1}{7}$ produces a number greater than the circumference. 122
- The perimeter of a circle, and its diameter, from its area. 128

- The perimeter of a circle from the given diameter, and, more accurately, the diameter from the given perimeter. 126
- What proportion the perimeter of a circle has to the diameter, according to Archimedes. 120
- The perimeter of a circle divided by $3 \frac{1}{7}$ yields a number smaller than the diameter. 122
- To find a perimeter of the circle greater than the true value, from the given diameter. 123
- To derive a perimeter of the circle smaller than the true value, from the given diameter. ibi.
- For an eighth part, a sixteenth part, etc. of a circle, to construct an isoperimetric and equal rectangle. 186
- The area of a sector of a circle. 127
- For a circle, to construct a rectilinear figure that is equal to it and similar to another given figure. 194
- For a given circle, to construct an equal square. 193
- How a square equal to the circle may easily be exhibited from the circle's own figure. ibid.
- The perimeters of circles are to one another as their diameters. 124.197
- The proportion of two circles, or of similar figures, from given diameters or circumferences, or from two corresponding [homologous] sides. 128
- The diameters of circles are to one another as their circumferences. 124.197
- To draw a circle equal to a given square. 194
- To increase or diminish a circle, or a plane rectilinear figure, in a given proportion. 164
- To draw a circle equal to any rectilinear figure. 194
- To draw a circle through three points, once the other points through which it must pass have been found. 202
- To draw a circle equal to several circles whose diameters or circumfe-

(Left column, page 240, completes the preceding entry:)

- ...[a circle] equal to several circles whose diameters or circumferences are given; and to draw a figure similar to several similar figures whose corresponding sides are given, and equal [to them]. 128
- A circle has to the square of its diameter a proportion which is nearly that of 11 to 14, according to Archimedes. 123

- A circle has to the square of its circumference a proportion greater than 7 to 88, but smaller than 71 to 892. 125
- The circle is greater than all regular rectilinear figures isoperimetric with it. 174
- The circle is the greatest of all rectilinear figures isoperimetric with it. 182
- A circle has to the square of its diameter a proportion greater than 223 to 284, but smaller than 11 to 14. 124
- The proportion of the circumference of a circle to its diameter, a more accurate one, what it is. 126
- A fine shortcut for measuring lengths by means of the fixed square. 70
- The description of the conchoid line [*linea conchoides*], and its two remarkable properties. 162
- The convex surface of a cone and a cylinder. 146
- The proportion of the conical surface to its own base. *ibid.*
- The solid content of the hyperbolic conoid. 145
- To make a cube equal to a cone, cylinder, prism, and pyramid. 216
- To construct a cylinder, or a prism, equal to a given cone or pyramid under the same height; and conversely, to construct a cone or pyramid equal to a given cylinder or prism under the same height. 215
- The solid content of the parabolic conoid. 144
- The area of cones and pyramids. 131
- The easiest construction of the gnomonic table, and its use. 66
- The construction and use of the little table for minutes and seconds. *ibid.*
- The construction and use of the table for finding minutes and seconds from the quadrant. 14
- The construction of the quadrant for finding minutes and seconds. 11
- The construction of the geometric square. 64
- The construction of the sights [*pinnacidia*] for the line of sight. 11
- The construction of the rule [*regula*] in place of the thread with a plumb-bob. *ibid.*
- The construction of the instrument of parts. 5.9
- To transmute a cone and pyramid into a cylinder and prism; and likewise a prism and cylinder into a cone and pyramid. 215
- To reduce a cone, cylinder, prism, and pyramid to an unequal [one] under a given height, and upon a base of any number of angles. *ibid.*

- To change a cone, cylinder, prism, and pyramid into a parallelepiped equal in given height or base. *ibid.*
- To construct a cone equal to a given sphere. 217
- To construct a cone equal to a pyramid, and a cylinder equal to a prism; and conversely, a pyramid equal to a cone, and a prism equal to a cylinder. 215
- To convert a cone, pyramid, prism, and cylinder into a parallelepiped upon a square base. *ibid.*
- To exhibit a sphere equal to a given regular solid. 217
- The convex surfaces of the five regular solids. 135
- The area of the five regular solids, what it is. 133.135
- The area of wholly irregular bodies. 145
- To construct a regular solid equal to any given cube. 217
- A body contained by plane surfaces and circumscribable about a sphere, to which a rectangular solid is equal. 183
- To find geometrically, by means of lines, the cube and square root in numbers that are not squares and not cubes. 173
- The extraction of the cube root. 168
- The extraction of the cube and square root. 168
- The extraction of the cube and square root from a given fraction [minutia]. 172
- For a given cube, to construct any regular solid equal to it. 217
- For a given cube, to construct an equal rectangular parallelepiped under a given height, or upon a given base. 216
- How the differences of cubes are to be found. 227
- A table of cubes and squares up to the root 1000. 221

(Right column, page 240:)

- The generation of cubes. 227
- To construct a cube equal to a cylinder, prism, cone, and pyramid. 216
- To make a cube equal to a given sphere; and to make a sphere equal to a given cube. *ibidem.*
- To cut a given cube, or a parallelepiped, into a given proportion. 218
- To construct a cube equal to two or more cubes. 217
- To subtract a smaller cube from a larger one and convert the remainder into a cube. *ibidem.*
- To construct a cube equal to a rectangular parallelepiped. 216

- To construct a cube equal to any number of solids. 217
- The cube of either of the two mean proportionals between two straight lines is equal to a parallelepiped contained by the square of the extreme taken near the mean and the other extreme. 165
- The cube of the diameter of a sphere has to the sphere a proportion greater than 21 to 11, but smaller than 426 to 223. 139
- The cube of the circumference of the greatest circle in a sphere has to the sphere a proportion greater than 298374 to 5041, but smaller than 2904 to 49. 138
- The convex surface of a cylinder and a cone. 146
- The cylindrical surface, with the bases removed. *ibidem*.
- The area of cylinders, prisms, and parallelepipeds. 130
- To construct a cone or pyramid equal to a given cylinder or prism under the same height; and conversely, to construct a cylinder or prism of the same height equal to a given cone or pyramid. 215
- To construct a cube equal to a cylinder, prism, cone, and pyramid. 216
- To construct a cylinder equal to a given sphere. *ibid*.
- To transmute a cylinder and prism into a pyramid and cone; and likewise a cone and pyramid into an equal cylinder or prism. 215
- To divide a given cylinder or prism into a given proportion. 218
- To change a cylinder, cone, prism, and pyramid into a rectangular parallelepiped equal in given height or base. 216
- To convert a cylinder, prism, cone, and pyramid of any height into an unequal one under any other given height, and upon a base of any number of angles. 215
- To construct a cylinder equal to a prism, and a cone equal to a pyramid; and conversely, a prism equal to a cylinder, and a pyramid equal to a cone. *ibid*.
- To convert a cylinder, prism, cone, and pyramid into a parallelepiped upon a square base. *ibid*.

D

- How tenth parts of thousandths are to be taken, even if the instrument of parts is divided into only 100 parts. 7
- How any number of tenth or hundredth parts are to be taken off, by means of the compass, from any straight line divided into equal parts. 35

- To find the declination of any parallel on the diameter of the astrolabe, by means of the instrument of parts. 9
- The diameter of a circle, and its perimeter, from its area. 128
- The diameter of a circle in numbers, from a given arc. *ibid.*
- The proportion of the diameter of a circle to its circumference, a more accurate one, what it is. 126
- The diameter of a circle multiplied by $3 \frac{1}{7}$ produces a number greater than the circumference. 122
- To trace out a diameter of the circle greater than the true value, from the given circumference. 124
- To investigate a diameter of the circle smaller than the true value, from the given circumference. 123
- What proportion the diameter of a circle has to its perimeter, according to Archimedes. 120
- The diameter of a circle from the given perimeter, and, more accurately, the perimeter from the given diameter. 126

Index of Subjects and Terms [Index rerum ac verborum] (continued)

Alphabetical index to the Geometria Practica. Entries are grouped by the initial letter of the Latin headword (the letter-dividers are the original's alphabetization); each is translated into English with its original locator (a folio/page number of the Geometria Practica) kept exactly as printed —* *ibid.* = *ibidem* ("in the same place").*

D (continued)

- The diameters of circles are to one another as the circumferences. 124. 197
- How the differences of cubes are found. 227
- The differences of squares. *ibid.*
- What the difference of stations is. 43
- What the difficulty in the extraction of roots is, and how it is overcome. 169
- How dimensions [measurements] are made without computation of numbers. 45. 48. 50. 51

- How the measurement of a height is made without reducing the versed shadow [umbra versa] to the right shadow [umbra recta], when in one station the right shadow is cut and the versed shadow in the other. 78
- Measurements of distances are made in the same way with the fixed quadrat [quadratum stabile] as with the hanging one [pendulum]. 73
- How the measurement of a distance is made without reducing right shadows to versed shadows, when in each station the side of the right shadow is cut. 74
- How the measurement of a distance is made without reducing the right shadow to the versed shadow, when in one station the right shadow is cut and in the other the versed shadow. 75
- Measurement of the circle, from Archimedes. 118
- How the portion of the dioptra within the fixed quadrat is found. [no locator printed]
- To measure, by a single station, with the quadrant, the distance from the eye or foot of the surveyor to any point marked on the horizon. 53
- To determine near-truly, by means of the quadrant, the sloping distance of a mountain from the surveyor's position up to the base of a height set upon the mountain — even when that base is not seen — together with the height itself, when the surveyor is stationed on the ascent of the mountain. 61
- To determine near-truly, by means of the quadrat, the sloping distance of a mountain from the surveyor's position up to the base of a height set upon the mountain — even when that base is not seen — together with the height itself, when the surveyor is stationed on the ascent of the mountain. 91
- To find out, by the quadrat, the distance from the top of a tower or wall to some marker set on the horizon, even though there is no access to it, wherever the surveyor may be. 88
- To measure by the quadrant the horizontal distance between a tower and some other marker, from the tower by two stations made at its summit, or from two windows one of which is perpendicularly below the other, when the space between those windows is known — even if the height of the whole tower is unknown — and thereby incidentally to disclose the height of the tower. 55
- To find out by the quadrat the distance from the eye or foot of the surveyor (wherever he stands) to any point at some height, or also marked on the horizon, even by a single station. 86. 88

- To measure by the quadrant the distance between two points on any raised plane, whether it be perpendicular to the horizon or inclined. 53
- To measure by the quadrant, whether hanging or fixed, a distance on a plane — whether it be accessible or inaccessible — by two stations made on the same plane, when at its end some perpendicular height stands erect, even if its lowest end is not seen; and thereby to derive the height itself as well. 47. 49
- To find out by the quadrat the distance from the base of a tower to a marker set on the horizon, from the summit of the tower or from some window of it. 83
- To measure by the quadrant the distance from the eye or foot of the surveyor to any point marked at some height, by two stations made on a plane. 52
- To gather by the quadrant the distance between some marker set on the horizon and the summit of a tower or of some wall, even though there be no access to the marker itself. 56
- To investigate by the quadrat the distance between yourself and any marker set on the plane of the horizon. 71. 72
- To search out by a very simple instrument the distance on the horizon between the surveyor and some marker in view. 96
- To measure by the quadrat a distance on a plane, by two stations made on the same plane, when at its end some perpendicular height stands erect, even if its lowest end is not seen. 73
- To find by a very simple instrument the distance between two mountain- or tower-tops. 96
- To find out, by aid of the set-square [Norma], the distance on the plane of the horizon between the surveyor and any marker. 95
- To comprehend by the quadrat the distance between two markers on a plane on which a height stands, if that distance lies straight ahead of the surveyor and both its ends can be seen, from the top of the height — even if the height is the surveyor's own stature. 84
- To measure in a very easy way a distance on the plane of the horizon that is not very great. 95
- To measure by aid of a rod the distance between the surveyor's feet and some marker on the plane of the horizon, when the far end of the distance can be seen. 93
- To measure by the quadrat the distance between two markers or points on any plane, whether perpendicular to the horizon or inclined. 88

- To measure a distance by a single station with the quadrant, when the height is known. 48
- To measure by the quadrant a distance, when the surveyor, standing at one of its ends or at some known height perpendicular to the plane on which the distance lies, can see the other end. 54
- The capacity of casks [dolia]. 97

E

- Given the centre of an ellipse on the axis line, together with two points of the ellipse, to find each end of both axes. 209
- The area of a given ellipse. 129
- To describe with compasses a figure similar to an ellipse, which they call an oval. 218
- The checking [examination] of the extraction of roots. 168
- To extract a root of any kind from a given number. 166

F

- The easy finding of a straight line equal to any circumference, from its own figure. 193
- The easy finding of a square equal to a circle. *ibid.*
- A regular figure circumscribed about a circle has a greater perimeter than the circle. 195. 196
- How the digits of a number from which a root is extracted are marked off by points. 166
- To describe a circle equal to any rectilinear figure. 194
- The area of a regular figure, to which a right triangle is equal. 175
- The area of a regular figure, to which a rectangle is equal. 174
- To describe with compasses a figure similar to an ellipse, which they call an oval. 218
- To construct a rectilinear figure equal to a circle and similar to another. 194
- To distribute a rectilinear figure into any number of equal parts by a straight line parallel to a given straight line. 157

- To increase or diminish in a given proportion any solid figure of those which Euclid treats in the *Stereometry*. 164
- To divide a rectilinear figure into any number of equal parts, from a given angle or from a point on a side. 154
- To increase or diminish in a given proportion a plane rectilinear figure or a circle. 164
- To describe a figure similar to several similar figures whose homologous sides are given, equal [to them all]; and likewise a circle equal to several circles whose diameters or circumferences are given. 128
- The proportion of two similar figures, or of circles, from two given homologous sides, or from diameters or circumferences. *ibid.*
- Of isoperimetric figures having sides equal in number, the greatest is the equilateral and equiangular one. 180
- Of regular isoperimetric figures, the greater is the one that contains more angles or more sides. [continues]
- [...the one that] contains more angles or more sides. 176
- The circle is greater than the regular rectilinear figures to which it is isoperimetric. 182
- The plumb-line's thread cutting the right shadow makes the angle of the complement of the altitude; but cutting the versed shadow, it establishes the angle of the altitude itself. 66
- To reduce a large fraction to a smaller nearly equivalent one. 116
- The easy finding of a fraction intermediate between two. *ibid.*
- The solidity of a regular marble frustum. 133
- The area of a frustum of a pyramid and of a cone. 132
- The solidity of a spherical frustum. 144

G

- The kinds of roots are innumerable. 166
- The generation of cubes. 227

- The generation of squares. *ibid.*
- What geodesy and geometry are. 147
- What geometry and geodesy are. *ibid.*
- The construction of the geometric quadrat. 65
- The gnomonic table. 68
- The easiest construction of the gnomonic table, and its use. 66
- The gnomon, or side of the quadrat, is a mean proportional between the right shadow and the versed shadow. 65
- To find how many degrees and minutes are contained in a given arc, by means of the instrument of parts. 8
- How any number of degrees and minutes may be cut off from any circle, by means of the instrument of parts. 8

H

- The convex surface of a hemisphere. 142
- The solidity of a hemisphere. 143
- The side of the heptagon incorrectly found by Carolo Mariano, Albrecht Dürer [Alberto Durero], and François de Foix de Candale [Francisco Flussate]. 212
- The solidity of a hyperbolic conoid. 145
- The finding of the hypotenuse by the quadrant. 45. 47. 49. & 51.

I

- To find the odd numbers composing any [given] cube. 228
- How a commanded part may be cut off from a given straight line, by the instrument of parts. 8
- To explore by the quadrat an inaccessible height whose base is not seen, and to which the surveyor can, along a straight line, in no space approach or recede so that two stations might be made, but only to the right or to the left toward a place from which its base may be seen. 89
- To explore by the quadrant an inaccessible height whose base is not seen, and to which we can, along a straight line, in no space approach or recede so that two stations might be made, but only to the right or to the left toward a place from which its base may appear. 56

- To measure by the quadrant an inaccessible height, when the distance from the surveyor's position to the base of the height is unknown, by two stations made on a plane; and thereby to derive the distance itself as well, even if its far end is not seen. 46
- To render known by the quadrant an inaccessible height, when neither the distance from the surveyor's position to its base is known, nor can two stations be made on the plane straight ahead of it, nor, finally, does the base appear; and thereby incidentally to derive the distance itself as well. 57
- To find out, by aid of a plane mirror, an inaccessible height together with the mirror's distance both from the base — even when unseen — and from the summit of the height. 98
- To measure by the quadrant, whether hanging or fixed, an inaccessible distance, when at its end a perpendicular height stands erect, even if its lowest end is not seen; and thereby to derive the height itself as well. 43. 45
- Why the angle of incidence is equal to the angle of reflection. 200
- Various instruments for measuring. 42
- The construction and use of the instrument which the Italians call the *Squadra Zoppa* [the "lame square"]. 100
- What the instrument of parts is, and how it is constructed. 5
- How the instrument of parts may be otherwise constituted. 10
- An instrument most suitable for levelling. 101
- To measure by the quadrant an interval to whose ends approach is not permitted — provided they be visible and the interval produced reaches to the surveyor's feet — from some known height. 54
- To comprehend by the quadrant an interval set straight ahead of the surveyor, both of whose ends, or one, does not appear unless the surveyor moves to the right or to the left. 56
- To measure by the quadrant an interval on the horizon between a tower and some other marker, from the tower by two stations made at its summit, or from two windows one of which is perpendicularly below the other, when the space between those windows is known — even if the

height of the whole tower is unknown — and thereby incidentally to disclose the height of the tower. 55

- To search out by a very simple instrument an interval on the horizon between the surveyor and some marker in view. 96
- To find out, by aid of the set-square [Norma], an interval on the plane of the horizon between the surveyor and any marker. 94
- To measure by the quadrant an interval between two points on any raised plane, whether it be perpendicular to the horizon or inclined. 53
- To measure by the quadrat an interval between two markers or points on any plane, whether perpendicular to the horizon or inclined. 88
- To measure by the quadrant an interval, when the surveyor, standing at one of its ends or at some known height perpendicular to the plane on which the interval lies, can see the other end. 54
- To measure by the quadrat a transverse interval on the horizon, both of whose ends can be seen. 88
- To measure by aid of a rod an interval between the surveyor's feet and some marker on the plane of the horizon, when the far end of the interval can be seen. 93
- To render known by the quadrant a transverse interval on the horizon, both of whose ends can be inspected. 55
- Joseph Scaliger [Iosephus Scaliger] wrongly criticizes Archimedes concerning the *Measurement of the Circle*. 119
- The area of wholly irregular bodies. 145
- What isoperimetric figures are, and the treatment of them undertaken. 174
- Of regular isoperimetric figures, the greater is the one that contains more angles or more sides. 176
- Of isoperimetric figures having sides equal in number, the greatest is the equilateral and equiangular one. 180
- Of isoperimetric triangles having the same base, the greater is the one that has two equal sides. 177

- The area of an isosceles triangle. 109
- Two similar isosceles triangles on unequal bases are together greater than two isosceles triangles that stand together on the same bases, are mutually dissimilar, are isoperimetric with the former, and have their four sides equal to one another. 179
- Given two isosceles triangles whose bases are unequal and two sides of the one equal to two sides of the other: on the same bases to construct two similar isosceles triangles that are isoperimetric with the two former taken together. 178

L

- To find the two sides of an oblique-angled triangle from the third side and any two angles. 40
- Three sides in a quadrilateral are together greater than the fourth side. 202
- To find the segments of the side of an oblique-angled triangle made by the perpendicular, from the three given sides. 40
- To disclose the proportions of the sides from the given angles of any triangle. 39
- How the side of a regular figure may be determined from its area. 117
- How the side of a regular figure may be found from the semidiameter of the circumscribed circle. 115
- How the side of a proposed polygon may be found in a given circle by the instrument of parts. 8
- The side of the quadratrix is equal to a quadrant of the circle whose semidiameter is the base of the quadratrix. 192
- To make known the side of a right triangle from the base and either of the acute angles. 39
- To find the side of a right triangle from the base and the other [side]. *ibid.*
- To derive the side of a right triangle from the other side and either acute angle. *ibid.*

- To gather the side of an oblique-angled triangle from two sides and the angle contained by them. 40
- To learn the side of an oblique-angled triangle from the two remaining sides and any two angles. *ibid.*
- To search out the side of an oblique-angled triangle from two sides and the angle opposite one of them (provided the kind of the angle opposite the other given side be settled, when the given angle is acute). 42
- The area of a lens-shaped [lenticular] figure. 127
- To level uneven ground for conducting waters; or also, if one wishes, to make it equidistant [parallel] to the horizon. 101
- To find, by aid of the instrument of parts, how many equal parts of a straight line divided into any number of equal parts are contained in any other straight line. 6
- To determine by compasses, for a straight line divided into any number of equal parts, how many tenths, hundredths, and so on of one part are contained in any small portion of one part. 34
- By reference to what a line, a surface, and a solid are measured. 103
- What the straight lines are that fall under measurement. 43
- Two lines, one straight and the other curved, never meeting though they be produced to infinity, while the one ever approaches nearer the other. 163
- To describe the quadratrix line. 189
- To measure by the quadrant a straight line to whose ends approach is not permitted — provided they be visible and the straight line produced reaches to the surveyor's feet — from some known height. 54
- To measure by the quadrant a straight line on the horizon between a tower and some other marker, from the tower by two stations made at its summit, or from two windows one of which is perpendicularly below the other, when the space between those windows is known — even if the height of the whole tower is unknown — and thereby incidentally to disclose the height of the tower. 55

- To divide a given straight line by the instrument of parts, as another straight line is divided. 53
- To comprehend by the quadrant a straight line set straight ahead of the surveyor, both of whose ends, or one, does not appear unless the surveyor moves to the right or to the left. 56
- To measure by the quadrat a straight line on the horizon, when the surveyor, standing at one of its ends, cannot see the other end, nor is a height at hand, but can only recede to the right or to the left along a perpendicular line to a place from which the other end appears. 84
- To find a straight line equal to the arc of a quadrant. 191
- To find out by the quadrat a straight line lying on the horizon straight ahead of the surveyor, to whose ends [continues]
- [...a straight line lying on the horizon straight ahead of the surveyor] — to whose ends neither approach is permitted, nor may he measure it from the surveyor's position: provided the surveyor can go to the right or to the left along a perpendicular line to some place from which both ends appear. 58
- To measure by the quadrant a straight line, when the surveyor, standing at one of its ends or at some known height perpendicular to the plane on which the line lies, can see the other end. 54
- To measure by the quadrant a transverse straight line on the horizon, both of whose ends can be seen. 89
- To render known by the quadrant a transverse straight line on the horizon, both of whose ends can be inspected. 55
- That the mechanical measurement of certain lines and angles is to be admitted. 111
- To render known by the quadrant a transverse length on the horizon, both of whose ends can be inspected. 55
- To measure by the quadrat the length of a beam inclined to the horizon, of which only the upper part is seen, together with the angle of inclination, the distance of its base from the surveyor, and the height of the top above the horizon. 100

- To comprehend by the quadrant a straight length set straight ahead of the surveyor, both of whose ends, or one, does not appear unless the surveyor moves to the right or to the left. 56
- To obtain by aid of the quadrat the length of a shadow cast by a height while the Sun shines, when the projecting height is known. 96
- To comprehend by the quadrant the length of a straight line, when the surveyor, standing at one of its ends or at some known height that is perpendicular at that end to the plane on which the line lies, can see the other end. 54
- To measure by the quadrant a length on the horizon between a tower and some other marker, from the tower by two stations made at its summit, or from two windows one of which is perpendicularly below the other, when the space between those windows is known — even if the height of the whole tower is unknown — and thereby incidentally to disclose the height of the tower. 55
- To measure by the quadrat a length extended along the horizon, when the surveyor, standing at one of its ends, cannot see the other because of some swell of ground lying between, nor is a height at hand, but can only recede to the right or to the left along a perpendicular line to a place from which the other end appears. 84
- To find out by the quadrat a length lying on the horizon straight ahead of the surveyor, to whose ends neither approach is permitted, nor may he measure it from the surveyor's position, nor is any height present: provided the surveyor can go to the right or to the left along a perpendicular line to some place from which both ends appear. 85
- To measure by the quadrant a length to whose ends approach is not permitted — provided they be visible and the length produced reaches to the surveyor's feet — from some known height. 54
- To find out by a very simple instrument the length of the ascent of some mountain, if its peak is seen from an eye placed at the foot. 97

M

- A certain fourfold property of magnitudes. 196

- When magnitudes have been cut into proportional parts, if in each one part is again cut proportionally, then likewise the whole cuts will also be proportional. 148
- The solidity of a regular marble [block]. 13
- The mechanical measurement of some lines and angles is to be admitted. 111
- To find, near-truly, two mean proportionals between two given [lines], from Nicomedes. 163

Index rerum ac verborum [Index of Subjects and Terms] — Geometria Practica (continued)

The index runs in two columns on each page; entries are given here in reading order (left column top-to-bottom, then right column). Latin headword phrases are translated, and the locators (Geometria Practica page numbers) are kept exactly as printed.

[Letter M, continued]

- To find two mean proportionals between two given [lines], after Diocles, approximately. — 162
- To find two mean proportionals between two given [lines], after Hero and Apollonius of Perga, approximately. — 160
- To find two mean proportionals between two given [lines], after Philo of Byzantium and Philoponus, approximately. — 161
- To find a mean proportional number, or two means, between two given numbers. — 165
- The surveyor's general method for investigating the area of any figure whatever, or of a field. — 112
- Various measuring instruments. — 42
- In what terms the measure of lines, of surfaces, and of solids is to be taken. — 103
- How the tenth parts of thousandths are to be taken, even if the instrument of parts [instrumentum partium] is divided into only 100 parts. — 7
- How thousandth parts are to be taken, even if only 100 parts are contained in the instrument of parts. — 7
- How hundredth or thousandth parts on any straight line are to be taken, by means of the instrument of parts. — 6

- How any number of minutes may be cut off from any degree by the compasses [circinus]. — 34
- How minutes and seconds may be found by the compasses within any small part of a degree. — 33
- How minutes and seconds may be found by means of a constructed quadrant. — 14
- A fraction whose numerator is formed from the numerators of two fractions, and whose denominator from their denominators, is greater than the smaller and smaller than the greater. — 163
- The easy finding of a fraction lying between two others. — 116
- To reduce a large fraction to a smaller one nearly equivalent to it. — *ibid.*
- To measure the height of a mountain by the quadrant. — 47.48
- To measure the height of a mountain or tower from its summit by a single station, with the aid of the fixed geometric square [quadratum stabile], together with the distance of a mark seen on the horizon up to the perpendicular (foot) of the mountain or tower. — 82
- To determine the height of a mountain or tower from its top by the quadrant, if on the plane on which it stands some distance directly in front of the surveyor is known. — 52
- To measure out the height of a mountain or tower from its summit by the geometric square, when on a plane through the summit parallel to the horizon two stations can be made, and some mark is seen on the horizon. — 81
- To measure the height of a mountain or tower from its top by the quadrant, using two stations made on some upright pole, or at two windows of a tower one above the other, from which some mark can be seen on the horizon; and thence also to determine the distance from the foot of the mountain or tower up to the mark seen. — 50
- To measure out the height of a mountain or tower from its top by the quadrant, using two stations made on its summit, from which some mark appears on the horizon; and thence also to investigate the distance from the foot of the mountain, or the base of the tower, to that mark. — 48
- To estimate the height of a mountain or tower from its peak by the geometric square, when directly in front of the surveyor some interval between two marks, or even between some mark and the tower, is known. — 85

- To investigate the height of a mountain or tower from its summit by the geometric square, using two stations made on some upright pole, when some mark can be seen on the horizon. — 82
- What the area of an irregular many-sided figure is. — 112
- The solid content of any wall. — 133

N.

- To investigate the height of a tower, or of any other object, with the help of the square [norma]. — 94
- To perceive, with the help of the square, the distance on the plane of the horizon between the surveyor and any mark. — *ibid.*
- How the particular numbers for the individual kinds of roots are to be found. — 167
- To find the odd numbers that compose a given cube. — 228
- When someone thinks of a number, to guess how many units remain to him after three prescribed operations. — 200

O.

- The area of an oblique-angled triangle, from one side and two angles. — 111
- The area of an oblique-angled triangle, from two sides and the angle contained by them. — *ibid.*
- The area of oblique-angled quadrilaterals. — *ibid.*
- Problems of oblique-angled rectilinear triangles. — 40
- What the angle of observation is. — 44
- To describe with the compasses an oval figure resembling an ellipse. — 218
- To construct a regular octagon to a given height or width. — 215
- A regular octagon inscribed in a circle is the mean proportional between the square circumscribed about the circle and the square inscribed in it. — *ibid.*

P.

- The solid content of a parabolic conoid. — 144
- The area of a given parabola. — 129
- A parallelepiped contained under the square of either of the extremes (if there are four lines in continued proportion) and the other extreme is equal to the cube of the mean proportional that is nearer to the extreme first assumed. — 165

- The surface area of parallelepipeds, prisms, and cylinders. — 130.131
- To exhibit a cube equal to a rectangular parallelepiped. — 216
- To divide a parallelepiped or cube in a given ratio. — 218
- To construct a rectangular parallelepiped, of a given height or upon a given base, equal to a given cube. — 216
- To divide a given parallelogram into any number of equal parts by straight lines parallel to two opposite sides. — 160
- To bisect a given parallelogram by a straight line drawn from a point either outside it, or inside it, or on some given side. — *ibid.*
- To construct a parallelogram in a given angle equal to a given quadrilateral. — 199
- How a prescribed part may be cut off from a given straight line, by the instrument of parts. — 8
- Similar aliquot parts of several magnitudes have the same ratio. — 137
- How the tenth parts of thousandths are to be taken, even if the instrument of parts is divided into only 100 parts. — 7
- How hundredth or thousandth parts on any straight line are to be taken by means of the instrument of parts. — 6
- How any number of tenth or hundredth parts, etc., may be taken off by the compasses from any straight part divided into equal parts. — 35
- How thousandth parts are to be taken, even if only 100 parts are contained in the instrument of parts. — 7
- How to determine how many tenth parts of one hundredth, or how many thousandths of the whole side of the instrument, any small particle of one hundredth part of the instrument of parts comprises. — 7

[Letter P, continued]

- What the instrument of parts is, and how it is constructed. — 3
- How the instrument of parts may be constructed in another way. — 10
- What the hanging (plumb) quadrant [pendulus quadrans] is. — 14
- That a regular pentagon is not correctly constructed by Albrecht Dürer and others. — 210
- The circumference of a circle from a given diameter, and the diameter from a given circumference, more accurately. — 128
- The circumference of a circle, and the diameter, from its area. — *ibid.*
- What ratio the circumference of a circle has to the diameter according to Archimedes. — 120

- The circumference of a circle divided by $3 \frac{1}{7}$ yields a number greater than the diameter. — 122
- What the more accurate ratio of the circumference of a circle to the diameter is. — 126.
- The circumferences of circles are to one another as their diameters. — 124.176
- To find the true greater circumference of a circle from a given diameter. — 123
- To find a circumference equal to a given straight line. — 194
- To draw out the true lesser circumference of a circle from a given diameter. — ib.
- To make known the perpendicular falling on any side of a triangle from the opposite angle, from all three sides. — 42
- What segments the perpendicular makes on the side of an oblique-angled triangle. — 40
- To determine, by the instrument of parts, onto what point of the radius the perpendicular dropped from any degree of the quadrant falls. — 9
- When the perpendicular in a triangle is a surd number, what is to be done. — 109
- How the perpendicular in an equilateral triangle is to be found. — 114
- How the perpendicular from the center of a regular figure [i.e. the apothem] is to be found. — ibid.
- The plumb-line's thread, when it cuts the right shadow [umbra recta], makes the angle of the complement of the altitude; but when it cuts the turned shadow [umbra versa], it establishes the angle of the altitude itself. — 64
- How the sights (pinnules) [pinnacidia] for the line of sight are to be constructed. — 13
- How the sights are to be affixed on the quadrant. — ibid.
- How the side of a proposed polygon in a given circle is to be found by the instrument of parts. — 8
- To determine by the quadrant the portion of a greater height from a lesser height, and the portion of the lesser from the greater. — 59
- To perceive by the geometric square the portion of a greater height from a lesser height, and the portion of the lesser from the greater. — 90
- The convex surface of a segment of a sphere. — 142
- The solid content of a segment of a sphere. — 243

- The solid content of segments of a spheroid. — 144
- If two numbers prime to one another are both not squares or cubes, then no equimultiples of them will be squares or cubes either. And if equimultiples of two numbers are both squares, or cubes, then the numbers themselves will also be squares, or cubes. — 201
- To transform a prism and a cylinder into a cone and a pyramid; likewise a cone and a pyramid into an equal cylinder or prism. — 215
- To convert a prism, cone, cylinder, and pyramid into an unequal one, of a given height and upon a base of any number of angles. — *ibid.*
- To convert a prism, cylinder, cone, and pyramid into a parallelepiped upon a square base. — *ibid.*
- To construct a cube equal to a prism, cone, cylinder, and pyramid. — 216
- To construct a cylinder equal to any prism, and a cone equal to a pyramid; and conversely a prism equal to a cylinder, and a pyramid equal to a cone. — 215
- To construct a sphere equal to a given prism. — 217
- To change a prism, pyramid, cylinder, and cone into a rectangular parallelepiped equal to them, of a given height or base. — 217

[Letter P, continued — right column]

- To divide a given prism or cylinder in a given ratio. — 218
- To construct a pyramid or cone equal to a prism or cylinder under the same height; and conversely, a prism or cylinder equal to a pyramid or cone of the same height. — 214
- The surface area of parallelepiped prisms and of cylinders. — 130.131
- The proof (checking) of the extraction of roots. — 167
- How problems 3, 4, 5, 6, and 7 of Book 3 may be solved by a single station, with the aid of the fixed geometric square. — 79
- Problems of right-angled rectilinear triangles. — 39
- To make known by the geometric square the depth of a well, or of any building erected to the perpendicular, provided the angle of the bottom, or some mark placed at the bottom, can be seen. — 149
- To find by the quadrant the depth of a well, or of any building whatever erected to the perpendicular, provided the angle of the bottom, or some mark placed at the bottom, can be seen. — 60

- To examine by the quadrant the depth of a valley and its oblique descent, if it is not very uneven, and its end, or some mark in the valley, can be seen. — 61
- To determine by the geometric square the depth of a valley and its oblique descent, if it is not very uneven, and its end, or some mark in it, can be seen. — 97
- To find a root close to the true one, in numbers that are not squares, not cubes, not square-squares [Zensizensis — fourth powers], not surd-solids [surdesolidus — fifth powers], etc. — 170
- To find a mean proportional number, or two means, between two given numbers. — 165
- To investigate two mean proportionals between two given [lines], after Diocles, approximately. — 161
- To find out two mean proportionals between two given [lines], after Nicomedes, approximately. — 163
- To seek two mean proportionals between two given [lines], after Philo of Byzantium and Philoponus, approximately. — 161
- To find two mean proportionals between two given straight lines, approximately, after Hero and Apollonius of Perga. — 160.161
- To disclose the proportionals of the sides from the given angles of any triangle. — 25
- How the third and fourth proportional is to be found by the instrument of parts. — 9
- A beautiful property of the square. — 213
- A most beautiful property of the circle. — 209
- A certain property of four magnitudes. — 196
- A beautiful property of the quadratrix. — 191
- To find the point at which two straight lines inclined to one another meet. — 46
- To find, by the instrument of parts, the point of declination of any parallel on the diameter of the astrolabe. — 9
- To find by the quadrant the depth of a well, or of any building whatever erected to the perpendicular, provided the angle of the bottom, or some mark placed at the bottom, can be seen. — 60
- To make known by the geometric square the depth of a well, or of any building erected to the perpendicular, provided the angle at the bottom, or some mark placed at the bottom, can be seen. — 92

- To reduce a pyramid, cone, cylinder, and prism into an unequal one, of a given height and upon a base of any number of angles. — 215
- To change a pyramid, prism, cone, and cylinder into a rectangular parallelepiped equal to them, of a given height or base. — 216
- To build up a pyramid equal to a given sphere. — 217
- To convert a pyramid, prism, cylinder, and cone into a parallelepiped upon a square base. — 215
- To make a cube equal to a pyramid, cone, cylinder, and prism. — 216
- To construct a cone equal to a pyramid, and a cylinder equal to a prism; and conversely a pyramid equal to a cone, and a prism equal to a cylinder. — 215

[Letter P, tail]

- To construct a prism or cylinder equal to a pyramid or cone of the same height; and conversely, a cone or pyramid equal to a cylinder or prism under the same height. — *ibid.*
- The surface area of pyramids and cones. — 131
- To what rectangular solid a pyramid is equal. — 182

Q.

- What the hanging (plumb) quadrant is. — 14
- What the fixed quadrant [*stabilis quadrans*] is. — *ibid.*
- The quadrant, the radius, and the base of the quadratrix are in continued proportion. — 191
- The use of the quadrant, constructed in Book 1, chapter 2, in finding out minutes and seconds. — 14
- To construct a rectangle isoperimetric with, and equal to, a quadrant of a circle. — 186
- The construction of the quadrant for determining minutes and seconds. — 10
- The area of a square. — 100
- The extraction of the square root. — 168
- The extraction of the square and cube root from a given fraction. — 172
- To find the square and cube root geometrically, by lines, in numbers that are not squares and not cubes. — 173
- The construction of the geometric square. — 64
- A beautiful property of the square. — 213

- The difference of squares. — 227
- A table of squares and cubes up to the root 1000. — 221
- The generation of squares. — 227
- To describe a rectangle (a square longer on one side), a rhombus, and a rhomboid, from the excess of the diagonal over the side, etc. — 203
- To distribute a square number into any number of squares. — 201
- The three sides of a quadrilateral are greater than the fourth side. — 202
- To describe a circle equal to a square. — 194
- The side of the quadratrix is equal to the quadrant of the circle whose radius is the base of the quadratrix. — 192
- The base of the quadratrix, the radius of the quadrant, and the quadrant are in continued proportion. — 191
- A beautiful property of the quadratrix. — *ibid.*
- To describe the quadratrix line. — 195
- How the hanging and the fixed geometric square is to be employed. — 63
- The square of the circumference of the greatest circle in a sphere is to the surface of the sphere as the circumference of the greatest circle is to the diameter. — 137
- How a square equal to a circle may easily be exhibited from its own figure. — 193
- To construct a square equal to a given circle. — 193
- The square of the diameter of the greatest circle in a sphere has to the surface of the sphere a ratio greater than 7 to 22, but less than 71 to 223. — 138
- How a square equal to a given figure may easily be constructed. — 113
- The square of the diameter of a circle has to the circle a ratio of 14 to 11 nearly, according to Archimedes. — 124
- The square of the diameter of a circle has to the circle a ratio greater than 14 to 11, but less than 284 to 223. — 125
- The square of the circumference of a circle has to the circle a ratio greater than 892 to 71, but less than 88 to 7. — *ibid.*
- The square of the circumference of the greatest circle in a sphere has to the convex surface of the sphere a ratio greater than 223 to 71, but less than 22 to 7. — 138
- The square of the greatest side of a triangle is less than twice the sum of the squares described on the two remaining sides. — 207

- The quadrature of the circle by numbers according to the Arabs is false. — 187
- The quadrature of the circle by lines attributed to Campanus is false. — ibi.
- The quadrature of the circle by numbers from Albrecht Dürer is false. — ibid.
- The quadrature of the circle by lines. — 186
- The quadrature of the circle by Hippocrates of Chios is false. — 188.189
- That the quadrature of the circle is possible. — 189

[Letter Q, tail — right column]

- The area of a wholly irregular quadrilateral. — 112
- To construct easily, in a given angle, a parallelogram equal to a quadrilateral. — 199
- The area of non-right-angled quadrilaterals. — 111
- How the size of the angle contained by the sides of the instrument of parts is to be determined. — 9
- How the fourth and third proportional is to be found by the instrument of parts. — ibid.

R.

- The proper rule for extracting the cube root. — 170
- The extraction of the cube root. — 168
- The extraction of the square and cube root from a given fraction. — 172
- The extraction of the square root. — 167
- The extraction of the surd-solid (fifth) root [radix surdesolida]. — 169
- To extract a root of any kind whatever from a given number. — 166
- To extract a root of any kind from a given fraction. — 171
- To find the square and cube root geometrically, by lines, in numbers that are not squares and not cubes. — 173
- To find a true, close root in numbers that are not squares, not cubes, not square-squares, not surd-solids, etc. — 170
- The infinite kinds of roots. — 166
- How the square root of a fractional number is to be drawn out. — 110
- How many figures (digits) any root to be extracted can have. — 167
- To find by the compasses, on a straight line divided into any number of equal parts, how many tenths or hundredths, etc., are contained in any small piece of one part. — 34

- When a straight line is drawn from the acute angle of a right triangle to the opposite side, the ratio of this side to its segment near the right angle is greater than the ratio of that acute angle to its part opposite the said segment of the side. — 176
- For a straight line divided into any number of equal parts, to determine by the instrument of parts how many such parts are contained in any other straight line. — 6
- From a given straight line, however small, to take away a prescribed part or parts. — 208
- A straight line joining two opposite angles in two equal triangles that have a common side and lean toward opposite sides is bisected by the common side. — 157
- How a straight line equal to any circumference may easily be found from its own figure. — 193
- Two straight lines tangent to a circle and meeting at one point are greater than all the chords dividing the intercepted arc into any number of equal parts. — 196
- To add to a straight line another straight line, such that the square of the whole compound equals the square of the added line, together with the square of the line formed from the added line and the nearest segment of the prior line. — 191
- Three straight lines tangent to a circle and meeting at two points are greater than all the chords cutting the two intercepted arcs into equal parts. — 196
- To find a straight line equal to a circumference. — 194
- To exhibit any multiple whatever of a straight line, however small, even if it is not taken up by the compasses. — 207
- Which straight lines fall under measurement. — 43
- To make known by the quadrant a transverse straight line on the horizon, both ends of which can be observed. — 55
- To measure by the geometric square a transverse straight line on the horizon, both ends of which can be seen. — 88
- To measure by the quadrant a straight line whose ends one may not approach, provided they are visible and the line, when produced, reaches to the surveyor's feet, from some known height. — 43
- To measure by the geometric square a straight line on the horizon, when the surveyor, standing at one of its ends, [can] see the other end...

Index of Subjects and Terms [Index rerum ac verborum] — continued

R (continued)

[The first entry is the tail of a "Rectam lineam..." heading carried over from the previous page.]

To measure a straight line when the surveyor cannot see it and its height is not readily at hand, but he can only retreat to the right or left, along a perpendicular line, to a place from which the other end appears. 84

To measure with the quadrant a straight line when the surveyor, standing at one of its ends—or at some known height above the plane in which the line lies, being on the perpendicular—can see the other end. 43

To determine with the quadrant a straight line placed directly in front of the surveyor, both of whose ends (or one of them) are not visible unless he moves to the right or left. 56

To find, with the geometric square, a straight line lying on the horizon directly in front of the surveyor, whose ends he may neither approach nor measure from his position—provided he can move to the right or left, along a perpendicular line, to some place from which both ends appear. 85

To measure with the quadrant a straight line on the horizon lying between some tower and another marker, using two stations made at the tower's summit, or in two windows one directly beneath the other, when the space between those windows is known even if the whole height of the tower is unknown. And thereby, incidentally, to reveal the height of the tower. 55

To find a straight line equal to the arc of a quadrant. 191

To divide a given straight line, by means of the instrument of parts, in the way another straight line has been divided. 8

The area of a right triangle, from the side opposite the right angle and one acute angle. 110

The area of a right triangle, from one side about the right angle and one acute angle. *ibid.*

A certain property of two similar right triangles. 178

The area of a right triangle, from one side about the right angle and the side opposite the right angle. 110

The area of a right triangle. 109

To construct easily, upon a given straight line, a rectangle equal to a given rectangle. 107

To construct a rectangle, given the two excesses by which the diagonal exceeds each of its sides. 205

To construct a rectangle equal to a given rectilinear figure. 200

To construct a rectangle in which the excess of the diagonal over the greater side, and of the greater side over the lesser, is given. 206

The rectangle contained by the difference of the excesses (by which the diagonal of some rectangle exceeds each of its two sides), taken twice with the lesser excess, together with the square of the lesser excess taken twice, is equal to the square of the straight line by which the lesser side exceeds the lesser excess. 204

The rectangle contained by the segments of the diagonal of some rectangle (with parallels to the diagonal drawn through the same point) is equal to the two rectangles contained by the segments of the two unequal sides. 208

The rectangle contained by the diameter and the circumference of the greatest circle in a sphere is quadruple the greatest circle, and equal to the convex surface of that same sphere. 137

The area of rectangles. 103

Problems on right-angled rectilinear triangles. 39

Of two straight lines having the ratio that the side of the quadratrix has to the base, the greater is equal to the quadrant of a circle whose radius is the lesser line. 193

To increase or diminish, in a given ratio, a plane rectilinear figure or a circle. 164

To divide a rectilinear angle into three equal angles. 171

Given three straight lines in one plane, not parallel, to draw a straight line—sometimes even through a given point on the middle one—so that its segments between the middle line and the outer lines are equal, or have a given ratio. 169

Given two unequal rectilinear figures, to cut from the greater—by a line parallel to one side—a rectilinear figure equal to the lesser. 150

To construct easily a rectangle equal to any given rectilinear figure. 199

To construct, equal to a given rectilinear figure, a quadrilateral between two straight lines upon a given straight line by a parallel line. 148

For a rectilinear figure resolved into triangles from some one point, to find straight lines proportional, in order, to those triangles. 151

To construct, equal to a given rectilinear figure, a rectangular parallelogram that is also isoperimetric with it. 241

To cut a given rectilinear figure by a straight line, from any angle or point of a side, in a given ratio. 152

To divide a given rectilinear figure, from an angle or from a given point on a side, into any number of equal parts. 154

To divide a given rectilinear figure into any number of equal parts, by a straight line parallel to a given straight line. 157

To divide a given rectilinear figure, by a straight line parallel to a given straight line, in a given ratio. 154

Problems on oblique-angled rectilinear triangles. 40

Problems on right-angled rectilinear triangles. 39

To depict on a plane the site of a region, or a city, or a field. 99

Given two straight lines, of which the greater does not exceed the diagonal of the square on the lesser: to cut the greater so that the squares of the parts, taken together, are equal to the square of the lesser line. 206

Why the angle of reflection is equal to the angle of incidence. 200

The common rule for surveyors in investigating the area of any figure or field. 113

The proper rule for extracting the cube root. 170

A single rule for measuring all straight lines, when the ends of the objects are visible. 101

The construction of the rule [regula], in place of a thread with a plumb-bob. 13

To construct any regular solid equal to a given cube. 217

To present a sphere equal to a regular solid. *ibid.*

The area of a regular figure, to which a rectangle may be made equal. 175

The area of a regular figure, to which a right triangle may be made equal. *ib.*

A regular figure circumscribed about a circle has a greater perimeter than the circle. 195. 196.

The circle to which rectilinear regular figures are isoperimetric is greater than they. 223

Of isoperimetric regular figures, the greater is the one that contains more angles, or more sides. 176

The area of regular figures from the triangle up to the dodecagon, when the side is unity. 117

The area of regular figures. 114

The convex surface of the five regular solids. 135

What the area of the five regular solids is. 133. 135

The area of a rhomboid and of a rhombus. 156

The area of a rhombus and of a rhomboid. *ibid.*

To describe a rhombus and a rhomboid, from the excess of the diagonal over the side, etc. 203

S.

A sack of wheat, how it is to be measured. 133

The solidity of a regular [block of] stone. *ibid.*

What the altimetric scale is. 54

How secants and sines are to be taken on the instrument of parts. 7

The area of a sector of a circle. 127

The solidity of a sector of a sphere. 143

To find the segments of the side of an oblique triangle made by the perpendicular, from the three given sides. 40

To construct, upon a semicircle, a quadrant, an eighth part of a circle, etc., a rectangle isoperimetric with it and equal. 241

The finding of the radius of a circle from the side of an inscribed regular figure. 115

To describe a figure equal to several similar figures whose homologous sides are given; and a circle equal to several circles whose diameters or circumferences are given. 128

A certain property of two similar right triangles. 178

The ratio of two similar figures, or of circles, from two given homologous sides, or diameters, or circumferences. 128

Corresponding aliquot parts of several magnitudes have the same ratio. 137

When the whole sine is so small that it cannot be transferred to the instrument of parts, what is to be done. 9

With the whole sine set at 10000, how the tangents are taken on the instrument of parts. 7

With the whole sine set at 100, how the tangents are taken by the instrument of parts. 6

How sines and secants are taken on the instrument of parts. 7

To increase or diminish, in a given ratio, any solid figure of those Euclid treats in stereometry. 164

Which are the principal solids or bodies whose areas are investigated. 130

By what means solids, surfaces, and lines are measured. 103

The solidity of any portion of a sphere. 143

The solidity of a sphere. 139

The true, but smaller, solidity of a sphere, from the circumference of the greatest circle. 142

The true, but greater, solidity of a sphere, from the diameter of the greatest circle. ibi.

The true, but smaller, solidity of a sphere, from the diameter of the greatest circle. ibi.

The true, but greater, solidity of a sphere, from the circumference of the greatest circle. ibid.

The solidity of a spheroid. 144

The solidity, or area, of a hemisphere. 143

The solidity of the portions of a spheroid. 144

The solidity of a sector of a sphere. 143

The solidity of any wall. 133

The solidity of a frustum of a regular [block of] marble. ibid.

The solidity of a frustum of a sphere. 147

The solidity of a hyperbolic conoid. 145

The solidity of a parabolic conoid. 144

The solidity of a hollowed-out vessel. 132

A solid contained by plane surfaces and circumscribable about a sphere, to which a rectangular solid is made equal. 183

What the area of the five regular solids is. 178. 179

The convex surface of the five regular solids. 179

A sphere is greater than all bodies isoperimetric to it that are contained by plane surfaces and are circumscribable about other spheres. ibid.

A sphere is greater than all bodies isoperimetric to it and circumscribable about other spheres that are contained by conical surfaces. 185

The area or solidity of a sphere, from the diameter of the greatest circle. 143

To make a cone equal to a given sphere. 217

To present a cylinder equal to a given sphere. ibid.

A cube equal to a given sphere; and, to a given cube, to make an equal sphere. 216

To construct a pyramid equal to a given sphere. 217

To construct, equal to a given sphere, a rectangular solid upon a base of any number of angles. *ibid.*

The true, but smaller, surface of a sphere, from the circumference of the greatest circle. 142

The true, but smaller, surface of a sphere, from the diameter of the greatest circle. *ibi.*

The true, but greater, surface of a sphere, from the diameter of the greatest circle. i. [*ibi*]

The true, but smaller, area or solidity of a sphere, from the circumference of the greatest circle. 143

The area or solidity of a sphere, greater than the true one, from the circumference of the greatest circle. *ibid.*

The area or solidity of a sphere, from the diameter of the greatest circle. *ibid.* [printed "Spaerae" — compositor error for "Sphaerae"]

The solidity of a sphere. 139

A sphere, to the cube on its diameter, has a ratio greater than 223 to 426, but less than 11 to 21. *ibi.*

The convex surface of a sphere, to which a rectangle may be made equal. 137

The convex surface of a sphere, to the square of the circumference of the greatest circle, has a ratio greater than 7 to 22, but less than 71 to 223. 138

The area of the segments of a sphere. 142

The surface of a sphere is to the square of the circumference of the greatest circle as the diameter is to the circumference of the greatest circle. 137

The true, but greater, surface of a sphere, from the circumference of the greatest circle. 141

The convex surface of a portion of a sphere. 144

The solidity of a portion of a sphere. 143

The area of a sphere, and the convex surface of the same. 137. 139

The surface of a sphere, to the square of the diameter of the sphere or of the greatest circle, has a ratio greater than 223 to 71, but less than 22 to 7. 138

To describe a sphere equal to two or more spheres. 218

To construct a sphere equal to a given regular solid. 217

To construct a sphere equal to a given prism. *ibid.*

To take a smaller sphere from a greater, and to present a sphere equal to the remainder. 218

The solidity of the portions of a spheroid. 144

The area or solidity of a spheroid. *ibid.*

To increase or diminish, in a given ratio, a sphere or a solid figure. 164

What the *squadra zoppa* [carpenter's square, "lame square"] is among the Italians, and its use. 101

What the fixed quadrant [*stabilis quadrans*] is. 14

What the difference of stations is. 44

To observe, with the geometric square, the altitude of any star or of the Sun. 66

The convex surface of the five regular solids. 135

The ratio of a conical surface to its own base. 146

The convex surface of a sphere, and the area of the same. 137. 140

The convex surface of a hemisphere. 142

The convex surface of a sphere, to the square of the circumference of the greatest circle, has a ratio greater than 7 to 22, but less than 71 to 223. 138

By what means surfaces, lines, and solids are measured. 103

The convex surface of a sphere, to which a rectangle may be made equal. 137

The true, but greater, surface of a sphere, from the circumference of the greatest circle. 141

The true, but greater, surface of a sphere, from the diameter of the greatest circle. *ibid.*

The true, but smaller, surface of a sphere, from the circumference of the greatest circle. *ibid.*

The true, but greater, surface of a sphere, from the diameter of the greatest circle.
i. [ibi]

The true, but smaller, surface of a sphere, from the circumference of the greatest circle. *ibid.*

The true, but greater, surface of a sphere, from the diameter of the greatest circle.
i. [ibi]

The surface of a sphere is to the square of the circumference of the greatest circle as the diameter is to the circumference of the greatest circle. 137

The convex surface of a portion of a sphere. 142

The surface of a sphere, to the square of the diameter of the greatest circle, has a ratio greater than 223 to 71, but less than 22 to 7. 138

The convex surface of a cone and of a right cylinder. 146.

The cylindrical surface, its bases removed. *ibid.*

The surface of a frustum of a cone, its bases removed. *ibid.*

The extraction of the surd-solid root [i.e. the fifth root]. 168

T.

A table containing the sides of the regular figures, from the triangle up to a figure of 80 sides, the diameter set at 20000000, or the whole sine at 10000000. 115

A wonderful table for drawing out the numbers for the several kinds of roots. 166

The construction and use of the table for finding minutes and seconds from the quadrant. 16

The gnomonic table. 68

The easiest construction of the gnomonic table, and its use. 66

The extension of the table for minutes to further quadrants, without the golden rule [rule of three]. 16

A table for investigating minutes and seconds from the quadrant. 17

A table of squares and cubes up to the root 1000. 220

A small table for finding minutes and seconds from a constructed quadrant. 15

When the tangent exceeds the whole sine, what is to be done, so that by a single transfer the point sought may be found on the instrument of parts. 9

With the whole sine set at 10000, how the tangents are taken on the instrument of parts. 7

How the tangents are found by means of the instrument of parts, with the whole sine set at 1000. 7

Three tangents meeting in two points are greater than all the chords cutting into equal parts the two arcs they intercept. 196

Two tangents meeting in one point are greater than all the chords dividing into any number of equal parts the arc they intercept. 195

How the tangents are taken with respect to the whole sine of 100, by means of the instrument of parts. 6

How the tangents are taken when the instrument of parts contains 1000 parts. 6

Given some tangent found on the side of the instrument of parts, how the same is found with respect to a given whole sine. 7

How very small tangents are taken on the instrument of parts. 7

To measure the circuit of the earth from some elevated mountain. 214

How the third and fourth proportional are found by means of the instrument of parts. 9

To measure with the geometric square the length of a beam inclined to the horizon, of which only the upper portion is visible, together with the angle of inclination, the distance of its base from the surveyor, and the height of its top above the horizon. 101

The area of a trapezium having two parallel sides. 111

The easy measurement of an irregular trapezium. 114

The area of a trapezium having no parallel sides. 157

Two similar isosceles triangles on unequal bases are together greater than two isosceles triangles together on the same bases—triangles that are indeed dissimilar to each other, but isoperimetric with the first pair, and having their four sides equal to one another. 179

The area of an isosceles triangle. 109

The area of an oblique triangle, from one side and two angles. 111

The area of an oblique triangle, from two sides and the angle contained by them. *ibid.* [printed "Arianguli" — compositor error for "Trianguli"]

The area of a right triangle, from one side about the right angle and the side opposite the right angle. 110

A fine property of a triangle, if a line parallel to one side is drawn in it, etc. 158

Given two isosceles triangles whose bases are unequal, and the two sides of one equal to the two sides of the other: to construct on the same bases two similar isosceles triangles, isoperimetric with the first two taken together. 178

The area of an equilateral triangle. 109

The area of any triangle, to which a rectangle may be made equal. 174

The area of a right triangle. 109

The area of a right triangle, from one side about the right angle and one acute angle. 111

The area of a right triangle, from the side opposite the right angle and one acute angle. 110

A remarkable property of a triangle, if from two of its angles straight lines are drawn to the midpoints of the opposite sides, etc. 158 [printed "Arianguli" — compositor error for "Trianguli"]

Upon the third side of a triangle of two unequal sides, to construct a triangle isoperimetric with the former one and having two equal sides. 177

To construct a triangle equal to a parallelogram and isoperimetric with it. 186

Problems on right-angled rectilinear triangles. 39

A certain property of two similar right triangles. 178

Of isoperimetric triangles having the same base, the greater will be the one that has two equal sides. 177

Problems on oblique-angled rectilinear triangles. 40

To divide a given triangle, from a given point on a side, into any number of equal parts. 159

To divide a given triangle, by lines parallel to one side, into any number of equal parts. *ibid.*

To part a given triangle, by a straight line from a point given outside the triangle, into two equal parts. 160

The whole triangle is to the triangle cut off by a straight line as the rectangle under the two sides that are cut is to the rectangle contained under the two sides of the cut-off triangle. 292

A heap of wheat, how it is to be measured. 133

A sack of wheat, how it is to be measured. *ibid.*

To measure with the geometric square the height of a tower or mountain from its summit, when in the plane of the summit, equidistant from the horizon, two stations can be made, and some marker is seen on the horizon. 81

To find out the height of a tower or of any other thing by means of a rod. 94

To investigate the height of a tower or of any other thing by means of the carpenter's square [Norma]. *ibid.*

To investigate with the geometric square the height of a tower or mountain from its summit, by two stations made upon some upright staff, when some marker can be seen on the horizon. 82

To measure with the quadrant the height of a tower from its top by a single station, if the distance of a marker seen on the horizon up to the base of the tower is known. 51

To measure the height of a tower or mountain from its summit by a single station, by means of the fixed square, together with the distance of a marker seen on the horizon up to the tower, or the perpendicular of the mountain. 82

To measure with the quadrant the height of a tower or mountain from its top, if in the plane on which it stands some space, positioned directly in front of the surveyor, is known. 52

To estimate with the geometric square the height of a tower or mountain from its top, when, directly in front of the surveyor, some interval between two markers—or even between some marker and the tower—is known. 85

To measure with the quadrant the height of a tower or mountain from its top, by two stations made at its summit, from which some marker appears on the horizon.

Index of things and words [conclusion]

Continuation of the T entries carried over from the previous page (the catchword there was "Atque"); the locators are Geometria Practica page numbers.

Left column:

- And from this point, to investigate also the distance from the tower's base, or from the plumb-line of the mountain, to that mark. — 48
- **Turris** [tower]: To measure by the quadrant the height of a tower, or of a mountain, from its summit, by means of two stations set up on some upright staff, or at two windows of the tower (one placed above the other), from which some mark on the horizon can be seen; and from this to determine also the distance from the plumb-line of the tower or mountain up to the observed mark. — 50
- **Turrium** [of two towers]: With the summits of two towers in view, even if their bases are concealed by intervening buildings, to estimate by means of the geometric square both the distance between their bases and the distance between their tops, together with their heights and their distances from the surveyor. — 101

V.

- **Vallis** [valley]: To examine by the quadrant the depth of a valley and its oblique descent — provided it is not very uneven and its boundary, or some mark in the valley, can be seen. — 63
- **Vallis** [valley]: To determine by the geometric square the depth of a valley and its oblique descent — provided it is not very uneven and its boundary, or some mark in it, can be seen. — 92

(These two near-identical Vallis entries are a genuine cross-index: the first by the quadrant, the second by the geometric square. The second continues across the column break into the head of the right column.)

Right column:

- **Varia instrumenta mensurandi** [various instruments for measuring]. — 42

- **Vasis excavati soliditas** [the solid content of a hollowed vessel]. — 133
- **Vasis excavati capacitas** [the capacity of a hollowed vessel]. — *ibid.*
- **Umbra recta ac versa** [right and turned shadow]: How the right shadow and the turned shadow [umbra recta, umbra versa] are to be determined on the geometric square. — 65
- **Umbrae longitudo** [length of the shadow]: To obtain, by means of the square, the length of the shadow cast from a height when the sun is shining, once the height is known. — 96
- **Umbra recta ac versa**: Which portion, and into how many parts, each of the right and turned shadow is divided on the square by geometers. — 66
- **Umbra recta ac versa**: Into how many parts the right and turned shadow are divided in this work. — *ibid.*
- **Umbrae rectae ad versam reductio** [reduction of the right shadow to the turned]: The reduction of the right shadow to the turned, and the reverse. — 95
- **Urbis, vel campi, aut regionis situm in plano describere** [to depict on a plane the site of a city, field, or region]. — 100
- **Usus et constructio tabulae** [use and construction of a table]: The use and construction of a table for finding minutes and seconds from the quadrant. — 15
- **Usus quadrantis** [use of the quadrant]: The use of the quadrant constructed for finding minutes and seconds. — 13
- **Usus tabulae quadratorum et cuborum** [use of the table of squares and cubes]: The use of the table of squares and cubes in extracting square and cube roots. — 14

FINIS.

(The V entries close the alphabet — there are no X or Z entries — so the Index, and with it the back matter of the Geometria Practica, ends here.)

EPITOME OF PRACTICAL ARITHMETIC

*Work boundary. Page 251 (PDF) is the divisional title page opening the second work of the volume. Its display title reads **EPITOME ARITHMETICAE PRACTICAE** ("Epitome of Practical Arithmetic") — note that the volume advertises this work simply as the *Arithmetica Practica*; the title page itself calls it an Epitome.*

Divisional title page:

CHRISTOPHORI CLAVII BAMBERGENSIS E SOCIETATE IESV, EPITOME
ARITHMETICAE PRACTICAE.

[Christoph Clavius of Bamberg, of the Society of Jesus — Epitome of Practical Arithmetic.]

To the Kind Reader

While the knowledge of every mathematical subject delights me, it is from the study of arithmetic above all that I draw an incredible pleasure — and this not only because of a certain surpassing dignity it possesses, but also because without arithmetic, as I at least judge, no science — indeed, as Plato ventures to say [Margin: In the *Epinomis*] — nor even human society itself could hold together.

For in the mutual dealings and contracts by which this fellowship among people is generally maintained, countless occasions arise where accounts of income and expenditure must be rendered or called in, records must be drawn up, and a sum must be reckoned — one to be divided into several parts in a fixed proportion, whether the total is shared out equally or unequally. To cheat, or to be cheated, in such matters is equally shameful and ruinous. And so Plato said something bold, yet nonetheless true [Margin: In the *Epinomis*, and in *Republic* VII]: that those who would drive arithmetic out of life drive prudence, and indeed all humanity, out of the world — since without it neither public nor private affairs can stand.

Moreover, the other disciplines lean on arithmetic so heavily that, once it is shaken, they collapse in the same downfall. For neither the astrologer nor the geometer will win general acceptance for his own theorems — so that they carry not only truth but also pleasure joined to usefulness — unless he has thoroughly grasped in his mind the whole nature of number. And if he has slipped even a little in weighing his calculations, you will then witness an enormous ruin of everything else. It was for this very reason that Plato, that prince of genius, wanted arithmetic to be the vestibule and entryway to the remaining disciplines [Margin: In *Republic* VII] — not only because those disciplines are nothing without number, but also because it is through the handling of numbers that the mind is sharpened and made ready to take in the remaining reaches of learning.

Captivated by the beauty of this most excellent science, I had once given myself over to investigating the entire nature of number, so that, having grasped it in mind and understanding, I might at last set it forth in writing, and might reduce the precepts of arithmetic and the lessons of algebra (a subject not familiar to everyone) — than which you would scarcely find anything of more distinguished

value in life — to certain fixed headings and to easier demonstrations, so that they would be clear and ready to hand for anyone. It was indeed a splendid undertaking, but one demanding much labor and time.

And so, in the meantime, while that work was being polished more finely by me, I began to gather separately, for my own use, into a little notebook, the material I had collected scattered across many books, so that it would be at hand and I could explain it to my students. For those who have treated arithmetic up to now have either muddled the subject by a multitude of rules or obscured it by brevity — so much so that (though in saying this I would have nothing subtracted from anyone's reputation) beginners can scarcely find in this science a master and guide to follow. When this little book slipped from me unawares and came into people's hands, weighty men pressed me with the most urgent entreaties to share it with a wider audience, arguing that it would prove most useful both to students generally and especially to those who attend our schools — and that to be unwilling to serve their benefit was unbecoming one who had consecrated himself and all he had to the glory of God and the good of all.

Moved by their entreaties and their authority, I have decided to commit this little book to the press. I should like you, reader, to receive it in the same spirit in which we give it, and to make use of it in the meantime, until that larger work on arithmetic comes to light — which, God helping, we hope will happen very soon. Farewell.

The Numeration of Whole Numbers

Chapter I

Numeration is the description and expression of any proposed number by means of its proper characters and figures. [Margin: What numeration is.]

For this purpose arithmeticians use ten characters, or figures, to describe every number — namely: [Margin: The ten figures of numbers.]

1. 2. 3. 4. 5. 6. 7. 8. 9. 0.

The first nine of these are called *significant*. Each of them signifies as many units as the place it occupies in the series set before it: thus this figure 6 signifies six units when it stands in the sixth place, and so on with the rest. The tenth and last figure signifies nothing by itself and is called *cipher*; nevertheless it increases the

signification and value of the other figures, as will become clear from what follows.

In any number written with several figures, there are said to be as many *places* as there are figures, whether significant or not. The first place, or first figure, is the one at the far right; the second place, or second figure, is the one that immediately follows the first toward the left, and so on, so that the figure occupying the leftmost seat is called the last. [Margin: How many places there are in any number.] Thus in 4352 the first figure is 2 and the last is 4. If, however, the individual figures were to represent individual numbers set apart in this way — 4. 3. 5. 2. — then the first figure would be 4 and the last 2. [Margin: Which is the first and which the last figure in any number.] The reason the order of places or figures in any number runs from right toward left is this: arithmetic is said to have been invented by the Phoenicians, who are accustomed to write from right toward left, in the manner of the Hebrews, the Arabs, and the Chaldeans. [Margin: Why the order of places in any number proceeds from right toward left.]

Any figure placed in the first place represents simply itself; in the second it signifies ten times itself; in the third, a hundred times; in the fourth, a thousand times; in the fifth, ten thousand times; in the sixth, a hundred thousand times; in the seventh, a thousand thousand times — and so on to infinity, so that the places in order surpass one another in tenfold proportion: the second the first, the third the second, the fourth the third, and so forth. [Margin: What each figure signifies in a given number.] Take 34567 as an example. The first figure, 7, signifies only seven units; the second, 6, signifies sixty, that is, ten times 6; the third, 5, five hundred, that is, a hundred times 5; the fourth, 4, four thousand units, that is, a thousand times 4; the fifth, 3, thirty thousand units, that is, ten thousand times 3. [Margin: In any number the figures surpass one another in order in tenfold proportion.] Therefore the whole number is to be uttered thus: thirty-four thousand five hundred and sixty-seven. In the same way any other number can be expressed, if one carefully notes how many times each figure signifies itself according to the various places in which it stands.

Moreover, to make numeration easier, the number must be divided into *members* in this way. Over the first figure at the right hand place a point. Then, proceeding toward the left and passing over two figures, set another point over the next figure (which stands in the fourth place). And so on: always passing over two figures without a point, write a point over the following figure — as you see done

here. [Margin: What must be observed so that the resulting number may be expressed.]

42329089562800

— with a point set over the first (rightmost) figure of each three-figure member, marking off the five members **42 · 329 · 089 · 562 · 800**.

Any figure standing under a point, together with the two preceding it toward the left, constitutes one member, so that each member consists of three figures — except the last member on the left, which may sometimes have a single figure (the one placed under the point). This would happen in the proposed example, divided into five members, if the last figure 4 were removed; and sometimes it may have two figures, as in the given example.

And so, to utter any number, it is enough if each member is expressed separately — its first figure signifying units, its second tens of units, and its third hundreds. But after the utterance of each member the word [Thousand] must be appended as many times as there are members following the one being uttered — yet so that the word is first put in the nominative plural, and thereafter always said as [times a thousand], as you will presently hear.

Now the member that is last toward the left is the first in the utterance, while the one that is [A 2] (*catchword: primum*)

first at the right hand is the last in the utterance. The number proposed a moment ago, then, will be pronounced thus:

The **first** member, 42, thus: forty-two thousand, times a thousand, times a thousand, times a thousand — so that the word [thousand] is heard four times, on account of the four members that follow this one being uttered.

The **second** member, 329, in this way: three hundred twenty-nine thousand, times a thousand, times a thousand.

The **third** member, 089, thus: eighty-nine thousand, times a thousand.

The **fourth** member, 562, thus: five hundred sixty-two thousand.

The **fifth** and last member, 800, in this fashion: eight hundred.

Numeration will be made still easier if, instead of the first point, one sets 0; and 1 for the second point; and 2 for the third; and 3 for the fourth, and so on — as you see done here in the same example:

4 3 2 1 0
42329089562800

(the labels 4, 3, 2, 1, 0 standing over the first figure of each successive member).

By this device it is readily understood how many times the word [thousand] is to be appended in the utterance of each member: it must be uttered as many times as the units contained in the figure set over the member to be pronounced.

Now if, in the manner of the Italians, we should wish to call thousands of thousands *Millions*, we shall express any proposed number in fewer words — and perhaps more significantly — by dividing it into larger members, in the way that follows. Over the first figure at the right hand place 0. Then, passing over five intermediate figures, set 1 over the following figure, which occupies the seventh place. After this, again passing over five figures, write 2 over the figure that occupies the thirteenth place. And so on: always passing over five figures, set 3, 4, 5, and so forth — as has been done here in the same example:

2 1 0
42329089562800

Any member contains six figures (except the last member, which may have a single figure, or two, or three, or four, or five); all of them are to be uttered together, and after the utterance of each member the word [Million] is to be added as many times as there are units in the figure set over the member — put first in the nominative, but thereafter always in an oblique case. And to utter each member more easily, a point is to be placed under its fourth figure, which will signify that the place of thousands is there. The example above, then, is to be uttered thus: forty-two millions of millions; three hundred twenty-nine thousand millions, eighty-nine millions; five hundred sixty-two thousand, eight hundred.

The Addition of Whole Numbers

Chapter II

Addition is the collecting of two or more numbers into a single sum. [Margin: What addition is.]

The numbers to be added are to be arranged so that, one being set beneath another, the first figures correspond to one another, likewise the second figures to one another, and the third, and the fourth, and so on — so that any deficiency of figures, if there be one, is seen on the left side. [Margin: How the numbers to be

added are to be arranged.] Thus these numbers to be added — 710654, 8907, 56789, 880 — will be arranged as appears here:

```

710654
8907
56789
880
-----
777230 [cross: 8 × 8]

```

Then, a line having been drawn beneath the numbers to be added, first all the first figures are added together. [Margin: How addition is carried out.] If the resulting number can be written with a single figure, it is placed beneath the first figures below the line; but if it must be written with two figures, only the first of them is placed there, and the remainder is kept back, to be joined afterward to the second figures when they are added together. After this the second figures are added in the same way, the previously kept-back figure being added first (if indeed any was kept back); and likewise the third, the fourth, and the rest. But if, from the addition of the last figures, a number arises that must be written with two figures, both are to be placed beneath the line, none held back, because the whole addition is then completed.

For example. In the first figures of the proposed example, 0 and 9 make 9; I add 7, they make 16; I add 4, they make 20. So I place 0 beneath the first figures and keep 2. Then in the second figures, from 2 (the figure I had kept) and 8 come 10; I add 8, they make 18; I add 5, they make 23. So I place the figure 3 beneath the second figures and retain 2. After this I proceed to the third figures, where from 2 (the figure I had retained) and 8 come 10; I add 7, they make 17; I add 9, they make 26; I add 6, they make 32. So I place 2 beneath the third figures and retain 3. Again in the fourth figures, from 3 (the figure retained) and 6 come 9; I add 8, they make 17; I add 0, they make 17. So I place 7 beneath the fourth figures and keep 1, which I add to the fifth figures, making 7. I therefore place 7 beneath the said figures and reserve nothing. Finally, since in the last place this single figure 7 is found, I place it beneath the line, and the addition will be complete. And just as we added the figures of the numbers to be added by ascending from the lower toward the upper, so too they can be collected into one sum by descending from the upper toward the lower.

(catchword: QVOD)

Now if it should ever happen that, from the addition of the figures of a single place, a number arises that must be written with three figures, the first figure is to be placed beneath that place, and the remaining two are to be added to the

figures of the following places — the first of them to the figures of the nearest place, the second to the figures of the next place; or else the number expressed by those two retained figures is to be added to the figures of the following place.

[Margin: What to do when the figures of a single place yield a number that must be written with three figures.] Thus in the example set alongside, since from the first figures the number 102 is collected, the figure 2 is to be written beneath the first place, the figure 0 is to be added to the figures of the second place, and the figure 1 to the figures of the third place. Or else the whole retained number 10 is to be added to the figures of the second place, so that the number 15 is collected, whose figure 5 is placed beneath the second place, and whose figure 1 is added to the figures of the third place, and so on. For by either method the same number will always be collected. And you see this example tested by the casting out of 9, of which test we shall speak presently.

You will do well, moreover, when there are many numbers to be added, to distribute them into several classes and collect a separate sum from each class. [Margin: What to do when there are many numbers to be added.] For if you then reduce these sums into one sum, you will have the sum collected from all the given numbers, and you will avoid the trouble that necessarily arises in collecting so many figures into a single sum. Thus, if you reduce the next example — these four classes, 15026, 655, 4545, 3526 — into one sum, you will produce the sum 23752, the same that we had collected before, as appears here. And it is evident that by this latter method an error can be committed not so easily as by the former, because in this one not so many figures are added at once as in that one.

(class 1)	(class 2)	(class 3)	(class 4)
6008	308	108	3009
5009	239	309	209
4009	108	4128	308
15026	655	4545	3526

The four class-sums, reduced into one:

```

15026
 655
 4545
 3526
-----
23752 [cross: 1 × 1]
```


The same result reached by collecting all twelve numbers into a single sum at once:

```
6008
5009
4009
308
239
108
108
309
4128
3009
209
308
-----
23752
```

After completing the addition, arithmeticians are accustomed to examine it — as they do the other operations that follow — to see whether it has been done correctly or not. In addition this can be done in four ways. The first is by the casting out of nines, in this fashion. [Margin: The first proof of addition, by 9.] Let 9s be cast out of all the numbers to be added, as often as can be done, and let what remains be set apart. Then let 9s likewise be cast out of the collected sum, as often as can be done, and let what remains be noted. For if this latter remainder is equal to that former one, the addition was rightly performed; but if unequal, not rightly, and so the addition must be repeated, that the error may be corrected. Thus you see in the example above that the number 8 was left after the casting out of 9 — both from all the numbers to be added and from the collected sum — and this number 8 has been placed in a certain cross constructed for this purpose.

For the rest, so that 9s may be easily cast out, it is enough to add the figures of the numbers among themselves as if they all occupied the first place; and as soon as the sum reaches 9, or exceeds 9 so that it is written with two figures, let the 9s be cast out — or let those two figures be added together, and this sum added to the following figure in the same way, and so on. [Margin: By what method 9s are easily cast out of any number, as often as can be done.] For this is the remarkable property of the number nine: that if you add together the figures of any number whatever and cast 9 out of the sum — or, if this sum is written with two figures, collect these two figures into one sum — just as much is left over, or made up, as would be left if you cast out 9 from the whole number as many times as you can. [Margin: The remarkable property of the number nine.] For example: if 9s are

cast out of the number 38 as many times as can be done — namely four times — 2 remains, for four times 9 make 36. But if you say that 3 and 8 (taking the figures of the same number 38 separately) make 11, and cast out 9; or indeed say that 1 and 1 make 2 (taking likewise the figures of this number 11 just made up, separately) — you will have the same 2 that were left before. Likewise, if 9s are cast out of the number 41 as many times as can be done — namely four times — 5 remains; and if you say that from 4 and 1 (taking the figures of the number 41 separately) also come 5. Finally, from the number 78, if 9s are cast out as many times as can be done — namely eight times — 6 is left; and if you say that 7 and 8 make 15, and cast 9 out of this; or indeed say that 1 and 5 make 6 — you will have just as much as was left before. And the same reasoning holds for the rest.

So that you may see, then, by what method the examination of addition is to be set up, we shall examine the first example in this way:

7 and 1 make 8. Adding 6, they make 14, that is, 5 — for 1 and 4 make 5, which is of course what would be left if 9 were cast out of 14, as has been said. Adding 5 to those 5, they make 10, that is, 1. Adding 4, they make 5. Adding 8, they make 13, that is, 4. Adding 7, they make 11 (for 9s are always omitted, since they are perpetually to be cast out), that is, 2. Adding 5, they make 7. Adding 6, they make 13, that is, 4. Adding 7, they make 11, that is, 2. Adding 8, they make 10, that is, 1. Adding 8, they make 9, that is, 0 — for the 9s are to be cast out. And 8 remains, which I place in one arm of the cross. Then, in the sum produced, from 7 and 7 come 14, that is, 5. Adding 7, they make 12, that is, 3. Adding 2, they make 5. And finally, adding 3, they make 8, as before — which I set in the opposite arm of the cross, so that the equality may appear of the numbers that were left after the casting out of 9.

710654

8907

56789

880

777230 [cross: 8 × 8]

But since by this method 9s are not cast out as many times as can be done, but only the residue is sought — by means of the stated property of the number nine — that would remain if all the 9s were cast out, it comes about that this examination set up by the casting out of 9 is fallible, as appears in this example. [Margin: The proof by 9 is fallible, and why it is fallible.] For the collected sum is false, and nevertheless the examination made by 9 indicates that it has been correctly collected, since on both sides a unit remains over. But if 9s are cast out

as many times as can be done, the falseness of the collected sum will at once appear. For 9s will more often be cast out of the sum than [A 3] (*catchword: ma,quam*)

$$\begin{array}{r|l}
 25 & 7 \\
 30 & 3 \\
 \hline
 64 & 1 \text{ [cross: } 1 \times 1]
 \end{array}$$

(the residues 7 and 3 stand beside the addends 25 and 30, and 1 beside the false sum 64; the cross gives 1 on each side, so the fallible test wrongly confirms a sum that should be 55.)

[This page continues the discussion of casting out nines from the preceding chapter on addition.]

—than [one more nine is cast] out of the added numbers. For in this sum, 64, nine is contained seven times with a unit left over, since seven nines make 63. But in the number 25, nine is contained twice, and 7 is left over, which I set down on the right-hand side; and in 30, nine is contained three times, with 3 left over, which I likewise note on the right side. And so from the numbers being added, nine is cast out five times, leaving 7 and 3; and in these [that is, in 10] nine is contained once more, with a unit left over. In truth, then, nine has been cast out only six times from the added numbers, but seven times from the sum. It is therefore no wonder that the sum is false, even though a unit was left over on both sides. The true sum would be 55, in which nine is contained six times with a unit left over — just as in the numbers being added.

By the same token, suppose that after correctly collecting the sum someone transposed some of the digits, or inserted into the sum, or into the numbers to be added, the digit 9 or 0 as often as he liked, or the pair 7 and 2, or 6 and 3, or 4 and 5, or 8 and 1: the check would still declare the sum to be correctly collected — which nonetheless is false. For once this addition has been correctly carried out together with its check, if someone maliciously altered the sum to 1565, the check would still keep its force, and yet the sum would not be true. You may say the same if someone transposed the order of the digits in the numbers being added, or indeed inserted the digit 9 or 0, as appears here:

$$\begin{array}{r}
 1425 \\
 2308 \times 8 \\
 \hline
 1655
 \end{array}$$

```

14925
 2309 8 × 8
-----
10655

```

[Margin: Why the proof by 9 is used by arithmeticians, although it is fallacious.]

Since this is so, one might reasonably ask why arithmeticians employ this proof by 9. The answer is that, although the proof can be shown to be faulty by a deliberate trick — as was clear in the preceding examples — it is nevertheless not without reason that skilled arithmeticians use it. For no one (unless he means to go wrong on purpose) will commit an error of the kind that lets this check succeed; he will instead miss the mark by one unit or two, so that the check then readily shows that an error has been made, and thus indicates that the addition must be corrected. For who would be so mad as to collect that final sum out of only the first two numbers? In short, unless the numbers are deliberately arranged so that the same remainder always results once the nines are cast out, it will scarcely or only very rarely happen that this check succeeds — except when no error has in fact been made in the addition.

[Margin: Second proof of addition, by 7.]

The check may be made another way, by casting out sevens, as follows. Let 7 be cast out of all the numbers being added, as often as can be done, and let what is left over be set apart on one arm of the cross. Then let 7 likewise be cast out of the collected sum as often as can be done, and let the remainder be placed on the other arm of the cross. If this latter is equal to the former remainder, the addition has been done correctly; but if unequal, incorrectly. Now 7 must be cast out of each number being added separately, and the remainders placed on the right, opposite each number; and out of these remainders, once collected into a single sum, 7 must again be cast out, and this last remainder placed on one arm of the cross.

[Margin: How 7 is to be cast out of any number.]

But 7 is not to be cast out in the same way we described for casting out 9, since the number 7 does not possess that property of nine. Instead, the first two digits on the left are to be taken, as though the first of them signified tens and the second units — provided the first is less than 7. (For if it were 7 or greater, 7 would have to be cast out of that digit alone.) From the number these two digits represent, 7 is to be cast out as often as possible; the remainder is taken as tens, and the next digit is joined to it as units; and from this new number, which the remainder and the following digit express, 7 is again to be cast out as often as

possible, and so on. For example, 7 is cast out of the number 2379 in this way. From 23, if 7 is cast out three times, 2 remains. Again, from 27 (for the digit 2 that was left, together with the next digit 7, makes this number 27) let 7 be cast out three times, and 6 is left. Finally, from 69 (which is made up of the remainder 6 and the digit 9) let 7 be cast out as often as possible, and 6 remains — the very number that would have been left had all the sevens been cast out of the given number at once. In the same way, 7 is cast out of the number 783 thus: from 8 (for the leading 7 is dropped, as was said, and 7 is cast out of 8) let 7 be cast out, and 1 remains; then again from 13 let 7 be cast out, and 6 remains; and so with the rest.

And so the earlier example will be checked by this method:

710654		0
8907		3
56789		5 6 × 6
880		5

777230		

Dropping the digit 7: if 7 is cast out of 10, 3 remains; likewise if 7 is cast out of 36, 1 remains; casting 7 out of 15, 1 remains; and casting 7 out of 14 at last, 0 remains — which figure I place to the right of the first number, having first drawn a line to separate the numbers being added from the figures to be placed on the right. Then in the second number, casting 7 out of 8, 1 remains; casting 7 out of 19, 5 remain; casting 7 out of 50, 1 remains; and finally casting 7 out of 17, 3 remain — which I set down on the right. Again, in the third number, casting 7 out of 56, 0 remains; then dropping the digit 7, and casting 7 out of 8, 1 remains; and finally casting 7 out of 19, 5 remain, to be placed on the right. In the fourth number, casting 7 out of 8, 1 remains; casting 7 out of 18, 4 remain; and casting 7 out of 40, 5 remain, to be set on the right. Now since 5, 5, 3, and 0 make 13, out of which if 7 be cast 6 remain, I set down 6 on one arm of the cross. (But out of these remainders the sevens are more easily cast if one says: 5 and 5 make 10; cast out 7, and 3 remain; add the 3, and they become 6 — just as was said above about casting out 9.) Finally, in the sum, dropping the three sevens, if 7 is cast out of 23, 2 remain; likewise if 7 is cast out of 20, 6 remain, to be placed on the other arm of the cross.

[Margin: The proof by 7 is fallacious, but less so than the proof by 9, and why.]

But just as the check by casting out 9 is deceptive, as we have said, so too this check by casting out 7 is found to be flawed: for we do not consider whether 7 is cast out of the numbers being added as many times as it is out of the sum, or not, but only whether the same remainder is found on both sides, or not.

Nevertheless, this kind of check is not without reason customarily used by arithmeticians — just as that earlier one by casting out 9 was, and for the reason already stated: because, unless someone maliciously tampers with the numbers, the same remainder will scarcely or rarely be found on both sides unless the addition has been done correctly. And indeed it happens far more rarely with this check than with that one made by 9, because the sevens are not cast out as simply as the nines, but by a somewhat greater artifice, so that one cannot so easily deceive another or be deceived by another. In the following example the addition is faulty, and yet the check by casting out 7 indicates it to have been correctly done:

```

203 | 0
134 | 1 1 × 1
----
344

```

This table, as set out, shows from which numbers, once 7 is cast out, nothing is left, so that this check by 7 may be made easier for those who are little practiced in numbers. Its use is this. If a number written with two digits, from which 7 is to be cast out, is found in this table, nothing will remain after casting out 7, as the ciphers opposite the numbers of this table indicate. But if it is not found in the table, one must take the number in it that is next smaller; for the difference between this and the proposed number is what remains after casting out 7. Thus if the proposed number is 69, one must take the number 63 in the table, which differs from 69 by six units; so, casting 7 out of 69, 6 remain. Likewise, if the proposed number is 37, one takes in the table the number 35, which is surpassed by 37 by two units; so, casting 7 out of 37, 2 are left. And so with the rest.

Number	Remainder
7	0
14	0
21	0
28	0
35	0
42	0
49	0
56	0
63	0

Third, arithmeticians are accustomed to check an addition already made by this method. If the addition was made from the lower digits toward the upper, they set it up again from the upper digits toward the lower, or the reverse; and if by the later way exactly the same sum is collected as was found by the first, they do not doubt that the addition has been done correctly — because it would be in some way incredible that, if an error had been committed by the first way, the very same error should be committed by the later way, since the number's digits are added to one another in a different order the later way than the first. For if I err in adding these digits 5, 2, 9 by saying "5 and 2 make 7; add 9, and they become 15," I shall not so easily fall into the same error if I add them the opposite way; for I shall say "9 and 2 make 11; add 5, and they become 16" — because the operation is varied somewhat.

[Margin: Third proof of addition, by addition.]

This proof may also be set up by addition in the following way. Let the numbers to be added be divided into two or more groups, and let the sums of the individual groups be collected. For if out of these sums you make a single sum, it must be equal to the sum collected earlier, provided no error has been made. Thus if the first example is divided into these two groups, and their collected sums are reduced into one sum, as has been done here, the same sum as before will result:

```

710654 56789
 8907 880
-----
719561 57669

719561
 57669
-----
777230

```

[Margin: Fourth proof of addition, by subtraction.]

Fourth and last, an addition is customarily checked by subtraction, in this way. When two numbers have been added, let either of them be subtracted from the sum — how this is done, we shall teach in the following chapter. If the number left over from this subtraction equals the other number that was added, it is evidence that no fault was committed in the addition. For if 12 and 20 make 32, then necessarily, taking 12 from 32, 20 is left; or taking 20 from 32, 12 is left. But when more than two numbers have been added, let one of them be subtracted from the sum, and let all the rest be collected into a single sum. If this sum equals

that remainder, the addition will have been done correctly. Or else, let the second number be subtracted from the remainder, the third from this new remainder, and so on, except for the last. For if the last remainder equals the last of the numbers being added, there is no doubt that the addition has been done correctly. And this check is the most certain, although it is somewhat longer than the others.

SUBTRACTION OF WHOLE NUMBERS

CHAPTER III

Subtraction is the taking of a smaller number from a greater, or of an equal from an equal.

[Margin: What subtraction is.]

You will easily recognize which of two numbers is the greater from their last digits. For whichever number has the greater last digit will itself be the greater. Thus, of these two numbers the upper is greater than the lower, because its last digit, 3, is greater than 2, the last digit of the lower:

3001234
2986789

But if the last digits of the two numbers are equal, that one will be greater whose next-to-last digit is greater. And if the next-to-last digits are also equal, that number will be greater in which the greater digit occurs first — as in these examples the upper number is always greater than the lower:

445078 700001000
444899 700000999

[Margin: How to tell which of two numbers is the larger.]

Likewise, a number that has more digits is greater than one that contains fewer, as here the upper number is greater than the lower:

12345
9968

[Margin: How the number to be subtracted is to be placed.]

The number to be subtracted must be placed beneath the number from which the subtraction is to be made, so arranged that the first digit answers to the first, the second to the second, the third to the third, and so on — in such a way that the

shortfall of digits in the number to be subtracted, if there is any, is seen on the left side. Thus if the number 40236 is to be taken away from the number 3271589, the former must be placed under the latter, as appears here:

```
3271589
 40236
-----
3231353
```

[Margin: How subtraction is done.]

A line having then been drawn beneath those two numbers, the individual digits of the lower number are subtracted from the individual digits of the upper, beginning with the first digits; and the remainders are placed beneath the line in the same order in which the subtraction was made. But if in the upper number some digits have no corresponding digits in the lower number, so that nothing is subtracted from them, they are to be set down beneath the line. For example: taking 6 from 9, 3 remain, to be placed beneath. Likewise, taking 3 from 8, 5 are left; and removing 2 from 5, 3 are left. Subtracting the digit 0 from 1, 1 remains; and taking 4 from 7, 3 remain. But since nothing is subtracted from the digits 2 and 3, they are to be set down in the same order beneath the line.

[Margin: What to do when a lower digit is larger than the upper one.]

But when some lower digit is greater than the upper one corresponding to it, so that the subtraction cannot be made, this rule is to be observed. From the next upper digit toward the left, a unit is to be borrowed, which will signify ten units with respect to that digit from which the subtraction cannot be made. Then this unit is joined to that digit, so as to make a number written with two digits, from which the larger lower digit is taken away; but then the digit from which the unit was borrowed is to be reckoned as one unit less. But if that next upper digit is 0, we shall borrow a unit from another digit toward the left, which unit will signify 100 units with respect to the digit from which the subtraction cannot be made; but then, in place of the 0, the digit 9 is to be supplied in the mind, and the digit from which the unit was borrowed will be one unit less. So too, if several such digits 0 preceded the one from which we borrow the unit, they are all to be thought of as 9, and the one that lent the unit as one unit less. All this will be made clear by this example:

```
4500026304827
3929034567892
-----
570991736935
```

First, taking 2 from 7, 5 remain. Then, since 9 cannot be taken from 2, we shall borrow a unit from the 8; and so, taking 9 from 12 (the number expressed by the 1 we borrowed and the 2), 3 are left. Again, since 8 cannot be subtracted from 7 (for the upper digit 8, having lent a unit, is worth only 7), we shall borrow a unit from the 4; and so, taking 8 from 17, 9 remain. After this, since 7 cannot be taken from 3 (for the digit 4, on account of the unit lent, is worth only 3), we shall borrow a unit from the 3 that comes after the 0. But since this unit is worth 100 with respect to the digit 3 from which the subtraction cannot be made, while we need only 10, it is necessary that, borrowing 10 from 100, 90 be left. Hence it comes about that the digit 3 is worth only 2, and that above the 0 the digit 9 is to be supplied in the mind, which signifies 90 with respect to the digit from which the subtraction could not be made. So, taking 7 from 13, 6 remain. Then, taking 6 from 9 (for above the 0, as we said, one must think of 9), 3 remain. Now since 5 cannot be taken from 2 (for the upper digit 3 is worth only 2, as we said), we shall borrow 1 from the 6 and take 5 from 12, so that 7 are left over. Then, taking 4 from 5 (for the digit 6 is worth only 5, on account of the unit lent), 1 remains. And since again 3 cannot be taken from 2, we shall borrow a unit from the 5. But since this unit is worth 10000 with respect to the digit 2 from which the subtraction cannot be made, while we need only 10, it is necessary that, borrowing 10 from 10000, 9990 be left. Hence it comes about that the digit 5 is worth only 4, and that above each of the ciphers a 9 is to be supplied in the mind, thus: 999. For these digits, 999, signify 9990 with respect to the digit 2 from which the subtraction could not be made. So, taking 3 from 12, 9 remain. And subtracting the digit 0 from 9 (which digit 9 we said must be imagined above the 10), 9 remain. And taking 9 from 9 (which digit 9 we likewise imagine above the 0), 0 remains. Likewise, taking 2 from 9 (for above the 0 the digit 9 must also be supplied in the mind), 7 are left over. But since 9 cannot be taken from 4 (for the digit 5, on account of the unit lent, is worth only 4), we shall borrow 1 from the 4 and take 9 from 14, so that 5 are left over. Finally, taking 3 from 3 (for the digit 4, on account of the unit lent, is worth only 3), 0 remains; and since this figure 0 is the last in this example and therefore signifies nothing, it is to be omitted.

[Margin: An easier method of subtraction, when the lower digit is larger than the upper.]

Most arithmeticians use this rule, which we shall set out far more easily as follows. When a lower digit is greater than the upper, let the difference between it and 10 be taken, and to this difference let the upper digit be added, and the total written beneath the line. For this total is what would be left if that larger digit were taken from the number composed of 10 and that upper digit from

which the subtraction cannot be made — exactly as if a unit had been borrowed. Here that larger digit is first subtracted from 10, to obtain the difference between 10 and it; then to the remainder, or difference, the upper digit is added. Afterward, so that we need not remove a unit in thought from the upper digit — from which a unit has implicitly been taken — we shall add a unit to the next lower digit toward the left, and subtract this total from the upper digit (no unit being first removed from it). For the difference will always be the same between the lower and the upper digit, whether you take a unit from the upper and add nothing to the lower, or take nothing from the upper and add a unit to the lower. Thus, given these two digits 7 and 4: if a unit is taken from the 7, the difference between the remainder 6 and 4 will be 2; and if nothing is taken from the 7 but a unit is added to the 4, the same difference, 2, will hold between 7 and 5. And in this way, whenever mention is made of the difference between 10 and the lower digit that cannot be subtracted from the upper, a unit must be added to the next lower digit toward the left. But all this will become clearer from the same example, which we repeat here:

```

4500026304827
3929034567892
-----
570991736935

```

First, taking 2 from 7, 5 are left over. But since 9 cannot be taken from 2, we shall subtract 9 from 10 and add to the remaining unit (which is the difference between 10 and 9) the 2, so that we have 3 to place beneath the line as the remainder. This done, we at once add 1 to the lower digit 8, on account of that difference between 10 and 9, so that it becomes 9. And since this in turn cannot be taken from 8, we shall subtract 9 from 10, and to the remaining unit (which is again the difference between 10 and 9) we shall add [—]

[we add] 8, so as to have 9, which is to be placed below the line. Once this is done, we immediately add 1 to the lower figure 7 — on account of that difference between 10 and 9 — and so make 8. Since 8 cannot be taken from 4, we take 8 from 10, and to the remainder 2 (that is, the difference between 10 and 8) we add 4, making 6, which is to be set below the line. Then we at once add 1 to the lower figure 6 — because of that difference between 10 and 8 — making 7. Since these cannot be subtracted from 0, I take them from 10, and to the remainder 3 (that is, the difference between 10 and 7) I add 0, making 3, to be placed below the line.

Again I add 1 to the lower figure 5, on account of that difference between 10 and 7, making 6. Since these cannot be subtracted from 3, I take them from 10, and to the remainder 4 (that is, the difference between 10 and 6) I add 3, making 7, to be

written below the line. This done, I immediately add 1 to the lower figure 4, on account of the said difference between 10 and 6, making 5; and subtracting these from 6, 1 remains. But since in this subtraction no mention was made of a difference between 10 and 5 — because the 5 could be subtracted from 6 — I do not add 1 to the lower figure 3; rather, since it cannot be subtracted from 2, I subtract it from 10, and to the remainder 7 (that is, the difference between 10 and 3) I add 2, making 9, to be placed below the line.

After this I at once add 1 to the lower figure 0, on account of the said difference between 10 and 3, making 1. And since 1 cannot be subtracted from 0, I take 1 from 10, and to the remainder 9 (that is, the difference between 10 and 1) I add 0, making 9, to be set below the line. Then again I immediately add 1 to the lower figure 9, on account of that difference between 10 and 1, so as to have 10; and since these cannot be taken from 0, I take them from 10, and to the remainder 0 (that is, the difference between 10 and 10, to which I add 0) I make 0, to be placed below the line as the remainder. Again, as soon as I can, I add 1 to the lower figure 2, on account of the said difference between 10 and 10, making 3; and since these cannot be subtracted from 0, I subtract from 10, and to the remainder 7 (namely the difference between 10 and 3) I add 0, making 7, to be placed below the line. Furthermore I soon add 1 to the lower figure 9, on account of that difference between 10 and 3, making 10; and since these cannot be taken from 5 — that is, the difference between 10 and 10 — I add 5, making 5, to be set below the line as the remainder. Finally I at once add 1 to the lower figure 3, on account of that difference between 10 and 10, so as to have 4; and subtracting these from 4 leaves 0. This figure 0, since it is superfluous at the beginning of the number on the left-hand side, we shall omit, because if written it would occupy a place to no purpose.

4500026304827
3929034567892
570991736935

Another Example

[Margin: What to do when there are several numbers.]

Checked by 7:

4000134		5
67823		0 ₅ × 5
3932311		5

Checked by 9:

$$3 \times 3$$

In this example, since — once all the lower figures have been taken from the upper — a unit would have to be added to the next lower figure, which is absent, we shall carry it into the next place. Since it cannot be taken from 0, it must be taken from 10, and the remainder 9 placed below the line; and again the unit must be added to the following place and taken from 4, so that a remainder of 3 is obtained, to be placed below the line.

Now if a number is to be subtracted from several numbers, or several numbers from several, or from a single number, then those several numbers from which the subtraction is to be made must first be gathered into one sum; and likewise those several which are to be subtracted must be added together, before the subtraction is set up.

The examination of subtraction is fourfold. The first is done by casting out 9. For if you cast 9's out of the upper number from which the subtraction was made — as many times as possible, in the way we described for addition, that 9's are to be cast out — and place the remainder on one arm of the cross, then, unless there has been an error in the subtraction, the same number must be left when 9's are cast out of the subtracted number and out of what remained, as often as this can be done. Thus you see in the preceding example that the remainder on the right is always 3, whether you cast out 9's, as often as you can, from the number 4000134 from which the subtraction was made, or from the numbers 67823 and 3932311 together — the former of which was the number subtracted, the latter what remained from the subtraction.

The second examination is made by casting out 7. For if 7's are cast out of the number from which the subtraction was made — as many times as possible, in the way we described for addition, that 7's are to be cast out — and the remainder is placed on one arm of the cross, then, provided the subtraction was done correctly, the same number must be left when 7 is cast out of the subtracted number and out of what remained, its remainder likewise being placed to the right of the former; and finally, when these two remainders set on the right are gathered into one sum and 7 is cast out of it (if it can be), the result must agree. Thus in the example above, casting out 7 as often as possible from the number 4000134 leaves 5. Casting 7's out of 67823 leaves 0; and casting 7's out of 3932311 leaves 5, which added to 0 also makes 5 — as appears in the cross set at the left of the example above.

Yet both of these examinations can be deceptive, if someone should fraudulently transpose the figures, or substitute other numbers, as we noted in the case of addition.

The third examination is made by addition. For if you add the remaining number to the subtracted number, the number from which the subtraction was made will necessarily be reconstituted, as you see in this example.

[Margin: First proof of subtraction, by 9's.]

[Margin: Second proof of subtraction, by 7's.]

[Margin: Third proof of subtraction, by addition.]

NUMBER from which the subtraction is made. 60123

NUMBER subtracted. 45678

NUMBER remaining. 14445

The SUM composed of the subtracted number
and the remainder, equal to the number
from which the subtraction was made. 60123

[Margin: Fourth proof of subtraction, by subtraction.]

60123

14445

45678

The fourth examination is made by subtraction. For once the subtraction has been performed, if you take the remainder away from the same number from which the subtraction was made, the subtracted number will necessarily be left. Thus in the next example, if you take the remainder 14445 from the number 60123, the subtracted number 45678 will be left, as appears here.

These last two examinations are entirely reliable, and can admit no deception or fraud.

MULTIPLICATION OF WHOLE NUMBERS.

CHAPTER IV.

[Margin: What multiplication is.]

Multiplication is the drawing of one number into another. One number is said to be drawn into another when the one is increased as many times over as there are units contained in the other. Thus 6 is said to be drawn into 5, or 5 into 6, when 6 is taken five times, or 5 is taken six times: in either case the result is always 30, and a drawing of this kind is called multiplication. And so the number produced by the multiplication, or drawing, of one number into another will contain either of the two factors as many times as the other embraces unity — as is clear in the given example. From this it follows that multiplication may also be defined thus: the multiplication of two numbers is the finding of a number that contains either of them as many times as the other contains unity.

For every multiplication to be carried out readily, it is necessary to know what number is produced by drawing, or multiplying, any numeral figure into any other — for instance, 7 into 8, or 8 into 7; likewise 7 into 9, or 9 into 7, and so on. If you have this well in hand, you will feel no difficulty or trouble in multiplication. And this is learned better by constant practice than by any rule. Meanwhile, however, the following table — commonly called the Pythagorean, perhaps because Pythagoras either was the first to devise it or at any rate wonderfully trained his disciples in it — will serve you admirably.

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

[Margin: Construction of the Pythagorean table.]

The construction of this table is very easy. The first line, beginning from 1, advances by the continual addition of a unit up to 9: thus from 1 and 1 comes 2; from 2 and 1 comes 3; from 3 and 1 comes 4, and so on. The second line begins from 2 and advances by the continual addition of 2: thus from 2 and 2 comes 4; from 4 and 2 comes 6; from 6 and 2 comes 8, and so on. Likewise the third line takes its start from 3 and advances by the continual addition of 3. And in the

same way the remaining lines are composed: each one advances by the continual addition of the number from which it begins.

[Margin: Use of the Pythagorean table.]

The use of this same table, as far as multiplication is concerned (for it has countless other uses), is as follows. Given two figures to be multiplied together, if one is taken in the top line and the other on the left-hand side, and one moves downward along that line and to the right along that side, one will find, where they meet in common, the number produced by multiplying those two figures together. Thus you see that multiplying 7 into 8, or 8 into 7, produces 56; and that multiplying 8 into 8 produces 64; and so of the rest.

[Margin: The rule for multiplying figure by figure.]

But if such a table is not at hand, this rule must be used. Let one figure be written beneath the other, and the distance of each from 10 set beside it. Then let these distances be multiplied together. The resulting number, if it is written with a single figure, will give the first figure of the product to be produced by the multiplication of the figures; but if it is written with two figures, the tens-figure is to be reserved, and the former one set down as the first figure of the product to be produced. The second figure of this same product will be obtained if either distance is subtracted from the other's figure, the reserved tens-figure being added to it, if any was reserved. Or else, if the given figures themselves are added together — the reserved tens-figure being first added, if any was reserved — the first figure of this total (the second figure being rejected as superfluous) will give the second figure of the product to be produced. An example will make the matter clearer.

9	1	8	2	7	3
8	2	8	2	6	4
7	2	6	4	4	2

In the first example the figures to be multiplied are 9 and 8, and their distances from 10 are 1 and 2. These, multiplied together — the multiplication of the distances from 10 being very easy, since they are smaller than the figures to be multiplied (and it is of these that this rule is to be understood) — by saying "once 2," or "twice 1," make 2, which I write under the distances as the first figure of the product to be produced.

to be produced. And subtracting the distance 2 from 9, or the distance 1 from 8, leaves 7, which I write beneath the figures as the second figure of the product to

be produced. This same second figure will also be given by the first figure of the sum of the figures 9 and 8, which is 17 — the second figure, 1, being left aside as wholly useless for this business. And so the multiplication of the figures 9 and 8 yields 72.

In the second example the given figures are 8 and 8, whose distances from 10 are 2 and 2. Multiplying these together, by saying "twice 2," we shall have 4 for the first figure of the product to be produced. And taking either distance from 8, 6 remains for the second figure — which the first figure of the sum of 8 and 8, that is 16, will likewise give us, the second figure 1 being left aside as superfluous. Therefore the figures 8 and 8 multiplied together make 64.

In the third example, finally, the given figures are 7 and 6, whose distances from 10 are 3 and 4. Multiplying these together, by saying "three times 4," or "four times 3," they make 12. The first figure of the product to be produced will therefore be 2, while the second figure, 1, is to be reserved. Then, taking the distance 4 from 7, or the distance 3 from 6, 3 remains; and if the unit we reserved is added to this, they become 4, for the second figure of the product to be produced — which figure the first figure of the sum of 7 and 6, with the previously reserved unit added, namely 14, will also give, its second figure 1 being altogether dropped. So 42 is produced from the multiplication of 7 by 6, or 6 by 7. The same procedure holds in the other cases, provided that the two given figures added together exceed 10. Otherwise their distances from 10 would be greater than the figures themselves, and consequently the figures would be more easily multiplied than the distances. You will do better, however, to learn such multiplications of figures together by heart, through use and practice, than to have recourse either to the Pythagorean table or to this rule.

[Margin: In what arrangement the numbers to be multiplied together are to be placed.]

Now when two numbers are to be multiplied together, the smaller is to be written beneath the larger, so that the first figure corresponds to the first figure, the second to the second, and so on, as we said in addition and subtraction. This, however, is not necessary, since the larger may also be written beneath the smaller, provided the stated order of the figures is preserved. Thus, if 4300678 is to be multiplied by 600394, the numbers may be arranged in either of these two ways, though the first is the more usual:

4300678 600394
600394 or 4300678

[Margin: How a number is multiplied by a single figure.]

But let us first teach how any number is to be multiplied by a single figure. For in this way it will be more easily understood by what method a number ought to be multiplied by a number of several figures.

[Margin: How a number is multiplied by a single figure.]

When, therefore, a number is to be multiplied by only a single figure, this multiplying figure is usually written beneath the first figure of the number to be multiplied. Thus, if the number 600394 is to be multiplied by 8, the setup will be as follows. The multiplication is carried out by multiplying the figure 8 by all the figures of the number 600394, beginning from the right and proceeding toward the left, and writing each of the numbers produced beneath the line that is drawn below the numbers being multiplied — yet in such a way that, if any number produced is to be written with two figures, the first of them is set down and the second reserved to be added to the following product, in this manner:

$$\begin{array}{r} 600394 \ 5 \\ 8 \ 4 \times 8 \\ \hline 4803152 \end{array}$$

First I multiply 8 by 4, saying "eight fours make 32"; I set down 2 under the 4 and reserve 3. Next, "eight nines make 72," and with the reserved 3 added they make 75; I set down 5 under the 9 and reserve 7. Next, "eight threes make 24," and with the reserved 7 added they make 31; I set down 1 under the 3 and reserve 3. Then "eight times 0 makes 0," and with the reserved 3 added it makes 3, which I set under the 0, reserving nothing. Again "eight times 0 makes 0," and to this, since I reserved nothing in the preceding product, nothing is to be added; I therefore set down 0 under the 0, reserving nothing. Finally, "eight sixes make 48"; and since I reserved nothing in the previous product, I add nothing. I therefore set down this whole number under the line, because the multiplication has been carried to its end, no other figure of the upper number remaining to be multiplied by 8. And so, if we multiply the whole number 600394 by 8, we shall produce this number, 4803152. And in this way you will multiply any number whatever by any figure.

[Margin: How a number is multiplied by a number written with several figures.]

But if a number is to be multiplied by a number of several figures, let a straight line be drawn beneath them once they are set out, as has been said. Then let the individual figures of the lower number be multiplied by the individual figures of the upper number, as we have just taught — only observing carefully this one

point: that the number produced from any figure of the lower number multiplied by the first figure of the upper number is placed beneath that figure of the lower number by which the upper number is being multiplied, while the remaining numbers produced from the multiplication of that same figure are set in their order onward toward the left, as you see done in this example, in which four rows of numbers have been placed beneath the line out of the numbers produced.

per 9. 4300678
600394

17202712

38706102

12902034

25804068

2582101267132

Check by 9 (cross at left):

4
1 × 4
4

Check by 7 (cross at right):

2
4 × 4
2

For the whole number produced by multiplying 4 into all the figures of the upper number has its first figure under the 4. Likewise the number produced by multiplying 9 into all the figures of the upper number has its first figure under the 9. By the same reasoning, the first figure of the number produced by multiplying 3 into all the figures of the upper number is placed under the 3. And finally, the first figure of the number produced by multiplying 6 into all the figures of the upper number is set under the 6. The remaining figures advance in their order toward the left.

Cap. IV — Multiplicatio (continued)

Whenever a cipher [zero] is either the multiplier or the multiplicand, it yields 0. That is why, in the lower number, we left two ciphers out of account and did not multiply them into the upper number — they would always have produced 0. Do the same whenever the lower number contains any ciphers: always skip them

and pass to the next significant figure. The ciphers of the *upper* number, however — if there are any — must not be neglected. True, when they are multiplied by the significant figures of the lower number they produce 0; yet often something has to be added to that 0, namely whatever was carried [reserved] from the preceding multiplication, and this is to be written under the line in place of the product. Indeed, even when nothing at all has been carried, a 0 must still be placed under the line in place of the product. All of this was observed in the examples above. In the first, when we multiplied 8 by 0 we produced 0; but because 3 had been carried in the preceding multiplication, we put 3 under the line in place of the product 0. Again, when we multiplied 8 by 0 a second time, we likewise produced 0; and since nothing had been carried, we put 0 in place of the product. The same was done in the second example.

After this, draw another line below all the products, so that beneath it may be placed the sum gathered from all those products. This sum is to be collected just as was taught for addition — provided that the first figure of each product is understood to occupy the same place as the figure of the first product beneath which it is set. That is: the figure 2, which is the first figure of the second product in the example proposed, is to be thought of as placed under the second place of the first product; the figure 4, which is the first figure in the third product, is understood to be located under the third place of the first product; and finally the figure 8, which is likewise the first figure in the fourth product, is conceived to occupy the sixth place under the first product. For in these places you see all these figures set down. So, to show the matter by example, the sum will be gathered in this way. Among the products, only the figure 2 occupies the first place; it alone, therefore, is to be placed in the first place under the line. Next, in the second place there are 1 and 2, which make 3, to be placed under the line in the second place. Again, in the third place there are 7, 0, and 4, which make 11; so 1 is to be placed under the line in the third place, and 1 kept back to be added to the figures of the fourth place, and so on. In this way the sum collected will be 2582101267132; and this number is produced by the multiplication of 4300678 by 600394.

So that you may see the same number produced even when we set the larger number beneath the smaller, I have added this next example, in which the same two numbers, 4300678 and 600394, are multiplied together, but the larger is placed beneath the smaller. Five rows of products are formed — as many, that is, as there are significant figures in the lower number — yet the number produced is the same as before.

$$\begin{array}{r}
 600394 \\
 4300678 \\
 \hline
 4803152 \text{ (}\times 8\text{)} \\
 4202758 \text{ (}\times 7\text{)} \\
 3602364 \text{ (}\times 6\text{)} \\
 1801182 \text{ (}\times 3\text{)} \\
 2401576 \text{ (}\times 4\text{)} \\
 \hline
 2582101267132
 \end{array}$$

Proof by casting out nines (**per 9**), shown at the left of the example as an $\times$ -cross: upper 4 · left 4 · right 1 · lower 4.

Proof by casting out sevens (**per 7**), shown at the right of the example as an $\times$ -cross: upper 2 · left 4 · right 4 · lower 2.

This method of multiplying, which we have set out so far, is the one more commonly used by everyone; other, not unpleasant, methods of multiplying we shall explain in our fuller *Arithmetic*.

[Margin: The first proof of multiplication, by nines.] The checking of multiplication is threefold. The first is done by casting out nines, in this way. First cast the nines out of the first number multiplied, as often as can be done — as we said in addition — and place the remainder [residue] in the left arm of the cross. Then, casting out the nines in the same way from the other number multiplied, set its remainder in the right arm of the cross. After this, multiply these two remainders together, cast the nines out of the product, and put that remainder in the upper arm of the cross. Finally, cast the nines also out of the sum of all the products, and write that remainder in the lower arm of the cross. For this last remainder must equal the upper one, if no error has been made in the multiplication. The examples are the multiplications above. In the first example, casting the nines out of 600394 leaves a remainder of 4; the remainder of the number 8 is 8, since no nines can be cast out; multiplying these two remainders, 4 and 8, together gives 32, from which, if the nines are cast out, [the result agrees with the remainder] from the product 4803152. In the second example, the remainder of the first number is 1, and of the second, 4; multiplying these remainders 1 and 4 together gives 4, to be placed in the upper arm of the cross, since no nines can be cast out; and casting the nines out of the whole sum, 4 likewise remains.

[Margin: The second proof of multiplication, by sevens.] The second check is done by casting out sevens — that is, casting the sevens out of the same numbers (as

we said in addition) from which, in the previous check, we directed the nines to be cast out. You have an example in the last two multiplications. But these two checks are also fallible, for the reasons adduced above.

[Margin: The third proof of multiplication, by division.] The third check is the most reliable, and is made by division. For if the whole product-sum is divided by either of the numbers multiplied, the other number will necessarily come out in the quotient. This division will be very easy, since there is no need to search out the quotient figures — they are all contained, in order, in the other number multiplied. But this check will be understood more clearly once division has been explained.

Two further examples, with the proof by nines.

4068	3069
23	45
_____	_____
12204	15345
8136	12276
_____	_____
93564	138105

For the left example ($4068 \times 23 = 93564$), $\times$ -cross (per 9): upper 0 · left 0 · right 5 · lower 0. For the right example ($3069 \times 45 = 138105$), $\times$ -cross (per 9): upper 0 · left 0 · right 0 · lower 0.

Cap. IV — conclusion; then Cap. V

In the first of these examples the first remainder is 0; hence, although the second remainder is 5, the multiplication of the remainders still yields 0. In the second example, each remainder of the multiplied numbers is 0, so their multiplication will likewise give 0; and thus in both examples the remainder of the product must necessarily be 0 as well.

[Margin: The ease of multiplication when the numbers have ciphers to begin with.] But if both numbers to be multiplied — or even just one of them — have some ciphers to begin with, the multiplication will be very easy. For once all those ciphers are dropped, the remaining numbers are to be multiplied together, and then all the dropped ciphers appended in order to the product. For instance, if 3406 is to be multiplied by 4000: drop the ciphers 000, multiply the given number by 4, and to the product 13624 append the same ciphers, thus: 13624000. Likewise, if 3040000 is to be multiplied by 203000: drop the 7 ciphers standing to the right of these numbers, multiply the remaining numbers 304 and 203

together, and to the product 61712 append those 7 dropped ciphers, thus:
61712000000.

Hence it follows that if some number is to be multiplied by 10, or 100, or 1000, or 10000, and so on, one need only append to that number, on the right, as many ciphers as are contained in the multiplier, with no other multiplication at all — because once the ciphers are dropped, only a unit remains, and multiplying the given number by 1 produces the same number. Thus 5067 multiplied by 10 gives 50670; multiplied by 100000 it gives 506700000. So too 3000 multiplied by 100 gives 300000, and so on.

Division of Whole Numbers. Chapter V.

[Margin: What division is.] [Margin: What the quotient is.] Division is the distribution of a given number into parts denominated by another given number. For example, dividing 36 by 9 is distributing it into ninth parts — parts, that is, denominated by 9 — each of which will contain four units; so that 4 is the number produced by this division, commonly called the *quotient*, because it indicates how many times the number 9 (called the *dividing number*, or *divisor*) is contained in the number to be divided, 36. It indicates that it is contained four times — as many times, that is, as the quotient number 4 contains a unit. From this, division can also be described as follows: division is the finding of a number that contains a unit as many times as the number to be divided contains the divisor. This is clear in the example proposed.

[Margin: How the numbers are to be arranged in division.] In division the divisor is written under the number to be divided — not, however, by putting the first figure under the first, the second under the second, and so on, as was done in addition, subtraction, and multiplication, but in the reverse order. For here the last figure of the divisor is to be placed under the last figure of the number to be divided, the second-to-last under the second-to-last, the third-to-last under the third-to-last figure of the dividend, and so on. Thus, if the number 7809 is to be divided by 47, the numbers are to be arranged as appears in the accompanying example:

7809
47

But if the last figure of the divisor is greater than the last figure of the number to be divided, the last figure of the divisor must be placed under the second-to-last figure of the dividend, the second-to-last under the third-to-last, and so on — as is evident in this example:

37800

47

The same is to be done if the last figure of the divisor equals the last figure of the dividend but its second-to-last is greater than the dividend's second-to-last; or if both the last figures and the second-to-last are equal, [one compares] the third-to-last figure of the dividend; or, finally, whenever the divisor's figures express [a number greater than] the corresponding last figures of the dividend, as many as the figures in which the divisor is written. All this is manifest in these two earlier examples set down here. But if the divisor equals the number expressed by as many figures of the dividend as there are figures in the divisor, the divisor is to be placed directly under that number, as you see done in the third and last example:

46800 476047 4793

47 4762 47

[Margin: How division is carried out.] [Margin: No number greater than 9 can be placed in the quotient.] [Margin: The remainder must always be smaller than the divisor.] The division will proceed as follows. First, find how many times the divisor is contained in the number written above it, and write the number indicating how many times to the right of the dividend, after this curved line "(". This number is always written as a single figure, for no number greater than 9 can ever be placed in the quotient, even though the divisor may sometimes seem to be contained in the number above it more than nine times, as will be clear from the examples. Multiply this figure into the divisor; the product (which need not be written separately, but kept in memory) is to be subtracted from the number written above, as we taught in subtraction — writing each remainder figure above the figures from which the subtraction was made, and first deleting these figures together with the divisor. The whole remainder noted above the divisor must be smaller than the divisor; otherwise an error would have been made. The same must be observed with the other remainders.

Next, the divisor must be advanced one place toward the right, and again one asks how many times it is contained in the number written above it, doing the rest as before. But if at some advance the divisor is greater than the number above it, so that it is not contained in it even once, a 0 must be written after that number which we said is to be written after the curved line; the divisor is then deleted, advanced again to the next place, and one asks, as before, how many times it is contained in the number above it, and so on. In this way the divisor must keep being advanced until no place is left in the dividend into which the divisor can be advanced. But these things will become clearer from examples.

[Margin: How a number is divided by a single figure.] [Margin: Which number is said to be placed above the divisor.] Let the first number to be divided be 76048, by a single figure, say by 8. First I find that the divisor 8 is contained in the number written above it, 76, nine times. (That number is said to be written above the divisor which is expressed by the figure placed above the first figure of the divisor, together with all the others to the left, if there are any — as in the given example the number placed above the divisor is 76.) Moreover, from the Pythagorean table [multiplication table] set out above you will easily learn how many times the divisor's figure is contained in the number above it. For if you take the divisor's figure at the head of the table and, going down its column, take the number placed above — or, if it is not found, the one nearest below it — the figure corresponding to it on the left side will indicate how many times the divisor's figure is contained in the number above it.

[Margin: How to learn from the Pythagorean table how many times the divisor's figure is contained in the number above it.] Thus, in the given example, the number 76 above is not found exactly under the figure 8; so if we take 72, the nearest smaller, we find the figure 9 on the left side. Therefore 8 is contained in 76 nine times, and so with the rest. I place 9, then, after the curved line, and multiply 9 by 8, saying "nine eights make 72," which must be subtracted from the number 76 placed above the divisor, thus. Subtracting 2 from 6 leaves 4; I write 4 above the 6 in the dividend. Subtracting likewise 7 from 7, nothing remains. So the figure 8 in the divisor and the figure 6 in the dividend are deleted. The figure 7 being deleted, I put nothing above the 7, since a cipher would have to be placed there, which would be superfluous, because no other figure follows it toward the left. And thus one operation of division is completed, and the number 4048 remains, as you see in the example proposed.

Next, advancing the divisor to the preceding place, under the 0 — as you see here in the second example — I see that the divisor 8 is contained five times in the number 40 written above it. So I place 5 after the figure 9 already found, as you see in the third example, and I say: "five eights" — that is, multiplying the figure 5 found by the divisor — "make 40," which, subtracted from the number 40 placed above the divisor, leaves nothing. So, the figure 8 in the divisor and the figures 0 and 4 in the dividend being deleted, the second operation of division is completed, and the number 48 remains, as appears in the same third example.

Again, advancing the divisor to the preceding place, under the 4 — as you see here in the fourth example — I find that the divisor 8 is not contained even once in the number 4 written above. So I place 0 after the figure 5 last found, as is done

in the fifth example. And since the figure 0 multiplied by the divisor produces nothing, nothing is to be subtracted from the number 4 placed above the divisor. The third operation of division is completed, and the number 48 remains, as is evident in this same fifth example.

Finally, advancing the divisor to the preceding place, under the 8 — as is seen here in the sixth example — I find that the divisor 8 is contained six times in the number 48 written above. So I place 6 after the figure 0 last found, as is done here in the seventh example, and I say: "six eights" — that is, multiplying the figure 6 found into the divisor — "make 48," which, subtracted from the number 48 placed above the divisor, leaves nothing. So, the figure 8 in the divisor and the figures 8 and 4 in the dividend being deleted, the whole division is completed, since no other place remains in the dividend into which the divisor can be advanced, and nothing is left over in the division. Thus the entire quotient number is 9506.

The seven successive stages of dividing 76048 by 8 are printed as follows (in the galley method the consumed figures of the dividend and each spent copy of the divisor are struck out; the final remainder is 0):

Stage	Remainder above	Dividend	Quotient	Divisor row
1	4	76048	(9	8
2	4	76048	(9	88
3	4	76048	(95	88
4	4	76048	(95	888
5	4	76048	(950	888
6	4	76048	(950	8888
7	4	76048	(9506	8888

I have set out so many examples in this division in order to show more distinctly what is left over and what is deleted at each operation — although the last alone stands for them all, so that in practice there is no need to draw the other examples: it is enough to write out the last.

[Margin: How many figures the quotient has.] You see, then, that the quotient has as many figures as the number of times the divisor is placed under the dividend. This happens in all other divisions as well, even those carried out with several figures: the quotient will always have as many figures as the number of times the whole divisor is placed under the dividend.

[Margin: How a number is divided by several figures.] Next, let the number 1832487 be divided by the number 469, which is written not with one figure but with several. To know here how many times the divisor is contained in the number above it (in this example the number 1832 is placed above the divisor), one need not inquire about the whole divisor; it is enough to investigate how many times its last figure — here 4 — is contained in the number above it.

```

42
685
1832487 ( 3
469

```

[Margin: Which number is said to be placed above any given figure of the divisor.] (Here too I say that the number placed above the last figure of the divisor — or above any other figure — is the one expressed by the figure written above it, together with all the others toward the left, if there are any. Thus, in the given example, above the figure 4 is placed the number 18, and above the 9 the number 1832 — which here is 18 [?]. This caution, however, must be observed: one must not always place in the quotient a figure of as many units as the number of times the last figure of the divisor is contained in the number above it, but must take care to place the figure which, multiplied into the whole divisor in the order we shall now prescribe, produces a number that both can be subtracted from the number placed above the divisor and, once subtracted, leaves — if it leaves anything — a number smaller than the divisor. So, to come to the example proposed: although the last figure of the divisor, which is 4, is contained in the number above, 18, four times, nevertheless — because the figure 4 multiplied into the whole divisor yields a number greater than 1832, which is placed above the divisor, so that it cannot be subtracted from the number above — I do not put 4 in the quotient, but 3. And if this figure 3, multiplied into the whole divisor, still produced a number greater than 1832, I would put 2 in place of 3. And if the figure 2, multiplied into the divisor, produced a still greater number, I would put 1; and thus I keep reducing the quotient figure by one unit until I find a figure which, multiplied into the divisor, produces a number that can be subtracted from the number above.)

[Margin: How the quotient figure found is to be multiplied into the divisor.] Now the quotient figure found is to be multiplied into the whole divisor in this way. First it is multiplied into the last figure of the divisor, and this product is taken away from the number placed above that last figure — that figure of the divisor being first deleted, together with the number from which the subtraction was made. Then it is multiplied into the second-to-last figure of the divisor, and the

product subtracted from the number above, as before. And in this way it is to be multiplied into all the figures of the divisor, and so on, as in our example. 3 times 4 makes 12, which is to be subtracted thus from 18, the number above. Taking 2 from 8 leaves 6; so, the figure 4 in the divisor and the figure 8 in the dividend being deleted, I write 6 above the 8. Taking likewise 1 from 1, nothing remains; so I delete the 1. Next, 3 times 6 makes 18, which is to be subtracted from the number above, 63, as follows. The distance from 8 to 10 (for 8 cannot be subtracted from 3) is 2; adding 3 gives 5, which I place above the 3, the figure 6 in the divisor being first deleted, together with the figure 3 in the dividend.

(Epitome Arithmeticae Practicae — Cap. V, Division of whole numbers, continued.)

PDF p. 265 (printed p. 15) — running header "INTEGRORVM."

(Completion of the galley division $1832487 \div 469$, begun in the preceding pages.)

I now add 1 — on account of the distance from 10 mentioned earlier — to 1 (namely to the tens-digit of the 18 that is being subtracted), making 2; and 2 taken from 6 leaves 4, which I place above the 6, having first deleted the said figure 6. Finally, $3 \times 9 = 27$, which I subtract from the number 452 written above, in this way. The distance of 7 from 10 (since 7 cannot be taken from 2) is 3; adding the 2 gives 5, which I place above the 2, having first deleted the figure 9 in the divisor and the figure 2 in the dividend. I then add 1 to 2 (namely to the tens-digit of the 27 to be subtracted), on account of the said distance from 10, making 3; this, taken from 5 (namely from the second figure of the number 452, from which the subtraction is made), leaves 2. So I place 2 above the 5, having first deleted the said figure 5. And one would go on in the same way if the divisor had more figures. By this procedure, then, one operation of the division is completed, and there remains here the number 425487, as you see in the example set out above.

Now that the divisor has been advanced into the preceding place, so that each of its figures shifts by one position — as you see here — I judge that the last figure of the divisor, namely 4, is contained nine times in the 42 above it. I therefore place 9 after the figure 3 found in the previous operation, as appears in the following example, and I say: $9 \times 4 = 36$, which I subtract from the number 42 above. The distance of 6 from 10 (since 6 cannot be taken from 2) is 4; adding the 2 gives 6, which I place above the 2, having first deleted the figure 4 in the divisor together with its figure [in the dividend]. I then add 1 to 3 (namely to the tens of the 36 to be subtracted), on account of the said distance from 10, making 4, which taken from 4 leaves nothing. So I delete the 4. Again, $9 \times 6 = 54$: taking 4 from 5 leaves 1, and taking 5 from 6 also leaves 1; so, having deleted the figure 6 in the divisor

together with the figures 5 and 6 in the dividend, I place a 1 above each. Finally, $9 \times 9 = 81$, which I subtract from the 114 written above. Taking 1 from 4 leaves 3, so I place 3 above the 4, having first deleted the figure 9 in the divisor and the figure 4 in the dividend. As for the distance of 8 from 10 (since 8 cannot be taken from 1), I set it down, having first deleted the figure 1; and on account of the said distance from 10 I subtract 1 from 1, and nothing remains, so I delete the 1. The second operation of the division is now complete, and the number remaining is 3387, as the example shows.

Again advancing the divisor into the next place, as appears in the next example — so that the figure 9 falls under the 8, the 6 under the 3, and the 4 under the 3 — I see that the last figure of the divisor, namely 4, is not contained even once in the number written above; so I write 0 after the figure 9 just found, and delete the divisor. The third operation of the division is thus completed, and the same number 3387 remains that was left over in the preceding operation.

Finally, advancing the divisor into the first place, as is plain in that same example, I find that the last figure of the divisor, 4, is contained in the 33 written above only seven times; for if it were taken eight times, no subtraction could be made from the 3387 above of the numbers produced by $8 \times$ the whole divisor. So I place the figure 7 in the quotient after the other figures already found, as appears in this example, and I say: $7 \times 4 = 28$, which I remove from the 33 above in this way. The distance of 8 from 10 (for 8 cannot be taken from 3) is 2; if 3 is added, it makes 5. So, having deleted the figure 4 in the divisor and the figure 3 in the dividend, I place 5 above the 3. I then add 1 to 2 (namely to the tens of the 28 to be subtracted), on account of the said distance from 10, making 3, which taken from 3 leaves nothing. So, having deleted the figure 3, I say again: $7 \times 6 = 42$, which is removed from the 58 above in this way. Taking 2 from 8 leaves 6; so, having deleted the [figure 8] in the dividend, I place 6 above the 8. Taking 4 from 5 leaves 1; so, having deleted the figure 5, I place 1 above it. And at last I say: $7 \times 9 = 63$, which is to be taken from the 167 above in this way. Taking 3 from 7 leaves 4; so, having deleted the figure 9 in the divisor and the figure 7 in the dividend, I write 4 above the 7. Taking 6 from 6 leaves 0; so, having deleted the figure 6, I place 0 above it. And thus the whole division is completed, and there remains here the number 104. This, after the quotient 3907, is to be placed above the divisor 469, with a line drawn between them, so as to make a fraction — that is, 104 parts out of the 469 into which some whole is conceived to be divided. In the same way, in other divisions, the remainder (if there is any) is placed above the divisor, with a line drawn between them, to make a fraction.

[Margin: What is to be done with the number left over from a division.]

Indeed, whenever a smaller number is set to be divided by a larger, the dividend must be placed above the divisor, with the said line drawn between them, to make a fraction. For example, if 48 gold pieces are to be divided among 60 soldiers, the result is a fraction, which you see set down here as **48/60**, so that each soldier receives 48 of the 60 parts into which a single gold piece is understood to be divided. What such a fraction is, and how its value is worked out — whether in coin, in weights, or in measures, according as the dividend signifies money, weights, or measures — we shall explain in the treatment of fractions.

[Margin: When a smaller number is set to be divided by a larger, what is to be done.]

Some people multiply the quotient figure they have found into the whole divisor in another way. First they multiply it into the first figure of the divisor and take the product away from the number above that figure; then they carry the same figure into the second figure of the divisor, and so on through the rest, until they reach the last, removing the products from the numbers written above. For instance, if the number 3387 is to be divided by 469 (just as was done in the last operation of the example above): after they have found that the last figure of the divisor, namely 4, is contained seven times in the 33 above (for it cannot be taken eight times, as we said just now), they place the figure 7 in the quotient. They do not say " $7 \times 4 = 28$," as we did, but: " $7 \times 9 = 63$, which from the number 3387 above...

Worked division on p. 265 (galley method) — reproduced as printed

$1832487 \div 469 = 3907$, remainder 104 (i.e. $3907 \text{ } 104/469$).

The page prints the scratch (galley) tableau in three successive states. Struck-through galley digits are reproduced as read; digits that cannot be resolved on the leaf are marked [?].

State 1 (quotient figure 3):

```
  42
 655
1832487 (3
 4699
  46
```

State 2 (quotient 390):

1
 63
 421
 6553
 1832487 (390
 469999
 4666
 44

State 3 (quotient 3907, remainder 104):

631
 42150
 655364
 1832487 (3907 · 104/469
 469999
 4666
 44

Abbreviated ("multiply into the whole divisor") method, $3387 \div 469 = 7$,
remainder 104:

10
 2924
 3387 (7
 469

PDF p. 266 (printed p. 16) — running header "DIVISIO"

ro 3387" — are to be subtracted. Taking 3 from 7 leaves 4; so, having deleted the figure 9 in the divisor and the figure 7 in the dividend, they place 4 above the 7. Taking 6 from 8 leaves 2, to be placed above the 8, previously deleted. Then again they say: $7 \times 6 = 42$, which they take from the number 332 [numeral: ?] above. Taking 2 from 2 leaves nothing; so, having deleted the figure 6 in the divisor together with the figure 2 in the dividend, they place 0 above the 2. And since the 4 (namely the other figure of the product 42) cannot be taken from 3, they take the distance of 4 from 10, namely 6, to which they add 3, making 9, to be placed above the 3, previously deleted. On account of that distance from 10 they take 1 from the last figure 3, and place the remaining 2 above the 3, having first deleted the figure 3. Finally they say: $7 \times 4 = 28$. Taking 8 from 9 leaves 1, to be placed above the 9, provided the figure 9 is first deleted in the dividend. Taking 2 from 2 also leaves nothing, and the operation is complete. By this procedure it usually happens that not as many figures are written above the dividend as are set down in that earlier way of multiplying the quotient figure into the divisor — where it is carried first into the last figure of the divisor, then into the next-to-last, and so

on, as we explained above. You will learn this from examples. But that earlier method is more in use among mathematicians and merchants, and in it an error can be corrected more easily whenever too large a figure has been placed in the quotient, as we shall show shortly.

[Margin: In what the difficulty of division consists.]

Once this example that we have explained is thoroughly grasped, there will be no difficulty in dividing any number whatever, of however many figures. For the whole labor seems to lie in this: in finding out how many times the last figure of the divisor is to be taken in the number written above — so that this quotient figure, multiplied into all the figures of the divisor, may produce a number that can be subtracted from the number above, and the number left after this subtraction may be smaller than the divisor.

[Margin: When the figure taken in the quotient is too small or too large, what is to be done.]

Now if it should happen (as not rarely befalls those little practiced in this business) that such a figure is placed in the quotient that, when it is multiplied into all the figures of the divisor and the products are subtracted from the numbers set above, the number left is larger than the divisor — or at least not all the products can be subtracted — then, if this happens at the beginning of the division, the error will be easily corrected by taking a larger or a smaller figure in the quotient, as the case requires; for the figures of the dividend set above the divisor are then still recognizable, even if deleted, so that from them the numbers produced by multiplying the new quotient figure into the figures of the divisor can easily be subtracted again — especially if those deleted figures of the dividend are written in order above the other deleted figures of the divisor, and the divisor too is replaced beneath the deleted divisor, so that the deleted figures may not trouble us. But if it happens in the middle of the operation, or near the end, one cannot so easily correct the error, since the figures of the dividend set above the divisor in that operation can scarcely be told apart from the others, being already deleted and mixed with others and set above the dividend. So, lest we then be forced to repeat the whole division (which everyone teaches must necessarily be done — a thing that would be most tiresome, especially if several operations of the division had already been completed), we have devised the following remedy.

If the quotient figure taken is too small — so that the number left, after subtracting the numbers produced by multiplying that figure into all the figures

of the divisor, is larger than the divisor — we shall subtract the divisor from the number left as many times as we can, until a number smaller than the divisor is left; and as many times as it has been subtracted, so many units we shall add to that quotient figure. But if the quotient figure taken is too large — so much so that, after subtracting some of the numbers produced by multiplying that figure into some of the figures of the divisor, we come upon a product that can no longer be subtracted — we shall multiply that quotient figure into the deleted figures of the divisor (those whose products have already been subtracted), and shall write the products, together with the figures of the number left previously added, in order above those figures of the divisor, deleting the earlier figures left. For by this procedure the number is restored that stood above the divisor before that operation. We shall therefore divide it again by the divisor (renewed, however, as regards the deleted figures, lest they cause confusion), taking another figure in the quotient one unit smaller than the one taken before. And if this figure too is too large, we shall in the same way restore the number placed above the divisor, and take another, smaller figure — until we find one which, multiplied into all the figures of the divisor, produces numbers that can both be subtracted and leave a number smaller than the divisor. But all this will become clearer by the following worked example.

[Margin: An example of correction, when too small a figure has been taken in the quotient.]

Let the number 1623149 be divided by 2899. With the divisor placed under the dividend, suppose someone insufficiently practiced has taken the figure 4 in the quotient. If then we say: $4 \times 2 = 8$, which subtracted (in the way we explained in the preceding example) from 16 leaves 8; then $4 \times 8 = 32$, which subtracted from 82 leaves 50; again $4 \times 9 = 36$, which subtracted from 503 leaves 467; and finally $4 \times 9 = 36$, which subtracted from 4671 leaves 4635 — this number is larger than the divisor. Therefore the figure 4 taken is too small. So, deleting this remainder together with the figure 4 taken, we shall write those deleted figures of the dividend, 16231, above the other deleted figures, and place the deleted divisor, renewed, under the divisor, as you see done in the next example. In this way the whole dividend 1623149 is restored, together with the divisor, as though the division had not yet been begun. We shall then place the figure 5 in the quotient, one unit greater than 4, as you see in this second example, and say: $5 \times 2 = 10$, which subtracted from 16 leaves 6. Then, having deleted the figure 2 in the divisor and the figure 1 in the dividend (which signifies ten with respect to the figure 6), we say again: $5 \times 8 = 40$, which subtracted from 62 leaves 22; likewise $5 \times 9 = 45$, which subtracted from 223 leaves 178; and finally $5 \times 9 = 45$, which

subtracted from 1781 leaves 1736 — a number smaller than the divisor. So the figure 5 has been rightly taken.

Worked correction-example on p. 266 — reproduced as printed

1623149 ÷ **2899** (first quotient figure: trial 4 too small → 5). Galley scratch tableaux, struck digits reproduced as read; ambiguous struck quotient markers flagged.

Trial with figure 4 (too small):

```
4
563
8075
1623149. (4
2899
```

Number restored, figure 5 tried:

```
6
423
5631
18075
1623149 (4[?]
2899
2889
```

Figure 5 confirmed:

```
1
23
678
4236
5631
18075
1623149 (45[?]
28999
2899
289
```

PDF p. 267 (printed p. 17) — running header "INTEGRORVM."

[Margin: An example of correction, when too large a figure has been taken in the quotient.]

But so that you may also have an example of the case when too large a figure is taken, let us suppose the figure 6 has been placed in the quotient. Multiplied into 2 it makes 12, which subtracted from 16 leaves 4. Then, since $6 \times 8 = 48$, which

cannot be taken from 42, it follows that the figure 6 taken is too large. Therefore, deleting this remainder 4 together with the figure 6 taken, we shall write the figures 1 and 6 of the dividend, which were deleted, above the same places, and the figure 2 in the divisor, which was deleted, below the same — so that the whole dividend originally proposed is restored, together with the divisor, as though the division had not yet been begun, as you see done in this example set beside. We shall then place, as appears in the following example, the figure 5, one unit smaller than 6, in the quotient, and say: $5 \times 2 = 10$, which subtracted from 16 leaves 6. Then, having deleted the figure 2 in the divisor [and the figure 1 in the dividend], which signifies ten with respect to the figure 6, we say again: $5 \times 8 = 40$, which subtracted from 62 leaves 22; likewise $5 \times 9 = 45$, which subtracted from 223 leaves 178; and finally $5 \times 9 = 45$, which subtracted from 1781 leaves 1736. So all the products could be subtracted, and the number left is smaller than the divisor. Therefore the figure 5 has rightly been taken in the quotient. From this you will easily understand what to do when, at the beginning of a division, too small or too large a figure has been taken.

Now attend to how the error is corrected if, in the middle of the operation, too large or too small a figure has been taken in the quotient.

Let the divisor, then, be advanced in the example above — the one in which too small a figure was taken at the start of the division — as you see there in the third position of the example. Let us imagine that the last figure of the divisor, 2, is contained seven times in the number 17 written above, so that after the figure 5 already found, 7 is written in the quotient. On this supposition we shall say: $7 \times 2 = 14$, which subtracted from 17 leaves 3, which I write above the 7, having first deleted the figure 2 in the divisor together with the figures 7 and 1 in the dividend. Then again we shall say: $7 \times 8 = 56$, which cannot be taken from 33. So the figure 7 taken is too large. Now, in order to restore the number 17 from which the subtraction was made — in case it cannot be told apart among so many deleted figures — the figure 7 taken must be multiplied into the figure 2 deleted in the divisor, and to the product must be added the figure 3 set above that figure of the divisor. Thus, since $7 \times 2 = 14$, if 3 is added it makes 17. So, deleting the figure 3, we shall write above it the number 7, and above the deleted figure 1 we shall place 1; and thus the number 17, from which the subtraction was made, is restored, as appears in this example set beside. And, the figure 2 having been placed under the figure 2 deleted in the divisor, so that the divisor too may be restored — as is plain in this same example — let us conceive that the last figure of the divisor, 2, is contained in 17 not seven times but six; and so, deleting the figure 7 in the quotient, let 6 be placed, as appears in this other example. On this

supposition we shall say: $6 \times 2 = 12$, which subtracted from 17 leaves 5. So, having deleted the figure 2 in the divisor together with the figures 7 and 1 in the dividend, we shall write 5 above the 7, and again say: $6 \times 8 = 48$, which subtracted from 53 leaves 5. So, having deleted the figure 8 in the divisor together with the figures 3 and 5 in the dividend, we shall write 5 above the 3, and again say: $6 \times 9 = 54$, which subtracted from 56 leaves 2. So, having deleted the figure 9 in the divisor together with the number 56 in the dividend, we shall place 2 above the 6, and finally say: $6 \times 9 = 54$, which cannot be taken from 24. So the figure 6 in the quotient is also too large.

Therefore, in order to know what number was placed above the divisor before we began this operation, we shall multiply the said figure 6 into the deleted figures of the divisor, as was said. Thus $6 \times 9 = 54$; adding the 2 set above the deleted divisor figure 9 gives 56. So, deleting the figure 2, we shall write 6 above it and keep back 5. Then $6 \times 8 = 48$; adding the 5 we kept gives 53. So we shall write 3 above the 5 and keep back 5. Finally $6 \times 2 = 12$; adding the 5 we reserved gives 17, which we shall place above the 15. And thus the number that stood above the divisor before this operation is restored. Then, the three figures 2, 8, 9 deleted in the divisor having likewise been restored, and the figure 6 in the quotient deleted, let us place 5 in its stead, as you see in this example. Now since $5 \times 2 = 10$, which subtracted from 17 leaves 7, we shall delete the figure 2 in the divisor together with the figure 1 in the dividend (which signifies ten with respect to the figure 7), and say: $5 \times 8 = 40$, which subtracted from 73 leaves 33. So, having deleted the divisor figure 8 together with the figure 7 in the dividend, we shall write 3 above it, and again say: $5 \times 9 = 45$, which subtracted from 336 lea—

Worked mid-operation correction tableaux on p. 267 — reproduced as printed

Galley scratch tableaux (dividend 1623149, divisor 2899), showing the trial second-quotient figures 7 (too large) and 6 (too large) and their restorations. Struck digits reproduced as read; the small struck quotient markers to the right are hard to resolve and are flagged.

Figure 6 tried (too large), then figure 5:

```

4
1623149 (6
2899
6
14
1623149 (6[?]
```

2899
2

Mid-operation trial (figure 7, then restoration):

1
2
63
14786
1623149 (6·5[?])
2899
2

1
233
678
4236
5631
18075
1623149 (45·7[?])
28999
2899
289

2
15
178
233
6782
4236
5631
18075
1623149 (45·76[?])
28999
289

2
17
183
178
2336
6782
4236
5631
18075
1623149 (45·76[?])
28999
2899
289
289
2

Chapter V. Division [Divisio] (continued)

Correcting a quotient figure that is too small — worked example

$$1623149 \div 2899$$

the remainder is 291. So, striking out the divisor's figure 9 together with the dividend-figure 336, we shall put 291 in its place. Then finally we say: 5 into 9 makes 45; subtracting this from 2914 leaves 2869, a number smaller than the divisor. Therefore the figure 5 was rightly taken.

Now advancing the divisor at last into the next place — that is, into the final place, as you see in the preceding example — let us reckon that the divisor's last figure, 2, is contained seven times in the 28 written above it. So having placed the figure 7 in the quotient, as you see in this other example set beside it, we say: 7 into 2 makes 14; taking this from 28 leaves 14. Again, 7 into 8 makes 56; subtracting this from 146 leaves 90. Again, 7 into 9 makes 63; taking this from 909 leaves 846. Again, 7 into 9 makes 63; subtracting this from 8469 leaves 8406 — a number larger than the divisor. Consequently the figure 7 that was taken is too small.

We shall therefore subtract the divisor from the said remainder as many times as we can, and we shall write in the quotient a figure greater than 7 by as many units as the number of times the divisor has been subtracted. This is how we subtract the divisor in the following example, the divisor having first been restored: taking 2 from 8 leaves 6; again, taking 8 from 64 leaves 56; again, subtracting 9 from 560 leaves 551; finally, taking 9 from 5516 leaves 5507 — a number still larger than the divisor. Then, taking 2 from 5 leaves 3; taking 8 from 35 leaves 27; again, subtracting 9 from 270 leaves 261; and lastly, subtracting 9 from 2617 leaves 2608 — a number now smaller than the divisor. Since, then, the divisor was subtracted twice, we shall write in the quotient — the figure 7 having first been deleted — the number 9, that is, a figure greater than 7 by two units. And so the whole quotient is 559.

We have been obliged to set this out with so many examples so that it might be understood more clearly what is left over at each single operation, although only this last one serves as a model for all the rest. And although we have expounded this remedy in so many words, practice will nonetheless easily teach that the matter is shorter and easier than can be expressed in words.

If, then, this remedy is applied whenever the figure taken in the quotient is greater or smaller than it ought to be, it is incredible how readily any number

whatever may be divided by any number whatever. For by this method there is no need to be over-anxious about what figure we ought to write in the quotient at each operation, since with that remedy we can correct any error committed with almost no effort. Thus this method of dividing, which we have laid down up to now, is the most excellent and most expeditious among all those usually explained by other authors; and therefore the greatest diligence and care should be applied, so that anyone who wishes to be outstanding and excellent in the art of computation may train himself in it.

[Margin: An easier method of division (*Facilior ratio divisionis*).]

But since, if the aforesaid remedy is applied, the division swells to too great a bulk, we shall be able to use this other device. Once the first operation of the division is finished, let the remainder be written separately together with the next figure of the dividend, and beneath this number let the divisor be placed again — beginning from the last figure on the right and proceeding toward the left. Then let the second operation of the division be carried out; and once it is done, let the remainder again be written separately with the following figure of the dividend, and so on to the end. For by this method all confusion of deleted figures will be avoided. For whenever, in any of these partial divisions, a figure too great or too small has been taken for the quotient, the error will easily be corrected if that partial division is repeated — which will be no trouble, since there is no need to repeat the whole division from the beginning. You have an example in the division of the number 2596860019 by 38748. So that you may know which figure of the dividend to write after each remainder, you will do rightly if, beneath the last figure of the first operation — as in the given example, beneath the 6 — you place a dot, to indicate that in the second operation the figure after that dot is to be taken. Beneath this figure another dot is to be marked, to signify that in the third operation the figure after that dot is to be taken, and so on, just as you see observed here.

The galley (scratch-division) figure printed in the right-hand column shows the successive states of the same division, $1623149 \div 2899$ (quotient 559), with the deleted figures the text describes. Transcribed top to bottom (the printed layout is a two-dimensional galley with struck-through digits that cannot be fully recovered from a linear reading):

2
38
279
1536
1751

2336
6782
4236
5631
180759
1623149 (45765 [?] ← quotient-position marker (struck trial digits; final
quotient narrated as 559)
289999
28999
289
289
228

8
19
24
364
1790
1536
1751
2336
6782
42360
36316
1807596
1623149 (457657 [?]
289999
28999
289
289
228

2
3
5
66
87
198
246
384
1790
15360
17511
23360
67821
423608
563167
1807596
1623149 (4576579 [?]

289999
 28999
 2899
 289
 228
 28

Assembling the quotient; proof by casting out nines — $2596860019 \div 38748$

The first worked example (galley form, $2596860019 \div 38748$, done as five partial divisions):

Operatio 1. Operatio 4.

2 1 3565
 3729 46763
 71448 74401 (1
 259686 (6 38748
 38748

Operatio 5.

Operatio 2. 7

7 1880
 184 84277
 65004 356539 (9
 271980 (7 38748
 38748

Operatio 3. Dividendus 2596860019

7440 (0 Divisor.... 38748

38748 Quotiens 67019

Totus Quotiens = $67019 \overline{) 2596860019} / 38748$

[Margin: Proof by 9 of the whole example (*Probatio per 9 totius exempli*).]

The proof-by-nines cross (casting out nines):

1
 3 X 5
 1

(residues placed in the four arms of the cross: dividend 1, divisor 3, quotient 5, product 1)

So, all the quotient figures having been set down in order, the whole quotient will be 67019. This device has the advantage that, if the proof does not turn out satisfactorily, there is no need to repeat the whole division: it is enough for the individual operations — which are few — to be examined, and for the one in

which the error was detected to be repeated. This device is easy and quick, and therefore I judge it fit to be used by beginners.

As for the fact that others multiply the figure placed in the quotient into the whole divisor and place the resulting number beneath the divisor (the first figure placed under the first, the second under the second, and so on) so as to subtract it from the number written above the divisor — that method is indeed reliable and easy, but it makes the division longer than it need be and holds up the one dividing not a little. For example, to divide the number 40689 by 1298: after they have placed the figure 3 in the quotient in the first operation, they multiply it into the divisor, first into the figure 8, saying: 3 into 8 makes 24, so they write 4 under the 8 and keep 2. Then 3 into 9 makes 27; adding the 2 that was kept makes 29; so, placing 9 under 9, they keep 2, and so on. After this, the divisor being deleted, they take 4 from 8 and place the remainder 4 above the 8, the figures 4 and 8 having first been deleted, and so forth. Then, the divisor being advanced, they proceed in the same way. We, however, dispatch this more briefly, since we do not write the product beneath the divisor. Yet that method has this advantage: that from the operation itself it is easily understood whether the figure taken in the quotient is too great or not. For if the number produced by multiplying that figure into the divisor can be subtracted from the number placed above the divisor, and leaves a number smaller than the divisor, then that figure was rightly taken; if not, there is no doubt that an error was made.

This method will become quicker if, in place of the divisor, one puts the number produced from the quotient-figure into the divisor, and carries out partial divisions, as we said a little earlier — as is shown here in the same example.

[Margin: How others carry out division (*Quomodo alij divisionem absolvant*).]

The galley for $40689 \div 1298$ (quotient 31):

```

  45
 1741
40689 ( 31
12988
 3894
  129
```

[Margin: The advantage of this method of dividing (*Commoditas in hoc modo dividendi*).]

[Margin: Another easy method of division (*Alia ratio divisionis facilis*).]

The same division, $2596860019 \div 38748$, carried out by partial subtractions (in place of the divisor, the product of the quotient-figure and the divisor is set down and subtracted):

Operation	Working	Quotient figure
Operatio 1	$259686 - 232488 = 27198$	6
Operatio 2	$271980 - 271236 = 744$	7
Operatio 3	7440 (too small)	0
Operatio 4	$74401 - 38748 = 35653$	1
Operatio 5	$356539 - 348732 = 7807$	9

Dividendus 2596860019 · Divisor 38748 · Quotiens 67019 (remainder 7807).

For example, in the first operation the dividend is this number, 259686. If, then, the divisor were written beneath it, the figure 8 would fall under the 6, and the figure 3 under the 5. And since 3 is contained in the 25 written above it six times (for the figure 7 would be too great), we say: 6 into 8 makes 48; we put 8 under the 6, keeping 4. Then 6 into 4 makes 24; adding the 4 we kept makes 28; so, placing 8 under 8, we keep 2, and so on, until the quotient-figure has been carried into all the figures of the divisor. Once the subtraction is made, the remainder 27198 results, and so on. In the third operation, since the dividend-number 7440 is smaller than the divisor, the figure 0 is placed in the quotient. But if at any time the number produced [shows the] quotient-figure taken to be too great, then that partial division must be repeated. If the division is set up in this way, there will be no confusion among the deleted figures, though the subtracting is somewhat longer than the method we handed down at the beginning.

[Margin: Another way of carrying out division (*Alius modus absolvendi divisionem*).]

As for the fact that others first multiply the divisor by each of the significant figures, writing each product-number next to the multiplying figure, so as to seek among these products the number placed above the divisor — and, this being found, or, if it is not found, the next smaller one being taken, they place in the quotient the multiplying figure ascribed to it, and subtract the number taken from the number placed above the divisor; and this too...

[this method] is easy and convenient, especially for beginners and those little practiced in this art, but too long and troublesome. For, to divide (for example) this number 97086 by 37: they place the divisor beside 1, then its double beside 2,

its triple beside 3, and so on. Afterward, among these numbers they look for the number 97 written above the divisor; and since they do not find it, they take 74, the next smaller, and place the figure 2 standing opposite it in the quotient, and take 74 from 97, writing the remainder 23 above 97, the figures 7 and 9 having first been deleted together with the divisor. Then, the divisor being advanced, they seek among the same numbers this number, 230, written above the divisor; and not finding it, they take the next smaller, 222, and place the figure 6 standing opposite it in the quotient, and finally subtract the same number 222 from 230. Proceeding in this way they complete the whole division. But who does not see that by this method the division is drawn out longer than it need be, especially if the divisor is written with four or five or even more figures? Likewise, when the quotient has only two or three figures, I do not see why the divisor should be needlessly carried into all nine figures.

The table of multiples of the divisor 37 ($37 \times 1 \dots 9$):

Multiple	Value
1	37
2	74
3	111
4	148
5	185
6	222
7	259
8	296
9	333

The accompanying galley for $97086 \div 37$ (quotient beginning 26...):

```

  23
97086 ( 26
  377
  3

```

[Margin: Division "per Dandam" (*Divisio per Dandam*).]

Nevertheless the method of dividing that I approve most is the one the Italians call "per Dandam," which is the safest: because the operation can be corrected at once, if by chance a figure too great or too small has been placed in the quotient, and because the figures in the dividend are never deleted. This method is

somewhat different from the one others use, but in my judgment more convenient and shorter.

Let the number to be divided be (for example) 1904639 by 2978. The number having been set down with a curved line, let the divisor be written above the place where the quotient is usually put, so that the multiplications may be made easier. Then, beneath the 6 — where the divisor's figure 8 would stand if it were placed beneath the dividend, as we said at the beginning of this chapter — let a dot be placed to indicate that the subtraction is to begin there. So then, since the divisor's last figure, 2, is contained in 19 six times (because nine, or eight, or seven times, the whole divisor cannot be contained in the number 19046), the figure 6 must be placed in the quotient; it must be carried into the divisor, and the subtraction made at the same time, in this order.

6 into 8 makes 48, which must be subtracted from the number 19046; but since the figure 8 is greater than 6, we shall take 8 from 10, and to the remainder 2 we add 6, and we shall place the sum 8 under the 6, keeping 5 in mind (namely 4 on account of the 40, and 1 on account of the 10 from which the 8 was subtracted). Then 6 into 7 makes 42, and adding the 5 kept makes 47; subtracting then 7 from 10 leaves 3, and adding 4 makes 7, which is to be placed under the 4, keeping 5 meanwhile on account of the 40 and 10. Again, 6 into 9 makes 54, and adding the 5 kept makes 59; subtracting then 9 from 10 (since they cannot be subtracted from 0) leaves 1, and the figure 0 being added, it becomes 1, to be written under the 0, keeping 6 meanwhile on account of the 50 and 10. Lastly, 6 into 2 makes 12, and adding the 6 we kept makes 18; subtracting this from 19 leaves 1, to be placed under the 9.

This operation completed, let two dots be placed under the figure 3, to indicate that the subtraction is to begin there. And since 2 is contained in 11 three times, the figure 3 must be placed in the quotient (because the figure 5 or 4 would be too great); and it must again be carried into the whole divisor in the same order, namely: 3 into 8 makes 24; subtracting 4 from 10 (since they cannot be subtracted from 3) leaves 6, and adding 3 makes 9, to be written under the 3, keeping 3 in mind on account of the 20 and 10. Further, 3 into 7 makes 21, and adding the 3 we retained makes 24. Then, taking 4 from 8 leaves 4, to be placed under the 8, keeping 2 on account of the 20. Again, 3 into 9 makes 27, and adding the 2 retained makes 29. Taking 9 from 10 (since they cannot be taken from 10) leaves 1, and adding 7 makes 8, to be written under the 7, keeping 3 meanwhile on account of the 20 and 10. Lastly, 3 into 2 makes 6, which with the 3 kept makes 9; taking this from 11 leaves 2, to be placed under the 1.

These things done, let three dots be placed under the 9, so that we may know that the subtraction is to begin there. And since 2 is contained in 28 nine times, we shall write in the quotient the figure 9, which will be carried into the whole divisor in this order. 9 into 8 makes 72; subtracting then 2 from 9 leaves 7, to be written under the 9, keeping 7 meanwhile on account of the 70. Then 9 into 7 makes 63, which with the 7 kept makes 70; subtracting then the figure 0 from 9 leaves 9, to be written under the 9, keeping 7 meanwhile on account of the 70. Further, 9 into 9 makes 81, and adding the 7 kept makes 88; taking 8 from 10 leaves 2 [additisque... scribantur], keeping 9 in mind on account of the 80 and 10. Lastly, 9 into 2 makes 18, which with the 9 kept makes 27; taking this from 28 leaves 1, to be placed under the 8. Therefore the whole quotient is 639, and the last remainder, 1697, is what remains from the division — just as, in the partial operations above, the last remainder 7807 is what is left over in that division.

The "per Dandam" galley for $1904639 \div 2978$ (quotient 639, remainder 1697), shown in its two successive states. The left column holds the running residues; the right column, headed *producti*, holds the products of the divisor by each quotient-figure ($2978 \times 6, \times 3, \times 9$):

Divisor 2978 · Dividend 1904639 · Quotient 639, remainder 1697 (printed as the fraction $\frac{1697}{2978}$ above the quotient — see note).

Residues	Products (<i>producti</i>)
1178..	17868 (= 2978 × 6)
2849.	8934 (= 2978 × 3)
1697	26802 (= 2978 × 9)

So, as you see, the whole difficulty of this kind lies in keeping in memory the tens of the numbers produced from the quotient-figure into the figures of the divisor. And indeed this method is most beautiful, because the residues of the operations are seen distinctly; so that, if by chance a figure too great or too small had been taken in the quotient, that figure, together with the false residue, can easily be deleted and another figure assumed without any confusion.

But if it seems troublesome to keep in mind those tens of the numbers produced, the products can be written separately, as you see in the example beneath the quotient. For 6 into the divisor makes the number 17868, and 3 into the same divisor the number 8934, and 9 into the same divisor the number 26802. For thus the subtraction will be easy, since to the preceding figure only 1 must sometimes be added — namely, when the subtraction has been made from 10. Nor indeed

does it please me [to place the products among the residues] (as some do), because confusion would arise among the numbers, while yet the subtractions are not made much easier.

[Margin: The first proof of division, by 9 (*Prima probatio divisionis per 9*).]

It remains for us to hand down the checking of division, which is threefold. The first is done by casting out nines, in this way. Nines having been cast out from the divisor as many times as possible — as we taught in addition — let the remainder be placed in the left part of the cross. Nines likewise having been cast out from the quotient as many times as possible, let the remainder be placed in the right part of the cross. Then, these two remainders having been multiplied by each other, and nines cast out from the product as many times as possible, let this remainder be placed [above], if in the division...

nothing remained over, it is placed in the upper part of the cross. But if some remainder was left in the division, that last remainder must be joined with the figures of this remaining division, and 9 is always to be cast out; and this residue is to be placed in the upper part of the cross. Finally, casting 9 out of the dividend as many times as it can be done, let the residue be placed in the lower part of the cross. For if this residue equals the residue placed in the upper part of the cross, the division has been done correctly; if not, incorrectly.

So this division will be checked as follows. Casting 9 out of the divisor 23 leaves 5. Casting 9 likewise out of the quotient 176 also leaves 5. Multiplying these two residues, 5 and 5, together yields 25; and casting 9 out of that leaves 7 — which (since nothing was left over from the division) is to be placed in the upper part of the cross. And because casting 9 out of the dividend 4048 also leaves 7, the division has been done correctly.

1
131
273
4048 (176
2333[?]
22

Cross-check by casting out 9 (of $4048 \div 23 = 176$):

7
5 × 5
7

This other division will be checked by the same reasoning. Casting 9 out of the divisor 236 leaves 2. Casting 9 likewise out of the quotient 139 [recte: 193] leaves

4. Multiplying these residues, 2 and 4, together makes 8, from which 9 cannot be cast. This 8, then, if nothing had been left over in the division, would have had to be placed in the upper part of the cross; but since 130 remains, we shall say: 8 and 3 make 11; casting out 9 leaves 2; adding 1 makes 3, to be placed in the upper part of the cross. And since casting 9 out of the dividend 45678 also leaves 3, the division has been done correctly.

```

1
2
183
434
22030[?]
45678 (193
23666[?]
233
2

```

Cross-check by casting out 9 (of $45678 \div 236 = 193$, remainder 130):

```

3
2 × 4
3

```

[Margin: The second proof of division, by [casting out] 7.]

The second check is made by casting out 7, as we taught under Addition — provided that from the remainder of the division, if there was any, 7 is likewise cast out in the same way, and that residue is added to the residue which, in the check by casting out 9, we said had to be added to the remainder of the division; and 7 is then cast out of the total.

Thus the earlier of the two nearest divisions will be checked as follows. Casting 7 out of the divisor 23 leaves 2. Likewise, casting 7 out of the quotient 176 leaves 1. Multiplying these two residues, 2 and 1, together produces 2, to be placed in the upper part of the cross, since nothing was left over from the division. And because casting 7 out of the dividend 4048 also leaves 2, the division has been done correctly.

Cross-check by casting out 7 (of $4048 \div 23 = 176$):

```

2
2 × 1
2

```

The later division will be checked in this way. Casting 7 out of the divisor 236 leaves 5. Likewise, casting 7 out of the quotient 193 leaves 4. Multiplying these

two residues, 5 and 4, together, and casting 7 out of the product 20, leaves 6 — which, if nothing had remained in the division, would have had to be placed in the upper part of the cross. But since the remainder was the number 130, from which, casting out 7, 4 is left over: this 4, added to the last residue kept, 6, makes 10; and casting 7 out of that leaves 3, to be placed in the upper part of the cross. And just as much remains if 7 is cast out of the dividend 45678. Therefore the division has been done correctly. Yet both of these checks can be fallacious, for the reason stated above.

Cross-check by casting out 7 (of $45678 \div 236 = 193$, remainder 130):

3
5 × 4
3

[Margin: The third proof of division, by multiplication.]

The third check — which is certain, and to which no fallacy can attach — is made by multiplication. For if the divisor and the quotient are multiplied together and the remainder of the division (if there was any) is added to the resulting number, the dividend will be reproduced, provided no error was made in the division. Thus the later of the two nearest divisions will be checked as follows. After the divisor 236 has been multiplied by the quotient 193, before the resulting partial products are added together, the remainder of the division, 130, is written beneath them — the first figure under the first place, the second under the second, and so on. For if we gather the partial products and this remainder into a single sum, as we taught under Multiplication, the number 45678, which was divided, will be produced.

236
193
———
708
2124
236
130
———
45678.

[Margin: It is sometimes useful to check a division that is not yet complete.]

It is sometimes useful — when, after carrying out some operation in a division, you are in doubt whether you have gone wrong in something or not — to check the division as far as it has been carried, so that you do not go on further in vain if some error has perhaps already been committed. You will check that portion of

the division no differently than any other division, provided you leave out the figures of the dividend under which the divisor has not yet been placed. Thus in this division, once the first operation has been done, you will check it by casting out 9. Casting 9 out of the divisor 2898 leaves 0. Likewise, casting 9 out of the quotient 2 leaves 2. Multiplying these residues, 0 and 2, together produces 0 — a figure that would have to be placed in the upper part of the cross, unless something were left over from the division. But since 913 remains, 9 must be cast out of this remainder. When that is done, 4 will remain, to be placed in the upper part of the cross. And just as much remains if 9 is cast out of the number 6709 divided so far, the figures 456 being left out, under which the divisor has not yet been placed.

```

191
2123
6709456 (2
2898

```

Cross-check by casting out 9 (partial division of $6709456 \div 2898$):

```

4
0 × 2
4

```

[Margin: The ease of division when the divisor has, at the outset, some zeros.]

But if the divisor has, at the outset, some zeros, the division will be easy: remove from the dividend as many figures on the right as the divisor has zeros, and divide the remaining number by the divisor once those zeros have first been dropped. Then the remainder of this division, if there is any, must be prefixed on the left to the removed figures, so as to form the numerator of a fraction; and the denominator will be the whole divisor together with its zeros. And if nothing remained in the division, the removed figures themselves are to be set down as the numerator of the fraction.

```

2
278
456
13946 (367 [fraction: 7693/38000000 – see following prose]
3888
33

```

For example, if the number 13946007693 is to be divided by 38000000, we shall take from it the first six figures on the right, 007693 — namely, as many as there are zeros at the outset of the divisor — and we shall divide the remaining number, 13946, by 38, those six zeros being omitted, as has been done in the

example set alongside. Now since nothing was left over in the division, we shall write above the divisor the removed number, 7693; for those two [leading] zeros on the left signify nothing and are therefore to be omitted.

Again, if the same number 13946007693 is to be divided by 300800000, we shall take from it the first five figures, 07693 — namely, as many as there are zeros at the outset of the divisor — and we shall divide the remaining number, 139460, by 3008, those five zeros being omitted, as has been done in this other example. And because in the division this number, 1092, is left over, if we prefix it on the left to these removed figures, 07693, we shall place above the whole divisor this number, 109207693, as you see here.

```

69
11142
139460 (46 [fraction: 109207693/300800000]
30288
300

```

Hence it follows that, if the last figure of the divisor is 1 and all the rest are zeros, the quotient is the dividend itself, with as many figures first removed from its right as there are zeros in the divisor; the numerator of the fraction will be the removed number, and the denominator the divisor itself. Thus if the number 4780920345 is to be divided by 100000, the quotient will be 47809 and 20345/100000. So too, if the number 9700203 is to be divided by 10000, the quotient will be 970 and 203/10000. And so with the rest.

[Margin: The division is sometimes made easy when the dividend has several zeros at the beginning.]

Nor should this be passed over: if the dividend has several zeros at the beginning, and before the whole division is completed no significant figure is left in the division, then all the zeros of the dividend not yet cancelled must be placed after the quotient. Thus if the number 1863000000 is proposed to be divided by 345 — since after the second operation nothing remained in the division — if, after the quotient 54 has been found, the five zeros of the dividend not yet cancelled are set down, the whole quotient will become 5400000, and the division will be complete.

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112
338
1863000000 (5400000
3455
34

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[Margin: Addition, subtraction, multiplication, and division are the foundations of everything taught in arithmetic.]

From what has been said about the addition, subtraction, multiplication, and division of whole numbers, everything taught in the whole of arithmetic depends as though upon its elements — so much so that everything is accomplished by means of them, and nothing else need be laid down for solving arithmetical problems than that numbers be added, subtracted, multiplied, or divided. And so, unless one is thoroughly practiced in those four arithmetical operations, he will turn in vain to the other matters we are about to treat.

THE NUMBERING OF FRACTIONAL NUMBERS. CHAPTER VI.

Just as, in the preceding chapters, we numbered whole numbers, gathered several proposed ones into a single sum, subtracted one from another, multiplied any two together, and finally divided one by another, so in what follows we shall do the same with fractional numbers — which by another name are commonly called *minutiae*, or fractions.

[Margin: What a fractional number, or fraction or *minutia*, is.]

A fraction, *minutia*, or fractional number is one part, or several parts, of some whole that has been divided into several equal parts. For example, if some whole is cut into five equal parts and someone takes one of them, that fifth part will be called a fractional number. Likewise, if someone takes two, three, or four of them, those two, three, or four fifths will be called a fractional number.

[Margin: What the numerator and denominator of a fraction are.]

Every *minutia* consists of two numbers, which are expressed in stating it. The first is called the numerator, because it numbers how many parts the proposed fraction contains of those into which the whole (of which it is a fraction) has been divided. The other is called the denominator, because it denominates those parts of the fraction — that is, it indicates into how many parts the whole is understood to have been divided. Thus, when a fraction containing three fifths is proposed, the numerator is 3, because it signifies that three parts of the whole are contained in that fraction; and the denominator is 5, because it indicates that those three parts are not just any parts, but fifths.

[Margin: How any fraction is to be written and pronounced.]

Any fraction is written this way: the denominator is placed directly under the numerator, with a small line drawn between the two numbers. Thus three fifths

are written like this, $3/5$. Both numbers are pronounced in the nominative, the numerator first, however. So the said *minutia* is to be read out thus: "three fifths." This one, $25/48$, is read "twenty-five forty-eighths," and it signifies that some whole has been divided into forty-eight equal parts and twenty-five of them have been taken.

[Margin: Where fractions come from. When a smaller number is divided by a larger, a fraction results.]

Fractional numbers usually arise from the remainder of the division of whole numbers. For when something has been left over in a division, from it is formed the numerator of a fraction whose denominator is the divisor, as we said above. Thus, when 46 is divided by 7, the quotient is 6 and 4 are left over; so a fraction of this kind, $4/7$, results, so that the whole quotient is $6 \frac{4}{7}$. Likewise, when a smaller number is set to be divided by a larger, a fraction is produced whose numerator is the dividend and whose denominator is the divisor. Thus, if 4 is to be divided by 7, the fraction $4/7$ results, and it signifies that 4 has been divided by 7, so that this *minutia*, $4/7$, is the seventh part of the number 4 — a part, I say, denominated from the divisor 7. For just as, when 12 is divided by 3, the number 4 is found, which is the third part of the divided number 12 (a part denominated from the divisor 3), so also, when we divide 4 by 7, the quotient $4/7$ results, which is the seventh part of the divided number 4, a part denominated from the divisor. By the same reasoning, every other *minutia* is a part of the numerator, denominated from the denominator. Thus this *minutia*, $3/4$, is the fourth part of three; for if 3 is divided by 4, the quotient is $3/4$. And so, if the *minutia* $3/4$ is taken four times, it makes $12/4$, which is equivalent to three, as will be clear from what we shall write a little further on. And so with the rest.

THE ESTIMATION, OR VALUE, OF FRACTIONAL NUMBERS. CHAPTER VII.

[Margin: How the value of *minutiae* is increased.]

The estimation, or value, of any *minutia* increases when, the numerator remaining the same, the denominator is decreased; or when, the denominator remaining the same, the numerator is increased. Thus in these fractions — $1/6$, $1/5$, $1/4$, $1/3$, $1/2$ — or in these — $1/7$, $2/7$, $3/7$, $4/7$, $5/7$, $6/7$ — each later one is greater than the one before, as will be clear from what follows: in the former set the denominator is decreased while the numerator always stays the same; in the latter, the numerator is increased while the denominator always stays the same.

[Margin: How the value of *minutiae* is decreased.]

The estimation, or value, of any *minutia* decreases, on the other hand, when, the numerator remaining the same, the denominator is increased; or when, the denominator remaining the same, the numerator is decreased. Thus in these fractions — $1/2$, $1/3$, $1/4$, $1/5$, $1/6$ — or in these — $7/8$, $6/8$, $5/8$, $4/8$, $3/8$ — each later one is smaller than the one before, as will be established by what follows: in the former set the denominator is increased while the numerator always stays the same; in the latter, the numerator is decreased while the denominator always stays the same.

[Margin: *Minutiae* whose numerators have the same ratio to their denominators are equal.]

Next, any *minutiae* such that the numerator of one has the same ratio to its denominator as the numerators of the others have to their denominators — each to each — are equal to one another. Thus these *minutiae* — $1/2$, $8/16$, $50/100$, $100/200$, $500/1000$ — are all equal to one another, because the numerator of each has to its denominator a subduple ratio, that is, it is half of it. So too these — $3/4$, $6/8$, $36/48$, $90/120$ — for the numerator of each has to its denominator a subsesquitertian ratio, that is, it contains three quarters of it.

[Margin: If the numerator and denominator of any fraction are multiplied or divided by any number, a fraction of the same value is produced.]

Since, moreover, if two numbers are multiplied or divided by the same number, the resulting numbers keep the same ratio as those two numbers multiplied or divided, it comes about that, when the numerator and denominator are multiplied or divided by any number, another *minutia* of the same value is produced, even though it has larger or smaller numbers. Thus, given this *minutia*, $6/9$, if each of its numbers is multiplied by 3, the *minutia* $18/27$ will be produced, of the same value. Likewise, if each number is divided by 3, the *minutia* $2/3$ results, of the same value.

And although all this could be demonstrated from Book 7 of Euclid, we shall be content to illustrate the matter with a single example, drawn from these two *minutiae*, $2/3$ and $6/9$, in which the truth of the thing plainly appears. For if we take the number 9 — which indeed has parts denominated from the denominators of the said *minutiae*, namely a third and a ninth — its two thirds will be equal to its six ninths. For since its third is 3, two thirds will be 6; and since its ninth is 1, its six ninths will likewise be 6. Therefore these *minutiae*, $2/3$ and $6/9$, are equal; and so with the rest.

[Margin: Which *minutia* is equivalent to one whole.]

Further, when the numerator of some *minutia* is equal to the denominator, that *minutia* is equivalent to one whole. Thus each of these *minutiae* — $2/2$, $6/6$, $20/20$, $1000/1000$ — constitutes one whole, that is, the whole thing which has been divided into the parts denominated from the denominators; because the numerator contains all the parts into which the whole is divided.

[Margin: Which *minutia* is smaller than one whole.]

But when the numerator of a *minutia* is smaller than the denominator, that *minutia* is smaller than one whole — such as these *minutiae*: $2/3$, $4/5$, $9/10$. For in each of them, as many of the parts denominated from its denominator are lacking to make up the whole as there are units by which the numerator falls short of the denominator. Thus this *minutia*, $2/3$, lacks $1/3$; this one, $4/5$, lacks $1/5$; and this, $9/10$, lacks $1/10$.

[Margin: Which *minutia* is greater than one whole.]

Finally, when the numerator of a *minutia* is greater than the denominator, that *minutia* is greater than one whole — such as these *minutiae*: $4/3$, $24/7$ [?], $100/4$ [?]. For in each numerator more parts are contained than those into which the whole is divided.

[Margin: How to find out which of two *minutiae* is the greater.]

Now, given two *minutiae*, to find out which of them is greater, keep to this rule. Setting the *minutiae* in order, multiply their numbers crosswise — that is, the numerator of the first into the denominator of the second, and the numerator of the second into the denominator of the first — placing the resulting products above the numerators. For the one whose numerator has yielded the larger number will be the greater *minutia*. But if the two products come out equal, the proposed *minutiae* will likewise be equal. Thus in the first of these three examples the later *minutia*, $6/8$, is greater than the earlier, $2/3$, because the number 18 — produced from 6, the numerator of the later *minutia*, into 3, the denominator of the earlier — is greater than the number 16, produced from 2, the numerator of the earlier *minutia*, into 8, the denominator of the later. In the second example, the *minutia* $1/2$ is greater than $20/41$. And in the third, finally, the *minutiae* $3/4$ and $12/16$ are equal, as is clear from the crosswise multiplications performed.

16. 18. 41. 40. 48. 48.

$$\frac{2}{3} \times \frac{6}{8} \quad \frac{1}{2} \times \frac{20}{41} \quad \frac{3}{4} \times \frac{12}{16}$$

The reason for this rule is that, when the numerators multiplied crosswise into the denominators produce equal numbers, the numerators keep the same ratio to the denominators, as is established from Proposition 19 of Book 7 of Euclid. And so, as we said above, the *minutiae* will be equal. Hence it follows that the numerator which yields the larger number has the larger ratio to its denominator, and therefore that *minutia* is the greater, as was said above. But so that you may learn this by an example too — that the *minutia* $\frac{6}{8}$ is greater than $\frac{2}{3}$ — let us take the number 48, which has parts denominated from the denominators of these *minutiae*, namely an eighth and a third. And so, since one eighth of this number 48 is 6, six eighths will be 36; and since one third of the same number is 16, two thirds will be 32, which number is smaller than 36.

[Margin: How the value of a given *minutia* is to be explored in a smaller coin, weight, or measure.]

But if a *minutia* of some coin, or of some larger weight or measure, is given, and you wish to explore its value in a smaller coin, weight, or measure — that is, to reduce it to a smaller coin, and so on — you will do it in this way. Multiply the numerator by the number that indicates how many times the smaller coin, to which the fraction is to be reduced, is contained in the larger, and divide the resulting number by the denominator. For the quotient will indicate the value of the given *minutia* in that smaller coin. Understand the same of weights and measures. For example, if this *minutia* is given — $\frac{4}{7}$ of a gold coin [aureus] — which signifies, as we said above in Chapter 6, four gold coins among seven... distributed into seven equal parts, and it must be reduced to giulii, baiocchi, and quattrini. (In this Arithmetic of ours we use examples drawn from Roman coinage, in which — at the time we first composed this Arithmetic — four quattrini [quattrini] made one baiocco [baiochus], ten baiocchi one giulio [Iulius], and ten giulii one gold scudo [aureus nummus].) We will multiply the numerator 4 by 10 — since 10 giulii make one gold scudo — so as to reduce those 4 gold scudi, divided into 7 parts, to 40 giulii; and we will divide the resulting product 40 by the denominator 7. The quotient will give $5 \frac{5}{7}$ giulii.

But if we wish to reduce this fraction of giulii — $\frac{5}{7}$, which signifies 5 giulii divided into 7 equal parts — to baiocchi, we will likewise multiply the numerator 5 by 10 (since 10 baiocchi also make one giulio), so as to reduce those 5 giulii, distributed into 7 equal parts, to 50 baiocchi; and we will divide the product 50 by the same denominator 7. The quotient will give $7 \frac{1}{7}$ baiocchi.

Then, if we wish to reduce this fraction of baiocchi — $1/7$, which signifies one baiocco divided into 7 equal parts — to quattrini, we will multiply the numerator 1 by 4 (since 4 quattrini make one baiocco), so as to reduce that baiocco, divided into 7 equal parts, to 4 quattrini; and we will divide the product 4 by the denominator 7, obtaining $4/7$ of one quattrino — that is, a little more than half a quattrino. Thus $4/7$ of a gold scudo contain 5 giulii, 7 baiocchi, and $4/7$ of a quattrino.

But if instead we should wish to reduce $4/7$ of a gold scudo directly to baiocchi, we will multiply the numerator 4 by 100 — since 100 baiocchi make up one gold scudo — so as to reduce those 4 scudi, distributed into 7 parts, to 400 baiocchi; and we will divide the product 400 by the denominator 7, obtaining $57 \frac{1}{7}$ baiocchi.

Again, suppose we must find out how many paces, feet, palms, or fingers are contained in $5/8$ of one Italian mile [*milliarium*], taking one mile to consist of 4000 [recte: 1000] geometric paces, one pace of 5 feet, one foot of 4 palms, one palm of 4 fingers, and one finger of 4 barleycorns. We will multiply the numerator 5 by 1000, so that 5 miles, distributed into 8 equal parts, are reduced to 5000 paces; and we will divide the product 5000 by the denominator 8, obtaining 625 paces.

In the same way, if we wish to reduce $10/13$ of one pace to feet, we will multiply the numerator 10 by 5 and divide the product 50 by the denominator 13, obtaining $3 \frac{11}{13}$ feet. If we then multiply this numerator 11 by 4 and divide the product 44 by the denominator 13, we obtain $3 \frac{5}{13}$ palms. Likewise, if we multiply this numerator 5 by 4 and divide the product 20 by the denominator 13, we find $1 \frac{7}{13}$ fingers. Finally, if we multiply this numerator 7 by 4 and divide the product 28 by the denominator 13, we find $2 \frac{2}{13}$ barleycorns. Thus $10/13$ of one pace contain 3 feet, 3 palms, 1 finger, and $2 \frac{2}{13}$ barleycorns.

Likewise, suppose the fraction $3/4$ of one pound [*libra*] must be reduced to ounces [*unciae*]. Since 12 ounces make one pound, we will multiply the numerator 3 by 12 and divide the product 36 by the denominator 4, obtaining 9 ounces.

Finally, suppose we must inquire how many minutes are contained in $5/6$ of one degree. Since 60 minutes make 1 degree, we will multiply the numerator 5 by 60 and divide the product 300 by the denominator 6, obtaining 50 minutes.

FRACTIONS OF FRACTIONAL NUMBERS

Chapter VIII

[Margin: Whence fractions of fractions arise. Fraction. What a fraction is. How fractions of fractions are pronounced and written.]

Not only is a whole thing divided into any number of equal parts, so as to produce simple fractions (which are our subject here); sometimes the fractional numbers themselves are understood to be distributed into several equal parts, as if they were certain whole things. From this arise fractions of fractional numbers — that is, fractions of fractions [*minutiæ minutiarum*]. For just as, when I take four parts of some whole divided into seven parts, I produce the simple fraction $\frac{4}{7}$, which signifies four-sevenths of the whole itself: so too, when I understand this simple fraction $\frac{4}{7}$ to be divided into five equal parts and I take three parts, I make a fraction of that fraction, namely three-fifths of the four-sevenths of some whole. It is to be written just as simple fractions are for the first (leading) one, while the latter is pronounced in the oblique case, and is marked without an intervening line, so that it may be distinguished from the others. Thus the aforesaid fraction of a fraction is to be written $\frac{3}{5} \cdot \frac{4}{7}$ and pronounced thus: *three-fifths of four-sevenths of one whole*.

This fraction of fractions, $\frac{2}{3} \cdot \frac{3}{4} \cdot \frac{1}{6} \cdot \frac{1}{2}$, will be expressed thus: *two-thirds of three-quarters of one-sixth of one-half of some whole*. It signifies that from the half of some whole one sixth part has been taken (the half being divided into six equal parts); and from this sixth part, divided into four equal parts, three quarters have been taken; and finally, from these three quarters, divided into three equal parts, two thirds have been taken. The same rule holds in the other cases.

But in what way the estimation, or value, of fractions of fractional numbers is to be known, we shall teach at the end of chapter 10, where we shall convert them back to simple fractions.

REDUCTION OF FRACTIONAL NUMBERS TO LOWEST NUMBERS, OR TERMS

Chapter IX

[Margin: Why fractions are reduced to lowest terms.]

It sometimes happens that a fraction is written with numbers so large that it could conveniently be expressed with smaller ones, without any change to its value or worth. For example, this fraction $36/72$ is equivalent to this one, $1/2$, expressed (as you see) in the smallest numbers. It is worthwhile to reduce a fraction written in larger numbers to the smallest numbers, or terms, for many reasons. First, because any fraction is more easily understood when expressed in smaller numbers than when written in larger ones. Who, indeed, does not grasp $1/2$ more readily than $36/72$, or than $500/1000$, or than $826/1652$ — though they all signify exactly the same thing? Next, because the operation of fractions is made easier if they are reduced to their lowest terms, as will become clear from what follows. Third, so that the books of mathematicians, who usually

note fractions with the smallest numbers, may be understood. For if, for instance, someone were to find something written by another — this number, 2528, divided by 48 giving the quotient $52 \frac{2}{3}$ — and he himself wished to check it, he would find the quotient $52 \frac{32}{48}$, which appears to differ from the former, though it is in fact the same. For this fraction $32/48$, reduced to its lowest terms, makes $2/3$. Therefore, before he judges that either he or that writer has erred (because he sees his own fraction differing from the other's), the fraction he has found, expressed in larger terms, must first be reduced to the smallest numbers, or terms.

[Margin: By what art fractions are reduced to the smallest numbers.]

By this art, then, any fraction written in larger numbers will be reduced to its lowest terms. Both the numerator and the denominator are to be divided by their greatest common measure — that is, by the largest number that measures both. For the quotient numbers (if you make the quotient of the numerator the numerator, and the quotient of the denominator the denominator) will give a fraction equivalent to that one, expressed in the smallest numbers. For since, when two numbers are divided by one and the same number, the quotients have the same ratio as those numbers, and since the quotients thus found are the smallest of all — because the numbers of the proposed fraction have been divided by the largest number that measures them, so that they cannot be divided by a larger one without leaving some remainder — it is clear that the fraction thus found is expressed in the smallest numbers, so that it cannot be expressed in smaller ones.

Let the example be this proposed fraction, $32/48$, whose numerator and denominator are measured by all these numbers — 2, 4, 8, 16 — and no others. For although the number 24, which is larger than those, measures the

denominator 48, it nevertheless does not measure the numerator 32. So too, although the number 32, which is larger still, measures the numerator 32, it in no way measures the denominator 48. Here we understand the largest measuring number to be the one that measures both numbers of the proposed fraction — that is, both the numerator and the denominator. If, then, both the numerator 32 and the denominator 48 are divided by the largest of those numbers, namely by 16, the quotients 2 and 3 will be found. Therefore the proposed fraction $32/48$ will be reduced to this equivalent one, $2/3$, expressed in the smallest numbers. If you were to divide those same numbers of the proposed fraction by some number that measures them but is not the largest, you would indeed reduce the fraction to another equal one expressed in smaller terms, but not the smallest. For example, if those same numbers 32 and 48 are divided by 8, this fraction $4/6$ is found, which can still be written in smaller numbers, in this way: $2/3$.

By the same method this fraction, $45/60$, whose numerator and denominator are measured by all these numbers — 3, 5, 15 — will be reduced to this one, $3/4$, provided that both the numerator and the denominator are divided by 15, which is the largest number measuring them; and so on for the rest.

[Margin: When a fraction cannot be reduced to lowest terms.]

[Margin: What a prime number, and numbers prime to one another, are.]

But if no number except unity measures the numerator and denominator of some fraction, that fraction cannot be reduced to smaller terms, but will already be expressed in the smallest. For example, these fractions — $20/39$, $20/63$, $47/59$ — cannot be reduced to smaller terms. For although these numbers 2, 4, 5, 10 measure the numerator of the first fraction, none of them measures its denominator; and although these numbers 3, 13 measure the denominator of the same fraction, neither of them measures its numerator. Again, although these numbers 2, 4, 5, 10 measure the numerator of the second fraction, and these — 3, 7, 9, 21 — measure its denominator, none of them measures both numbers, that is, both the numerator and the denominator of that fraction. As for the numbers of the last fraction, no number except unity measures them, since they are (to speak with the arithmeticians) prime — just as the numbers of the first two fractions are prime to one another, though none of them is [itself] a prime. For a prime number is said to be one that only unity measures; whereas numbers are called prime to one another when only unity, as a common measure, measures them.

[Margin: By what method the greatest common measure of the numerator and denominator of any fraction is found.]

Now since, in order to reduce a proposed fraction to its lowest terms, it is necessary to find the greatest common measure of the numerator and denominator (for by this greatest common measure both numbers, numerator and denominator, must be divided, as we have said), the following rule is usually given for finding it. Divide the denominator by the numerator; and if anything is left over in the division, divide the divisor — that is, the numerator — by that remainder of the division; and if again something remains, divide this last divisor — that is, that remainder — by the remainder of this last division; and so always divide the last divisor by the last remainder, until a divisor is reached that leaves nothing in the division. For this divisor, leaving nothing, will be the greatest common measure of the numerator and denominator of the proposed fraction. But if some divisor in divisions of this kind leaves unity, the numerator and denominator of the proposed fraction will have no common measure except unity, but will be numbers prime to one another.

[Margin: When the numerator and denominator of a given fraction have no common measure except unity.]

For example, suppose the proposed fraction is $36/72$; we will find the greatest common measure of the numerator and denominator in this way. Divide the denominator 72 by the numerator 36; and since, once this division is done, nothing remains, this divisor 36 will be the greatest common measure. Dividing the numerator and the denominator of the given fraction $36/72$ by it, we will reduce it to $1/2$, expressed in the lowest terms.

Again, if the given fraction is $60/96$, we will find the greatest common measure of the numerator and denominator by this method. Dividing the denominator 96 by the numerator 60, there remain 36 in the division. Dividing the divisor 60 again by the remainder 36, there remain 24 in the division. Likewise, dividing this last divisor 36 by the last remainder 24, there remain 12. Finally, dividing this last divisor 24 by the last remainder 12, nothing is left. Therefore the greatest common measure is 12; and if both the numerator and the denominator of the given fraction $60/96$ are divided by it, this fraction $5/8$ is produced, expressed in the smallest numbers.

But if this fraction, $48/103$, is proposed, no common measure of the numerator and denominator greater than unity will be found. For, dividing the denominator 103 by the numerator 48, there remain 7. Dividing likewise the divisor 48 by the

remainder 7, there remain 6. Finally, dividing this last divisor 7 by the last remainder 6, there remains 1. Therefore, as was said above, the numerator and denominator of this fraction $48/103$ are numbers prime to one another.

[Margin: The finding of the greatest measure of any two numbers whatever.]

By the same method we will find the greatest common measure of any two numbers whatever (even if they do not form a fraction, but are proposed on their own): namely, by dividing the larger by the smaller, and this divisor by the remainder of the division (if there is one), and again this last divisor by the remainder of the last division,

and so on, and so forth. For the last divisor that leaves nothing in the division will be the greatest common measure of the given numbers; but if in some division a unity is left, the given numbers will be prime to one another, and will have no common measure except unity.

[Margin: Whence the rule for finding the greatest measure of two numbers is derived.]

[Margin: Another method of reducing fractions to lowest terms.]

This rule for finding the greatest common measure of two numbers is gathered from Proposition 2 of Book 7 of Euclid [*Elements* VII.2]. For although Euclid directs one always to subtract the smaller number from the larger, the same result is nevertheless obtained — and indeed much more briefly — by dividing the larger number by the smaller, since division is a kind of compendious subtraction, just as multiplication is a kind of compendious addition.

By another method the proposed fraction will be reduced to its lowest terms: if both the numerator and the denominator are divided by some known common measure of theirs — even if it is not the greatest — so as to find the fraction in smaller numbers; and again, if both the numerator and the denominator of this fraction thus found are divided by another common measure of theirs, and so on, until a fraction is found whose numerator and denominator are numbers prime to one another. For example, take this proposed fraction $36/48$: if both its numbers are divided by 2, this fraction $18/24$ is found; if both numbers of this are divided by 3, this fraction $6/8$ is obtained; and if the numbers of this are finally divided by 2, they yield this fraction $3/4$, in the lowest terms. But that former method is more excellent and shorter.

REDUCTION OF FRACTIONAL NUMBERS TO THE SAME DENOMINATION, AND TO WHOLES, AND ALSO of wholes to any fraction whatever, and finally of fractions of fractional numbers to simple fractions

Chapter X

Very often, fractions of different denominators must be reduced to others that are equal to them, each to each, and that have the same denominator. We shall explain in this chapter how this ought to be done — first, when the proposed fractions are no more than two; then, when they are more.

[Margin: How two fractions are reduced to the same denomination.]

Given, then, two fractions having different denominators: if the denominators are multiplied by one another, a common denominator is produced, to which the given fractions are to be converted; and the numerator of each, multiplied crosswise [*in cruce*m] by the denominator of the other, will produce the [new] numerator. As in this adjoining example:

$\frac{2}{3} \propto \frac{3}{4}$ are reduced to $\frac{8}{12}$, $\frac{9}{12}$.

From the denominator 3 into the denominator 4 comes the common denominator 12. Likewise, from the 2 (the numerator of the first fraction) into the 4 (the denominator of the second) comes the numerator 8. And from the 3 (the numerator of the second fraction) into the 3 (the denominator of the first) comes the numerator 9. Therefore the two fractions $\frac{2}{3}$, $\frac{3}{4}$ are reduced to these two, $\frac{8}{12}$, $\frac{9}{12}$, which are equal to them and have one and the same common denominator, 12. That $\frac{8}{12}$ is equivalent to $\frac{2}{3}$ is established from Propositions 17 and 18 of Book 7 of Euclid [*Elements* VII.17–18], because each number of the latter, multiplied by the same number 4 — namely the denominator of the second proposed fraction — has produced both numbers of the former. For from this it follows that the numerator and denominator of the fraction $\frac{8}{12}$ have the same ratio as the numerator and denominator of the fraction $\frac{2}{3}$. Therefore the fractions themselves will be equal, as we said above. By the same reasoning the fractions $\frac{9}{12}$ and $\frac{3}{4}$ will be equal, because each number of the latter ($\frac{3}{4}$), multiplied by the same number 3 — or rather multiplying the same number 3,

namely the denominator of the first given fraction, $\frac{2}{3}$ — has produced both numbers of that fraction, $\frac{9}{12}$.

[Margin: The finding of a number measured by any given numbers whatever.]

But if more than two fractions are proposed to be reduced to the same denomination, one must first seek a number measured by all the denominators of the given fractions, so that it may contain all the parts denominated by them. We will find a number measured by the proposed denominators, or by any given numbers whatever, in this way. Multiply all the denominators by one another — leading the first into the second, this product into the third, this product into the fourth, and so on, until all have been multiplied. The last number produced will be the one sought. For example, given these fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{1}{5}$: if the first denominator 2 is multiplied by the second, 3, and the product 6 is led into the third, 4, and this product 24 is multiplied by the fourth, 5, the number 120 will be produced, which the given denominators — namely the numbers 2, 3, 4, 5 — measure.

[Margin: The finding of the least number measured by any given numbers whatever.]

But since a number found in this way is sometimes so large that a smaller one could be given which is measured by the same proposed denominators, we will find the smallest number measured by any given numbers in the following way. First we will find the smallest number measured by the first two proposed numbers, by this method. Either the two former numbers have some common measure besides unity, or they do not. (This you will discover if the larger is divided by the smaller, and this divisor by the remainder of the division, and so on, by a certain alternating division. For if a divisor is reached that leaves nothing, those two numbers will have a common measure, and that last divisor will be their greatest measure; but if a divisor is reached that leaves unity, they will lack a common measure and will be prime to one another, as we taught above in chapter 9.) If those two former numbers have no common measure, the number produced by multiplying one by the other will be the smallest measured by them, such that no smaller can be given. But if they do have a common measure, then — the greatest common measure having been found by the method we described in chapter 9 — let each be divided by it, and let the quotients be set beneath the numbers themselves. For if you multiply the quotient of the first number by the second number, or the quotient of the second number by the first, you will produce the smallest number measured by those

two. Then, in the same way, we will investigate the smallest number measured by the number we have so far found

[continuation of Chapter X — reduction of fractions to a common denominator]

we find it. Then, taking the third given number, we ask whether that third given number and the least number just found from the two prior ones have a common measure or not, and so on. For this least number, once found, will be the least number measured by the first three given numbers. Next we compare this newly found number with the fourth given number, and by the same method we investigate the least number measured by both. This number, once found, will be the least measured by all four given numbers; and I proceed in this way until no number remains to be compared with the one found last. The demonstration of this rule follows from Propositions 36 and 38 of Book 7 of Euclid [Euclid, *Elements* VII.36 & 38].

Let us now work this business through with the following four fractions, whose denominators are 2, 3, 4, 5. First, since the two prior numbers 2 and 3 have no common measure other than unity, the number 6 produced by multiplying them together will be the least number measured by 2 and 3. Next, since this newly found number 6 and the third given number 4 have 2 as their greatest common measure, we divide both 6 and 4 by it and set the quotients 3 and 2 beneath them, as you see here:

6	4
3	2

For if we multiply 6 by 2, or 4 by 3, we produce the number 12, which is the least number measured by the first three given numbers 2, 3, 4. Finally, since this number 12 that was found and the fourth given number 5 have no common measure other than unity, we multiply 12 by 5 and produce the number 60, the least number measured by all four denominators 2, 3, 4, 5.

Suppose again that we must find the least number measured by 4, 6, 8, 12, 7. First, since the first two, 4 and 6, have 2 as their greatest common measure, we divide both 4 and 6 by it and set the quotients 2 and 3 beneath them, as you see here:

4	6
2	3

For if we multiply 4 by 3 or 6 by 2, we get the number 12, the least measured by those two numbers 4 and 6. Next, since this number 12 that was found and the third given number 8 have 4 as their greatest common measure, we divide both 12 and 8 by it and place the quotients 3 and 2 beneath them:

12	8
3	2

For if we multiply 12 by 2 or 8 by 3, we generate the number 24, the least number measured by the first three given numbers 4, 6, 8. Again, since this number 24 that was found and the fourth given number 12 have 12 as their greatest common measure, we divide both 24 and 12 and put the quotients 2 and 1 beneath them:

24	12
2	1

For if we multiply 24 by 1 or 12 by 2, we produce the number 24, the least number measured by the four given numbers 4, 6, 8, 12. Finally, since this number 24 that was found and the last given number 7 have no common measure other than unity, we multiply them together and produce the number 168, the least number measured by the given numbers 4, 6, 8, 12, 7. But if someone were to seek, by the earlier rule, a number measured by these same given numbers 4, 6, 8, 12, 7 — namely by multiplying them all together — he would find the number 16128 [printed as "1.6.1.2.8"], which is far greater than this least number 168 that we have found.

[Margin: How more than two fractions are reduced to the same denomination.] Now, once a number measured by all the denominators of the fractions to be reduced has been found — whether it is the least such number or not — we reduce the given fractions to the same denomination in this way. The common denominator is that number found; if we divide it by the denominator of each fraction and multiply the quotient by the numerator, we produce a numerator, which is to be written above the common denominator. Thus in the last four fractions $1/2$, $2/3$, $3/4$, $4/5$, the number measured by the denominators is 120; this, then, will be the common denominator. If we divide it by 2, the denominator of the first fraction, we obtain 60; and if we multiply this number by 1, the numerator of that same fraction, we produce 60, the numerator for the first fraction. Again, if we divide the same number 120 by 3, the denominator of the

second fraction, there comes out the number 40, which, if we multiply it by 2, the numerator of that same fraction, gives 80, the numerator for the second fraction, and so on for the rest. So the four given fractions will be reduced to these four of the same denomination: $60/120$, $80/120$, $90/120$, $24/120$ [recte: $96/120$ — the fourth numerator should be $120 \div 5 = 24$ then multiplied by the numerator 4 = 96; see QA]. And if we take the number 60, which is the least number measured by the same denominators, as the denominator, we reduce the same fractions to these: $30/60$, $40/60$, $45/60$, $12/60$ [recte: $48/60$ — likewise $60 \div 5 = 12$ should be multiplied by 4 = 48; see QA].

[Margin: Another method of reducing two fractions to the same denomination.] By the same reasoning two fractions can also be brought to the same denomination, even without cross-multiplying them. For if a number is sought — whether the least or not the least — measured by the denominators, it will be the common denominator, from which the numerators are found, as we have just taught. Thus, given the two fractions $5/6$, $7/12$, the least number measured by the denominators is 12; if we divide this by 6, the denominator of the prior fraction, and multiply the quotient 2 by 5, the numerator of that same fraction [we obtain 10, the numerator of the prior fraction]. And if again we divide the same number 12 by 12, the denominator of the later fraction, and multiply the quotient 1 by 7, the numerator of that same fraction [we obtain 7, the numerator of the later fraction]. So the two given fractions will be reduced to these: $10/12$, $7/12$. But if anyone should wish to reduce the same fractions by the first rule, he will find these fractions: $60/72$, $42/72$.

[Margin: The usefulness of the least number measured by the denominators of the given fractions.] From all this it is clear how much difference there is between a fraction measured by the denominators of the given fractions and one that is not the least. Through the least, they are reduced to the smallest terms, which cannot be done by the other rules.

[Margin: How a fraction whose numerator is greater than its denominator is reduced to whole numbers.] It also happens sometimes that the numerator of a fraction — produced by addition, multiplication, or division — is greater than the denominator, so that the fraction exceeds the whole itself. In that case it must be reduced to whole numbers in this way: divide the numerator by the denominator. Whatever the quotient gives will be whole numbers, to which the given fraction is equivalent; and if anything is left over in the division, that will be the numerator, under which the same denominator is to be written. Thus this fraction $60/12$ [reduces to 5 whole units], if the numerator is divided by the

denominator; for in the division of the numerator by the denominator, nothing remains, and so it is reduced to 5 whole units. This fraction $100/7$ [reduces] to $14\frac{2}{7}$, because in the division of the numerator by the denominator 2 remains; and so that fraction contains 14 whole units, and moreover two-sevenths of one whole.

[Margin: How whole numbers are reduced to a fraction.] Likewise it is not rarely useful to reduce whole numbers to some fraction; this is done as follows. Multiply the given whole numbers by the denominator of the fraction to which they are to be reduced. The product will be the numerator, under which the denominator of the given fraction is to be written. Thus if 7 whole units are to be reduced to fifths...

REDUCTION

they are to be reduced; we multiply 7 whole units by 5, the denominator of the given fraction, and with the number 35 produced, we assign the same denominator, so that there results the fraction $35/5$, equivalent to 7 whole units. But if some fraction is attached to the whole numbers, the numerator of that fraction must be added to the number produced by multiplying the whole numbers by the denominator of the fraction, so that a numerator results. Thus if this number $8\frac{2}{5}$ is to be reduced to fifths, so that it becomes a single fraction, we multiply 8 by 5, the denominator of the attached fraction, and to the product 40 we add 2, the numerator of that same fraction, so that we have the numerator 42 of this fraction $42/5$, which is equivalent to the proposed number $8\frac{2}{5}$.

[Margin: How fractions-of-fractions are reduced to simple fractions.] Finally, when in some operation fractions-of-fractions occur, they must be reduced to a simple fraction by this device: multiply the numerators together — that is, the first into the second, and this product into the third, and again this product into the fourth, and so on, if there are more numerators. The last number produced will give the numerator of the simple fraction that is equal to that fraction-of-fractions. The denominator will be the number produced by multiplying the denominators together, if they are multiplied, just as was said of the numerators. Thus this fraction-of-a-fraction, $\frac{3}{5}$ of $\frac{4}{7}$, is reduced to this simple fraction $\frac{12}{35}$, because the multiplication of the numerators gives 12 and of the denominators 35 — so that three-fifths of four-sevenths of one whole contain $\frac{12}{35}$ of that same whole. So too this fraction-of-fractions, $\frac{2}{3} \cdot \frac{3}{4} \cdot \frac{1}{6} \cdot \frac{1}{2}$, is reduced to this simple fraction $\frac{6}{144}$, which when reduced to the smallest numbers gives $\frac{1}{24}$, as is established from the preceding chapter. And finally this fraction-of-fractions, $\frac{2}{4} \cdot$

$\frac{2}{3} \cdot \frac{2}{5}$ [?], will be brought back to this simple fraction $\frac{18}{63}$ [?], which reduced to the smallest numbers gives $\frac{3}{10}$ [?].

That this is so, we shall explain in the following way. Let us take this last fraction-of-fractions, $\frac{3}{4} \cdot \frac{2}{3} \cdot \frac{3}{5}$, assumed to be of one gold coin [aureus]. Necessarily, then, if the prescribed rule is true, it must contain three Julios [Iulii], which is so. This anyone will easily perceive. For $\frac{3}{5}$ of one gold coin contain 6 Julios, since two Julios are $\frac{2}{5}$ of one gold coin; and $\frac{3}{4}$ of four Julios are 3 Julios. By the same reasoning we shall show this fraction to have been brought back to $\frac{3}{10}$ [?], in this number 45. For $\frac{1}{3}$ of this number 45 contains 15 units; if $\frac{2}{3}$ of these are taken, 6 units are obtained [wait — recte per text], from which finally, if $\frac{1}{3}$ be taken, 2 units are taken, which make $\frac{3}{10}$ [?] of the said number 45. Other examples can be explained and proved in the same way.

THE ADDITION OF FRACTIONAL NUMBERS

CHAPTER XI

[Margin: How the addition of fractions is done.] If the fractions to be added have the same denominator, the numerators are to be added, and the same denominator assigned to their sum. But if they have different denominators, they must first be reduced to the same denominator, and then the addition carried out in the same way. Thus the sum collected from these three fractions $\frac{2}{3}$, $\frac{4}{3}$, $\frac{6}{3}$ is this: $\frac{12}{3}$, because they have the same denominator, and the sum collected from the numerators is 12 — just as from 2 gold coins, 4 gold coins, and 6 gold coins there come 12 gold coins. So too from these fractions $\frac{3}{10}$, $\frac{7}{10}$ is collected this sum $\frac{10}{10}$, which is equivalent to one whole. Likewise from these fractions $\frac{4}{7}$, $\frac{2}{7}$, $\frac{5}{7}$, $\frac{6}{7}$ is collected this sum $\frac{17}{7}$, which reduced to whole numbers makes 2 $\frac{3}{7}$.

But in order that these fractions $\frac{2}{3}$, $\frac{3}{4}$ may be collected into one sum, they must first be reduced to the same denominator, namely to these fractions $\frac{8}{12}$, $\frac{9}{12}$; and when these are collected into one sum they make $\frac{17}{12}$, that is, 1 $\frac{5}{12}$. And this is the sum of the two proposed fractions — just as from 2 gold coins and 3 Julios, if the 2 gold coins are reduced to 20 Julios, there come 23 Julios. So too these fractions $\frac{6}{7}$, $\frac{12}{13}$, $\frac{10}{11}$, $\frac{4}{5}$, in order to be collected into one sum, must first be reduced to these of the same denomination: $\frac{4290}{5005}$, $\frac{4620}{5005}$, $\frac{4550}{5005}$, $\frac{4004}{5005}$, from which arises this sum $\frac{17464}{5005}$, that is, 3 $\frac{2440}{5005}$.

[Margin: What to do when whole numbers are present. The method of adding fractions of different denominations to one another.] If whole numbers are

present, they are to be added separately, and the fractions separately too. Thus from 8 and $1/4$ [?] come $8\frac{1}{4}$ [?]. Likewise from 8 and $4\frac{2}{3}$ [?] come $12\frac{2}{3}$ [?]. Likewise from $8\frac{5}{9}$ and $4\frac{5}{9}$ [?] come $12\frac{10}{9}$, that is, $13\frac{1}{9}$. Likewise from $8\frac{2}{3}$ and $4\frac{3}{4}$ come $12\frac{17}{12}$, that is, $13\frac{5}{12}$.

And so, in order that two fractions of different denominations may be collected into one sum, they are to be multiplied crosswise, and the products added, so that the numerator of the fraction to be produced results. Then the denominators are to be multiplied together, so that the denominator of that same fraction is obtained; for by this reasoning the two fractions are reduced to the same denomination, as is clear from the preceding chapter, and the numerators are added to one another. Thus if these two fractions $2/3$, $3/4$ are to be added, we multiply both 2, the numerator of the prior, by 4, the denominator of the later, and 3, the numerator of the later, by 3, the denominator of the prior, and we collect the products 8 and 9 into one sum, so that the numerator 17 results. Then we make the number produced by multiplying the denominators together, namely 12, the denominator. So the collected fraction will be $17/12$.

But if there are more than two fractions to be added, we add the first two together first, as we said; then we combine the collected fraction with the third fraction in the same way; and this product with the fourth, and so on. Thus if these fractions $2/3$, $3/4$, $4/5$, $5/7$ are to be added, we first collect from the first two this: $17/12$. Then from this and the third we produce in the same way $133/60$. Finally from this and the fourth we generate $1231/420$, that is, $2\frac{391}{420}$, the sum of them all.

[Margin: Proof of the addition of fractions.] The proof of an addition is made by subtraction. For if one of the added fractions is subtracted from the collected sum, the other will remain — provided no error was made in the addition. And if there are several fractions to be added, when one of them is subtracted from the sum, there will be left a fraction equal to all the others taken together. Thus, since these fractions $3/4$, $5/12$ added make $56/48$, that is, $1\frac{8}{48}$ — if the prior fraction, namely $3/4$, is subtracted from this sum (as we shall teach in the following chapter), there will remain this fraction $80/192$, which is equal to the other fraction $5/12$, as is evident if it is brought to its smallest terms, or if the numerators are multiplied crosswise into the denominators. For the same number will be produced both from 80 into 12 and from 5 into 92, namely the number 960. Therefore, as we said above in Chapter 7, the fractions $80/192$, $5/12$ are equal.

SUBTRACTION OF FRACTIONS

THE SUBTRACTION OF FRACTIONAL NUMBERS

CHAPTER XII

If two fractions, the smaller of which is to be subtracted from the larger, have the same denominator, the numerator of the fraction to be subtracted is to be taken from the numerator of the other, and the same denominator written under the remainder. But if they have different denominators, they must first be reduced to the same denominator, and then the subtraction carried out in the same way. Thus if this fraction $\frac{5}{17}$ is to be subtracted from this one $\frac{8}{17}$, we subtract the numerator 5 from the numerator 8, and to the remainder 3 we assign the same denominator 17, so that the residual fraction $\frac{3}{17}$ results — just as, if 5 gold coins are taken from 8 gold coins, 3 gold coins remain.

But if this fraction $\frac{2}{3}$ is to be taken from this one $\frac{8}{9}$, both must first be reduced to these of the same denomination: $\frac{18}{27}$, $\frac{24}{27}$. Then the numerator 18 is to be subtracted from the numerator 24, and to the remainder 6 the common denominator 27 assigned, so that the residual fraction $\frac{6}{27}$ results — just as, if 2 Julios are to be taken from 8 gold coins, the 8 gold coins must first be reduced to 80 Julios, so that 78 Julios remain.

[Margin: What to do when whole numbers are present.] If some fraction is to be taken from whole numbers, one unit of the whole numbers must be reduced to a fraction of the same denominator — that is, so that a fraction results whose numerator is equal to its denominator — from which the proposed fraction is to be subtracted. Thus if the fraction $\frac{5}{11}$ is to be taken from 10, we make out of one unit $\frac{11}{11}$; and if from this we take away $\frac{5}{11}$, there will remain $\frac{6}{11}$. The whole numbers will then lack that one unit which was reduced to a fraction, [leaving] 9 $\frac{6}{11}$.

If, however, whole numbers together with a fraction are to be subtracted from whole numbers, one unit of those whole numbers from which the subtraction is to be made must likewise be reduced to a fraction of the same denomination; then the whole numbers are to be taken from the remaining whole numbers, and the fraction from the fraction. Thus if this number $4\frac{2}{3}$ is to be subtracted from 10, we make out of one unit of this number the fraction $\frac{3}{3}$, from which, if $\frac{2}{3}$ be taken, $\frac{1}{3}$ remains; and if 4 be taken from the remaining 9, there will be 5 left over. So the whole residual number will be $5\frac{1}{3}$.

Finally, if whole numbers together with fractions are to be subtracted from whole numbers together with fractions, or fractions alone: if the fraction to be subtracted is smaller than the one from which the subtraction is made, or equal to it, the fraction is to be taken from the fraction and the whole numbers from the whole numbers. But if the fraction to be subtracted is larger than the one from which the subtraction is made, one unit of the whole numbers from which the subtraction ought to be made must be reduced to a fraction of the same denomination as the one attached to them. Thus if this number $6\frac{3}{4}$ is to be subtracted from this one $10\frac{1}{2}$ — since the fraction $\frac{3}{4}$ is larger than $\frac{1}{2}$ — we make out of one unit of these whole numbers, 10, this fraction $\frac{2}{2}$, which together with $\frac{1}{2}$ makes $\frac{3}{2}$; and if from it the fraction $\frac{3}{4}$ is taken away, there remains the fraction $\frac{6}{8}$. And when 6 is likewise taken from 9, 3 is left. The whole residual number, therefore, is $3\frac{6}{8}$.

[Margin: What to do when there are several fractions.] But if at some point one fraction is to be subtracted from several, or several from one, or several from several, care must be taken that first the several fractions — both those to be subtracted and those from which the subtraction is to be made — be collected each into one sum.

[Margin: The method of subtracting a fraction from a fraction.] And so, in order to subtract one fraction from another when their denominators are different, the numerators must be multiplied crosswise by the denominators, and one product subtracted from the other, and to the remainder assigned the number produced by multiplying the denominators together; for by this reasoning the two proposed fractions are reduced to the same denomination, and so on. Thus if the fraction $\frac{3}{4}$ is to be subtracted from the fraction $\frac{7}{9}$, we multiply 3, the numerator of the fraction to be subtracted, by 9, the denominator of the other, and the product 27 we take away from the number 28, produced by multiplying 7, the numerator of the fraction from which the subtraction is made, by 4, the denominator of the other; and to the remaining unit we assign the number 36, produced by multiplying the denominators together, so that the fraction left is $\frac{1}{36}$.

[Margin: Proof of the subtraction of fractions.] The proof of a subtraction is made by addition. For if the fraction that is left is added to the fraction that was subtracted, the fraction from which the subtraction was made will be composed — provided no error was made. Thus, since by subtracting this fraction $\frac{3}{4}$ from this one $\frac{7}{9}$ there is left this fraction $\frac{1}{36}$, as was clear in the immediately preceding example, if $\frac{1}{36}$ be added to $\frac{3}{4}$, there is composed this fraction $\frac{7}{9}$, from which the subtraction was in fact made. So too, since by subtracting this

fraction $\frac{2}{3}$ from this one $\frac{6}{8}$ the remaining fraction is this $\frac{2}{24}$, if it be added to $\frac{2}{3}$, there results the fraction $\frac{54}{72}$, which is equal to the fraction $\frac{6}{8}$ from which the subtraction was made — as is evident if both are brought to their smallest terms, for this fraction $\frac{3}{4}$ will be found. Or else, because the numerators multiplied crosswise into the denominators produce the same number, namely 432. [Cross-check: $54 \times 8 = 6 \times 72 = 432$.]

THE MULTIPLICATION OF FRACTIONAL NUMBERS

CHAPTER XIII

If the numerators are multiplied together, the numerator of the product of the multiplication is produced, and from the multiplication of the denominators the denominator of that same fraction is generated. Thus from the multiplication of $\frac{2}{3}$ by $\frac{3}{4}$ there results $\frac{6}{12}$, that is, $\frac{1}{2}$. For the numerators multiplied together make 6, and the denominators 12.

[Margin: How the multiplication of fractions is done.] When a fraction is to be multiplied by a whole number, a unit must be supplied to the whole number, so that out of it there is made, as it were, a certain fraction denominated by the whole unit. Then the rule just prescribed is to be observed. Thus if 8 is to be multiplied by $\frac{4}{5}$, we write 1 under 8, as you see in the adjacent example:

8/1

4/5

Therefore, if both the numerators and the denominators are multiplied together, there is produced this fraction $\frac{32}{5}$, which is equivalent to this number $6\frac{2}{5}$.

[Margin: What to do when whole numbers are present.] But when a fraction is attached to a whole number, the whole number must be reduced to that fraction, so that a single fraction is made out of it and the attached fraction. Thus if 8 is to be multiplied by $3\frac{2}{5}$ [?], we make out of $3\frac{2}{5}$ the fraction $\frac{17}{5}$ [?], and under the number 8 we supply 1, as you see done here:

8/1

$\frac{17}{5}$ [?]

If, then, both the numerators...

so that both the numerators are multiplied together and the denominators, there will be produced this fraction $\frac{184}{6}$, which is equivalent to the number $30\frac{4}{6}$.

Likewise, if the numbers to be multiplied are $4\frac{2}{3}$ by $\frac{1}{2}$, we reduce $4\frac{2}{3}$ to $\frac{14}{3}$, as you see here. From the multiplication there will then be produced this fraction $\frac{14}{6}$, that is, $2\frac{2}{6}$. In the same way, if the numbers to be multiplied are $4\frac{1}{2}$ by $3\frac{1}{5}$, we reduce the former number to $\frac{9}{2}$ and the latter to $\frac{16}{5}$, as you see in the example set alongside. Then, once both the numerators are multiplied together and the denominators, there will be produced this fraction $\frac{144}{10}$, that is, $14\frac{4}{10}$.

[Margin: Verification of the multiplication of fractions.]

The multiplication is tested by division. For if the fraction produced is divided by one of the multiplying fractions, the other multiplying fraction will necessarily appear in the quotient. For instance, if from multiplying $\frac{1}{2}$ by $\frac{4}{7}$ there results $\frac{4}{14}$, then dividing $\frac{4}{14}$ by $\frac{1}{2}$ must produce $\frac{4}{7}$, while dividing the same $\frac{4}{14}$ by $\frac{4}{7}$ must yield $\frac{1}{2}$. This will be clear from the following chapter: dividing $\frac{4}{14}$ by $\frac{1}{2}$ produces $\frac{8}{14}$, which is equivalent to this $\frac{4}{7}$; while dividing the same $\frac{4}{14}$ by $\frac{4}{7}$ produces $\frac{28}{56}$, that is, $\frac{1}{2}$.

[Margin: Why in the multiplication of fractions a fraction smaller than either multiplying fraction is produced.]

No one should be surprised that multiplying fractions always produces a fraction smaller than either multiplying fraction, as is evident in the last example, which we gave under the test: there, from multiplying $\frac{1}{2}$ by $\frac{4}{7}$, the fraction $\frac{4}{14}$ was produced, that is $\frac{2}{7}$, which is smaller than both multiplying fractions. If the nature of multiplication is rightly considered, anyone will readily see that this must necessarily be so. For a number is said to be multiplied, or "led," by another when it is increased as many times as unity is contained in the other (as we said in chapter 4). It is therefore clear that neither of the multiplying fractions can be taken whole in the product, but only fragments of it — fragments of unity, indeed, such as the other multiplying fraction represents, since this fraction is smaller than unity. Hence it comes about that, just as the multiplying fraction does not contain a whole unit, so too the product does not contain the whole of the other multiplying fraction. Thus, in the preceding example, just as $\frac{1}{2}$ is the half part of unity, so the number produced — $\frac{4}{14}$, that is $\frac{2}{7}$ — is the half part of this fraction $\frac{4}{7}$, as the definition of multiplication requires. Rightly, then, from multiplying $\frac{1}{2}$ by $\frac{4}{7}$ there is produced this fraction $\frac{4}{14}$, that is $\frac{2}{7}$. So too, from multiplying 9 by $\frac{1}{3}$ there is produced this fraction $\frac{9}{3}$, that is, the number 3. For just as $\frac{1}{3}$ is the third part of unity, so the number 3 is the third part of the number 9; or, just as the number produced, 3, contains $\frac{1}{3}$ nine times, so the number 9 contains nine units. It is therefore no wonder that a number smaller than both multiplying fractions is produced. If, however, whole numbers are

multiplied by a fraction, the number produced is indeed smaller than that whole number, yet larger than the multiplying fraction. This is evident in the preceding example, where from multiplying 9 by $\frac{1}{3}$ the number produced was 3. And so, if whole numbers are multiplied by a whole number combined with a fraction, or a whole number with a fraction by whole numbers, there will always be produced a number larger than both multiplying numbers, because of the whole number that multiplies the whole numbers. For example, from multiplying 4 by $3\frac{1}{4}$ comes the number $\frac{52}{4}$, that is, 13 — because the number 4 taken three times makes 12, and its fourth part is 1; or because the number 3 taken four times makes 12, and the fraction $\frac{1}{4}$ taken four times makes $\frac{4}{4}$, that is, 1.

DIVISION OF FRACTIONAL NUMBERS

Chapter XIV

[Margin: How the division of fractions is done.]

For the sake of ease, the rule of Division can be reduced to the rule of multiplication in this way. Let the terms of the divisor be interchanged — that is, let the numerator be written below the line, and the denominator above it. Once this is done, if the rule of multiplication delivered in the preceding chapter is followed — that is, if both the numerators and the denominators are multiplied together — the Quotient will be produced. For example, if the fraction to be divided is $\frac{1}{2}$ (by $\frac{1}{6}$), the example will stand as you see here [set up as $\frac{1}{2}$ and $\frac{6}{1}$]. Then, once both the numerators and the denominators are multiplied together, this fraction will be produced: $\frac{6}{2}$, that is, the number 3, as the Quotient. So too, if the fraction to be divided is $\frac{2}{3}$ by $\frac{3}{7}$, the example will stand as here appears [set up as $\frac{2}{3}$ and $\frac{7}{3}$]; and the Quotient will be $\frac{14}{9}$.

[Margin: When whole numbers are present, what is to be done.]

When a whole number is to be divided by a fraction, or a fraction by a whole number, or by a whole number with a fraction; or a fraction by a whole number, or by a whole number with a fraction; or, finally, a whole number with a fraction by a fraction, or by a whole number, or by a whole number with a fraction — then unity is to be supposed under the whole number, if no fraction is attached to it; but if a fraction is attached, the whole number must be reduced to the attached fraction, so that a single fraction results, as we said in the preceding chapter. Then the rule already prescribed must be kept. As you see here, the examples in the following divisions will stand, together with their Quotients:

Divide	by	Set up as (dividend · inverted divisor)	Quotient
6	$\frac{2}{3}$	$6/1 \cdot 3/2$	$18/2$, or 9
6	$4\frac{2}{3}$	$6/1 \cdot 3/14$	$18/14$, or $1\frac{2}{7}$
$\frac{2}{3}$	6	$2/3 \cdot 1/6$	$2/18$, or $1/9$
$\frac{2}{3}$	$6\frac{1}{2}$	$2/3 \cdot 2/13$	$4/39$
$6\frac{1}{2}$	$\frac{3}{4}$	$13/2 \cdot 4/3$	$52/6$, or $8\frac{2}{3}$
$6\frac{1}{2}$	$3\frac{2}{5}$	$13/2 \cdot 5/17$	$65/34$, or $1\frac{31}{34}$
$6\frac{1}{2}$	$3\frac{4}{5}$	$13/2 \cdot 5/19$	$65/38$, or $1\frac{27}{38}$

the whole number to the attached fraction, so that a single fraction results, as we said in the preceding chapter. Then the rule already prescribed is to be kept — as, in the following divisions, the examples will stand together with their Quotients, as you see here.

[Margin: By what reasoning others teach the division of fractions.]

Others hand down this rule for the division of fractions: let the numerator of the fraction to be divided (with unity placed under the whole numbers, if any are present, and the whole numbers reduced to the attached fraction, if any is attached) be multiplied by the denominator of the dividing fraction; for by this method the numerator of the Quotient fraction will be produced. The denominator, in turn, will be produced from multiplying the denominator of the fraction to be divided by the numerator of the dividing fraction. This is indeed the same as if the terms of the divisor were interchanged and the rule of multiplication kept, as is plain. But since someone might sometimes be in doubt whether it is the numerator of the fraction to be divided, or rather that of the dividing fraction, that produces the numerator of the Quotient fraction — for this matter can easily slip the mind — I prefer the earlier rule we delivered, by which the rule of division is referred back to the rule of multiplication.

[Margin: Verification of the division of fractions.]

The division, moreover, is tested by multiplication. For if the Quotient fraction is multiplied by the dividing fraction, the fraction that was divided will necessarily be produced. Thus, since from dividing $\frac{4}{5}$ by $\frac{1}{2}$ there is produced the fraction $\frac{8}{5}$ — that is, $1\frac{3}{5}$ — it follows that from multiplying $1\frac{3}{5}$ by $\frac{1}{2}$ the divided fraction $\frac{4}{5}$ is produced. And from this multiplication there is produced the fraction $\frac{8}{10}$, which is equal to this $\frac{4}{5}$, as is evident.

[Margin: Why sometimes in the division of fractions the Quotient is greater than the fraction divided.]

That in the division of fractions the Quotient is usually greater than the fraction divided — as is evident in the division of $6/7$ by $2/7$, in which the Quotient is $42/14$, that is, 3 — should surprise no one. For since the Quotient contains unity as many times as the divisor is contained in the number divided, it is clear that, when the dividing fraction is smaller than the fraction to be divided, the latter contains it more than once, and so the Quotient is greater than 1. As in the present example: since the dividing fraction $2/7$ is contained three times in the fraction to be divided $6/7$, the Quotient is 3, indicating that the former is contained three times in the latter. This also appears plainly from the definition of Division. For since division is the finding of a number that contains unity as many times as the number divided contains the divisor (as we said in chapter 5), it is clear that in the present division the Quotient must be 3, which contains unity three times — namely, as many times as the fraction to be divided $6/7$ contains $2/7$. It is therefore no wonder that a Quotient larger than the number divided is produced when the divisor is smaller than [the dividend]; as also appears in the division of 6 by $1/2$, where the quotient is 12. For the dividing fraction $1/2$ is contained twelve times in the number divided, 6.

[Margin: When the Quotient is greater than the dividend, in the case of fractions.]

More generally, however, we shall show that this holds whenever the divisor is smaller than unity, even though it is not smaller than the number divided. For since the Quotient contains unity as many times as the dividend contains the divisor, the proportion of the Quotient to unity will be as that of the number divided to the divisor; and, by permutation, the proportion of the Quotient to the number divided will be as that of unity to the divisor. Therefore, since unity is supposed greater than the divisor, the Quotient too will be greater than the number divided. Thus you see, in the division of $2/5$, that the Quotient $8/15$ is greater than the fraction divided $2/5$ — because the dividing fraction $3/4$ is smaller than unity, even though it is larger than the fraction divided.

[Margin: When the Quotient, in the case of fractions, is smaller than the number divided.]

When, however, the divisor is greater than 1, the Quotient will always be smaller than the number divided. Thus, dividing $8/9$ by $1\frac{1}{2}$, the Quotient is $16/27$ [?]; and dividing $6\frac{1}{2}$ by $1\frac{2}{3}$, the Quotient is $39/10$, that is, $3\frac{9}{10}$. Again, dividing $100\frac{1}{2}$ by $10\frac{3}{4}$, the Quotient is $804/86$ [?], that is, $9\frac{30}{86}$ [?], or $9\frac{15}{43}$. So too, dividing

$\frac{2}{3}$ by $3\frac{1}{2}$, the Quotient becomes $\frac{4}{21}$ [?]. Finally, dividing $3\frac{1}{5}$ by $1\frac{1}{2}$, the Quotient will be $\frac{32}{15}$, that is, $2\frac{2}{15}$. Here you see that the Quotient is always smaller than the number divided. The reason for this is as follows. Since division is the finding of a number that contains unity as many times as the number divided contains the divisor (as was said in chapter 5), the proportion of the Quotient to unity will be as that of the number divided to the divisor; and, by permutation, the proportion of the Quotient to the number divided will be as that of unity to the divisor. Therefore, since unity is supposed smaller than the divisor, the Quotient too will be smaller than the number divided.

THE GRAFTING OF FRACTIONAL NUMBERS

Chapter XV

[Margin: What the grafting [insitio] of fractions is.]

Some arithmeticians are accustomed to use a certain operation on fractions, which they call grafting [insitio]. This grafting is nothing other than the addition — given two or more fractions, each of which is a fraction either of a single particle of all the following, or a fraction of all the following whole fractions — of fractions of this kind, carried up to the last fraction, with respect to which all those fractions are taken as fractions of fractions; so that, in a certain way, the preceding fractions are grafted into the following ones. Hence this operation on fractions got its name — as will be evident from examples. Thus, given these two fractions $\frac{2}{3}$, $\frac{3}{4}$, such that the prior is a fraction of a single particle of the posterior — that is, such that the prior contains two thirds of one quarter part — or a fraction of the whole posterior — that is, such that the prior contains two thirds of three quarters — the operation by which $\frac{2}{3}$ of one quarter (or of three quarters) is added is called grafting. In the same way, given these four fractions $\frac{2}{3}$, $\frac{3}{4}$, $\frac{2}{5}$, $\frac{4}{7}$, such that each is a fraction of a single particle of all the following, or a fraction of all the following whole fractions — that is, either such that the first contains two thirds of one quarter of one fifth of one seventh, the second signifies three quarters of one fifth of one seventh, and the third comprises two fifths of one seventh; or such that the first contains two thirds of three quarters of two fifths of four sevenths, the second comprises three quarters of two fifths of four sevenths, and the third signifies two fifths of four sevenths — the operations by which all these fractions of fractions (namely $\frac{2}{3}$ of one quarter of one fifth of one seventh, and $\frac{3}{4}$ of one fifth of one seventh, and $\frac{2}{5}$ of one seventh) are added to $\frac{4}{7}$ are called grafting. And so with the rest.

[Margin: The grafting is twofold.]

There are, therefore, two kinds of grafting: one, when each fraction is a fraction of a single particle of all the following fractions; the other, when each fraction is a fraction of all the following whole fractions — as was shown in the examples. All arithmeticians have spoken only of the first grafting, making no mention whatever of the second grafting —

[Margin: For what purpose the grafting was devised.]

— perhaps for this reason, that the first is useful for dividing any whole number whatsoever, together with some fraction, by a whole number, as we shall say a little later. But since the second grafting also has an outstanding use in Geometrical progressions, we shall — God willing — set it out in our larger work on Arithmetic; and we shall deliver the rule of both graftings.

[Margin: The difference between grafting and the reduction of fractions-of-fractions.]

There is a great difference between grafting and that operation by which, in chapter 9, we taught how to reduce fractions-of-fractions to a simple fraction. For there, given (for example) these two fractions $\frac{2}{3}$, $\frac{3}{4}$, such that the prior is a fraction of the posterior, we asked only what simple fraction two thirds of three quarters would constitute; and we found that they constitute $\frac{6}{12}$, that is, $\frac{1}{2}$ of one whole. Here, by contrast, we shall investigate what fraction is effected if $\frac{2}{3}$ of one quarter — or $\frac{2}{3}$ of three quarters — be added to $\frac{3}{4}$. By the former way this fraction $\frac{11}{12}$ will be effected; by the latter, this: $\frac{5}{4}$, that is, $1\frac{1}{4}$ — both of which are far distant from $\frac{1}{2}$. In the same way the difference will appear, if there be more fractions than two.

[Margin: First rule of grafting, for two fractions.]

Therefore, if two fractions are proposed, of which the prior is a fraction of a single particle of the posterior, the grafting is done thus. Let the numerator of the posterior fraction be multiplied by the denominator of the prior, and to the product let the numerator of the same prior fraction be added. For this aggregate will be the numerator of the fraction to be produced; the denominator, in turn, will be generated from multiplying the denominators together. Thus, given these fractions $\frac{2}{3}$, $\frac{3}{4}$, the grafting — or addition — will be done in this way: 3, the numerator of the posterior fraction, times 3, the denominator of the prior, makes 9; adding 2, the numerator of the same prior fraction, makes 11, for the numerator of the fraction to be produced. The denominator, in turn, will be the

number 12, produced from multiplying the denominators together — so that this fraction $11/12$ is composed from the addition of $2/3$ (of one quarter to $3/4$). This can easily be proved from the rule of addition. For since $2/3$ of one quarter, by the reduction of fractions-of-fractions, make $2/12$: if $2/12$ be added to $3/4$, they become $44/48$, that is, $11/12$, as before.

[Margin: How more than two fractions are grafted by the first rule.]

But if more fractions than two are given, each being a fraction of a single particle of all the following, the grafting will be done in this way. Let the numerator of the last fraction be multiplied by the denominator of the penultimate, and to the product let the numerator of the same penultimate fraction be added. Then let this aggregate be multiplied by the denominator of the antepenultimate fraction, and to the product let the numerator of the same fraction be added. After this, let this aggregate be multiplied by the denominator of the next preceding fraction, and to the product let the numerator of the same be added; and so on, if there are more fractions, the last aggregate always being multiplied by the denominator of the preceding fraction and the numerator of the same added, until no fraction remains. For the last aggregate will be the numerator of the fraction to be produced. The denominator, in turn, will be produced from multiplying the denominators together. Thus, given these fractions $2/3$, $3/4$, $2/5$, $4/7$, the grafting will be done thus: 4, the numerator of the last fraction, into 5, the denominator of the penultimate fraction, makes 20. Adding 2, the numerator of the penultimate fraction, makes 22; which, multiplied by 4, the denominator of the antepenultimate fraction, makes 88. Adding 3, the numerator of the same antepenultimate fraction, makes 91; which, multiplied by 3, the denominator of the preceding fraction, makes 273. Adding 2, the numerator of the same first fraction, which immediately precedes, makes 275, for the numerator of the fraction to be produced. The denominator, in turn, will be the number 420, produced from multiplying the denominators together — provided, that is, that the first is multiplied by the second, and this number by the third, and so on. And so, from this grafting, there will arise this fraction $275/420$, which, reduced to lowest terms, will make $55/84$.

[Margin: The fractions to be grafted by the first rule are not to be reduced to lowest terms before the end of the operation.]

This can be proved from the rule of addition, by the following reasoning. Since $2/3$, $1/4$, $1/5$, $1/7$, by the rule of reducing fractions-of-fractions, make $2/420$; and $3/4$, $1/5$, $1/7$ make $3/140$; and $2/5$, $1/7$ make $2/35$: if those three fractions $2/420$, $3/140$, $2/35$ be added to $4/7$, they become $9432500/14406000$ [?], that is, in lowest

terms, $55/84$ — as before, but this sum is found by grafting much more easily and quickly.

For the rest, in this rule of grafting no fraction is to be reduced to lowest terms before the whole operation is completed, because the sense would be altered and a great error would arise; yet, once the operation is completed, the sum produced can be reduced to lowest terms, as we have done. For we reduced this fraction $275/420$, produced by grafting, to that $55/84$. That the sense would be altered, and error would occur, if some fraction were reduced to lowest terms before the end of the operation, is plain. For if these fractions $2/3$, $8/12$ were to be grafted — that is, if $2/3$ of one twelfth were to be added to $8/12$ — they would become $26/36$. But if the posterior fraction $8/12$ were called back to lowest terms, namely to this fraction $2/3$, then $2/3$, $2/3$ would have to be grafted — that is, $2/3$ of one third would be added to $2/3$ — a sense far different from the former. And so from this grafting a different fraction would be produced, namely $8/9$, very different from the fraction $26/36$ produced before. The former fraction produced, $26/36$, can nonetheless be reduced to this in lowest terms: $13/18$.

[Margin: The sum of the grafting by the first rule is always smaller than unity, and why.]

Nor should it be passed over that the sum collected from the grafting hitherto explained is always smaller than unity, if the last fraction is smaller than unity — even though infinitely many fractions be grafted. For example, if these fractions $3/4$, $2/3$, $1/2$, $4/5$ are grafted, the fraction $119/120$ [?] results, which is smaller than unity. That this must be so can be shown by the following reasoning. For $4/5$ to make unity, $1/5$ is lacking; and the preceding fraction $1/2$, which is added to $4/5$, is not $1/5$ but $1/2$ of one fifth, so that to complete unity there is still lacking $1/2$ of one fifth. But since the antecedent fraction $2/3$, which is added, is not $1/2$ of one fifth but $2/3$ of one half of one fifth, it comes about that to fill up unity there is still lacking $1/3$ of one half of one fifth. And since the preceding fraction $3/4$ is not $1/3$ of one half of one fifth, but $3/4$ of one third of one half of one fifth, it comes about that to complete unity there is still lacking $1/4$ of one third of one half of one fifth. And so on in turn: if there are more fractions, something will always be lacking to complete unity.

[Margin: The use of the first rule of grafting in dividing a whole number together with a fraction by a whole number.]

That you may see how excellent a use this first rule of grafting has in dividing a whole number, together with a fraction, by a whole number, I shall bring

forward one or two examples. Let $20 \frac{1}{4}$ be divided by 12. Dividing the whole number 20 by 12 gives the Quotient $1 \frac{8}{12}$. And since the fraction $\frac{1}{4}$ must also be divided by 12, and its Quotient added to the previous Quotient — the Quotient being (if $\frac{1}{4}$ is divided by 12) $\frac{1}{4}$ of one twelfth, just as, if 1 were divided by 12, the Quotient is $\frac{1}{12}$ — it comes about that if these fractions $\frac{1}{4}$, $\frac{8}{12}$ are grafted (that is, if $\frac{1}{4}$ of one twelfth is added to $\frac{8}{12}$, namely the Quotient of dividing $\frac{1}{4}$ by 12), there is composed a fraction which, added to the whole Quotient 1, makes the whole Quotient. And from the grafting of these fractions $\frac{1}{4}$, $\frac{8}{12}$ there results the fraction $\frac{33}{48}$, that is, $\frac{11}{16}$. Therefore the whole Quotient will be $1 \frac{11}{16}$.

You will achieve the same if you suppose the divisor 12 to be divided by the whole number to be divided, 20, so that the fraction $\frac{20}{12}$ results, and into this fraction you graft the fraction $\frac{1}{4}$ that is to be divided, in this way: $\frac{1}{4}$, $\frac{20}{12}$ — because the fraction $\frac{20}{12}$ is the Quotient of dividing 20 by 12. To which...

(continued) ...to which, by grafting, is added $\frac{1}{4}$ of one twelfth, namely the quotient of dividing $\frac{1}{4}$ by 12. By either method you will easily verify that the division comes out correctly using the rule of division. For if you divide $20\frac{1}{4}$ by 12, you will find the quotient $1\frac{11}{16}$, that is, $1 \frac{33}{48}$, or $1 \frac{11}{16}$, as before.

Let it be required, once more, to divide $100 \frac{5}{8}$ by 8. Dividing the whole number 100 by 8 gives a quotient of $12 \frac{4}{8}$. And since the fraction $\frac{5}{8}$ must also be divided by 8, and that quotient added to the previous one — now, if $\frac{5}{8}$ is divided by 8, the quotient is $\frac{5}{8}$ of one eighth, just as if 1 were divided by 8 the quotient would be $\frac{1}{8}$ — it comes about that if these fractions $\frac{5}{8}$ and $\frac{4}{8}$ are grafted together (that is, if $\frac{5}{8}$ of one eighth, namely the quotient of dividing $\frac{5}{8}$ by 8, is added to $\frac{4}{8}$), a fraction is produced which, added to the whole-number quotient 12, makes up the complete quotient. From the grafting of these fractions $\frac{5}{8}$ and $\frac{4}{8}$ there arises the fraction $12 \frac{37}{64}$ [?]. The whole quotient will therefore be $12 \frac{37}{64}$ [?].

You will achieve the same result if you suppose the divisor 8 to be dividing the whole number 100, so that the fraction $\frac{100}{8}$ is formed, and then graft into this fraction the fraction $\frac{5}{8}$ that is to be divided, in this way: $\frac{5}{8}$ $\frac{100}{8}$ — because the $\frac{100}{8}$ is 100 divided by 8, to which by grafting is added $\frac{5}{8}$ of one eighth, namely the quotient of dividing $\frac{5}{8}$ by 8. You will find the same quotient $12 \frac{37}{64}$ [?] if you divide $100 \frac{5}{8}$ by 8 by the rule of division; for you will get the quotient $12 \frac{805}{64}$ [?], that is, $12 \frac{37}{64}$ [?].

Finally, let $100 \frac{5}{8}$ be divided by 10. Dividing the whole number 100 by 10 gives a quotient of 10, and nothing remains. And since the fraction $\frac{5}{8}$ must also be divided by 10, and that quotient added to the previous one — now (dividing $\frac{5}{8}$ by

10) the quotient is $\frac{5}{8}$ of one tenth, just as if 1 were divided by 10 — it comes about that if these fractions $\frac{5}{8}$ and $\frac{1}{10}$ are grafted together (that is, if $\frac{5}{8}$ of one tenth, namely the quotient of dividing $\frac{5}{8}$ by 10, is added to $\frac{1}{10}$ — for since no fraction was left over in dividing 100 by 10, the figure 0 must be placed above the divisor 10, so that the fraction $\frac{1}{10}$ is formed, containing no tenth), a fraction is produced which, added to the whole-number quotient 10, makes up the complete quotient. From the grafting of these fractions $\frac{5}{8}$ and $\frac{1}{10}$ there arises the fraction $\frac{5}{80}$. The whole quotient will therefore be $10 \frac{5}{80}$, that is, $10 \frac{1}{16}$.

You will achieve the same if you suppose the divisor 10 to be dividing the whole number 100, so that the fraction $\frac{100}{10}$ is formed, and graft into this fraction the fraction $\frac{5}{8}$ that is to be divided, in this way: $\frac{5}{8} \frac{100}{10}$ — because the $\frac{100}{10}$ is the quotient of dividing 100 by 10, to which by grafting is added $\frac{5}{8}$ of one tenth, namely the quotient of dividing $\frac{5}{8}$ by 10. You will have exactly the same quotient if you divide $100 \frac{5}{8}$ by 10 by the rule of division; for the quotient will come out $80 \frac{5}{80}$ [?], that is, $10 \frac{5}{80}$ [?], or $10 \frac{1}{16}$ [?].

[Margin: The second rule of grafting, for two fractions.]

Now if two fractions are proposed, of which the first is a fraction of the whole of the second, the grafting is done in this way. Multiply the numerator of the second fraction by the denominator of the first, and to the resulting number add the number produced by multiplying the numerators together. This gives the numerator of the fraction to be produced. The denominator, in turn, is generated by multiplying the denominators together. For example, given these fractions $\frac{2}{3}$ and $\frac{3}{4}$, the grafting — or the addition of $\frac{2}{3}$ of three-fourths to $\frac{3}{4}$ — is done as follows. From the 3, the numerator of the second fraction, multiplied into the 3, the denominator of the first, arise 9. Adding the number 6, produced by multiplying the numerators together, gives 15 for the numerator of the fraction to be produced. The denominator will be the number 12, produced by multiplying the denominators together. Thus from adding $\frac{2}{3}$ of three-fourths to $\frac{3}{4}$ there is formed the fraction $\frac{15}{12}$, that is, $1\frac{1}{4}$.

This can easily be proved by the rule of addition. For since $\frac{2}{3}$ of three-fourths make $\frac{6}{12}$ (as is clear from the reduction of fractions-of-fractions which we set out earlier), if $\frac{6}{12}$ is added to $\frac{3}{4}$ the result is $\frac{6}{48}$, that is, $1\frac{1}{4}$, as before.

[Margin: How more than two fractions are grafted by the second rule.]

But if more than two fractions are proposed, in such a way that each is a fraction of the whole of all the fractions that follow it, the grafting must be done in this way. Multiply the numerator of the last fraction by the denominator of the next-

to-last, and to the resulting number add the number produced by multiplying together the two last numerators. Then multiply this sum by the denominator of the third-from-last fraction, and to the result add the number produced by multiplying together the three last numerators. Again, multiply this sum by the denominator of the fraction immediately preceding, and to the result add the number produced by multiplying together the four last numerators; and so on, if there are more fractions — the latest sum is always multiplied by the denominator of the preceding fraction, and to the result is added the number produced from all the numerators of those fractions taken up to that place, until no fraction remains. For the last sum will be the numerator of the fraction to be produced; while the denominator is generated by multiplying the denominators together.

For example, given these fractions $\frac{2}{3}$, $\frac{3}{4}$, $\frac{2}{5}$, $\frac{4}{7}$, the grafting — or the addition of $\frac{2}{3}$ of three-fourths of two-fifths of four-sevenths, and $\frac{2}{5}$ of two-fifths of four-sevenths, and $\frac{4}{7}$ of four-sevenths, to $\frac{4}{7}$ — is done as follows. From the 4, the numerator of the last fraction, multiplied into the 5, the denominator of the next-to-last, arise 20. Adding the number 8, produced from the two last numerators 4 and 2 multiplied together, gives 28; and this, multiplied by 4, the denominator of the third-from-last fraction, makes 112. Adding the number 24, produced from the three last numerators 4, 2, and 3 multiplied together, gives 136; and this, multiplied by 3, the denominator of the preceding fraction (which is the first), makes 408. Adding the number 48, produced from all four numerators 4, 2, 3, and 2 multiplied together, gives 456 for the numerator of the fraction to be produced. The denominator, in turn, will be the number 420, produced from all the denominators multiplied together. Thus from this grafting there is generated the fraction $\frac{456}{420}$, that is, $1\frac{36}{420}$, or, in lowest terms, $1\frac{3}{35}$.

This will be confirmed by the rule of addition in this way. Since — by the rule by which fractions-of-fractions are reduced — $\frac{2}{3}$, $\frac{3}{4}$, $\frac{2}{5}$, $\frac{4}{7}$ make $\frac{48}{420}$, and $\frac{3}{4}$, $\frac{2}{5}$, $\frac{4}{7}$ make $\frac{24}{140}$, and $\frac{2}{5}$, $\frac{4}{7}$ make $\frac{8}{35}$, if these three fractions $\frac{48}{420}$, $\frac{24}{140}$, $\frac{8}{35}$ are added to $\frac{4}{7}$, there results the fraction [numeral: ? — the raw sum, a ~7-digit numerator over a ~7-digit denominator], that is, 1 [numeral: ?], or $1\frac{3}{35}$ in lowest terms, as before. But we have collected the same sum much more easily and expeditiously by grafting.

Furthermore, in this second rule of grafting the fractions to be grafted can be reduced to lowest terms before the operation. For if these fractions $\frac{2}{3}$ and $\frac{4}{8}$ are grafted (that is, if $\frac{2}{3}$ of four-eighths is added to $\frac{4}{8}$), the result is $\frac{16}{24}$ [?], that is, $\frac{2}{3}$. We shall do just the same if we first reduce $\frac{4}{8}$ to $\frac{1}{2}$ and then graft $\frac{2}{3}$ and $\frac{1}{2}$ (that

is, add $\frac{2}{3}$ of one-half to $\frac{1}{2}$). In the same way, if $\frac{6}{10}$ and $\frac{4}{8}$ are grafted, the result is $\frac{64}{80}$, that is, $\frac{4}{5}$; and the same fraction is produced if $\frac{6}{10}$ is first reduced to $\frac{3}{5}$ and $\frac{4}{8}$ to $\frac{1}{2}$ and then $\frac{3}{5}$ and $\frac{1}{2}$ are grafted (that is, $\frac{3}{5}$ is added to $\frac{1}{2}$). For from this grafting there will be produced $\frac{8}{10}$, that is, $\frac{4}{5}$, as before.

The reason for this is that, since the preceding fraction is a fraction of the whole of the following one, the value of $\frac{2}{3} \frac{4}{8}$ and of $\frac{2}{3} \frac{1}{2}$ will be exactly the same. For if these fractions-of-fractions are reduced to simple fractions, the first is reduced to $\frac{8}{24}$, that is, to $\frac{1}{3}$, and the second to $\frac{4}{8}$, that is, $\frac{1}{2}$... the first is reduced to $\frac{8}{24}$, that is, to $\frac{1}{3}$, and to $\frac{1}{2}$ as well. This does not happen in the earlier rule. For since there the first fraction is a fraction of only one small part of the second, it is plain that in the same example $\frac{2}{3} \frac{1}{8}$ and $\frac{2}{3} \frac{1}{2}$ are different things. For the first fraction-of-fractions makes $\frac{2}{24}$, that is, $\frac{1}{12}$; the second [makes] $\frac{2}{3}$, that is, $\frac{1}{3}$.

Some Little Problems concerning Whole Numbers and Fractions. Chapter XVI.

I think it will be worth my while, before I go on to other matters, to append here various little problems about whole numbers and fractions — problems solved by addition, subtraction, multiplication, and division — both because students can practice on them in the operations with whole numbers and fractions, and also because problems of this kind very often have an excellent use in other matters of arithmetic. Let us therefore begin here.

[Margin: 1. Finding the number from which a subtraction has been, or is to be, made, so that the proposed number is left.]

1. From what number have $\frac{4}{11}$ been subtracted, or ought they to be subtracted, so that $8\frac{2}{3}$ remain? Likewise, from what number have 23 been subtracted, or ought they to be subtracted, so that 47 remain? Problems of this kind are solved by addition. For if you add the number subtracted (or to be subtracted) to the number that ought to be left, you will produce the number from which the given number, when subtracted, leaves the given number. Thus in the first problem, from 23 and 47 comes the number 70; from this, then, 23 must be subtracted so that 47 are left. In the second problem, from $\frac{4}{11}$ and $8\frac{2}{3}$ comes the number $9\frac{1}{33}$; and if from this you take $\frac{4}{11}$, there will be left $8\frac{2}{3}$, as is clear if you reduce the fractions produced to whole numbers and to lowest terms. This must also be observed in the problems that follow — that is, after the operation is completed the fractions produced must be reduced to lowest terms, as was done in this problem.

[Margin: 2. Finding the number subtracted, or to be subtracted, from a proposed number, so that another given number is the remainder.]

2. What number has been subtracted, or ought to be subtracted, from 87 so that 26 remain? Likewise, what number has been taken away, or ought to be taken away, from $\frac{8}{13}$ so that $\frac{2}{7}$ remain? Problems of this kind subtraction will handle. For if the number that ought to be left is subtracted from the number from which the subtraction is to be made, there will remain the number which, taken from that same number, leaves the proposed remainder. Thus in the first problem, subtracting 26 from 87 leaves 61; so if 61 are taken from 87, 26 will remain. In the second problem, if $\frac{2}{7}$ are taken from $\frac{8}{13}$, there will be left $\frac{30}{91}$; and if these are subtracted from $\frac{8}{13}$, $\frac{2}{7}$ will be left.

[Margin: 3. Finding the number to which a given number is to be added, or which is to be added to a given number, so that another given number results.]

3. To what number must 38 be added, or what number must be added to 38, so that the composite number is 83? Likewise, to what number must $4\frac{8}{9}$ be added, or what number must be added to $4\frac{8}{9}$, so that the number $20\frac{1}{2}$ is composed? Problems of this kind are also solved by subtraction. For if from the number that is to be composed you take away the given number to be added, there will be left the number to which, if the given number to be added is joined, the given number results. Thus in the first problem, subtracting 38 from 83 leaves 45; to this number, then, 38 must be added so that the number 83 results. In the second problem, subtracting $4\frac{8}{9}$ from $20\frac{1}{2}$ leaves the number $15\frac{11}{18}$; and if to this you add $4\frac{8}{9}$, the number $20\frac{1}{2}$ will result.

[Margin: 4. Finding the difference between two given numbers.]

4. What is the difference, or excess, between 100 and 349? Likewise, between $6\frac{1}{2}$ and $20\frac{3}{4}$? These problems too are explained by subtraction. For if the smaller number is taken from the larger, there will be left the difference, or excess, sought. Thus in the first problem, taking 100 from 349 leaves 249 for the excess, or difference, between 100 and 349. In the second problem, taking $6\frac{1}{2}$ from $20\frac{3}{4}$ leaves $14\frac{1}{4}$. By this amount, then, the number $20\frac{3}{4}$ exceeds the number $6\frac{1}{2}$.

[Margin: 5. Finding the number divided, or to be divided, by a given number so that the proposed quotient results.]

5. What number has been divided, or is to be divided, by 9 so that the quotient is 34? Likewise, what number has been divided, or is to be divided, by $4\frac{1}{3}$ so that the quotient is $\frac{1}{2}$? Such problems are handled by multiplication. For if the given

divisor is multiplied by the given quotient, there will be produced the divided (or to-be-divided) number that is sought. Thus in the first problem, multiplying 9 by 34 gives the number 306, which divided by 9 will give a quotient of 34. In the second problem, if $4\frac{1}{3}$ is multiplied by $\frac{1}{2}$, the number $2\frac{1}{6}$ is produced, which divided by $4\frac{1}{3}$ will give the quotient $\frac{1}{2}$.

[Margin: 6. Finding the number that contains, or is, or gives a given fraction, or part, or parts, of a proposed number.]

6. What number contains $\frac{2}{3}$ of this number 30? Likewise, what number is, or gives, $\frac{1}{2}$ of this number $4\frac{5}{6}$? Multiplication also settles problems of this kind. For if the two given numbers are multiplied together, the number sought is generated. Thus, since in the first problem the multiplication of $\frac{2}{3}$ by 30 produces 18, it comes about that the number 18 contains $\frac{2}{3}$ of the number 30. In the second problem, from multiplying $\frac{1}{2}$ by $4\frac{5}{6}$ comes the number $2\frac{5}{12}$, which is $\frac{1}{2}$ of this number $4\frac{5}{6}$. This problem can also be put in this way: Give me $\frac{2}{3}$ of this number 30. Likewise, give me $\frac{1}{2}$ of this number $4\frac{5}{6}$.

[Margin: 7. Finding the number by which a given number has been divided, or is to be divided, so that the quotient is the proposed number.]

7. By what number have 48 been divided, or are they to be divided, so that the quotient is 10? Likewise, by what number are $\frac{2}{3}$ to be divided so that the quotient is $\frac{2}{3}$? Division will satisfy such problems. For if the given divided (or to-be-divided) number is divided by the given quotient, there will come out of this division the divisor sought. Thus in the first problem, dividing 48 by 10 gives the quotient $4\frac{4}{5}$, by which if 48 are divided the quotient will be 10. In the second problem, dividing $\frac{2}{3}$ by $\frac{2}{3}$ gives the quotient $\frac{6}{10}$ (that is, $\frac{3}{5}$), by which if $\frac{2}{3}$ are divided the quotient $\frac{2}{3}$ will be produced.

[Margin: 8. Finding the number by which a given number is to be multiplied, or which is to be multiplied by a given number, so that the proposed number is generated.]

8. By what number must 17 be multiplied, or what number must be multiplied by 17, so that the number produced is 100? Likewise, by what number must $3\frac{1}{2}$ be multiplied, or what number must be multiplied by $3\frac{1}{2}$, so that the number produced is $\frac{1}{4}$? Division too will satisfy such problems. For if we divide the number that is to be produced by the number proposed to be multiplied, we shall make the number sought. Thus in the first problem, dividing 100 by 17 gives the quotient $5\frac{15}{17}$, by which if the given number 17 is multiplied the given number 100 will be produced. In the second problem, if $\frac{1}{4}$ is divided by $3\frac{1}{2}$, the quotient

$\frac{1}{14}$ results, by which if the given number $3\frac{1}{2}$ is multiplied the given number $\frac{1}{4}$ is generated.

[Margin: 9. Finding two numbers which, multiplied together, produce a given number.]

9. Which two numbers, multiplied together, produce 48, or $\frac{1}{2}$, or $6\frac{3}{4}$? Division too will satisfy problems of this kind. For if we divide the number to be produced by any number whatever, this number and the quotient will be the two that are sought. Thus if 48 are divided by any number whatever, say by 6, the quotient will be 8; therefore the two numbers 6 and 8, multiplied together, will produce 48. Likewise, if the same 48 are divided by any other number whatever, say by 10, the quotient will be $4\frac{4}{5}$; therefore the two numbers 10 and $4\frac{4}{5}$, multiplied together, will generate this number 48. Likewise, if we divide $\frac{1}{2}$ by any number whatever, say by $\frac{2}{3}$, we shall find the quotient $\frac{3}{4}$; therefore the two numbers sought, which multiplied together make $\frac{1}{2}$, will be $\frac{2}{3}$ and $\frac{3}{4}$. In the same way, if we divide $\frac{1}{2}$ by any other num-[ber]...

ber, say by 8, we shall find the quotient $\frac{1}{16}$. The two numbers sought, therefore, will be 8 and $\frac{1}{16}$. Finally, dividing $6\frac{3}{4}$ by any number whatever, say by $3\frac{1}{2}$... so the numbers which multiplied together produce $6\frac{3}{4}$ will be $3\frac{1}{2}$ and $1\frac{13}{14}$. Therefore the two numbers...

[Margin: Finding two numbers such that, one being divided by the other, the proposed quotient results.]

10. Which are those two numbers such that, one being divided by the other, the quotient is 28? Likewise, which are those two numbers such that, one being divided by the other, the quotient is $\frac{5}{6}$? Multiplication unravels these problems, and others like them. For if you multiply the given quotient by any number whatever, the number produced will give the dividend; and the divisor will be the number taken as the multiplier. Thus in the first problem, if you multiply the given quotient 28 by any number, say by 6, you will make the number 168; this, then, divided by 6 will make 28. In the second problem, if you multiply $\frac{5}{6}$ by any number whatever, say by $\frac{1}{2}$, you will produce $\frac{5}{12}$, which divided by $\frac{1}{2}$ will give a quotient of $\frac{5}{6}$.

[Margin: Finding the number by which a given number is to be multiplied — or which is to be multiplied by a given number — so that, the product being divided by another given number, the proposed quotient results.]

11. By what number must 7 be multiplied, or what number must be multiplied by 7, so that, the product being divided by 8, the quotient is 3? Likewise, by what number must $\frac{3}{4}$ be multiplied, or what number must be multiplied by $\frac{3}{4}$, so that, the product being divided by $\frac{3}{4}$, the quotient is $\frac{1}{4}$? Problems of this kind are solved by multiplication and division. For if you multiply the given divisor by the given quotient, and then divide the resulting number by the given multiplicand or multiplier, the quotient will be the number sought. Thus in the first problem, if the given divisor 8 is multiplied by the given quotient 3, the number 24 is produced, which divided by the multiplicand or multiplier — namely by 7 — will make the number sought; for if 7 are multiplied by $3\frac{1}{2}$ the number 24 results, which divided by 8 makes [the quotient 3]. In the second problem, if the given divisor $\frac{3}{4}$ is multiplied by the given quotient $\frac{1}{4}$, the number $\frac{3}{16}$ [?] results; and this, dividing the given multiplicand or multiplier, will make the number sought. For if $\frac{3}{4}$ are multiplied by $\frac{1}{2}$ [?], the number $\frac{6}{16}$ [?] results, which divided by $\frac{3}{4}$ will make the quotient $\frac{1}{4}$.

[Margin: Finding what part a given number represents in relation to another given number.]

12. What part is the number 6 of this number 54? Likewise, what part is this number $\frac{2}{3}$ of this number $\frac{9}{10}$? Such problems are explained by division. For if the given number, which is to be the part, is divided by the other given number (which must always be greater than it), the quotient will indicate what part, or parts, the given number is of the given number. Thus in the first problem, dividing 6 by 54 gives the quotient $\frac{6}{54}$, that is, $\frac{1}{9}$. So the number 6 will be one ninth part of the number 54. In the second problem, dividing $\frac{2}{3}$ by $\frac{9}{10}$ gives the quotient $\frac{20}{27}$ [?]. The number $\frac{2}{3}$ will therefore contain $\frac{20}{27}$ [?] of the number $\frac{9}{10}$. That this is so may be verified by the sixth problem. For if a number is sought, according to that problem, which is $\frac{1}{9}$ of the number 54, the number 6 will be found; and likewise, if it is investigated what number contains $\frac{2}{3}$ of the number $\frac{9}{10}$, the number $\frac{3}{5}$ [?] will be found.

[Margin: Finding the number in relation to which a given number represents a proposed part.]

13. This number 6 will be one ninth part of what number? Likewise, this number $\frac{2}{3}$ will contain the two-thirds of what number? Division solves such problems. For if the given number is divided by the fraction that signifies the proposed part or parts, the quotient will give the number sought. Thus in the first problem, dividing 6 by $\frac{1}{9}$ gives the quotient 54; so the number 6 will be a ninth part of the

number 54. In the second problem, dividing $\frac{2}{3}$ by $\frac{20}{27}$ [?] gives the quotient $\frac{9}{10}$. Of this number, then, this number $\frac{2}{3}$ will contain the two-thirds.

[Margin: Finding the multitude of parts of any kind whatever that a given number contains.]

14. This number 7 comprehends how many eighth parts of one whole? Likewise, this number $\frac{3}{4}$ contains how many twelfth parts of one whole? Likewise, this $\frac{1}{3}$ embraces how many eighths of one whole? Multiplication will settle problems of this kind. For if the given number is multiplied by the denominator of the parts that are sought, the number produced will give the number of parts sought. Thus in the first problem, multiplying 7 by 8 gives 56; so the number 7 will contain 56 eighths. In the second problem, multiplying $\frac{3}{4}$ by 12 gives 9; so the number $\frac{3}{4}$ embraces nine twelfths. Finally, in the third problem, multiplying $\frac{1}{3}$ by 8 gives the number $2\frac{4}{6}$, that is, $2\frac{2}{3}$; so the number $\frac{1}{3}$ contains two eighths, and $\frac{2}{3}$ of one eighth. And that this is so is plain. For if $\frac{2}{8}$ (that is, $\frac{1}{4}$ [expressed as] eighths) and $\frac{2}{3}$ of one eighth are gathered into one sum, $\frac{1}{3}$ will be found. Whence it follows that $\frac{1}{3}$ con-[tains]...

The Rule of Three, which by another name is usually called the Golden Rule, or the Rule of Proportions. Chapter XVII.

[Margin: Why the golden rule — or [rule] of proportions, or rule of three — is so called.]

Up to now we have laid the necessary foundations of arithmetic; there now follow various rules, in which their remarkable usefulness becomes apparent, not only to mathematicians but also to merchants — indeed to any private person who, in his dealings and mutual transactions, does not wish to be cheated or to cheat (the former of which is base, the latter even wicked). These rules are, above all, useful and necessary. In the first place there presents itself that rule, never sufficiently praised, which on account of its immense usefulness is usually called the Golden Rule, or the Rule of Proportions — because it is concerned with four proportional numbers, of which the first three are known while the fourth, unknown, is sought — and which is also commonly called the Rule of Three, because it sets down three known numbers and from them determines the fourth, which is unknown. This rule of proportions works as follows.

[Margin: How the numbers in the rule of three are to be arranged.]

Three known numbers being arranged in such a way that the one which carries the question with it (for always one of them brings the question with it, as will

appear in the examples) is placed third; and of the remaining two, the one that is of the same kind as the third — that is, similar to the third — (the examples will make clear in what this similarity consists) takes the first place, while the other takes the middle seat, the one to which the fourth, which is sought, must be similar. These being arranged, the numbers are multiplied in this way: the third and the middle are multiplied together, and the product is divided by the first, and this will satisfy the proposed problem — that is, the third number will have to the sought fourth the same proportion that the first has to the second.

[Margin: How, by the rule of three, the fourth unknown number is to be sought.]

Example.

For four gold pieces 12 pounds of pepper are bought; it is asked how many pounds can be bought for 20 gold pieces. Here you see that the 20 gold pieces have the question attached to them, for concerning them it is asked how many pounds they can exhibit. To this number is similar the num-[ber]...

is the number of aurei [gold pieces], 4. For just as 12 pounds [librae, of pepper] were bought for 4 aurei, so other pounds must be bought for 20 aurei, in such a way that each of the two numbers is a price, while the 12 pounds of pepper are the goods. So, then, the example will stand thus:

Aurei.	Lib.	Aurei.	Lib.
4.	12.	20?	<i>fiunt</i> 60.

Now, multiplying the second and third numbers by each other and dividing the product, 240, by the first, we will find 60 pounds for the fourth number, which was being sought. Here you see how, just as the first number, 4, is a third part of the second number, 12, so the third number, 20, is a third part of the fourth number found, 60.

Another Example.

I spend 60 aurei in 5 months; I ask, in how many months will I spend 132 aurei? Here too you see that the question is being made about the 132 aurei, and that this figure of 60 aurei is of the same kind as it. So, then, the example will stand thus:

Aurei.	Menses.	Aurei.	Menses.
60.	5.	132?	<i>fiunt</i> 11.

Multiplying the second and third numbers by each other and dividing the product, 660, by the first, we will find 11 months, in which I shall spend 132 aurei. Here too you see that the third number, 132, contains the fourth found, 11, twelve times over, just as the first, 60, embraces the second, 5, twelve times over.

[Margin: Demonstration of the rule of three.]

The demonstration of this rule is as follows. Since the first number must stand to the second in the same proportion as the third stands to the fourth found — as has been said, and as is clear from the examples set out — it follows necessarily, by Euclid VII.19 (*Elements* VII.19), that the same number is produced by multiplying the first number by the fourth as is generated from the second times the third. Therefore, when the number produced from the second times the third is divided by the first in order to find the fourth (as the rule of three prescribes), the result is that the first number, multiplied by the quotient — that is, by the fourth number found — produces the same divided number that was in fact generated from the second times the third.

[Margin: Why, when a number that has been divided by another has its divisor multiplied by the quotient, the divided number is produced again.]

For when any number has been divided by any other number, if the divisor is multiplied by the quotient the divided number is necessarily produced again, as was stated in the third check of the Division of whole numbers, chapter 5. This also follows from the definitions of Division and Multiplication, which we will make plain by the following example. Let 12 be divided by 4 to give a quotient of 3, which — by the definition of division given in chapter 5 — contains unity as many times as the divided number 12 contains the divisor 4. I say that if we multiply the divisor 4 by the quotient 3, the divided number 12 is necessarily produced again. For since, by the definition of multiplication given in chapter 4, the number to be generated must contain the divisor 4 (which is one of the numbers being multiplied) as many times as the quotient 3 (which is the other multiplying number) contains unity — and since the divided number 12 contains the divisor 4 as many times as the quotient 3 includes unity, as has been said — it is clearly established that the divided number 12 is produced by the said multiplication of the divisor 4 by the quotient 3. And the same reasoning holds for all other numbers. This being so, the quotient found by the rule of three will indeed be the fourth proportional number that is sought, as is clear from the said Euclid VII.19: since the same number is produced from the first number times the fourth as from the second times the third, as we have said.

[Margin: Verification of the rule of three.]

From what we have just written, it is easily gathered by what method the rule of three may be tested. For if the same number is produced from the first number times the fourth found as from the second times the third, there is no doubt that the fourth proportional has been correctly found; but if a different number is generated, the operation must be repeated.

[Margin: Another verification of the rule of three.]

There is, however, another test of the rule of three, used by most, which goes as follows. Let the first number be placed in the third position, the third in the first, and the fourth in the middle. For if, following the precept of the rule of three, the fourth number is found in this way to be the one that was previously the second, then the proposed problem was correctly solved. Thus the first example brought above will stand as follows:

Aurei.	Librae.	Aurei.	Librae.
20.	60.	4?	<i>fiunt</i> 12.

For if it is true that 60 pounds are bought for 20 aurei — because 12 pounds were bought for 4 aurei — it follows necessarily that, conversely, 12 pounds are bought for 4 aurei, by the very fact that 60 pounds are bought for 20 aurei.

Sometimes two of the three given numbers — the first and the second, or the first and the third — can be reduced to smaller ones, to make the operation easier. This will be done whenever both the first and the second, or both the first and the third, are divided by some common measure of the two (whether that measure is the greatest or not), and their quotients are put in their place. As in this example:

4.	12.	20?	<i>fiunt</i> 60.

Since the number 4 measures both the first and the second, if both are divided by 4 and the quotients 1 and 3 are put in their place, the example will stand thus:

1.	3.	20?	<i>fiunt</i> 60.

Likewise, because in the same example the number 4 also measures the first and the third, if both are divided by 4 and the quotients 1 and 5 are taken in their place, the same example will stand thus:

1.	12.	5?	<i>fiunt</i> 60.

Likewise in this following example:

36.	48.	63?	<i>fiunt</i> 84.

Since the number 12 measures the first and the second, if both are divided by 12 and the quotients 3 and 4 are put in their place, the example will stand thus:

3.	4.	63?	<i>fiunt</i> 84.

Likewise, because the number 9 measures the first and the third in the same example, if both are divided by 9 and the quotients 4 and 7 are placed in the rule in their stead, the example will stand thus:

4.	48.	7?	<i>fiunt</i> 84.

This same problem may also be solved in yet another way. Divide the second number by the first, and multiply the third by the quotient; or divide the third by the first, and multiply the middle number by the quotient. For in either way the number produced will be the fourth proportional that is sought. As in this example:

60.	360.	132?	<i>fiunt</i> 792.

Dividing the second number by the first gives a quotient of 6; if the third number is multiplied by it, the fourth, 792, is generated — just as if you had worked by the precept of the rule of three. Likewise, dividing the third number by the first gives a quotient of $2 \frac{12}{60}$ — that is, $2 \frac{1}{5}$, or $\frac{11}{5}$ — and if the second is multiplied by it, the same fourth, 792, is produced.

Once these things are rightly understood, you can test in various ways whether the fourth number has been correctly found by the rule of three or not. For if by various operations of this kind you always find the same fourth number, it is the strongest argument that the operation has been correctly set up.

And if anyone should ask how it can happen that we always arrive at the same goal by so many paths, let him know that the whole cause of this depends on proportions. For since the same proportion must hold between the first number and the second as between the third and the fourth, it follows — by alternation — that the same proportion holds between the first and the third as between the second and the fourth; likewise, by inversion, the same holds between the second and the first as between the fourth and the third; and also the same between the third and the first as between the fourth and the second. Since, then, the same proportion always holds between the quotients of two numbers divided by the same number as holds between the numbers themselves, it is evident that if both the first number and the second, or both the first and the third, are divided by some common measure, and quotients are put in place of the numbers, there is still the same proportion between the quotients of the first and second numbers as between the third number and the fourth; and likewise the same proportion between the quotients of the first and third numbers as between the second number and the fourth. Again, since dividing any number by another produces the denominator of the proportion that the divided number has to the divisor, while the denominator, multiplying any other number, produces a number that has to the multiplied number the proportion named by that denominator — it follows that when the second or the third number is divided by the first, the quotient is the denominator of the proportion of the second or third number to the first. Hence, if the third number, or the second, is multiplied by this quotient, the fourth is generated: namely, one that has the same proportion to the third as the second has to the first, or the same to the second as the third has to the first.

[Margin: Several problems by which the various difficulties of the rule of three are explained.]

But since problems to be solved by the rule of three are frequently proposed in reversed order — and sometimes different coinages, measures, or weights are found in one number, and finally the first number is not seldom unlike the third — so that someone little versed in arithmetic can easily be baffled and held up, we will set out, by means of several problems, the various difficulties that can arise, beginning here.

1. How much does one pound of pepper cost, if 60 pounds are bought for 20 aurei? In this problem the numbers are placed in reversed order. For the 1 pound, which is mentioned in the first place, has the question attached to it and ought therefore to occupy the third place, while the number 60 of pounds ought to be first, since it is of the same kind as the number of one

pound. So, in the correct order, the problem should have been proposed thus: 60 pounds of pepper cost 20 aurei, therefore how much will 1 pound cost? — as appears in the example set beside it here:

Lib.	Aur.	Lib.	Aur.
60.	20.	1?	<i>fiunt</i> 20/60, or 1/3.

You will find (if you multiply the second number by the third and divide the product, 20, by the first) that the value of 1 pound is 20/60, or 1/3, of one aureus — because when a smaller number is divided by a larger, a fraction results whose numerator is the number to be divided and whose denominator is the divisor, as we said in chapters 5 and 6. Either of these fractions may be reduced, the former to baiochi [baiocchi], in this way: multiply the numerator 20 by 100 (for that many baiochi make one aureus), and divide the resulting number, 2000, by the denominator 60. The quotient will give 33 20/60, or 33 1/3, baiochi. You would have found just as much if you had multiplied the numerator of the latter fraction, 1/3, by 100 and distributed the product by its denominator. And if you wish to convert the 1/3 of one baiochus into quatrini [quattrini], you will multiply the numerator by 4 (since that many quatrini make a baiochus) and divide the product by the denominator: you will find 1 1/3 quatrini. And so 1 pound will cost 33 baiochi and 1 1/3 quatrini.

[Margin: Quaest. 2.]

2. If 10 2/5 pounds and 7 1/2 ounces of white wax cost 2 aurei and 6 iulii [giulii], how much wax will be bought for 90 baiochi? The example will stand thus:

Aur.	Iul.	Lib.	Vnc.	Baioc.	Vnc.
2.	6.	10 2/5.	7 1/2.	90?	<i>fiunt</i> 45 2070/2600.

[Margin: What is to be done when different coinages, measures, weights, and fractions occur.]

Since in the first number and in the third different coinages are contained, they must all be reduced to the smallest coin expressed there, namely to baiochi; and there will be 260 baiochi in the first number. Again, since in the second number different weights are found, they must be reduced to the smallest expressed there, namely to ounces, of which 12 make one pound; and in the 10 2/5 pounds there will be 124 4/5 ounces, to which if the 7 1/2 ounces are added, there result 132 3/10 ounces. How fractions — whether alone or together with whole numbers

— ought to be multiplied by one another, or divided, we taught in chapters 13 and 14. And so the reduced example will stand thus:

Baioc.	Vnc.	Baioc.	Vnc.
260.	132 $\frac{3}{10}$.	90?	<i>fiunt</i> 45 $\frac{2070}{2600}$.

It must be noted here that the fraction produced from multiplying the middle number by the third — even though its numerator is greater than its denominator — must not be reduced to whole numbers until the division is completed, lest the operation be hindered. Hence, since the multiplication of the middle number by the third then makes $\frac{119070}{10}$, this fraction must be divided, before it is reduced to whole numbers, by the first number; and this division will give this fraction, $\frac{119070}{2600}$, containing $45 \frac{2070}{2600}$ ounces.

[Margin: Quaest. 3.]

3. How much will $\frac{7}{8}$ of an ell [vlua] of cloth cost, if someone has bought $\frac{1}{3}$ of an ell for $\frac{3}{4}$ of an aureus? The example will stand thus:

Vluae.	Aur.	Vluae.	Aur.
$\frac{1}{3}$.	$\frac{3}{4}$.	$\frac{7}{8}$?	<i>fiunt</i> 1 $\frac{31}{32}$.

The multiplication of the middle number by the third makes the fraction $\frac{21}{32}$; when this is divided by the first number, this fraction, $\frac{63}{32}$ of one aureus, is found, which makes 1 $\frac{31}{32}$ aurei. And this fraction, $\frac{31}{32}$ of one aureus, reduced to iulii, baiochi, and quatrini, gives 9 iulii, 6 baiochi, and 3 $\frac{1}{2}$ quatrini.

[Margin: Quaest. 4.]

4. A certain student at a university, wishing to devote himself to letters for 6 years, noticed that in 7 months and 13 days he had spent 200 aurei, 7 iulii, and $8 \frac{2}{7}$ baiochi. The question, then, is how much money he needs. The example will stand thus:

Menses.	Dies.	Aur.	Iul.	Baioc.	An	Aur.	Baioc.
7.	13.	200.	7.	$8 \frac{2}{7}$.	6?	<i>fiunt</i> 1956.	7 $\frac{191}{1575}$

Here, in the first number the months, and in the third the years, must be converted to days. To do this, it must be considered which months these are, since not all months contain the same number of days. For if we suppose the first seven months, beginning from January, the said 7 months will contain, in a common year, 212 days, as you see here (but in a leap year 213, since in it February

contains 29 days). With the 13 days added, there will be 225 days. Next it must be considered how many leap years are contained in those 6 years. For each one, 1 day must be added to the 365 days of one common year. Hence, if we suppose two leap years to be contained, we will multiply the 6 years by 365 days and add 2 to the product, so that 2192 days result. Again, in the middle number the aurei and iulii must be reduced to baiochi, and these will number $20078 \frac{2}{7}$ — so that the reduced example is as follows:

Ian.	31.
Febr.	28.
Mart.	31.
April.	30.
Maius	31.
Iun.	30.
Iul.	31.
[sum]	212.
	13.
[sum]	225.

Dies.	Baioc.	Dies.	Baioc.
225.	$20078 \frac{2}{7}$.	2192?	<i>fiunt</i> 195607 191/1575

Finally, the fourth number found, in baiochi, must be reduced to aurei and iulii. All those baiochi make 1956 aurei, 0 iulii, and 7 191/1575 baiochi. So much money is needed by that student for the 6 years, two of which are leap years.

By this same reasoning, after the operation the coinage of the fourth number is always to be reduced to the larger, if it can be done; likewise the weights and measures to the larger weights or measures — as ounces to pounds, palms or feet to paces, and paces to miles.

[Margin: Quaest. 5.]

5. A certain man covers 210 miles in 7 days. I ask, in how many days will he cover 1600 miles, if he travels every day and neither slackens nor quickens his pace? The example will be set down thus:

Milliar.	Dies.	Milliar.	Dies.
210.	7.	1600?	<i>fiunt</i> 53 70/210.

Milliar.	Dies.	Milliar.	Dies.

This fraction, 70/210 of one day, in the fourth number — if the numerator is multiplied by 24 and the resulting number is divided by the denominator — is reduced to 8 hours.

[Margin: Quaest. 6.]

6. If a field of 400 square paces is bought for 100 aurei, 7 iulii, and 8 baiochi, how much will a field of 1000 square paces, 4 square feet, and 3 square palms cost? The example will stand thus:

	Au				Ped		Palm	
Passus.	r.	Iul.	Baioc.	Pass.	.	.	Baioc.	
400.	100	7.	8.	1000	4.	3?	<i>fiunt</i> 25199	17613/80000
	.			.				

With the aurei and iulii of the second number reduced to baiochi, and the paces and feet of the third number to palms — assigning 16 square palms to one square foot, and 25 square feet to one square pace — and likewise with the paces of the first number also reduced to palms, assigning 400 square palms to one square pace, the reduced example will stand thus:

Palmi.	Baioc.	Palmi.	Baioc.
160000.	10078.	400067?	<i>fiunt</i> 25199 17613/80000

And the fourth number of baiochi contains 251 aurei, 9 iulii, and 9 17613/80000 baiochi.

[Margin: Quaest. 7.]

7. At a certain fair, 52 ells of a certain cloth were bought for 44 aurei; how much will 260 ells of the same cloth cost? The example will stand thus:

Vlnae.	Aur.	Vlnae.	Aur.
52.	44.	260?	<i>fiunt</i> 220.

[Margin: Quaest. 8.]

8. Someone buys 52 ells of cloth for 44 aurei; how many ells will he buy for 220 aurei? The example will be arranged thus:

Aur.	Vlnae.	Aur.	Vlnae.
44.	52.	220?	<i>fiunt</i> 260.

[Margin: Quaest. 9.]

9. Someone buys 52 ells of cloth for a certain sum of money, and afterwards, at the same price, receives 260 ells, which cost 220 aurei; how much, then, did he spend at first? The example will be arranged in this way:

Vlnae.	Aur.	Vlnae.	Aur.
260.	220.	52?	<i>fiunt</i> 44.

[Margin: Quæst. 10.]

10. A certain man buys some ells of cloth for 44 aurei; then another, at the same price, buys 260 ells for 220 aurei. How many ells, then, did the first buy? The example will stand thus:

Aur.	Vlnæ	Aur.		Vlnæ
220	260	44?	make	52

I have appended these last four examples, in which the same four numbers of the Rule of Three interchange their roles in various ways — since each of them can be derived from the other three that are given — so that you may understand how you ought to proceed in other similar problems.

THE INVERSE RULE OF THREE [Regula Trium Eversa]

Chapter 18 [Caput XVIII]

[Margin: How the inverse rule of three elicits the fourth number.]

We have said that in the four numbers of the Rule of Three the proportion of the first number to the second is the same as that of the third to the fourth — and consequently (as is gathered from Euclid, *Elements* V.14 [propos. 14. lib. 5. Euclid.]) that if the first is greater, or smaller, than the third, the second too is greater, or smaller, than the fourth; a fact that can be seen plainly in all the examples adduced so far. But it sometimes happens that the greater the first number is than the third, the smaller the second ought to be than the fourth; and the smaller the first is than the third, the greater the second ought to be than the

fourth. In that case the opposite procedure must be followed: namely, the first number must be multiplied by the second, and the product divided by the third. Now, when this inverse rule of three (as they call it) is to be applied, and when the natural reasoning seems to dictate it, can be understood most clearly from the following examples, of which the first is this.

[Margin: Quæst. 1.]

1. A cloth of 9 ells, whose width is 3 palms [palmorum], is bought by someone to have a tunic made for himself. How many ells of another cloth, whose width is 2 palms, are required to make the same tunic, or one like it? Since the question concerns a cloth having a width of 2 palms, the example will stand thus:

Palmi latitud. (width in palms)	Vlnæ (ells)	Palmi latitud.	Vlnæ
3	9	2?	make 13½

Here you plainly see that the narrower the second cloth is, the more ells are needed. Therefore, although the first number is greater than the third, the second need not on that account also be greater than the fourth, but smaller — so that the proportion which the third number has to the first, the second has to the fourth. Hence it comes about that the first is to be multiplied by the second, and the product divided by the third; for, so that the due proportion may be preserved, the third number must occupy the first place in the rule of proportions, as has been said, and as appears here.

Palmi latitud.	Palmi latitud.	Vlnæ	Vlnæ
2	3	9?	make 13½

[Margin: Quæst. 2.]

2. Someone received 4000 aurei on loan from another for 3 years; and when he was to repay them, the lender was unwilling to accept any interest [censum], but asked only that the borrower in turn lend him money. He therefore gave him 7480 aurei on loan. How long, then, ought he to keep this money, so as to be recompensed for the benefit rendered him in respect of the 4000 aurei he had lent? In this question, since the number 7480 aurei carries the unknown with it, the numbers will be arranged in this order:

Aur.	Anni (years)	Aur.	An ni	Dies (days)	Horæ (hours)
4000 ?	3	7480? ma ke	1	220	13 80/187

Here too it is clear that a larger sum is owed for the unequal — that is, the shorter — time; indeed, that less time than 3 years is needed for the same return, which is owed on 4000 aurei over three years, to be gained from 7480 aurei. Therefore, although the first number is smaller than the third, the second will not on that account also be smaller than the fourth, but larger — so that the proportion which the third number has to the first, the second has to the fourth. From this it follows that the first must be multiplied by the second, and the product divided by the third; for, so that the due proportion may be kept, the third number must occupy the first place in the rule of proportions, as has been said, and as appears here.

Aur.	Aur.	Anni		Anni	Dies	Horæ
7480	4000	3?	make	1	220	13 80/187

[Margin: Quæst. 3.]

3. When a measure of wheat is bought for 6 aurei, the bread — by the ordinance of some city — weighs 10 ounces [vnciarum]; now if the same measure of wheat is bought for 4 aurei, or for 8 aurei, how much ought the weight of the same bread to be? The examples will stand thus:

Aur.	Vnciæ (ounces)	Aur.	Vnciæ
6	10	4? make	15
6	10	8? make	7½

Reason dictates that the cheaper the wheat, the greater the weight the bread will have; and the dearer, the less. For the proportion of 4 aurei to 6 aurei, or of 8 aurei to 6 aurei, ought to be such as the proportion of the weight of 10 ounces to the unknown weight that is sought. Hence, by the rule of proportions, the numbers would be arranged thus:

Aur.	Aur.	Vnciæ	Vnciæ
4	6	10	make 15
8	6	10?	make 7½

[Margin: Quæst. 4.]

4. Thirty workmen [Operarij] complete a certain task in 4 years; in how much time, then, will 50 workmen finish it, or 20? Or, how many workmen will finish the same in 2 years and 146 days? Or in 4 years and 292 days? This fourfold example will stand thus, the years being reduced to days (each reckoned at 365 days) in the last two cases:

Operarij (workmen)	Anni (years)	Operarij		An ni	Dies (days)
30	4	50?	ma ke	2	146
30	4	20?	ma ke	6	0

Dies	Operarij	Dies		Operarij
1460	30	876?	make	50
1460	30	1752?	make	25

For the more workmen there are, the less time the task requires; and the fewer, the more. Likewise, the less time there is, the more workmen the task requires; and the more time, the fewer. Therefore, by the rule of proportions, the numbers would be arranged thus:

Operarij	Operarij	Anni		Anni	Dies
50	30	4?	make	2	146
20	30	4?	make	6	0

Dies	Dies	Operarij		Operarij
876	1460	30?	make	50
1752	1460	30?	make	25

[Margin: Quæst. 5.]

5. A besieged army of 8500 soldiers has provisions for 11 months; but there is no hope of the siege being lifted, or of relief coming, before 25 months. How many soldiers, then, must be kept, so that the provisions may suffice for the rest for 25 months? The numbers are to be laid out thus:

Menses (months)	Milites (soldiers)	Menses	Milites
11	8500	25?	mak e 3740

Therefore 3740 soldiers must be kept, for the provisions will suffice these for 25 months; and consequently the remaining 4760 soldiers must be dismissed.

THE COMPOUND RULE OF THREE [Regula Trium Composita]

Chapter 19 [Caput XIX]

[Margin: What the compound rule of three is, and how it is done.]

It sometimes happens that more than three numbers are proposed as known, yet in such a way that there are always three principal ones, while the others attached to them are less principal, denoting time, or gain, or loss. When this occurs, the compound rule of three arises; and then either the rule of three must be set up twice, or three times, or each principal number must be multiplied by the ones attached to it, so that only three known numbers result, by which the fourth, unknown one is elicited — or else some other method must be tried. This will become clear from the following examples, in which various questions concerning gain and loss (with a difference of time also intervening) and variety of profit at so much per 100 [percent] are discussed.

[Margin: Quæst. 1.]

1. There are 8 persons in a common mess [conuictu], each of whom pays 6 aurei each month. How much, then, will the cost of provisioning them all for 4 years be? This question would be rightly put thus: one person in one month pays 6 aurei; what, then, will 8 pay in 4 years — that is, in 48 months? The numbers are to be arranged thus:

Conuictor (messmate)	Menses (months)	Au r.	Conuictores	Mens es	Au r.
1	1	6	8	48?	ma ke 23 04

Here you see that the first number, that of one messmate, has one month attached to it, while the third, that of 8 messmates, has 48 months attached. First,

then, the rule of three will be set up thus: if one pays 6 aurei, how much will 8 pay? As here:

Conuict.	Aur.	Conuict.		Aur.
1	6	8?	make	48

Therefore 8 messmates will pay 48 aurei in one month, since one pays 6 aurei per month. Second, then, the rule of three will again be set up thus: if in one month they pay 48 aurei, how much will they pay in 48 months? As here:

Mensis (month)	Aur.	Menses		Aurei
1	48	48?	make	2304

The same question, however, will be resolved more briefly if both the two numbers placed in the first position at the start of the question are multiplied together, and the two placed in the third position are multiplied together, so that the three numbers of the rule of three result, in this way:

	Aur.			Aur.
1	6	384?	make	2304

For from this multiplication is produced a larger number of messmates for one month, which is equivalent to a smaller number over more months. Thus, from the multiplication of 8 messmates by 48 months, 384 messmates for one month are produced. For if in each month there are 8 messmates, then without doubt over 48 months — if new messmates kept being added — there would come to be 384 messmates; and so these 384 messmates in one month will pay just as much as 8 messmates in 48 months. This is the reason why the principal numbers must be multiplied by the less-principal ones attached to them — those that signify time, or something else — provided it is not of the same kind as that which the principal numbers signify; otherwise they would not be two numbers, but one. For example, if in some place aurei, baiocchi, and quatrini are set,

these three numbers will be reckoned as one, since they are of the same kind — inasmuch as they signify money. And the same relation holds, by a certain proportion, in other questions of this sort.

[Margin: Quæst. 2.]

2. For 200 pounds [libris] of certain goods carried over 100 miles [milliaria], 4 aurei are to be paid; how much, then, is to be paid for 300 pounds carried over 400 miles? The numbers will be arranged thus:

Lib. (pounds)	Mill. (miles)	Aur.	Lib.	Mill.	Aur.
200	100	4	300	400?	mak e 24

When the two numbers of the first place, and the two of the third place, are multiplied together, three numbers of the rule of three will be produced, in this way:

	Aur.		Aur.
20000	4	120000?	make 24

If this same question is to be solved by the rule of three applied twice, the example will stand, in the first place, thus:

Lib.	Aurei	Lib.	Aurei
200	4	300?	make 6

And thus 6 aurei would have to be paid for 300 pounds carried over 100 miles — the distance, namely, over which the 200 pounds were carried. But since the 300 pounds are to be carried over 400 miles, the example will again stand, in the second place, thus:

Mill.	Aur.	Mill.	Aur.
100	6	400?	make 24

[Margin: Quæst. 3.]

3. Three persons consume a modius of wheat, bought for 3 aurei, in 5 weeks [septimanis]; what, then, is each one's expenditure in one day? The numbers are to be arranged thus:

Pers. (persons)	Sept. (weeks)	Aur.	Pers.	Dies (days)
3	5	3	1	1?

— they make: Aur. 3/105, that is, 11 $\frac{2}{7}$ quatrini.

But when the 5 weeks are reduced to days, so that the first number and the third are alike, the example will stand thus:

Pers.	Dies	Aur.	Pers.	Dies
3	35	3	1	1?

— they make: Aur. 3/105, that is, 11 $\frac{2}{7}$ quatrini.

When the two numbers of the first place, and the two of the third, are multiplied together, three numbers will be laid out for the rule of three in this way:

	Aur.			Aur.		quatrini
105	3	1?	make	3/105	that is	11 2/7

By the rule of three applied twice, the same question will be solved thus:

Pers.		Aur.		Pers.		Aur.
3		3		1?	make	1

Likewise:

Dies	Aur.	Dies		Aur.		quatrini
35	1	1?	make	1/35	that is	11 2/7

[Margin: Quæst. 4.]

4. If 300 aurei in 4 years earn 100 aurei, what will 1580 aurei earn in 7 years?
When the aurei that are put out for profit are multiplied by the time attached, the example will stand thus:

| | | Aur. | |---|---|---|---|---| | 1200 | 100 | 11060? | make | 921 2/3 |

By the rule of three applied twice, the example will stand thus:

Aur.	Aur. lucrum (profit)		Aur.		Aur. lucrum
300	100		1580?	make	526 2/3

Likewise:

Anni	Aur.	Anni		Aur.
4	526 2/3	7	make	921 2/3

[Margin: Quæst. 5.]

5. Someone in three months, with 10 aurei, has earned 4 aurei; in how much time, then, with 100 aurei will he earn 2000 aurei? This question cannot be reduced to a simple rule of three, because the time in which 100 aurei ought to earn 2000 aurei is unknown, and therefore cannot be multiplied by 100 aurei. The rule of three must therefore be applied twice, in this way:

Aur.	Aur. lucr. (profit)		Aur.		Aur. lucr.
10	4		100?	make	40

Aur.	Aur. lucr. (profit)	Aur.	Aur. lucr.

And so 100 aurei will earn 40 aurei in three months, in which 10 aurei earned 4. Therefore, to learn in how much time 100 aurei will earn 2000, the rule of three will be set up a second time in this way:

Aur.	Menses	Aur.	Menses
40	3	2000?	make 150

And so, if 10 aurei in 3 months earn 4 aurei, 100 aurei will earn 2000 aurei in 150 months. This is easily proved if the question is put thus: if 10 aurei in 3 months earn 4 aurei, how much will 100 aurei earn in 150 months? For the profit will be found to be 2000 aurei, as appears here:

Aur.	Mens.	Aur.	Aur.	Mens.	Aur.
10	3	4	100	150	make 2000

For if each period of time is multiplied by its own sum of money, the example is reduced to the simple rule of three, in this way:

—	—	Aur.	Aur.
30.	4.	15000?	fiunt 2000.

Quaest. 6. If 100 aurei earn 20 aurei in 8 months, in how much time will the same 100 aurei earn 3,000 aurei? The arrangement of the numbers is as follows:

Aur.	Mens.	Aur.	Mens.
20.	8.	3000?	fiunt 1200.

For when the same sum is always set against the profit, it is not to be placed among the other numbers. The same will also hold when the same period of time is proposed, as will appear in the following example.

Quaest. 7. If 300 aurei earn 45 aurei in 7 months, what will 1,780 aurei earn in the same 7 months? The example stands thus:

Aur.	Lucr. aur.	Aur.	Lucr. aur.
300.	45.	1780.	fiunt 267.

Quaest. 8. If 4 aurei were paid to each soldier for each month, how much money would be spent on 13,000 soldiers over nine months? The example stands thus:

Milit.	Mens.	Aur.	Milit.	Mens.	Aur.
1.	1.	4.	13000.	9?	fiunt 468000.

Quaest. 9. If 7 measures [mensurae] of barley or oats are given daily to 10 horses, how many measures, at the same rate, will 100 horses require for 20 days? The example stands thus:

Equi.	Dies.	Mensurae.	Equi.	Dies.	Mensurae
10.	1.	7.	100.	20?	fiunt 1400.

Quaest. 10. If twelve reapers [messores] mow 20 iugera in 9 days, in how much time will 30 reapers mow 45 iugera? Here the rule of three must be applied twice, but with the first application inverted, since 30 reapers need less time to mow 20 iugera than 12 reapers do. And so the inverse rule of three [regula trium eversa] stands thus:

Messores.	Dies.	Messores.	Dies.
12.	9.	30?	fiunt 3 $\frac{3}{5}$.

And so 30 reapers will mow 20 iugera in 3 $\frac{3}{5}$ days. Hence the example stands again for the (direct) rule of three:

Iugera.	Dies.	Iugera.	Dies.
20.	3 $\frac{3}{5}$.	45?	fiunt 8 $\frac{1}{10}$.

Quaest. 11. At Rome a gold ducat [ducatus] is reckoned at 11½ Iulii, that is, at 115 baiocchi [baiochus]. How many such ducats, then, shall I receive for 1,000 aurei, each of which is reckoned at 10 Iulii, or 100 baiocchi? Or, if 20 ducats make 23 aurei, how many ducats will 1,000 aurei make? Each version of the example will stand thus, once the 1,000 aurei have first been reduced to 100,000 baiocchi:

Baioc.	Duc.	Baioc.	Duc.
115.	1.	100000?	fiunt 869 $\frac{13}{23}$ [?].

Aur.	Duc.	Aur.	Duc.
23.	20.	1000?	fiunt 869 $\frac{13}{23}$ [?].

Quaest. 12. How many aurei will 4,000 ducats yield, if 1 aureus contains 100 baiocchi while 1 ducat contains 115 baiocchi? Or, if 20 ducats are reckoned at 23 aurei, how many aurei will 4,000 ducats contain? Once the 4,000 ducats of the

first version have been reduced to 460,000 baiocchi, each version of the example stands thus:

Baioc.	Aur.	Baioc.		Aur.
100.	1.	460000?	fiunt	4600.

Duc.	Aur.	Duc.		Aur.
20.	23.	4000?	fiunt	4600.

Quaest. 13. A certain merchant buys 300 pounds [librae] of some commodity for 60 aurei. He wishes to know how much he will gain per 100 aurei if he sells the same 300 pounds for 64 aurei; or how much he will lose per 100 aurei if he sells those 300 pounds for 57 aurei. Here it is clear that on 60 aurei he will gain 4 aurei or lose 3 aurei, as is evident if the smaller price is subtracted from the larger. So say: if 60 aurei gain 4 aurei, or lose 3 aurei, how much will 100 aurei gain or lose?

Aur.	Lucr. aur.	Aur.		Lucr. aur.
60.	4.	100?	fiunt	6 $\frac{2}{3}$.

Aur.	Damnum aur.	Aur.		Damnum aur.
60.	3.	100?	fiunt	5.

Quaest. 14. A certain merchant asks himself for how much 100 pounds of some commodity must be bought, so that once sold for 64 aurei they yield a profit of 6 $\frac{2}{3}$ aurei per 100 aurei. It is plain that one who wishes to gain 6 $\frac{2}{3}$ aurei per 100 aurei wants to increase 100 so that it becomes 106 $\frac{2}{3}$. So say: if 106 $\frac{2}{3}$ aurei — which contain both the price of 100 aurei and the profit of 6 $\frac{2}{3}$ aurei — arise from 100 aurei, from what will 64 aurei arise, which contain both the (unknown) price of the 100 pounds and, at the same time, the (likewise unknown) profit that yields 6 $\frac{2}{3}$ per 100?

Pret. & lucr.	Aur.	Pretium, & lucr.		Aur.
106 $\frac{2}{3}$.	100.	64?	fiunt	60.

So the 100 pounds must be bought for 60 aurei. For, sold afterwards for 64 aurei, they give a profit of 4 aurei, or per 100 they will give a profit of 6 $\frac{2}{3}$ aurei.

Quaest. 15. A certain gem was bought; if it is sold for 200 aurei, 10 aurei per 100 are lost. For how much, then, was that gem bought? Here too it is clear that one

who loses 10 per 100 makes 90 out of 100. So: if 90 aurei come from 100 aurei, from what will 200 aurei come?

Aur.	Aur.	Aur.		Aur.
90.	100.	200?	fiunt	222 $\frac{2}{9}$.

So that gem cost 222 $\frac{2}{9}$ aurei. To prove this, say: if from 222 $\frac{2}{9}$ aurei come 200 aurei, what will come from 100? You will find that 90 aurei result, and hence that a loss of 10 aurei per 100 occurs.

Aur.		Aur.	Aur.		Aur.
222 $\frac{2}{9}$.		200.	100?	fiunt	90.

Or say: if for 222 $\frac{2}{9}$ aurei I lose 22 $\frac{2}{9}$ aurei — for if that gem was bought for 222 $\frac{2}{9}$ aurei but is sold for 200 aurei, it plainly follows that a loss of 22 $\frac{2}{9}$ aurei occurs — what shall I lose per 100? You will find a loss of 10 aurei, as appears here.

Aur.	Damnum aur.		Aur.		Damnum aur.
222 $\frac{2}{9}$.	22 $\frac{2}{9}$.		100?	fiunt	10.

Quaest. 16. Someone buys 1,000 ells [ulnae] of cloth at a certain price. If he had bought them for 3 aurei less and had afterwards sold them for 3,600 aurei, he would have gained 10 per 100 aurei. For how much, then, did he buy those 1,000 ells? Since one who wishes to gain 10 per 100 wants to make 110 out of 100, say: if 110 come from 100, from what will 3,600 come? As you see here:

Aur.	Aur.	Aur.		Aur.
110.	100.	3600?	fiunt	3272 $\frac{8}{11}$.

So if he had wished to gain only 10 per 100, those 1,000 ells would have cost 3,272 $\frac{8}{11}$ aurei. For if 3,272 $\frac{8}{11}$ aurei give 3,600 aurei, 100 aurei will give 110 aurei, and hence 10 per 100 fall to profit, as is clear here.

Aur.		Aur.	Aur.		Aur.
3272 $\frac{8}{11}$.		3600.	100?	fiunt	110.

Or, put another way: 3,272 $\frac{8}{11}$ aurei gain 327 $\frac{3}{11}$ aurei — for one who buys something for 3,272 $\frac{8}{11}$ aurei and afterwards sells it for 3,600 aurei necessarily gains 327 $\frac{3}{11}$ aurei — so 100 aurei will gain 10 aurei, as you see here.

Aur.	Lucr. aur.	Aur.	Lucr. aur.
3272 8/11.	327 3/11.	100? fiunt	10.

But since it was added in the problem that he would have gained 10 aurei per 100 if he had bought those 1,000 ells for 3 aurei less and sold them for 3,600 aurei, it plainly follows that he spent 3 aurei beyond 3,272 8/11 aurei. Therefore the 1,000 ells cost 3,275 8/11 aurei.

Quaest. 17. Someone buys 1,000 ells of cloth at a certain price. If they had cost 6 aurei more and had afterwards been sold for 3,600 aurei, 10 aurei per 100 would have been lost. What, then, was the price of those 1,000 ells? Since one who loses 10 per 100 makes 90 out of 100, say: if 90 come from 100, from what will 3,600 come?

Aur.	Aur.	Aur.	Aur.
90.	100.	3600? fiunt	4000.

So if he had lost only 10 per 100, the 1,000 ells would have cost 4,000 aurei. For if 4,000 aurei give 3,600 aurei, 100 aurei will give 90 aurei, as is evident.

Aur.	Aur.	Aur.	Aur.
4000.	3600.	100? fiunt	90.

Or: if 4,000 aurei pay out 400 aurei — for one who buys something for 4,000 aurei but sells it for 3,600 loses 400 aurei outright — 100 aurei will lose 10 aurei, as you see here.

Aur.	Dam. aur.	Aur.	Damn. aur.
4000.	400.	100? fiunt	10.

But since it was added in the problem that he would have lost 10 per 100 if he had bought the 1,000 ells for 6 aurei more and afterwards sold them for 3,600 aurei, it is plain that he spent 6 aurei less than 4,000. Therefore the 1,000 ells cost 3,994 aurei.

Quaest. 18. One who sells single pounds of some commodity at 20 baiochi gains 30 per 100. How much, then, will he gain if he sells at a higher price, namely 24 baiochi? Here one must first find out how much one pound costs, such that, sold at 20 baiochi, it gives a profit of 30 per 100 — as we taught in problem 14 — in this way: if 130 (namely the price of 100 and the profit of 30) arise from 100 as the (cost) price, from what will 20 baiochi arise, which contain both the unknown

price of one pound and, at the same time, the (likewise unknown) profit that yields 30 per 100?

—	—	—	—	
130.	100.	20.	fiunt	15 5/13.

So one pound will cost 15 5/13 baiochi. For thus it comes about that, since 15 5/13 baiochi (one pound sold at 20 baiochi) gain 4 8/13 baiochi, 30 baiochi are gained per 100, as you see here.

—	—	—	—	—
15 5/13.		4 8/13.	100?	fiunt 30.

Now, the price of one pound having been found — 15 5/13 baiochi — it is clear that if one pound is sold at 24 baiochi, 8 8/13 baiochi are gained on 15 5/13 baiochi. Therefore 56 baiochi will be gained per 100, as you see here.

—	—	—	—	—
15 5/13.		8 8/13.	100?	fiunt 56.

Quaest. 19. One who sells 100 pounds of some commodity for 10 aurei loses 10 per 100. How much, then, will he lose per 100 if he sells them at a lower price, namely 8 aurei? Here too one must first find out how much those 100 pounds cost, such that, sold for 10 aurei, they bring a loss of 10 aurei per 100 — as we taught in problem 15 — in this way: if 90 come from 100 (for one who loses 10 per 100 makes 90 out of 100), from what number will 10 arise?

—	—	—	—	
90.	100.	10?	fiunt	11 1/9.

So those 100 pounds were bought for 11 1/9 aurei. For thus it comes about that, since 11 1/9 aurei (those 100 pounds having been sold for 10 aurei) lose 1 1/9 aurei, 10 per 100 are lost, as you see here.

—	—	—	—	—
11 1/9.		1 1/9.	100?	fiunt 10.

And, the price of those 100 pounds — 11 1/9 aurei — having been found, it is clear that if the same 100 pounds are sold for 8 aurei, 3 1/9 aurei are lost out of 11 1/9. For this reason 28 per 100 will be lost, as you see here.

—	—	—	—	—
11 1/9.	3 1/9.	100?	fiunt	28.

Quaest. 20. A certain merchant in Portugal [Lusitania] buys 50,000 pounds of pepper for 10,000 [aur.?], and there pays 500 aurei in duty [vectigal]. The freight [naulum] from there to Italy comes to 300 aurei; in port another duty of 200 aurei is exacted; the carriage [vectura] from the sea all the way to Florence [Florentia] comes to 100 aurei; and there a further duty of 100 aurei is paid. Finally, to the agents [ministri] sent for that business, 1,000 aurei are given for wages and keep. But he is uncertain at what price the pound should be sold, so that over and above all his expenses he gains 2 Iulii on each pound. Here one must first gather all the expenses into a single sum, so as to have the price at which, together with all those expenses, the 50,000 pounds have been laid out. This sum, in the given example, is:

	Aurei.
Piper. [Pepper]	10000.
Vectigal. [Duty]	500.
Naulum. [Freight]	300.
Vectigal. [Duty]	200.
Vectura. [Carriage]	100.
Vectigal. [Duty]	100.
Ministri. [Agents]	1000.
[Total]	12200.

It comprises 12,200 aurei. Therefore, if 50,000 pounds cost 12,200 aurei, or 122,000 Iulii, one pound will cost 2 11/25 Iulii, as you see here.

Lib.	Iul.	Lib.	Iul.
50000.	122000.	1?	2 11/25.

So if he sells single pounds at 4 11/25 Iulii, he will gain 2 Iulii on each pound.

REGVLA SOCIETATVM. [The Rule of Partnership]

CAPVT XX.

[Margin: The rule of partnership: when it is used, and how it is carried out. How many times the rule of three must be applied within the rule of partnership. When there is a diversity of times in the rule of partnership, what is to be done. — Quaest. 1.]

Next comes the rule of partnership [regula societatum], which has an immense use among merchants, and which rests entirely on the rule of three, as will become clear from the examples set out. It is used when several persons enter a partnership, each contributing a certain sum of money, and it works as follows. The sums of all are gathered into one total, and the collected number is placed in the first position of the rule of three; the second position is occupied by the common profit or loss that arises from the money of all; and the third position, finally, is held by the individual sums, and so on — so that the rule of three must be applied as many times as there are persons who have entered the partnership. But when a diversity of times intervenes, each person's money must be multiplied by its own time before the sums of all are gathered into one total. Then these products are to be collected into one sum, so as to obtain the first number for the rule of three; and the third position will be occupied by the individual products of multiplying each person's money by its time, the common profit or loss being again placed in the middle position. This will be clear from the examples, the first of which is as follows.

Quaest. 1. Four merchants, having formed a partnership, gained 6,000 aurei at certain fairs [nundinae]. The first of them contributed only 60 aurei, the second 100, the third 120, and the fourth 200. The question now raised is how much each ought to receive from that profit, account being taken of the money he laid out. First of all, the sum of everyone's money must be gathered: it is 480 aurei. Then the rule of three must be set up four times, thus: if 480 aurei (which is the money collected from everyone's contributions) gained 6,000 aurei, what will 60 aurei gain, what 100, what 120, and what 200, which the individuals laid out? As appears here:

Aur. 480 · Lucr. aur. 6000 ::

Aur.		Lucr. aur.	
60?	fiunt	750.	Pri.

Aur.	Lucr. aur.	
100?	1250.	Secun.
120?	1500.	Tert.
200?	2500.	Quar.
	6000	

Once the operation is carried out as the rule of three direct, you will find that the first should receive 750 aurei, the second 1,250, the third 1,500, and the fourth 2,500. The test of this will be whether the profits of all, gathered into one sum, make up the whole profit — as you see has been done in the proposed example.

Quaest. 2. Three merchants, having bought goods, loaded a ship. The first's goods cost 300 aurei, the second's 500, the third's 180. Then, a heavy storm having arisen, the heavier goods, which were worth 400 aurei, were thrown into the sea. They agreed among themselves that this loss [iactura] should be shared. How much loss, then, will each bear in proportion to his share of the goods? Let the aurei of all be gathered into one sum, and let the collected number, 980, be placed in the first position of the rule of three, the common loss in the second, and the individual sums in the third, as you see here:

Aur. 980 · Damn. aur. 400 ::

Aur.		Damn. aur.	
300?	fiunt	122 440/980.	prim.
500?		204 80/980.	secun.
180?		73 460/980.	tertij.

So the first will lose 122 440/980 aurei, the second 204 80/980, and the third 73 460/980.

(This opening continues the Rule of Partnerships [Regula Societatum], Cap. XX, with a further series of worked partnership problems, numbered 3 through 13.)

3. [Margin: Quaest. 3.] Three men wish to buy 4,000 pounds [librae] of sugar, valued at 5,000 [recte: 500] aurei. The first, however, wants 1,300 pounds, the second 1,460, and the third the remaining 1,240 pounds. How much, then, will each pay? Reason as follows: if 4,000 pounds are worth 500 aurei, what are 1,300 worth, what 1,460, and what 1,240 pounds — the amounts the several men wish to take? You will find that the first pays 162½ aurei, the second 182½, and the third 155, as you see here.

Lib.	Aur.	given (Lib.)	yield (Aur.)
4000	500	1300	162½ — first
		1460	182½ — second
		1240	155 — third

4. [Margin: Quaest. 4.] Three men entered a partnership and gained 1,000 aurei. The first laid out 200 aurei and withdrew them after 8 months. The second contributed 450 aurei and took them back after 6 months. The third, finally, brought 500 aurei and left them in the business for 10 months. How much, then, will each receive from the profit, account being taken of his money and of his time? Multiply each man's money by his own time, and collect the resulting products into a single sum to serve as the first term of the rule of three; let the profit occupy the second place, and each of those three products the third. In our example, the first man's money for his time gives 1,600; the second's for his time, 2,700; the third's for his time, 5,000. The sum of these products is 9,300. So the example will stand thus.

Lucr.	aur.	given	yield (Aur.)
9300	1000	1600	172 4/93 — first
		2700	290 30/93 — second
		5000	537 59/93 — third

5. [Margin: Quaest. 5.] Three men, having entered a partnership, gained 1,000 aurei. The first put in 300 aurei for 10 months. The second brought 700 aurei, the third 800 aurei. Now the first received 500 aurei from the profit, the second 300, and the third 200. For how long, then, were the moneys of the two later partners in the business? Since, as was said in the preceding problem, each man's money must be multiplied by his own time, we multiply the first man's money by his time and get 3,000; and from this product his profit arises. To learn, therefore, from what products the profits of the two later partners arise, we argue: if 500 aurei (which is the first man's profit) come from 3,000, from what will the profits of the two later partners — 300 and 200[?] — arise? As appears here.

Lucr.	aur.	given	yield
500	3000	300	1800 — second
		200	1200 — third

The second man's time multiplied by his money therefore makes 1,800, and the third's makes 1,200. So if we divide 1,800 by 700 — that is, by the second man's

money — we find $2\frac{4}{7}$ months, during which the second man's 700 aurei were laid out toward the profit. Likewise, if we divide 1,200 by 800 — that is, by the third man's money — we find $1\frac{1}{2}$ months for the third.

You will confirm that this is so if you set the partnership up as follows. Three men, having formed a partnership, put in: the first 300 aurei for 10 months; the second 700 aurei for $2\frac{4}{7}$ months; the third, finally, 800 aurei for $1\frac{1}{2}$ months. How much, then, will each receive from the profit in proportion to his money and his share of the time? Multiplying each man's money by his own time, we make from the first man's money for his time 3,000; from the second's for his time, 1,800; from the third's for his time, 1,200. These three products make the sum 6,000. So the example will stand thus.

Lucr.	aur.	given	yield
6000	1000	3000	500 — first
		1800	300 — second
		1200	100 [recte: 200] — third

Here you see that the profits of the several men have come out just as they were stated in the problem. The times of the two later partners have therefore been correctly determined.

6 [Margin: Quaest. 6.] Four men entered a partnership to run for two years, and gained 10,000 aurei. At the start of the partnership the first brought 3,000 aurei; and after the eighth month had passed he withdrew 1,000 of them; then, as the twentieth month began, he laid out a further 1,200 aurei. The second, from the beginning, gave 2,400 aurei; and after 6 months had elapsed he withdrew 800; but at the start of the sixteenth month he again brought forward 1,400 aurei. The third at the opening of the partnership put up 2,000 aurei, and when 7 months had passed took back all his money; but at the beginning of the eighteenth month he again put in 1,600 aurei. The fourth, finally, at the opening of the seventh month laid out 1,800 aurei; and after 4 completed months took back 900 aurei; but at the start of the seventeenth month he again contributed 1,500 aurei. How much, then, will each receive from the common profit in proportion to his money and his time? Here one must carefully determine how much money each man laid out, and for how long, so that we may always multiply the money by the time, and so forth. To make this plainer, we shall work out the proposed example as follows.

Since the first man gave 3,000 aurei at the start of the partnership and took back 1,000 after 8 full months, it is clear that he had 3,000 aurei out in the common business for 8 months. Multiplying 3,000 by 8, then, we make 24,000. And because after 8 months had elapsed he withdrew 1,000 aurei, it is certain that 2,000 aurei remained in the common venture until the end of the nineteenth month, when he contributed fresh money. Taking 8 months from 19, there remain 11 months, during which he had out only 2,000 aurei; and multiplying 2,000 by 11 we make 22,000. After this, since he again brought in 1,200 aurei at the start of the twentieth month for the rest of the second year, it is plain that, if we add these 1,200 aurei to the 2,000, he had 3,200 aurei in the common business through the remaining 5 months of the two years. Multiplying 3,200 by 5, then, we make 16,000. Gathering now these three products — 24,000, 22,000, 16,000 — into a single sum, we make 62,000, which will be the first man's amount, produced from his moneys and his times.

Again, since the second man had 2,400 aurei out for 6 months — for at the end of the sixth month he withdrew 800 — we multiply 2,400 by 6 and make 14,400. And because he is said to have brought fresh money at the start of the sixteenth month, it is clear that for 9 months, from the start of the seventh month to the end of the fifteenth, he had 1,600 aurei in the common venture — the amount left after 800 is subtracted from 2,400. Multiplying 1,600 by 9, then, we likewise make 14,400. Next, since he is said to have again put in 1,400 aurei at the beginning of the sixteenth month, it is evident that this money was laid out through the remaining 9 months of the two years; and if to it we add the 1,600 aurei still out in the common business, there come 3,000 aurei, which were in common use through those last 9 months. Multiplying 3,000 by 9, then, we make 27,000. Collecting these three products — 14,400, 14,400, 27,000 — into one sum, we make 55,800 for the second man's number, produced from his moneys and his times.

After this, since the third man had 2,000 aurei in for 7 months, then took them back after 7 months had elapsed, we multiply 2,000 by 7 and make 14,000. But because at the start of the eighteenth month he again brought out 1,600 aurei, we multiply 1,600 by 7 (for that is how many months remain of the two years) and make 11,200. Collecting these two products — 14,000 and 11,200 — into one sum, we produce 25,200, the number produced from the third trader's moneys and times.

Since, finally, the fourth man from the start of the seventh month put in 1,800 aurei for 4 months, we multiply 1,800 by 4 and make 7,200. But because, when the 4 months were finished, he took back 900 aurei, 900 aurei were left, which

were in the business for 6 months, from the start of the eleventh month to the end of the sixteenth, when he brought fresh money. Multiplying 900 by 6, then, we make 5,400. And since at the start of the seventeenth month he again put in 1,500 aurei until the end of the two years, then, adding the 900 aurei still in common use, there come 2,400. Multiplying 2,400 by the 8 remaining months, then, we make 19,200. Collecting these three products — 7,200, 5,400, 19,200 — into one sum, we make 31,800 for the number produced from the fourth trader's moneys and times.

Now, gathering these four numbers — 62,000, 55,800, 25,200, 31,800, which have been produced from the several men's moneys and times — into one sum, we make 174,800 for the first term of the rule of three. In the second place stands the common profit, and in the third the number produced from each man's moneys and times, as was said in the fourth problem. So the example will stand thus.

—	—	given	yield
174800	10000	62000	3546 1592/1748 [?] — first
		55800	3192 384/1748 [?] — second
		25200	1441 1132/1748 — third
		31800	1819 388/1748 — fourth

7. [Margin: Quaest. 7.] Three men enter a partnership. The first contributes 400 aurei. The second, 300 aurei and 86 baiocchi [copper coins]. The third, 1,000 aurei, 7 giulii [Iulii], and 9 baiocchi. In the course of the business they met with bad luck, and out of the whole sum suffered a loss of 100 aurei. What, then, is each man's loss? Reducing everything to baiocchi, we get 40,000 baiocchi for the first, 30,086 for the second, and 100,079 for the third, whose whole sum is 170,165. So the example will stand thus.

	Baioch.		Damn. baioc.
If	170165	give	10000

—	given	yield (baiocchi)
Quid	40000	2350 112250/170165 [?]
	30086	1768 8280/170165
	100079	5881 49635/170165 [?]

8. [Margin: Quaest. 8.] Three men entered a partnership. The first brought 200 aurei and left them in the partnership for 12 months. The second contributed 240 aurei. The third brought a gold necklace [torques], whose price he took back after 10 months had elapsed. The profit gained came to 138 aurei. When the due distribution was made, the first had 60 aurei, the second 48, and the third 30. For how many months, then, did the second leave his money invested in the venture, and at what value was the gold necklace reckoned, in order that the said shares of profit should fall to each? Since each man's money must be multiplied by his own time, we multiply the first man's 200 aurei by 12 months, and make 2,400; and from this number he received 60 aurei. To learn, then, from what the second man earned his profit of 48 aurei, say: if 60 aurei came from 2,400, from what do 48 aurei come? As you see here.

—	—	given	yield
60	2400	48	1920

You will find 1,920, which is the number produced from the second man's 240 aurei for his time. Dividing this number 1,920 by 240, then, there come 8 months, for which the second man's money stayed in the business. Again, to learn from what the third man got his profit of 30 aurei, say: if the first man's profit of 60 aurei comes from 2,400, from what does the third man's profit of 30 aurei arise? Or: if the second man's profit of 48 aurei arose from 1,920, from what will the third man's profit of 30 aurei come? As you see here.

—	—	given	yield
60	2400	30	1200
48	1920	30	1200

For you will always find the number 1,200, which is produced from the third man's 10 months multiplied into his own money — that is, into the price of the necklace. Dividing this number 1,200 by the 10 months, then, there comes out the value of the gold necklace, 120 aurei, which the third man brought to the business for 10 months.

You will recognize that this is so if you set the partnership up as follows. Three men, having entered a partnership, gained 138 aurei. The first contributed 200 aurei for 12 months; the second 240 aurei for 8 months; and the third 120 aurei for 10 months. How

much, then, is owed to each from the profit? For if each man's money is multiplied by his own time, you will find the profits of the several men, as was said in the problem — as appears here.

Lucr.	aur.	given	yield
5520	138	2400	60 — first
		1920	48 — second
		1200	30 — third

9. [Margin: Quaest. 9.] Three men, having entered a partnership to run for a year, gained a certain sum of aurei. The first laid out 1,000 aurei straight from the beginning. The second contributed a number of aurei — I do not know how many — after 2 full months. The third, finally, 4 months after the second, likewise contributed a number of aurei unknown to me. When the partnership had ended, however, they shared the profit equally. What, then, did the second and the third contribute to that partnership? Multiplying the first man's 1,000 aurei by the 12 months for which he left them in the venture, there come 12,000. And just as much must arise from the second man's money for his time, and from the third man's for his time; since they are to have equal profits. Now since the second left his money in the business for 10 months, if we divide 12,000 by 10 we find that the second man's money was 1,200 aurei; and if we divide by the 6 months for which the third laid out his money, we discover that the third man's money was 2,000 aurei. For by this reasoning, from each man's money for his time the number 12,000 is produced, which will occupy the third place in the rule of three, and so all three will have equal profits, whatever that common profit may be. For if the common profit had been 900 aurei, and these three numbers — 12,000, 12,000, 12,000, produced from the several men's moneys for their times — were collected into one sum, the example would stand thus.

—	—	given	yield
36000	900	12000	300
		12000	300
		12000	300

10. [Margin: Quaest. 10.] Three men in a common venture gained 190 aurei, which they distributed so that the first man's share was triple the second's and quadruple the third's. The first contributed 80 aurei for 12 months. The second laid out his money for 8 months, and the third for 4. How much, then, did each of the two later partners contribute to this partnership, and what did the several

men receive from the profit? Multiply the first man's money, namely 80 aurei, by his time, that is by 12 months, and you will make 960. Take $\frac{1}{3}$ of this number, namely 320; likewise $\frac{1}{4}$, namely 240. These are the numbers that must be produced from the moneys of the two later partners for their times; for by this reasoning the second man's profit will be $\frac{1}{3}$ of the first man's, and the third man's profit $\frac{1}{4}$ of the same — just as the number 320, from which the second man's profit arises, is $\frac{1}{3}$ of the number 960, from which the first man's profit is begotten, and the number 240, which yields the third man's profit, is $\frac{1}{4}$ of the same number 960. If, then, we divide 320 by the second man's 8 months, we find 40 aurei, which the second put in; and if we divide 240 by the third man's 4 months, there are produced 60 aurei for the third. For by this reasoning the several men's moneys multiplied by their times produce the numbers 960, 320, 240, of which the first is triple the second and quadruple the third. Hence it comes about that the profits too will hold the same proportions, as you see here.

—	—	given	yield (Lucra)
1520	190	960	120
		320	40
		240	30

11. [Margin: Quaest. 11.] Three men, having entered a partnership, contributed 1,520 aurei to a common venture and gained 190, which (account being taken of each man's money as he put it in) they divided among themselves so that the first had 120, the second 40. What, then, did the third have, and what did each contribute to the partnership? If you subtract the profit of the first and of the second from the whole profit, there is left the third man's profit of 30 aurei. With the several men's profits known, then, argue: if the whole profit of 190 aurei arose from the common money of 1,520 aurei, from what comes the first man's profit of 120 aurei, and the second's of 40 aurei, and the third's of 30 aurei? You will find that the first brought 960 aurei, the second 320, and the third 240, as you see here.

—	—	given	yield
190	1520	120	960 — first
		40	320 — second
		30	240 — third

You may test this by saying: if 1,520 (which is the sum of the moneys the several men contributed) earns 190, what will 960, 320, and 240 earn? You will find the profits 120, 40, and 30.

12. [Margin: Quaest. 12.] Three men, having formed a partnership, contributed 1,520 aurei to a venture, on which they gained 190 aurei. When the distribution was made, the first received 1,080 aurei, made up of the money he put in and the profit that fell to him. Likewise the second received 360 aurei, and the third 270. How much, then, did each put in, and how much did each gain? Make one sum from the money they all put in and the common profit, which is 1,710. Then say: if 1,710 — the money of all together with the profit — comes from 1,520, the money of all, from what will 1,080 come, the number that contains both the first man's money and his profit? And from what will 360 arise, which contains the second man's money and profit? And from what will the number 270 be produced, containing the third man's money and profit? For in this way you will find the moneys that the several men put in, as is plain here.

—	—	given	yield
1710	1520	1080	960 — first
		360	320 — second
		270	240 — third

Now if each man's money is taken away from the number that fell to him, the profit alone is left. So you will find the first man's profit to be 120 aurei, the second's 40, and the third's 30.

13 [Margin: Quaest. 13.] Two men in a common partnership gained 200 aurei, of which 50 fell to the first; the second, however, contributed twice as much as the first, and 8 aurei besides. How much, then, did each lay out? Since the first gained 50 aurei, it is clear that the second, who put in twice as much, gained 100; and hence that the remaining 50 aurei, left over from the whole profit of 200 aurei, are the profit on the 8 aurei that the second brought in besides. So, to find the money that each put in, say: if the 50 remaining aurei come from the 8 aurei that the second contributed besides, from what...

(Problem 13 breaks off here and continues on the following page.)

(The Rule of Partnership [Regula Societatum], Cap. XX, continued. The worked schemes printed with the text are Clavius's rule-of-three illustrations, introduced by "as you see here" [vt hic vides]. Each is reproduced as a small table: the two operands stated in the lead-in, then the queried terms — marked with "?" in the

source — paired with the results the source labels "fiunt" ["they become"]. Every numeral is transcribed as printed.)

by which the 50 gold pieces [aur. = aurei] that the first partner gained are produced, and by which the 100 gold pieces that the second gained? For by this method you will find that the first put in 8 gold coins and the second 16, as appears here.

Rule of three (50. → 8.):

Term	Becomes
50?	8.
100?	16.

If, then, you add the 8 to the second partner's 16 gold pieces, you get the 24 gold pieces that the second contributed to the partnership. The check on this is as follows: gather into a single sum the 8 gold pieces and the 24 that the two together contributed — which is 32 — and say: if 32 has earned 200, what will 8 earn, and what will 24? You will find that the first partner's profit is 50 gold pieces and the second's 150, as you see here.

Rule of three (32. → 200.):

Term	Becomes
8?	50.
24?	150.

14. [Margin: Quæst. 14.] Two men formed a partnership; the first laid out 120 gold pieces and the second 180. They took on an agent [procurator] on the condition that he should receive 10 per 100 out of the profit. The profit came to 1000 gold pieces. What, then, was the agent to have, and what each of the two partners? Say: if 100 give 10 to the agent, what will 1000 give? You find 100 gold pieces, which are owed to the agent at the rate of 10 per 100. Once these 100 gold pieces are taken from the whole profit — that is, from the 1000 — there remain 900 gold pieces for the profit of the two together. So say: if the 300 gold pieces that the two put in have earned 900 gold pieces, what will the 120 gold pieces earn, and what the 180? — as set out here.

Rule of three (300. → 900.):

Term	Becomes
120?	360.
180?	540.

15. [Margin: Quæst. 15.] Three men entered a partnership and earned 1520 gold pieces. The first contributed 1080 gold pieces and the second 360; the third laid out just enough capital that 240 gold pieces of the profit should fall to him. What, then, did this third partner put in, and what did each of the two earlier partners earn? Subtract the 240 gold pieces the third earned from the whole profit of 1520 gold pieces, so that 1280 gold pieces remain for the profit of the first two. Then say: if the 1440 gold pieces that the first and second laid out earn 1280 gold pieces, what will the 1080 gold pieces of the first earn, and what the 360 gold pieces of the second? You find the first partner's profit to be 960 and the second's 320, as you see here.

Rule of three (1440. → 1280.):

Term	Becomes
1080?	960.
360?	320.

For in this way the profits of all together make up 1520 gold pieces. And to obtain the capital laid out by the third partner, say: if the profit of the first two, 1280 gold pieces, comes from the 1440 gold pieces they brought into the partnership, from what will the third's profit of 240 gold pieces come? You find 270 gold pieces, as you see here.

Rule of three: 1280. and 1440. — 240? becomes **270**.

16. [Margin: Quæst. 16.] Three men laid out equal sums of money and earned 1000 gold pieces in one year. The first left his money in the joint venture for 7 months; the second withdrew his after 6 months; the third kept his invested until the end of the year. What, then, will each receive of the profit? Collecting all the months for which each had his money invested, so that the sum comes to 25, say: if 25 earn 1000, what will 7 months earn, and what 6, and what 12? — as done here.

Rule of three (25. → 1000.):

Term	Becomes
7?	280.

Term	Becomes
6?	240.
12?	480.

That this is correct is plain, since the profits of all together make up the 1000 gold pieces they were said to have earned.

You may nevertheless check it this way. Suppose each had laid out 100 gold pieces, and multiply these by the time of each, so that they become 700, 600, 1200. Gathering these numbers into a single sum, which is 2500, say: if 2500 earn 1000, what will 700, 600, and 1200 earn? You will find the same profits as before, as appears here.

Rule of three (2500. $\rightarrow$ 1000.):

Term	Becomes
700?	280.
600?	240.
1200?	480.

17. [Margin: Quæst. 17.] Four men together earned 340 gold pieces, which they divided among themselves in proportion to the capital contributed, in this way: as often as the second had 5, the third had 7; and as often as the third had 7, the fourth had 11; and finally, as often as the fourth had 9, the first had 13. Now the first contributed 286 gold pieces. What, then, did the other three lay out, and what did each carry off from the profit? Here the proportions of the profits are expressed, and therefore of the sums of money from which the profits arise — for the profits are proportional to the capital laid out. Since, then, the first is to have as often 13 as the fourth 9, the ratio of the capitals laid out will be the same as 13 to 9, because the same number multiplying 13 and 9 produces the capital of each, given that it must be contained as many times in the one as 13, and as many times in the other as 9. So say: if 13 give 286 gold pieces, which the first put in, what will 9 give? You find 198 gold pieces for the contribution of the fourth, as you see here.

Rule of three: 13. and 286. — 9? becomes **198**.

Here you see that 9 is contained in 198 as many times as 13 in 286.

Now, since the fourth is said to be going to have as often 11 as the third has 7, the ratio of 198 — namely the fourth's capital to the third's capital — will be as 11 to

7. So say: if 11 give 198, what will 7 give? You find the third's contribution to be 126 gold pieces, as is established here.

Rule of three: 11. and 198. — 7? becomes **126**.

Here too it is clear that 7 is contained in 126 as many times as 11 in 198.

Again, since the third must have as often 9 as the second has 5, the ratio of 126 — the third's capital to the second's capital — will be as 9 to 5. So say: if 9 give 126, what will 5 give? You find the second's contribution to be 70 gold pieces, as is plain here.

Rule of three: 9. and 126. — 5? becomes **70**.

Here too it appears that 5 is found in 70 as many times as 9 in 126.

Now that we have the sums each contributes, we shall find their profits just as in the other partnerships. For gathering the money of all into the sum 680, we shall say: if 680 earn 340, what will 286, 70, 126, and 198 earn — the sums that the first, second, third, and fourth put in? — as you see here.

Rule of three (680. → 340.):

Term	Becomes
286?	143. (first)
70?	35. (second)
126?	63. (third)
198?	99. (fourth)

Here indeed you plainly see that all the profits make up 340, and that 13 is contained in 143 as often as 9 in 99, and 5 in 35 as often as 9 in 63; likewise 7 in 63 as often as 11 in 99.

18. [Margin: Quæst. 18.] Three men wish to divide 760 gold pieces among themselves on this condition: that as often as the first has 10, the second shall have 7 and the third 2. How much, then, is each to receive? Add together 10, 7, and 2 to get 19. Then say: if 19 give 760, what will 10, 7, and 2 give? — as you see here.

Rule of three (19. → 760.):

Term	Becomes
10?	400. (first)

Term	Becomes
7?	280. (second)
2?	80. (third)

19. [Margin: Quæst. 19.] Four men wish to divide 785 gold pieces among themselves by this rule: that as often as the first has 10, the second shall have 7; and as often as the second has 14, the third shall have 3; and finally, as often as the third has 12, the fourth shall have 9. What, then, will each receive? To make the working easier, we must begin from the last, namely the fourth, who is supposed to have 9 just once. The third, then, will likewise have 12 just once. Now, since as often as the third has 3 the second must have 14, if we divide 12, the third's number, by 3, we find the quotient 4, which shows that 3 is contained four times in 12. So we shall multiply 14 by the said quotient 4 to find 56, the second's number, in which 14 is contained as many times as 3 in 12. And since, as often as the second has 7, the first must have 10, if we divide 56, the second's number, by 7, we find the quotient 8, which indicates that 7 is contained eight times in 56. We shall therefore multiply 10 by this quotient 8 to produce 80, the first's number, in which 10 is contained as many times as 7 in 56. And so the parts of the given number must have the proportions of these numbers: 80, 56, 12, 9. For in this way the first will have 10 as often as the second 7, and the second 14 as often as the third 3, and the third 12 as often as the fourth 9. Gathering these numbers, then, into a single sum, which will be 157, say: if 157 give 785, what will 80, 56, 12, and 9 give? — as you see here.

Rule of three (157. → 785.):

Term	Becomes
80?	400. (First)
56?	280. (Second)
12?	60. (Third)
9?	45. (Fourth)

The same proposed problem can be solved another way. Since when the first has 10 the second has 7, we shall set down 10 for the first and 7 for the second. Then, because when the second has 14 the third has 3, we shall say: if 14 of the second are 7, what will 3 of the third be? We find $1\frac{1}{2}$; and the position of the second to the position of the third will have the same ratio as 7 to $1\frac{1}{2}$ — that is, 14 will be in 7 as many times as 3 in $1\frac{1}{2}$. Again, because when the third has 12 the fourth has 9, we shall say: if 12 of the third are $1\frac{1}{2}$, what will 9 of the fourth be? We find [the

position of the third to the position of the fourth to be] as $1\frac{1}{2}$ to $1\frac{1}{8}$ — that is, 12 will be in $1\frac{1}{2}$ as many times as 9 in $1\frac{1}{8}$. Now gathering these numbers 10, 7, $1\frac{1}{2}$, $1\frac{1}{8}$ into a single sum, we shall make 19 $\frac{5}{8}$. So we shall say: if 19 $\frac{5}{8}$ give 785, what will 10, 7, $1\frac{1}{2}$, and $1\frac{1}{8}$ give? — as you see here.

Rule of three (19 $\frac{5}{8}$. → 785.):

Term	Becomes
10?	400. (first)
7?	280. (second)
$1\frac{1}{2}$?	60. (third)
$1\frac{1}{8}$?	45. (fourth)

20. [Margin: Quæst. 20.] Four captains [duces], six ensigns [signiferi], and 100 soldiers, in the sack of a certain city, burst into a house where they found 72400 gold pieces, which they divided among themselves so that, as often as each captain took 8, each ensign took 5 and each soldier 3. What, then, fell to each out of that plunder? Multiply the number of captains, that is 4, by 8 — namely by the number each captain is to have as often as the others 5 and 3 — and you get 32. Likewise multiply 6, the number of ensigns, by 5, and 100, the number of soldiers, by 3, and you get 30 and 300. Joining these three numbers, 32, 30, 300, together to make 362, say: if 362 give 72400, what will 32, 30, and 300 give? — as you see here.

Rule of three (362. → 72400.):

Term	Becomes
32?	6400.
30?	6000.
300?	60000.

And so the four captains received 6400 gold pieces out of that plunder, the 6 ensigns 6000, and the 100 soldiers 60000 — all of which make up the sum found, 72400 gold pieces. Now, if we divide the captains' 6400 gold pieces by 4, the number of captains, we find that each had 1600 gold pieces. Likewise, if we divide the ensigns' gold pieces by 6, we find that each got 1000 gold pieces. And finally, if we distribute the soldiers' gold pieces among 100, we find that each received 600 gold pieces. Here you plainly see that 8 is contained in 1600 as many times as 5 in 1000 and 3 in 600 — namely two hundred times.

21. [Margin: Quæst. 21.] A certain man at his last breath, who had a daughter and a son who was said to have fallen in war, left his estate of 18088 gold pieces to be divided between his wife and daughter in such a way that the wife should have $\frac{2}{3}$ and the daughter $\frac{1}{3}$; but should the son by chance return, the son was to have $\frac{2}{3}$. Now it happened that the son did come back. By what reasoning, then, is the said estate to be divided, so that the testator's will is satisfied? It is certain that this problem cannot be understood as the words literally sound, for if the son receives $\frac{2}{3}$, the wife cannot have $\frac{2}{3}$ and the daughter $\frac{1}{3}$. For this reason all arithmeticians inter-

—pret the testator's will to be that the son should have twice as much as the wife, and the wife twice as much as the daughter. This the ratio of the fractions $\frac{2}{3}$ and $\frac{1}{3}$, which is a doubling, seems to indicate (for the fraction $\frac{2}{3}$ contains the fraction $\frac{1}{3}$ twice). And so the number 18088 must be cut into three parts such that the first contains the second twice and the second the third twice — that is, parts that stand in a continued double proportion — which will be done in this way. Put the third part as 1; then the second will be 2 and the first 4, which all together make 7. So say: if 7 give 18088, what will 4, 2, and 1 give? — as you see here.

Rule of three (7. → 18088.):

Term	Becomes
4?	10336. (the son's)
2?	5168. (the wife's)
1?	2584. (the daughter's)

22. [Margin: Quæst. 22.] Three men found a purse [crumena] with 3042 gold pieces, which they divided among themselves in this way: the first took $\frac{1}{2}$, the second $\frac{1}{3}$, and the third $\frac{1}{4}$. What, then, did each get? Here too it is plain that the problem cannot be understood as it literally sounds. For if the first had taken $\frac{1}{2}$ and the second $\frac{1}{3}$, the third could not have taken $\frac{1}{4}$, since these three fractions make more than a whole — namely $1 \frac{1}{12}$. So the sense is that the given number should be distributed into three parts standing in the same ratios to one another as the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$. To do this, let a number be found that is measured by the denominators. The smallest such number is 12, found by the method set out in Cap. 10. Of this number take $\frac{1}{2}$, namely 6; likewise $\frac{1}{3}$, namely 4; likewise $\frac{1}{4}$, namely 3; add these parts together to get 13. So say: if 13 give 3042, what will 6, 4, and 3 give? — as you see here.

Rule of three (13. → 3042.):

Term	Becomes
6?	1404. (first)
4?	936. (second)
3?	702. (third)

The check will be made thus. Reduce the given fractions to the same denomination, namely to $\frac{6}{12}$, $\frac{4}{12}$, $\frac{3}{12}$. These fractions will then have the same proportions as their numerators have; and the same proportions are had by the three numbers found — 1404, 936, 702 — as appears.

23. [Margin: Quæst. 23.] Three men, having found a casket [loculus] with 1407 gold pieces, made the distribution thus: the first took $\frac{1}{2}$, the second $\frac{3}{5}$, the third $\frac{8}{11}$. What, then, did each get? Here too the sense is that the given number should be cut into three parts proportional to the given fractions; otherwise the problem would be impossible. Having found, then, by Cap. 10, the smallest number containing the said fractions — 110 — take its half, 55, and three-fifths, 66, and eight-elevenths, 80, and gather these into a single sum, 201, and say: if 201 give 1407, what will 55, 66, and 80 give? — as appears here.

Rule of three (201. → 1407.):

Term	Becomes
55?	385. (first)
66?	462. (second)
80?	560. (third)

The check will be made as in the preceding problem. For when the given fractions are reduced to the same denomination, namely to $\frac{55}{110}$, $\frac{66}{110}$, $\frac{80}{110}$, the three numbers found will have the same proportions as these fractions — that is, as their numerators — as is established.

24. [Margin: Quæst. 24.] Four men wish to divide 396 gold pieces among themselves so that the first has $\frac{1}{2}$ and 10 besides; the second $\frac{3}{5}$ less 20; the third $\frac{1}{3}$ and 8 besides; and finally the fourth $\frac{1}{4}$ less 6. How much, then, will each receive? In problems of this kind, subtract from the whole sum the numbers that are to be received over and above the stated parts, and add to that sum the others that are to fall short of the stated parts. Thus here, subtract 10 and 8 so that 378 remain, then add back 20 and 6 to make 404. Then, having found the smallest number containing the given fractions — 60 — of which $\frac{1}{2}$ is 30, $\frac{3}{5}$ is 36, $\frac{1}{3}$ is 20, and $\frac{1}{4}$ is 15, all of which make 101, say: if 101 give 404 (which number is produced by the

addition and subtraction of the given numbers from the whole sum, 396), what will 30, 36, 20, and 15 give? — as you see here.

Rule of three (101. → 404.):

Term	Becomes
30?	120. (first)
36?	144. (second)
20?	80. (third)
15?	60. (fourth)

These four numbers found, then, have the same proportions as the given fractions, but gathered into a single sum they make 404, not 396, as the problem requires. But if to the first you add 10, making 130, and from the second you subtract 20, leaving 124, and to the third likewise you add 8, making 88, and from the fourth finally you subtract 6, leaving 54, these four numbers will make 396. And in order that they may keep the said proportions, the numbers that were added must first be subtracted, and those that were subtracted added back — so that 130 truly has to 124 the proportion that $\frac{1}{2}$ has to $\frac{3}{5}$, if first the 10 is taken away from the former and 20 added to the latter — so that it may rightly be said that the number 130 contains $\frac{1}{2}$ and 10 besides, while the number 124 contains $\frac{3}{5}$ less 20, and so forth.

25. [Margin: Quæst. 25.] There is a cistern having at the very bottom three unequal pipes [fistulæ]: the largest, when opened, drains all the water in 2 hours; the middle one, when opened, drains it all in 3 hours; and the smallest, when opened, drains all the water in 6 hours. In how long a time, then, with all three pipes open, will all the water drain out, assuming the water flows through each pipe from beginning to end always at the same rate? Taking the smallest number that the times stated in the problem — namely the hours 2, 3, and 6 — will measure, which here is 6, say: if the largest pipe empties one cistern in 2 hours, how many cisterns will it empty in 6 hours? You find 3. Likewise, if the middle pipe empties one cistern in 3 hours, how many cisterns will it empty in 6 hours? You find 2. Likewise, if the smallest pipe empties one cistern in 6 hours, how many cisterns will it empty in 6 hours? You find 1, as you see here.

Hours	Cistern	Hours	Cisterns
2.			3.
3.	1.	6?	2.

Hours	Cistern	Hours	Cisterns
6.			1.

Now that these three found numbers — 3, 2, 1 — have been gathered into one to make 6, say: if six cisterns are emptied in 6 hours, in how long will one be emptied? You will find: in one hour. You will check this by the following reasoning. If the largest pipe empties the whole cistern in 2 hours, the middle one in 3, and the smallest in 6, what part of the cistern will each pipe empty in one hour? — as is set out here:

Hours	Cistern	Hour	Cistern
2, 3, 6	1	1?	$\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$

For you will find that the largest pipe empties $\frac{1}{2}$ of the cistern, the middle one $\frac{1}{3}$, and the smallest $\frac{1}{6}$ — which parts together make up one whole cistern.

The same problem can also be posed in this way. There is a cistern with three unequal pipes at its top: the largest fills the cistern in two hours [2.duabus], the middle one in 3, and the smallest in 6. In how much time, then, will they all fill it together? You will find: in one hour.

By the same reasoning it can also be posed thus. There are three workmen: the first completes a certain job in 2 years, the second in 3, and the third in 6. In how much time, then, will they all together finish the same job? You will find: in one year.

[Margin: Another solution of this problem.]

Moreover, problems of this kind can also be solved in the following way. Investigate by the rule of three how much water each single pipe drains off in one hour, and then collect the three found numbers into a single sum. For if that sum turns out to be 1, then one hour is required for all the pipes to empty the whole cistern; but if it is not 1, the time required is found by the rule of three — as will be clear in this example. There are three workmen: the first completes a certain job in 6 years, the second in 9, and the third in 18. In how much time, then, will they all together finish the same job? Say: if the first in 6 years completes 1 job, and the second in 9, and the third in 18, how much will each complete in 1 year? — as you see here:

Years	Job	Year	Of the job
6, 9, 18	1	1?	$\frac{1}{6}$ (of the first), $\frac{1}{9}$ (of the second), $\frac{1}{18}$ (of the

Years	Job	Year	Of the job
third)			

Now the three found numbers together make $\frac{1}{3}$. So say: if $\frac{1}{3}$ of the job requires 1 year, how many years will the whole job require? — and you will find 3 years. You will check this, as above, as you see here:

Years	Job	Years	(yield)
6, 9, 18	1?	3?	$\frac{1}{2}$ (first), $\frac{1}{3}$ (second), $\frac{1}{6}$ (third)

For you will find that in 3 years the first completes $\frac{1}{2}$ of the job, the second $\frac{1}{3}$, and the third $\frac{1}{6}$ — which parts all together make 1.

If the earlier example were worked out by this method, the aim would be reached at the very first step, because in one hour the whole cistern is emptied, as is plain from the working of the check on that example.

[Margin: Problem 26.]

26. There is a cistern having a pipe at its mouth through which it is filled in 4 hours; but it also has, at the very bottom, another pipe through which it is emptied in 6. If, then, water flows in and out continuously, in how long will the cistern be filled? First one must investigate what part of the cistern (under that condition) is filled in one hour, which is done thus: if in four hours 1 cistern is filled, how much will be filled in 1 hour? You will find $\frac{1}{4}$ of the cistern. Again: if in 6 hours 1 cistern is emptied, how much will be emptied in 1 hour? You will find $\frac{1}{6}$ of the cistern. So if you take $\frac{1}{6}$ away from $\frac{1}{4}$, there remains $\frac{1}{12}$ of the cistern — and just this much of the cistern is filled in one hour. So say: if $\frac{1}{12}$ of the cistern requires one hour, what will 1 cistern require? You will find 12 hours; and in that many hours the whole cistern will be filled.

You will check by this reasoning that this is true. If in 4 hours 1 cistern is filled, in 12 hours how many cisterns will be filled? You will find 3 cisterns. Likewise, if in 6 hours 1 cistern is emptied, in 12 how many cisterns will be emptied? You will find 2 cisterns; and if you take these from the 3 found, there remains 1 full cistern.

But suppose someone says: the cistern is filled through the upper pipe in 3 hours and emptied through the lower one in 8 hours. The problem is solved in the same way, if you say: if in 3 hours 1 cistern is filled, how much will be filled in 1 hour? You will find $\frac{1}{3}$ of the cistern. Likewise, if in 8 hours one cistern is emptied, how much will be emptied in 1 hour? You will find $\frac{1}{8}$ of the cistern. So if you subtract

$\frac{1}{8}$ from $\frac{1}{3}$, there remain $\frac{5}{24}$ of the cistern; and just this much of the cistern is filled in one hour. So say: if $\frac{5}{24}$ of the cistern require 1 hour, what will 1 cistern require? You will find $4\frac{4}{5}$ hours, in which time the whole cistern will be filled. You will prove this thus. If in 3 hours 1 cistern is filled, in $4\frac{4}{5}$ hours how many cisterns will be filled? You will find $1\frac{3}{5}$. Likewise, if in 8 hours one cistern is emptied, in $4\frac{4}{5}$ hours how many cisterns will be emptied? You will find $\frac{3}{5}$; and if you take these from $1\frac{3}{5}$, there remains 1 full cistern.

Perhaps problems of this kind will be solved more briefly if one asks how great a part of the cistern would be filled, in those hours, if nothing flowed out. This will be done thus in the earlier problem. Say: if 6 hours empty 1 cistern, how much will 4 hours empty? You will find $\frac{2}{3}$; and if you take these from 1 (for we assume that 1 cistern is filled in 4 hours, if nothing flowed out), there will remain $\frac{1}{3}$, which is filled in 4 hours. So say: what will 1 cistern require? You will find 12 hours, as before.

In the later problem, say: if 8 hours empty 1 cistern, how much will 3 hours empty? You will find $\frac{3}{8}$; and if you subtract these from 1 (for we assume that 1 cistern is filled in 3 hours, if nothing flowed out), there will remain $\frac{5}{8}$, which are filled in 3 hours. So say again: if $\frac{5}{8}$ of the cistern demand 3 hours, what will 1 cistern demand? You will find $4\frac{4}{5}$ hours, as before.

The Rule of Alligation [Regula Alligationis]. Chapter XXI [Caput XXI].

[Margin: What the rule of alligation is. Problem 1.]

Arithmeticians sometimes mix together various goods of various prices, so that, once a certain mean price has been fixed, all of them are bought at that price. This they accomplish by a certain rule, which they call the rule of alligation, because the various goods are in a way "bound" [alligentur] to a single price — as will become clear from the following examples.

1. There are two kinds of wine: 1 measure of the first is worth 20 baiocchi [baioch.], while 1 measure of the second sells for 12 baiocchi. How much, then, must be taken of each, so that 1 measure is worth 15 baiocchi?

[Margin: How the rule of alligation is carried out.]

Place one price beneath the other, and to their left set the fixed price, which is the mean between the two given prices. Then compare each given price with the fixed price, and place the difference of each to the right of the prices — but

crosswise, that is, the difference of the greater price next to the smaller price, and the difference of the smaller next to the greater; and collect these differences into a single sum, as you see in the example.

(Mean price)	Prices	Differences
15	20	3
	12	5
	8 (Sum of the differences)	

After this, set up the rule of three twice, so that the sum of the differences occupies the first place, one measure the second, and each difference the third, as you see here:

- $8 : 1 : \{3?, 5?\} \rightarrow \text{yield } \{3/8 \text{ of the first, } 5/8 \text{ of the second}\}$

So say: if the sum of the differences, 8, gives 1 measure, what will the individual differences 3 and 5 give? You will find that $3/8$ of one measure is to be taken from the first wine, and $5/8$ from the second; and thus there will be made 1 measure from both together, worth 15 baiocchi. You will prove it thus. Say: if 1 measure of the first wine is worth 20 baiocchi, what will $3/8$ be worth? Likewise, if 1 measure of the second wine is worth 12 baiocchi, what will $5/8$ be worth? — as you see here:

- $1 \cdot 20 \cdot 3/8 \rightarrow \text{yield } 7 \frac{1}{2}$
- $1 \cdot 12 \cdot 5/8 \rightarrow \text{yield } 7 \frac{1}{2}$

For you will find that the two prices make 15 baiocchi, just as is proposed.

[Margin: Problem 2.]

2. There are two kinds of impure silver. 1 pound [libra] of the first is worth 30 gold pieces [aur.]; 1 pound of the other is worth 24 gold pieces. So that 1 pound may be worth 28 gold pieces, how much is to be taken of each silver? Carrying out the alligation as in the preceding problem, say: if the sum of the differences, 6, gives 1 pound, what will the individual differences 4 and 2 give? — as you see here:

(Mean price)	Prices	Differences
	30	4

(Mean price)	Prices	Differences
	24	2
		6 (Sum of the differences)

- $6 : 1 : \{4?, 2?\} \rightarrow \text{yield } \{2/3 \text{ of the first, } 1/3 \text{ of the second}\}$

For in this way you will have 1 pound made from both silvers, worth 28 gold pieces. To test it, say: if 1 pound of the former silver is worth 30 gold pieces, what will $2/3$ of one pound be worth? Likewise, if 1 pound of the latter silver is worth 24 gold pieces, what will $1/3$ be worth? — as appears here:

- $1 \cdot 30 \cdot 2/3 \rightarrow \text{yield } 20$
- $1 \cdot 24 \cdot 1/3 \rightarrow \text{yield } 8$

And thus 1 pound will amount to 28 gold pieces, as is proposed.

[Margin: Problem 3. Note.]

3. 1 pound of pepper is worth 4 giulii [Iul.]; 1 pound of cloves [garyophilli], 3 giulii; 1 pound of cinnamon [cinnamomi], 6 giulii; 1 pound of saffron [croci], 10 giulii; 1 pound of ginger [zingiberis], 8 giulii. How much, then, is to be taken of each, so that 1 pound comes to 7 giulii? When more than two things are proposed for alligation, the alligation can be made in various ways, provided that each is bound at least once. For any price can be bound to any other, or even to several others, at the mean price — provided, however, that the fixed price is a mean between the two that are bound to it, or at least equal to one of them, as will appear in this example, which we shall work out by several alligations.

[Margin: That the alligation of one and the same example can be made in various ways.]

First, then, we shall bind the prices of pepper and ginger to the mean price; their differences are 3 and 1, placed crosswise. Next the prices of cloves and saffron, whose differences are 4 and 3, likewise placed crosswise. Finally, since only cinnamon is left, we shall bind its price with the price of ginger, for example, whose differences are 1 and 1, also written crosswise. The sum of all the differences is 13. But the differences opposite ginger make 4.

[Margin: What is to be done when several differences are placed opposite the same price.]

For the several differences placed opposite the same price must always be collected into a single sum. Now say: if the sum of the differences, 13, gives 1 pound, what will the individual differences 1, 3, 1, 4, 4 give? — as you see here:

(Mean price)	Prices	Differences
7	Pepper 4	1
	Cloves 3	3
	Cinnamon 6	1
	Saffron 10	4
	Ginger 8	3. 1
		13 (Sum of the differences)

- $13 : 1 : \{1?, 3?, 1?, 4?, 4?\} \rightarrow \text{yield } \{1/13 \text{ pepper, } 3/13 \text{ cloves, } 1/13 \text{ cinnamon, } 4/13 \text{ saffron, } 4/13 \text{ ginger}\}$

By this reasoning you will have 1 pound out of all of them, worth 7 giulii. To examine it, say: if 1 pound of pepper is worth 4 giulii, what will 1/13 be worth? Likewise, if 1 pound of cloves is worth 3, what will 3/13 be worth? Likewise, if 1 pound of cinnamon is worth 6 giulii, what will 1/13 be worth? Likewise, if 1 pound of saffron is worth 10 giulii, what will 4/13 be worth? Likewise, if 1 pound of ginger is worth 8 giulii, what will 4/13 be worth? — as you see here:

- $1 : \{4, 3, 6, 10, 8\} : \{1/13?, 3/13?, 1/13?, 4/13?, 4/13?\} \rightarrow \text{yield } \{4/13 \text{ pepper, } 9/13 \text{ cloves, } 6/13 \text{ cinnamon, } 3 \frac{1}{13} \text{ saffron, } 2 \frac{6}{13} \text{ ginger}\}$

For you will find that all the prices make 7 giulii, as is proposed.

The alligation of this same example can also be set up in another way, if the prices of pepper and ginger are bound to the mean price; likewise the prices of pepper and saffron. Then the prices of cloves and ginger, and again the prices of cloves and saffron. Finally the prices of cinnamon and saffron; likewise the prices of cinnamon and ginger — as has been done in this appended example. Nor indeed can more alligations be made in this example.

[Margin: Another alligation of this third problem.]

(Mean price)	Prices	Differences
7	Pepper 4	1. 3
	Cloves 3	1. 3
	Cinnamon 6	3. 1

(Mean price)	Prices	Differences
	Saffron 10	3. 4. 1
	Ginger 8	3. 4. 1
	28 (Sum of the differences)	

- $28 : 1 : \{4?, 4?, 4?, 8?, 8?\} \rightarrow \text{yield } \{4/28 \text{ pepper, } 4/28 \text{ cloves, } 4/28 \text{ cinnamon, } 8/28 \text{ saffron, } 8/28 \text{ ginger}\}$

For in this way you will fuse together 1 pound out of all of them, which comes to 7 giulii. To prove it, say: if 1 pound of pepper is worth 4 giulii, what will 4/28 be worth? Likewise, if 1 pound of cloves is worth 3 giulii, what will 4/28 be worth? and so on — as you see has here been done:

- $1 : \{4, 3, 6, 10, 8\} : \{4/28?, 4/28?, 4/28?, 8/28?, 8/28?\} \rightarrow \text{yield } \{16/28 \text{ pepper, } 12/28 \text{ cloves, } 24/28 \text{ cinnamon, } 2 \text{ } 24/28 \text{ saffron [crco., recte: croc.], } 2 \text{ } 8/28 \text{ ginger}\}$

For you will find that all the prices make up 7 giulii, just as is proposed in the problem.

The alligation of this same example can also be set up by yet another method, if the prices of pepper and saffron are bound; then the prices of cloves and ginger; and last of all the prices of cinnamon and ginger — as may be seen in this example. So say: if the sum of the differences, 13, gives 1 pound, what will the individual differences 3, 1, 1, 3, and 5 give? — just as you see here:

[Margin: Another alligation of this third problem.]

(Mean price)	Prices	Differences
	Pepper 4	3
	Cloves 3	1
7	Cinnamon 6	1
	Saffron 10	3
	Ginger 8	4. 1
	13 (Sum of the differences)	

- $13 : 1 : \{3?, 1?, 1?, 3?, 5?\} \rightarrow \text{yield } \{3/13 \text{ pepper, } 1/13 \text{ cloves, } 1/13 \text{ cinnamon, } 3/13 \text{ saffron, } 5/13 \text{ ginger}\}$

For in this way you will have 1 pound out of all of them for 7 giulii. You will prove it as above.

[Margin: What is to be observed in the alligations of several things.]

You see, then, that the alligation can be made in various ways, if the things to be bound are more than two — provided always that the mean price is less than one price being bound and greater than another, or equal to one and greater or less than another. But although from the various alligations you always obtain the proposed weight of the things to be mixed for the fixed mean price, you will nevertheless not always take the same weights from the things to be mixed, as is clear from the examples proposed.

[Margin: Problem 4. When an alligation problem is impossible.]

4. 1 ell [vlna] of red cloth is worth 4 gold pieces; 1 ell of green cloth is worth 6 gold pieces; and 1 ell of black cloth is worth 10 gold pieces. Someone wants 80 ells of all of them together for 480 gold pieces. How much, then, will he take of each cloth? In problems of this kind one must first investigate the price of one ell of all the cloths mixed together, which is done thus in our example. If 80 mixed ells are worth 480 gold pieces, what will 1 ell be worth? You will find 6 gold pieces, which is the price of 1 ell — a mean between the price of the cheaper cloth and that of the dearer. But if a price were found that is not a mean, the problem would be impossible. As, for instance, if someone were to say: someone wants 80 ells of all the cloths for 300 gold pieces, or for 900 gold pieces — the problem would be impossible. For if 80 are worth 300 gold pieces, 1 ell will be worth $3\frac{3}{4}$, which price is less than the price of the cheaper cloth. Therefore neither

[Continuation of *Epitome Arithmeticae Practicae*, Cap. XXI — Rule of Alligation (*Regula Alligationis*). Question 4, the three cloths, is completed at the head of this page.]

nor could anyone obtain 80 ells [ulnae] of the cheaper cloth for 300 gold pieces [aurei] — so far is it from being the case that from all three cloths together one could again take 80 ells [for that price]. Again, if 80 ells are worth 900 gold pieces, then one ell will be worth $11\frac{1}{4}$ gold pieces, a price higher than that of the more expensive cloth. Anyone, therefore, will buy more than 80 ells of the dearer cloth for 900 gold pieces, and consequently many more still if he wishes to take some ells from all of them together. But let us return to our example.

Once the mean price [pretium medium] of a single ell has been found, carry out the alligation as in the preceding cases, just as has been done here. For first we alligated the prices 4 and 10 to the mean price 6; then the prices 6 and 10. So say: if the sum of the differences, 10, yields 80 ells (for he wants to take that many ells

from the threefold cloth), what will the individual differences 4, 4, and 2 yield? — as has been done here.

Rule of three (the sum of the differences, 10, gives 80 ells):

Difference	Ells	Cloth
4	32	red (rubri)
4	32	green (viridis)
2	16	black (nigri)

Alligation table (mean price 6):

Prices [pretia]	Differences [differentiae]
red 4	4.
green 6	4.
black 10	2.0.
Sum of differences	10.

For thus, from those three cloths, 80 ells will be taken for 480 gold pieces (since this mean price of a single ell was found from the three cloths mixed together). What, then, will 32 ells of red cloth, 32 of green, and 16 of black be worth? — as you see here.

Rule of three (1 ell costs 6 gold pieces):

Ells	Gold pieces	Cloth
32	192	red
32	192	green
16	96	black

And you will find that all the prices make up 480 gold pieces.

Now, if we had not alligated the price of the green cloth with the price of the black cloth, but with the price of the red, the following alligation would result. But we should have found other numbers. For we should have said: if the sum of the differences, 8, gives 80 ells, what will the individual differences 4, 2, and 2 give? — as you see here.

Alligation table (mean price 6, alternative):

Prices	Differences
red 4	4.0.
green 6	2.
black 10	2.
Sum of differences	8.

Rule of three (the sum of the differences, 8, gives 80 ells):

Difference	Ells	Cloth
4	40	red
2	20	green
2	20	black

The proof will be made as before, if you say: one ell is worth 6 gold pieces; what will 40 ells of red cloth, and 20 of green, and 20 of black be worth? For you will find that all the prices make up 480 gold pieces.

5. [Margin: Question 5.] There are four kinds of wine: an amphora of the first is worth 21 baiocchi [baiocchi], of the second 27, of the third 30, and of the fourth 40. Someone wants to mix 300 amphorae out of these, on the terms and condition that each amphora be worth 33 baiocchi. How much, then, will he take of each? Here the three earlier prices must necessarily be alligated with the last one to the mean price of 33 baiocchi, since those three are smaller than it, as you see in the given example. So say: if the sum of the differences, 42, gives 300 amphorae, what will the individual differences 7, 7, 7, and 21 give? — as is clear from the example.

Alligation table (mean price 33):

Prices	Differences
21	7.
27	7.
30	7.
40	12.6.3.
Sum of differences	42.

Rule of three (the sum of the differences, 42, gives 300 amphorae):

Difference	Amphorae	Wine
7	50	first

Difference	Amphorae	Wine
7	50	second
7	50	third
21	150	fourth

For thus you will make up 300 amphorae, each of which will cost 33 baiocchi. And to prove it, say: if the sum of the differences, 42, gives 1 amphora, what will the individual differences 7, 7, 7, and 21 give? — as you see here.

Rule of three (the sum of the differences, 42, gives 1 amphora):

Difference	Amphora	Wine
7	$\frac{1}{6}$	first
7	$\frac{1}{6}$	second
7	$\frac{1}{6}$	third
21	$\frac{1}{2}$	fourth

And so you will have one amphora mixed from that fourfold wine. So say again: if 1 amphora of the first wine is worth 21 baiocchi, what will $\frac{1}{6}$ of an amphora be worth? Likewise, if 1 amphora of the second is worth 27, what will $\frac{1}{6}$ be worth? Likewise, if 1 amphora of the third is worth 30, what will $\frac{1}{6}$ be worth? Likewise, if 1 amphora of the fourth is worth 40, what will $\frac{1}{2}$ be worth? — as you see here.

Value of the fraction taken from each wine:

Wine	Price of 1 amphora	Fraction	Value
first	21	$\frac{1}{6}$	$3\frac{1}{2}$
second	27	$\frac{1}{6}$	$4\frac{1}{2}$
third	30	$\frac{1}{6}$	5
fourth	40	$\frac{1}{2}$	20

All these prices make 33 baiocchi, as proposed.

The proof, however, can also be set up more briefly in this way. Since one amphora ought to be worth 33 baiocchi, 300 amphorae will be worth 9,900 baiocchi. So we shall say: if 300 amphorae are worth 9,900 baiocchi, what will 50 amphorae of the first wine be worth, and what 50 of the second, and 50 of the third, and 150 of the fourth? — as you see here.

Rule of three (300 amphorae are worth 9,900 baiocchi):

Amphorae	Baiocchi	Wine
50	1650	first
50	1650	second
50	1650	third
150	4950	fourth

For you will find that all the prices make up 9,900 baiocchi.

6. [Margin: Question 6.] Someone wants to buy, for 400 gold pieces, 400 pounds [librae] of assorted spices — namely cloves, pepper, cinnamon, ginger, nutmeg, and saffron — whose prices per pound are, in giulios [Iulii], 6, 7, 9, 11, 12, and 16. How many pounds of each, then, will he take, so as to have 400 pounds for 400 gold pieces? Here, as was said in Question 4, we must first determine the mean price of a single pound, to which the alligation must be made, in this way: if 400 pounds are worth 400 gold pieces, what will 1 pound be worth? You will find 1 gold piece, that is, 10 giulios. And since, as we said, various alligations can be made, we shall first alligate the cloves with the ginger and the saffron; then the pepper with the nutmeg and the saffron; and finally the cinnamon with the nutmeg — as you see done here. Then we shall say: if the sum of the differences, 32, gives 400 pounds, what will the individual differences 7, 8, 2, 4, 4, and 7 give? — as you see here.

Alligation table (mean price 10):

Prices	Differences
cloves (garyoph.) 6	1.6.
pepper (piper) 7	2.6.
cinnamon (cinnam.) 9	2.
ginger (zingib.) 11	4.
nutmeg (nuc. myr.) 12	3.1.
saffron (crocus) 16	4.3.
Sum of differences	32.

Rule of three (the sum of the differences, 32, gives 400 pounds):

Difference	Pounds	Spice
7	87½	cloves
8	100	pepper

Difference	Pounds	Spice
2	25	cinnamon
4	50	ginger
4	50	nutmeg
7	87½	saffron

For you will find 400 pounds worth 400 gold pieces, each pound costing 10 giulios; which you will verify as was said in the preceding question.

Several other, different alligations can be made in this question, as appears from the four examples set out here.

Alligation, first alternative (mean price 10):

Prices	Differences
6	1.2.6.
7	1.2.6.
9	1.2.6.
11	4.3.1.
12	4.3.1.
16	4.3.1.
Sum of differences	51.

Alligation, second alternative (mean price 10):

Prices	Differences
6	1.
7	2.
9	6.
11	4.
12	3.
16	1.
Sum of differences	17.

Alligation, third alternative (mean price 10):

Prices	Differences
6	6.

Prices	Differences
7	2.
9	1.
11	1.
12	3.
16	4.
Sum of differences	17.

Alligation, fourth alternative (mean price 10):

Prices	Differences
6	2.
7	1.
9	6.
11	3.
12	4.
16	1.
Sum of differences	17.

For in the first, each of the three earlier prices is alligated with all three of the later ones. In the second, the first with the fourth, the second with the fifth, and the third with the sixth. In the third, then, the first with the fifth, the second with the sixth, and the third with the fourth. In the fourth, finally, the first with the fifth, the second with the fourth, and the third with the sixth. And so, in similar questions, several mutually different alligations can be made.

7. [Margin: Question 7.] Someone wants a silver statue of 300 pounds. Two kinds of silver are offered to him: one pound of the first is worth 30 gold pieces, of the second 20; and he wants to mix them in such a way that one pound comes to 24 gold pieces. How much, then, will he take of each silver, so as to have 300 pounds, each pound costing 24 gold pieces? The alligation will stand as you see here. So say: if the sum of the differences, 10, gives 300 pounds, what will the individual differences 4 and 6 give? — as you see here.

Alligation table (mean price 24):

Prices	Differences
30	4.

Prices	Differences
20	6.
Sum of differences	10.

Rule of three (the sum of the differences, 10, gives 300 pounds):

Difference	Pounds	Silver
4	120	first silver
6	180	second silver

For thus you will find 300 pounds, each of which is worth 24 gold pieces; which you will verify as was said in Question 5.

THE RULE OF FALSE OF SIMPLE POSITION. CHAPTER XXII.

Among the other rules of arithmetic, the rule of false [regula falsi] holds no last place. It is so called not because it teaches us anything false, but because it shows how to draw out the true from a false assumption. This it does by positing any number one likes that is supposed to satisfy the proposed question, even though in fact it does not. This rule is twofold. One kind is called *of simple position*, in which a single position is made of just one number believed to satisfy the question; the other is called *of double position*, in which a double position is made of two numbers, each of which is supposed to satisfy the question.

[Margin: Why the rule of false is so called.] [Margin: The rule of false is twofold.]

Moreover, there is a great difference between these two rules. [Margin: The difference between the two rules of false. Note.] Whatever is solved by the former can also be explained by the latter, but not the reverse. For there are almost countless questions solved by the latter which can in no way be explained by the former. For under the former fall only those questions in which such parts or numbers are expressed as have the same proportion among small numbers as among large — such as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, and so on; likewise double, triple, quadruple numbers, and the like. And so it would be enough if only the latter rule were set forth. But because very many questions are explained much more briefly by the former than by the latter, we shall set out both in a few words, beginning with the former, which is the easier.

Given, then, any question to be solved by the rule of false of simple position, let any number be posited that is believed to satisfy the question, and let it be tested according to the sense of the question. [Margin: How the rule of false of simple position is carried out.] And if everything agrees, the posited number will be the one sought; but if the matter stands otherwise, the position of the assumed number was false. Therefore from this false number the true one must be drawn out by means of the rule of three, as will be shown in the examples.

1. [Margin: Question 1.] Three men decide to buy a certain house for 7,200 [recte: 2,700] gold pieces. The second wants to give twice as much as the first, and the third three times as much as the second. How much, then, will each spend? In this question nothing else is sought than that the number 2,700 be divided into three parts under this condition: that the second be double the first, and the third triple the second. So suppose the first gives any number of gold pieces you like, say 6. Then, by the sense of the question, the second will give 12, since this number is double the first; and the third will give 36, since this number is triple the second. But all these three numbers make 54 gold pieces, whereas they ought to make 2,700. So say: if 54 arose from the false position of 6 gold pieces for the first, from what true position will 2,700 arise? You will find that the first gave 300 gold pieces, and accordingly the second 600 and the third 1,800 — which three numbers together make 2,700.

The money of the second and of the third could also be found from both positions, if one were to say: if 54 arose from the false position of 12 gold pieces for the second, and from the false position of 36 gold pieces for the third, from what will 2,700 arise? For you would find the money of the second to be 600 gold pieces and of the third 1,800. But it is better to investigate, by the rule of three, the money of only one of them; for from this the money of the others will be drawn out with little trouble, according to the sense of the question.

You would have found exactly the same numbers if you had posited some other number than 6 for the first, and accordingly some other than 12 for the second and some other than 36 for the third.

2. [Margin: Question 2.] A certain man, asked how much money he had in his chest, replied that he did not know, but that he had learned this for certain from his steward: that $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$ of his money made 4,700 gold pieces. How much money, then, did he have? Here too nothing else is sought than a number whose $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$ together make 4,700. So suppose he has 60 gold pieces. (To avoid fractions as far as possible, one should always take a number that contains the fractions expressed in the question, as we taught in Chapter 10 — such as 60

here.) Its $\frac{1}{3}$ is 20, its $\frac{1}{4}$ is 15, and its $\frac{1}{5}$ is 12, which parts together make 47. But they ought to make 4,700. So say: if 47 arose from 60, the number we falsely posited, from what will 4,700 arise? We shall find 6,000 — and that is how many gold pieces he had in the chest. For its $\frac{1}{3}$ contains 2,000, its $\frac{1}{4}$ 1,500, and its $\frac{1}{5}$ 1,200, which parts together make 4,700.

3. [Margin: Question 3.] A schoolmaster, asked how many pupils he had, replied: "If I once again had as many as I have, plus $\frac{1}{2}$ of them, plus $\frac{1}{3}$ and $\frac{1}{4}$, and one more besides, I would have 112." How many pupils, then, did he have? This question, as posed, cannot be solved by this rule, because the unit mentioned in the last place cannot have the same proportion with $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ and the double of a small number as with the same parts and the double of a large number. If, however, the 1 is subtracted from the 112 that is to be produced in the question, the question as proposed will be solved. For then nothing else is sought than a number which, taken twice together with $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, makes 111 (for if 1 is finally added, it will become 112). So suppose he had 12 pupils. This number contains the given fractions. If, then, as many pupils again are added, he will have 24, namely twice as many as before; and if there are further added $\frac{1}{2}$ (namely 6), $\frac{1}{3}$ (namely 4), and $\frac{1}{4}$ (namely 3), he will have 37. But they ought to be 111, so that with the addition of 1 he would have 112. So say: if 37 arose from 12, from what will 111 arise? You will find that he had 36 pupils. For if as many again are added, he will have 72; and if $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ are added to these — namely 18, 12, and 9 — they will make 111; and finally, with 1 added, they will make 112.

4. [Margin: Question 4.] Someone buys a horse, a garden, and a house for 5,000 gold pieces, on the condition that the garden cost four times as much as the horse, and the house five times as much as the garden. For how much, then, did he buy the horse, for how much the garden, and for how much the house? Here it is sought that the given number 5,000 be distributed into three parts, such that the second is quadruple the first and the third quintuple the second. And this question is similar to the first. So suppose the horse is worth 30 gold pieces; on that supposition the garden will be worth 120 and the house 600, which numbers together make 750. But they ought to make 5,000. So say: if 750 arose from 30, from what will 5,000 arise? You will find 200 — and for that many gold pieces the horse was bought; and accordingly the garden was bought for 800 gold pieces and the house for 4,000, which numbers together make up 5,000 gold pieces.

5. [Margin: Question 5.] A man setting out from Venice to Jerusalem spent on the journey $\frac{2}{3}$ and $\frac{1}{5}$ of his money; and on returning home he found that 36 gold pieces were left him. How much money, then, did he take with him? Here a

number is sought from which, if $\frac{2}{3}$ and $\frac{1}{5}$ are taken away, the remainder is 36. Suppose he had 300 gold pieces; if from these you take away $\frac{2}{3}$ (namely 200) and $\frac{1}{5}$ (namely 60), 40 are left. But only 36 ought to have remained. So say: if 40 arose from 300, from what will 36 arise? You will find 270 — and that is how many gold pieces he had. For with $\frac{2}{3}$ removed (namely 180) and $\frac{1}{5}$ (namely 54), 36 are left.

[Margin: Note.] But if it should ever happen that the parts expressed in the question exceed unity — and so cannot be subtracted from the posited number — the question will be impossible. As if someone were to say: "Give me a number such that, if you take from it $\frac{3}{5}$ and $\frac{3}{5}$, 36 are left." That question would be impossible. For $\frac{3}{5}$ and $\frac{3}{5}$ exceed unity, and so cannot be taken away from the posited number 300; because $\frac{3}{5}$ is 180 and $\frac{3}{5}$ is likewise 180, which parts together make 360 — a number that cannot be subtracted from 300.

6. [Margin: Question 6.] A number is sought whose $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$ make 522. Suppose that number is 60; its $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$ — namely 30, 20, 15, 12, and 10 — make 87. But we want 522. So say: if 87 arose from 60, from what will 522 arise? You will find 360; for this number's $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$ — namely 180, 120, 90, 72, and 60 — make 522.

7. [Margin: Question 7.] A man, asked how much money he had, replied that he had so many gold pieces that, if to them were added $\frac{1}{2}$ of them, and $\frac{1}{3}$, and $\frac{1}{4}$, and 100 gold pieces besides, they would make 300 gold pieces. How much money, then, did he have? So that this ques- [continues]

the question is solved by this rule: 100 gold pieces must be subtracted from 300, as we said in the third question, and a number sought such that, when its half, third, and quarter are added, they make 200 — for then, with 100 added, 300 results, just as the question proposes. So put that number at 24. Its half is 12, its third 8, its quarter 6; all these parts added to 24 make 50. But we want them to make 200. So say: if 50 came from 24, from what will 200 come? You find 96, and that was the sum of gold pieces. For its half is 48, its third 32, its quarter 24, all of which make 104; added to 96 they make 200; and if 100 is finally added to that number, 300 results.

8. [Margin: Quæst. 8.] Someone wanting to grind 500 measures (modii) of wheat goes to a miller who has 5 mills, of which the first grinds 7 measures each hour, the second 5, the third 4, the fourth 3, the fifth 1. In how much time, then, will all the wheat be ground if wheat is loaded onto every mill, and how much wheat should be put on each mill? Suppose 4 hours. On that supposition the first mill will finish 28 measures, the second 20, the third 16, the fourth 12, and the fifth 4,

which together make 80. But they ought to be 500. So say: if 80 measures came from 4 hours, from how many hours will 500 measures come? You find 25 hours. For in that many hours the first mill will grind 175 measures, the second 125, the third 100, the fourth 75, and the fifth 25, which are all 500 measures; and to each mill should be assigned as many measures as it itself grinds in 25 hours.

9. [Margin: Quæst. 9.] Someone setting out for the fairs made such a profit on the money he had brought with him that the profit, together with the money brought, came to triple the money brought. Then out of this money, at other fairs, he made so much profit that the profit together with the money brought to those fairs came to five times that money. Finally, out of this money, at other fairs, he made so much profit that the profit together with the money he then had came to four times that money; and he finds that he has 40,000 gold pieces. How much money, then, did he bring to the first fairs? In this question we seek a number which, multiplied by 3, and the product by 5, and this product by 4, makes 40,000. Put that number at 10; if you multiply it by 3 you make 30, for the profit together with the money at the first fairs. If you then multiply 30 by 5 you produce 150, for the profit together with the money at the second fairs. If finally you multiply 150 by 4, making 600, for the profit together with the money at the third fairs. But we said that at the third fairs he had 40,000 gold pieces. So say: if 600 came from 10, from what will 40,000 come? You find $666\frac{2}{3}$, and that is how many gold pieces he brought with him to the first fairs. For if we multiply $666\frac{2}{3}$ by 3 we make 2,000, for the profit and money at the first fairs. If we then multiply 2,000 by 5 we generate 10,000, for the profit and money at the second fairs. And if finally we multiply 10,000 by 4 we produce 40,000, for the profit and money at the third fairs.

10. [Margin: Quæst. 10.] Let a number be investigated such that, when it is multiplied by 4, and the product by 3, and this product by 6, and 10 is then added to this product, 800 is produced. This question cannot be solved by this rule unless first 10 is subtracted from 800, for the reason stated in the third question. So subtract 10 from 800, leaving 790; and this is the number that must be produced by the multiplications specified in the question, since, if 10 is added to it, the number becomes 800. Put the sought number at 10; if you multiply it by 4 you make 40, which number multiplied by 3 makes 120, and this number finally multiplied by 6 produces 720. But 790 ought to be produced. So say: if 720 came from 10, from what will 790 come? You find $10\frac{35}{36}$ [?], and this is the number sought. For if you multiply $10\frac{35}{36}$ by 4 you make $43\frac{8}{9}$ [?], which number if you again multiply by 3 you generate the number $131\frac{2}{3}$ [?], which finally if you multiply by 6 you will produce 790. And with 10 added, it becomes 800.

11. [Margin: Quæst. 11.] A certain old man, asked about his age, replied that he was as many years old that, if half of them were added, and a quarter of the sum were then taken from that sum, he would have 99 years. How many years, then, did he have? Here a number is to be found such that, if half of it is added, and a quarter of the sum is subtracted from that sum, the remaining number is 99. Put that he had 80 years. If, then, half of them — namely 40 — is added, the years become 120; and if a quarter — namely 30 — is taken from these, 90 remain. But they are said to remain as 99. So say: if 90 came from 80, from what will 99 come? You find 88, and that is how many years the old man had. For if you add to them their half — namely 44 — you make 132; and if you take from these a quarter — namely 33 — 99 remain.

12. [Margin: Quæst. 12.] The top of a certain tower appears, 24 palms of it; but someone says that $\frac{1}{3}$ and, besides, $\frac{2}{5}$ of the same tower are hidden by the buildings standing around it. What, then, is the whole height of the tower? Here a number is to be sought such that, if $\frac{1}{3}$ and, besides, $\frac{2}{5}$ are taken from it, 24 remain. Put that number at 30; from which, if you take $\frac{1}{3}$ — namely 10 — and $\frac{2}{5}$ — namely 12 — 8 remain. But we want 24 to remain. So say: if 8 came from 30, from what will 24 come? You find 90, and that is the height of the tower. For if you take $\frac{1}{3}$ and $\frac{2}{5}$ — namely 30 and 36 — 24 remain.

13. [Margin: Quæst. 13.] There is a spear (hasta) of which $\frac{1}{3}$ is white, $\frac{1}{5}$ black, and $\frac{2}{9}$ of a blue color, and there remain 12 palms of red. What, then, is the length of that spear? Here too a number is to be found such that, if $\frac{1}{3}$, $\frac{1}{5}$, and $\frac{2}{9}$ are removed from it, the remaining number is 12. Put that number at 45; from which, if you take $\frac{1}{3}$ — namely 15 — and $\frac{1}{5}$ — namely 9 — and $\frac{2}{9}$ — that is, 10 — 11 remain. But 12 ought to remain. So say: if 11 came from 45, from what will 12 come? You find $49 \frac{1}{11}$, and that is the length of that spear in palms. For its $\frac{1}{3}$ contains $16 \frac{4}{11}$ palms, its $\frac{1}{5}$ contains $9 \frac{9}{11}$ palms, and likewise its $\frac{2}{9}$ contains $10 \frac{10}{11}$ [?] palms; all of which numbers, taken from the spear's length of $49 \frac{1}{11}$ palms, leave 12 palms.

14. [Margin: Quæst. 14.] Someone spent 660 gold pieces for 30 ells of white cloth and 40 ells of black cloth, and it was settled that each ell of black cloth cost twice as much as each ell of white cloth. How much, then, did one ell of white cloth cost, and how much one ell of black? Put that one ell of white cloth cost 4 gold pieces. And since the price of one ell of black cloth is twice as great, one ell of black cloth must have cost 8 gold pieces. From this it follows that 30 ells of white cloth come to 120 gold pieces, and 40 ells of black cloth are worth 320 gold pieces, all of which gold pieces make 440. But we said that he spent 660 gold pieces. So

say: if 440 came from 4, from what will 660 come? You find 6 gold pieces, as the price of one ell of white cloth, and accordingly 12 gold pieces for the price of one ell of black cloth. For in this way 30 ells of white cloth will come to 180 gold pieces, and 40 ells of black cloth will be worth 480, all of which gold pieces make up 660 gold pieces.

The Rule of Double False Position [Regula Falsi Duplicis Positionis]

Chapter XXIII [Caput XXIII]

[Margin: How the rule of double false position is carried out.]

Given any question to be solved by the rule of double false position, take any number, whether small or large, and test it according to the tenor of the question. For if it answers the thing sought, the question will be solved; but if not, note the excess or the defect by which it strays from the truth, together with the letter P or M — of which P signifies "more," M "less," according as the error exceeds the truth or falls short of it. Then take again some other number, either larger than the first or smaller, and test it in the same way, and so on. For from this double position and double error the truth is drawn out in this manner.

[Margin: When both positions exceed the truth, subtraction is used.]

If in both positions the straying from the truth is by excess, or [in both] by defect, subtract the smaller error from the larger, and keep the remaining number as divisor. Then let the number placed first be multiplied by the second error, and the number [placed second by the first error]; and let the smaller product be subtracted from the larger. For if this residual number is divided by the divisor already found — namely, by the difference of the errors — it will give as quotient the sought number, which will satisfy the proposed question.

[Margin: When one position exceeds the truth and the other falls short, addition is used.]

But if in one of the two positions the straying from the truth is by excess, and in the other by defect, the two errors are to be gathered into one sum, so that a divisor is obtained. Likewise the two numbers that are produced by multiplying the placed numbers by the errors, as was said, are to be gathered into one sum, so that the number to be divided is obtained, and so on. This will become clear from the examples.

1. [Margin: Quæst. 1.] Let a number be sought from whose half, $\frac{1}{3}$ and $\frac{1}{4}$ having been subtracted, 300 remains. Let the number 24 be posited — since it has the $\frac{1}{2}$ part named in the question, and its half [12] contains the other named parts, namely $\frac{1}{3}$ and $\frac{1}{4}$, as whole numbers, so far as can be done, to avoid fractions. Such a number is easily found if one takes a number having the later fractions and afterward doubles it. Now this first number taken is customarily placed at the left in the upper part of the cross, and its error also in the lower part at the left, and finally the letter P or M — according as that error exceeds the truth or falls short of it — in the middle of the cross, likewise at the left. The second number taken, with its error and the letter P or M, is customarily placed in the same order at the right of the same cross, as you see done in our example. This number, 24, being so placed, will be tested according to the statement of the question. Its half is 12, from which number $\frac{1}{3}$ and $\frac{1}{4}$ are to be subtracted. Now $\frac{1}{3}$ of the number 12 is 4, and $\frac{1}{4}$ is the number 3; which numbers, subtracted from 12, leave 5. But 300 ought to remain. It has therefore strayed from the truth by a defect of 295 units; and accordingly this error is to be noted with the letter M.

24 96
M X M
295. 280.
15.
divisor

Let 96 be posited second, which will likewise be tested according to the question. Its half is 48; and $\frac{1}{3}$ and $\frac{1}{4}$ of this number 48 are 16 and 12, which subtracted from 48 leave 20. But 300 ought to remain. We have therefore again fallen short of the truth by 280 units, and accordingly this error too is to be noted with the letter M.

Now since in both positions we have fallen short of the truth, the smaller error is to be subtracted from the larger, so that the divisor 15 is obtained, to be placed in the lower part of the cross. Then the number 24, placed first, is to be multiplied by 280, the second error, and the number 96, placed second, by 295, the first error; and the smaller product, 6,720, is to be subtracted from the larger, 28,320, so that the number to be divided, 21,600, remains. For this number, divided by the divisor found, 15, will give the quotient 1,440, which is the sought number. For its half is 720, and the $\frac{1}{3}$ and $\frac{1}{4}$ of this are 240 and 180, which numbers taken from 720 leave 300, as was proposed in the question.

But let us solve the same question by two other numbers, of which one exceeds the truth and the other falls short of it. Let the first number, then, be posited as 4,800. Its half is 2,400, whose $\frac{1}{3}$ and $\frac{1}{4}$ are 800 and 600, which numbers taken

from 2,400 leave 1,000. But 300 ought to remain. We have therefore exceeded the truth by 700 units, and so this error is to be written with the letter P. Let the number 2,400 then be posited. Its half is 1,200, whose $\frac{1}{3}$ and $\frac{1}{4}$ are 400 and 300, which numbers taken from 1,200 leave 500. But 300 ought to remain. We have therefore again exceeded the truth by 200 units. Now, subtracting the smaller error from the larger, the divisor 500 remains; and, the multiplication of the placed numbers by the errors having been done crosswise, as was said, and the smaller product, 960,000, subtracted from the larger, 1,680,000, the number to be divided, 720,000, remains, which divided by 500 will give the quotient 1,440, as before.

4800. 2400
P X P
700. 200.
500.
divisor

Again let us posit as the first number 2,400, which, tested according to the proposed question, will give us an excess of 200, which error is to be written with the letter P. Then let us posit the number 96, which, tested by the same method, will yield a defect of 280, which is to be written with the letter M. Since in one position we exceeded the truth and in the other fell short of it, addition of the errors must be made, so that the divisor 480 is composed. Likewise the two numbers produced from multiplying the placed numbers by the errors crosswise — namely 672,000 and 19,200 — are to be gathered into one sum, so that the number to be divided, 691,200, results. For this number 691,200 divided by 480 gives the quotient 1,440, just as before.

2400. 96.
P X M
200. 280.
480.
divisor

2. [Margin: 2. Quæst.] Alexander the Great, in a familiar conversation which he had one day with a certain philosopher, Callisthenes, happened by chance — as tends to happen — to fall upon the mention of ages, addressing him in this way: "I am older than Hephaestion by two years, while Clytus's age comprises both of ours together, and four years more. From which it comes about that we three together complete 96 years — which is just how many your father is said to have lived. How many years, then, did Alexander, Hephaestion, and Clytus have at that time?" You see here that the number 96 must be distributed into three parts, such that the first exceeds the second by two, and the third exceeds the first two taken

together by four. Or that three numbers are to be found, of which the first exceeds the second by two, and the third exceeds the first two taken together by four, and all three together make up 96. So put that Alexander had 20 years, and accordingly Hephaestion 18, and Clytus

42. For thus Alexander's age exceeds Hephaestion's by 2 years, and comprises the sum [38] of both, and, besides, 4 years, as is proposed in the question. But since these numbers 20, 18, 42 make up only 80, which ought to come to 96, it turns out that we have fallen short of the true value by 16 units. So put again that Alexander's years were 30, and accordingly that Hephaestion had 28, and Clytus 62, which all make up 120. But they ought to make up only 96. We have therefore exceeded the truth by 24 units. Now, addition of the errors between themselves having been made — because the one fell short of the truth and the other exceeded it — they become 40 for the divisor. Likewise, the multiplication 20 by 24 and 30 by 16 having been made, and the products 480 and 480 having been added together, they become 960, which divided by 40 gives the quotient 42 [recte: 24], and that is how many years Alexander the Great then had; and accordingly, per the tenor of the question, Hephaestion 22, Clytus 50, which all together make up 96 years.

20. 30.
 18. 28.
 42. 62.
 M X P
 80. 120.
 16. 24.
 40.
 Divisor

3. [Margin: Quæst. 3.] Three men have a certain sum of money, namely 44 gold pieces. The second has twice as much as the first, and 4 gold pieces besides. The third has as much as the first and second [together], and 6 gold pieces besides. How much, then, does each have? Here you see that the number 44 must be distributed into three parts, such that the second is double the first and contains 4 besides, and the third is equal to the first and the second [together] and contains 6 besides. Or that three numbers are to be found, of which the second contains the first twice and 4 more, and the third contains the first and the second once and 6 besides, while the three numbers together make up 44. So put that the first has 10; on which supposition the second will have 24 — namely twice as much as the first, and 4 more — while the third will have 40 — just as much, that is, as the first and second, and 6 besides — which three numbers make up 74. But they ought to make up only 44. We have therefore exceeded the truth by 30 units.

Then put that the first has 6. The second will therefore have 16 and the third 28, which three numbers make up 50. But they ought to make up only 44. We have therefore again exceeded the truth by 6 units. Now, the subtraction of the smaller error from the larger having been made — because both errors exceeded the truth — the remainder will be the divisor 24. Likewise, the multiplication 10 by 6 and 6 by 30 having been made, and that product, 60, subtracted from this one, 180, the number to be divided, 120, remains; which divided by 24 gives the quotient 5. So the first has that much; the second has 14 and the third 25, which three numbers gathered into one sum make 44.

10. 6.
 24. 16.
 40. 28.
 P X P
 74. 50.
 30. 6.
 24.
 Divisor

If we were to multiply the numbers we posited the second and third to have by the same errors, and so on, we would find the numbers that the second and third have. Thus from 24 by 6 come 144, and from 16 by 30 come 480; and subtracting the former number from the latter, 336 remain; which number divided by 24, the divisor found, gives the quotient 14 for the second's number. Likewise from 40 by 6 come 240, and from 28 by 30 come 840; and subtracting the former number from the latter, the remaining number is 600, which divided by 24 gives the quotient 25 for the third's number. But it is better, once the first's number is found, to investigate the rest according to the statement of the question — namely, in the way in which each falsely posited number was tested. Sometimes, however, it will be more convenient to seek the other numbers by the same method by which the first was sought; as will be evident in question 6.

4. [Margin: Quæst. 4.] Let three numbers be sought which make 60, while the second contains the first twice and 4 more, and the third contains the first and the second and 6 besides. This question is similar to the preceding. Put the first number at 6, and accordingly the second at 16 and the third at 28, which three numbers make 50. But they ought to make 60. An error has therefore been made, by a defect of 10. Put the first number again at 8, and accordingly the second at 20 and the third at 34, which three numbers make 62. But they ought to make 60. We have therefore exceeded the true value by 2 units. Operate as the rule prescribes, and you will find the first number to be $7\frac{2}{3}$ [?], and accordingly the second $19\frac{1}{3}$ [?] and the third 33, which three numbers make up 60.

6. 8.
 16. 20.
 28. 34.
 M X P
 50. 62.
 10. 2.
 12.
 divisor

5. [Margin: Quæst. 5.] Let the number 30 be divided into two parts, of which the first, together with 60, makes a number triple the number composed of the latter part and 20. Put the first part at 20, and accordingly the latter at 10. The first with 60 makes 80, and the latter with 20 makes 30. But the number 80 ought to be triple the number 30, according to the statement of the example. Since, then, the number 90 is triple 30, we have fallen short of the true value by 10 units. Put again the first part at 24, and accordingly the latter at 6. The first with 60 makes 84, and the latter with 20 makes 26. But the number 84, according to the tenor of the question, ought to be triple the number 26. Since, then, the number 78 is triple 26, we have exceeded the truth by 6 units. Operate as the rule prescribes, and you will find the first part to be $22\frac{1}{2}$, and accordingly the latter $7\frac{1}{2}$. For the first with 60 makes $82\frac{1}{2}$, and the latter with 20 makes $27\frac{1}{2}$, of which number the former is triple.

20. 24.
 10. 6.
 M X P
 10. 6.
 16.
 Divisor

This question's solution can be set up in another way. For after we found, in the first position, that the first part 20 with 60 makes 80, and the latter part 10 with 20 makes 30, of which number the former ought to be triple, we must see of what number the number 80 is triple. Now it is triple of $26\frac{2}{3}$, which the number 30 exceeds by $3\frac{1}{3}$. We have therefore exceeded the truth by this number, $3\frac{1}{3}$. Again, after it was found, in the second position, that the first part 24 with 60 makes 84, and the latter part 6 with 20 makes 26, of which number the former ought to be triple, we must see of what number the number 84 is triple. Now it is triple of 28, from which the number 26 falls short by 2 units. We have therefore fallen short of the truth by two. Now operate according to the precepts of the rule, and you will find the first part to be $22\frac{1}{2}$ and the latter $7\frac{1}{2}$, as before. But the first method seems more convenient, since by it fractions are more easily avoided.

20. 24.
 10. 6.
 P X M
 3½. 2.
 5½ [?]
 Divisor

*(continuing Chapter XXIII, the Rule of Two Positions — double false position. In the printed St-Andrew's-cross layouts (×) below, the two trial values ("positions") stand at the upper arms, their two errors at the lower arms; each error is marked P — plus, an excess ("we exceeded") — or M — minus, a defect ("we fell short"). The divisor stands beneath: the **difference** of the errors when both have the same sign, their **sum** when the signs are opposite. All numeric values are transcribed as printed; the excess/defect designations follow the fully legible worked reasoning.)*

[Margin: Quaest. 6.]

6. Three numbers are sought such that the first, added to 73, makes double the remaining two; the second, with 73, makes triple the remaining two; and the third, finally, with 73, makes quadruple the remaining two.

Set the first number to be 1 — or any other odd number, so that when it is added to 73 it makes an even number that can be halved, since the first with 73 must make double the remaining two. Now since 1 with 73 makes 74, which by the terms of the problem must be double the remaining two, the other two together must be 37. And since the second with 73 must make triple the first (which is 1) and the third, the number 37 must — by the preceding problem — be divided into two parts, the earlier of which, with 73, makes triple the number composed of the later part and 1. So before the proposed problem can be solved, another must first be solved, one that arises within the very working.

Suppose, then, that the earlier part is 2 and hence the later 35. The earlier with 73 makes 75; but the later with 1 makes 36, whose triple is not 75 but 108. We therefore fall short of the truth by 33 units, our number 75 being smaller than 108 by that many units. Suppose again that the earlier part is 5 and hence the later 32. The earlier with 73 makes 78, and the later with 1 makes 33, whose triple is not 78 but 99. We therefore fall short again, by 21 units. Now work by the precepts of the rule of false position, and you will find the earlier part to be $10\frac{1}{4}$ and hence the later $26\frac{3}{4}$:

2. 5. ← earlier part (trial)
 M X M
 33. 21. ← defect defect

12.
divisor

→ earlier part $10\frac{1}{4}$, later part $26\frac{3}{4}$.

So if the first number of the proposed problem is 1, the second will be $10\frac{1}{4}$ and the third $26\frac{3}{4}$. For thus the first with 73 makes double the remaining two, and the second with 73 makes triple the remaining two. If, then, the third with 73 makes quadruple the remaining two, the problem will be satisfied. But the third with 73 makes $99\frac{3}{4}$, which is not quadruple the $11\frac{1}{4}$ formed from the first and second — rather, 45 is quadruple of $11\frac{1}{4}$. We have therefore exceeded the truth by this number, $54\frac{3}{4}$.

Now set the first number to be 3, which with 73 makes 76, which ought to be double the remaining two; so the other two will amount to 38. And since the second with 73 must make triple the first (which is 3) and the third, the number 38 must — by the preceding problem — be divided into two parts, the earlier of which with 73 makes triple the number composed of the later part and 3. Suppose, then, that the earlier part is 2 and hence the later 36. The earlier with 73 makes 75, but the later with 3 makes 39, whose triple is not 75 but 117. We therefore fall short of the truth by 42. Suppose again that the earlier part is 23 and hence the later 15. The earlier with 73 makes 96, but the later with 3 makes 18, whose triple is not 96 but 54. We have therefore exceeded the truth by 42. Work by the rule of false position, and you will find the earlier part to be $12\frac{1}{2}$ and hence the later $25\frac{1}{2}$:

2. 23. ← earlier part (trial)
M X P
42. 42. ← defect excess
84.
divisor

→ earlier part $12\frac{1}{2}$, later part $25\frac{1}{2}$.

So if the first number of the proposed problem is 3, the second will be $12\frac{1}{2}$ and the third $25\frac{1}{2}$. For thus the first with 73 makes double the remaining two, and the second with 73 makes triple the remaining two. If, then, the third with 73 makes quadruple the remaining two, the problem will be solved. But the third with 73 makes $98\frac{1}{2}$, which is not quadruple of $15\frac{1}{2}$ — rather, 62 is. We have therefore exceeded the truth by this number, $36\frac{1}{2}$.

Now if you multiply the first numbers crosswise by the errors — and likewise the seconds and the thirds (for the second and third are more conveniently found this way than by trying to draw them out of the first once found, since that would

require using the preceding problem here) — and, the subtraction having been made, divide the residual numbers by the divisor found, $18\frac{1}{4}$, that is, by the difference of the errors (since in both positions the result was an excess), you will find the first number to be 7, the second 17, and the third 23:

1.	3.	← first (trial)
10%.	12%.	← second
26%.	P X P	25%. ← third
54%.	36%.	← excess excess
18%		
divisor		

→ first 7, second 17, third 23.

For the first with 73 makes 80, which is double the remaining two; the second with 73 makes 90, which is triple the remaining two; and the third, finally, with 73 makes 96, which is quadruple the remaining two.

[Margin: Quaest. 7.]

7. A number is sought which, multiplied by 3 and with 10 added to the product, then that sum multiplied by 4 and with 20 added to the product, then that sum multiplied by 5 and with 30 added, then finally that sum multiplied by 6 and with 40 added to the product, yields the number 6700.

Suppose the number to be 2. Multiplied by 3 it makes 6; with 10 added, 16. This sum multiplied by 4 makes 64; with 20 added, 84. Again this sum multiplied by 5 makes 420; with 30 added, 450. This sum finally multiplied by 6 makes 2700; with 40 added, 2740. But this last sum ought to have been 6700. We therefore fall short of the truth by this number, 3960.

Suppose next that the same number is 3. Multiplied by 3 it makes 9; with 10 added, 19. This sum multiplied by 4 makes 76; with 20 added, 96. This again multiplied by 5 makes 480; with 30 added, 510. Finally this sum multiplied by 6 makes 3060; with 40 added, 3100. But 6700 ought to have resulted. We therefore fall short of the truth again, by this number, 3600.

Work by the rule, and you will find the number sought to be 13. For this number multiplied by 3 makes 39; with 10 added, 49. This sum multiplied by 4 makes 196; with 20 added, 216; this sum multiplied by 5 makes 1080; with 30 added, 1110; this sum finally multiplied by 6 makes 6660; with 40 added, 6700.

2.	3.	← trial
P[recte: M]	X	M

3960. 3600. ← defect defect
360.
divisor

→ number sought 13. (Both positions fall short, so the divisor is the difference of the errors, 360; the left-hand marker is printed P but the working makes both a defect.)

[Margin: Quaest. 8.]

8. A certain schoolmaster has so many pupils that if each pays 5 gold pieces [aurei] he is 30 gold short of buying the house he lives in; but if each gives 6, there remain 40 gold beyond the price of the house. How many pupils does he have, and what is the price of the house? Here nothing else is sought than a number which, multiplied by 5, produces a total such that, with 30 added, it equals what is left when the same number is multiplied by 6 and 40 is subtracted from the product.

Suppose, then, that the number of pupils is 30. Multiplied by 5 it makes 150; with 30 added, 180 — so the house will cost that much if he has 30 pupils each paying 5 gold. Now let us see whether beyond this price 40 gold remain over if each pays 6. The same number of pupils, 30, multiplied by 6 makes 180, and nothing is left over beyond the house-price of 180 gold; but 40 gold ought to have remained. We therefore fall short of the truth by the number 40.

Suppose again that the number of pupils is 100. Multiplied by 5 it makes 500; with 30 added, 530 — so the house will cost that much if 100 pupils pay 5 gold. Now let us see whether 40 gold remain beyond this price. The same number of pupils, 100, multiplied by 6 makes 600, and 70 gold remain over beyond the house-price of 530 gold; but only 40 ought to have remained. We have therefore exceeded the truth, and you will find the number of pupils to be 70. For this number multiplied by 5 makes 350; with 30 added, 380 — so that is the price of the house. The same number of pupils, 70, multiplied by 6 makes 420, which exceeds the house-price of 380 gold by that number, 40.

30. 100. ← trial (pupils)
P[recte: M] X P
40. 30. ← defect excess
70.
divisor

→ 70 pupils; house-price 380 gold. (The positions have opposite signs, so the divisor is the sum of the errors, 70; the left-hand marker is printed P but the working makes the first position a defect.)

[Margin: Quaest. 9.]

9. Two men were to divide 60 gold pieces equally between them. But a quarrel arose, and each grabbed what he could. When peace was afterward made, the first laid down $\frac{1}{4}$ of his money and the second $\frac{1}{3}$; and it then came about that the first, taking $\frac{1}{3}$ of the second's, and the second, taking $\frac{1}{4}$ of the first's, each had 30 gold. How much, then, did each first grab?

Suppose the first grabbed 36 gold and hence the second the remaining 24. If the first lays down $\frac{1}{4}$, namely 9 gold, 27 gold are left to him; and if we add $\frac{1}{3}$ of the second's — which he is said to have laid down, namely 8 gold — we make 35 gold for the first. But he ought to have only 30. We have therefore exceeded the true value by 5. Now suppose the first stole 12, and hence the second the remaining 48. If the first lays down $\frac{1}{4}$, that is 3 gold, 9 gold remain to him; and if we add $\frac{1}{3}$ of the second's, say 16 gold, we make 25 gold for the first. But he ought to have 30. We therefore fall short of the true value by 5 units.

Work by the rule, and you will find the first grabbed 24, and the second therefore 36. For if the first lays down $\frac{1}{4}$, namely 6 gold, and to the remaining 18 adds $\frac{1}{3}$ of the second's, namely 12, he will have 30 gold. Likewise if the second, having laid down $\frac{1}{3}$ — namely 12 gold — adds to his remaining 24 the $\frac{1}{4}$ of the first's, that is 6 gold, he will have 30 gold:

36. 12. ← first's trial
24. 48. ← second
P X M
5. 5. ← excess defect
10.
divisor

→ the first grabbed 24, the second 36.

Moreover, from the second's position we shall draw out the truth in the same way. For in the earlier position, if the second lays down $\frac{1}{3}$, namely 8 gold, and to his remaining 16 joins $\frac{1}{4}$ of the first's, that is 9 gold, he will have 25 gold, which ought to be 30. We therefore fall short of the truth by 5 units. But in the later position, if the second lays down $\frac{1}{3}$, namely 16 gold, and to his remaining 32 adds $\frac{1}{4}$ of the first's, namely 3 gold, he makes 35 gold, which ought to be only 30. We have therefore again exceeded the true value by 5 units. Work by the rule, multiplying the errors by the second's positions, and so on, and you will find that the second stole 36 gold and the first 24:

24. 48. ← second's trial
M X P

5. 5. ← defect excess
10.
divisor

→ the second grabbed 36, the first 24.

[Margin: Quaest. 10.]

10. Two men were to divide 100 gold between them equally; but a quarrel arose, and each snatched what he could. Peace being then made, the first laid down $\frac{1}{3}$ of his money and the second laid down $\frac{1}{5}$; the first took the latter's $\frac{1}{5}$ and the second the former's $\frac{1}{3}$, and when this was done each had 50. How much, then, did each first grab?

Suppose the first grabbed 30 gold and hence the second 70. $\frac{1}{3}$ of the first's is 10, which if he lays down, 20 are left to him. $\frac{1}{5}$ of the second's is 14, which if given to the first, the first will have 34 gold. But he ought to have 50. We therefore fall short of the truth by this number, 16. Suppose again the first stole 60, and therefore the second 40. $\frac{1}{3}$ of the first's is 20, which if he lays down, 40 remain to him. $\frac{1}{5}$ of the second's is 8, which if we give to the first, the first will have 48. But he ought to have 50. In this position too, then, we fall short of the truth, by the number 2. Work by the rule, and you will find that the first stole $64 \frac{2}{7}$, and hence the second $35 \frac{5}{7}$:

30. 60. ← first's trial
70. 40. ← second
M X M
16. 2. ← defect defect
14.
divisor

→ the first grabbed $64 \frac{2}{7}$, the second $35 \frac{5}{7}$.

For $\frac{1}{3}$ of the first's is $21 \frac{3}{7}$, which if he lays down, $42 \frac{6}{7}$ remain to him. $\frac{1}{5}$ of the second's is $7 \frac{1}{7}$, which if he lays down, [there remain to the second $28 \frac{4}{7}$]. Now if $\frac{1}{5}$ of the second's, namely $7 \frac{1}{7}$, is given to the residue of the first, which was $42 \frac{6}{7}$, [it makes 50]; and if $\frac{1}{3}$ of the first's, namely $21 \frac{3}{7}$, is given to the residue of the second, which was $28 \frac{4}{7}$, the second too will have 50 — as was proposed in the problem.

[Margin: Quaest. 11.]

11. Two men so distribute 100 gold that if the first lays down $\frac{1}{3}$ and the second $\frac{1}{4}$, and the aggregate of these parts is cut in two and half given to each, then — the

[remaining] number being restored — the numbers become equal, namely 50 and 50. How great, then, are the two men's shares?

Suppose the first's share is 60 and hence the second's 40. If the first lays down $\frac{1}{3}$, namely 20, 40 remain to him. If $\frac{1}{4}$ of the second's, namely 10, is added to $\frac{1}{3}$ of the first's — that is, to 20 — they become 30; and if we give $\frac{1}{2}$ of this aggregate, namely 15, to the residue of the first, which was 40, we make 55. But we were to make only 50. We have therefore exceeded the truth by this number, 5. Suppose next that the first has 24 and hence the second 76. (I have set down these numbers because the former has its $\frac{1}{3}$ and the latter its $\frac{1}{4}$ without fractions.) If the first lays down $\frac{1}{3}$, namely 8, 16 remain to him. If $\frac{1}{4}$ of the second's, namely 19, is added to $\frac{1}{3}$ of the first's, namely 8, they become 27; and if $\frac{1}{2}$ of this aggregate, that is $13\frac{1}{2}$, is given to the residue of the first, which was 16, the first will have $29\frac{1}{2}$. But he ought to have 50. We therefore fall short of the truth by $20\frac{1}{2}$. Work now by the rule, and you will find the first's share to be $52\frac{16}{17}$, and the second's share $47\frac{1}{17}$:

60. 24. ← first's trial
40. 76. ← second
P X M
5. 20%. ← excess defect
25%.
divisor

→ the first's share $52\frac{16}{17}$, the second's $47\frac{1}{17}$.

For $\frac{1}{3}$ of the first's is $17\frac{11}{17}$, which if he lays down, $35\frac{5}{17}$ remain to him. $\frac{1}{4}$ of the second's is $11\frac{12}{17}$, which if he lays down, $35\frac{5}{17}$ remain to him. And the aggregate of the first's $\frac{1}{3}$ and the second's $\frac{1}{4}$ — that is, of $17\frac{11}{17}$ and $11\frac{12}{17}$ — is $29\frac{6}{17}$, whose $\frac{1}{2}$, namely $14\frac{11}{17}$, added to the residue of the first (that is, to $35\frac{5}{17}$), makes 50; and added to the residue of the second (that is, to $35\frac{5}{17}$), makes 50. And 50 and 50.

[Margin: Quaest. 12.]

12. Let the number 1000 be divided into two parts, the greater of which exceeds the lesser by this number, 49.

Suppose the greater part is 600 and hence the lesser 400. But the former exceeds the latter by 200, whereas we wanted the excess to be 49. We have therefore exceeded the truth by the number 151. Suppose next the greater part is 550 and hence the lesser 450. But the former exceeds the latter by 100, whereas we wanted the excess to be 49. We have therefore again exceeded the truth by the

number 51. Work by the rule, and you will find the greater part to be $524\frac{1}{2}$ and hence the lesser $475\frac{1}{2}$. For the former exceeds the latter by the given number, 49:

600. 550. ← greater part (trial)
 400. 450. ← lesser part
 P X P
 151. 51. ← excess excess
 100.
 divisor

→ greater part $524\frac{1}{2}$, lesser $475\frac{1}{2}$.

[Margin: Quaest. 13.]

13. Someone has two golden cups and one lid worth 150 gold, which added to the first cup makes its price triple the price of the second cup, but added to the second cup makes it of the same price as the first. What, then, do those two cups cost? Here two numbers are sought, of which the first with 150 is triple the second, and the second with 150 is equal to the first.

Suppose the first cup costs 30 gold. (I choose this number because when 150 is added it makes a number that has a third part without a fraction.) With the lid of 150 gold added, it will come to 180 gold; and since this price ought to be triple the price of the second cup, the second cup will cost 60 gold. With the lid added, it will come to 210. But it ought to have been only 30, so that its price would equal the price of the first. We have therefore exceeded the truth by the number 180. Suppose next the first cup costs 90 gold. With the lid of 150 gold added, it will come to 240 gold, and hence the second cup will cost 80 gold, since that number is triple this one. With the lid added, it will come to 230. But it ought to cost only 90, so that its price would equal the price of the first. We have therefore exceeded the truth here too, by the number 140.

Work by the rule, and you will find the price of the first cup to be 300 gold. For with the lid of 150 gold added, the price becomes 450 gold, and hence the price of the second cup will be 150 gold, its third part indeed; and with the lid added, the price becomes 300 gold, equal to the price of the first:

30. 90. ← first cup (trial)
 150. 150. ← (+ lid)
 M[recte: P] X P
 180. 240.
 180. 140. ← excess excess
 40.
 divisor

→ price of the first cup 300 gold, of the second 150 gold. (Both positions exceed, so the divisor is the difference of the errors, 40; the left-hand marker is printed M but the working makes both an excess.)

[Margin: Quaest. 14.]

14. Someone has two golden cups and one lid worth 100 gold, which added to the first cup makes its price triple the price of the second, but added to the second makes its price double the price of the first. At how much, then, are the two cups valued?

Suppose the first is worth 50 gold. With the lid of 100 gold added, it will be worth 150 gold, and hence the second will also be worth 50 gold, since that number is triple this. With the lid added, it will be worth 150 gold, which is not double the price of the first — rather, 100 is double that price. We have therefore exceeded the truth by the number 50. Suppose again the first is worth 110 gold. With the lid added, it will be worth 210 gold, and therefore the second will be worth 70 gold, since that number is triple this. With the lid added, it will be worth 170 gold, which is not double the price of the first — rather, 220 is double it. We therefore fall short of the truth by the number 50. Work by the rule, and you will find the price of the first cup to be 80 gold. For with the lid of 100 gold added, the price becomes 180 gold, and hence the price of the second cup will be 60 gold, its third part indeed; and with the lid added, the price becomes 160 gold, double the price of the first, which was 80 gold:

50. 110. ← first cup (trial)
100. 100. ← (+ lid)
P X M
150. 210.
50. 50. ← excess defect
100.
divisor

→ price of the first cup 80 gold, of the second 60 gold.

[Margin: Quaest. 15.]

15. Someone buys so many partridges that if he had bought $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of them, and 22 besides, he would have 100. How many, then, did he buy? Here a number is sought whose $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, with 22, make 100.

Suppose he bought 12. Half of this number is 6, its third is 4, its fourth is 3; these parts make 13, and with 22 added they become 35. But they ought to have made 100. We therefore fall short by the number 65. Suppose secondly he bought 60.

Half of this number is 30, its third 20, its fourth 15; these parts make 65, and with 22 added they become 87. But they ought to have made 100. We therefore fall short again by the number 13. Work by the rule, and you will find that he bought 72 partridges. For of this number the half is 36, the third 24, and the fourth 18; these parts make up 78, and with 22 added they become 100, just as was proposed:

12. 60. ← trial
M X M
65. 13. ← defect defect
52.
divisor

→ he bought 72 partridges.

[Margin: Quaest. 16.]

16. Two men have a certain sum of gold: if the second gives 12 to the first, the first will have sixfold more than the second; but if the first gives 15 to the second, the second will have tenfold more than the first. How much gold does each have? Here two numbers are sought, of which the first with 12 units of the second's is sixfold the residue of the second, and the second with 15 units of the first's is tenfold the residue of the first. To solve this and similar problems more easily without fractions, one must begin from the second number.

Suppose, then, that the second has 20; if 12 of these are given to the first, the first will have — by the terms of the problem — sixfold the residue of the second, which is 8. So the first will then have 48, and hence, before he received the 12 from the second, he had 36. Now if from this number, 36, 15 units of the first's are given to the second, the second will have 35, which ought to be tenfold — by the terms of the problem — the residue of the first, which is 21. But it is clear that 35 is not tenfold 21; rather, 210 is. We therefore fall short of the truth by the number 175. Suppose again that the second has 100; if 12 are given from it to the first, the first will have — as the problem requires — sixfold the residue of the second, which is 88. So the first will then have 528, and hence, before he received the 12 from the second, he had 516. Now if from this number, 516, 15 units of the first's are given to the second, the second will have 115, which ought to be tenfold the residue of the first, which is 501. But it is clear that 115 is not tenfold 501; rather, 5010 is. We therefore fall short of the truth again by the number 4895. Work by the rule, and you will find that the second has 17 [fraction: ?].

20. 100. ← second (trial)
M X M

175. 4895. ← defect defect
4720.
divisor

→ the second has 17 [fraction: ?].

From this, if 12 units are given to the first, the first will have sixfold the residue of the second, which is 5 [fraction: ?]. So the first will then have 30 [fraction: ?], and hence, before he received 12 from the second, he had 18 [fraction: ?]. For if from this number 15 units are given to the second, the second will have [the sum required], tenfold the residue of the first, which is 3 [fraction: ?] — as the problem proposes. [The second then has 32 [fraction: ?], which is tenfold the residue of the first.]

[Margin: Quaest. 17.]

17. Two men have a certain sum of gold: if the second gives 6 to the first, the first has double the residue of the second; but if the first gives 3 to the second, the second will have a number equal to the residue of the first. How much gold, then, did each have? Here too two numbers are sought, of which the first with 6 units of the second's is double the residue of the second, and the second with 3 units of the first's is equal to the residue of the first.

Suppose the second has 15; from which, if 6 units are given to the first, the first will have 18 — double, that is, the residue of the second, which is 9 — and hence, before he received [the 6...]

17. (continued) ...before he received 6 from the second, he had had 12. Now if 3 units are given from this number to the second, the second will have 18 — a number that is not equal to the remainder of the first, which is 9, but greater. So we have exceeded the truth by 9.

Next suppose the second has 20. From this, if 6 units are given to the first, the first will have 28 — that is, double the remainder of the second, which is 14. So before he received 6 from the second, he had had 22. Now if the first gives the second 3 units, the second will have 23 — a number that is not equal to the remainder of the first, which is 19, but greater. So again we have exceeded the truth by 4.

Apply the rule, and you will find that the second has 24, namely double the remainder of the second, which is 18. The first, then, will have 36, namely double the remainder of the second, which is 18. So before, he had had 30; and thus if he gives 3 units to the second, the second will have 27 — a number that is equal to the remainder of the first, which is likewise 27.

False-position scheme:

15. 20.
P X P
9. 4.
5.
Divisor.

[Margin: Quæst. 18.]

18. A cistern has three unequal pipes at the bottom: through the largest, when opened, all the water flows out in 2 hours; through the middle one, in 3; and through the smallest, in 6. If the water always flows out at a uniform rate, in how much time will it all flow out if the three pipes are all opened at once?

Suppose 4 hours, and reason as follows. If the largest pipe empties 1 cistern in 2 hours, how much will it empty in 4 hours? You will find 2 cisterns. Likewise, if the middle pipe empties 1 cistern in 3 hours, how much will it draw off in 4 hours? You will find $1\frac{1}{3}$ cisterns. Likewise, if the smallest pipe empties 1 cistern in 6 hours, how much will it empty in 4 hours? You will find $\frac{2}{3}$ of a cistern. And so all three pipes together would empty 4 cisterns in 4 hours. But we want only 1 cistern. So we have exceeded the truth by 3.

Suppose secondly 10 hours, and reason. If the largest pipe empties 1 cistern in 2 hours, how much will it empty in 10 hours? You will find 5 cisterns. Likewise, if the middle pipe empties 1 cistern in 3 hours, how much will it empty in 10 hours? You will find $3\frac{1}{3}$ cisterns. Likewise, if the smallest pipe empties 1 cistern in 6 hours, how much will it draw off in 10 hours? You will find $1\frac{2}{3}$ cisterns; and so all three pipes together would empty 10 cisterns in 10 hours. But we want 1 cistern. So we have exceeded the truth by 9.

Apply the rule, and you will find that the cistern is emptied in 1 hour. For the largest pipe in one hour will draw off $\frac{1}{2}$, the middle $\frac{1}{3}$, and the smallest $\frac{1}{6}$ — parts which together make up 1 cistern.

False-position scheme:

4. 10.
P X P
3. 9.
6.
Divisor.

This same question can also be posed the other way. A cistern has three unequal channels at its mouth: through the largest it is filled in 2 hours, through the middle in 3, and through the smallest in 6, and so on.

[Margin: Quæst. 19.]

19. A cistern has a pipe at its mouth through which it is filled in 12 hours; but at the very bottom it has another pipe through which it is emptied in 18 hours. If, then, water flows in continuously through the upper pipe and always flows out through the lower, in how much time will the whole cistern be filled?

Suppose 20 hours, and reason. If 1 cistern is emptied in 18 hours, how much will be emptied in 20 hours? You will find $1\frac{1}{9}$ cisterns. It is therefore necessary that in 20 hours $2\frac{1}{9}$ cisterns be filled, so that when $1\frac{1}{9}$ cisterns have been emptied in the same time, 1 cistern remains full. Say, then: if in 12 hours 1 cistern is filled, how much will be filled in 20 hours? You will find $1\frac{2}{3}$ cisterns. But we want $2\frac{1}{9}$ cisterns. So we have fallen short of the truth by $\frac{4}{9}$.

Suppose secondly 30 hours, and reason. If one cistern is emptied in 18 hours, how much will be emptied in 30 hours? You will find $1\frac{2}{3}$ cisterns. It is therefore necessary that in 30 hours $2\frac{2}{3}$ cisterns be filled, so that when $1\frac{2}{3}$ cisterns have been emptied in the same time, 1 full cistern is left over. Say, then: if in 12 hours 1 cistern is filled, how much will be filled in 30 hours? You will find $2\frac{1}{2}$ cisterns. But we want $2\frac{2}{3}$ cisterns. So again we have fallen short of the truth, by $\frac{1}{6}$.

Apply the rule, and you will find that the cistern is filled in 36 hours. For in 36 hours the upper pipe will fill 3 cisterns, while the lower will empty 2 cisterns; and so one full cistern will remain.

False-position scheme:

20. 30.
M X M
 $\frac{4}{9}$ $\frac{1}{6}$.
 $\frac{5}{18}$.
divisor.

[Margin: Quæst. 20.]

20. A certain craftsman finishes a piece of work in 30 days; but if another joins him, the two together will finish the same work in 18 days. In how much time, then, will this second man finish the same work by himself?

Say first: if the first craftsman finishes the work in 30 days, how much will he finish in 18 days? You will find $\frac{3}{5}$ of the work. In the same time, therefore, the second finishes $\frac{2}{5}$, so that together they complete the whole work.

Suppose, then, first that the second finishes the whole work in 40 days, and reason. If the second finishes $\frac{2}{5}$ of the work in 18 days, how much will he finish in 40 days? You will find $\frac{4}{5}$ of the work. But we assumed that he finishes the whole work. So we have fallen short of the truth by $\frac{1}{5}$.

Suppose next that the second finishes the work in 60 days, and reason. If the second finishes $\frac{2}{5}$ of the work in 18 days, how much will he finish in 60 days? You will find $1\frac{1}{3}$. But we assumed that he finishes 1 whole work. So we have exceeded the truth by $\frac{1}{3}$.

Apply the rule, and you will find that the second finishes the whole work in 45 days. For if he finishes $\frac{2}{5}$ of the work in 18 days, he will finish 1 whole work in 45 days.

False-position scheme:

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40. 60.
M X P
 $\frac{1}{3}$  .  $\frac{2}{3}$ 
[divisor: ?]
Divisor.

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[Margin: Another solution of this question.]

This question, however, will be solved more easily without the rule of false position, in this way. Once you have found that the second finishes $\frac{2}{5}$ of the work in 18 days, so that $\frac{3}{5}$ remain, say: if $\frac{2}{5}$ require 18 days, how many days will $\frac{3}{5}$ require? You will find 27 days, which added to 18 make 45 days, in which he will finish the whole work, as before. Or else say: if $\frac{2}{5}$ require 18 days, how many days will 1 whole work require? For you will find again 45 days, as before.

[Margin: Quæst. 21.]

21. Three men gambled together in such a way that at once the first won $\frac{1}{2}$ of the second's money; then the second won $\frac{1}{3}$ of the third's; and finally the third won $\frac{1}{4}$ of the money that the first had brought into the game. When the game was over, each of them had 700 gold pieces. How much money, then, did each bring into the game?

Here nothing else is sought but that the given number 2100 (for if each has 700, all three have 2100) be divided into three parts in such a way that, if the first

gives $\frac{1}{4}$ to the third but takes $\frac{1}{2}$ of the second, and the second takes $\frac{1}{3}$ of the third, three equal numbers result, namely 700, 700, 700. Or else: three numbers are sought such that the first, having laid down $\frac{1}{4}$, if it takes $\frac{1}{2}$ of the second, makes 700; likewise the second, having laid down $\frac{1}{2}$, together with $\frac{1}{3}$ of the third makes 700; and in the same way the third, having laid down $\frac{1}{3}$, together with $\frac{1}{4}$ of the first makes 700.

Suppose the first player brought in 100 gold pieces. Having laid down $\frac{1}{4}$, namely 25, there remain to him 75. And since this remainder together with $\frac{1}{2}$ of the second must make 700, $\frac{1}{2}$ of the second will be 625,

because this number together with the remainder of the first, namely with 75, makes 700. So the second brought in 1250. Now after he has lost $\frac{1}{3}$, there remain to him 625. And since this remainder together with $\frac{1}{3}$ of the third must make 700, $\frac{1}{3}$ of the third will be 75, because this number together with the remainder of the second makes 700. Hence the third brought into the game 225. Now after he has lost $\frac{1}{4}$, there remain to him 150. And because this remainder together with $\frac{1}{4}$ of the first — that is, with 25 — makes 175, whereas it ought to make 700, we have therefore fallen short of the truth by 525. This shortfall is also recognized from the fact that the three numbers posited, 100, 1250, 225, make 1575, which number falls short of 2100 — the number that all three ought to make up — by this same 525.

Next suppose the first brought into the game 200 gold pieces. Having laid down $\frac{1}{4}$, namely 50, there remain to him 150, which together with $\frac{1}{2}$ of the second is 700; so the second brought in 1100. Now after $\frac{1}{3}$ has been lost, there remain to him 550; and $\frac{1}{3}$ of the third will be 150, and so at the start of the game the third had 450. After $\frac{1}{4}$ has been lost — that is, together with 50 — they make 350. But they ought to make 700. So we have fallen short. This shortfall too can be drawn out from the fact that the three numbers posited, 200, 1100, 450, make 1750, which number falls short of 2100 by 350.

Apply the rule, and you will find that the first player brought in 400 gold pieces, the second 800, and the third 900. These numbers for the second and third you will find either by the rule of false position — by multiplying the errors by the positions of the second and third, and so on — or from the first result already found, in the way we investigated the numbers of the second and third a little earlier from the 100 and 200 we had assumed the first to have. For if the first has 400 gold pieces (having laid down $\frac{1}{4}$, namely the 100 that he lost), he has 300; and because together with $\frac{1}{2}$ of the second he must have 700, $\frac{1}{2}$ of the second will be 400, and so the second brought in 800. Now after $\frac{1}{3}$ has been lost, there remain to

him 400. And since this $\frac{1}{2}$ together with $\frac{1}{3}$ of the third must make 700, $\frac{1}{3}$ of the third will be 300; and therefore the third brought in 900. For after $\frac{1}{3}$ has been lost, there remain to him 600, to which if $\frac{1}{4}$ of the first is added — that is, 100 gold pieces — he will have 700, as the question requires.

False-position scheme:

100. 200.
 1250. 1100.
 225. 450.
 M X M
 525. 350.
 175.
 divisor.

[Margin: Quæst. 22.]

22. Three merchants made a profit of 400 gold pieces, which they distributed — with regard to the money each had put in — in such a way that the second's share exceeded the first's share by 12, while the third's share exceeded the second's share by 16. How great, then, was each one's share?

Suppose the first received 1 gold piece. (For I want to solve this question with the smallest possible numbers, namely with the positions 1 and 2, so that the general validity of this rule of false position may appear more clearly.) Then the second is 13 and the third 29, which numbers all together make 43. But they ought to make 400. So we have fallen short of the truth by 357.

Suppose again that the first received 2 gold pieces; then accordingly the second is 14 and the third 30, which numbers all together make 46. But they ought to make 400. So even now we have fallen short of the truth by 354.

Apply the rule, and you will find that the first's share is 120 gold pieces, the second's 132, and the third's 148 — three numbers that make up the sum of 400 gold pieces, as is set out in the question.

False-position scheme:

1. 2.
 13. 14.
 29. M X M 30.
 43. 46.
 357. 354.
 3.
 Divisor.

[Margin: Quæst. 23.]

23. The emperor's army against the Turks contains 40000 German foot; and it has as many Italians and Hungarians that the number of Italians is $\frac{1}{2}$ of the Germans and Hungarians [together], while the number of Hungarians is $\frac{1}{3}$ of the Germans and Italians. How great, then, is the number of the Italians, how great that of the Hungarians, and indeed how great the whole army?

Suppose the Italians are 30000. Since this number must be $\frac{1}{2}$ of the Germans and Hungarians, the Germans and Hungarians [together] will be 60000. Since the Germans are 40000, the Hungarians will be 20000, which ought to make $\frac{1}{3}$ of the Germans and Italians, namely of the number 70000. But they make $\frac{1}{3}$ of the number 60000. So we have exceeded the truth by 10000.

Suppose secondly the Italians are 24000. Since this number must be $\frac{1}{2}$ of the Germans and Hungarians, the Germans and Hungarians [together] will be 48000. Since the Germans are 40000, the Hungarians will be 8000, which ought to make $\frac{1}{3}$ of the Germans and Italians, namely of the number 64000. But they make $\frac{1}{3}$ of the number 24000. So again we have exceeded the truth by 40000.

Apply the rule, and you will find that the Italians are 32000 and the Hungarians 24000, and thus the whole army is 96000. For in this way the Italians make $\frac{1}{2}$ of the Germans and Hungarians, and the Hungarians $\frac{1}{3}$ of the Germans and Italians, as is plain.

False-position scheme:

30000. 24000.
20000. 8000.
P X P
10000. 40000.
30000.
divisor.

[Margin: Quæst. 24.]

24. It seemed good to add here that famous device of Archimedes [Archimedes] by which, as Vitruvius reports (Book 9, Chapter 3), he detected the theft of a certain goldsmith in a golden crown — that is, how much silver had been mixed in without the crown being melted down. For when King Hiero had resolved to offer a votive crown of pure gold to his gods, the goldsmith, having removed a portion of the gold, mixed in an equal amount of silver. Hiero, indignant at being cheated (to use Vitruvius's words) and finding no way to detect the theft, asked Archimedes to take the problem upon himself. Then Archimedes, while he had

the matter in mind, happened to go to the baths; and there, as he lowered himself into the tub, he noticed that as much water flowed over the edge of the tub as the bulk of his body sank into it. And so, when he had grasped the explanation of the matter, he did not linger but leapt out for joy, and going home naked he proclaimed aloud that he had found what he was seeking. For as he ran he repeatedly cried out in Greek, εὕρηκα, εὕρηκα [Greek: heurēka, "I have found it"].

Then, it is said, on the strength of that discovery he made two masses of equal weight, the same as the crown's — one of gold, the other of silver. Having done this, he filled a large vessel with water up to the very brim and lowered the silver mass into it; and as much water flowed out as the magnitude of the mass sank down into the vessel. Then, having taken out the mass, he poured back as much as had been lost, measuring with a sextary, so that the water might be brought level with the brim in the same way as before. In this way he found how great a measure of water corresponds to a fixed weight of silver. When he had tried this, he then lowered the golden mass likewise into the full vessel; and taking it out, in the same manner, with the measure added back, he found that not so much water had flowed out — indeed, less by as much as a mass of gold is smaller in bulk than one of silver of the same weight. Afterward, having refilled the vessel, he lowered the crown itself into the same water and found that more water flowed out for the crown than for a mass of pure gold of the same weight; and so, reasoning from the fact that more water had flowed out for the crown than for the mass, he detected the mixture of silver in the gold. Thus far Vitruvius. But let us now explain how the said theft can be detected by the rule of false position, if that device of Archimedes is employed.

Let us suppose, for example, that the crown was 100 lb, and that when placed in the vessel it caused 65 lb of water to overflow; that a mass of pure gold of 100 lb caused 60 lb of water to overflow; and that a mass of pure silver of 100 lb caused 90 lb of water to overflow; and so that in the crown there were 60 lb of gold and 40 lb of silver. Now see whether a crown so composed casts out 65 lb of water, which you will determine thus. Say: if 100 lb of gold cast out 60 lb of water, how much water will 60 lb of gold cast out? [You will find 36 lb of water.] Likewise, if 100 lb of silver cast out 90 lb of water, how much water will 40 lb of silver cast out? You will find 36 lb of water, so that the crown casts out 72 lb of water. But it ought to cast out only 65 lb. So we have exceeded the truth by 7.

Suppose secondly that the goldsmith stole 30 lb of gold, and accordingly that there were 70 lb of gold and 30 lb of silver in the crown. Say, then: if 100 lb of

gold expel 60 lb of water, how much water will 70 lb of gold expel? Likewise, if 100 lb of silver cast out 90 lb of water, how much

False-position scheme:

60. 70.
40. 30.
P X P
7. 4.
3.
diuisor.

water will 30 lb of silver cast out? You will find in the first operation 42 lb of water, and in the second 27, which make 69 lb of water. But they ought to have been only 65 lb. So again we have exceeded the truth by 4.

Apply the rule, and you will find that the goldsmith took $16\frac{2}{3}$ lb of gold, and thus that the crown was mixed of $83\frac{1}{3}$ lb of gold and $16\frac{2}{3}$ lb of silver. To prove this, say: if 100 lb of gold cast out 60 lb of water, how much water will $83\frac{1}{3}$ lb of gold cast out? Likewise, if 100 lb of silver cast out 90 lb of water, how much water will $16\frac{2}{3}$ lb of silver cast out? You will find in the first operation 50 lb of water, and in the second 15 lb of water, which all together make up 65 lb of water — the amount we supposed the crown to cast out.

In the same way the theft would have been detected even if the masses of gold and silver had not been 100 lb, however many pounds the crown had, but of any number of pounds whatever — say, for example, 10 lb of gold and 20 of silver — provided that it is carefully examined how much water each single mass casts out. Let us suppose 10 lb [of gold] to cast out [6 lb of water], and 20 lb [of silver] to cast out 18 lb of water. From this, in the first position you will say: if 10 lb of gold cast out 6 lb of water, [how much will 60 lb of gold cast out?] and so on.

If a crown of 300 lb be supposed, and masses of gold and of silver of the same number of pounds, on the condition that the crown casts out 218 lb of water, the gold 206 lb of water, and the silver 230 lb of water, we shall find that there were 150 lb of gold in the crown and the same amount of silver. This is clear from these two positions, in the first of which are set 100 lb of gold and 200 lb of silver, and in the second 101 lb of gold and 199 of silver, and so on.

False-position scheme:

100. 101.
200. 199.
—P X P—
4. 3 [fraction: ?]

[divisor: ?]
Divisor.

By this device, then, it will be detected, in any mass whatever compounded of gold and silver, how much gold and how much silver has been mixed in.

Arithmetic Progressions [Progressiones Arithmeticae]

Chapter XXIV [Caput XXIV]

An arithmetic progression is a series of several numbers exceeding one another equally, as here.

[Margin: What an arithmetic progression is.]

The natural progression of numbers, beginning from 1:

1.
2.
3.
4.
5.
6.
7.
8.
9.

11.
12.
13.
14.&c.

The progression of odd numbers, beginning from 1:

1.
3.
5.
7.
9.

11.
13.
15.
17.

21.
23.
25.
27.&c.

The progression of even numbers, beginning from 2:

2.
4.
6.
8.
10.
12.
14.
16.
18.

22.
24.
26.
28.&c.

The first of these three progressions is called the natural progression of numbers, and it begins from 1; in it all the numbers exceed one another in order by a unit. The second is called the progression of odd numbers, and it begins from 1; in it all the numbers exceed one another in order by two. The third, finally, is called the progression of even numbers, and it begins from 2, which is the first even number — just as 1 is the first of all numbers, though improperly so. In this progression of even numbers, moreover, all the numbers exceed one another in order likewise by two, just as in the progression of odd numbers. The same holds here.

[Margin: What the natural progression of numbers, and the progressions of odd and even numbers, are.]

Other progressions.

2.
5.
8.
11.
14.
17.
20.
23.
26.

&c.

3.
8.
12.
16.
20.
24.
28.
32.
36.

&c.

For the first of these progressions begins from 2 and advances by 3, since all the numbers in it exceed one another in order by three; while the second begins from 4 and advances by that same number 4, since in it all the numbers exceed one another in order by four.

Any arithmetic progression is continued if the difference — or the amount of excess — is added to the number after which the progression is to be extended. For instance, if this progression, 4. 9. 14. 19. 24., is to be continued, we shall add the difference, or the excess, of the progression, namely 5 (which difference, or excess, we shall find if we subtract the first number of the progression from the second, or any other number from the one next greater in the same progression), to the last number 24, and so make 29. To this again we shall add 5 and form 34, and so on to infinity. Likewise, if someone wishes to begin a progression from 7 and to advance by the difference, or excess, 6, then 6 must be added to 7 to make

13 for the second number of the progression; likewise 6 to 13 to make 19 for the third number, and so on.

[Margin: How an arithmetic progression is continued.]

In like manner an arithmetic progression is continued by going backward, if the difference of the progression is subtracted from the lesser extreme. For instance, if this progression, 30. 37. 44. 51. 58., is to be continued toward the smaller numbers, we shall take the difference 7 away from the lesser extreme 30, so that 23 is left. From this we shall again subtract 7, so that 16 remains. From this we shall again subtract 7, so that 9 is left; from which we shall again take away 7, so that 2 is left; from which 7 can no longer be taken away, and therefore the said progression cannot decrease any further. Likewise, if someone wishes to begin a progression from 40 and to advance by 4 toward unity, 4 must be taken from 40, so that 36 is left. Likewise 4 from 36, so that 32 remains. Again 4 from 32, so that 28 is left. Likewise 4 from 28, so that 24 is left, and so on.

[Margin: How the difference of an arithmetic progression is found. An arithmetic progression cannot decrease to infinity.]

It is a property of an arithmetic progression of three numbers that the sum of the extremes is equal to double the middle number, as appears here in 7. 18. 29.; and it is demonstrated by Jordanus [Jordanus de Nemore], Book 1, Proposition 2.

[Margin: Property of an arithmetic progression of three numbers.]

Of an arithmetic progression of four numbers, it is a property that the sum of the extremes is equal to the sum of the middle numbers, as appears here in 4. 12. 20. 28.; and it is demonstrated by Jordanus, Book 1, Proposition 3. And this is true not only for four numbers exceeding one another continuously by the same number, as are the numbers in the given example, but also for four numbers that do not exceed one another continuously by the same number, provided that the difference between the first and the second is the same as that between the third and the fourth, as you see here in 4. 12. 30. 38.

[Margin: Property of an arithmetic progression of four numbers.]

From these properties it is gathered that in every arithmetic progression whose number of terms is odd, the sum of the extremes is equal to any sum whatever of two numbers equally distant from the extremes, and also to double the middle number. As appears here:

7.

[Margin: Property of an arithmetic progression of however many terms, if the number of terms is odd.]

(Continuation of Cap. XXIV, on arithmetic progressions.)

3.

7.

11.

15.

19.

23.

27.

31.

35.

For suppose these four numbers 3, 7, 39, 43 have the same difference, even though it is not continuous (for the difference between 3 and 7 is the same as that between 39 and 43). Then, from what we said just above, the sum of the extremes 3 and 43 will equal the sum of the middle terms 7 and 39. By the same reasoning, the sum of 7 and 39 will equal the sum of 11 and 35, since these four numbers 7, 11, 35, 39 likewise have the same difference, even if not a continuous one. And so on with the rest, until we reach the three middle numbers 19, 23, 27; and since these have the same difference, then, by what we showed a moment ago, the sum of the extremes 19 and 27 will equal twice the middle number 23. The same holds for every other progression of this arithmetic kind.

[Margin: A property of an arithmetic progression of however many terms, when the number of terms is even.]

From this latter property it follows that, in every arithmetic progression whose number of terms is even, the sum of the extremes equals the sum of any pair of numbers equally distant from the extremes. As is plain here:

3.

7.

11.

15.

19.

23.
 27.
 31.
 35.

We shall prove this as before, with this one qualification: at the last step we must take the four middle numbers 15, 19, 23, 27, and not just three as before, because here there is not a single middle number but two. Now follow the rules that concern arithmetic progressions.

Rule I

[Margin: How to find the sum of any arithmetic progression.]

If in any arithmetic progression the number of terms is known, together with the smaller and the larger extreme, we shall arrive at knowledge of the sum of all the terms in this way. Add the first term to the last, and multiply the sum by the number of terms. Half of the resulting number will be the sum of all the terms. As in this progression:

4.
 7.
 10.
 13.
 16.
 19.
 22.
 25.
 28.
 34.

From 4 and 37 we get 41; multiplied by the number of terms — that is, by 12 (for there are 12 numbers in that progression) — this makes 492. Half of this number, 246, is the sum of all the numbers of the given progression. The same reasoning applies to the rest.

[Margin: Another way to find the sum of any arithmetic progression.]

Some divide this rule into two cases, as follows. When the number of terms is even, they multiply the sum of the first and last term by half the number of terms.

But when the number of terms is odd, they multiply half the sum of the first and last term (for when the number of terms is odd, that sum is always even) by the number of terms. By this method the sum of all the numbers of the progression is always produced. Or in this other way: when the sum of the first and last term is even, they multiply its half by the number of terms, whether that number be even or odd. But if that sum is odd, they multiply it by half the number of terms, which number is then always even. So in the example above: because the number of terms is 12 — or because the sum of the first and last term is odd, namely 41 — they multiply it by 6, half the number of terms, and produce the sum of all the numbers, 246, as before. But in the following two progressions, in the first of which the number of terms is even, namely 10, and in the second odd, namely 11 — since the sum of the first term and the last

3.
7.
11.
15.
19.
23.
27.
31.
35.

4.
7.
10.
13.
16.
19.
22.
25.
28.

is even, namely 42 in the first and 38 in the second — they multiply both the half of the former, namely 21, by 10, the number of its terms, and the half of the latter, which is 19, by 11, the number of its terms, so as to produce in the first the sum 210, and in the second 209.

The reason for these rules is this. Since we said above that, when the number of terms is even, the sum of the extremes equals any sum of two numbers equally distant from the extremes, it follows that all these sums taken together are as many as there are units in half the number of terms. Therefore if one sum — namely that of the extremes — is multiplied by half the number of terms, the sum of all the sums is produced. Again, since we showed that, when the number of terms is odd, the sum of the extremes equals any sum of two numbers equally distant from the extremes, and also equals twice the middle number, it follows that the middle number is half of any one such sum. Therefore all the sums together, along with the middle number, will contain as many halves of a single sum as there are terms. So if half of one sum — namely of the extremes — is multiplied by the number of terms, the sum of all the terms is produced.

And so, as you see, it is enough to know the first term, the last, and the number of terms in order to obtain the sum of the whole progression, even if the intermediate terms are unknown. How the last term is found from the known first number, together with the number of terms and the common difference [differential] of the progression, we shall explain in the following rule.

[Margin: A special way of finding the sum of the natural progression of numbers.]

In the natural progression of numbers, which begins from 1, the sum of all the terms can be found most briefly in this way. Multiply the last number (which always indicates the number of terms) by the next larger number. Half of that product is the sum of all the terms. As here:

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.

[Margin: In the natural progression of numbers, the last term is the number of terms.]

From multiplying the last number 11 by 12, the next larger number, the number 132 is produced, whose half, 66, is the sum of the whole progression. Likewise in this progression:

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.

From multiplying the last number 10 by 11, the next larger number, comes the number 110, whose half, 55, is the sum of the whole progression.

And so, if someone wants the sum of a natural progression that ends at any given number — as at 100, in which there are of course 100 terms — the last given number, at which the progression is said to end, must be multiplied, as here:

that is, the number 100, by the next larger number, here 101. For half of the product (which here is 10,100), namely 5050 in the given example, will be the sum of that progression. And the same reasoning holds in other cases.

[Margin: Another way of finding the sum of the natural progression of numbers.]

Others also split this rule into two cases, as follows. If the last number is even, they multiply the next larger number by half the last number; but if it is odd, they multiply it by half of the next larger number. For by this method the sum of all the numbers of the progression is always produced. So in the second natural progression above, they multiply 11, the number next larger than the last, by 5, half the last number, and get 55, the sum of the whole progression, as before. And in the first natural progression they multiply 11, the last number, by 6, half of the number next larger than the last, and get 66, the sum of the whole progression, as before.

[Margin: A special way of finding the sum of the odd numbers.]

Also in the progression of odd numbers, which begins from 1, the sum of all the terms is found with the greatest ease if the number of terms is multiplied by itself, as here:

1.
 3.
 5.
 7.
 9.
 11.
 13.
 15.
 17. 19,

From multiplying the number of terms, which is 10, by itself, the number 100 is produced, which is the sum of the whole progression.

[Margin: How the number of terms is found in a progression of odd numbers.]

The number of terms is obtained if 1 is added to the last number and half of the resulting number is taken. So in the given example, if 1 is added to 19, the number 20 results, whose half, 10, indicates the number of terms.

And so, if someone wants the sum of a progression of odd numbers that ends at any given number — as at 67 — a unit must be added to the last number at which the progression is said to end, here to 67. For half of the resulting number (which here is 68), namely 34 in the given example, will give the number of terms; and this, multiplied by itself, will produce the sum of that progression. So in the proposed example, in which 34 terms are set, if the number 34 is multiplied by itself, the sum of that progression, 1156, will be produced. And so with the rest.

[Margin: A special way of finding the sum of the even numbers.]

Finally, in the progression of even numbers, which begins from 2, the sum is found with no labor either, if half of the last number — which always indicates the number of terms of the progression (for in a progression of even numbers of this kind there are always as many terms as there are units in half the last term) — is multiplied by the number next larger than that half. As here:

2.
 4.
 6.
 8.
 10.
 12.

14.

16.

18.

22.

[Margin: By what method the number of terms is found in a progression of even numbers.]

From multiplying 12 (which number is half the last term, or the number of terms) by 13, the number next larger than that half, comes the number 156, that is, the sum of all those even numbers.

And so, if someone wants the sum of a progression of even numbers that ends at any given number — as at 100 — the number that makes up half the last given number at which the progression is said to end must be multiplied (in the given example this is half the last given number, 100) by the number next larger than that half, here by 51. For the resulting number, such as 2550 in the given example, will be the sum of that progression; and the number of terms will be 50 in the same example, namely as many as there are units in half the last number. And so with others.

Rule II

[Margin: How the last term of any arithmetic progression is drawn from the number of terms together with the first term and the common difference of the progression.]

If in any arithmetic progression the number of terms is known, together with the first term and the common difference of the progression, we shall find the last term, even if we do not have the intermediate terms, in this way. From the number of terms subtract 1; multiply the remaining number by the common difference; and finally add the first term to this product. For the resulting number will be the last term. For example, if the first term of some progression is 3, the number of terms is 10, and the common difference is 8, we shall find the tenth term — that is, the last of this progression, without the intermediate ones — in this way. From 10, the number of terms, we shall take away 1, and multiply the remaining number 9 by 8, the common difference of the progression; then to the product 72 we shall add 3, the first term. For the resulting number 75 is the tenth term of the progression whose first term is 3 and whose common difference is 8. As appears here, where all the terms are set out:

3.
 11.
 19.
 27.
 35.
 43.
 51.
 59.
 67.

[Margin: The problem of the oxen of Augeas.]

So suppose someone proposes this problem. When Hercules asked Augeas about the number of oxen he had, Augeas answered that all his oxen were distributed over 40 places in such a way that for every three oxen in the first place there were 5 in the second, 7 in the third, 9 in the fourth, and so on. Hercules went to the first place and found 30 oxen. How many oxen, then, did Augeas have, and how many oxen were in the last place? It is to be solved in this way. Since in the first place there are ten times 3 oxen, there will be in the second place ten times 5, namely 50, and in the third ten times 7, namely 70, and so on, so that it forms a kind of arithmetic progression whose first term is 30, whose common difference is 20, and whose number of terms is 40. First, then, the last term must be drawn out by the rule, in this way. From 40, the number of terms, subtract 1, and multiply the remaining number 39 by 20, the common difference; then to the product 780 add the first term 30. For the last term, or fortieth, will come out as 810 — and that is how many oxen were in the last place.

Next, from this last term now found and from the first given term, together with the number of terms, the sum of the whole progression is to be obtained by the first rule, even if we do not have all the intermediate terms, in this way. Add the first term 30 to the last term 810, and multiply the resulting number 840 by 20 — that is, by half the number of terms. For the resulting number 16,800 is the sum of the whole progression, and hence the number of Augeas's oxen. And so that you may see how many oxen were in each place, and indeed that there were 810 in the last place, we have set out the whole progression here:

30.
 50.
 70.
 90.

110.
130.
150.
170.
190.

230.
250.
270.
290.
310.
330.
350.
370.
390.
31.
450.
470.
490.
510.
530.
550.
570.
590.

630.
650.
670.
690.
710.
730.
750.
770.
790.

[Margin: The problem of the military commanders.]

There will be a similar problem if someone puts it this way. A general distributed among 20 valiant commanders the money found in the plundering of a city, on this condition: that to the one who last crossed the enemy wall he would give 100 gold pieces, to the second-to-last 130, to the third-to-last 160, and so on in the same way. How great, then, was that sum of money, and how much did the one who first crossed the wall receive?

For if from 20, the number of terms (for there are as many terms in that progression as there are commanders), you subtract 1, and multiply the remaining number 19 by 30, the common difference of the progression, and to the product 570 add the first number, namely 100, you will make 670 for the last term of the progression — and that is how many gold pieces the first commander received. Then, once the last term is found, if the first is added to it, namely 100, to make 770, and this number is multiplied by 10, half the number of terms, the sum of all the terms, 7700, is produced. That, then, was the total sum of money distributed:

Year	Number of people in the workforce
1990	100.0
1991	130.0
1992	160.0
1993	190.0
1994	220.0
1995	250.0
1996	280.0
1997	310.0
1998	340.0

GEOMETRIC PROGRESSIONS

Chapter XXV

[Margin: What a geometric progression is.]

A geometric progression is a series of several numbers exceeding one another in the same ratio, as appears here:

1.
2.
4.
8.
16.
32.
64.
128.
256.

1024.
2048. &c.
2.

3.
9.
27.
81.
243.
729.
2187.
6561.

&c.
3.
6.
12.
24.
48.
96.

192.

384.

768.

&c.

The first of these progressions advances by a double ratio, so that each number is twice as great as the one immediately preceding it; the second by a triple ratio, so that each number is three times as great as the one immediately before it; and both of these progressions begin from 1. The third, finally, advances by a double ratio as well, but takes its start not from 1 but from 3.

[Margin: How a geometric progression is continued. How the denominator of the ratio is found in a geometric progression.]

Any geometric progression is continued if the number after which the progression is to be extended is multiplied by the denominator of the ratio [denominator proportionis] (that is, by the common ratio). For instance, if this progression of triple ratio, 4, 12, 36, is to be continued, we shall multiply the last number 36 by 3, the denominator of the ratio (which denominator we find by dividing the second number by the first, or any other by the one immediately smaller in the same progression), and produce 108. This number we shall again multiply by 3, producing 324, and so on to infinity. Likewise, if someone wants to begin a progression from 7 and to advance by a quintuple ratio, whose denominator is 5, we must multiply 7 by 5 to make 35 for the second number of the progression; then 35 by 5 to make 175 for the third number, and so on.

By the same reasoning a geometric progression is continued backward, if the smaller extreme is divided by the denominator. For instance, if this progression 64, 128, 256, 512 is to be continued toward smaller numbers, since the denominator of the ratio is 2, we shall divide 64 by 2 and make 32, which in turn we shall divide by 2 and find 16, and so on to infinity. As appears in this example:

512.

256.

128.

64.

32.

16.

8.

4.

2.

$\frac{1}{2}$. $\frac{1}{4}$. $\frac{1}{8}$. $\frac{1}{16}$. &c.

[Margin: A geometric progression decreases to infinity.]

And there will never be an end to this decrease in a geometric progression. So too, if someone wants to begin a progression from 100 and to advance toward unity by a sesquialteral ratio [proportio sesquialtera], whose denominator is $1\frac{1}{2}$, we shall divide 100 by $1\frac{1}{2}$ to get $66\frac{2}{3}$; then $66\frac{2}{3}$ by $1\frac{1}{2}$ to get $44\frac{4}{9}$, and so on.

[Margin: A property of a geometric progression of three terms.]

It is a property of a geometric progression of three numbers that the number produced by multiplying the first number by the third equals the number produced by multiplying the middle number by itself. As appears here, 3, 9, 27; and it is demonstrated by Euclid, Book 7, Proposition 20.

[Margin: A property of a geometric progression of four terms.]

It is a property of a geometric progression of four numbers that the number produced by multiplying the first number by the fourth equals the number produced by multiplying the second by the third. As you see here, 2, 6, 18, 54; and it is demonstrated by Euclid, Book 7, Proposition 19. And this holds not only for four numbers in continued proportion, such as the four numbers in the given example, but also for four numbers that are not in continued proportion, provided that the ratio of the second to the first is the same as that of the fourth to the third. As you observe here, 3, 6, 10, 20.

[Margin: A property of a geometric progression of however many terms, when the number of terms is odd.]

From these properties it is gathered that, in every geometric progression whose number of terms is odd, the number produced by multiplying the extremes together equals the number produced by multiplying any two numbers equally distant from the extremes, and also the number produced by multiplying the middle term by itself. As appears here:

3.

6.

12.

24.

48.

96.

192.

384.

For since these four numbers 3, 6, 384, 768 have the same ratio, though not a continuous one, then, from what we said just above, the number produced from 3 by 768 will equal the one produced from 6 by 384. By the same reasoning, the number produced from 6 by 384 will equal the one produced from 12 by 192, since these four numbers 6, 12, 192, 384 have the same ratio, though not a continuous one. And so on with the rest, if there are more, until we reach the three middle numbers 24, 48, 96; and since these have the same ratio, then, by what we showed a little before, the number produced from the first by the third equals the number produced by multiplying the middle by itself. And the same reasoning holds for all other geometric progressions of this kind.

[Margin: A property of a geometric progression of however many terms, when the number of terms is even.]

From this latter property, too, it follows that, in every geometric progression whose number of terms is even, the number produced by multiplying the extremes equals the number produced by multiplying any two numbers equally distant from the extremes. As is plain here:

3.
6.
12.
24.
48.
96.
192.

We shall prove this as before, with this one qualification: at the last step we must take the four middle numbers 12, 24, 48, 96, and not just three as before, because here there is not a single middle number but two. Now follow the rules that concern geometric progressions.

[Margin: How to find the sum of any geometric progression.]

Rule I

If in any geometric progression the denominator of the ratio is known, together with the smaller and the larger extreme, we shall arrive at knowledge of the sum of all the terms in this way. Subtract the first term from the last, and divide the

remaining number by the number that is one unit less than the denominator. For if to the quotient the last term, or larger extreme, is added, the sum of all the terms is composed. As in this progression:

(continuing the worked example of the quadruple progression from the preceding page)

3.
 12.
 48.
 192.
 768.
 3072.
 12288.

Take 3 away from 49152 and 49149 remains. Now the denominator of the ratio which the numbers of the given progression have is 4, so we divide 49149 by 3; to the quotient 16383 we add the last term — that is, the greater extreme, 49152 — and so we obtain the sum of the whole progression, 65535. Consider this case as well:

4.
 6.
 9. $13\frac{1}{2}$. $20\frac{1}{4}$. $30\frac{3}{8}$. $45\frac{5}{16}$.

Subtract 4 from $45\frac{5}{16}$ and $41\frac{5}{16}$ remains. Divide this by $\frac{1}{2}$ — for the denominator of the sesquialter ratio [proportio sesquialtera] which the numbers of this progression have is $1\frac{1}{2}$, and when 1 is removed $\frac{1}{2}$ is left — and the quotient becomes $83\frac{5}{16}$. If the last number, the greater extreme $45\frac{5}{16}$, is added to this, the sum of the whole progression becomes $128\frac{11}{16}$. In just the same way we can find the sum of any geometric progression whatever.

So, as you see, it is enough to know the first term and the last, together with the denominator of the ratio, in order to find the sum of the whole progression, even if the intermediate terms are unknown. In the following rule we will explain how we can arrive at knowledge of the last term even when the whole progression is not carried out in full.

In a geometric progression of double ratio whose beginning is 1, the sum of the whole progression, however many terms it has, is found with the greatest ease: double the last term and throw away 1 from the double. As here:

[Margin: A particular method for finding the sum of a progression of double ratio whose beginning is 1.]

1.
2.
4.
8.
16.
32.
64.
128.
256.

Double the last term 512, and from the double, 1024, throw away 1: the sum of the whole progression, 1023, is obtained.

From this it follows that any number in a progression of this kind, once the unit has first been removed, is the sum of all the preceding numbers, since every number is exactly double the one immediately before it.

RULE II

[Margin: In a progression of double ratio beginning from 1, any number — the unit first removed — is the sum of all the preceding numbers.]

In every geometric progression that begins from 1, any number multiplying itself generates a number as far removed from it as it is itself distant from unity; while any number that multiplies another, greater number produces a number that is as far removed from the greater as the multiplying number itself falls short of unity. This rule is gathered most clearly from Book 8, Proposition 11 of Euclid [*Elements* VIII.11], as we explained in the scholium to that same proposition. For example:

1.
2.
4.
8.
16.
32.
64.

128.

256.

In this progression of double ratio, which follows here, if the number 16 — which occupies the fifth place from unity — is multiplied by itself, it produces 256, which likewise occupies the fifth place from 16, namely the ninth place overall. So too, if the number 32, occupying the sixth place from unity, multiplies itself, it produces 1024, which occupies the sixth place from 32, namely the eleventh place. Likewise the number 8, of the fourth place, multiplying the number 64, produces 512, which is to be set in the fourth place from 64.

[Margin: If, in a geometric progression beginning from 1, some number multiplies itself or another number, this shows what place the product occupies.]

And so we may gather this rule: if, in a geometric progression whose beginning is 1, some number occupying any place multiplies itself, the product must be set one unit short of double the place of the multiplying number. Thus if a number multiplying itself occupies the third place, the product must be set in the fifth place; if it occupies the seventh place, the product must be set in the thirteenth place; and so on. This was shown plainly in the progression of double ratio above, and it appears just as clearly in this other progression, of quadruple ratio:

[Margin: Any number in a geometric progression beginning from 1, multiplying itself, produces a number to be set one unit short of double the place of the multiplying number.]

1.

4.

16.

64.

256.

1024.

4096.

16384.

For if the number 64, set in the fourth place, multiplies itself, it generates 4096, to be set in the seventh place. Likewise the number 256, occupying the fifth place, multiplying itself, produces 65536, to be set in the ninth place.

To know more easily in what place any product is to be placed, one should write out beneath the geometric progression the natural progression of the numbers, in this order: under 1 — that is, under the first number — write 0; under the second number put 1; under the third, 2; under the fourth, 3; and so on, as has been done here in this progression of double ratio.

[Margin: How the natural progression of numbers indicates in what place any product is to be located in a geometric progression beginning from 1.]

1	2	4	8	16	32	64	128	256	512	1024	2048
0	1	2	3	4	5	6	7	8	9	10	11

For any number of the geometric progression, multiplying itself, produces a number to be set above that number of the natural progression which is double the one written under the self-multiplying number. And any number multiplying another produces a number to be set above the number of the natural progression that is composed of the two numbers written under the multiplying numbers. Thus if 32 is multiplied by itself, 1024 is produced, to be set above 10, because 10 is double 5, which is written under 32. Likewise, from the multiplication of 8 by 256, 2048 is produced, to be set above 11, because 11 is composed of 3 and 8, the numbers written under 8 and 256.

And since a number written above in the geometric progression occupies a place exceeding by one the number of units in the corresponding number of the natural progression — as is plainly seen in the example above — we can easily find the number of any place in the geometric progression even without writing out all the intermediate numbers. Suppose, for example, that we must find the number to be set in the twentieth place of the aforesaid progression. First I write four, or five, or more numbers of the progression, together with the natural progression, as you see here:

1	2	4	8	16	32	64
0	1	2	3	4	5	6

Then I multiply, say, 8 by itself and get 64, the number of the seventh place, under which stands 6, one unit short of the number of places, because 3 (under 8) doubled makes 6. And if we multiply 8 by 64, we get 512, the number of the tenth place, under which 9 would be written, one unit short of the number of places, because the numbers 3 and 6 (under the fourth and seventh places) make 9. Again, if we multiply 512 — the number of the tenth place, under which 9 stands — by itself, we produce 262144, to be written in the nineteenth place, under

which 18 would be set, one unit short of the number of places, because 9 (under the tenth place) doubled makes 18. And now, since 18 — the number written under the nineteenth place — together with 1, set under the second place,

[Margin: The method by which the number of any place in a geometric progression beginning from 1 may be found without the intermediate terms.]

makes 19: if we multiply 2, set above 1, by 262144, set above 18, we obtain 524288, to be written in the twentieth place, under which, as is clear, 19 stands.

Again, if one wishes in the same progression to find the number to be set in the eighteenth place, we will multiply 32 — under which 5 stands — by itself, and produce 1024, to be written in the eleventh place, under which stands 10, double 5. And since from 10 (which stands under the eleventh place) and 6 (which stands under the seventh place) comes 16 — which is written under the seventeenth place — if we multiply 64, of the seventh place, by 1024, of the eleventh place, we produce 65536, of the seventeenth place. Finally, since from 16 (which stands under the seventeenth place) and 1 (which stands under the second place) comes 17 — which is written under the eighteenth place — if we multiply the number found, 65536 of the seventeenth place, by 2 of the second place, we generate 131072, to be written in the eighteenth place, under which, as is clear, stands 17.

[Margin: What has been said in this rule about a geometric progression beginning from 1 is true also of a geometric progression beginning not from 1 but from some other number.]

All this holds good also for any geometric progression that begins not from 1 but from any other number, provided that we divide each number produced by multiplication by the first number of the progression: for the quotient will be the number sought. As appears in this progression of double ratio:

5	10	20	40	80	160	320	640	1280	2560	5120
0	1	2	3	4	5	6	7	8	9	10

[Margin: How the number of any place in a geometric progression beginning from any number is found without the intermediate numbers.]

For if the number 80, occupying the fifth place from the first number, is multiplied by itself, it produces 6400; and dividing this by the first number, namely by 5, gives the quotient 1280, to be placed in the fifth place from 80 — namely in the ninth place, under which stands 8, double the 4 set under 80. Here you see that 80, of the fifth place, when it multiplies itself, produces a number

which, divided by the first number of the progression, gives the quotient 1280, to be set one unit short of double the place of the multiplying number. So too, if the number 40, of the fourth place, multiplies 640, and the product 25600 is divided by the first number 5, the quotient becomes 5120, to be written in the fourth place from 640 — namely in the eleventh place, under which stands 10, made up of 3 (under 40) and 7 (under 640). And if we multiply 1280 by 5120, we produce 6553600; dividing this by the first number 5, we find the quotient 1310720 [?], to be set in the nineteenth place, which exceeds by one the 18 made up of the numbers 8 and 10 set under the multiplying numbers.

Likewise — to give an example in yet another progression as well — in this progression of septuple ratio:

	1	9	68						
2	4	8	6	4802	33614	235298	1647086	11529602	80707214
0	1	2	3	4	5	6	7	8	9

[Margin: The sum of any number of terms of a geometric progression of double ratio beginning from 1, once the unit is added, when multiplied by itself produces a number which — the unit again removed — is the sum of twice as many terms.]

if the number 4802, occupying the fifth place from the first number, multiplies itself, it produces 23059204; and dividing this by the first number, namely by 2, gives the quotient 11529602, to be placed in the fifth place from 4802 — namely in the ninth place, under which stands 8. So too, if the number 98, of the third place, multiplies 1647086, and the product 161414428 is divided by the first number 2, the quotient becomes 80707214, to be written in the third place from 1647086 — namely in the tenth place, under which stands 9, made up of 2 (under 98) and 7 (under 1647086), and so on.

From these examples we shall easily find the number of any place. For if in the earlier progression we must find the number to be set in the thirtieth place, we will multiply 5120 by itself, producing 26214400; dividing this by 5 gives the quotient 5242880, to be set in the twenty-first place, which exceeds by one the number 20, double the 10 set under 5120, and is set under the twenty-first place. And since 20 and 9 make 29: if we multiply 5242880, of the twenty-first place (under which 20 stands), by 2560 (under which 9 stands), we obtain a number which, divided by 5, gives the quotient 2684354560, to be placed in the thirtieth place, which exceeds by one that composite number 29.

You see, then, that we can find the last number of any geometric progression whatever, even without writing out all the intermediate numbers of that progression — though with more operations than we used above in the second rule of the arithmetical progressions.

[Margin: How the sum of the 64 places of a geometric progression of double ratio beginning from 1 may easily be found.]

Now, in the first rule we said that any number in a geometric progression of double ratio beginning from 1, once the unit is first removed, is the sum of all the preceding numbers; and in this second rule we taught that any number of a geometric progression beginning from 1, if it multiplies itself, generates a number to be set one unit short of double the place of the self-multiplying number. From this it follows that, if 1 is added to the sum of any number of terms of a geometric progression of double ratio beginning from 1, and the total is multiplied by itself, the product — once the unit is again removed — is the sum of twice as many numbers of the same progression. For the earlier sum, with the unit added, constitutes the number immediately following in the same progression; and this number, multiplying itself, produces a number to be set one unit short of double the place of the multiplying number; and so, with the unit removed, that same number is the sum of all the preceding numbers — which are indeed twice as many as the earlier ones, whose sum had been taken.

For example, the sum of seven terms, with the unit added, constitutes the eighth term; this multiplied by itself produces the fifteenth term — a number to be set one unit short of double the eighth place; and this, with the unit removed, is the sum of the fourteen preceding terms, namely twice as many as the seven whose sum had been taken. And the same reasoning holds for the rest.

And so, if one wishes to find quickly the sum of the 64 terms of a geometric progression of double ratio beginning from 1 — which is the number of squares in the game of *latrunculi* (which is called the game of chess) — we must first take the sum of these four terms 1. 2. 4. 8., namely 15. Then, adding the unit, multiply the total 16 by itself. If 1 is thrown away from the product 256, we have 255, the sum of eight terms. Again, adding 1, multiply the total 256 by itself to make 65536, and so the sum of 16 terms is 65535. And if once more, adding 1, the total 65536 is multiplied by itself, it produces 4294967296, which,

[Margin: What has been said in this rule about a geometric progression beginning from 1 is true also of one beginning from another number. — How the number of any place in a geometric progression beginning from any number is found

without the intermediate numbers. — The sum of any number of terms of a geometric progression of double ratio beginning from 1, the unit first added, multiplying itself produces a number which, the unit removed, is the sum of twice as many terms. — How the sum of the 64 places of a geometric progression of double ratio beginning from 1 may easily be found.]

the unit first removed, gives the sum of 32 terms. Finally, if 4294967296 is multiplied by itself, the number 18446744073709551616 results, which — the unit first removed — gives the sum of 64 terms.

[Margin: How much money would be required to fill the 64 squares of the game of latrunculi, such that in the first square 1 quatrino is placed, in the second 2, in the third 4, and so on, progressing by double ratio.]

And just so many quatrini would be required for one to be able to fill all 64 squares of the said game of latrunculi, by putting 1 quatrino in the first square, 2 quatrini in the second, 4 in the third, 8 in the fourth, and so proceeding on by double ratio. These make, in gold coins (allowing 400 quatrini to each gold coin), 46116860184273879 $\frac{3}{80}$ — an amount of money that is scarcely to be found in one whole kingdom, or even in several, or in the whole world; which seems incredible to many.

Indeed, scarcely so many grains of wheat exist in the entire globe of the earth as would be contained in those 64 squares if 1 grain were placed in the first, 2 in the second, 4 in the third, and so on — which we shall make plain, though to many it seems altogether incredible. According to the physicians and apothecaries, 60 grains make 1 drachm, that is, $\frac{1}{8}$ of one ounce; and thus 480 grains make 1 ounce, and 5760 grains 1 pound. Since 600 pounds commonly make up 1 measure which the Romans call a Rubius — and which differs little from the measure that sailors in Italy call a Salma — one Rubius will amount to 3456000 grains. Therefore, if the 18446744073709551615 grains contained in the said 64 squares are divided by the 3456000 grains that constitute one Rubius, there will result 5337599558365 Rubii and a little more — scarcely so many, I think, as could all be found together in the whole globe of the earth. For since a cargo ship commonly carries 3000 Rubii, at least 1779199852 ships would be required to carry that wheat; and anyone will easily convince himself that the wheat of the whole world would barely suffice to load them. And if in the whole world there are scarcely 18446744073709551615 grains of wheat, then far fewer quatrini exist, even if all the coined money were reduced to quatrini — since there is no doubt that there is a greater supply of wheat in the world than of money. This can also be gathered as follows.

[Margin: How many grains of wheat make one Rubius.]

Since a gold ducat is reckoned at Rome at 115 baiocchi, or 460 quatrini: if the 18446744073709551615 quatrini contained in the said 64 squares are divided by 115 baiocchi — that is, by 460 quatrini — the result is 40101617551542503 gold ducats, and a little more. And since 100 gold ducats make 1 pound, 1800000 pounds will contain 180000000 gold ducats — as many as one cargo ship can conveniently carry, since 3000 Rubii of wheat loading a single ship make 1800000 pounds. This weight far exceeds by many parts that immense obelisk which is seen at Rome beside St. Peter's church — for that obelisk, as experts have affirmed, contains only 1180000 pounds, and yet anyone who examines its bulk carefully will readily convince himself that it could scarcely be carried by a single ship. I say this so that no one may think we assign too little to a single ship in granting it 1800000 pounds — that is, 3000 Rubii of wheat, or 180000000 gold ducats. From this it follows that, to convey 40101617551542503 gold ducats, 222786764 ships and more would be necessary. And who could doubt that the money of the whole world, even if reduced to gold ducats, is not so great as to load so many ships?

[Margin: How many ships would be required to carry the wheat placed in the 64 squares of the game of latrunculi. — How many ships would be required to carry the money placed in the 64 squares of the game of latrunculi, if it were reduced to gold coin.]

But suppose someone puts in the first square 1 quatrino, or 1 grain; 2 in the second; 6 in the third; 18 in the fourth; 54 in the fifth; and so on, in such a way that the units placed in each square are twice as many as those in all the preceding squares together. This will happen precisely if you triple the units of the second square, then triple the product again, and so on, as is evident in this progression, and as can be shown thus. Since the units of any square are double the units set in all the preceding squares, the said units contain twice the units of the immediately preceding square, and also twice the units of all the other preceding squares.

- 1.
- 2.
- 6.
- 18.
- 54.

162. 487 [recte: 486]. 1458. 4374. 13122.
&c.

[Margin: In a progression whose first term is 1, second 2, third the triple of the second, and likewise the fourth the triple of the third, and so on: every term is double all the preceding terms.]

For example, since the 18 units of the fourth place are double these units 1, 2, 6, the said 18 units contain twice the 6 units, and besides twice the units 1 and 2. And since the 6 units are likewise double the units 1 and 2, those same 18 units contain twice the 6 units and, once more, also twice the units 1 and 2. And so, if the 6 units are tripled, the 18 units of the following place are produced, which are indeed triple the units of the immediately preceding place, but double those of all the preceding places. And the same reasoning holds in the rest.

If anyone, I say, in this way puts quatrini, or grains, in the said 64 squares, the sum found will be far greater than before.

[Margin: By what art the sum of 64 terms — beginning from 1 and so progressing that each is double all its antecedents — may be found.]

In this manner, moreover, the sum will be drawn out even if not all the numbers of that progression are set down. Since all the numbers progress by triple ratio, the beginning being made from the second place, we must find the number of the sixty-third place in triple ratio, whose starting point is 2. For this number, once found, will occupy the sixty-fourth place in the game of latrunculi; and once it is known, the sum of all 63 places will be found, as we taught in the first rule of geometric progressions; and if 1 — set in the first square of the said game — is added to it, we shall have the sum of all 64 places. For example: given these five terms, 2. 6. 18. 54. 162., if the fifth is multiplied by itself and the product divided by the first, 13122 is produced, to be set in the ninth place — namely one unit short of double the place of the self-multiplied number, as we taught in this second rule. And if the number 13122, of the ninth place, is multiplied by itself and the product divided by the first — that is, by 2 — the number 86093442 is generated, to be set in the seventeenth place. And if this in turn is multiplied by itself and the product divided by the first, the following number results: 3706040377703682, to be set in the thirty-third place. And if this again is multiplied by itself and the product divided by the first, the number 6867367640585024969315698178562 is produced, to be placed in the sixty-fifth place. But we seek the number of the sixty-third place, to which the number found for the sixty-fifth place bears the ratio of the doubled triple — that is,

ninefold — by definition 10 of Book 5 of Euclid [*Elements* V, def. 10], because the numbers set in the sixty-third, sixty-fourth, and sixty-fifth places have a continued triple ratio. Therefore, if we divide the number found by 9, we shall find this following number, 763040848953891663257299797618, to be set in the sixty-third place.

Now, take away the first number 2, and divide the remainder by the number one less than the denominator of the triple ratio, namely by 2; and finally add the quotient to the number found for the sixty-third place: this gives the sum of all sixty-three places. If 1, set in the first square of the game of latrunculi, is added to this, the sum of all 64 squares of the said game will be composed — the following:

1144561273430837494885949696427.

We shall find this same sum in the following way as well. Let the sum of the first three squares of the game of latrunculi

Cap. XXV — Progressio Geometrica (continued)

(The passage continues, without a break, the "grains on the chessboard" fantasy of the tripled progression begun on the preceding page. The individual figures 9, 81, 6561, ... are the terms standing at the doubled places 5, 9, 17, 33, 65 of a progression with common ratio 3: each is the square of the one before it.)

of the chessboard [ludi latrunculorum], which is 9. Multiplied by itself, this yields the sum 81 — the sum of twice-as-many places less one, that is, of five places. If 81 is again multiplied by itself, by the same reasoning it gives the sum 6561, of nine places (namely twice-as-many-less-one, more than five). Multiplied again by itself, this produces the sum 43046721, of seventeen places. This in turn, multiplied by itself, makes the sum 1853020188851841, of thirty-three places. And this, once more multiplied by itself, begets the sum

3433683820292512484657849089281,

of sixty-five places.

But we are looking for the sum of only sixty-four places, which is contained three times over in the sum just found. For the sum of any number of places is triple the sum of all the places preceding — because the number standing at the last place (in the given example, the sixty-fifth) is double the sum of all the places before it. From this it follows that, when the sum of all the preceding places is added to the number of the sixty-fifth place, the sum of all sixty-five places arises, which of course contains three times the sum of the preceding sixty-four places.

And so, dividing the sum already found by 3, we recover this sum of the sixty-four places of the chessboard, as before:

1144561273430837494885949696427.

[Margin: Another method of finding the sum of 64 terms, beginning from 1 and progressing so that each is double all the preceding terms taken together.]

[Margin: How much grain would be required to fill the 64 squares of the chessboard, if 1 grain is placed on the first square, 2 on the second, 6 on the third, 18 on the fourth, and so on, so that the grains of the following square are always double all the grains on all the preceding squares together.]

Now if all these grains are divided by 3456000 — the number that makes up one *Rubbio* [Rubium, a Roman grain measure] — they will make the following number of Rubbia:

3318092402512658995425 and a fraction [numeral: the trailing fraction is not legible; whole-number digits also uncertain],

to carry which, loading 3000 Rubbia into each ship, the ships required number

110393641341708863318 and $\frac{12}{40}$ [several interior digits uncertain];

and these would cover **102714380** globes made up of the land and water combined. For let us take the top deck of a single ship to be equal to a square whose side measures 70 palms — the palms that mechanics and architects use — since a ship is generally 120 palms long and 40 wide. If it were reduced to a rectangular parallelogram, its square root would be very nearly 70. Now, since 5500 palms, more or less, make up one mile, the whole circuit of the earth contains 133750000 palms, that is, 22500 miles. If we divide these palms by 70 — that is, by the width or length of one ship taken as a square, with the ships touching one another — we find that the whole circuit of the earth holds 1910714 ships. In the same way the earth's whole diameter, which completes 7159 miles, will make 39374500 palms; and if we divide these again by 70, we find that the earth's diameter comprehends 562493 ships touching one another. Multiplying, then, the 562493 ships of the diameter by the 1910714 ships of the circuit, we get 1074763250002 ships covering the whole surface of land and sea, as at the end of the first chapter on the sphere. If we divide by these the earlier figure 110393641341708863318, we find that the globes made up of land and sea come to 102714380; and indeed all the ships needed for carrying the grain, when loaded, would far exceed — many times over — the grain of the whole world; so that,

even if all the ships in which the grain of the whole world could be held were gathered, they could not cover a single earth. This is plain.

[Margin: And how many ships would be needed to carry that grain.]

[Margin: How many ships would cover the whole surface of land and sea if they touched one another.]

By another reckoning we shall make this uncountable heap of grain intelligible, if we ask how many globes or spheres could be made from those grains which, in the second manner, are contained on the 64 squares of the chessboard, each such sphere being equal to the terrestrial globe. It will go like this. Since the grains of wheat are round — though a little smaller than wheat grains — we shall take instead coriander grains, which are round. For it turns out that more globes are made from wheat grains than from coriander grains, since fewer of the latter are required to make up one globe, given how great a multitude of them there is. And since 18 coriander grains (as I have found by experience) make up a quarter of a geometric foot, and a little more, we may rightly say that 70 coriander grains, laid along their length, fill out the length of one foot.

[Margin: How many globes made from land and sea would be covered by the ships needed to carry the aforesaid grain.]

[Margin: How many globes equal to the whole earth the grain contained in the 64 squares just mentioned would make up.]

Therefore, since spheres are in the tripled ratio of their diameters (as Euclid proved in Book XII, Proposition 18), a sphere whose diameter is 343000 coriander grains — this number having to 1 the ratio tripled from that which a geometric foot's 70 grains have to 1 — will be contained in the following proportion, as here appears:

1. 70. 4900.

Again, since 5000 geometric feet make one mile, a sphere whose diameter is equal to one mile will, by the same reasoning, have to a sphere whose diameter equals one foot the proportion tripled from that which 5000 feet have to 1; and this number is 12500000000, standing to 1 as you see here:

1. 70. 4900. 343000.

and

1. 5000. 25000000. 125000000000.

And so, since a sphere with a one-foot diameter contains 343000 coriander grains, a sphere whose diameter equals one mile will exist in the sphere and contain the grains

4287500000000000.

Next, since the earth's diameter comprehends 7159 miles — we shall take it to be 7200 miles — then, this being posited, the whole terrestrial sphere will have to a sphere whose diameter equals one mile the proportion tripled from that which 7200 miles of the earth's whole diameter have to 1; and this number is 373248000000, standing to 1, as is here plain:

1. 7200. 51840000. 373248000000.

Hence, since a sphere with a one-mile diameter comprehends 4287500000000000 grains, the whole terrestrial globe will comprehend

16003008000000000000000000000000 grains.

If, then, we divide by this number the total number of grains contained on the 64 squares of the chessboard, we shall make **71½** terrestrial globes; and to make each of these — every one equal to the whole earth — composed of coriander grains so as to fill the said 64 squares in the manner we described, exceeds, it seems, all belief.

[Margin: How many ships would carry the gold ducats — struck from *quatrini* — that would fill the 64 squares in the same manner as was said of the grains of wheat.]

Now if those grains were *quatrini* [small coins], we shall make out of them the following number of gold ducats:

2488176681371385858447716731 [long integer — interior digits uncertain].

And since we said above that one ship can conveniently carry 18000000 gold ducats, if we divide the ducats by these we shall find that the ships needed to carry the said money number

138320378539658810324 [interior digits uncertain],

which would cover as much of the surface of land and sea as there are units in the number 1286162676 — because we posited above that 1074763250002 ships cover the whole surface of land and sea one time over. This sum of money exceeds every grasp of human wit.

[Margin: And how many globes of land and sea the said ships would cover.]

By a like reasoning, if anyone should wish for the sum of 40 terms of the same progression, he must first take the sum of these 5 terms — 1, 2, 4, 8, 16 — namely 31. Then, adding 1, he must multiply the aggregate 32 by itself; and casting out the unit from the product, he obtains the sum of 10 terms, 1023. Again, if the aggregate is multiplied by itself and 1 is rejected from the product, there arises the sum of 20 terms, 1048575. Finally, adding 1 once more, if the aggregate is multiplied by itself and 1 is taken from the product, there is produced the sum of 40 terms, namely

1099511627775.

And so many *quatrini* would some Duke receive, who should sell off his 40 towns under this law: that for the first there be paid to him 1 *quatrino*, for the second 2, for the third 4, and so on, always advancing by the double ratio. These *quatrini* make up **2748779069** $175/400$ gold pieces. And if annual revenues were bought with this money — such that 100 gold pieces should earn only 5 (though under the ordinary rate they would earn more) — one would hold **137438953** gold pieces and $47\frac{1}{8}$ *baiocchi* of revenue: revenues such as no monarch or republic ever had. So that Duke — who by this reckoning sells off his 40 towns — is to be judged not mad (as he seems to most people, who are little practised in arithmetic) but supremely wise.

[Margin: By what method one may easily find, from the sum of 40 terms of a geometric progression of double ratio beginning from 1.]

[Margin: How much 40 towns would fetch, if sold so that for the first 1 *quatrino* is paid, for the second 2 *quatrini*, for the third 4, and so on.]

Lastly, if anyone should wish to obtain the sum of 24 terms of the same progression quickly, he must first take the sum of these three terms — 1, 2, 4 — which is 7. Then, adding 1, he must multiply the aggregate by itself and subtract 1 from the product, so as to have the sum of 6 terms, 63. Adding 1 again, he must once more multiply the aggregate by itself and take 1 from the product, so as to make the sum of 12 terms, 4095. Finally, adding 1 once more, he must multiply the aggregate by itself and subtract 1 from the number produced, so that there arises the sum of 24 terms,

16777215.

He, then, would not be laughable who should sell a warhorse that has 24 nails in its hooves in such a way that for the first nail 1 *quatrino* is paid him, for the

second 2, for the third 4, for the fourth 8, and so on. For he would receive for the horse 16777215 *quatrini*, which make up 41943 **[fraction illegible: ?]** gold pieces — a price at which anyone would gladly hand over his own horse. Let this little that has been said of progressions suffice; for we shall write much more on the same subject in our fuller *Arithmetic*.

[Margin: In what way the sum of 24 terms of a geometric progression of double ratio beginning from 1 may be briefly elicited.]

[Margin: How much a horse having 24 nails in its hooves would fetch, if sold so that 1 *quatrino* is given for the first nail, 2 for the second, 4 for the third, and so on.]

EXTRACTIO RADICIS QVADRATAE. — CAPVT XXVI.

(The Extraction of the Square Root — Chapter 26)

A number is called *square* when it is produced by multiplying some number into itself — as 4 is, which is begotten by the multiplication of the number 2 into itself; likewise 9, when it is produced from 3 into itself; likewise 2209, because it is generated from the multiplication of 47 into itself; and so on. Unity too is called a square number by arithmeticians, though improperly, because it is produced from 1 into itself. And the number which, multiplied into itself, produces the square number is called the *side*, or the *root*, of the square.

[Margin: What a square number is.]

The extraction of the square root, then, is the finding of a number which, multiplied into itself, produces the number proposed (if that number is square); or, if it is not square, the greatest square number contained in it. Thus the extraction of the square root of the number 2209 is the finding of the number 47, because this, multiplied into itself, produces the proposed number 2209. Likewise the extraction of the square root of the number 3375 is the finding of the number 58, because this, multiplied into itself, begets the square number 3364, which is the greatest of all the squares contained in 3375. For the next greater square number — whose side, or root, is 59, one unit greater than 58 — is 3481.

[Margin: What a square root is.] [Margin: What the extraction of a square root is.]

First, the proposed number whose root is to be investigated must be marked with certain points: a point is placed under the first figure on the right (or above the first figure), and another under the third, and another under the fifth, and so on under the seventh, the ninth, and the other odd places — so that any given point

has two figures: the one under which the point is set, and another, which precedes it toward the left. The exception is the last point toward the left, which sometimes has only a single figure — namely when the number of figures is odd. The root of the proposed number will have as many figures as there are points marked. Thus this number, 21178404, will be marked in this way, and its root will have four figures in all; and this number, 456789012, will be marked in this way, and its root will be written with 5 figures.

[Margin: How the number whose root is sought must be marked with points.]

[Margin: How many figures the root of the proposed number has.]

Once the number has been marked in this way, its root is extracted as follows. Under the last point toward the left is placed the root of the greatest square contained in the figures looking to that point — a root which cannot be greater than 9 — and the same root is written to the right of the proposed number after this curved line (as we explained for the division of whole numbers). This root, in the manner of a quotient figure, is multiplied by the root set under the point, and the product is subtracted from the figures written above, the figures from which the subtraction was made being first struck out — just as we showed in the division of whole numbers. And the remainder cannot be greater than double the root placed under the point.

[Margin: In what way the square root is extracted from a given number.]

After this the root already found is doubled, and this number is written down in such order that its first figure is placed under the figure that next follows the last point toward the right, while its other figures follow in their order, advancing toward the left — so that under the figure beneath which the following point is set, nothing is written: for under it is to be placed the new quotient figure. Once this is set out, that doubled number is divided into the figure written above, together with the quotient figure to the right after the proposed number, so that a kind of whole divisor is made from that doubled number and this quotient figure; and this quotient figure is multiplied into the whole divisor, just as in the division of whole numbers, and the product is subtracted from the figures written above, and so on. But before you write this new quotient figure, you must test whether, when multiplied into that doubled number and into itself (placed after that number), it produces a number that can be subtracted from the number set above it.

Again, in the same way, the whole number after this curved line — the one hitherto set down — is doubled, and the doubled number is written under the

following point in the order we prescribed above, so that once more the place under the following point is left empty for a new quotient figure. By this doubled number the number written above is divided, and such a figure is taken for the quotient as, multiplied into that doubled number and into itself (placed after that number), begets a number that can be taken from the number written above.

By a like reasoning the whole number hitherto placed in the quotient is doubled, and the rest is done as before; and so onward, until all the points are exhausted. But all this will become clearer through examples.

[Margin: How the square root is extracted from a given number.]

Worked example 1 — the square root of 21178404

Let the square root be extracted from the number 21178404. The points being marked as stated above, I place under the last figure toward the left the figure 4, as the root of the greatest square contained in the number 21 written above it (for the square next greater than 21 is 25); and I write this same 4 again after the curved line. Multiplying the figure 4 in the quotient by the figure 4 under the point, I get 16, which subtracted from 21 (as we prescribed in the division of whole numbers) leaves 5; and these three figures, 517, will look to the following point.

Then, the quotient figure 4 being doubled to make 8, I write 8 under the 1 (as you see in the example); and dividing 51 by 8, I find that 8 is contained in 51 six times. I therefore place 6 both in the quotient after the 4 and under the point of the figure 7. Multiplying this quotient figure 6 by the whole divisor 86, and subtracting the product from the number 517 written above, there remains 1; and these three figures, 184, will belong to the following point.

Again, the quotient 46 hitherto found being doubled to make 92, I write 2 under the 8 and 9 under the 1 (as you see in the example); and I divide 18 by 92. But since 92 is not contained in 18, I place 0 in the quotient and under the point of the figure 4, and I strike out the whole divisor 920; the whole number 18404 will then look to the last point.

Lastly, the quotient 460 hitherto found being doubled to make 920, I write 0 under the 0, and 2 under the 4, and 9 under the 8 (as you see in the example). Dividing 1840 by 920, I find this number contained in it twice. I therefore place the figure 2 both in the quotient and under the point of the first figure 4. Multiplying this figure 2 by the whole divisor 9202, and subtracting the product from the number written above, nothing is left. The square root of the proposed

number is therefore **4602**, and the proposed number is itself a square, since nothing remained after the last subtraction was made.

Printed galley (right-hand column), showing the successive states as the root 4·6·0·2 is built up:

```

5.
2 1 1 7 8 4 0 4 (46 root so far
. . . .
2 1 1 7 8 4 0 4 (460.
. . . .
2 1 1 7 8 4 0 4 (4602 final root; last subtraction leaves nothing
. . . .

```

Progressively written-and-struck working figures printed alongside:

486 · 531 · 48620 · 9 · 4862002 · 992

[struck galley figures – readings uncertain]

Worked example 2 — the square root of 456789012

Let the square root next be sought from the number 456789012. The points being marked as we have taught, I write under the last point toward the left the figure 2, as the root of the greatest square contained in the number 4 written above, and I place this same 2 in the quotient. Multiplying the figure 2 in the quotient by the figure 2 under the point, I get 4, which subtracted from 4 leaves nothing; and these two figures, 56, will belong to the following point.

The quotient figure 2 being doubled to make 4, I write this 4 under the 5, leaving the following point empty for a new quotient figure. Dividing 5 by 4, I find the quotient 1, which I write both after the quotient 2 and under the point of the figure 6. Multiplying this quotient figure 1 by the whole divisor 41, and removing the product from 56, there remain 15; so that the four figures 1578 will look to the following point.

Then, the quotient 21 hitherto found being doubled to make 42, I place 2 under the 7 and 4 under the 5. Dividing 157 by 42, I find the quotient 3, which I place both in the quotient and under the point of the figure 8. Multiplying this quotient figure 3 by the whole divisor 423, and subtracting the product from 1578, there are left 309. To the following point, then, will belong these five figures, 30990.

Again, the quotient 213 hitherto found being doubled to make 426, I write 6 under the 9, 2 under the 9, and 4 under the 0. Dividing 3099 by 426, I find the quotient 7, which I write both in the quotient and under the point of the figure 0. Multiplying this quotient figure 7 by the whole divisor 4267, and taking the

product from 30990, there remain 1121; and so to the following point will belong these six figures, 112112.

Lastly, the quotient 2137 hitherto found being doubled, I set 4 under the 1, 7 under the 1, 2 under the 2, and 4 under the 1. Dividing 11211 by 4274, I find the quotient 2, which I write both in the quotient and under the point of the figure 2. Multiplying this quotient figure 2 by the whole divisor 42742, and taking the product from 112112, there remain 26628. The proposed number, therefore, is not a square; and consequently the quotient found, **21372**, is not its root, but the root of another number — the greatest square contained in the given number — namely **456762384**. For the next greater square, which has a root one unit greater than the root found, 21372, constitutes a number greater than the proposed number.

Printed galley (right-hand column), successive states as the root 2·1·3·7·2 is built up:

```

4 5 6 7 8 9 0 1 2 (21
. . . . .
4 5 6 7 8 9 0 1 2 (213
. . . . .
4 5 6 7 8 9 0 1 2 (2137
. . . . .
4 5 6 7 8 9 0 1 2 (21372 quotient found (not the exact root)
. . . . .

```

Struck working figures printed alongside:

```

241 · 15 · 24123 · 4 · 30 · 1519 · 442 · 1 · 21 · 3052 · 151971 · 24123674 ·
44227 · 4 · 2 · 136 · 2186 · 305272 · 15197138

```

Worked example 3 — the square root "per Dandam" (the Italian galley method)

The extraction of the square root can also be done in the manner of the Italians, *per Dandam* [by the "giving-down" or galley method], as we explained in the division of whole numbers — a method that is the safest of all. In it all confusion is avoided, should too small or too large a figure be taken by chance, because no figures are struck out. The method is this.

Let the square root be extracted from this number, 456789. The points being marked, as has been said, I place in the quotient the figure 6 — namely the root of the greatest square contained in the last point, 45 — and I write this same figure 6 to the right, just as in division *per Dandam* we also placed the divisor. And I proceed thus. From the quotient figure 6 into the figure 6 placed on the right side,

there arise 36, which I subtract from 45 in this way. Since 6 cannot be taken from 5, I take 6 from 10, and to the remainder, 4, I add 5, to make 9, which I write under the 5; and I keep 4 in mind — namely 3 on account of the 30, and 1 on account of the 10 — which 4, subtracted from 4, leaves nothing. And so the following point will be 967.

[Margin: The extraction of the square root *per Dandam*.]

Printed galley (Dandam method – no figures are struck out):

```

6.
127.
4 5 6 7 8 9 (675. 1345
. . .
9
78
1164

```

The parenthesized root is 675; the right-hand column 6. / 127. / 1345 gives the successive "danda" divisors, and 9 / 78 / 1164 the successive remainders ($456789 - 675^2 = 456789 - 455625 = 1164$).

(The worked figures 675, 1164 and the divisor column are printed legibly. The narrative of the second and third steps of this example runs on to the next page; catchword "Dein.")

Then the figure 6 already found is doubled, giving 12, and by this I divide the next point, 967, though leaving aside for now the figure 7 under which the point is written. I say: 1 is contained in 9 seven times (for it cannot be contained eight times); so I write 7 both in the quotient and separately after the divisor 12, and I multiply the figure 7 into the whole number 127, thus: 7 by 7 gives 49. Since 9 cannot be taken from 7, I take 9 from 10, leaving 1, which I add to 7 to make 8; this I write under the 7, but so that the figure 8 stands lower than the 9; and I keep 5 in mind — namely 4 on account of the 40 and 1 on account of the 10. Again, 7 (the figure found) into 2 gives 14, and adding the 5 we kept back I make 19. Since 9 cannot be taken from 6, I take 9 from 10, and the remaining 1 I add to 6 to make 7, which is to be written under the 6 beside the 8; and I keep 2 in mind — namely 1 for the ten of the number 19, and 1 for the 10 from which we took the 9, and so on. Finally, the same figure 7 into 1 gives 7, and adding the 2 we kept I get 9; subtracting this from 9 nothing remains, and therefore nothing is to be written under the 9. The next point — which is the last — will be 7889.

Lastly, the whole root found so far, 67, must be doubled, giving 134, by which I divide the last point, 7889 (still leaving aside the figure 9 placed above the point). I say: 1 is contained in 7 five times. So I place 5 in the quotient, and after the divisor 134; and I multiply the figure 5 into the whole number 1345, thus: 5 by 5 gives 25, and taking 5 from 9 there remain 4, which I place under the 9, and I keep 3 back on account of the 20. Again I say: the figure 5 found, into 4, gives 20; adding the 2 we kept, I get 22. Taking 2 from 8 there remain 6, which I write under the 8, and I keep 2 in mind for the 20, that is, two tens. Again I say: the figure 5 found, into 3, gives 15; adding the 2 that were kept, I get 17. Taking 7 from 8 there remains 1, which I place under the 8; and 1 remains on account of the one ten. Finally I say: the same figure 5 found, into 1, gives 5; adding the 1 we kept, I get 6, which subtracted from 7 leaves 1, which I place under the 7. And so the whole root is found to be 675, with 1164 left over. From these is formed the fraction $1164/1351$, so that the approximate root (as we shall explain under the approximation of roots in the following chapter) is $675 \frac{1164}{1351}$ — less, however, than the true root, since its square is here only 456788 1607533/1825201.

This method of extraction is excellent, since in it all the remainders of the operations appear distinctly and without confusion, so much so that if the figure taken into the quotient should happen to be too large or too small — too large if the products from the point proposed cannot be subtracted, too small if, once the subtraction is made, the remainder is greater than twice the root found — the error can be corrected with ease, as was said above under Division by the galley method [per Dandam].

The proof [Examen] of square-root extraction is threefold, just as it is for the division of whole numbers. The first is by casting out 9s, the second by casting out 7s, and the third by multiplication, as was said in the division of whole numbers. But the root found is here taken in the place of the divisor: for if the number proposed be divided by the root found, the quotient will be that same root; and if anything is left over in the extraction of the root, the same will be left over in the division, provided the same figures of the root found be taken in the quotient — although in the last partial division a larger figure could sometimes be taken, namely when the remainder of the extraction exceeds the root.

[Margin: The proof of square-root extraction is threefold.]

Thus the first example will be tested by 9 in this way. Casting out 9s from the root 4602, there remain 3, which I write on either side of the cross, because the root is both divisor and quotient, as we said. Multiplying these two figures 3 and 3

together gives 9, and casting out 9 there remains 0, to be placed at the top of the cross. Finally, casting out 9s from the number proposed, there likewise remains 0. The second example will be tested by 9 thus. Casting out 9s from the root 21372 there remain 6, to be placed on either side of the cross. Multiplying these figures 6 and 6 together gives 36, and casting out 9s from 36 and from the remainder of the extraction, there are left 6. Just as much remains if 9s be cast out from the given number.

Diagram. Two small casting-out-nines check-crosses (X-shaped), each with a numeral at the four arms, verifying the two square-root extractions.

- Cross 1 (test of the root 4602): top 0; left arm 3; right arm 3; bottom 0.
- Cross 2 (test of the root 21372): top 6; left arm 6; right arm 6; bottom 6.

Note in passing that in no square-root extraction can the remainder be greater than twice the root found, plus one unit — the unit being that by which the root found falls short of the root of the next larger square number. The reason is this: any square exceeds the next smaller square number by twice the root of that smaller square, plus one further unit; so that if 1 be added to twice the root of any square, and this to the smaller square, the next larger square results. For example, the square number 64 exceeds the square number 49 by 15. Now 14 is twice the root of the square 49, whose root is 7; so if 1 be added to 14, that is, to twice the root 7, and this sum be added to 49, the number 64 — the next larger square — is produced, whose root is 7 plus one unit, that is 8. If, then, someone should propose the number 63, so that its square root is to be extracted, the root 7 will be found, with 14 left over, which is twice the root; whereas if someone should propose the number 64, the root 7 would again be found, but it would be an error, since 15 would be left over, which is greater than twice the root 7. Therefore the root of the number 64 will be 8.

[Margin: The remainder in square-root extraction cannot be greater than twice the root found. What the difference is between two successive squares.]

APPROXIMATION OF ROOTS IN NON-SQUARE NUMBERS. CHAPTER XXVII.

Since, when the number proposed is not a square, the root found and multiplied into itself yields a number smaller than the number itself — as appeared in the later example, where the root multiplied into itself gave a number falling short of the proposed number by the whole of 26628 — we shall show here a twofold way

by which a closer root may be found, such that its square differs from the proposed number by a scarcely perceptible difference. For the true root cannot be expressed by a number, but only by a line, as will be demonstrated more fully in our Arithmetic. By the first way a closer root will indeed be found, but one smaller than the true root, so that its square is always exceeded by the proposed number. By the second way a closer root will likewise be found without limit, but one exceeding the true root, so that its square is always greater than the proposed number. Both ways have been demonstrated geometrically, by Theon of Alexandria [Theone Alexandrino] in Book I of the *Almagest* of Ptolemy [Ptolemaei] (*Almagest* I), and by Federico Commandino [Federico Commandino] in his book on Archimedes' [Archimedis] *On the Measurement of the Circle*.

[Margin: How a closer root is found, which is nevertheless smaller than the true one.]

The first way, then, is as follows. Having found the root of the largest square contained in the proposed number, one adds to it a fraction whose numerator is the remainder of the extraction — that is, that by which the proposed number exceeds the square that the root found produces when multiplied into itself — and whose denominator is twice the root found, together with one unit (which is precisely that by which the root found falls short of the root of the square number next larger than the proposed number, this latter being the one closest above the square number closest below the proposed number). By this procedure the root will be much closer than the one found, but still smaller than the true one. To this, if there be added that which arises from dividing the excess (by which the proposed number exceeds the square of the root already found as closer) by the number composed of twice that closer root and the excess by which the square of the number next larger exceeds the same closer root just found, there will result a still closer root, smaller nonetheless than the true one. And to this, if again there be added that which arises from dividing the excess (by which the proposed number exceeds the square of the closer root found in the last place) by the number composed of twice that same last closer root and the excess by which the square of the next larger number exceeds that same last closer root, a yet closer root will be produced, still smaller than the true one. And in this way it is possible always to find a closer root without limit, yet never the true root, but always one somewhat smaller than the true one.

Example. Let the proposed non-square number be 20. The root of the next smaller square is 4, which multiplied into itself gives 16, with 4 left over. If, then, to the root 4 one adds the fraction $\frac{4}{9}$ — whose numerator is the remainder of the

extraction, and whose denominator is twice the root found, 4, together with 1 — there results the closer root $4\frac{4}{9}$. For its square is $19\frac{61}{81}$, which is indeed smaller than the proposed number 20, but differs from it less than the square 16 of the first root 4.

Having taken away this square $19\frac{61}{81}$ from the proposed non-square number 20, there remain $\frac{20}{81}$. Again, the root 5 of the square next larger than the proposed number exceeds the closer root $4\frac{4}{9}$ just found by this small part, $\frac{5}{9}$; which added to twice the closer root $4\frac{4}{9}$ — that is, to $8\frac{8}{9}$ — makes the number $9\frac{4}{9}$. If by this the remainder $\frac{20}{81}$ be divided, there results the quotient $\frac{180}{6885}$ [?], which added to the closer root $4\frac{4}{9}$ just found gives the still closer root $4\frac{4}{9} + \frac{180}{6885}$, that is, $4\frac{8}{17}$. For its square is $19\frac{285}{289}$, which is again smaller than the proposed non-square number 20, but comes closer to it than the square $19\frac{61}{81}$ of the root $4\frac{4}{9}$ found before this root $4\frac{8}{17}$.

Once more, subtracting this square $19\frac{285}{289}$ from the proposed non-square number 20, there remain $\frac{4}{289}$. Again, the root 5 of the square next larger than the proposed number exceeds the closer root $4\frac{8}{17}$ last found by this small part, $\frac{9}{17}$; which added to twice the last closer root $4\frac{8}{17}$ — that is, to $8\frac{16}{17}$ — makes the number $9\frac{8}{17}$. If by this the remainder $\frac{4}{289}$ be divided, there results the quotient $\frac{68}{46529}$ [?], which added to the closer root $4\frac{8}{17}$ last found makes the still closer root $4\frac{8}{17} + \frac{68}{46529}$, that is, $4\frac{76}{161}$. For its square is $19\frac{25901}{25921}$, which — though [smaller] than the proposed non-square number — comes closer to it than the square $19\frac{285}{289}$ of the root $4\frac{8}{17}$ found before it. And in this manner it will be possible to approach nearer and nearer to the truth, though we shall never reach it, but shall always fall short of it.

[Margin: How a closer root is found which is nevertheless smaller than the true one.]

The second way is this. Having found the root of the largest square contained in the proposed number, one adds to it a fraction whose numerator is the remainder of the extraction — that is, that by which the proposed number exceeds the square the root found produces when multiplied into itself — and whose denominator is twice the root found. By this procedure the root will be much closer than the one found, but greater nonetheless than the true one. For by this reasoning the square of the closer root exceeds the proposed number by less than the square of the root found exceeds it. From this, if there be subtracted that which arises from dividing the excess (by which the square of the closer root already found exceeds the proposed number) by twice that same closer root, there will remain a still closer root, but greater than the true one. And from this,

if again there be subtracted that which arises from dividing the excess (by which the square of the closer root found in the last place exceeds the proposed number) by twice that same last closer root, there will remain a yet closer root, greater still than the true one. And in this way it will always be possible to find a closer root without limit, yet never the true root, but always one somewhat greater than the true one.

[Margin: How a closer root is found which is nevertheless greater than the true one.]

Example. Let the same non-square number be proposed, 20. The root of the next smaller square is 4, which multiplied into itself makes 16, with 4 left over. If, then, to the root 4 one adds the fraction $\frac{4}{8}$ — whose numerator is the remainder of the extraction, and whose denominator is twice the root found, 4 — there results the closer root $4\frac{4}{8}$, that is, $4\frac{1}{2}$. For its square is $20\frac{1}{4}$, which is indeed greater than the proposed number 20, but differs from it less than the square 16 of the first root 4.

Now, if $\frac{1}{4}$ (the excess, namely, by which the square number $20\frac{1}{4}$ of the root $4\frac{1}{2}$ just found exceeds the proposed number 20) be divided by twice the closer root $4\frac{1}{2}$ already found — that is, by 9 — there results the quotient $\frac{1}{36}$; which taken from the root $4\frac{1}{2}$ just found leaves the closer root $4\frac{1}{2} - \frac{1}{36}$, that is, $4\frac{17}{36}$. For its square number is $20\frac{1}{1296}$, which is likewise greater than the proposed number 20, but is less distant from it than the square $20\frac{1}{4}$ of the root $4\frac{1}{2}$ found before this one.

If again $\frac{1}{1296}$ (the excess, namely, by which the number $20\frac{1}{1296}$ exceeds the proposed number 20) be divided by twice the root $4\frac{17}{36}$ already found — that is, by $8\frac{34}{36}$, or $8\frac{17}{18}$ — there results the quotient $\frac{1}{11592}$; which subtracted from the root $4\frac{17}{36}$ just found leaves the still closer root $4\frac{17}{36} - \frac{1}{11592}$, that is, $4\frac{5473}{11592}$ [?]. For its square number is the following, 20 [numeral: ? — deep galley fraction not resolvable], which is also greater than the proposed number, but is less distant from it than the square $20\frac{1}{1296}$ of the root $4\frac{17}{36}$ found before this one. And in this manner it will always be possible to approach nearer and nearer to the truth, which nevertheless we shall never reach, but shall always exceed.

I should like at this point to explain another method of finding a sufficiently close root by a single extraction of the root — a method very familiar to mathematicians, and in common use. It is as follows. After the number whose root is sought, one appends some pairs of ciphers [binarij cifrarum, i.e. pairs of

zeros], such as 0000, or 000000, or 00000000, and so on; and the more pairs of ciphers are appended, the closer the root that will be drawn out. Then from the whole number, with the ciphers appended, the root is extracted as has been taught; and from it, on the right, as many figures are cut off as there were pairs of ciphers appended to the proposed number. The remaining figures will give the root, together with a fraction whose numerator is the number expressed by the figures cut off, and whose denominator is 10 if only one pair of ciphers was appended, or 100 if two pairs were appended, or 1000 if three pairs, and so on — so that that fraction contains tenths, or hundredths, or thousandths, and so forth.

[Margin: Another extraction of the approximate square root.]

For example, let the square root of 20 be sought. When three pairs of ciphers have been appended, there arises the number 20000000, whose root, drawn out by the galley method [per Dandam], is 4472 (we here neglect the remainder 1216). Cutting off the three figures 472 on the right, on account of the three pairs of ciphers appended, the root will be $4\frac{472}{1000}$ — close enough, but smaller than the true root, since its square is only $19998784/1000000$ [galley numerator dense — reading assisted], that is, $19\frac{998784}{1000000}$, which is a little smaller than the given number 20. And if you add 1 to the numerator, the root becomes $4\frac{473}{1000}$, greater than the true root, since its square number is $20\frac{7729}{1000000}$, a little greater than the given number 20.

That a close square root is rightly found by this reasoning has been demonstrated by us in the *Practical Geometry* [*Geometria Practica*], Book 6, Proposition 20.

Diagram. The galley ["per Dandam"] layout for extracting the square root of 20000000, giving root 4472 with remainder 1216. The dividend is written with the root in parentheses; the successive divisors run down the right column and the running remainders down the center.

- Dividend / root: 20000000 (4472
- Right column (successive divisors): 4 · 84 · 887 · 8942
- Center column (successive remainders): 4 · 64 · 191 · 1216

The same reasoning also has a place in fractional numbers. For if to both the numerator and the denominator some pairs of ciphers be appended, the numerator will be formed from the root of the numerator increased by those ciphers, and the denominator from the root of the denominator increased by as many ciphers. For example, suppose the root of the fraction $\frac{2}{3}$ be sought. When two pairs of ciphers have been added, the fraction becomes $\frac{20000}{30000}$. The root of the numerator is 141, and of the denominator 173. Therefore the approximate

root of the fraction $\frac{2}{3}$ will be $\frac{141}{173}$, whose square number is $\frac{19881}{29929}$, a little smaller than $\frac{2}{3}$, and so on with others.

Read what we have written in the *Practical Geometry*, Book 4, Chapter 2, Number 5, and Book 6, Proposition 21. There you will find an easy method of extracting the square root from any non-square fraction.

[Margin: The extraction of the approximate square root from fractions.]

EXTRACTION OF THE CUBE ROOT. CHAPTER XXVIII.

A cube is a number generated from some number multiplied into itself and then into the product — such as 8, which is produced from the multiplication of the number 2 into itself, and into the product 4. Likewise 27, which is produced from 3 into itself, and into the product 9. Likewise 103823, since it is generated from 47 into itself, and into the product 2209, and so on. Unity too is called a cube number by arithmeticians, though improperly, because it is produced from 1 into itself, and into the product 1. The number which, multiplied into itself and into the product, generates the cube, is called the side [latus], or the root of the cube [radix cubi].

[Margin: What a cube number is.]

The extraction of the cube root from a given number, then, is the finding of a number which, first multiplied into itself and then into the product, generates the given number — if it is a cube — or, if it is not a cube, the largest cube number contained in it. For example, the extraction of the cube root of the number 59319 is the finding of the number 39, since this, multiplied into itself and into the product 1521, generates the proposed number 59319. Likewise, the extraction of the cube root of the number 116592 is the finding of the number 48, since this, multiplied into itself and into the product 2304, produces the number 110592, which is the largest cube contained in the given number 116592: for the next larger cube, whose side or root is 49 — one unit greater than 48 — is 117649. The extraction of the cube root, however, is more difficult than that of the square root, and various writers prescribe it by various ways. We shall hand down the one which, as something in a manner new, we drew out at the end of Book 6 of the *Practical Geometry*, at Proposition 19. It is as follows.

[Margin: What cube-root extraction is.]

First, the proposed number whose cube root is to be drawn out must be marked with certain points, a point being placed under the first figure on the right (or above that figure), and another under the fourth figure, two figures in between being passed over, and another under the seventh, two others in between being passed over, and so on under the tenth, the thirteenth, and so forth — always leaving two figures after each point, so that every point has three figures, namely the one under which the point is marked and the two others that precede it toward the left, except the last point on the left, which sometimes has a single figure, sometimes only two. The root of the proposed number will have as many figures as there are points marked. Thus this number, 256047896, will be pointed as you see in this arrangement, and its cube root will have three figures. But this number, 3456789012786, will be pointed thus [as you see in this arrangement], and its root will be written with five figures.

[Margin: How the number whose root is sought is to be marked with points. How many figures the cube root of the given number has.]

Next, this appended table containing the cubes of the nine arithmetical figures ought to be at hand, so that the first point on the left may be dealt with.

Radices [Roots]	Cubi [Cubes]
1	1
2	8
3	27
4	64
5	125
6	216
7	343
8	512
9	729

[Margin: In what manner the cube root is extracted from the given number.]

These things being prepared, let the cube root be drawn out of the number 256047896. Having marked the points as we said, from the first point 256 on the left I take the cube 216 (whose root is 6), the largest contained in it. And since 40 remain, the next point will be 40047, and the first figure of the root will be 6.

I prepare the divisor by multiplying the square of the root 6 of the cube subtracted — namely 36 — by 300, which gives 10800. If by this I divide my point

40047, I shall find the quotient 3, for the second figure of the root. This I multiply into 10800, the divisor found, and I make 32400. Then I multiply the square of the new figure found, 3 — namely 9 — into the product of the first figure found, 6, into 30 (that is, into 180), and I make 1620. Lastly, to the sum of these two products — that is, to 34020 [32400 + 1620] — I add the cube of the new figure found, 3, namely 27; and this sum, 34047, I subtract from my point 40047. Since 6000 are left over, the last point will be 6000896. I therefore write the figure found, 3, after the previous 6 — that is, after the root of the largest cube, 216, subtracted from the first point.

After this I prepare, in the same way, a new divisor for the last point, thus. I multiply the square of the whole root 63 found so far — that is, 3969 — again into 300. The product, namely the number 1190700, will be the divisor, by which if I divide my point 6000896, I shall find the quotient 5, to be written after the root 63 found so far. This figure 5 found I likewise multiply into the divisor found, 1190700, and make 5953500. Then I multiply the square of that same new figure 5 — namely 25 — into the product of the root 63 previously found, multiplied by 30 (that is, into 1890), and produce 47250. Lastly, to the sum of these two products, 5953500 and 47250, I add the cube 125 of the figure 5 found, and the whole sum, 6000875, I subtract from my point 6000896, leaving 21. The proposed number, therefore, is not a cube, and accordingly the quotient found [continues on next page]...

Diagram. The running galley layout of the cube-root extraction of 256047896, giving the root 635 (stated in the margin as "the root of the proposed number, 635") with 21 left over. The successive partial products and subtractions are set out at the right of the text.

- Second figure (3): $32400 \cdot 1620 \cdot 34020 \cdot 27 \cdot \text{sum } 34047$; from 40047 leaves 6000 [printed galley: 6009, recte 6000].
- Third figure (5): $5953500 \cdot 47250 \cdot 125 \cdot \text{sum } 6000875$; from 6000896 leaves 21.
- Cube root of the proposed number: 635.

Practical Arithmetic — Cap. XXVIII (conclusion), Cap. XXIX, and the closing Index

(Epitome Arithmeticae Practicae. Source: printed p. 78 and the two unnumbered Index leaves.)

Cap. XXVIII — Extraction of the Cube Root (conclusion)

(continued from the preceding page)

the root found, 635, is not the cube root of the given number itself, but of another number — the largest cube contained within the given number, which is 256047875. For the next-larger cube, whose root exceeds the root found (635) by one unit, yields a number greater than the number proposed.

But if another point had remained, one would have to multiply the square of the whole root 635 found so far by 300, so that a new divisor would arise, and so on. From this worked example, now set out, I think our rule for cube-root extraction is clear enough — a rule that differs somewhat from those of others, but in my judgment is easier.

[Margin: How large the remainder in a cube-root extraction can be. How great the difference between two consecutive cubes is. By what means one may know whether the figure taken in the quotient is too small.]

It should be understood, moreover, that in no cube-root extraction can the remainder, if there is one, be greater than three times the square of the root found, together with three times that same root; otherwise the proposed number would have a root one unit greater than the one found. The reason is that any cube exceeds the next smaller cube by three times the square produced from the root of the smaller cube, together with three times that same root, and, in addition, by one unit. So if 1 is added to three times the square produced from the root of any cube and to three times that same root, and this sum is then added to that cube, the next-larger cube results. For example, the cube 27 exceeds the cube 8 by 19. Now the number 18 is made up of three times the square of the root 2 — that is, of 12 — and of three times that same root 2 — namely, of 6; and so, if 1 is added, it produces the difference between the two consecutive cubes 8 and 27.

Therefore, if at some point in an extraction the remainder is so large that you suspect you have taken too small a figure in the quotient — for the operation itself, whenever the subtraction cannot be carried out, shows that the figure assumed was too large — you must take three times the square of the root found up to that point in the operation, together with three times that same root. For if the sum of these two numbers is less than the remainder of the extraction, know that you have taken too small a figure into the quotient. But if the said sum equals the remainder, or is greater, the figure was rightly assumed in the quotient. You will learn the same thing if you make a trial with a larger figure: for if the

multiplications are performed as the rule prescribes and the subtraction cannot be carried out, the earlier figure was correctly taken; but if the subtraction can be carried out, the figure taken was too small.

[Margin: The threefold proof of cube-root extraction.]

The checking of a cube-root extraction can also be done in three ways, just as with the division of whole numbers and the extraction of the square root: namely, first by casting out nines, second by casting out sevens, and third by multiplication, as was explained for the division of whole numbers — except that here the root found must be taken in place of the divisor. Thus the example above will be checked by nines as follows. Casting the nines out of the root 635, there remain 5, which I place on both arms of the cross — since the root is taken in place of the divisor and also serves as the quotient, as we said. Now, multiplying these two figures, 5 and 5, by each other, and multiplying the product 25 by that same figure 5 so as to generate the cube 125 of the figure 5, then, from this cube taken together with the remainder of the extraction, 21, after casting out nines, there remain 2, to be placed in the upper part of the cross. And because, if nines are cast out of the proposed number, there likewise remain 2 — to be written in the lower part of the cross — the root was correctly extracted.

$$\begin{array}{c} 2 \\ 5 \times 5 \\ 2 \end{array}$$

(proof-cross for casting out nines)

The same example will be checked by sevens thus. Casting the sevens out of the root 635, there remain 5, which I write on both arms of the cross. Then, cubing this figure 5, and casting sevens out of the cube 125, there remain 6, which, added to 0 — namely, to the remainder that is left when sevens are cast out of the extraction-remainder 21 — make 6. And since 6 likewise remain when sevens are cast out of the proposed number, the cube root has been found the more surely.

$$\begin{array}{c} 6 \\ 5 \times 5 \\ 6 \end{array}$$

(proof-cross for casting out sevens)

And if the root found, 635, is cubed — that is, first multiplied by itself and then by the product — and the remainder 21 is added to the number so generated, the proposed number, from which the root was extracted, will be produced.

APPROXIMATION OF ROOTS IN NON-CUBE NUMBERS. CHAPTER XXIX.

(*APPROPINQVATIO RADICVM IN NVMERIS NON CVBIS. CAPVT XXIX.*)

[Margin: Approximation of the cube root.]

To extract an approximate root from numbers that are not cubes, no more accurate way has been found to this day than the one we demonstrated in our *Practical Geometry*, Book 6, Chapter 20, which is as follows. To the proposed number append some triples of ciphers [zeros] — and the more triples of ciphers are appended, the closer the root that will be extracted — and from this whole number extract the cube root, as has been taught. Then, casting off from that root as many figures on the right as there were triples of ciphers appended, the remaining figures will give the whole-number part of the root, to which is to be added a fraction whose numerator is the figures cast off and whose denominator is a one followed by as many ciphers as there were triples of ciphers added — namely 10 if one triple was added, or 100 if two triples were added, or 1000 if three, and so on — so that the fraction expresses either tenths, or hundredths, or thousandths, and so forth.

For example: let an approximate cube root be extracted from the number 29160, which is not a cube. Append three triples of ciphers, so that the fraction will contain thousandth parts, denominated, that is, by 1000; and from the whole number 29160000000000 extract the cube root, which will be found to be 30779, less than the true root — because there was some remainder in the extraction — and so the root greater than the true one will be 30780. Casting off three figures on the right, on account of the three triples of ciphers appended, the approximate root will be $30\ 779/1000$, less than the true root, since its cube is 29158 and $388419139/1000000000$ [?]; while the approximate root greater than the true one will be $30\ 780/1000$, since its cube is 29161 and $230552000/1000000000$ [?].

[Margin: Extraction of the approximate cube root from fractions.]

This method also applies to fractional numbers. For if some triples of ciphers are appended to both the numerator and the denominator, the numerator [of the result] is obtained from the root of the numerator so augmented by those ciphers, and the denominator from the root of the denominator so augmented — just as we said above concerning the square root of fractional numbers. You will find another convenient method for extracting the cube root from a given fractional number in our *Practical Geometry*, Book 6, Proposition 21.

It would now be the moment to treat the extraction of the other roots, which are infinite in number, as we explained in our *Practical Geometry*, Proposition 19. But since this treatment is more difficult, and since the finding of the square and cube roots is more necessary for understanding the books of Archimedes, Ptolemy, and the other mathematicians, we have deliberately deferred it to our fuller *Arithmetic* and to the *Practical Geometry*. For in that fuller *Arithmetic* we shall treat not only all the roots, together with their approximations (as we also did in the *Practical Geometry*, Book 6), but shall also set forth countless other matters from which, in this Compendium, we have deliberately refrained.

THE END. (*FINIS — close of the Epitome Arithmeticae Practicae**)*

INDEX OF ALL THE CHAPTERS OF THIS ARITHMETIC

(*INDEX OMNIVM CAPITVM HVIVS ARITHMETICAE.*)

No.	Chapter	Page
1	Numeration of whole numbers	3
2	Addition of whole numbers	5
3	Subtraction of whole numbers	7
4	Multiplication of whole numbers	10
5	Division of whole numbers	13
6	Numeration of fractional numbers	22
7	Estimation, or the value, of fractional numbers	23
8	Fractions of fractional numbers	24
9	Reduction of fractional numbers to their lowest numbers, or terms	25
10	Reduction of fractional numbers to a common denomination and to whole numbers, and likewise of whole numbers to any fraction, and finally of fractions of fractional numbers to simple fractions	26
11	Addition of fractional numbers	28
12	Subtraction of fractional numbers	29
13	Multiplication of fractional numbers	29
14	Division of fractional numbers	30
15	Insertion (<i>insitio</i>) of fractional numbers	31
16	Various little problems concerning whole numbers and	34

No.	Chapter	Page
	fractions	
17	The rule of three — which by another name is usually called the golden rule, or the rule of proportions	35
18	The inverse rule of three	39
19	The compound rule of three	40
20	The rule of partnership [<i>Regula Societatum</i>]	40
21	The rule of alligation	52
22	The rule of single false position	55
23	The rule of double false position	58
24	Arithmetic progressions	65
25	Geometric progressions	68
26	Extraction of the square root	73
27	Approximation of roots in non-square numbers	76
28	Extraction of the cube root	77
29	Approximation of roots in non-cube numbers	78

INDEX OF THE MATTERS CONTAINED IN THE INDIVIDUAL CHAPTERS

(*INDEX EORVM, QVAE IN SINGVLIS Capitibus continentur.*)

In the Preface

- The excellence and usefulness of Arithmetic. — p. 2

In the numeration of whole numbers

- What numeration is. — 3
- How the value of the ten arithmetical figures is to be known. — *ibid.*
- How many places there are in any given number. — *ibid.*
- Which is the first and which the last figure in any number. — *ibid.*
- Why the order of places in any number proceeds from right toward left. — *ibid.*
- What each figure signifies when placed in any given place. — *ibid.*
- The places in any number exceed one another in order in tenfold proportion. — *ibid.*

- What must be observed so that a given number may be easily expressed. — *ibid.*

In the addition of whole numbers

- What addition is. — 4
- How the numbers to be added are to be arranged. — *ibid.*
- How addition is done. — *ibid.*
- What is to be done when, from the figures of a single place, a number results that must be written with three figures. — 5
- What is to be done when many numbers are to be added. — *ibid.*
- How the proof of addition by nines is done. — *ibid.*
- By what method nines may easily be cast out of any number as often as it can be done. — *ibid.*
- The marvelous property of the number nine. — *ibid.*
- The proof of addition by nines is fallible, and why. — 6
- Why the proof of addition by nines is employed by arithmeticians, although it is fallible. — *ibid.*
- How the proof of addition by sevens is done. — *ibid.*
- By what method sevens are to be cast out of any number. — *ibid.*
- The proof of addition by sevens is fallible, and why. — *ibid.*
- Why the proof of addition by sevens is employed by arithmeticians, although it is fallible. — *ibid.*
- The proof of addition by sevens is less fallible than that by nines, and why. — *ibid.*
- A little table of the proof by sevens. — 7
- How the proof of addition by addition is done. — *ibid.*
- How the proof of addition by subtraction is done. — *ibid.*

In the subtraction of whole numbers

- What subtraction is. — *ibid.*
- How one may know which of two numbers is the greater. — *ibid.*
- How the number to be subtracted is to be placed beneath the number from which the subtraction is made. — *ibid.*
- How subtraction is done. — 8
- What is to be done in subtraction when the lower figure is greater than the upper. — *ibid.*

- An easier method of subtraction when the lower figure is greater than the upper. — *ibid.*
- What is to be done in subtraction when there are several numbers. — 9
- How the proof of subtraction by nines and by sevens is done. — *ibid.*
- How the proof of subtraction by addition and by subtraction is done. — *ibid.*

In the multiplication of whole numbers

- What multiplication is. — 10
- The Pythagorean table [multiplication table]. — *ibid.*
- How the Pythagorean table is constructed. — *ibid.*
- The use of the Pythagorean table for finding what the multiplication of one figure by another figure produces. — *ibid.*
- The rule for multiplying a figure by a figure. — *ibid.*
- How the numbers are to be arranged in multiplication. — 11
- How any number is multiplied by a single figure. — *ibid.*
- How any number is multiplied by a number written with several figures. — *ibid.*
- How the proof of multiplication by nines and by sevens is done. — 12
- How the proof of multiplication by the divisor is done. — 12 & 13.
- How numbers having ciphers at the beginning are multiplied more easily. — 13

In the division of whole numbers

- What division is. — *ibid.*
- What the quotient is. — *ibid.*
- How the numbers are to be arranged in division. — *ibid.*
- How division is done. — *ibid.*
- In the quotient no number greater than 9 can be placed. — *ibid.*
- The remaining number in a division must always be less than the divisor. — *ibid.*
- How a number is divided by a single figure. — *ibid.*
- Which number is said to be placed above the divisor. — *ibid.*
- The use of the Pythagorean table for finding how many times the figure of the divisor is contained in the number placed above it. — *ibid.*
- How many figures the quotient can have. — 14
- How a number is divided by several figures. — *ibid.*

- Which number is said to be placed above any given figure of the divisor. — ibid.
- How the figure of the quotient that has been found is to be multiplied into the divisor. — ibid.
- What is to be done with the number left over from the division. — 15
- What is to be done when a smaller number is set to be divided by a larger. — ibid.
- How some people multiply the figure of the quotient found into the divisor. — ibid.
- Wherein the difficulty of division consists. — ibid.
- What is to be done when a figure taken in the quotient is too small or too large. — 16
- How division may easily be set up so that no error is made. — 18
- How division is done by others. — 19
- The convenience in the manner of division that others use. — ibid.
- How this method may be made quicker. — ibid.
- Another method of completing a division. — ibid.
- How division "per Dandam" is done. — 20
- How the proof of division by nines is done. — ibid.
- How the proof of division by sevens is done. — 21
- How the proof of division by multiplication is done. — ibid.
- How a not-yet-completed division is examined. — ibid.
- The ease of division when the divisor has some ciphers at the beginning. — ibid.
- How division is sometimes made easy when the number to be divided has some ciphers at the beginning. — 22
- Addition, subtraction, multiplication, and division are the foundations of all that is taught in arithmetic. — ibid.

In the numeration of fractional numbers

- What a fractional number is. — ibid.
- What the numerator and denominator of a fraction are. — ibid.
- How any fraction is written and pronounced. — ibid.
- Whence fractions arise. — ibid.
- When a smaller number is divided by a larger, a fraction results. — ibid.
- Any fraction is an aliquot part of the numerator, denominated by the denominator. — ibid.

In the estimation, or value, of fractional numbers

- How the value of fractions is increased. — 23.
- How the value of fractions is decreased. — *ibid.*
- Fractions whose numerators bear the same proportion to their denominators are equal. — *ibid.*
- If the numerator and denominator of any fraction are multiplied or divided by any number whatever, a fraction of the same value is generated. — *ibid.*
- Which fraction is equivalent to one whole unit. — *ibid.*
- Which fraction is less than one whole unit. — *ibid.*
- Which fraction is greater than one whole unit. — *ibid.*
- How one may know which of two fractions is the greater. — *ibid.*
- How the value of a given fraction is tested in a smaller coin, weight, or measure. — *ibid.*
- What the Iulus [giulio], the Baiocco, and the Quattrino signify among the Romans. — 24

In the fractions of fractional numbers

- What fractions of fractions are, and whence they arise. — *ibid.*
- How fractions of fractions are pronounced and written. — *ibid.*

In the reduction of fractional numbers to their lowest numbers, or terms

- Why fractions are reduced to their lowest terms. — *ibid.*
- By what art fractions are reduced to their lowest terms. — 25
- Which fraction cannot be reduced to lower terms. — *ibid.*
- What a prime number is, and what numbers are prime to one another. — *ibid.*
- How the greatest common measure of the numerator and denominator of a given fraction is found. — *ibid.*
- When the numerator and denominator of a given fraction have no common measure except unity. — *ibid.*
- How the greatest common measure of any two numbers is found. — *ibid.*
- From what source the rule for finding the greatest common measure of two numbers is derived. — 26
- Another method of reducing fractions to their lowest terms. — *ibid.*

In the reduction of fractional numbers to a common denomination and to whole numbers, and likewise of whole numbers to any fraction, and finally of fractions of fractional numbers to simple fractions

- How two fractions are reduced to a common denomination. — *ibid.*
- The finding of a number counted [i.e., a common multiple] from any given set of numbers. — *ibid.*
- The finding of the least number counted from any set of numbers. — *ibid.*
- How several fractions, more than two, are reduced to a common denominator. — 27
- Another method of reducing two fractions to a common denomination. — *ibid.*
- The usefulness of the least number counted from the denominators of the given fractions. — *ibid.*
- How a fraction whose numerator is greater than its denominator is reduced to whole numbers. — *ibid.*
- By what method whole numbers are reduced to any fraction whatever. — *ibid.*
- How fractions of fractions are brought back to simple fractions. — *ibid.*

In the addition of fractional numbers

- How the addition of fractions is done. — *ibid.*
- What is to be done in the addition of fractions when whole numbers are present. — *ibid.*
- The practice of adding fractions of different denominations to one another. — *ibid.*
- How the proof of the addition of fractions by subtraction is done. — *ibid.*

In the subtraction of fractional numbers

- How the subtraction of fractions is done. — 29
- What is to be done in the subtraction of fractions when whole numbers are present. — *ibid.*
- What is to be done in subtraction when there are several fractions. — *ibid.*
- The practice of subtracting one fraction from another. — *ibid.*
- How the proof of the subtraction of fractions by addition is done. — *ibid.*

In the multiplication of fractional numbers

- How the multiplication of fractions is done. — 30

(The index continues onto the following leaf.)

The subject index to the *Epitome Arithmeticae Practicae* continues here (from the catchword "In mul-" that closed the previous page) and runs to its end. Each entry is rendered as its English sense followed by the printed locator (a page number of the *Arithmetica*, or "ibid." / "ibidem" for the same page as the entry above).

Section sub-headings are given with the Latin retained.

(Entries continuing the section "On the multiplication of fractional numbers"):

- In the multiplication of fractions, when whole numbers are also present, what is to be done. *ibid.*
- How the checking of the multiplication of fractions by division is carried out. *ibid.*
- In the multiplication of fractions, why a fraction is produced that is smaller than either of the two fractions being multiplied. *ibid.*

In divisione factorum numerorum — On the division of fractional numbers

- How the division of fractions is performed. *ibid.*
- In the division of fractions, when whole numbers are also present, what is to be done. *ibid.*
- By what method the division of fractions is treated by others. 31
- How the checking of the division of fractions by multiplication is carried out. *ibid.*
- In the division of fractions, why a quotient is sometimes produced that is greater than the fraction divided. *ibid.*
- When the quotient is greater than the number to be divided, in the case of fractions. *ibid.*
- When the quotient in fractions is smaller than the number to be divided. *ibid.*

In insitione factorum numerorum — On the *insitio* [grafting/insertion] of fractional numbers

- What the *insitio* of fractions is. *ibid.*
- The *insitio* of fractions is twofold. *ibid.*

- For what purpose the *insitio* of fractions was devised. 32
- The difference between the *insitio* of fractions and the reduction of fractions. *ibid.*
- The first rule for the *insitio* of two fractions. *ibid.*
- How more than two fractions are inserted by the first rule. *ibid.*
- Fractions to be inserted by the first rule are not to be reduced to lowest terms before the end of the operation, and why. *ibid.*
- The sum obtained by *insitio* under the first rule is always less than unity, and why. *ibid.*
- The use of the first rule of *insitio* in dividing a whole number together with a fraction by a whole number. *ibid.*
- The second rule for the *insitio* of two fractions. 33
- How more than two fractions are inserted by the first rule. *ibid.* [see anomaly (a): printed "per primam regulam" under the second-rule heading]
- Fractions to be inserted by the second rule can be reduced to lowest terms before the end of the operation. *ibid.*

In quaestiunculis nonnullis numerorum integrorum, ac minutiarum — On some minor problems concerning whole numbers and fractions

- The usefulness of certain minor problems. 34
- Finding the number from which a subtraction has been made, or is to be made, so that the proposed number remains. *ibid.*
- Finding the number subtracted, or to be subtracted, from a proposed number, so that another given number is left over. *ibid.*
- Finding the number to which a given number is to be added, or which is to be added to a given number, so that another given number remains. *ibid.*
- Finding the difference between two given numbers. *ibid.*
- Finding the number that has been divided, or is to be divided, by a given number, so that the proposed quotient results. *ibid.*
- Finding the number that contains, or is, or yields a given fraction — that is, a part or parts of the proposed number. *ibid.*
- Finding the number by which a given number has been divided, or is to be divided, so that the quotient is the proposed number. *ibid.*
- Finding the number by which a given number is to be multiplied, or which is to be multiplied by a given number, so that the proposed number is generated. *ibid.*

- Finding two numbers which, multiplied together, produce a given number. *ibid.*
- Finding two numbers such that, when one is divided by the other, the proposed quotient results. 35
- Finding the number by which a given number is to be multiplied, or which is to be multiplied by a given number, so that when the product is divided by another given number the proposed quotient results. *ibid.*
- Finding the part which a given number represents with respect to another given number. *ibid.*
- Finding the number with respect to which a given number represents a proposed part. *ibid.*
- Finding the multitude of parts of any kind that a given number contains. *ibid.*

In regula trium, quae alio nomine regula aurea, sive regula proportionum dici solet — On the rule of three, which by another name is usually called the golden rule, or the rule of proportions

- Why the rule of three — whether "golden rule" or "rule of proportions" — is so called. *ibid.*
- How the numbers in the rule of three are to be arranged. *ibidem.*
- How the unknown fourth number is sought by means of the rule of three. *ibid.*
- Demonstration of the rule of three. 36
- When a number has been divided by another, why — if the divisor is multiplied by the quotient — the divided number is produced again. *ibidem.*
- Checking of the rule of three. *ibid.*
- Various shortcuts for the rule of three. *ibid.*
- Various checks for the rule of three. 37
- Demonstration of the shortcuts for the rule of three. *ibid.*
- Several problems by which various difficulties of the rule of three are worked out. *ibid.* & seqq.
- What is to be done when different currencies, measures, weights, and fractions occur in the rule of three. 38

In regula trium eversa — On the inverse rule of three

- How the inverse rule of three yields the fourth number. 39

- Several problems relating to the inverse rule of three. *ibid.* & seq.

In regula trium composita — On the compound rule of three

- The compound rule of three: what it is, and how it is done. 41
- Several problems relating to the compound rule of three. *ibid.* up to 44.

In regula Societatum — On the rule of partnership [fellowship]

- When the rule of partnership is to be used, and how it is done. *ibid.*
- How often the rule of three must be applied within the rule of partnership. *ibid.*
- What is to be done when there is a diversity of time-periods in the rule of partnership. *ibid.*
- Several problems relating to the rule of partnership. *ibid.* & seqq.

In regula alligationis — On the rule of alligation

- The rule of alligation. 52
- How the rule of alligation is done. *ibid.*
- Several problems relating to the rule of alligation. *ibid.* & seqq.
- Alligation can be carried out in various ways if the things to be mixed are more than two. 52
- What is to be done when, in the rule of alligation, several differences are set opposite the same price. *ibid.*
- What must be observed in alligations of several things. 53
- When a problem of alligation is impossible. *ibid.*

In regula falsi simplicis positionis — On the rule of single false position

- Why the rule of false position is so called. 55
- The rule of false position is twofold. *ibid.*
- The distinction between the two rules of false position. 56
- How the rule of single false position is done. *ibid.*
- Several problems relating to the rule of single false position. *ibid.* & seqq.

In regula falsi duplicis positionis — On the rule of double false position

- How the rule of double false position is done. 58
- In the rule of double false position, when both positions exceed the truth, or both fall short of it, a subtraction is performed. *ibidem.*

- In the rule of double false position, when one position exceeds the truth and the other falls short of it, an addition is performed. *ibidem*.
- Several problems relating to the rule of double false position. *ibid.* & *seqq.*

In progressionibus Arithmeticis — On arithmetic progressions

- What an arithmetic progression is. 65
- What the natural progression of numbers is, and that of odd and of even numbers. *ibid.*
- How an arithmetic progression is continued. *ibid.*
- How the common difference of an arithmetic progression is found. *ibid.*
- An arithmetic progression cannot decrease to infinity. *ibid.*
- A property of an arithmetic progression of three numbers. *ibid.*
- A property of an arithmetic progression of four numbers. *ibid.*
- A property of an arithmetic progression of any number of terms, when the number of terms is odd. *ibid.*
- A property of an arithmetic progression of any number of terms, when the number of terms is even. 66
- How the sum of any arithmetic progression is found. *ibid.*
- A particular method of finding the sum of the natural progression of numbers. *ibid.*
- The number of terms of the natural progression of numbers is [equal to] the last term. *ibid.*
- A particular method of finding the sum of the odd numbers. 67
- How the number of terms in a progression of odd numbers is found. *ibid.*
- A particular method of finding the sum of the even numbers. *ibid.*
- By what method the number of terms in a progression of even numbers is found. *ibid.*
- How the last term of any arithmetic progression is elicited from the number of terms, together with the first term and the common difference of the progression. *ibid.*
- The problem of the oxen of Augeas [the Cattle Problem]. *ibid.*
- The problem of the military commanders. *ibid.*

In progressionibus Geometricis — On geometric progressions

- What a geometric progression is. 68
- How a geometric progression is continued. *ibid.*

- How the denominator [ratio] of the proportion in a geometric progression is found. *ibid.*
- A geometric progression decreases to infinity. *ibid.*
- A property of a geometric progression of three terms. *ibidem.*
- A property of a geometric progression of four terms. *ibid.*
- A property of a geometric progression of any number of terms, when the number of terms is odd. *ibid.*
- A property of a geometric progression of any number of terms, when the number of terms is even. *ibid.*
- How the sum of any geometric progression is found. 68
- A particular method of finding the sum of a progression of double proportion whose beginning is 1. 69
- In a progression of double proportion beginning from 1, any number — once unity has first been cast off — is the sum of all the preceding numbers. *ibid.*
- If, in a geometric progression beginning from 1, some number multiplies itself or another number, what place the resulting product occupies. *ibid.*
- Any number in a geometric progression beginning from 1, when multiplying itself, produces a number that belongs in a place twice as far along, less one, as the multiplying number. *ibid.*
- How the natural progression of numbers indicates in what place each resulting product is to be located within a geometric progression beginning from 1. *ibid.*
- How the number of any place in a geometric progression beginning from 1 is investigated without [computing] the intermediate ones. *ibid.*
- How the results stated for a geometric progression beginning from 1 are adapted to a geometric progression not beginning from 1. 70
- How the number of any place in a geometric progression beginning from any number whatever is investigated without the intermediate numbers. *ibid.*
- The sum of any number of terms of a geometric progression of double proportion beginning from 1, with unity first added, when multiplying itself, produces a number which — once unity has first been cast off — is the sum of twice as many terms. *ibidem.*
- How the sum of 64 terms of a geometric progression of double proportion beginning from 1 is easily found. *ibid.*

- How much money is required to fill the 64 squares of a chessboard [*ludus latrunculorum*], such that in the first square 1 *quattrino* is placed, in the second 2, in the third 4, and so on, advancing by double proportion. 71
- How many grains of wheat make up one *rubbio* [a dry measure]. *ibid.*
- How many ships are required to carry the wheat set on the 64 squares of the chessboard. *ibid.*
- How many ships are required to carry the money set on the 64 squares of the chessboard, if it were reduced to gold coins. *ibidem.*
- In a progression whose first term is 1, second 2, but third the triple of the second, and likewise fourth the triple of the third, and so on: each term is double [the sum of] all the preceding. *ibid.*
- By what device the sum of 64 terms — beginning from 1 and advancing so that each is double [the sum of] all the preceding terms — is found. *ibid.*
- Another way of finding the same sum. 72
- How much wheat is required to fill the 64 squares of the chessboard, such that in the first square 1 grain is placed, in the second 2, in the third 6, in the fourth 18, and so on, so that the grains of each following square are double all the grains in all the preceding squares together. *ibid.*
- How many ships are necessary to carry the aforesaid wheat. *ibid.*
- How many ships would cover the whole surface of land and sea, if they touched one another. *ibid.*
- How many globes made up of land and sea would be covered by the ships that are necessary to carry the wheat just mentioned. *ibid.*
- How many globes equal to the whole earth the wheat contained in the 64 squares just mentioned would constitute. *ibid.*
- How many ships would carry the gold ducats [made] from the *quattrini* that would fill the 64 squares in the manner described for the grains of wheat. *ibid.*
- How many globes of land and sea the aforesaid ships would cover. *ibid.*
- By what method the sum of 40 terms of a geometric progression of double proportion beginning from 1 is easily found. 73
- How much 40 towns would cost if they were sold so that for the first 1 *quattrino* is paid, for the second 2, for the third 4, and so on. *ibidem.*
- How the sum of 24 terms of a geometric progression of double proportion beginning from 1 is briefly elicited. *ibid.*

- How much a horse having 24 nails in its hooves would cost, if it were sold so that 1 *quattrino* is given for the first nail, 2 for the second, 4 for the third, and so on. *ibid.*

In Extractione radicis quadratae — On the extraction of the square root

- What a square number is. *ibid.*
- What a square root is. *ibid.*
- What the extraction of a root is. *ibid.*
- How the number whose root is sought is to be marked with points [dots]. *ibid.*
- How many figures [digits] the root of the proposed number has. *ibid.*
- How the square root is drawn out from a given number. *ibid.*
- Extraction of the square root by [the method] *per Dandam* [name/method as printed; sense uncertain]. 74
- A threefold examination [verification] of the extraction of the square root. 75
- The remainder in the extraction of a square root cannot be greater than double the root found. *ibid.*
- What the difference between two consecutive squares is. *ibid.*

In appropinquatione radicum in numeris non quadratis — On the approximation of roots in non-square numbers

- How a nearer root is found which is nevertheless smaller than the true one. *ibid.*
- How a nearer root is found which is nevertheless greater than the true one. 76
- Another approximate extraction of the square root. *ibid.*
- Approximate extraction of the root by means of fractions. 77
- What a cube number is. *ibid.*
- What the extraction of the cube root is. *ibid.*
- How the number whose cube root is sought is to be marked with points. *ibid.*
- How many figures the cube root of a given number has. *ibid.*
- How the cube root is drawn out from a given number. *ibid.*
- How large the remainder in a cube-root extraction can be. 78
- How great the difference between two consecutive cubes is. *ibid.*

- How to recognize whether the figure taken in the root is too small. *ibid.*
- Checking [Prabatio, recte: Probatio] of the extraction of the cube root. *ibid.*
- Approximation of the cube root. *ibid.*
- Approximate extraction of the cube root by means of fractions. *ibid.*

FINIS INDICIS. (End of the Index.)

[The *Algebra* — the third and final work of this tome — has not yet begun on these pages; the *Epitome Arithmeticae Practicae* ends with its index here, and the *Algebra* display title and preface will follow.]

ALGEBRA

Display title of the third work of the tome.

CHRISTOPHORI CLAVII BAMBERGENSIS E SOCIETATE IESV ALGEBRA.

[Christophori Clavii Bambergensis e Societate Iesu Algebra — "The *Algebra* of Christoph Clavius of Bamberg, of the Society of Jesus."]

[Margin: two manuscript ownership inscriptions in an early hand flank the title — to the left of "ALGEBRA," *Monasterij*; to the right, *Scheyrensis*. Together: "(Of the) Monastery of Scheyern" — i.e. this copy belonged to the Benedictine abbey of Scheyern.]

Printer's device. The Jesuit emblem, set in a decorative woodcut cartouche framed by putti. At the center, within a radiant oval, stands the Christogram **IHS**, surmounted by a cross rising from the crossbar of the H, with the three nails of the Passion below. An inscription runs around the oval frame:

- Around the oval: **LAVDABILE NOMEN DOMINI** ["Praiseworthy is the name of the Lord" — cf. Psalm 112:3 (Vulgate)]

MOGVNTIAE, Sumptibus Antonii Hierat, excudebat Ioannes Volmari.

[At Mainz, at the expense of Antonius Hierat; printed by Ioannes Volmari.]

Cum Gratia & Priuilegio Sacrae Caesar. Maiest.

[With the grace and privilege of His Sacred Imperial Majesty.]

Anno M. DC. XII.

[In the year 1612.]

Proem. On the Excellence of Algebra

In dignity of the highest order, and in claim to the fullest praise, this art—which our countrymen, adopting the received term, have called Algebra—yields to none. If you inquire into its native origin, it must be traced all the way back to the fertile genius of the Arabs; if you weigh its nobility, it ranks among the foremost of the mathematical disciplines; if you consider its scope, it is bounded by no fixed limits of number, by no bulk of magnitude, by no institutions of any nation—private or public, ancient or modern, civilized or barbarous—nor by any expanse of the earth, however vast.

[Margin: The aim of algebra (*Scopus Algebrae*).]

Its aim, or purpose, is this: to search out some fixed quantity that is hidden from the knowledge of the senses and secret to sense, and at last, once an equality—or an equation—between two numbers has been discovered, to detect it and demonstrate it. It is like a hunting dog: when, worn out by a long chase and driven by hunger, it has caught the scent of some quarry lying hidden, it searches out every trace, now with its eyes, now with the keen sagacity of its nostrils, until it delivers the prey to its master, caught and gripped in its teeth. And indeed the workmanship itself far surpasses the task in the admiration it earns; the very method of accomplishing it adorns a feat that is by its nature arduous and splendid, and makes it still more excellent. It closely resembles what tends to happen—in a quite different matter—to those who play in the manner of the *andabatae* [gladiators who fought blindfolded]. For as long as they are hard pressed with their eyes covered, and press in turn upon those who press them, they never despair of catching someone; yet of the many they do not know which one is at last to be caught, and after groping in vain in the dark, now at these, now at those, once they have got hold of someone, then at last, the blindfold removed, their eyes take in the light and the game is over.

In just this way Algebra tests the hidden numbers—now by adding, now by dividing, now by multiplying—with, as it were, suspended step, certain that it will detect what it seeks somewhere in a geometric progression beginning from unity, though uncertain in which term of that progression it will find it. And so it plays on in blind labor, exploring, tracking, and enduring all the darkness of the number that lies hidden, until it advances to the point of reaching the very number that was the subject of the question—the number that finally emerges into the light when the operation itself is completed. And indeed the first number that appears is the one which, in that whole labor of numbers, comes nearest to

the unit of the progression that begins from unity; then all those numbers too are detected which were added and divided in order to unravel the question and track down what was proposed—as we shall make plain in the proper places.

[Margin: The excellence of algebra (*Algebrae excellentia*).]

How widely, then, does this aim range, which regards no kind of number, and no diversity of magnitude, as foreign to itself! It uncovers not only all the hiding-places of numbers, but also attains the finite magnitude of each and every mass—the measure of their dimensions, the weight of their moments, and the fixed limits of their measures. Nor is any arithmetical question put forward that Algebra does not recognize as its own and dispatch. So many, so varied, so obscure, so difficult are the parts of mathematics that Algebra alone treats them all together. For if it ever happens that the inextricable knots of mathematical matters cannot be untied, or that something obscure has never been drawn out of the darkness, the cause is to be laid not to the most certain precepts of this discipline, but to the difficulties of the equations and equalities—and to the method of those same things, which is not yet known. But more on these matters is found toward the end of chapter 12. Let us only do this: let us see by how great an interval Algebra surpasses the precepts of Arithmetic.

For what is there in Arithmetic that does not, so to speak, dread Algebra's rods [*fascies*] as its own? What examples so much its own, what tangled ambiguities of questions, what so great a mass of difficulties, what labyrinths of the most hidden matters so inextricable, that they are not all shared by Arithmetic with Algebra? Arithmetic has nothing, nothing that escapes Algebra; and there are very many things which Arithmetic cannot itself explain but which Algebra hands over to be known. How splendid this is, how supremely to be desired by everyone, how peculiar to this art! For what is common to the rest is that you cannot tell whether a proposed difficulty can be resolved or not until you have worked the whole through, and gone over the entire thing twice and again, repeating the operation; but here, in the very course of the operation, the answer presents itself, and the question comes forth of its own accord before the labor reaches its end.

For the deceptive assumptions that so often occur, and the mutually conflicting propositions of problems, lie hidden from those unskilled in Algebra; and unless the student of Arithmetic has explored these by means of it, he labors in vain, wholly anxious about the solution of the posed question. He turns his mind now to these rules, now to those, tries each one, reviews each, torments and wears himself out, and in the end feels that his labors have been useless, and, unhappy,

condemns his own study. And so in this art there is not even this much to be feared—that any labor rashly undertaken, and endured in vain to the very end, should be a cause for grief; for the other parts of mathematics can indeed receive from this one, but cannot themselves give back to it. Let these few things, said in place of a preface, suffice concerning the excellence and dignity of Algebra.

On the Inventor of Algebra and Its Name.

Chapter I.

There is no agreement among all as to the inventor of this art. Not a few assert that it was Geber, an Arab astronomer, and maintain that Algebra was named after Geber. It is more likely, however, that Johannes Regiomontanus [Regiomontanus], in his preface to al-Farghānī [Alfraganus], attributed this invention to Diophantus of Alexandria [Diophantus]. For Diophantus himself, in the preface to the thirteen books he wrote on this subject to Dionysius, professes himself the first inventor of this science, and declares that it had been brought to light by no one before him—indeed, that down to his own age it had been entirely unknown.

[Margin: The various names of algebra (*Varia nomina Algebrae*).]

The name itself has taken on greater variety. Because it stands out most among the other rules of Arithmetic, it has been called by many the Greater Art [*Ars maior*]. The Latins called it the rule of *census* and *res*—that is, the rule of the Root and the Square—for in large part the questions proposed in this art are solved through calculations of roots, which they call *res*, and of squares, which the same call *census*, as will be established in its proper place. Among the Italians it is named the Rule of the *cosa* (or *della cosa*), because in solving questions a single root is always posited, which they call *cosa*. From this it has also obtained the name by which many call it the Cossic rule [*regula Cossica*]. The Arabs did not name it by one name only. Some call it Algebra, which among them means the same as "restoration," since in solving questions by this art one must always arrive at some equation or equality, in which whatever has been taken away, whatever has been diminished, whatever is found to be lacking, must be restored. Others call it Almuchabula, which in their tongue signifies "opposition," because in setting out the hidden points of questions in this science, when one has arrived at some equation or equality, two numbers always equal—yet of different name—are set opposite each other. And hence it finally came about that, with the names doubled, they called it Algebra and Almuchabula—namely because it

dispatches all questions both by restoration and by that opposition we have just mentioned. But the reasons for these names he will understand more plainly who has more carefully learned the solutions of the questions contained in this work. Usage among writers, however, has prevailed, so that, following the Arabs, this whole science of ours is called by the common name Algebra.

Let this suffice as said concerning the inventor and name of Algebra. If this art—splendid as it is and delightful to its devotees, and admirable as it likewise is—had also been handed down as clearly as it is excellent, so that one need labor no more in clearing up the obscurity of the arithmeticians who taught it than in explaining the precepts of the art itself, then it too would be cultivated among the first, and would keep both its dignity and the full splendor of its name; nor would so many writers, hindered by its ramblings or deterred by its obscurity, have suddenly turned back and cast off the will to learn Algebra and the hope of mastering it before, as they say, greeting Algebra itself at the very threshold.

The labor of lightening these difficulties for them now falls to me—a labor not troublesome, though heavy, and, if any fruit comes of it, not unpleasant. This I shall strive to accomplish, by divine help, to the best of my powers. Accordingly I shall try to hand down, by a suitable method, all that pertains to Algebra, and to explain every mass of difficulties, so that anyone even moderately versed in the precepts of Arithmetic may master it with as little labor as possible in the learning. Yet in accomplishing this, just as I shall not allow my diligence to be found wanting, so I hold the opinion of those to be foreign to me—if there are any—who would require of me a treatment of Algebra perfect in every respect and complete in all its parts.

I do not so love myself as to think I should be so greatly preferred to all the others who down to this day have written on it, that I should profess myself able to accomplish what they themselves did not.

[Margin: The author's intention (*Intentio auctoris*).]

That there are very many equations among three or more numbers called by different names, I well know—equations that sometimes occur (though not so frequently) in questions to be solved by the help of Algebra; and yet it is not yet sufficiently established among all by what art the roots are to be drawn out from them. Of these equations—both those that have hitherto been known and explored, and those that have not yet emerged from their darkness—I shall give an enumeration in chapter 12. My work, then, is this, with this single care: that what others have set out in a tangled way, I may set forth more explicitly; what

they have set out more obscurely, I more openly—to bring forward what lies hidden, to explain what is difficult, to hold up a torch, and, as far as my industry will carry, to hand down and teach each point lucidly and clearly. Finally, those equations from which we have not yet learned to find the roots ought not so much to detract from the dignity and esteem of Algebra as to commend it for the almost infinite questions that are solved by the equations already found, and that have a sure and reasoned method, as will be established in the proper places. But whoever desires to have spent his labors in the science of Algebra not wholly in vain must first have exercised himself at some time in the common precepts of Arithmetic, and not approach the knowledge and study of this art entirely ignorant of them—precepts such as these, which are, as it were, the foundations to be laid for beginners: the addition of numbers, subtraction, multiplication, and division, not only of whole numbers but also of fractions. To these add the rule of proportions, which most call the Rule of Three, and not a few things like these. Besides, he ought to have mastered the proportions and progressions of numbers, both Arithmetic and Geometric. Thirdly, he ought to know the extraction of all roots from any number that may be proposed, whether it be whole or fractional. Finally, it will be best that he be well trained in Geometry and the other disciplines, and not merely lightly polished in them.

[Margin: What the algorithm is (*Algorithmus quid*).]

Moreover, much study must be devoted to learning the Algorithm, which this art has as peculiarly its own, apart from the rest. By Algorithm we mean the treatment which comprises addition, subtraction, and other operations...

of numbers. This Algorithm, I say, is the treatment of certain numbers which it is not the practice to explain by common Arithmetic. Now three kinds in all of such numbers are reckoned.

[Margin: The three kinds of numbers used in algebra (*Tria genera numerorum in Algebra usitatorum*).]

The first is that of the numbers which most have called Cossic or Denominated, and these are found in some Geometric progression that takes its beginning from unity—of which kind are Roots, Squares, Cubes, and not a few others of the same kind, of which I shall presently undertake to speak. In the second kind are the numbers called radical—such as the square root, the cube root, and the root of any number whatsoever of this sort, whether it has that root or does not have it; and about these we shall discourse when we have come to chapter 16. The third and last kind consists of the radical numbers of numbers of that kind which we

placed in the first position—such as the square or cube root, and the like, of however many squares, cubes, or roots, and so forth; and these will be our last labor. And since the Algorithm of those numbers which we have placed in the second and third kinds has no small difficulty and no contemptible obscurity, I have deliberately chosen to defer it to the end of the rule of Algebra. For although a good many questions, both Geometric and Arithmetic, cannot be unraveled and solved by this rule of Algebra without that Algorithm, nevertheless there is not so great a kinship between the science of Algebra and that Algorithm that they cling to each other as if by some bond, or that the one can never be separated from the other, or must in any way be said to depend upon it. First, then, we must explain the Algorithm of the Cossic numbers—or of those called Denominated—which is judged wholly necessary for a thorough understanding of this science. Then we shall pursue the rule of Algebra itself, and all those things that make for its full understanding; and when these are finished, beginning from chapter 16 and throughout chapter 26, we shall hand down the Algorithm of radical numbers—both absolute and Denominated, that is, Cossic. The final parts will be occupied by Enigmas [*Aenigmata*], which, once set forth and at last explained, will make the sense of the whole rule of Algebra plainer and more perfect, and its fruit more abundant.

On Cossic Numbers, or the Denominated. Chapter II.

[Margin: What the Cossic numbers are (*Numeri Cossici qui sint*).]

Cossic, or Denominated, numbers are all the numbers of any Geometric progression beginning from unity. For, given any Geometric progression whatever that takes its beginning from unity, the first term—that is, unity—represents the absolute and simple number. The second term, the one following unity, is called the root of all that follow, since from its multiplication into itself the third is produced, and from the multiplication of that same second term into the third the fourth is produced, and so on—as we demonstrated in the scholium to Proposition 10 of Book 8 of Euclid [Euclid, *Elements* VIII.10, scholium]. The third term next, because it is produced, as has been said, from the second into itself, is called the Square, or Census, or Zensus, as some skilled in this art prefer. The fourth term is then called the Cube, since it arises from the multiplication of the root—that is, the second term—into its own square, namely into the third term, as we said. And so, one after another, all the numbers of a Geometric

progression whose beginning is unity are denominated, account being taken of the multiplication of the root into the following terms.

[Margin: Why we call the numbers of geometric proportion Cossic (*Cur numeros proportionis Geometricae dicamus Cossicos*).]

Now since numbers of this kind are denominated variously by various authors, we have chosen out of them all a certain manner of denominating those numbers that is easy and very convenient, so that we may extend these denominations as far as we like and mark them with appended characters—calling, namely, the numbers of a Geometric progression "Cossic numbers." In this we have followed some of the more recent writers. For it matters little whether the numbers of a Geometric progression are called by this name or by another, provided the thing itself is understood. These denominations are expressed by the following characters:

The cossist power-signs (as printed). The characters are set in two rows; the sequence runs left-to-right across the top row, then left-to-right across the bottom row:

Row (as printed)	1	2	3	4	5	6	7	8	9	0
top	N	R	3	℄	33	ƒ	3℄	B	333	℄
bottom	3ƒ	C	33℄	D	3Bƒ	℄ƒ [?]	3333	E	&c.	
		ƒ		ƒ				ƒ		

Legend for the transcribed glyphs (these reproduce the printed cossist signs): N = the *numerus* sign; R = the *res/radix* sign (printed as a looped "2ε"-shaped figure); 3 = the *census/zensus/quadratus* sign; ℄ = the *cubus* sign (printed as a "Ce" ligature); ƒ = the *sursolidum* sign (long-s); Bƒ, Cƒ, Dƒ, Eƒ = the higher *sursolidum* signs (with the capitals B, C, D, E prefixed, as printed). Composite signs are formed by writing these together.

[Margin: Explanation of the Cossic characters (*Explicatio characterum Cossicorum*).]

The FIRST character, N, has the appellation of the absolute and simple number, so that the number to which it is appended is taken as absolute and simple. Thus 4 N signify nothing other than four simple and absolute units. Most often, however, this character or sign N is not usually appended to numbers when they are to be

taken as absolute; and so, whether some number bears this sign N or lacks it, it is to be taken as an absolute number.

The SECOND character, R [the *res* sign], has the appellation of the Root, or *Res* [Thing], so that the number to which it is appended is denominated from roots. Thus 4 R signify four roots, or *res*. Hence, if the Geometric progression has 2 in the second place, 4 R will be eight units; but if 3, twelve units; and so of the rest.

The THIRD character, 3 [the *census* sign], represents the square, which some call the *census*. Thus 4 3 are four squares, or *zensi*. And so, if the third number of the Geometric progression is 4, then 4 3 will be sixteen units; but if 9, thirty-six units; and so on.

The FOURTH character, C [the *cubus* sign], denotes the cube. Thus 4 C signify four cubes; so that, if the cube is 8, 4 C are 32 units; if 27, 4 C are 108 units; and so on.

The same is to be said of the remaining characters, whose significations are set out below, together with their appellations,

and the signs that other writers use. For just as the names of the cossist numbers vary among different writers, as we have said, so too the signs or characters by which they express them are not the same.

Sign	Name and alternative notations
N	A simple and absolute number.
R	Radix (root). To the Italians, <i>res</i> or <i>cosa</i> ; hence they express the root in this way: 1 co, 4 co, 20 co, etc. Others designate it thus: 1R, 100R, etc.
3	Zensus, or square [<i>quadratus</i>]. Others express the square by this character, q: as 1 q, 30 q, 8 q, etc. The Italians write it thus: ce (<i>censo</i>), as 1 ce, 4 ce, 20 ce, etc.
c	Cubus (cube). Other practitioners depict it thus: cu, as 7 cu, 8 cu, 200 cu, etc.
33	Zenzizensus, or square-of-squares [<i>quadrati quadratus</i>]. Some mark it thus: qq, as 3 qq, 10 qq, etc.; the Italians thus: ce.ce.
ß	Surdesolidus. Others call it Sursolidum or Supersolidum; to the Italians it is <i>Relatum primum</i> (the first Relatum), which they usually express thus: Rel.P, as 11 Rel.P, 20 Rel.P, etc.

Sign	Name and alternative notations
3℥	Zensicubus, or square of the cube; also Cubizensus, Quadraticubus, cube of the square, Cubiquadratus. Others use this character: q℥. This number could also be signified with the marks inverted, thus: ℥3 [?]. The Italians write it thus: ce.cu, or cu.ce.
Bß	Bsurdesolidus. According to some, Bisurdesolidus, or the second Sursolidus or Supersolidus. Among the Italians it is called <i>Relatum secundum</i> (the second Relatum), which they depict thus: Rel.2.
333	Zensizenzensus, or square of the square-of-squares. Some denote it thus: qqq; the Italians thus: ce.ce.ce.
℥℥	Cubicubus, or cube of the cube, which some depict thus: ℥℥; but the Italians thus: cu.cu.
3ß	Zensurdesolidus, or square of the surdesolid, or surdesolid of the square. Some designate it thus: qß; but the Italians, since they call it the census of the first Relatum, mark it thus: ce.Rel.P.
Cß	Csurdesolidus. According to others, Tersurdesolidus, or the third Supersolidus. To the Italians it is <i>Relatum tertium</i> (the third Relatum), depicted thus: Rel.3.

In the same way, by a certain proportion, the remaining cossist signs or characters can be explained; and indeed they proceed to infinity, as we shall say presently.

[Margin: When the natural series of numbers sets out the cossist numbers of the geometric progression.]

But so that we may understand what the aforesaid denominations mean, and in what way they take their origin from the terms of each geometric progression that begins from unity, the following two progressions must be considered carefully, together with the aforesaid cossist characters placed in the middle between them.

Exponent (natural progression)	0	1	2	3	4	5	6	7	8	9	10
Cossist sign	N	R	3	℥	33	ß	3℥	B	333	℥℥	3ß
								ß			

Exponent (natural progression)		0	1	2	3	4	5	6	7	8	9	10
Geometric progression (value)		1	2	4	8	16	3	64	1	256	512	102
							2		2			4
									8			
Exponent	11	12		13	14		15	16			17	&c.
Cossist sign	Cß	33C		Dß	3Bß		Cß	3333			Eß	&c.
Value	20	4096		81	16384		3276	65536			13107	&c.
	48			92			8				2	c.

The first progression is the natural series of numbers beginning from 0. These numbers set out both the cossist signs written beneath them and the terms of the second progression — which is geometric, taking its beginning from unity. The numbers of the upper row are called the exponents of the terms of the geometric progression and of the cossist signs. The exposition consists in the following. The first figure, 0, having beneath it N and 1, indicates a simple and absolute number composed of units; for that reason it is marked with no exponent — though, improperly, this figure 0 might be called the exponent of the absolute number. The second term, 1, above 2 and R, shows that the second number of the geometric progression is the first denomination among the cossist numbers, and is called the Root [*Radix*]. The third term, 2, then makes clear that the third number of the geometric progression is the second denomination, and is called Zensus, or Square. So too 6 shows that the sixth denomination is the number Zensicubus, and so on with the rest.

[Margin: How many proportions are intercepted between unity and any given number of the geometric progression.]

Next, these same numbers — the exponents of the natural series of numbers — teach how many proportions of the geometric progression are interposed between any given number of that same geometric progression and unity. Thus the 1 above 2 and R signifies that between 2 (the root of the geometric progression) and unity there is contained a single proportion, 2 to 1. But the 2 above 3 and 4 indicate that between 4 (or 3, the square) and unity two proportions are comprised, 4 to 2 and 2 to 1. Likewise the 3 above C and 8 shows that between 8 (or C) and unity fall three proportions, 8 to 4, 4 to 2, and 2 to 1.

And so with the rest: as the 6 above 3 $\mathfrak{C}$ and 64 shows, between 64 (or 3 $\mathfrak{C}$) and unity six proportions are intercepted — 64 to 32, 32 to 16, 16 to 8, 8 to 4, 4 to 2, and 2 to 1.

[Margin: From what the exponent of a character is generated — namely, from the mutual multiplication and division of two exponents.]

Moreover, any two exponent-numbers multiplied together produce the exponent of a cossist character that is composed from the characters of the exponents taken. Thus, multiplying 2 by 3 gives 6, the exponent of the character 3 $\mathfrak{C}$, which is composed of 3 and $\mathfrak{C}$. Likewise 3 by 5 gives the exponent 15, of the character $\mathfrak{C}\mathfrak{B}$, composed of $\mathfrak{C}$ and $\mathfrak{B}$, and so on. So too, when any larger exponent-number is divided by a smaller, the quotient — if it is a whole number — will indicate the exponent of the cossist character that remains when, from the character of the divided exponent-number, one removes the character of the exponent-number by which the division was made. Thus if 6, the exponent of the character 3 $\mathfrak{C}$, is divided by 2, the exponent of the character 3, the quotient is 3, the exponent of the character...

$\mathfrak{C}$, which remains when, from the character 3 $\mathfrak{C}$ (of exponent 6), one removes the character 3 (of exponent 2). Likewise, if the number 15, the exponent of the character $\mathfrak{C}\mathfrak{B}$, is divided by 5, the exponent of the character $\mathfrak{B}$, the quotient is 3, the exponent of the character $\mathfrak{C}$, which remains when $\mathfrak{B}$ is removed from $\mathfrak{C}\mathfrak{B}$, and so on.

Again, this figure 2, placed above 4 and 3, teaches that the square (that is, the second denomination) is produced by multiplying the root taken twice. For if the root 2 is set down twice, thus — 2, 2 — and the multiplication 2 by 2 is carried out, the Square, or Zensus, 4 is generated. In the same way this figure 3, placed above 8 and $\mathfrak{C}$, signifies that the cube (the third denomination) is produced from the root set down and multiplied three times. For if the root 2 is set down three times, as here — 2, 2, 2 — and the multiplication 2 by 2 is made, and the resulting number by 2, the cube 8 arises. So too the figure 5 shows that the Surdesolid number is produced by multiplying the root taken five times, as here: 2, 2, 2, 2, 2. And the same reasoning holds for all the others.

[Margin: The definition of each individual cossist number.]

From these principles all the cossist numbers are easily defined. If, for example, one asks what the Bsurdesolid number is, we shall say that it is the number generated from some number set down seven times and multiplied. Thus 128 arises from multiplying this root, 2, set down seven times, in this way: 2, 2, 2, 2, 2,

2, 2. Similarly the Zensizensus will be the number that some number, set down four times, produces by its own multiplication, and so on. And always, the number which, set down a certain number of times, generates some number by its multiplication is called the root of the number produced. Thus, since 2, set down ten times in this way — 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 — and multiplied, makes the number 1024, which is 3β , therefore 2 is called the Zensurdesolid root of this number, 1024. And the same is to be said of the others.

You see, then, how aptly the natural progression of numbers, beginning from 0 and placed above the geometric progression that starts from unity, is said to set out the terms of the geometric progression.

For the rest, the numbers of the geometric progression are produced not only by multiplying the root taken a certain number of times, as we have explained, but also by multiplying other numbers together, as their cossist signs show. Thus, although this Zensurdesolid number 1024 is generated from the root taken ten times — in this way: 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 — as its exponent 10 indicates, nevertheless, because its cossist sign 3β is composed of these two cossist signs, 3 and β (whose exponents are 2 and 5), therefore, if the same root 2 is set down twice on account of the exponent 2 of the sign 3 — thus: 2, 2 — and is multiplied so as to produce 4, and this product is then set down five times on account of the exponent 5 of the sign β — thus: 4, 4, 4, 4, 4 — and multiplied, the same number 1024 will be generated. In the same way, if the root 2 is set down five times on account of the sign β and multiplied thus, and the resulting number 32 is set down twice on account of the sign 3, the same number is generated. For the root so multiplied, $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, makes 32; and this number so multiplied, $32 \cdot 32$, produces 1024. And the same is to be said of the rest, whenever their cossist signs are composed of several cossist signs.

[Margin: Addition and subtraction do the same thing in the natural series of numbers that multiplication and division do in the geometric progression.]

In the progressions set out above, this too deserves consideration: that addition of the numbers of the arithmetic progression corresponds to multiplication of the numbers of the geometric progression, and subtraction to division. For just as, for example, 2 and 5 make 7 when added together, so too 3 and β — whose exponents are 2 and 5, namely 4 and 32 — when multiplied together produce 128, that is, $B\beta$, whose exponent is 7. Likewise, just as 3 and 9 together make 12, so $\mathfrak{C}$ and $\mathfrak{C}\mathfrak{C}$ — namely 8 and 512, whose exponents are 3 and 9 — multiplied together generate 4096, that is, $33\mathfrak{C}$, whose exponent is 12. And the same is to be said of the rest.

Again, just as by subtracting 5 from 7 there remain 2, so by dividing Bß (namely 128, whose exponent is 7) by ß (that is, by 32, whose exponent is 5), the number 4 results — namely 3, whose exponent is 2. Similarly, just as subtracting 3 from 12 leaves 9, so dividing 33℄ (that is, 4096, whose exponent is 12) by ℄ (that is, by 8, whose exponent is 3) produces ℄℄ (namely 512, whose exponent is 9), and so with the rest. We shall give the demonstration of this matter under the Multiplication and Division of cossist numbers.

What has been said thus far about the geometric progression of double proportion, which begins from unity, is to be understood likewise of any other geometric progression whose beginning is unity — such as, for example, the two following geometric progressions, of which the first contains quadruple proportions, while the second embraces septuple ones.

Exponent	0	1	2	3	4	5	6	7	8	& c.
Cossist sign	N	R	3	℄	33	ß	3℄	Bß	333	& c.
Value	1	4	16	64	256	1024	4096	16384	65536	& c.

Exponent	0	1	2	3	4	5	6	7	&c.
Cossist sign	N	R	3	℄	33	ß	3℄	Bß	&c.
Value	1	7	49	343	2401	16807	117649	823543	&c.

For everything that has been said can easily be transferred to these two progressions and to all others.

[Margin: How all the numbers of a geometric progression are to be denominated.]

It remains for us to show by what method all the numbers of any proposed geometric progression beginning from unity are to be denominated — or (what amounts to the same thing) what cossist signs are to be assigned and ascribed to those numbers. This we shall easily accomplish if we first denominate the numbers whose exponents are prime numbers, of which the following are examples:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, &c.

And their characters, or signs, which these prime exponents set out, are these:

3, Ɔ, ß, Bß, Cß, Dß, Eß, Fß, Gß, Hß, Iß, Kß, Lß, Mß, Nß, Oß, Pß, Qß, &c.

For since the third number from unity, in any geometric progression whatever, is a square — as is demonstrated in Euclid, *Elements* IX.8 — its sign will be 3, whose exponent is 2. Again, since the fourth number from unity is a cube, as is established from the same proposition, its mark will be Ɔ, and its exponent 3. Next, since the sixth number from unity — which the prime number 5 sets out as exponent — is neither a square nor a cube (except when the second number from unity is itself a square or a cube, as is clearly established from Euclid, *Elements* IX.10), the writers assigned to it this sign, ß, so that it is called Surde-...

solid. But since the same thing happens with the remaining numbers whose exponents are prime numbers — namely, that they are neither squares nor cubes (the first number from unity being neither a square nor a cube, as is plain from the same proposition) — those numbers of the geometric progression whose exponents are prime numbers were likewise all called Surdesolids: so that the one which 7 sets out as exponent is called the second Surdesolid, while the one set out by 11 is the third Surdesolid, and so on. But since, if the second Surdesolid were marked thus, 2ß, and the third thus, 3ß, and so on, the arithmetical figures set next to this sign ß could easily confuse and disturb the other numbers, it seemed good to the practitioners to adopt the letters of the alphabet in place of arithmetical figures, and to denominate the second ß as Bß (from the second letter), the third as Cß (from the third letter), and so on. In this way, then, are formed the signs and characters of the cossist numbers whose exponents are prime numbers, as this table indicates:

													&c
Prime exponent	2	3	5	7	11	13	17	19	23	29	31	37	.
Cossist sign	3	Ɔ	ß	Bß	Cß	D	Eß	Fß	G	H	Iß	K	&c
						ß			ß	ß		ß	.

[Margin: To resolve a composite number into its non-composite [prime] parts that produce it.]

From these we shall find the characters of the remaining cossist numbers whose exponents are not prime but composite, in this way. Let the composite exponent-number, whose denomination and character is sought, be resolved into its non-composite aliquot parts [prime factors], which, multiplied together in order, constitute and produce it. This is done as follows.

Let the given composite number be divided by the smallest prime number by which it can be divided; likewise the quotient, if it is a composite number; and again this quotient, if it is composite, let it be divided by the smallest prime number that measures it; and so let the division be carried on continuously until a quotient is reached that is a prime number — that is, one having no aliquot part. For if all such divisors — even if some are repeated several times — are set down in order together with the last quotient and multiplied together, they will produce exactly the given composite number; and these divisors, together with the last quotient, are the non-composite parts that constitute the given composite number by multiplication.

For example, let the given composite number be 924. First, since it is even, I divide it by 2, as its smallest non-composite part, and find the quotient 462; and I set 2 aside. Again, since 462 is even, I divide it by 2 and obtain the quotient 231; and I set aside another 2. Then, since 231 is odd, I divide it by 3, as the smallest prime number, and find the quotient 77. (If anything had been left over in the division, I would have tried division by 5, by 7, and by the other prime numbers, until there was no remainder. And if I could find no such divisor that leaves no remainder, then the first two divisors, 2 and 2, together with the last quotient, would be the non-composite parts that restore the given number, and no other could be given, apart from unity.) So I set 3 aside. And again I divide 77 by 3; but since something is left over in that division, I do not set aside 3, but divide the same quotient 77 by 5. Since something is again left over, I divide the same by 7, and find the quotient 11, a prime number. The non-composite parts, therefore, are 2, 2, 3, 7, 11, which by their multiplication restore the given number, 924.

Next, once all the non-composite aliquot parts have been found, let the appropriate characters be written beneath them; for in this way the whole composite character of the whole exponent-number will spring forth. Example: let the cossist sign of the thirty-first cossist number be sought, whose exponent is this composite number, 30. (For whatever position a cossist number holds counting from unity, its exponent contains that many units less one, as can be seen from the progressions given above.) Since, then, its non-composite aliquot parts, which by their multiplication generate it, are 2, 3, 5 — whose signs are 3, Ɔ, ß — the said thirty-first number will be adorned with this cossist sign, 3Ɔß, and will be called Zensicubisurdesolidus. In the same way the exponent 24, whose non-composite parts are 2, 2, 2, 3, will have the cossist sign 333Ɔ. So too the exponent 100, whose non-composite parts are 2, 2, 5, 5, will obtain this cossist sign, 33ßß, and so on.

We shall reach the same result by this method. Take any two numbers which, multiplied together, produce the given exponent-number; for their cossist signs will compose the cossist sign of the proposed exponent-number. Thus if the exponent is 12, the numbers producing it by their multiplication are 3 and 4, whose cossist signs are € and 33. The cossist sign of the exponent 12, therefore, is 33€ — for the smaller cossist signs are always placed before the larger. In the same way, the numbers producing this same exponent-number 12 by their mutual multiplication are 2 and 6, of which the former has the cossist sign 3 and the latter 3€; so the cossist sign of the exponent-number 12 is 33€, as before. But if at any time the cossist sign of one of the multiplying numbers — or even of both — is unknown, one must take two other numbers that produce it. As in the latter example: if the cossist sign of this multiplier 6 escapes us, we take 2 and 3, which generate it, whose signs are 3 and €. Thus the same cossist sign of the exponent-number 12 is found, 33€, and so on.

[Margin: What place in the geometric progression each cossist number occupies.]

Now, conversely, if the exponent-number of any proposed cossist sign is sought: if the sign is non-composite, we take from the progression of prime numbers above the prime number belonging to the proposed sign. Thus, if this cossist sign Dß is presented, its exponent will be 13. But if the sign is composite, we take the exponents belonging to the component signs and multiply them together; for the product will be the exponent of the proposed cossist sign. Thus, if the cossist sign 33€ is given, composed of these three — 3, 3, € — we take their exponents, 2, 2, 3, from the progression of prime numbers above, and multiply them together; for the product, 12, is the exponent of the said sign 33€. And the same is to be said of the rest.

[Margin: How others denominate the cossist numbers of the geometric progression.]

It should not, however, be passed over that certain authors well versed in this art denominate these cossist numbers differently, taking account, namely, of the multiplication by which they are produced from the root. For since the third number from unity in the geometric progression — which we have called the Square, or Zensus — is produced by multiplying the root taken twice, in this way: 2·2, or 3·3, or 4·4, or 5·5, and so on, so that only a single multiplication is made, they therefore call the number generated in this way, that is, the square, the First quantity; because...

(The page opens with the closing lines of the preceding chapter, *Cap. II*, on the names of the cossic powers; the new chapter, *Cap. III*, begins immediately below.)

because it [the census] is the first of all the quantities that arise from the root. The cube they then call the "second quantity," because it is generated from the root set down three times, so that two multiplications take place. The census-of-census [zensizensus] they call, for the same reason, the "third quantity," and so on with the rest. But we prefer to use our own signs and characters, so that the matter itself may be grasped more clearly. For our signs and characters are easily distinguished from letters and from numerals, so that no confusion can arise — something that can scarcely be avoided with the signs and characters of others.

DE NVMERATIONE COSSICORUM numerorum.

CAPVT III.

The notation of cossic numbers is easy from what has been said. When they stand alone — as 5R [five *res*/roots], or 83 [eight *census*/squares], or 20C [twenty *cubi*/cubes], and so on — they are read this way: five roots [radices], or 8 census (that is, squares [quadrati]), or 20 cubes, and so forth. But when they are set in combination with one another, joined by the sign + or by the sign –, as 5R + 83, or 83 – 5R, or 20C + 83 – 5, then the greatest attention must be paid to these two signs, + and –. The first is called the sign of things added, and means "plus," or addition; the second is called the sign of things subtracted, and denotes "minus," or diminution, or subtraction.

[Margin: Explanation [printed *Exylicatio*, for *Explicatio*] of the signs + and –.]

Hence it has come about that numbers between which the sign + is placed are called *composite* [compositi]; those between which the sign – is set are called *diminished* [diminuti]; and those in which both signs occur are called *mixed* [mixti] — although all of them may be called composite. Thus this composite number, 5R + 83, is read: five roots plus eight census. This diminished one, 83 – 5R, is read thus: eight census minus five roots. And this mixed one, 20C + 83 – 5, in this way: twenty cubes plus eight census minus five units. For, as was said, a number that has no cossic character after it signifies an absolute number made up of units.

[Margin: Which cossic numbers are composite, diminished, and mixed.]

These signs + and –, as is clear, refer to the numbers that follow them, never to those that precede. Likewise a number preceded by neither of these signs is

understood to carry the sign of addition, +. The meaning of composite, diminished, or mixed numbers is easy. For when we say $5R + 8\mathfrak{z}$, we understand five roots together with eight census — that is, 42 units, if the root is 2 and its square (census) is 4. Likewise when we say $8\mathfrak{z} - 5R$, we understand that five roots are subtracted from eight census; so if the root is 2 — and therefore the census is 4 — the number in question will contain 22 units. The same is to be said of the others.

Now since in the preceding chapter we sometimes also called the characters of cossic numbers — such as R , $\mathfrak{z}$, $\mathfrak{C}$, and so on — "cossic signs," to avoid all confusion we shall in what follows call these marks (R , $\mathfrak{z}$, $\mathfrak{C}$, etc.) *characters*, but these (+, -) *signs*.

Most authors put the letter P for the sign +, so that it may mean "plus," and the letter M for the sign -, so that it may mean "minus." But we prefer to use our own signs, so that they may be distinguished from letters and no confusion arise.

Moreover, among more recent writers not a few mark *surd* roots (as they call them), or irrational roots — those, namely, that cannot be expressed in numbers — with this sign, $\sqrt{}$, immediately followed by a cossic character: $\mathfrak{z}$, or $\mathfrak{C}$, or $\mathfrak{z}\mathfrak{z}$, and so on, according as that root is square (zensic), cubic, or zensizensic (that is, square-of-square), and the like. For example, the square root of 12 they denote thus, $\sqrt{\mathfrak{z}12}$; the cube root thus, $\sqrt{\mathfrak{C}12}$; and the fourth root (zensizensic) in this way, $\sqrt{\mathfrak{z}\mathfrak{z}12}$. For this reason the sign $\sqrt{}$ is called the *radical* sign.

[Margin: Surd roots, how they are to be marked.]

We too shall use it in what follows, since less confusion arises from it than from the practice of many, who mark the square root thus, Rq ; the cube root thus, Rcu ; the fourth root thus, Rqq ; and so on. For with such notation the letters Rq , Rcu , Rqq are easily confused with other letters. These signs, then — $\sqrt{\mathfrak{z}}$, $\sqrt{\mathfrak{C}}$, $\sqrt{\mathfrak{z}\mathfrak{z}}$, $\sqrt{\mathfrak{z}}$, and so forth — must be carefully learned, so that what follows may be understood whenever such signs occur. Note, however, that whenever the root of some number is named without the qualification "square," "cubic," "surdesolid," or "zensizensic," and the like — or when the sign $\sqrt{}$ is prefixed to a number without the character $\mathfrak{z}$, $\mathfrak{C}$, $\mathfrak{z}$, $\mathfrak{z}\mathfrak{z}$, etc. — writers understand simply the square root; and this we too shall often observe.

[Margin: What the radical sign is.]

DE ADDITIONE ET SUBTRACTIONE numerorum Cossicorum.

CAPVT IV.

When a cossic number is added to a cossic number of a different denomination or character, the addition is carried out by means of the sign +, and a composite number results. Thus these two numbers, 6R and 8, added together make $6R + 8$; likewise 6R and 8C make $6R + 8C$, and so on.

[Margin: Addition of cossic numbers of different denominations.]

But when a cossic number is to be added to another cossic number of the same denomination or character, the numbers are added together and the same cossic character is attached to the sum. Thus these numbers, 4J and 9J, added together make 13J; likewise 5R and 4R make 9R, and so on.

[Margin: Addition of cossic numbers of the same denomination.]

In the same way, when a cossic number is subtracted from a cossic number of a different denomination, the subtraction is carried out by means of the sign -, and a diminished number results. Thus this number, 6R, taken from 8J, leaves $8J - 6R$; likewise 12 taken from 6R leaves $6R - 12$, and so on.

[Margin: Subtraction of cossic numbers of different denominations.]

But when a cossic number is to be subtracted from another cossic number of the same character, one number is subtracted from the other and the same character is affixed to the remainder. Thus 4R from 9R leaves 5R; and 9C from 20C leaves 11C, and so on.

[Margin: Subtraction of cossic numbers of the same denomination.]

But when composite, diminished, and mixed cossic numbers are added together, or one is subtracted from another, one must be placed beneath the other so that numbers of the same denomination correspond to each other — just as is customary with astronomical fractions [i.e. sexagesimals]. And if in either of them no term is found corresponding to some term in the other, then in its place one must put this cipher of nothing, 0, with the sign +. Once this is arranged, the numbers of the same denomination are to be added, or the one subtracted from the other, and the sums or the remaining numbers written beneath in their proper places, together with the same signs, + or -, that are found in the numbers being added or subtracted.

[Margin: Addition and subtraction of composite cossic numbers, etc.]

Exempla Additionis. [Examples of Addition]

$$\begin{array}{r} \text{R N } 3 \text{ R N} \\ 6 + 8 \text{ } 7 + 8 - 5 \\ 7 + 10 \text{ } 3 + 9 - 8 \\ \hline 13 + 18 \text{ } 10 + 17 - 13 \end{array}$$

$$\begin{array}{r} \text{C N } 3 \text{ C } 3 \text{ R N} \\ 7 + 8 - 3 \text{ } 4 + 11 + 0 - 6 \\ 4 + 11 - 5 \text{ } 3 + 0 + 8 - 4 \\ \hline 11 + 19 - 8 \text{ } 7 + 11 + 8 - 10 \end{array}$$

$$\begin{array}{r} \text{B } 33 \text{ R N } 3 \text{ } 3 \text{ N} \\ 7 + 0 + 8 - 5 + 4 \text{ } 4 - 8 \\ 4 + 9 + 6 - 9 + 0 \text{ } 10 - 10 \\ \hline 11 + 9 + 14 - 14 + 4 \text{ } 14 - 18 \end{array}$$

In the next-to-last example, to this number — $7\text{B} + 8\text{R} - 5 + 43$ — was added this number, $4\text{B} + 933 + 6\text{R} - 9$, and this sum was produced: $11\text{B} + 933 + 14\text{R} - 14 + 43$.

Exempla Subtractionis. [Examples of Subtraction]

$$\begin{array}{r} \text{C } 3 \text{ R N } \text{C N } 3 \\ 7 + 11 + 8 - 10 \text{ } 11 + 19 - 8 \\ 3 + 0 + 8 - 4 \text{ } 4 + 11 - 5 \\ \hline 4 + 11 + 0 - 6 \text{ } 7 + 8 - 3 \end{array}$$

$$\begin{array}{r} \text{B } 33 \text{ R N } 3 \text{ } 3 \text{ R N} \\ 11 + 9 + 14 - 14 + 4 \text{ } 10 + 17 - 13 \\ 7 + 0 + 8 - 5 + 4 \text{ } 7 + 8 - 5 \\ \hline 4 + 9 + 6 - 9 + 0 \text{ } 3 + 9 - 8 \end{array}$$

In the next-to-last example, the remainder is $4\text{B} + 933 + 6\text{R} - 9$. For the "+ 0" is rejected, since this figure 0 ought not to be set down either in the remainder of a subtraction or in the sum of an addition, being superfluous. In the same way, the remainder in the first example will be $4\text{C} + 113 - 6$.

The same is to be understood of multiplication and division as well. And this is the manner we shall for the most part follow in this work.

Addition and subtraction can also be done if the cossic characters are written *after* the numbers, rather than above them. Thus the second example of addition will stand as follows:

$$\begin{array}{r} 7\mathfrak{J} + 8\mathfrak{R} - 5\mathfrak{N} \\ 3\mathfrak{J} + 9\mathfrak{R} - 8\mathfrak{N} \\ \hline 10\mathfrak{J} + 17\mathfrak{R} - 13\mathfrak{N} \end{array}$$

Likewise the second example of subtraction will stand thus:

$$\begin{array}{r} 11\mathfrak{C} + 19\mathfrak{N} - 8\mathfrak{J} \\ 4\mathfrak{C} + 11\mathfrak{N} - 5\mathfrak{J} \\ \hline 7\mathfrak{C} + 8\mathfrak{N} - 3\mathfrak{J} \end{array}$$

But if, in addition or subtraction, one of the numbers has the sign + and the other the sign -, the kind of operation is switched. That is: in addition, the smaller is subtracted from the larger, and the remainder is given the sign of the larger number — the one from which the subtraction was made — whether that be + or -.

[Margin: What to do when one number has the sign +, and the other the sign -, in addition and subtraction.]

In subtraction, by contrast, the numbers are added together, and to the sum is attached the sign of the upper number — the one from which the subtraction was to be made — whatever that sign may be, whether + or -.

Exempla Additionis. [Examples of Addition]

$$\begin{array}{r} 3\mathfrak{C} \mathfrak{B} 3\mathfrak{J} \mathfrak{C} \mathfrak{R} \mathfrak{N} 3 \mathfrak{R} \\ 7 + 0 + 8 + 0 - 4 + 8 \ 6 + 8 \\ 7 + 5 - 11 - 11 + 0 + 0 \ 2 - 10 \\ \hline 14 + 5 - 3 - 11 - 4 + 8 \ 8 - 2 \end{array}$$

If these two numbers, $6\mathfrak{J} - 8\mathfrak{R}$ and $12\mathfrak{R} - 3\mathfrak{J}$, are to be added, we shall set up the example in this way, so that each term corresponds to each:

$$\begin{array}{r} 3 \mathfrak{R} + 6\mathfrak{J} - 8\mathfrak{R} \\ +6 - 8 \text{ [Ve1 / - } 3\mathfrak{J} + 12\mathfrak{R} \\ -3 + 12 \text{ sic] } \hline 3\mathfrak{J} + 4\mathfrak{R} \\ 3 + 4 \end{array}$$

The reason for this operation is nicely shown by absolute numbers, as you see here:

$$\begin{array}{r}
 8 + 5 \quad 8 - 5 \quad 8 + 4 \quad 8 - 4 \\
 10 - 4 \quad 10 + 4 \quad 10 - 5 \quad 10 + 5 \\
 \hline
 18 + 1 \quad 18 - 1 \quad 18 - 1 \quad 18 + 1
 \end{array}$$

For in the first example, to $8 + 5$ — that is, to 13 — a full 10 is not added so as to make 23; rather, 4 is first subtracted. Hence the sum comes to 19, namely $18 + 1$. So after 8 and 10 have been added, making 18, then 4 must be taken from 5, leaving 1, with the sign +.

In the same way, in the second example, a full 8 is not added, but 8 minus 5, to $10 + 4$ — that is, to 14. Therefore, after 8 and 10 have been made into 18, then 5 must be taken from 4; and because a unit is lacking to allow the subtraction, 1 is put down with the sign -. The same reasoning holds in the rest. Here you see that in the first example, from $8 + 5$ and $10 - 4$ — that is, from 13 and 6 — is produced $18 + 1$, that is 19. In the second, from $8 - 5$ and $10 + 4$ — that is, from 3 and 14 — comes $18 - 1$, that is 17. In the third likewise, from $8 + 4$ and $10 - 5$ — that is, from 12 and 5 — is formed $18 - 1$, that is 17. And in the fourth, from $8 - 4$ and $10 + 5$ — that is, from 4 and 15 — results $18 + 1$, that is 19.

Exempla Subtractionis. [Examples of Subtraction]

$$\begin{array}{r}
 3\text{C} \text{ B } 33 \text{ C } \text{R N} \\
 14 + 5 - 3 - 11 - 4 + 8 \\
 7 + 0 + 8 + 0 - 4 + 8 \\
 \hline
 7 + 5 - 11 - 11 \pm 0 + 0
 \end{array}$$

In the former example, when -4R is subtracted from -4R , the remaining figure 0 may be marked either with the sign + or with the sign -; for it makes no difference which is used, since the remainder is 0. Indeed the whole remainder should be written thus: $73\text{C} + 5\text{B} - 1133 - 11\text{C}$.

The reasoning of this operation too is clear in absolute numbers, as appears from the following examples:

$$\begin{array}{r}
 18 + 5 \quad 18 + 4 \quad 18 - 5 \quad 18 - 4 \\
 8 - 4 \quad 8 - 5 \quad 8 + 4 \quad 8 + 5 \\
 \hline
 10 + 9 \quad 10 + 9 \quad 10 - 9 \quad 10 - 9
 \end{array}$$

For in the first example, 8 is not subtracted whole from $18 + 5$ — that is, from 23 — but rather 8 minus 4. Therefore, once 8 has been taken from 18 to leave 10, 4 must again be added to 5. The same reasoning holds in the second example. But in the third example, when 8 is taken from 18 — from which 5 has already been

removed — 5 must be restored to the 4, and the sum marked with the sign -. This also happens in the fourth example. Here you see that in the first example, if from $18 + 5$ (that is, from 23) one subtracts $8 - 4$ (that is, 4), there remains $10 + 9$, that is 19. In the second likewise, if from $18 + 4$ (that is, from 22) one subtracts $8 - 5$ (that is, 3), there remains $10 + 9$, that is 19. In the third, if from $18 - 5$ (that is, from 13) one subtracts $8 + 4$ (that is, 12), there remains $10 - 9$, that is 1. And in the fourth, if from $18 - 4$ (that is, from 14) one takes away $8 + 5$ (that is, 13), there remains likewise $10 - 9$, that is 1.

But should it happen that the same sign, + or -, is found both in the number to be subtracted and in the one from which the subtraction is made, yet the subtrahend is the larger, then the subtraction must be carried out in reversed order — the upper taken from the lower — and the remaining number must be given the opposite sign, so that from + comes -, and from - comes +. This the two following examples show:

[Margin: What to do in subtraction, when both numbers have the sign + or -, and the numbers are placed in reversed order.]

$$\begin{array}{r}
 3 \text{ R } N \text{ 6 } 3 \text{ N} \\
 9 + 4 - 5 \quad 6 - 5 + 3 \\
 4 + 7 - 8 \quad 7 - 9 + 10 \\
 \hline
 5 - 3 + 3 \quad -1 + 4 - 7
 \end{array}$$

(Continuing Cap. IV — Addition and Subtraction of cossic numbers.)

In the latter example, since 6℥ and 7℥ are understood to carry the sign +, the remaining number therefore takes the sign -. Hence it must be written thus: $43 - 1\text{℥} - 7$. For the first number is not correctly affected by the sign -. The truth of this operation is also clear in these two examples of absolute [ordinary] numbers, where you see that in the former example:

$$\begin{array}{r}
 9 + 4 - 5 \\
 4 + 7 - 8 \\
 \hline
 5 - 3 + 3 \\
 \\
 6 - 5 + 3 \\
 7 - 9 + 10 \\
 \hline
 - 1 + 4 - 7
 \end{array}$$

For from $9 + 4 - 5$ (that is, 8) subtract $4 + 7 - 8$ (that is, 3): there remains $5 - 3 + 3$, namely 5. In the latter example, however, if from $6 - 5 + 3$ one subtracts $7 - 9 + 10$

— that is, [subtracts] 8 from 4 — there remains $-1 + 4 - 7$, that is, -4 ; because indeed the upper number falls 4 units short for the lower to be subtracted from it.

Moreover, all the precepts handed down so far concerning addition and subtraction, as far as the signs $+$ and $-$ are concerned, are easily comprised in these two rules, offered as an aid to memory.

[Margin: Two rules to be observed in addition and subtraction.]

I.

The same signs give the same sign, except in subtraction when the numbers are placed in reverse order. For then the upper is subtracted from the lower, and $+$ becomes $-$, and $-$ becomes $+$.

II.

Different signs change the kind of operation. In addition, the sign of the larger number is set down; in subtraction, however, the sign of the upper number, from which the subtraction is made — whether it be larger, or smaller, or equal. These rules, once the precepts already given are understood, are not difficult.

[Margin: Testing of addition and subtraction.]

It remains for us to teach how the proof, or test, of Addition and Subtraction is to be set up. There are two ways in which one can verify whether or not the operation in Addition or Subtraction has been done correctly. First, Addition can be tested by subtraction, and subtraction by Addition, just as is customarily done with absolute numbers.

Secondly, let a table be constructed with some geometric progressions beginning from unity, as you see here.

N	R	3	¢	33	ß	3¢	Bß	333
1	2	4	8	16	32	64	128	256
1	3	9	27	81	243	729	2187	6561
1	4	16	64	256	1024	4096	16384	65536
1	½	¼	⅛	1/16	1/32	1/64	1/128	1/256

Next, let the cossic numbers to be added be resolved, according to one of these progressions, into absolute numbers, and let them be added together, or one taken from the other, according to the sign $+$ or $-$. After this, in the same way, let the cossic numbers of the collected sum be resolved. For if the addition has been

done correctly, the resolved numbers of the collected sum, taken together, will be equal to the resolved numbers to be added, taken together. As in this example:

°Ce. 3.

6 + 4

4 - 8

10 - 4

6¢ (if the proof is set up by the progression whose root is 2) make 48, and 43 make 16. Let 48 and 16 be joined together, on account of the sign +: they become 64. Likewise, 4¢ make 32; let these be added to 64, on account of the sign + (for a number marked with no sign is always understood to be marked with this sign +, as has been said): they become 96. Finally, 83 make 32, which, taken from 96 on account of the sign -, leave 64. See now whether 10¢ - 43, resolved, also make 64. It is clear that 10¢ make 80, from which, if 43 — that is, 16 — be deducted on account of the sign -, there remain 64, and so on.

Subtraction will be proved no differently, if its cossic numbers are converted into absolute numbers. For the numbers subtracted, together with the remainder, ought to equal the numbers from which the subtraction was made, and so on. As in this proposed example, if we set up the proof by the progression whose root is 3:

°Ce. 3.

10 - 4

6 + 4

4 - 8

10¢ make 270, from which, if 43 — that is, 36 — be deducted on account of the sign -, there remain 234. It is sufficiently clear that 6¢ + 43 and 4¢ - 83, resolved and added together, also make 234.

In the same way, if 6¢ + 43, resolved, are subtracted from 10¢ - 43, resolved, there will be left 4¢ - 83. Or indeed, if 4¢ - 83, resolved, are deducted from 10¢ - 43, there will remain 6¢ + 43, resolved.

The same example will be examined by the progression whose root is $\frac{1}{2}$ — somewhat more laboriously — in this way. 10¢ make $\frac{10}{8}$, from which, if 43 (namely $\frac{4}{4}$, that is $\frac{8}{8}$) be taken away, there remain $\frac{2}{8}$. It is therefore necessary that 6¢ + 43, resolved and subtracted from $\frac{2}{8}$, leave 4¢ - 83 resolved. And since 6¢ make $\frac{6}{8}$, and 43 make $\frac{4}{4}$ (that is, $\frac{8}{8}$) — which with $\frac{6}{8}$ make $\frac{14}{8}$, to be subtracted from $\frac{2}{8}$ — but since the numbers are placed in reverse order, $\frac{2}{8}$ is to

be subtracted from $14/8$, so that there remain $-12/8$, according to the first rule. And these ought to yield just as much as $4\text{C} - 8\text{J}$. Now 4C are $4/8$, from which, to take away 8J (that is, $8/4$, or $16/8$), $4/8$ must be taken from $16/8$ — because the numbers are placed in reverse order — so that there remain $-12/8$, according to the law of the first rule. Which was what was proposed.

DE MULTIPLICATIONE et Divisione numerorum Cossicorum. — CAPUT V.

On the Multiplication and Division of Cossic Numbers — Chapter V

[Margin: Multiplication and division of a cossic number by an absolute number.]

When a cossic number is multiplied or divided by an absolute number, there is produced — the multiplication or division being carried out as with absolute numbers — a number of the same cossic denomination. Thus, multiplying 4R or 8J by 3, there come out 12R or 24J , and so on. Likewise, dividing 12R or 24J by 3, there are produced 4R or 8J , and so on.

[Margin: Multiplication and division of a cossic number by a cossic number.]

But when a cossic number is multiplied or divided by a cossic number, there is produced — the multiplication or division being carried out as with absolute numbers — a number of a *different* cossic denomination. And so that it may be known which denomination is generated, any geometric progression beginning from unity displays this beautifully, together with the exponent numbers, of which we spoke in chapter 2, and which appear here.

0	1	2	3	4	5	6	7	8	9	&c.
N	R	3	C	33	ß	3C	Bß	333	CC	&c.
.										
1	2	4	8	16	3	64	12	256	512	&c.
					2		8			

For in multiplication the exponent number of the cossic denomination produced is formed from the addition of the exponents of the numbers — namely, of those that express the denominations which are multiplied. In division, however, the exponent number of the cossic denomination produced is found, being what is left when the exponent number of the dividing denomination is subtracted from

the exponent number of the denomination to be divided. Thus, from the multiplication 4R into 4R, or 4R into 7R, come 163 or 283; for unity, which is the exponent of this character R, added to unity makes 2, the exponent of the character 3. So too, from the multiplication 4R into 53 come 20C. And from 43 into 6ß come 24 Bß, and so on. Likewise, from the division 24 Bß by 43 come 6ß; and from the division 363C by 433 are produced 93; and from the division 1233 by 4C come 3R; and from 83 by 83 comes 1 N, because, subtracting 2 from 2, there is left 0, the exponent of an absolute number. So too, dividing 123C by 4ß, there come out 3R, and so on.

From these things it is easily understood how a cossic number is multiplied into itself — squared, cubed, and so on. For 5C into itself, squared, make 253C, since 3 and 3 make 6. But 5C into itself, cubed, make 125CC; for first 5C into itself make 253C, then 253C into 5C make 125CC, and so on.

[Margin: Demonstration of the multiplication and division of cossic numbers.]

The DEMONSTRATION of this is easy. Let the progression of cossic characters that name the numbers of the geometric progression be set out again, together with the natural progression of the exponent numbers, as here:

												&c					
0	1	2	3	4		5	6		7	8		9		10		11	.
N	R	3	C	33		ß	3C		B	333		CC		3ß		Cß	&c
.									ß								.

Since, then, as we taught in the scholium to proposition 11 of book 9 of Euclid, any number of a geometric progression beginning from unity, when it multiplies itself, produces a number of the same progression that is as far distant (exclusively) from it as it itself is distant (exclusively) from unity — it comes about that C, multiplying itself, produces 3C, which occupies the third place after C (exclusively), just as C holds the third place after N, that is, after unity (exclusively). Therefore, since the exponent of this character C — namely 3 — added to itself makes 6, the exponent which holds the third place after 3 (exclusively), because three units are contained in 3, by just so many intervals is the denomination C distant from N, or unity (exclusively) — for the exponent of each cossic denomination contains as many units as the number of places the denomination itself holds from unity, or from N (exclusively) — it is clear that the number 6, which arises from the addition of 3 to 3, is the exponent of this cossic character 3C, which is produced from the multiplication of C into itself. And the same reasoning holds for the other cossic characters that multiply themselves.

AGAIN, since — as is clear from the same scholium — if a smaller number of a geometric progression beginning from unity multiplies some larger number of the same progression, there is generated a number as far distant (exclusively) from the larger, in the same progression, as the smaller is distant (exclusively) from unity: it comes about that $\mathfrak{C}$, multiplying $3\mathfrak{C}$, produces $\mathfrak{C}\mathfrak{C}$, which occupies the third place after $3\mathfrak{C}$ (exclusively), just as $\mathfrak{C}$ holds the third place after N , that is, after unity. Therefore, since the exponent 3 of the character $\mathfrak{C}$, added to the exponent 6 of the character $3\mathfrak{C}$, makes 9 — the exponent which has attained the third place after 6 (exclusively), because in 3 are contained three units, by just so many places does this character $\mathfrak{C}$ stand (exclusively) from unity, or N — it is manifest that the number 9, which arises from the addition of 3 to 6, is the exponent of this cossic character $\mathfrak{C}\mathfrak{C}$, which is generated from the multiplication of $\mathfrak{C}$ into $3\mathfrak{C}$. For the same reason the same character $\mathfrak{C}\mathfrak{C}$ is generated from the multiplication of 33 into $\mathfrak{B}$, and so for the rest.

SINCE, moreover, just as Addition proves Subtraction and Subtraction proves Addition, so too Multiplication proves Division and Division proves Multiplication — it comes about that, just as from the addition of 3 to 3 come 6, and from the multiplication of $\mathfrak{C}$ into $\mathfrak{C}$ comes $3\mathfrak{C}$, so, subtracting 3 from 6, there are left 3, and, dividing $3\mathfrak{C}$ by $\mathfrak{C}$, there comes out $\mathfrak{C}$. Likewise, just as from the addition of 3 to 6 come 9, and from the multiplication of $\mathfrak{C}$ into $3\mathfrak{C}$ comes $\mathfrak{C}\mathfrak{C}$, so too, subtracting 3 from 9, there are left 6, and, dividing $\mathfrak{C}\mathfrak{C}$ by $\mathfrak{C}$, there is produced $3\mathfrak{C}$. And the same reasoning holds for the rest.

From this demonstration is clear what we handed down in chapter 2: namely, that Addition of numbers of an arithmetic progression corresponds to multiplication of numbers of a geometric progression, and Subtraction to Division. For we demonstrated that, just as from the addition of 2 to 4 come 6, so from the multiplication of 3 into 33 comes $3\mathfrak{C}$; and just as, subtracting 2 from 6, there are left 4, so, dividing $3\mathfrak{C}$ by 3, there comes 33 , and so on.

[Margin: Multiplication and division of a composite cossic number by an absolute number, or by a cossic number whether simple or composite.]

THEREFORE, from the multiplication of any cossic character into any other whatever, there is produced a cossic character as far distant (exclusively) from the larger as the smaller is distant from unity — that is, as it recedes exclusively from this sign N — or, [as far] as the exponent of the smaller cossic character is. Thus, from $\mathfrak{C}$ into 333 comes $\mathfrak{C}\mathfrak{B}$, and so on. But from division — that is, of any larger cossic character by any other smaller one whatever — there comes a cossic character as far distant (exclusively) from the larger, going backward however,

as the smaller is distant (exclusively) from unity, that is, from the character N; or as much as is the exponent of the smaller cossic character. Thus, from the division of C€ by 33 is produced ß, and so on.

WHEN, however, a cossic number — composite or diminished [i.e. containing subtracted terms] — is multiplied or divided by an absolute number, or by a cossic number whether simple or composite or diminished, then, besides the instruction already given, the greatest account must be taken of these signs + and -. For when the numbers multiplying or dividing each other have the same sign, + or -, there is always attached to the product this sign +. But when one of them has the sign + and the other this -, the product is always affected by this sign -. This is what other writers teach in these words: In multiplication and division, Plus by Plus, or Minus by Minus, produces Plus; but Plus by Minus, or Minus by Plus, produces Minus. And this will be committed to memory by this single rule:

[Margin: Rule to be observed in the multiplication and division of cossic numbers.]

The same signs give the sign of the added terms; different signs give the sign of the subtracted terms.

Exempla Multiplicationis — Examples of Multiplication

$$\begin{array}{r} 73 + 4R \\ 9N \\ \hline \end{array}$$

$$633 + 36R$$

$$\begin{array}{r} 73 - 4R \\ 9 \\ \hline \end{array}$$

$$633 - 36R$$

$$\begin{array}{r} 73 - 4R \\ 9R \\ \hline \end{array}$$

$$63€ - 363$$

$$\begin{array}{r} 83 + 9 \\ 83 + 9 \\ \hline \end{array}$$

$$\begin{array}{r} 723 + 81 \\ 6433 + 723 \\ \hline \end{array}$$

$$6433 + 1443 + 81$$

$$\begin{array}{r} 83 - 9 \\ 83 - 9 \end{array}$$

$$\begin{array}{r}
 \hline
 - 72\text{J} + 81 \\
 64\text{J}\text{J} - 72\text{J} \\
 \hline
 64\text{J}\text{J} - 144\text{J} + 81 \\
 8\text{C} - 4\text{J} \\
 6\text{R} \\
 \hline
 48\text{J}\text{J} - 24\text{C} \\
 6\text{J} + 8\text{R} - 6\text{N} \\
 2\text{J} - 4\text{N} \\
 \hline
 - 24\text{J} - 32\text{R} + 24\text{N} \\
 12\text{J}\text{J} + 16\text{C} - 12\text{J} \\
 \hline
 12\text{J}\text{J} + 16\text{C} - 36\text{J} - 32\text{R} + 24\text{N} \\
 9\text{J} + 8\text{N} - 3\text{R} \\
 7\text{C} - 4\text{J}\text{J} - 8\text{J} \\
 \hline
 - 72\text{J}\text{J} - 64\text{J} + 24\text{C} \\
 - 36\text{J}\text{C} - 32\text{J}\text{J} + 12\text{R} \\
 63\text{R} + 56\text{C} - 21\text{J}\text{J} \\
 \hline
 - 36\text{J}\text{C} + 75\text{R} - 125\text{J}\text{J} + 80\text{C} - 64\text{J}
 \end{array}$$

In this last example the larger cossic denominations always precede the smaller in the sum. But since the sign – is not correctly placed at the beginning, the produced sum must therefore be arranged thus:

$$75\text{R} - 36\text{J}\text{C} - 125\text{J}\text{J} + 80\text{C} - 64\text{J}$$

Exempla Divisionis — Examples of Division

$$\begin{array}{l}
 10\text{C} \\
 \text{per } 3\text{C} \left(3\frac{1}{3} \text{ N} \right)
 \end{array}$$

$$\begin{array}{l}
 10\text{C} \\
 \text{per } 3\text{R} \left(3\frac{1}{3} \text{ J} \right)
 \end{array}$$

$$\begin{array}{l}
 10\text{R} \\
 \text{per } 3\text{R} \left(3\frac{1}{3} \text{ N} \right)
 \end{array}$$

$$\begin{array}{l}
 9\text{J} + 4\text{R} \left(3\text{R} + 1\frac{1}{3} \text{ N} \right) \\
 3\text{R} \\
 3\text{R}
 \end{array}$$

$$\begin{array}{r}
 1833 - 123 \text{ (} 4\frac{1}{2} \text{ } \mathfrak{C} - 3\mathfrak{R} \\
 4\mathfrak{R} \\
 4\mathfrak{R}
 \end{array}$$

40

$$\begin{array}{r}
 303 - 58\mathfrak{R} + 24\mathfrak{N} \text{ (} 6\mathfrak{R} - 8\mathfrak{N} \\
 5\mathfrak{R} - 3\mathfrak{N} \\
 5\mathfrak{R} - 3\mathfrak{N}
 \end{array}$$

[Throughout this section the cossist powers are written with the printer's tokens, a coefficient preceding each: $\mathfrak{N}$ = *numerus* (the constant term); $\mathfrak{R}$ = *res / radix* (the first power, x); 3 = *census* (the square, x^2); $\mathfrak{C}$ = *cubus* (the cube, x^3); 33 = *census de censu* (the fourth power, x^4); $\mathfrak{B}$ = *surdesolidum* (the fifth power, x^5). Concatenation of tokens composes the powers additively.]

Chapter V (continued) — Multiplication and Division of Cossic Numbers

[This opening continues the last worked division carried over from the preceding page. The dividend $1233 + 16\mathfrak{C} - 363 - 32\mathfrak{R} + 24\mathfrak{N}$ (that is, $12x^4 + 16x^3 - 36x^2 - 32x + 24$) is divided by $23 - 4\mathfrak{N}$, yielding the quotient $63 + 8\mathfrak{R} - 6\mathfrak{N}$.]

$$\begin{array}{r}
 -123 \text{ } 0\mathfrak{R} \\
 1233 + 16\mathfrak{C} - 363 - 32\mathfrak{R} + 24\mathfrak{N} \text{ (} 63 + 8\mathfrak{R} - 6\mathfrak{N} \\
 23 + 0\mathfrak{R} - 4\mathfrak{N} \\
 23 + 0\mathfrak{R} - 4\mathfrak{N} \\
 23 + 0\mathfrak{R} - 4\mathfrak{N}
 \end{array}$$

In every division of cossic numbers the ranks of the powers must run on without a gap. For that reason, in the last example — where the divisor is 23 [*census*] $- 4\mathfrak{N}$ [*numerus*] — I inserted $0\mathfrak{R}$ [*res*] with a plus sign, so that the complete divisor reads $23 + 0\mathfrak{R} - 4\mathfrak{N}$. You can see the same thing done in the next example, where the number to be divided is $1\mathfrak{C}$ [*cubus*] $+ 1$ and the divisor is $1\mathfrak{R} + 1$.

[Here $x^3 + 1$ is divided by $x + 1$, giving $x^2 - x + 1$.]

$$\begin{array}{r}
 -13 + 1\mathfrak{R} \\
 1\mathfrak{C} + 03 + 0\mathfrak{R} + 1 \text{ (} 13 - 1\mathfrak{R} + 1\mathfrak{N} \\
 1\mathfrak{R} + 1 \\
 1\mathfrak{R} + 1 \\
 1\mathfrak{R} + 1
 \end{array}$$

These last three examples, in which the divisor is a compound or defective number [i.e. one containing subtracted terms], have been contrived deliberately. For with divisors of this kind the division usually cannot be carried out; instead

the divisor is written beneath the number to be divided, a line is drawn between them, and the result is a fraction — as here.

[Margin: When the division cannot be carried out.]

$8R$ divided by $2J + 4N$ || $8J - 9R$ divided by $4R + 3N$

The quotients are:

$$\frac{8R}{2J + 4N} \quad \frac{8J - 9R}{4R + 3N}$$

What is to be done with fractions of this sort I shall teach below, in Chapter 10, when I treat the reduction of equations among the fractional quantities that turn up. A fraction is formed in the same way when a cossic number, whether simple or compound, is divided by a simple cossic number of a higher rank — as $8J$ by $2C[?]$, which yields this fraction, $8J / 2C[?]$. Likewise, dividing $9J + 4$ by $3R$ gives this fraction, $(9J + 4) / 3R$; and so on.

It remains to explain how multiplication and division are to be checked. There are two ways. First, multiplication can be verified by division, and conversely, just as the rule prescribes for ordinary [absolute] numbers.

[Margin: The proof of multiplication and division.]

Secondly, the proof can be carried out by resolving the cossic numbers for some root of the geometric progression, as was said at the end of Chapter 4. For when two cossic numbers are multiplied together, their resolved values, multiplied together, must necessarily equal the resolved value of the cossic product taken for the same root. And the resolved value of the cossic dividend, when divided by the resolved divisor, must yield the same as the resolved value of the quotient for that same root. As appears here.

Example of Multiplication

$$\begin{array}{r} 6R - 8 \\ 5R - 3 \\ \hline - 18R + 24 \\ 30J - 40R \\ \hline 30J - 58R + 24 \end{array}$$

For the root 2, the factor $6R - 8$ comes to 4, and $5R - 3$ comes to 7. Multiplying 4 by 7 gives 28; and resolving the product $[30\bar{3} - 58R + 24]$ for the same root 2 gives exactly the same, 28 — as is plain.

Example of Division

$$\begin{array}{r} - 18R \\ 30\bar{3} - 58R + 24 \quad (\quad 5R - 3N \\ 6R - 8 \\ 6R - 8 \end{array}$$

The dividend $30\bar{3} - 58R + 24$, resolved for the root 2, gives 28; and the divisor $6R - 8$ gives 4. Dividing 28 by 4 gives 7. And it is clear that the quotient $5R - 3$, resolved for the same root, likewise gives 7.

As to why like signs — + and +, or — and — — always produce + in multiplication and division, while unlike signs — + and –, or – and + — always produce –: quite apart from the fact that the check of multiplication and division amply confirms it, the point is also plainly seen in ordinary numbers. In the following example, who does not see that multiplying $8 - 2$ (that is, 6) by $7 - 3$ (that is, 4) produces 24, namely $56 - 38 + 6$?

[Margin: That like signs yield the sign +, but unlike signs yield –.]

$$\begin{array}{r} 8 - 2 \\ 7 - 3 \\ \hline 56 - 14 \\ - 24 + 6 \\ \hline 56 - 38 + 6 \end{array}$$

For when 7 times 8 makes 56, since it is not the whole of this number 7 that is multiplied into the 8, but 7 with 3 first taken away, the product $56 - 14$ comes out larger than the true product. Hence multiplying -3 by 8 gives 24, to be marked with the sign –; and so again – into + yields –. But since we should not take away the whole of what comes from -3 into 8, but only what comes from -3 into 8 with 2 first removed, what was wrongly subtracted must be restored. And so, finally, -3 into -2 gives 6, to be marked with this sign +, which signifies addition, so that the 6 that had been taken away are restored. The same reasoning holds in the remaining examples of multiplication.

Now since multiplying 6 by -4 produces the number -24 , as was said, it follows necessarily that dividing -24 by -4 gives 6, and dividing -24 by 6 gives -4 . So in division too, – into – yields +, and – into + (or the reverse) yields –. Therefore, just

as the affirmation of an affirmation and the negation of a negation both affirm, so plus into plus, or minus into minus, whether multiplied or divided, produces plus. And just as the affirmation of a negation, or the negation of an affirmation, denies, so plus into minus, or minus into plus, whether multiplied or divided, produces minus.

On Fictitious Numbers, or Numbers Less Than Nothing — Chapter VI

[Margin: Fictitious numbers, or numbers less than nothing.]

Just as various roots are imagined for numbers that do not possess them — such as the square, cube, fourth [*Zensizensic*], or fifth [*Surdesolid*] root of the number 20 — a fiction adopted with the greatest usefulness and convenience for those engaged in mathematics, as I shall say in its place: so too the writers on algebra, not idly, imagine certain numbers less than nothing, such as $0 - 4$, that is, nothing with four units first taken away; or $6 - 10$, that is, six units less ten units. A single example of the following subtraction will make this clear.

$$\begin{array}{r} 8 - 2 \\ 10 - 5 \\ \hline - 2 + 3 \end{array}$$

Let $10 - 5$ be subtracted from $8 - 2$ — that is, 5 from 6. First I take -5 from -2 ; that is, since the upper number is the smaller, I take 2 from 5, and to the remainder, 3, I assign the opposite sign, $+$, following the law of subtraction given above. For in truth the number -2 is understood to exceed the number -5 by three units: since -2 falls two units below nothing, while -5 descends five units below that same nothing — that is, it recedes from nothing three units further than -2 does. This can also be understood thus: the number -2 lacks only two units to be equal to nothing, whereas -5 lacks five to come out equal to nothing.

Next I subtract $+10$ from $+8$ — a larger number from a smaller — which cannot be done. So I find no true number to set down as the remainder. For if the upper number 8 were larger than the lower 10, a true number would remain from that subtraction; and if the upper were equal to the lower, nothing would be left. But now, since the number to be subtracted, 10, is larger than the number 8 from which the subtraction is to be made, the result is a number below 0 — that is, less than nothing: namely the amount by which the upper falls short, such that, when the lower is taken from it, 0 remains. This is the kind of number that is left by subtracting the upper from the lower, marked with the sign $-$, which signifies a

deficiency: namely, -2 . So the whole remainder is $-2 + 3$. But since it is not correct to prefix this sign $-$ at the very start of a number, the remainder should instead be set down as $3 - 2$. In the same way, whenever some larger number is to be subtracted from a smaller (which truly cannot be done), in this science one puts down as the remainder the excess by which the lower number surpasses the upper, prefixed with the sign $-$: so that subtracting 20 from 12 leaves -8 , and so on.

You see, then, how elegantly this art, out of its boundless resources, makes use both of the things that exist and of the things that do not exist but are only imagined to exist. You will grasp this still more precisely from what follows.

Just as whole numbers are placed above unity, and fractions of unity below it — so that unity stands midway between whole and fractional numbers — so above 0, or nothing, unity is placed together with the rest of the true numbers, while below 0 unity is imagined together with the rest of the numbers, but always prefixed with this sign $-$; so that 0, or nothing, stands midway between the true numbers and the fictitious ones, that is, those less than nothing. The two progressions set side by side below show this: the upper is the natural series of numbers, the lower a geometric one.

&c																		&c
.	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	.		
&c	1/128	1/64	1/32	1/16	1/8	1/4	1/2	1	2	4	8	16	32	64	128	&c		
.												6	2	4	8	.		

The numbers of the one series serve as the exponents of the numbers of the other, as I said in Chapter 2. For here too the numbers of the upper progression do by addition and subtraction the same as the numbers of the lower progression do by multiplication and division. Thus, just as multiplying $1/4$ by 32 gives 8, so adding -2 to 5 gives 3. Again, just as multiplying $1/2$ by 8 gives 4, so adding -1 to 3 gives 2. Conversely, just as dividing 8 by $1/4$ yields 32, so subtracting -2 from 3 leaves 5. And just as dividing 8 by $1/2$ yields 16, so subtracting -1 from 3 leaves 4; and so on. This is clear from what we said about the addition and subtraction of cosmic numbers. Indeed, from this it readily follows that in addition and subtraction unlike signs change the kind of operation, and so on, as we taught in Chapter 4; otherwise addition and subtraction in the natural progression of numbers would not correspond to multiplication and division in the geometric progression.

From all this it is plainly established that such fictitious numbers — those less than nothing — are not devised in vain, since their addition to (or subtraction from) whole numbers corresponds so neatly to the multiplication and division of the whole numbers of the geometric progression by the fractional numbers of the same progression, as has been said.

It is also worth noting here that a cossic number can have two unequal square roots — or, what comes to the same thing, that two unequal numbers can be exhibited, each of which, multiplied into itself, produces the same number. For if the two numbers $2\bar{3} - 3\bar{R}$ and $3\bar{R} - 2\bar{3}$ are each multiplied by itself, they produce $4\bar{3}\bar{3} + 9\bar{3} - 12\bar{C}$, as you see here.

[Margin: That one and the same number can have two unequal square roots.]

$$\begin{array}{r|l}
 2\bar{3} - 3\bar{R} & || \quad 3\bar{R} - 2\bar{3} \\
 \hline
 - 6\bar{C} + 9\bar{3} & || \quad - 6\bar{C} + 4\bar{3}\bar{3} \\
 4\bar{3}\bar{3} - 6\bar{C} & || \quad 9\bar{3} - 6\bar{C} \\
 \hline
 4\bar{3}\bar{3} + 9\bar{3} - 12\bar{C} & || \quad 4\bar{3}\bar{3} + 9\bar{3} - 12\bar{C}
 \end{array}$$

— even though the two numbers themselves, when multiplied out, are very unequal. For the first, $2\bar{3} - 3\bar{R}$, resolved for the root 2, comes to $8 - 6$, that is 2; and this number multiplied into itself yields 4, which equals the product $4\bar{3}\bar{3} + 9\bar{3} - 12\bar{C}$, since that too, resolved for the same root 2, comes to 4. The second, $3\bar{R} - 2\bar{3}$, comes to $6 - 8$, that is -2 ; and this number multiplied into itself likewise produces 4, provided the rules of multiplication are observed. So both $2\bar{3} - 3\bar{R}$ and $3\bar{R} - 2\bar{3}$ are roots of the squared number $4\bar{3}\bar{3} + 9\bar{3} - 12\bar{C}$ — which may well seem strange and paradoxical, and yet that it is so is plainly confirmed by resolving both the roots and the resulting number.

The same is seen in ordinary numbers. For each of the numbers $4 - 1$ and $1 - 4$ — that is, 3 and -3 — multiplied into itself begets 9, as you see here.

$$\begin{array}{r|l}
 4 - 1 & || \quad 1 - 4 \\
 \hline
 - 4 + 1 & || \quad - 4 + 16 \\
 16 - 4 & || \quad 1 - 4 \\
 \hline
 9 & || \quad 9
 \end{array}$$

The same may be tried with countless other numbers. For if $4 - 1$ is multiplied into $1 - 4$, the number produced is $8 - 17$, that is -9 . And multiplying $2\bar{3} - 3\bar{R}$ into

$3\Re - 2\Im$ produces the number $12\text{C} - 4\Im\Im - 9\Im$, which resolved for the root 2 comes to $32 - 36$, that is -4 — even though each of them, multiplied into itself, produces 4. The cause of this seems to be traceable to the multiplication of cossic numbers and of the signs $+$ and $-$; and the weakness of the human mind is to be blamed, in that it cannot grasp how this can be true. For the correctness of the aforesaid multiplication is not to be doubted, since it has been confirmed by many examples.

On Fractions of Cossic Numbers — Chapter VII

The operations on fractions — that is, cossic fractions [*minutiae*] — are nearly the same as those on ordinary fractions, provided one applies what has so far been taught about whole cossic numbers as regards the cossic characters and the signs $+$ and $-$. To set down a few examples of reading them: the fraction $3\Re / 8\Re$ signifies that three units are divided by $8\Re$ [that is, $3 \div 8x$]. And $7\Im / 9$ denotes that 7 *censuses* [$7x^2$] are divided by 9 units. Likewise $(9\Im\Im + 8\Im) / 6\text{C}$ indicates that the number $9\Im\Im + 8\Im$ is divided by 6C . And $(9\Im\Im - 8\Im) / 6\text{C}$ means that the number $9\Im\Im - 8\Im$ is divided by 6C ; and so on.

[Margin: The reading of cossic fractions.]

Reduction can be done in two ways. Either only the numbers are reduced, as in common fractions, leaving the cossic characters untouched; or the cossic characters themselves are reduced as well. Thus this fraction, $8\Re / 18\text{C}[\text{?}]$, as far as the numbers go, reduces to $4\Re / 9\text{C}[\text{?}]$. And this one, $(9\Re + 36) / 81$, reduces to $(1\Re + 4) / 9$, because their greatest common measure is 9. The method of finding the greatest common measure of three or more numbers is drawn from Euclid, Book VII, Proposition 3 [*Elements* VII.3].

[Margin: The reduction of cossic fractions.]

For if there are three numbers, once the greatest common measure of two of them is found: if it also measures the third, it is itself the greatest measure of all; but if it does not measure the third, then the greatest common measure of that greatest measure and the third number is the greatest measure of all three. In the same way, given four numbers, once the greatest measure of three is found: if it also measures the fourth, it is itself the greatest measure of all four; but if it does not measure it, then the greatest common measure of that greatest measure and the fourth number is the greatest measure of all four. And so on for the rest.

Similarly, this fraction, $8\Re / (2\Re - 4)[\text{?}]$, reduces to $4\Re / (1\Re - 2)[\text{?}]$; and this one, $(18\Im - 9\Re) / (6\Re + 3\Im)$, reduces to $(6\Im - 3\Re) / (2\Re + 1\Im)$; and so on. [The small

compound cossic characters in the first abbreviation example are at the resolution floor — see QA.]

But the reduction of the cossic characters is done by subtracting the exponent of the lowest character from the exponents of the others. For if the remaining terms are then given their proper characters, the reduction — as regards the characters — is complete. Hence it is clear that in this reduction the term of the lowest cossic character becomes Absolute [a pure number, N]. Likewise, a fraction in which an Absolute number is found cannot be reduced with respect to its characters. Thus this fraction, 7 / [denominator: ?], cannot be reduced as regards the characters; but as regards the numbers ...

[Cossic-sign key used below (glossed once here): N = numerus, the constant term; R = res (radix), the unknown, x; 3 = census, x^2 ; C = cubus, x^3 ; 33 = census de censu, x^4 . Concatenation of signs composes powers.]

as regards the numbers, it will be reduced by cancellation to this: 1 / (283)[?]. But this fraction, 23 / (26R)[?], as regards the characters, reduces to this. Then, as regards the numbers, it reduces to this: 4 / (183)[?]. And this fraction, (183 - 9R) / (6R + 33)[?], is reduced to that one, as regards both numbers and characters: (6R - 3) / (2 + 2R)[?].

[Margin: Demonstration of the abbreviation [reduction to lowest terms] of cossic fractions.]

But the demonstration of this cancellation is easy. As regards the numbers: since the numbers of the given fraction are divided by the same number—namely by their greatest common measure—the proportion between the quotients will be the same as that between the numbers divided. Hence these fractions will be equal: 43 / (18R)[?]. And as regards the characters: since each character is lowered by the same exponent, so that the distance between the characters to which the reduction is made is exactly the same as between the characters originally proposed, the characters to which the reduction is made will have the same proportion as the characters originally proposed. And so these fractions will be equal: 43 / (18R)[?] and 4N / (183)[?]. [Margin: a — *Elements* VII.19.] For the proportion of 43 to 18R[?] is the same as 4 to 183[?], since the same number arises from multiplying 43, the first number, by 183[?], the fourth number, as arises from 4, the third number, [multiplied] by 18R[?], the second number. For in both cases this number is produced: 72C[?], and so on. Or indeed: since between 3 and 3R[?], and between N and 33, three proportional middle terms fall—because 3

and $3R[?]$ are equally depressed, namely by the exponent 2 of this lesser character 3—and so on.

[Margin: Reduction of cossic fractions.]

The **reduction** of fractions to the same denominator is accomplished by multiplying the numerators crosswise, and the denominators by one another, as with common fractions. Thus these fractions, $3R / 43[?]$ and $(1533) / (20333)[?]$, are reduced to these: $4R / (53R)[?]$ and $(163) / (20333)[?]$. For this number 43, multiplied by... [Margin: Demonstration of the reduction of cossic fractions. b — *Elements* VII.17.]

The reasoning behind this reduction is very easy. Since the same number, $5C[?]$, multiplied... produced $1533[?]$ and $20333[?]$, the same proportion will hold— 1533 to $20333[?]$ as $3R$ to $43[?]$ —and therefore these fractions, $3R / 43[?]$ and $(1533) / (20333)[?]$, will have the same value. In the same way these will be equal: $4R / (53R)[?]$ and $(163) / (20333)[?]$. For this number $43[?]$, multiplying the two numbers $4C[?]$ and $53C[?]$, produced these two: $163[?]$ and $20333[?]$.

But if some whole number and a fraction are to be reduced to the same denomination: suppose unity to be the denominator of the whole number, and carry out the rest as before. Thus, for 6 and $4R / 73[?]$, in this way: $6/1$ and $4R / 73[?]$. And these two, $53 / 1[?]$ and $4R / 33[?]$, are reduced to these: $(1533) / (33)[?]$ and $(4R) / (33)[?]$. For unity is supposed as the denominator for this cossic whole number $53[?]$.

But if some fraction is attached to whole numbers, the whole numbers must first be reduced to that fraction; which indeed is done by multiplying the whole numbers by the denominator of the fraction, and adding the resulting number to the numerator. Thus, if we wish to reduce $43 + (2R / 1R)[?]$ and $33 / 13[?]$ to the same denomination, we shall first multiply 43 by $1R[?]$, so that we may have these two fractions: $(43R + 2R) / (1R)[?]$ and $(33) / (13)[?]$, to which we shall reduce them; and so with the rest.

[Margin: Addition, subtraction, multiplication, and division of cossic fractions.]

Now the four remaining operations on cossic fractions—that is, Addition, Subtraction, Multiplication, and Division—do not differ from those which we handed down concerning common fractions, provided that account is always taken of the signs + and −, and also of the cossic characters, whose operations are contained in the precepts above. And so we shall settle the matter here by examples alone.

Examples of Addition

$3R / 2N \times 43 / 3R$ make $(933 + 83) / (6R)[?]$, that is, $(93 + 8N) / (6R)[?]$. Again, $[?]N / 73 \times [?]N / (12R - 33)$ make $([?]3 + [?]R) / ([?]R - [?]3)[?]$. Again, $(9R + 23) / (3C) \times (3133 - 83) / (3C)[?]$ make... [The cross-multiplied numerators and denominators here are dense and not fully legible even at high magnification; coefficients marked $[?]$ are not transcribed rather than guessed.]

For since in the latter example the denominators are exactly alike, only the numerators are added together, and the same denominator is written under the sum. This fraction, as regards both numbers and characters together, will be reduced to this: $(7R - 23 + 3N) / (123)[?]$.

Examples of Subtraction

$3R / 2N$ minus $43 / 3R[?]$ leave $(16R) / (123)[?]$, that is, $4N / 3R[?]$. Again, $3N / 43$ minus $33 / (2C)[?]$ leave $([?]) / ([?])[?]$, that is, $4N / 5R[?]$. Again, $33 / 6N...$ and $(93 + 8R) / (2C - 6N)$ from $([?] + [?]) / (3C - 6N)[?]$ leave $([?]R + 73) / (3C - 6N)[?]$.

For since in the latter example the denominators are exactly alike and the same, the numerator is to be subtracted from the numerator, and the same denominator written under the remaining number.

Examples of Multiplication

$(933 + 83) / (6C)[?]$ by $3R / 2N$ make $(273 + 24C) / (12C)[?]$. Again, $73 / (833)$ by $(42R - 8N)$ make $(28C - 563) / (833)[?]$. For here, for a whole number, unity is supposed as the denominator, so that the example is prepared for multiplication in this way: $73 / (833)$ by $(42R - 8N) / 1N$. Again, $(73 + 8R) / (5C - 11N)$ by $4R / 53$ minus $8N$ make $([?]) / (15R - 553)[?]$. For in this example the reduction of $-8N$ to the fraction to which it was attached was first made, namely by multiplying $-8N$ by the denominator 53 . Hence the fractions to be multiplied were these: $(73 + 8R) / (5C - 11N)$ by $(42R - 403) / (53)[?]$.

Examples of Division

$(273 + 24C) / (12C)[?]$ by $3R / 2N$ make $(543 + 48C) / (3633)[?]$. Again, $8R / 93$ by $3N / 3R[?]$ make $(243) / (183)[?]$, that is, $4/3$, or $1\frac{1}{3}$. Again, $(7R - 143) / (233)[?]$ by $73 / (833)$ make $(5633 - 112C) / (143C)[?]$.

Now since Addition and Subtraction, and likewise Multiplication and Division, prove one another, it comes about that if, from the addition of $3R / 2N$ and $43 / 3R$, there arise $(933 + 83) / (6R)[?]$ —that is, $(93 + 8N) / (6R)[?]$ —then, subtracting $3R /$

2N from $(933 + 83) / (6R)[?]$, $43 / 3R$ is left. And by the rule what is left is $(163) / (12R)[?]$, that is, in lowest terms, $4N / 3R[?]$. It is necessary, therefore, that these two fractions, $43 / 3R$ and $4N / 3R$, be equal—that is, that the proportion of 43 to 3R be the same as that of 4N to 3R. And this is indeed true; for if the said fractions are cross-multiplied, equal numbers will be produced—that is, as much will come from the first number into the fourth (namely from 43 into 3R) as from the second into the third (namely from 3R into 4N). And so, [it is] not...

necessary that exactly the same fraction should result; it is enough that an equal fraction be produced, one whose equality can easily be tested by cross-multiplication. The same is to be said of the examples of Multiplication and Division, as is clear from the examples above.

Certain abridged operations with fractions, very useful for solving a great many problems

[Margin: Certain shortcuts for fractions.]

I.

To add to a given number any part of it, or specified parts of it.

Let the given number be $(3R - 16N) / (2N)$, to which I wish to add $\frac{3}{4}$ of that same number. I use this shortcut, then, even though I do not know what $\frac{3}{4}$ of it is. To the part, or parts, to be added, I add unity (which is done briefly by adding the denominator to the numerator)—that is, to $\frac{3}{4}$ I make $7/4$. This fraction I multiply by the given number, and I obtain $(21R - 112N) / (8N)$, the sum sought.

Again, if $\frac{2}{3}$ is to be added to 123 : I add unity to $\frac{2}{3}$, making $5/3$. This fraction I multiply by 123 , and I make this sum: $(603) / (3N)$, that is, 203 . Similarly, if to the fraction $\frac{1}{3}$ its half is to be added: unity being added to $\frac{1}{2}$, this makes $3/2$. I multiply $3/2$ by $\frac{1}{3}$, and the total sum becomes $[?]$ [final fraction not legible].

II.

To add to any part or parts of a given number, [another] part or parts of the same number.

Let the given number be $(8R - 12N) / (7N)$, to whose $\frac{1}{3}$ I wish to add $\frac{3}{4}$ of the same given number. I add the proposed parts together, $\frac{1}{3}$ and $\frac{3}{4}$, making $13/12$. This fraction I multiply into the given number, $(8R - 12N) / (7N)$, and I obtain the sum $(104R - 156N) / (84N)$.

Similarly, suppose I wish to add $\frac{2}{3}$ of the number 35 to $\frac{4}{5}$ [?] of it. I first add $\frac{2}{3}$ and $\frac{4}{5}$ [?], making $\frac{34}{35}$ [?]; I multiply $\frac{34}{35}$ by 35, and obtain $(1190) / (35)$, that is, 34, and so on. [The two input parts and their sum do not reconcile arithmetically as read; flagged [?].]

III.

To subtract from a given number any part or parts of it.

Let the given number be $(7\bar{3} + 8R - 10N) / (12R)$, from which I wish to subtract $\frac{1}{3}$ and $\frac{1}{5}$ —that is, $\frac{8}{15}$ (for if there are several fractions, they must first be gathered into one sum). I take $\frac{8}{15}$ from unity (which is done briefly by subtracting the numerator from the denominator, for the remaining number is the numerator), and $\frac{7}{15}$ remains. I multiply $\frac{7}{15}$ by the given number, $(7\bar{3} + 8R - 10N) / (12R)$, and I make the remaining number $(49\bar{3} + 56R - 70N) / (180R)$.

Similarly, I wish to subtract $\frac{4}{5}$ of this number $\frac{9}{10}$ from the number itself. I take $\frac{4}{5}$ from unity, and $\frac{1}{5}$ remains. I multiply $\frac{1}{5}$ by $\frac{9}{10}$, and I make $\frac{9}{50}$ for the remaining sum, and so on.

IIII.

To subtract from any part or parts of a given number, [another] part or parts of the same number.

Let the given number be $(12R + 8N) / (5)$, from whose $\frac{1}{3}$ I wish to take $\frac{1}{4}$. I first take $\frac{1}{4}$ from $\frac{1}{3}$, and $\frac{1}{12}$ remains. I multiply $\frac{1}{12}$ by $(12R + 8N) / (5)$, and I make the sum that is to be left: $(12R + 8N) / (60)$.

Again, I wish to subtract $\frac{3}{4}$ of this number 36 from $\frac{4}{5}$ [?] of the same number. I take $\frac{3}{4}$ from $\frac{4}{5}$ [?], leaving [?]; I multiply [?] by 36, and I obtain $(180) / (36)$ [?], that is, 5, for the remaining number, and so on. [Intermediate fractions do not fully reconcile as read; flagged [?].]

V.

To find any part or parts of a given number.

Let the given number be $(12R + 8N) / (5)$. I want $\frac{1}{3}$ and $\frac{1}{4}$ of it—that is, $\frac{7}{12}$ (for when there are several fractions, they must first be collected into one sum). I multiply $\frac{7}{12}$ by the given number, $(12R + 8N) / (5)$, and I make $(84R + 56N) / (60)$; and this number constitutes $\frac{7}{12}$ —that is, $\frac{1}{3}$ and $\frac{1}{4}$ —of the given number $(12R + 8N) / (5)$.

Similarly, I want $\frac{3}{4}$ of this number 24. I multiply $\frac{3}{4}$ into 24, and I make $(72) / (4)$, that is, 18, and so on.

VI.

To find a number of which a named part—or parts—is given; that is, to find a number of which any given number is any assigned part or parts.

Let this number, $(12R - 4N) / (5)$, be two-thirds of some number. I want to find that number. I divide $(12R - 4N) / (5)$ by $\frac{2}{3}$, and I make $(36R - 12N) / (10)$; and this is the number of which two-thirds make $(12R - 4N) / (5)$.

By the same reasoning, I wish to find a number of which this number 8 is $\frac{4}{5}$. I divide 8 by $\frac{4}{5}$, and I make $(40) / (4)$, that is, 10; and this is the number of which $\frac{4}{5}$ make 8, and so on.

On the Rule of Algebra

Caput VIII

Now that the algorithm of cossic numbers has been completed, both order and occasion demand that we set forth the most celebrated rule of Algebra. For on the occasion of this rule, everything that has so far been handed down concerning cossic numbers will become clearer, and the rest—the things required for a more perfect knowledge of this rule—will have to be explained. For not everything can be explained, as it ought to be, except when the occasion is taken from the parts of the rule of Algebra, as we shall say in its proper place. The rule of Algebra, then—which other writers have split into several rules—is of the following kind.

[Margin: The rule of Algebra.]

For the unknown number in the problem, let a single root be posited, thus: $1R$. (Sometimes several roots may also be posited, thus: $2R$ or $3R$, and so on—or some other number, as suits the problem proposed.) Let this be examined according to the terms of the problem, until some equation is found. Let this equation be reduced, if reduction is needed. Then let the remaining number of the equation be divided by the number of the higher cossic character. For either the quotient itself will be the number that was sought—namely the value of the root posited at the outset—or else some root of the quotient number will make known the number that was sought. And the divisor, by its own cossic character, will nicely show whether a root is to be extracted from the quotient, and which root is to be extracted.

This rule has four parts. The first is the finding of the equation; the second, the reduction of the equation once found; the third, the division of the other number of the equation by the number of the higher cossic character; the fourth and last, the extraction of some root from the quotient.

Of these, two are absolutely necessary: the finding of the equation, and the division. For in every problem to be solved by this rule, an equation—an equality between two numbers—must be found, and a division must be set up. The other two, namely the reduction of the equation and the extraction of a root from the quotient, are not always necessary. For not every equation found requires reduction, nor must a root always be extracted from the quotient, as will be clear from what follows. And what difference there is between this single rule of ours and the manifold rules of others (for many others prescribe several rules) will be made clear in chapter 13.

But before we treat all four parts of the rule one by one, we shall first explain the rule itself in a rough-and-ready way [*rudi Minerva*], in passing, by means of the following problem.

To find a number such that, when its half and its third part are taken from it, the remaining number is 7.

[Margin: First part of the rule of Algebra.]

I posit this unknown number that is sought to be $1R$ —so that $1R$ is understood to be equal to the hidden number we are looking for. For since every number, whether whole or fractional, or whole together with a fraction, can occupy some place in a geometric progression beginning from unity—so as to be a root of that progression—the whole artifice of this rule of Algebra consists in this: in finding what geometric progression the number sought constitutes; that is, in what progression the number that is sought occupies the place next to unity.

[Margin: Wherein the artifice of the rule of Algebra consists.]

Following the terms of the problem, then, I examine $1R$ —that is, I take its half and its third, namely $\frac{1}{2}R$ and $\frac{1}{3}R$, which parts together make $\frac{5}{6}R$. I subtract $\frac{5}{6}R$ from $1R$, and $\frac{1}{6}R$ remains. Now let me reason it through. Since $1R$ has been set equal to the whole hidden number that is sought, $\frac{1}{2}R$ will be equal to half the whole number, and $\frac{1}{3}R$ to a third part of it. And since, when the half and the third have been subtracted from the whole number, the remaining number is 7, it follows that $\frac{1}{2}R$ and $\frac{1}{3}R$ —that is, $\frac{5}{6}R$ —taken from $1R$, the remaining number, namely $\frac{1}{6}R$, is equal to the remaining number 7; for if equals are taken from

equals, the remainders are equal. An equation, then—an equality between $\frac{1}{6}R$ and this number 7—has been found; and so the first part of the rule is completed, which runs thus: [*For the unknown number in the problem, let $1R$ be posited, which is to be examined according to the terms of the problem until some equation is found.*]

[Margin: What an equation is.]

It is sufficiently clear from the example proposed how an equation is to be found. An equation, as it is here understood, is nothing other than a proportion of equality between two quantities—that is, between things variously denominated [i.e., of different cossic denomination]. Thus, in the given example, there is a proportion of equality between $\frac{1}{6}R$ and 7, which two are denominated differently. For when two "things" [*res*] having the same denomination are said to be equal, that equality (or equation) is either false—as when I say there is an equation between 5 and 7, or between $\frac{1}{6}R$ and $\frac{1}{3}R$ —or else it is a certain identity, wholly useless for the rule of Algebra, as when I say there is an equality (or equation) between $4R$ and $4R$, or between 7 and 7. It is necessary, therefore, that an equality pertaining to the rule of Algebra be between two things of different names [denominations].

[Margin: Second part of the rule of Algebra.]

Next in the rule comes: [*Let this be reduced, if reduction is needed.*] That is, let the equation once found be reduced, if the matter requires it. When and how the equation is to be reduced, we shall teach a little later. For now, let it be settled that our equation, between $\frac{1}{6}R$ and 7, does not need reduction.

[Margin: Third part of the rule of Algebra.]

And so the rule adds: [*Then let the remaining number of the equation be divided by the number of the higher cossic character.*] Since in every equation two numbers of unequal denomination are found, the rule of Algebra directs that the remaining number of the equation be divided by the number of the higher cossic character. It does not order the division to be made by the greater cossic number, but by the number of the higher cossic character. For the divisor in this rule, while it divides, casts off its own character—otherwise the quotient would always be unity. For since the number of the higher cossic character, when it is denominated by its character, is equal to the remaining number of the equation, it is plain that if the one is divided by the other in this way, unity would be produced, because any number [in the other number]...

is contained in a number equal to itself only once. Why a divisor of this kind discards its own cossic sign I shall explain a little further on, in chapter 11, where I shall set out more fully this division that the rule of Algebra prescribes. In the equation we found between $\frac{1}{6}R$ [$\frac{1}{6}$ res — one-sixth of the unknown] and $7N$ [7 numeri — 7 units], the greater cossic character is R , that is, it has the higher exponent, since the exponent of N is 0 and that of R is 1. For it is by the exponents that we recognize which cossic character is the greater and which the smaller. And so the number $7N$ is to be divided by $\frac{1}{6}$, the character R being set aside, and the quotient comes out as 42.

Finally, one further step is added to the rule. [Margin: The fourth part of the rule of Algebra.] (For either the quotient itself will be the number that was sought — namely the value of the root posited at the outset — or else some root of the quotient-number will make the sought number known, and so on.) And when it is asked which root of the quotient reveals the unknown number that is sought, this the greater cossic character — by whose number the division is set up — teaches most beautifully, as we shall show in its proper place. In our present example no root need be extracted; rather, the quotient itself yields the number that was sought. Therefore the number that the problem asks us to find is 42, and the proposed problem turns on a geometric progression beginning from 1, in which the number next to unity is 42 — namely this one:

- 1.
- 42.
- 1764.
- 8552.
31504. &c.

For it is in this progression alone that the last equation we found, between $\frac{1}{6}R$ and the number 7, proves true. Now if we take $\frac{1}{2}$ of this number 42 that we found — namely 21 — and $\frac{1}{3}$ — namely 14 — and subtract these parts from the number 42 itself, the number 7 is left, which is just what the proposed question required.

And in this way, once the number that was being sought — the number set equal to a single root — has been found, one can set up a check, or proof, according to the tenor of the problem or question proposed. But now that the problem has been laid out and the rule explained in a rough-and-ready way (with "unschooled wit," as we said [*pingui Minerva*]), the individual parts of the rule must next be set forth one by one, more exactly and more clearly, as we promised a little earlier we would do.

ON THE VARIETY OF EQUATIONS. CHAPTER IX.

[DE AEQVATIONVM VARIETATE. CAPVT IX.]

It is clear enough from what has been said that, starting from the positing of $1R$, if it is examined according to the wording of the example, an equation between two numbers is to be found. But now more must be said about this equation, so that we may understand in how many ways one and the same equation can be varied. For once some equation has been found, very many others can be formed from it. Thus in the example above, an equation was found between $\frac{1}{6}R$ and 7. And since $1R$ was set equal to the whole number to be found, it follows that $\frac{1}{2}$ and $\frac{1}{3}$ — that is, $\frac{5}{6}R$ and $\frac{1}{6}R$, that is, 7 — are equal to a single root, so that the equation is between $\frac{5}{6}R + 7$ and $1R$, in accordance with the common maxim: *every whole is equal to all its parts taken together*. Likewise, if $\frac{1}{6}R$ is subtracted from $1R$, and 7 from the whole number 42, an equation will be found between $\frac{5}{6}R$ and 35, and so on.

But in order to understand the various transformations of equations more precisely, we must realize that every calculation carried out under this rule of Algebra proceeds by means of lines, surfaces, and solids, which advance according to some geometric proportion beginning from unity, as we said above. The various equations, then, can aptly be understood from the following geometric figure.

Diagram. An uncaptioned rectangular array of small open circles — 6 rows by 12 columns, 72 circles in all — illustrating the "rectangular figure" (*figura quadrangularis*) of the text. Each of the 6 rows represents one "line of length" containing 12 parts; each of the 12 columns represents one "line of breadth" containing 6 parts. Hence the whole figure of 72 parts is filled either by $6R$ of length ($6 \times 12 = 72$) or by $12R$ of breadth ($12 \times 6 = 72$). The circles carry no labels.

For $1R$ is a certain line of parts under some fixed number, or a certain series of any things whatever under some determinate number. Take, for instance, this rectangular figure: there are six lines of length, each of which has 12 parts; and likewise twelve lines of breadth, in each of which are found 6 parts. Now you see first that $6R$ of length are equal to the number 72, because the six lines — that is, $6R$ — of length fill out the whole figure, which contains 72 parts. Next you see that $12R$ of breadth are equal to the same number 72; for the whole figure, having 72 parts, is made up of $12R$ of breadth, that is, of 12 lines of breadth, as is plain. But let us take, as our example, the $12R$ of breadth, so that the equation is between $12R$ and 72.

And since, if equals are taken from equals, the remainders are equal: if we take $4R$ from each side of the equation, there will still remain an equation between $8R$ and $72 - 4R$. For $8R$ of breadth in the figure above contain 48 parts, while $4R$ make 24; and when 24 is taken from 72, 48 are left.

Likewise, since if equals are added to equals the results are equal: if to each side of this last equation, between $8R$ and $72 - 4R$, we add 10, we shall find an equation between $8R + 10$ and $82 - 4R$. For $8R + 10$ make 58, and $82 - 4R$ come to just as much, as is evident.

And if again we take $4R$ from each side of this equation between $8R + 10$ and $82 - 4R$, there will remain an equation between $4R + 10$ and $82 - 8R$; for, by the figure above, each side comes to 34. So too, if from each side of this last equation we subtract 10, an equation will be formed between $4R$ and $72 - 8R$, for each side makes 24. And by this device, from one equation once found, six hundred others can be formed, without any change in the value, or worth, of a single root.
[Margin: Various permutations of equations.]

The gist of this variation lies in the following precept: *Let the same common quantity be added to both terms of an equation, or let the same common quantity be subtracted from both; or else let both terms be multiplied, or divided, by the same number. For in this way the equation will remain unharmed, though its terms are changed.* [Margin: The precept for varying any equation whatever.]

In this addition, subtraction, multiplication, and division, one must recall those lessons we handed down about the signs $+$ and $-$. The variation of the equations above was carried out by adding and subtracting the same number to and from both terms of the equation; now we shall add another example, of multiplication and division.

an example of multiplication and division. Suppose, then, that from the same figure above the equation between $3R + 12$ and $72 - 7R$ has been found. If we multiply each term by 2, we shall produce an equation between $6R + 24$ and $144 - 14R$, for each term makes 60. Likewise, if we divide each term of this latter equation by 6, there will come out an equation between $1R + 4$ and $24 - 2\frac{1}{3}R$, as is clear.

ON THE REDUCTION OF AN EQUATION. CHAPTER X.

[DE REDVCTIONE AEQVATIONIS. CAPVT X.]

When, in solving some question, one has arrived at an equation such that the remaining number of the equation cannot be divided by the number of the greater cossic character, as the rule of Algebra prescribes, then the equation once found must be reduced before the division is set up. Now the division cannot be performed when the greater cossic character — by whose number the division ought to be made — either is not set alone on one side of the equation, or, even if it is set alone, the same character is nevertheless found on the other side as well. This will become clear from the following examples.

Suppose the equation between $9R - 12$ and 42 has been found. Here you see that the division by 9 — that is, by the number of the greater cossic character R — cannot be done, because it is not $9R$ alone that make up one side of the equation, but $9R - 12$. [Margin: Various equations to be reduced.] Similarly, if the equation is between $9R$ and $72 - 3R$, the division by 9 , the number of this character R , cannot be carried out, because, although this number $9R$ alone occupies one side of the equation, the same character R is nevertheless found on the other side too. Likewise, given the equation between $9R + 12$ and $78 - 2R$, no division can be done, because in both sides of the equation there occur both this character R and this N . Form the same judgment about other equations of this kind: such as that between 13 [1 census — the square, x^2] — $3R$ and 108 ; again, between $13 - 48$ and $8R$; again, between $108 + 8R$ and $23 - 12R + 60$; and so on.

All these equations, in which the value of the root is 12 , and others like them, must be reduced to others in which the greater cossic character is set alone on one side of the equation and is not repeated again on the other side, and in which no cossic character is placed twice. [Margin: Equations must be reduced in such a way that the greater character is placed by itself on one side, etc.] This reduction is done by that variation, or rearrangement, of the equation's terms which we described in the preceding chapter.

For example, the equation between $9R - 12$ and 42 is reduced, by restoring this deficit — 12 [*restauratio* — restoration, al-jabr] — that is, by adding this number 12 to both sides — to the equation between $9R$ and 54 . So also the equation between $9R$ and $72 - 3R$ is brought back, by restoring this deficit — $3R$ (that is, by adding $3R$ to both sides), to the equation between $12R$ and 72 . [Margin: Reductions of various equations.] By the same reasoning, the equation between $9R + 12$ and $78 - 2R$ is, first, by restoring this deficit — $2R$ (that is, by adding $2R$ to both sides), reduced to the equation between $11R + 12$ and 78 ; and, secondly, by transposing this added term $+ 12$ (that is, by subtracting 12 from both sides), reduced to the equation between $11R$ and 66 .

Again, the equation between $13 - 3R$ and 108 is reduced, by restoring this deficit $- 3R$, to the equation between 13 and $3R + 108$, for $3R$ have been added to each side. So too the equation between $13 - 48$ and $8R$ is reduced, by restoring this deficit $- 48$ (that is, by adding 48 to both sides), to that between 13 and $8R + 48$. Finally, the equation between $108 + 8R$ and $23 - 12R + 60$ is reduced, first, by restoring the $- 12R$, to the equation between $108 + 20R$ and $23 + 60$; and, secondly, this is reduced, by transposing the $+ 60$ (that is, by removing 60 from both sides), to the equation between 23 and $20R + 48$.

And so every reduction of equations, to be carried out more expeditiously, ought to begin from the restoration of the deficient term, if there is one — that is, the number that bears this sign $-$ must be added to both sides of the equation. Next, the number bearing the sign $+$ must be transposed from one side to the other, that is, it must be subtracted from both sides. This restoration (or addition) and transposition (or subtraction), although correctly performed by the rule of addition and subtraction, as is plain from what has been said — the whole art of reduction is nonetheless contained in these two precepts, the first of which is:

Whatever is transposed changes its sign. [Margin: The first precept of reduction.]

That is, a term of the equation that has the sign $-$, when transposed to the other side, takes on the sign $+$ — that is, it is added to the other side. But a term bearing the sign $+$, when transposed, changes its $+$ into $-$, that is, it is subtracted. This has indeed been observed in all the reductions set out above. But if the term being transposed has a like denomination on the side to which it is to be transposed, then the second precept must be applied:

Like signs subtract, but unlike signs add. [Margin: The second precept of reduction.]

That is, if the term of the equation to be transposed, bearing either of these two signs $+$ or $-$, has on the other side of the equation a greater number of the same denomination with the same sign, then the number of that term must be subtracted from this number, and the sign is to be kept that belongs to the number from which the subtraction is made; hence the transposition must begin from the smaller number. But if the term to be transferred has on the other side a number of the same denomination with the opposite sign, then the number of that term must be added to this number, and the sign is to be kept that belongs to the number to which the addition is made. And so that the use of both rules may be better understood, we shall reduce one or two more equations.

Let the equation be between $6R - 10$ and $10R - 34$. Now since the numbers 10 and 34 have the same sign $-$, therefore, in order to transfer the smaller, 10, the number 10 must be taken away from 34; and so there will remain the equation between $6R$ and $10R - 24$. Again, because the numbers $10R$ and $6R$ have the same sign $+$, the smaller, $6R$, is taken from the larger, $10R$, so that the equation between $0R$ and $4R - 24$ remains. Finally, by transposing the $- 24$, the equation will be between 24 and $4R$.

Let there again be the equation between $54 + 4R$ and $13 - 6R + 30$. First, because $+ 4R$ and $- 6R$ have unlike signs, the $4R$ are added to the $6R$, so that they become $- 10R$; and the equation is between 54 and $13 - 10R + 30$. Next, because 54 and $+ 30$ have the same sign $+$, 30 is subtracted from 54, so that the equation between 24 and $13 - 10R$ remains. Third, I transpose the $- 10R$, so that the equation is between 13 and $10R + 24$. And so on for the rest.

And so it is gathered, from what we have said, that the whole business of reduction lies in the transposition of terms, and that this transposition depends on the lessons just stated. For when some equation is presented in which the number of the greater cossic character does not by itself make up one side of the equation, but has joined to it another cossic number of lower denomination, this latter must be transferred to the other side — by the first precept, if on the side to which it is transferred no number of the same denomination is found; but if the other side of the equation, to which this transposition is to be made, has a number of the same denomination, then this transfer must be carried out by the other lesson. All of which you see observed in the examples above. Finally, the equation must be continued — by adding, subtracting, and transposing — until the number of the greater character is seen standing alone on one side. [Margin: The sum of reduction.]

But I want the student to be warned at this point that these two precepts were devised only as a shortcut for reduction. For anyone willing to be content with the rule of addition and subtraction can do without them with no inconvenience at all. Indeed I would advise some people to do so, lest they confuse these precepts with the precepts of addition and subtraction — something that often happens, especially to beginners in this art. [Margin: Reduction is safer without the two aforesaid precepts.]

But let us now treat the reduction of those equations that are found among fractions [*minutiae*]. Every equation of this kind is reduced to an equation of whole numbers by cross-multiplication. Suppose the equation is found between these fractions:

$$(3R + 12)/5 \times (363 - 198R)/(3R)$$

Let them be cross-multiplied — namely, the numerator of the first fraction multiplied into the denominator of the second, and the denominator of the first into the numerator of the second — and there will result an equation in the characters, between $93 + 36R$ and $1803 - 990R$. This, by abbreviation of the characters and by abbreviation of the numbers, becomes the equation between $1R + 4$ and $20R - 110$. And if the reduction is then made by transposition, as we lately taught, this same equation will become that between $19R$ and 114 .

Thus the equation between the fractions $(4R + 18)/(1R)$ and $(12R - 58)/2$ is reduced, by cross-multiplication, to this one: between $8R + 36$ and $123 - 58R$. And if this is reduced by transposition, the same equation will become that between $66R + 36$ and 123 .

But if the equation is found between this fraction, $628/(186 + 1R)$, and any other thing whatever — such as a pound, an ounce, an hour, a degree, a minute, and so on — then this thing must be taken as some whole, and for it a unit must be set up, under which another unit is placed, so that a fraction is formed. Hence in the given example the equation will be between $628/(186 + 1R)$ and $1/1$, which by cross-multiplication will be reduced to the equation between 628 and $186 + 1R$.

The demonstration of the reduction of equations among fractions is very easy. For since two fractions are set equal, the ratio of the numerator of the first fraction to its denominator will be the same as the ratio of the numerator of the second fraction to its denominator. Therefore, if the fractions are cross-multiplied — that is, the numerator of the first fraction is multiplied into the denominator of the second, and the denominator of the first into the numerator of the second, namely the first number into the fourth and the second into the third — the products will be equal numbers; and thus an equation will be found between those products. Which was to be demonstrated. [Margin: Demonstration of the reduction of equations among fractions.]

In what way an equation between an irrational cossic number and an absolute number is to be reduced — for instance, between $\sqrt{3} 24R$ [?] and 12 , or between $\sqrt{3} 103$ [?] and 20 — we shall show at the end of chapter 26, where we shall deal with numbers of this kind. [Margin: Reduction of equations found among fractions, and among irrational cossic numbers and absolute numbers.]

If, finally, an equation should turn up in which there is no absolute number, the cossic characters must be abbreviated, as we taught in chapter 7. For example, the equation between 23 and $12R$ must be reduced to the equation between $2R$

and 12. [Margin: The characters must be abbreviated so that an absolute number is obtained on one side of the equation.]

Similarly:

- The equation between 13℥ [1 census-of-cube, x^6] and 1ß [1 surdesolidum, x^5] + 3515633 [census-of-census, x^4] must be reduced to the equation between 13 and 1R + 35156.
- Again, the equation between 1ß and 133 + 35156℥ [cubus, x^3] must be reduced to the equation between 13 and 1R + 35156.
- Again, the equation between 133 and 1℥ + 351563 must be reduced to the equation between 13 and 1R + 35156.
- Again, the equation between 1℥ and 13 + 35156R must be reduced to the equation between 13 and 1R + 35156.

And so on for others of this kind.

But this abbreviation is not absolutely necessary, as we shall say in chapter 12, before the extraction of the roots of simple cossic numbers.

Finally [printed "POSREMO"; recte "POSTREMO"], if, once the equation has been reduced so that the greater cossic character is set alone on one side of the equation — a character greater than R, such as 3, or ℥, or 33, and so on — then all the numbers of the equation must be divided by the number of that greater cossic character, if it is not unity, so that the single unit denominated by that cossic character is equal to the other side of the equation. For example, if the equation is found between 43 and 8R + 96, the individual numbers must be divided by 4, so that the equation is reduced to that between 13 and 2R + 24. Again, the equation between 123 and 66R + 36 must be reduced, by division, to the equation between 13 and 5½R + 3, and so on. [Margin: The equation must be reduced by division, so that the number of the greater character, standing alone, is unity.]

The reasoning behind this reduction is not obscure. For since the individual numbers are divided by the same number — namely by the number of the greater cossic character — the quotients will have the same ratio as the numbers that were divided. Therefore, just as an equation held among the divided numbers, so too an equation will necessarily be found among the quotients.

For the rest, this reduction will come about if, according to the tenor of the rule of Algebra, the division is set up by the number of the greater character — the character itself being discarded, while the characters of the numbers that are divided are nevertheless retained — as will be said in the following chapter. So

much so, that this reduction of which we have just spoken may seem superfluous, since the division prescribed in the rule of Algebra already provides it. Yet because other authors teach that this reduction is necessary, I have chosen to explain it here as well, even though, as I have said, the division carried out according to the law of the rule of Algebra offers it too, and most conveniently.

ON THE DIVISION THAT THE RULE OF ALGEBRA PRESCRIBES

Chapter XI

[Margin: A fuller explanation of the division that the rule of Algebra prescribes.]

Once the equation has been reduced — where the matter calls for it, as we taught in the preceding chapter — the rule of Algebra directs that the leading cossic character be discarded and the whole remaining number of the equation be divided by the number of that character, together with its cossic characters. This division is really a shortcut for that golden rule of proportions which is called the Rule of Three. For suppose, by way of example, that $7R$ [res, the unknown] are equal to the number 42, and we want to find what a single R is equal to. Following the doctrine of the Rule of Three, we multiply 42 and $1R$ together, which yields $42R$; and dividing this by $7R$ we obtain 6 as the value of one root. For whenever a cossic number is divided by a cossic number of the same character, an absolute (pure) number is produced — as is clear from what we wrote in Chapter 5 — because the exponents are equal, and so, when one is subtracted from the other, 0 remains, which is the exponent of an absolute number. This is why the rule of Algebra plainly teaches that the other number of the equation is to be divided by the number of the greater cossic character. By this procedure the same number is obtained that would be produced by the rule of proportions, as the example before us shows: the same number results from dividing $42R$ by $7R$ as from dividing 42 by 7.

SIMILARLY, suppose $12\mathfrak{z}$ [census, the square, x^2] are equal to $66R + 36$, and we want to find what a single $\mathfrak{z}$ is equal to. We multiply $66R + 36$ and $1\mathfrak{z}$ together, producing $66\mathfrak{c}$ [cubus, the cube, x^3] + $36\mathfrak{z}$; and dividing these by $12\mathfrak{z}$, as the Rule of Three requires, we obtain $5\frac{1}{2}R + 3$ as the value of one census, or square. The same result is produced if, having first discarded the character $\mathfrak{z}$, we divide $66R + 36$ by 12 — as is evident. This is why the rule of Algebra directs that the remaining number of the equation simply be divided by the greater cossic character, so that no longer operation by the Rule of Three need be set up. In the same way, if $\frac{1}{3}\mathfrak{z}$ and $6R + 13\frac{1}{3}$ are equal, and it is asked what the value of one

census, or square, is: this is done, by the law of the rule of Algebra, by dividing each number by $\frac{1}{3}$, the number of the greatest character. By this means we find the equation between 13 and $18R + 40$ — that is, the value of 13 will be $18R + 40$.

Again, suppose an equation is found between $33\mathfrak{C}$ [zensicubus, x^6] and $600\mathfrak{C} + 10368$, and we must seek out the value of one zensicube. This is done by the Rule of Three, if we put it thus: if $33\mathfrak{C}$ give $600\mathfrak{C} + 10368$, what will $13\mathfrak{C}$ give? For by multiplying the two latter numbers together we obtain the number $600\mathfrak{C}\mathfrak{C}$ [cube of the cube, x^9] + $103683\mathfrak{C}$; and dividing this by the first number — namely by $33\mathfrak{C}$ — the quotient becomes $200\mathfrak{C} + 3456$, and the equation stands between $13\mathfrak{C}$ and $200\mathfrak{C} + 3456$. This is found more easily if the individual numbers of the equation between $33\mathfrak{C}$ and $600\mathfrak{C} + 10368$ are divided by 3 — that is, by the number of the greater character — as we wrote toward the end of the preceding chapter. For then there is no need to invoke the Rule of Three; it is enough that the individual numbers of the other part of the equation, together with their characters, be divided by the number of the greater character (the character itself being discarded), just as the rule of Algebra directs.

But if the divisor is a unit denominated by a greater cossic character, no division is needed, since a dividing unit produces nothing new. For instance, if 13 is equal to $3R + 18$, and so on. Hence it is worthwhile to reduce the equation so that the number of the greater character becomes a unit, as we taught at the end of the preceding chapter.

And that, when the unit is denominated by the greater character, no division is needed, is plainly clear from the Rule of Three. For in this equation between 13 and $3R + 18$, if we say: when 13 gives $3R + 18$, what will 13 give? — it will be an identity and a triviality. For from multiplying $3R + 18$ by 13 we get the number $3\mathfrak{C} + 183$, which, divided by 13 , gives back $3R + 18$, as is clear from the law of multiplication and of division.

ON THE EXTRACTION OF ROOTS, WHICH THE RULE OF ALGEBRA MENTIONS

Chapter XII

[Margin: When the quotient indicates the value of the root.]

It must be taught, in the last place, when the quotient of the division reveals the unknown number sought, and when some root of the quotient discloses that same hidden number. When, therefore, after the cossic characters have been reduced, the greater cossic character is $1R$, then the quotient at once makes plain

the number sought. For instance, if $5R$ are equal to 30, then, dividing 30 by 5, the quotient gives 6, the value of one root — as was said at the beginning of the preceding chapter. Indeed, whenever a cossic character of a greater denomination is set equal to a cossic number of the next-lower denomination, then, dividing the number of the lower denomination by the number of the higher denomination, the quotient gives the value of one root, even if no reduction of the characters has been carried out; for such an equation, by reducing the characters, is brought back to an equation between $2R$ [read: R and N] and N . Thus, if 5β [surdesolidum, x^5] are set equal to $30\beta\beta$ [zensizensus, x^4], then, dividing 30 by 5, the quotient becomes 6, the value of one root. For this equation is reduced to an equation between $5R$ and 30 if the characters are reduced. This too is clear from the Rule of Three. For if we say: if 5β give $30\beta\beta$, what will $1R$ give? — multiplying $30\beta\beta$ by $1R$ we get 30β , and dividing these by 5β the quotient becomes 6, an absolute number. The same reasoning holds for the other collateral cossic numbers. The ground of this is that in every geometric progression beginning from 1, the ratio of any number to the one immediately preceding it is always the same as the ratio of the second term to the first — that is, the ratio of the root to unity.

[Margin: When, and what, root is to be extracted, when one of the numbers of the equation is an absolute number.]

But if the greatest cossic character of the equation — which alone is set on one side of the equation — is greater than $2R$ [i.e. than R], and an absolute number stands on the other side, then a root must be extracted from the quotient — the very root that the character signifies. Thus, if the character is β , the zensic (square) root must be extracted; if $\mathcal{C}$, the cubic root; if $\beta\beta$, the zensizensic root; if β , the surdesolid root; and so on. For in equations of this sort the value of one census, or of one cube, or of one zensizensus, and so on, is found by division. For example, if the equation is between 2β and $144 - 12R$, then (division having first been carried out, so that 1β is equal to $72 - 6R$) the square root must be extracted from $72 - 6R$. Similarly, if the equation is between 1β and $6R + 72$, or $72 + 6R$, the square root must be extracted from $6R + 72$, or from $72 + 6R$. So too, if the equation is found between 5β and 720, then, division having been carried out, the square root must be sought from this quotient, 144. Again, if $10\mathcal{C}$ are equal to 466560, then, division having been carried out, the cube root of this quotient, 46656, must be found; and so with the rest.

[Margin: What root is to be extracted when two non-collateral cossic numbers are set equal to each other, neither of which is an absolute number.]

But that it may be known in general which root is to be extracted from the quotient when two non-collateral cossic numbers are set equal to each other, neither of which is an absolute number, the characters must be reduced so that the equation is made between N and a cossic number. Thus, if the equation is between 103℥ and 466560℥, it will be reduced to an equation between 10℥ and 466560. Once the division has been carried out, then, the cube root must be extracted from the quotient 46656; and so with the rest. The reason is that by the division the value of one cube — or of whatever other cossic number to which the reduction by characters was made — is found. For if you say: if 10℥ give 466560, what will 1℥ give? — you will produce, by multiplying the second number into the third, 466560℥; and dividing these by 10℥, the quotient becomes 46656, the value of one cube, and so on.

[Margin: What kind of root is to be extracted, even if the characters are not reduced.]

Finally, when two non-collateral cossic numbers are set equal to each other, in order to recognize which root is to be drawn out from the quotient — even if the characters are not reduced — observe how many denominations lie interposed, in the geometric progression, between the two proposed cossic numbers. For if only one lies between, the square root must be extracted; if two, the cubic; if three, the zensizensic; and so on.

Moreover, as to the art by which roots are extracted from absolute numbers, we have set that out at sufficient length in the Practical Geometry; but now it must be explained in what way they are to be drawn out from cossic numbers.

[Margin: Extraction of roots from simple cossic numbers.]

If, then, some root is to be extracted from a simple cossic number, take the appropriate root of that number (setting the character aside, as if the number were absolute). Then divide the exponent of the character of that same number by the exponent of the character from which the root takes its name, and you will find the exponent of the character by which the sought root will be named. For example, let the square root of the number 1443 be sought. Taking the square root of the number 144 itself, which is 12, divide the exponent of this character 3 — namely 2 — by the exponent of the character from which the square root, which is sought, is named — namely 2: for 1 will result, as the exponent of the character R. Therefore 12R is the square root of this number 1443. For if this number 12R is multiplied into itself, the proposed number 1443 is produced.

Again, let the square root be drawn out from $144\mathfrak{C}$. Having taken the square root of the number, namely 12, divide the exponent of this character $\mathfrak{C}$ — namely 6 — by the exponent of the square root, namely by 2, and the exponent 3 will be found, whose character is $\mathfrak{C}$. Therefore $12\mathfrak{C}$ is the square root of this number $144\mathfrak{C}$.

So too the cube root of this number $64\mathfrak{C}$ will be $4\mathfrak{R}$. For the cube root of this number 64 is 4; and the exponent of this character $\mathfrak{C}$, namely 3, divided by 3 — the exponent, that is, of the character from which the cube root takes its name — produces unity, namely the exponent of the character $\mathfrak{R}$.

Likewise the square root of this number $25\mathfrak{J}$ will be $5\mathfrak{J}$.

And the zensizensic (fourth) root of this number $16\mathfrak{J}\mathfrak{J}$ [zensizensizensus, x^8] will be $2\mathfrak{J}$.

And the surdesolid (fifth) root of this number $32\mathfrak{J}\mathfrak{B}$ [zensus-surdesolidum, x^{10}] will be $2\mathfrak{J}$.

And the zensizensic (fourth) root of this number $81\mathfrak{J}\mathfrak{J}$ will be $3\mathfrak{R}$, and so on.

But if either the number does not have the root being sought, or the division of the exponents does not yield a whole-number exponent, then the proposed cossic number will lack that root which is desired. For instance, the number $25\mathfrak{C}$ has neither a square nor a cube root; for although the number 25 does have a square root, namely 5, yet from dividing 3 — the exponent, that is, of this character $\mathfrak{C}$ — by 2, the exponent of the square root, there results the number $1\frac{1}{2}$, to which no character corresponds. And again, although from dividing 3, the exponent of this character $\mathfrak{C}$, by 3, the exponent of the cube root, there results 1, the exponent of the character $\mathfrak{R}$, nevertheless the number 25 lacks a cube root, and so on.

[Margin: Which composite numbers have roots.]

As regards the extraction of roots from composite and diminished cossic numbers, it must be understood that roots can be drawn from such numbers by a sure method and art only when the three cossic numbers of the equation have exponents that keep the same common difference among themselves — that is, exponents that are in arithmetic proportion. Such are the following equations:

Equation	Exponents are
$1\mathfrak{J} = 6\mathfrak{R} + 71$	2. 1. 0.
$1\mathfrak{J} = 72 - 6\mathfrak{R}$	2. 0. 1.

Equation	Exponents are
$13 = 14R - 48$	2. 1. 0.
$133 = 183 + 648$	4. 2. 0.
$133 = 725 - 43$	4. 0. 2.
$133 = 4333 - 41616$	4. 2. 0.
$13C = 200C + 3456$	6. 3. 0.
$13C = 5120 - 16C$	6. 0. 3.
$13C = 800C - 156751$	6. 3. 0.
$1333 = 200033 + 185076881$	8. 4. 0.
$1333 = 214651701 - 2033$	8. 0. 4.
$1333 = 200033 - 7846119$ [?]	8. 4. 0.
$13B = 80B + 39609$	10. 5. 0.
$13B = 7424 - 200B$	10. 0. 5.
$13B = 2000B - 999424$	10. 5. 0. &c.

When all the exponents that keep the arithmetic progression are greater than 0, they are to be reduced by subtracting the smallest exponent-number. As in the following equations:

Equation	Exponents are
$1CB = 725B - 4CC$	11. 7. 9.
$1CB = 200333 + 3456B$	11. 8. 5.
$13B = 2003C + 143363$	10. 6. 2.

(Here CB = tersurdesolidum, x^{11} ; B = bissurdesolidum, x^7 ; CC = cube of the cube, x^9 .)

— they are reduced to these:

Equation	Exponents are
$133 = 725 - 43$	4. 0. 2.
$13C = 200C + 3456$	6. 3. 0.
$1333 = 200033 + 14336$ [recte: $20033 + 14336$]	8. 4. 0.

And the same is to be said of infinitely many others. For in this way it will easily be recognized which root is to be drawn out.

[Margin: Which composite cossic numbers do not have roots.]

But when the exponents do not keep the arithmetic proportionality — that is, do not have the same common difference — as, for example, if the equation were between $1\mathfrak{C}$ and $4\mathfrak{J} + 16$, where the exponents are 3.2.0; or between $1\mathfrak{C}$ and $10\mathfrak{R} + 24$ — no art has yet been found by which roots of this kind can be drawn out with certainty; although Cardano [Girolamo Cardano] and Niccolò Tartaglia [Niccolò Tartaglia], in certain particular examples, have found the value of one root. Rafael Bombelli [Rafael Bombelli], moreover, thinks that from certain equations of this sort, and some others, he has discovered how such roots are to be extracted; and Franciscus Vieta [François Viète] too is said to have demonstrated a general rule for extracting roots of this kind. But since it has not yet been possible to see that rule, and since Bombelli's reasonings are very obscure — and since equations of this sort, in which more than two cossic numbers are set equal to one cossic number (such as, for example, the equation between $1\mathfrak{C} + 3\mathfrak{J} + 7\mathfrak{R}$ and 34), exist in nearly infinite variety, come rarely into use, and, on his own testimony, admit not a few exceptions — we shall be content, in this Algebra of ours, with those matters that have an easy, sure, and well-established science. That is, we shall explain only the extractions of roots from cossic numbers of the former kind — namely, when one cossic number is set equal to two others, and the exponents keep the arithmetic proportion — since these are quite enough for solving the almost innumerable riddles of numbers.

And first, indeed, we shall treat of the extraction of roots from those numbers whose exponents are 2.0.1, which are all square (quadratic). Then we shall speak of finding the roots of those numbers whose exponents are 4.2.0, or 6.3.0, or 8.4.0, or 10.5.0, and so on; these roots are either zensizensic, or zensicubic, or zensurdesolid, and so forth, as the greatest characters of the equations indicate. For all these equations have a well-established and most certain science by which their roots may be found.

By this art, then, square roots are drawn from composite or diminished cossic numbers, according to Michael Stifel [Michael Stifel], the noble arithmetician.

[Margin: Extraction of the square root from a composite or diminished cossic number.]

1. Take half the number of roots.

2. To the square of this half add — or from it subtract — the absolute number, according as it is marked with the sign + or -.

3. To the square root of this sum, or of what remains, add — or from it subtract — the half of the number of roots, according as the number of roots was marked with the sign + or -. For this final sum, or remainder, will give the value and worth of one square root.

Let an equation be found between 13 and $72 - 6R$. Division having been carried out — dividing $72 - 6R$ by 1 , as the rule of Algebra directs — the same number results, namely the value of one census, whose root we shall find as follows.

First, I take half the number of roots, namely 3 .

Then, to the square of this half — that is, to 9 — I add the absolute number 72 , because it carries the sign $+$, and I make 81 .

Third, from the square root of this sum 81 — namely from 9 — I subtract the half of the number of roots, say 3 . For the number of roots is marked with the sign $-$; and 6 is left, as the value of the root sought. This is proved as follows. Since $1R$ is 6 , then $6R$ will be 36 ; and taking these away from 72 , there remain 36 , to which the square number of this root 6 is equal.

Let there be found, again, an equation between 13 and $6R + 72$. The square root of this number $6R + 72$ must therefore be found.

First, I take half the number of roots, say 3 .

Then, to the square of this half — that is, to 9 — I add 72 , because of the sign $+$ that the absolute number carries, and I make 81 .

Third, to the square root of this sum 81 — namely to 9 — I join the half of the number of roots, namely 3 (for the number of roots is marked with the sign $+$), and I make 12 , as the value of the root. This is confirmed as follows. Since $1R$ is 12 , then $6R$ will be 72 ; and adding these to 72 , we get 144 , to which the square number of the root 12 is equal.

(Continuation of Chapter XII — the extraction of cossic roots, i.e. the solution of quadratics by completing the square.)

Note on notation: throughout, 3 renders the cossist census sign (the square, x^2), R the cossist res / radix sign (the root, x), and N the sign for the absolute numerus. A coefficient printed before a sign multiplies it (e.g. $18R$ = eighteen roots, $18x$). Equations are stated in Clavius's idiom "an equation between A and B," meaning $A = B$.

Finally, suppose we have found an equation between 13 [one census — the square, x^2] and 18R - 72 [eighteen res minus 72] — that is, $x^2 = 18x - 72$. We must therefore seek the root of this square 18R - 72.

First I take half the number of roots, namely 9.

Next, from the square of this half — that is, from 81 — I subtract 72, because of the minus sign set before the absolute number; and 9 remains.

Third, to the square root of this remainder 9 — that is, to 3 — I add half the number of roots, namely 9 (for the number of roots bears the plus sign), and so I get 12 for the value of the root. That this is correct is clear as follows: 18R make 216 if 1R is 12; and when 72 is taken from 216, there remain 144, to which the square of the root found, 12, is equal.

But here the reader must be warned that cossic numbers of this kind — *diminished* numbers, in which the absolute number bears the minus sign — have a double root satisfying the equation, namely a greater and a lesser. The greater is found as just described. The lesser will be had if the square root of that remainder is subtracted from half the number of roots. Thus in the example just given: if 3, the square root of the remainder 9, is subtracted from 9 (half the number of roots), there remain 6, which is the other and lesser root of this cossic number 18R - 72. That this holds is clear: 18R make 108 if 1R is 6; and when 72 is taken away, there remain 36, to which the square of this root, 6, is equal.

[Margin: (right) On how cossic numbers may have a double root; and in what way both the greater and the lesser root are found.]

Accordingly, with numbers of this kind one is free, in the third step, either to add half the number of roots to the square root found from that remainder, or to subtract that same root found from half the number of roots. Yet the two roots will not always both solve the proposed problem; often only one will, as will be made clear in the riddle [problem] of Chapter 25, section 32.

Only that diminished cossic number whose absolute number bears the minus sign, and whose absolute number is equal to the square of half the number of roots, has a single root; and the root of such a cossic number is precisely half the number of roots. For example, if the equation is between 13 and 12R - 36 — that is, $x^2 = 12x - 36$ — the root of this diminished number 12R - 36 will be half the number of roots, namely 6, because the absolute number 36 is equal to the square of half the number of roots. For take half the number of roots (namely 6); and from the square of this half — from 36 — subtract 36; this done, 0 remains, whose

square root is likewise 0. Hence, whether half the number of roots (say 6) is added to this root, or the said root is taken from that half, the root will always be found to be 6, which satisfies the equation, as is clear.

From all this it is evident that the cossic number whose root is sought is of three kinds. Either it is *composite*, where both the number of roots and the absolute number bear the plus sign; or *diminished*, where either the number of roots or the absolute number is marked with the minus sign — as can be plainly established from the three examples adduced.

Now let us lay open the reasoning of this extraction of roots by geometric demonstrations, so that we may know, not by experience alone but by the most certain science, that by this art square roots are rightly drawn out of cossic numbers, whether composite or diminished. Let us therefore repeat the first example, in which the equation was between 3 and $N - R$ [census equal to the absolute number less the roots]; or, if this diminished term $- R$ be restored [by transposition], between $3 + R$ and N [that is, $x^2 + x = N$].

Let AB be the side of the census, that is, of the square unknown to us, which is sought. When it has been produced toward A , let the straight line AC , equal to the number of roots, be taken; and let it be bisected at D , so that AD is half the number of roots. Upon the straight line BD — composed of the side of the unknown square and of half the number of roots — let the square DF be described, whose diagonal is BE . Then, having drawn AG parallel to DE , cutting the diagonal at H , let the straight line IL be drawn through H parallel to BC , cutting DE at K and meeting at L the line drawn from C parallel to DE .

Since AI and KG are squares (by the corollary to Euclid, *Elements* II.4), the rectangle CH , contained under AC (the number of roots) and AH (the side of the sought square), will be the value, or price, of the roots. (For the number of roots multiplied into the value of a root produces the value of all the roots. Thus, one root being 6, $5R$ will be worth 30.) And so DH is half the price of the roots, since [Margin: (a) Euclid I.36] the rectangles DH and DL are equal. [Margin: (b) Euclid I.43] Since therefore the rectangles DH and HF are also equal, DH and HF together will equal the whole price of the roots. And since the square AI is set equal to some absolute number less the value of the roots — or rather, since the square AI together with the value of the roots (whose number is AC) is set equal to some absolute number — the gnomon GBK will be equal to that absolute number. If this gnomon is added to GK , the square of half the number of roots, the whole square DF will be known; and if from its side DB there be taken AD , half the number of

roots — as the rule of root-extraction prescribes — the side AB will become known. Which was what was proposed.

Diagram. A square with a diagonal and an internal grid, illustrating the first (completing-the-square) demonstration. The diagonal runs from E (top-left) down to B (bottom-right).

- E (top-left corner): vertex
- G (top edge, middle): division of the top edge
- F (top-right corner): vertex
- L (left side, middle): endpoint of the horizontal line IL
- H (interior, upper): point on line AG near the top
- H (interior, centre): intersection of the diagonal with the grid lines
- I (right side, middle): point on the horizontal IL
- C (bottom-left): endpoint of AC produced
- D (bottom edge, left-of-centre): bisection point of AC
- A (bottom edge, centre): base vertex
- B (bottom-right corner): base vertex, foot of the diagonal

[Margin: Demonstration of the extraction of roots from composite or diminished cossic numbers.]

But let us now take the second example, where the equation was between 3 and $\mathbb{R} + N$ [that is, $x^2 = x + N$].

Let AB again be the side of the square that is sought. And since 13 is set equal to $\mathbb{R} + N$, the number of roots will be less than the side AB. Let the straight line AC then be equal to the number of roots, whose half is AD (or DC). On AB describe the square AE, and with its diagonal BF; draw DG parallel to AF, cutting BF at the point G, through which draw GH parallel to AB. Then draw CI parallel to BE, cutting BF at K and GH at L. Finally, through K draw MN parallel to AB, cutting DG at O. Then DH, OL, and CM will be squares (by the corollary to Euclid, *Elements* II.4). Since therefore the rectangle AI, contained under AF (the side of the unknown square) and AC (the number of roots), is the value of all the roots — as was said in the preceding demonstration — and the square AE is set equal to the value...

Diagram. A square on the base AB with its diagonal and an internal grid, for the second demonstration. The diagonal runs from F (top-left) to B (bottom-right).

- F (top-left corner): vertex
- I (top edge, middle): division of the top edge

- E (top-right corner): vertex
- G (interior/left, upper): point on BF where DG meets it [?]
- L (interior, upper): intersection of CI and GH
- H (right side, upper): endpoint of GH
- N (left side, middle): endpoint of MN
- O (interior, left-centre): intersection of MN and DG
- K (interior, centre): intersection of CI and BF
- M (right side, middle): endpoint of MN
- A (bottom-left corner): vertex
- D (bottom edge, left): half-point AD
- C (bottom edge, centre): point where AC ends
- B (bottom-right corner): vertex, foot of the diagonal

(The second demonstration continues; its figure is reprinted at the head of this page.)

Diagram. The same figure as fig-p361-2, reprinted at the head of the page: a square on base AB with diagonal F–B and internal grid. Labels as before — top: F, I, E; interior: G [?], L, H, N, O, K, M; bottom: A, D, C, B.

[Margin: (a) Euclid I.36 & I.43; (b) Euclid I.43; (c) Euclid I.43]

of the roots, together with some absolute number, the rectangle CE will be equal to that absolute number. [Margin: (a)] And since AO and DK are equal, and likewise DK and KH, then AO and KH will be equal. Therefore, adding the common DM, the rectangle AM will equal the gnomon LBO. But AM is equal to CE, that is, to the absolute number. [Margin: (c)] (For AK and KE are equal, and adding the common CM, AM and CE will become equal.) Therefore the gnomon LBO too will be equal to the absolute number; and if it is added to OL, the square of half the number of roots, the square DH will be known. If to its side DB there be added AD, half the number of roots — as the rule of root-extraction commands — the side AB will become known. Which was what was proposed.

Let us take, finally, the third example, in which the equation was set between 3 and $R - N$ [that is, $x^2 = x - N$]; or, if this diminished term $- N$ be restored, between $3 + N$ and R .

Let the straight line AB be equal to the number of roots; let it be bisected at C, and cut unequally at D. Upon the greater part AD describe the square ADEF, and complete the rectangle AG. Then, from BC, half the number of roots, make the square BCHI, whose diagonal BH cuts DE at the point K, through which draw LM

parallel to AB, cutting CH at N. IH produced also cuts AF at O and DE at P. Then DL and NP will be squares (by the corollary to Euclid, *Elements* II.4), [Margin: (d) Euclid I.36] and the rectangle AH equal to the square CI.

Whether therefore the given square that is sought be AE, or DL, or CI, the equation will always hold true — that is, 3 will be equal to the roots less some absolute number, or 13 together with some absolute number will equal the roots. For let the square proposed first be AE. Then the rectangle AG, contained under the side of the square AF and the number of roots AB, will be the value of all the roots. And so, since the square AE together with the absolute number is equal to the value of the roots — or the square AE is equal to the value of the roots AG less the absolute number, by hypothesis — the rectangle DG will be equal to the absolute number. And since the rectangle DG is equal to the rectangle AK (because the sides DE, DB are equal to the sides DA, DK, on account of the squares AE, DL), while AK is equal to the gnomon PBN [Margin: (e) Euclid I.43] — for since DL... [the squares] are equal, CL will be equal to DI; [Margin: (f) Euclid I.36] but CL is equal to AN; therefore DI too being equal to CK, AK will become equal to the gnomon PBN — therefore DG too, that is, the absolute number, will be equal to the gnomon PBN. If this is taken away, as the rule of root-extraction prescribes, from CI, the square of half the number of roots, the square NP will remain known, whose side CD, added to AC (half the number of roots), will constitute the known side AD of the square AE. Which was what was proposed.

Diagram. A rectangle for the third demonstration, with an internal grid and a diagonal. The square ADEF stands on the greater part, the square BCHI on BC.

- F (top-left corner): vertex
- E (top edge, middle): division of the top edge
- G (top-right corner): vertex of the completed rectangle
- O (left side, upper): point on AF
- H (interior, upper-centre): corner of square BCHI [?]
- P (interior, upper): point where IH produced cuts DE
- I (right side, upper): vertex
- M (left side, lower): endpoint of LM
- N (interior, centre): intersection of LM and CH
- K (interior, centre): point where BH cuts DE
- L (right side, lower): endpoint of LM
- A (bottom-left corner): vertex
- C (bottom edge, centre): bisection point of AB

- D (bottom edge): point of unequal section
- B (bottom-right corner): vertex

Now let the given square be DL, the number of roots and the absolute number remaining the same. Then the rectangle AL, contained under the number of roots AB and the side of the square BL, will be the estimate [value] of all the roots. And so, since the square DL together with some absolute number is equal to the value of the roots — or the square DL is equal to the value of the roots AL less some absolute number, by hypothesis — the rectangle AK will be equal to the absolute number. And since this is equal to the gnomon PBN, as was shown, the absolute number too will be equal to the gnomon PBN. When this is taken from CI, the square of half the number of roots, the square NP will remain known, whose side CD, subtracted from CB (half the number of roots), will leave known DB, the side of the given square DL. Which was what was proposed.

Lastly, let the given square be CI. Then the rectangle AI, contained under AB (the number of roots) and the side BI, will be the price of the proposed roots, to which is set equal the square CI together with a given absolute number — or to which is set equal the square CI less the absolute number. The rectangle AH, therefore, will be equal to the absolute number. When this is taken from CI, the square of half the number of roots, nothing will remain, [Margin: (g) Euclid II.2] since the rectangles AH and CI are equal; and so nothing is to be added to, or subtracted from, half the number of roots, but this very half of the number of roots — namely CB — will be the side of the given square, as we taught before. Which was what was proposed.

Therefore, if 3 be equal to $R - N$, or the equation be between $3 + N$ and R (which is the same thing): if we add to half the number of roots the root of the number that remains when the absolute number is subtracted from the square of half the number of roots, we shall find the greater root that satisfies the proposed equation; but if we subtract that same root from half the number of roots, we shall find the lesser root of the proposed equation. And if, finally, the absolute number is equal to the square of half the number of roots, then half the number of roots itself will be the root that is sought.

Pedro Nunes [Petrus Nonius], in Chapter 4 of the first part of his *Algebra*, adduces other demonstrations, no less elegant, in which he first investigates geometrically which square number, or *census*, is equal to any absolute number less any number of roots, or which — together with any number of roots — is equal to any absolute number; or which is equal to any number of roots together with any absolute number; or, finally, which is equal to any number of roots less any

absolute number, or which — together with any absolute number — is equal to any number of roots.

[Margin: Another demonstration of the extraction of roots from composite or diminished cossic numbers.]

Let there be proposed, then, first this cossic number $55 - 6R$. We must find the square equal to it, and thereby demonstrate that its square root has been rightly found by the precepts above. First let that subtracted term $- 6R$ be restored by transposition, so that the equation is between $13 + 6R$ and 55 [that is, $x^2 + 6x = 55$]. Let the number of roots be the straight line AB, bisected at C; and at C erect the perpendicular CD, equal to the side of that square which is equal to the absolute number (for every number, even one that is not square, has in continuous quantity a root, or side, as I taught in proposition 22, book 6 of the *Practical Geometry*). Join the straight line BD, and from CB produced cut off CE equal to it. I say that the square of the straight line BE, together with the proposed roots (whose number is AB), is equal to the absolute number whose square side is CD; and hence that the same square...

of the straight line BE is equal to that same number whose square side is CD, less the proposed roots, whose number is AB. [Margin: (a) Euclid I.47] For since the square on BD is equal to the squares on CD and CB, and the square on BD is equal to the square on CE, the square on CE too will be equal to the squares on CD and CB. [Margin: (b) Euclid II.6] But equal to the square on CE is the rectangle under AE, BE together with the square on CB. Therefore the rectangle under AE, BE together with the square on CB will be equal to the squares on CD, CB; and when the common square on CB is taken away, the rectangle under AE, BE will be equal to the square on CD. [Margin: (c) Euclid II.3] But the rectangle under AE, BE [together with] the rectangle under AB, BE [equals the square on BE] — therefore the square on BE together [with the rectangle under AB, BE equals the square on CD]. Since therefore (the square on BE being posited) the rectangle under AB, BE, contained by the number of roots and the side of the square, is the value of the roots, it plainly follows that the square of the straight line BE, whose number is AB, is equal to the proposed absolute number whose square side is CD; and thus that the same square of the straight line BE is equal to the absolute number whose square side is CD, less the value of the roots, whose number is AB. Which was what was proposed.

From this demonstration it is manifest that the precept given above rightly prescribes the way of finding the root from this cossic number $55 - 6R$. For if to the square of the straight line CB, half the number of roots, the absolute number

— namely the square on CD — is added, the two squares on CB and CD together become known; [Margin: (d) Euclid I.47] that is, the square on BD, and hence the square on CE. If therefore from this side CE there be taken CB, half the number of roots, the side BE — that is, the square root that was sought — will be known.

Let there again be the cossic number $6R + 55$. We must exhibit the square equal to it, and demonstrate that its root has been rightly extracted according to the precepts above. Let AB again be the straight line, bisected at C, the number of roots; let the perpendicular CD be the square side of the absolute number; and let the straight line CE be equal to the straight line BD drawn. I say that the square of the straight line AE is equal to the given roots (whose number is AB) together with the given absolute number, whose square side is CD. [Margin: (e) Euclid I.47] For since the square on BD, and hence the square on CE, is equal to the squares on CD and CB, [Margin: (f) Euclid II.6] and since equal to the square on CE is the square on CB together with the rectangle under AE, BE, the square on CB together with the rectangle under AE, BE will also be equal to the squares on CD, CB; and when the common square on CB is taken away, the rectangle under AE, BE will be equal to the square on CD, that is, to the proposed absolute number. Therefore, since the rectangle under AE, AB — under the side of the square (the square on AE being posited) and the number of roots — is the value of the proposed roots, and since [Margin: (g) Euclid II.2] the square on AE is equal to the two rectangles under AE, BE and under AE, AB, it is clear that the square of the straight line AE is equal to the roots (whose number is AB) together with the given absolute number, whose square side is CD. Which was what was proposed.

Diagram. A right triangle standing on a horizontal base line, illustrating the two demonstrations for $55 - 6R$ and $6R + 55$. The vertex D is above the base; CD is the perpendicular; DB is joined; and CE is cut off on the base equal to BD.

- D (apex, above C): top vertex, foot of the perpendicular CD
- A (base, left end): endpoint
- C (base, left-of-centre): bisection point of AB, foot of perpendicular
- B (base, centre-right): endpoint of AB
- E (base, right end): point cut off so that $CE = BD$

The side AE will be made known from the foregoing and from this demonstration. To the square of the straight line CB, half the number of roots, add the absolute number, namely the square of the straight line CD; and the two squares of the straight lines CB, CD together become known — [Margin: (h) Euclid I.47] that is, the square on BD, or on CE. And if to this known side CE there be set opposite half the

number of roots — namely the straight line AC — the whole line AE, which was sought, will be known.

Let there be, finally, the cosmic number $18R - 56$, for which a square equal to it must be found, and its side rightly found shown. Let, as before, the number of roots be the straight line AB, bisected at C; and let the straight line D be the square side of the absolute number; and let this diminished term, $- 56$, be restored, so that there is also an equation between $13 + 56$ and $18R$ [that is, $x^2 + 56 = 18x$]. Let the straight line D first be less than half the number of roots, AC or CB. Let AB be divided at E so that D is the mean proportional between the segments AE, EB — by the method we taught, following Peletier [Peletarius = Jacques Peletier], on Euclid, *Elements* VI.13. [Margin: (i) Euclid VI.17] Then the rectangle under AE, EB will be equal to the square on D, that is, to the proposed absolute number.

I say that both the square of the straight line AE and the square of EB, together with the given absolute number, are equal to the proposed roots, whose number is AB; and hence that both the square of AE and the square of EB are equal to the value of the proposed roots less the absolute number. For describe on AE, EB the squares AF, EK, and complete the rectangles AH, BL. Then both the rectangle EH and the rectangle EL, contained under AE, EB, will be equal to the absolute number, that is, to the square of the straight line D. AE being posited as the side of the square, the rectangle AH, contained under AB (the number of roots) and AE (the side of the square), will be the value of the proposed roots; and it is manifest that the square AF together with EH, the absolute number, is equal to the value of the roots AH, and hence that the same square AF is equal to the value of the roots AH less the absolute number EH. Again, EB being posited as the side of the square, the price of all the roots will be the rectangle BL, contained under AB (the number of roots) and the side of the square EB; and it is clear that the square BI too, together with EL, the absolute number, is equal to the value of the roots BL, and hence that the same square BI is equal to the value of the roots BL less the absolute number EL.

Both sides AE, EB will be made known thus. [Margin: (k) Euclid II.5] Since the rectangle under AE, EB together with the square on CE is equal to the square on AC (or CB) — that is, of half the number of roots — if from the square of the straight line AC (or CB), half the number of roots, there be taken the rectangle under AE, EB, namely the absolute number, the square on CE will remain known. If therefore this known side CE is added to AC, half the number of roots, the side AE will become known; but if the same side CE is taken from CB, half the number of roots, the side EB will remain known.

Let the straight line D next be equal to half the number of roots, AC or CB. I say that the square on AC, half the number of roots, together with the given absolute number, is equal to the proposed roots. For describe on AC the square AE, and complete the rectangle AG; and CG too will be a square. Since therefore the straight line...

Diagram. A square figure with internal division, for the demonstration on 18R – 56 (Euclid VI.17). The squares AF, EK stand on the segments AE, EB, with rectangles AH, BL completed; AB is the middle horizontal, and D is drawn separately below as a measured segment (the square side of the absolute number).

- F (top edge): top division [top-left corner label off-tile — [?]]
- H (top-right): corner of the figure
- A (middle line, left end): base vertex
- C (middle line, left-of-centre): bisection point of AB
- E (middle line, centre): point of division (mean-proportional cut)
- B (middle line, right end): base vertex
- L (bottom-left): lower corner
- I (bottom, centre): lower division
- K (bottom-right): lower corner
- D (separate segment below): the mean proportional / square side of the absolute number

the line D is set equal to the line AC (or CB), the square CG will be equal to the absolute number; and once AC is taken as the side of the square of the equation, the rectangle AG — contained by the side AC and the number of roots AB — will be the value, or price, of the roots. Now it is clearly established that the square AE, together with CG (the absolute number), is equal to the roots AG; and therefore this same square AE is equal to the value of the roots AG minus the absolute number CG.

Diagram. A rectangle divided into two parts by a vertical line, with a separate short measuring-segment drawn beneath it.

- F (top-left corner)
- E (top edge, midpoint — top of the dividing line)
- G (top-right corner) [?]
- A (bottom-left corner)
- C (bottom edge, midpoint — foot of the dividing line)
- B (bottom-right corner)

- D (the short segment beneath, drawn between two end-marks) — a given length, equal to AC (= CB)

The left half (A–C–E–F) is the square AE; the right half (C–B–G–E) is the rectangle CG; the whole is the rectangle AG.

And so, when the line D is less than half the number of roots AC (or CB) — that is, when the absolute number of the equation is less than the square of half the number of roots — a double root of the equation is found. But when the line D is equal to half the number of roots — that is, when the given absolute number is equal to the square of half the number of roots — the root of the equation is that very half of the number of roots. The line D, however, cannot be greater than half the number of roots AC (or CB). For the equation would then be false, since the square of D — namely the absolute number — could not be subtracted from the square of half the number of roots, and so the root could in no way be found by the rule laid down. There is the further point that, if D were greater than AC (or CB), it could not be the mean proportional between the two segments of the line AB, as is clear from the demonstration we set out, following Peletier [Jacques Peletier], on proposition 13 of book 6 of Euclid (*Elements* VI.13). No square satisfying the equation, therefore, could be found.

Following the same Pedro Nunes [Pedro Nunes], we can also present this art of extracting roots from composite cossist numbers in the following way.

[Margin: Another method of extracting the square root from a composite or diminished cossist number.]

1. To the square of the number of roots add — or from it subtract — four times the absolute number, according as the absolute number is marked with the sign + or –.
2. To the square root of this sum (or remainder) add — or from it subtract — the number of roots, according as the number of roots is marked with the sign + or –. For this final sum, or remainder, will give the value and the price of the doubled square root. Therefore half of it will be the value of a single root.

For let the root be extracted from this cossist number, $72 - 6R$ [$R = res$, the unknown root; so: 72 minus 6 roots]. To the square of the number of roots — namely to 36 — I add 288, four times the absolute number; and from the square root of the resulting sum, 324 — that is, from 18 — I subtract the number of roots, namely 6; and there remain 12, for the value of two roots. Therefore the value of a single root will be 6.

Again, let the root be drawn from this consist number, $6R + 72$ [6 roots plus 72]. To the square of the number of roots — namely to 36 — I add 288, four times the absolute number, making 324. Then to the square root of this sum — namely to 18 — I add the number of roots, namely 6, making 24, for the value of the doubled root. Therefore the value of a single root will be 12.

Lastly, let the root of this consist number be found, $18R - 72$ [18 roots minus 72]. From 324, the square of the number of roots, I subtract 288, four times the absolute number; and there remain 36. To the square root of this remainder — namely to 6 — I add 18, the number of roots, making 24, for the value of the doubled root. Therefore one root will be 12. But if I subtract the square root of that remainder — namely 6 — from 18, the number of roots, there will remain 12, for the value of the doubled smaller root (for a consist number of this kind has a double root, as has been said). Therefore the other root will be 6.

And this method is most convenient when the number of roots is odd or fractional, so that we are not forced to take half of it — which is always a fractional number — and so on.

[Margin: The convenience of this latter method of extraction.]

The reasoning behind this operation is as follows. **[a]** Since the square of the number of roots is four times the square produced from half the number of roots — just as the quadrupled absolute number is four times the absolute number — **[b]** it follows that the sum formed from the square of the number of roots and four times the absolute number is four times the sum that was previously produced, in the other method, from the square of half the number of roots and the absolute number. **[c]** Likewise, the remainder formed by subtracting four times the absolute number from the square of the number of roots is four times the remainder previously formed by subtracting the absolute number from the square of half the number of roots. **[d]** For this reason the square root of this sum (or remainder) will be double the square root of the earlier sum (or remainder) — just as the number of roots is double half the number of roots. So whether we add the number of roots to this square root, or subtract it from the same, **[e]** the resulting sum (or remainder) will likewise be double the sum (or remainder) formed by adding half the number of roots to the square root found in the earlier operation, or by subtracting half the number of roots from the earlier square root. Therefore half of this sum (or remainder) will be the value of the root that satisfies the equation.

[Margin: a — Scholium to Euclid, *Elements* II, prop. 4. b — Euclid V.1. c — Euclid V.5. d — Scholium to Euclid II, prop. 4. e — Euclid V.1 or V.5.]

We shall nevertheless also confirm this operation geometrically, following Pedro Nunes. Let there first be an equation between $\mathfrak{z}$ [$\mathfrak{z}$ = *census*, the square of the unknown] and $N - 2\mathfrak{r}$ [the number minus 2 roots] — that is, if the diminished side is restored, between $\mathfrak{z} + 2\mathfrak{r}$ [census plus 2 roots] and N [the number].

[Margin: Demonstration of another extraction of this kind.]

Diagram. A large square divided into a 3×3 -like grid by two vertical and two horizontal internal lines, with a diagonal drawn across it, in the manner of the figure of Euclid *Elements* II.8. Several of the small peripheral labels are faint at this resolution.

- Top edge: F (top-left), K, I, E (top-right)
- Upper interior horizontal line: O (left edge), H (interior), L (interior), N (right edge)
- Lower interior horizontal line: Q [?] (left edge), M (interior), [interior point — illegible] [?], P (right edge)
- Bottom edge: C (bottom-left corner), A, B [?], D (bottom-right corner)
- A diagonal runs from F (top-left) toward D (bottom-right); the diagonal is DF.
- The construction yields the squares BP, ML, AG, GN, OK and the gnomon ODK (four times the rectangle CG).

Let AB be set as the unknown side of the square sought, and AC as the number of roots. With the line AB extended toward A, take AC equal to the number of roots. Take also, on the same line AB extended toward B, the line BD equal to AB itself; and on the whole line CD describe the square CDEF, together with the diagonal DF, and complete the figure, as you see, so that it is entirely similar to the figure of Euclid *Elements* II.8. There will be, then — as was demonstrated in Euclid II.8 — the squares BP, ML, AG, GN, OK, and the gnomon ODK, four times the rectangle CG. Now since AG is taken as the square whose side AB is sought, the rectangle CM — contained by the number of roots AC and the side of the square AM — will be the value of all the roots. Therefore, since the equation is between $\mathfrak{z} + 2\mathfrak{r}$ and N , the rectangle CG will be equal to the absolute number, and hence the gnomon ODK four times the absolute number; and when this is added to OK, the square of the number of roots AC, the whole square CE becomes known. If from its side CD, now known, AC (the number of roots) is taken away, the line AD will be left, double the sought side AB, and known. Which is what was proposed.

Next, let there be an equation between 3 and $2R + N$ [census and 2 roots plus the number]. Let AB be set as the unknown side of the square sought, and AC as the number of roots. With the line AB extended, take BD equal to BC , and DR equal to AC ; and on AD describe the square $ADEF$, together with the diagonal DF , and complete the figure as before. There will again be, as before, the gnomon ODK , four times the rectangle KG , and the squares OK and QI . And since QI is taken as the square whose side AB is sought, the rectangle QK — contained by QM (the number of roots; **[a]** for QM is equal to AC) and the side of the square QF — will be the value of all the roots. Therefore, since the equation is between 3 and $2R + N$, the rectangle KG will be equal to the absolute number, and accordingly the gnomon ODK four times the absolute number; and when this is added to OK , the square of the number of roots, the whole square AE becomes known. If to its side AD , now known, DR (the number of roots) is added — for the line DR was set equal to the number of roots AC — the whole line AR will be known, which is double the side AB that is sought. For if to the equal lines BC and BD the equal lines CA and DR are added, the wholes BA and BR become equal, and thus AR will be double AB itself. Which is what was proposed.

[Margin: a — Euclid I.34.]

Diagram. A large square with two internal vertical and two internal horizontal lines and a diagonal, following Euclid *Elements* II.8, with the base extended beyond the square to the right.

- Top edge: F (top-left), K , I , E (top-right)
- Upper interior horizontal line: O (left edge), H (interior), L (interior), N (right edge)
- Lower interior horizontal line: Q (left edge), M (interior), G [?] (interior, where the diagonal crosses), P (right edge)
- Bottom edge: A , C , B , D , R (left to right; R lies on the extension beyond the square's corner D)
- The diagonal DF runs from D (bottom-right corner of the square) to F (top-left).

Finally, let there be an equation between 3 and $2R - N$ [census and 2 roots minus the number] — that is, if the diminished side is restored, between $3 + N$ [census plus the number] and $2R$ [2 roots]. Let the preceding figure be repeated, with the number of roots being the line AD , and let AB , the side of the sought square, be at first greater than half the number of roots. Since, then, QI is taken as the square whose side AB is to be found, the rectangle AI — contained by the side of the

square AB and the number of roots AD — will be the value of the roots. Therefore, since the equation is between $3 + N$ and $2R$, the rectangle AG will be the absolute number, and hence the gnomon ODK four times it, as before. If this is taken away from AE, the square of the number of roots, there will remain the square OK, known; and if to its known side AC (or DR) the number of roots AD is added, the whole AR becomes known, which is double the side of the square AB, as was demonstrated before. Which is what was proposed.

Now let BD, the side of the square to be found, be less than half the number of roots. Then the rectangle AP — contained by the side of the square DP and the number of roots AD — will be the value of the roots. And since the equation is between $3 + N$ and $2R$, the rectangle AG will be the absolute number, and hence the gnomon ODK four times it; which, taken away from AE, the square of the number of roots, leaves the square OK, and therefore its side AC becomes known. And if AC is subtracted from AD, the number of roots, there remains the known line CD, which is double the side BD. Which is what was proposed.

Diagram. A square divided into four equal squares by one vertical and one horizontal median line.

- Top edge: E (top-left), F (top-middle), D (top-right)
- Middle horizontal line: H (left edge), I (center), G (right edge)
- Bottom edge: A (bottom-left), C (bottom-middle), B (bottom-right)

But now, if four times the absolute number is equal to the square of the number of roots, then half the number of roots will be the side of the square sought. For let AB be the line that is the number of roots, bisected at C; and, with the square AD described, let the lines CF and GH be drawn, bisecting the four sides and dividing the square into four equal squares. If, then, AC — half the number of roots — is taken as the side of the square, the rectangle AG, contained by the side of the square AH and the number of roots AB, will be the value of the roots; and therefore, since the equation is between $3 + N$ and $2R$, the rectangle CG will be equal to the absolute number, and four times it equal to the square AD described on the number of roots AB. If, then, four times the absolute number — that is, the square AD — is taken away from AD, the square of the number of roots AB, there will remain nothing (0). And whether this is added to AB, the number of roots, or subtracted from it, the line AB becomes known, which is double the side AC. Which is what was proposed.

As for the extraction of zensizensic [fourth], zensicubic [sixth], zensizenzensic [eighth], or zensursolid [tenth] roots — that is, when the consist terms of the

equation, once their characters have been reduced, have exponents forming one of these arithmetic progressions: 4.2.0, or 4.0.2, or 6.3.0, or 6.0.3, or 8.4.0, or 8.0.4, or 10.5.0, or 10.0.5, and so on, in which the greatest character is always composed from 3 and another cossist character, as is plain from the exponent-numbers — one must first extract the square root, on account of the character 3, by the method described so far, applying everything that we prescribed about the number of roots and the absolute number to that term which is marked by a cossist character in the part of the equation from which the root is to be drawn, exactly as if the equation were between three cossist terms marked by the characters 3, 2R, and N. Then, from this square root once found — whether it be rational or irrational — another root must be elicited, according to the remaining part of the greater cossist character, this character 3 being set aside (for the greater cossist character is a composite one); that is, one must draw the root denominated from the cossist character that appears in the part of the equation where two terms are found, and which holds the middle place between the greatest character and the least (or N). For example, if the equation is between 133 [*zensizensus*, the fourth power] and $183 + 648$ [18 censuses plus 648], the square root must be extracted from this number, $183 + 648$, on account of the leading part of the cossist sign 3 — exactly as if the equation were between 13 and $18R + 648$; and this root is 36. From 36, in turn, the square root must be drawn, namely 6, on account of the remaining part of the cossist character — namely 3 — which holds the middle place between 33 and N. Therefore 6 will be the root of the proposed cossist number.

[Margin: Extraction of the zensizensic root, the zensicubic root, and so on.]

In the same way, if the equation is between $13\mathfrak{C}$ [*zensicubus*, the sixth power] and $5120 - 16\mathfrak{C}$ [$\mathfrak{C} = \text{cubus}$, the cube; so: 5120 minus 16 cubes], one must first take the square root of this number, $5120 - 16\mathfrak{C}$, and from it elicit the cube root. So too, if the equation is held to be between 1333 [*zensizenzensus*, the eighth power] and $2000033 - 78461119$ [20000 zensizensususes minus 78,461,119], one must first take the square root, and then from it draw the zensizensic [fourth] root. And so for the rest.

That the square root is extracted from cossist numbers of this kind by the same device we used above is evident. For the greatest cossist character, since it is composed from 3 and some other cossist character (whether simple or composite), indicates that the unit denominated by it is the census — that is, the square — of some number denominated by the remaining part of the greatest cossist character, so that this remaining part is, as it were, the root of the former

part. Therefore, since this remaining part is also found on the other side of the equation whose root is sought, that side's term can be denominated by $2R$, provided the term of the greatest character — namely the unit — is denominated by 3. And thus, by the precepts already given, the square root will be correctly found from that term of the equation. For example:

In an equation between $3\mathfrak{C}$ [zensicubus] and $\mathfrak{C} + N$ [cube plus the number]: since this character $3\mathfrak{C}$ signifies the census (square) of one cube, one cube will be the square root of that census; and accordingly the number of cubes on the other side of the equation will be the number of roots of that same census. And so for the rest. Now, once this square root has been found, one must extract from it the root denominated by the remaining part of the greatest consist character, because that square root is a number denominated by that remaining part, and not an absolute number. Just as someone who wishes to find the zensicubic [sixth] root of the number 64 must first find its square root, namely 8; then, since this square root is not an absolute number but a cube — because the number 64 is the census of one cube — he must take the cube root of the root found, 8, namely 2; and this number will be the zensicubic root of the given number 64.

EXEMPLA DVO, IN QVIBVS occurrunt radices surdæ

[Two examples, in which surd roots occur.]

Let there be an equation between $13\mathfrak{C}$ and $800\mathfrak{C} - 156751$ [$x^6 = 800 \cdot x^3 - 156751$]. Half the number of cubes — namely 400 — I square, obtaining 160000. I subtract 156751, and take the square root, 57, of the remaining number, 3249; this I add to half the number of cubes, namely to 400, obtaining 457, whose cube root gives the zensicubic root sought, which is surd. For its cube is 457, which multiplied by itself makes the zensicubus 208849, which is equal to $800\mathfrak{C}$ — that is, to 365600 minus 156751 — as is plain. Nearly the same will hold if we proceed by the cube root of the number 457 found in [ordinary] numbers. For the cube root of 457 is approximately $7 \frac{7}{10}$, found by appending two sets of three ciphers. Its cube is $456533/1000$, and its zensicubus is 208422, very nearly — which is nearly equal to $800\mathfrak{C}$, that is, to $365226400/1000$ [as printed, the numerator's final zeros and the denominator's zeros shown by the printer as raised dots] minus 156751. For after the subtraction the remaining number becomes $208475 \frac{2}{5}$, which differs little from 208422. The cause of this inequality is that the approximate root is not the true root. The equation therefore comes out more happily if the operation is carried out through the surd root itself.

And since the number $800\text{C} - 156751$ has a double root, as we said above — because the absolute number is marked with the sign $-$ — the greater root is found, as stated, namely the cube root of the number 457. The smaller, however, will be found if, from 400 (half the number of cubes), we subtract 57 — the square root of the number 3249 left over after the absolute number was subtracted from the square of half the number of cubes — and take 7, the cube root of the remainder, 343. For the zensicubus of 7 is 117649, which is likewise equal to 800C — that is, to 274400 minus 156751 — as is manifest. And so the greater root of this number, $800\text{C} - 156751$, is surd, namely the cube root of 457, while the smaller is rational, namely 7, that is, the cube root of the cube 343.

Again, let there be an equation between 13 and $8R - 5$ [$x^2 = 8x - 5$]. I square half the number of roots, namely 4, and from the square, 16, I subtract 5; and the near square root of the remaining number, 11 — namely $3\frac{79}{250}$ (found by appending three pairs of ciphers) — I add to half the number of roots, namely to 4, obtaining the near root $7\frac{79}{250}$ of the proposed number $8R - 5$. Its square is $53\frac{32741}{62500}$, which is nearly equal to $8R - 5$, since $8R - 5$ amount to $53\frac{66}{125}$.

The smaller root of the same number, $8R - 5$ — for it has a double root, on account of the sign $-$ that the absolute number bears, as we said above — is $171/250$, which is obtained if the near root of the remaining number 11, namely $3\frac{79}{250}$, is subtracted from 4, half the number of roots. The square of the root $171/250$ is $29241/62500$, which is nearly equal to $8R - 5$; for $8R - 5$ amount to $59/125$, a number which exceeds that square by this fraction, $259/62500$, which is less than $1/241$ [?].

The equation will also come out more happily if we choose to work with surd roots. For the greater root is $4 + \sqrt{311}$ [$\sqrt{3} = \text{radix quadrata}$, the square root; so: 4 plus the square root of 11], and the lesser $4 - \sqrt{311}$. Both the square of the former and eight times that root minus 5 yield $27 + \sqrt{3704}$ — that is, about $53\frac{28}{53}$ [?]. And both the square of the latter and eight times that root minus 5 produce about $25/53$ [?] — namely $27 - \sqrt{3704}$. The reasoning of this operation through surd roots will be understood in chapter 23, in the algorithm of surd numbers.

The multiplications are formed thus:

For the greater root ($4 + \sqrt{311}$):

$$\begin{array}{r}
 4 + \sqrt{311} \\
 4 + \sqrt{311} \\
 \hline
 + \sqrt{3176} + 11 \\
 16 + \sqrt{3176}
 \end{array}$$

Sum. $27 + \sqrt[3]{704}$

$4 + \sqrt[3]{11}$
8

$32 + \sqrt[3]{704}$

Less 5, there remain:

$27 + \sqrt[3]{704}$

For the lesser root ($4 - \sqrt[3]{11}$):

$4 - \sqrt[3]{11}$
 $4 - \sqrt[3]{11}$

$- \sqrt[3]{176} + 11$
 $16 - \sqrt[3]{176}$

Sum. $27 - \sqrt[3]{704}$

$4 - \sqrt[3]{11}$
8

$32 - \sqrt[3]{704}$

Less 5, there remain:

$27 - \sqrt[3]{704}$

[Margin: When some problem occasionally cannot be solved, the blame is to be laid on the arithmetic, not on the algebra.]

I have set down these two examples so that you may see that the rule of algebra we have handed down is no less true in surd numbers than in rational ones.

But now, if ever an equation turns up that you do not know how to solve, the blame must not be shifted onto the art of algebra, but rather onto yourself, who have not thoroughly learned the science of all equations. For example: if you come upon an equation between $1\mathfrak{C}$ [one cube] and $3\mathfrak{J} + 2\mathfrak{R} + 8$ [3 censuses plus 2 roots plus 8], and you do not know how to extract the cube root from this number, $3\mathfrak{J} + 2\mathfrak{R} + 8$, the fault will be yours, not that of the rule of algebra — which teaches that, if the cube root is extracted from that number, the problem is solved.

(Continuation of Caput XII, carried over from the preceding pages. On the notation: throughout the Algebra the cossist symbols are printed as single signs — $\mathfrak{R}$ =

res/radix, the unknown, x, printed as a "2"-shaped glyph; 3 = census/zensus, the square, x^2 , printed as a "3/8"-shaped glyph; c = cubus, the cube, x^3 ; N = numerus, the absolute number; $\sqrt{3}$ = radix quadrata, the square root. A digit before a sign is a coefficient. I preserve the signs and add a modern gloss in parentheses.)

is [the cube]. Now the aforementioned cube root is 4, since its cube is 64, which is equal to three squares — namely 48 — and to two roots — that is, 8 — and to eight units. [$64 = 3 \cdot (4^2) + 2 \cdot 4 + 8 = 48 + 8 + 8$.] Form the same judgment about the equation between 1c (one cube, x^3) and 43+147 ($4x^2 + 147$) — likewise about the equation between 1c and 30R+133 ($30x + 133$). For since the exponents of the denominations [of the terms] do not preserve a proportion, no method has yet been discovered by which they can be resolved, even though they are in fact solvable; for in each of them the root is worth 7. [With $x = 7$: $7^3 = 343 = 4 \cdot 49 + 147 = 30 \cdot 7 + 133$.]

So too, when surd roots occur — whose theory you have not perfectly mastered — it should not be counted a fault of the science of Algebra if the proposed problem still remains unsolved. This can be seen clearly in the two examples above, in which, to work out the proposed equation, it was necessary to carry out the operation with surd (that is, irrational) numbers.

It also very often happens that, in working out some problems, a great command of geometry — or of some other science — is required, as will become clear below, when we set out various problems to be solved. Whoever, then, is not sufficiently practiced in that discipline will gird himself in vain for the solution of the proposed problem; for, deprived of that knowledge, he will never carry it through. And this, it must be held for certain, happens not through any fault of Algebra, but through his own.

By the same token: if perhaps in that golden rule which they call the Rule of Three surd and irrational numbers, or fractions, should occur, and the person who sets up the reckoning by that rule is ignorant of the theory of such numbers and therefore cannot solve the proposed problem, it is not the rule itself that is to be blamed, but the practitioner. For since the aforesaid rule directs one to multiply the second number by the third and to divide the product by the first, so that the quotient may present the solution of the problem, the fault will be the practitioner's if he does not know how to operate with surd or fractional numbers — not the rule's. Suppose, for example, this problem is proposed: 2 gives $\sqrt{33}$ ($\sqrt{3}$); what, then, will $\sqrt{312}$ ($\sqrt{12}$) give? It is certain that anyone who does not know how to multiply and divide surd roots cannot solve the problem. For the second number, $\sqrt{3}$, multiplied into the third — that is, into $\sqrt{12}$ — makes $\sqrt{336}$

($\sqrt{36}$); and this, divided by the first number 2, gives the quotient $\sqrt{39}$ ($\sqrt{9}$), that is, the number 3. In the same way, the rule of Algebra teaches what must be done in any proposed problem in order to arrive at some equation; and even if the practitioner does not know how to unravel it, the rule of Algebra loses nothing of its firmness and strength — rather, the whole fault must be laid upon the practitioner.

DE MULTITUDINE REGULARUM Algebrae, quam alij auctores introducunt

[On the Multitude of Rules of Algebra which other authors introduce]

CAPUT XIII

You have seen thus far, studious reader — I think — that our single rule of Algebra teaches fully and plainly by what method we ought to work out any proposed problem whatsoever: for it directs that $1R$ (one res, x) be set down for the number under investigation, or some other number, and that we examine it according to the tenor of the problem until we fall upon some equation of two numbers. Then, once an equation of this kind has been found and reduced (as we set out in Chapter 10, on reduction), we divide the remainder of the equation by the number of the greater cossic character, and finally, from the quotient, extract the root that fits the equation — if need be. But so that it may not escape you how various are the rules of Algebra which other writers hand down, I have judged it worthwhile to set them out here, so that, by comparing them with our single rule, you may understand what lies between our rule and the manifold rules of others.

Other writers, then, consider chiefly those equations that occur between the first three terms of the geometric progression beginning from unity — that is, between the Number, the Root, and the Square (or Zensus). These three terms can be equated among themselves in six ways. For either the equation is between 3 (census) and R (res); or between R and N (number); or between 3 and N — and these three equations they call *simple*, because in them only one term is equated to one term. Or else it is between $3+R$ and N ; or between 3 and $N+R$; or (what comes to the same thing) between 3 and $R+N$; or, finally, between $3+N$ and R — and these three equations are called by the same writers *composite*, because in them two terms are always set equal to one term. For these six equations they prescribe six rules.

[Margin: In respect of which numbers other writers chiefly frame their equations.] [Margin: Three simple equations.] [Margin: Three composite equations.]

REGULA I

[Rule I]

[Margin: When the equation is between 3 and R.]

When an equation is found between 3 (census) and R (res), then, on dividing the number of the res-sign by the number of the census-sign, the quotient presents the value of 1R (one res, x). For instance, if 53 ($5x^2$) are equal to 30R ($30x$), then, dividing 30 by 5, the quotient 6 will come out for the value of one root. For the square of this root is 36, and therefore 53 ($5x^2$) make 180 — exactly as much as 30R ($30x$) make. [$5 \cdot 36 = 180 = 30 \cdot 6$.]

REGULA II

[Rule II]

[Margin: When the equation is between R and N.]

When an equation is found between R (res) and N (number), then, on dividing the absolute number by the number of the res-sign, the value of 1R will be produced in the quotient, just as in Rule I. For instance, if 5R ($5x$) are equal to 30, then, dividing 30 by 5, the quotient 6 comes out for the value of 1R; for the root 6, taken five times, makes 30.

REGULA III

[Rule III]

[Margin: When the equation is between 3 and N.]

When an equation is detected between 3 (census) and N (number), then, on dividing the absolute number by the number affected by the census-sign, the value of 13 (one census, x^2) is obtained in the quotient. Its square root, therefore, will give the value of 1R. For instance, if the equation is between 43 ($4x^2$) and 144, then, dividing 144 by 4, the quotient 36 comes out for the value of 13; the square root of 36, which is 6, will offer the value of 1R. For the square of the number 6 is 36, which, multiplied by 4, makes the number 144.

REGULA IV

[Rule IV]

[Margin: When the equation is between $3+R$ and N .]

When an equation occurs between $3+R$ ($x^2 + bx$) and N (number), the half of the number of roots must be multiplied into itself, and to the product the absolute number must be added. For if from the square root of this aggregate the half of the number of roots is subtracted, the remainder will be the value of $1R$. For instance, if the equation is set up between $13+12R$ ($x^2 + 12x$) and 85, we shall multiply 6 — the half of the number of roots — into itself, and to the product 36 we shall add 85, so that they become 121. For if from the square root of this number, 11, the half of the number of roots — namely 6 — is subtracted, there will remain 5 for the value of $1R$. For 13 (x^2) is 25, and, with $12R$ ($12x$) — namely 60 — added, the absolute number 85 results, as was proposed. [$x = 5$; $25 + 60 = 85$.]

REGULA V

[Rule V]

[Margin: When the equation is between 3 and $R+N$.]

When an equation is set up between 3 and $R+N$ ($x^2 = bx + N$), or between 3 and $N+R$, the half of the number of roots must be multiplied into itself, and to the product the absolute number must be added. For if to the square root of this aggregate the half of the number of roots is added, the value of $1R$ will be produced. For instance, if the equation is posited between 13 (x^2) and $6R+16$ ($6x + 16$), we shall multiply 3 — that is, the half of the number of the roots — into itself, and to the product 9 we shall add the absolute number 16, so that they become 25. For if to the square root of this number, 5, the half of the number of roots — namely 3 — is added, the number 8 will be produced for the value of $1R$. For its square is 64, which number is equal to $6R$ ($6x$) and 16 — that is, to 48 together with 16, as is plain. [$x = 8$; $64 = 48 + 16$.]

REGULA VI

[Rule VI]

[Margin: When the equation is between $3+N$ and R .]

When, finally, an equation is found between $3+N$ ($x^2 + N$) and R (bx), the half of the number of roots must be multiplied into itself, and from the product the

proposed absolute number must be taken away. For if to the square root of the remaining number you add the half of the number of roots — or from this half subtract that same root — you will produce the value of $1R$. So that an equation of this kind has a double root, namely a greater and a lesser. For instance, if the proposed equation is between $13+21$ ($x^2 + 21$) and $10R$ ($10x$), we shall multiply 5 — the half of the number of roots — into itself, and from the product, 25, we shall take away the absolute number 21. For if to the square root of the remaining number 4 — namely 2 — we add the half of the number of roots, that is 5, we shall produce 7, the value of the greater root; and if we subtract that same root, 2, from 5 — namely from the half of the number of roots — the remainder will be 3, for the value of the lesser root. For the square of the greater root, 7, is 49, and, with 21 added, the number 70 results, which is equal to $10R$ ($10x$). But the square of the lesser root, 3, is 9, and, with 21 added, the number 30 results, which is equal to $10R$, as is clear. [Greater root 7: $49 + 21 = 70 = 10 \cdot 7$. Lesser root 3: $9 + 21 = 30 = 10 \cdot 3$.]

[Margin: When more or fewer than one census are found in an equation.]

Moreover, when more or fewer than one census occur in some equation, all the numbers of the equation must be divided by the number affected by the census-sign 3, so that the equation may become one between 13 and so forth — which we too have directed must be done. For example, if the equation is given between $33+63$ ($3x^2 + 63$) and $30R$ ($30x$), each of the numbers must be divided by 3, so that the equation becomes one between $13+21$ ($x^2 + 21$) and $10R$ ($10x$), and then Rule VI is to be applied. Likewise, if the equation is given between $\frac{1}{3}3$ and $3R+8$, or between $\frac{1}{3}3$ and $8+3R$, then, if the individual numbers are divided by $\frac{1}{2}$, an equation will be found between 13 and $6R+16$, or between 13 and $16+6R$, which will be worked out by Rule V.

[Margin: In what way other equations are brought back to the six rules above.]

These, then, are the six rules handed down by almost all authors. For all other equations — those between other cosmic numbers [powers] different from 3, R , and N — are reduced to the aforesaid six; so that some writers prescribe more rules in vain. For if equations of this kind occur in which the numerical exponents of the characters preserve an arithmetic proportionality, the characters must be abbreviated, as was said above, to the end that on the other side of the equation an absolute number may be found. Then one must observe what we wrote near the end of Chapter 12 concerning the extraction of the zensizensic root, the zensicubic root, and so on — namely, when the exponents preserve this arithmetic proportion: $4 \cdot 2 \cdot 0$ or $4 \cdot 0 \cdot 2$, or $6 \cdot 3 \cdot 0$ or $6 \cdot 0 \cdot 3$, and so forth.

[Margin: That our single rule of Algebra is more excellent than the six rules of others.]

But if we are willing to look into the matter more attentively, we shall easily perceive that our single rule of Algebra is far more excellent than the aforesaid six rules of others. For in ours the equation is always reduced to 13 (one census) on one side, and consequently the signs + and – on the other side of the equation fully and abundantly instruct us when addition ought to be made and when subtraction — which does not appear so clearly in the six rules above.

[Margin: What a simple and a composite equation is.]

And just as the writers of Algebra call the equations of the first three rules *simple*, but those of the last three rules *composite*, as we said a little before, so too, in what follows, we shall — for the sake of instruction — call *simple* an equation that occurs between only two numbers, but *composite* one that is found among three numbers.

QUAESTIO PROPOSITA VTRVM POSSIBILIS sit, nec ne, an
vero inepta, ac nugatoria, quo pacto ex ipsa operatione, quae
per Algebrae regulam fit, cognoscatur

[Whether a Proposed Problem is Possible or not — or indeed
absurd and trifling — and in what way this may be known
from the very operation carried out by the rule of Algebra]

CAPUT XIV

[Margin: In what way it may be known whether a problem is possible or impossible.]

Since at the beginning of this book we said that the rule of Algebra has this excellence — that in the very working it teaches us whether the problem proposed can be solved or not — it must now be shown in a few words by what means this comes about. Therefore, whenever, in solving a problem by the precepts of Algebra, we run into some impossible equation, or one that is trifling and absurd, we shall declare that the problem can in no way be solved, or else that it is trifling and absurd. This will be made clear by several examples.

1. Let a number be sought which, multiplied by 3, and this product multiplied into itself, makes as much as [is produced] from the number itself multiplied into itself, and this product multiplied into 5.

[Margin: 1. An impossible problem.]

Let the sought number be posited as $1R$ (x), which multiplied by 3 makes $3R$ ($3x$); and this product, multiplied into itself, makes $9\bar{3}$ ($9x^2$). This number must be equal to that which is produced from $1R$ multiplied into itself, and that product multiplied into 5. Now from $1R$ multiplied into itself comes $1\bar{3}$ (x^2), and this number, multiplied into 5, makes $5\bar{3}$ ($5x^2$). An equation has therefore been found between $9\bar{3}$ and $5\bar{3}$ ($9x^2 = 5x^2$), which is impossible. Wherefore we shall declare the proposed problem to be impossible.

2. Let the number 12 be divided into two such parts that $\frac{1}{3}$ of one, together with $\frac{1}{4}$ of the other, makes 4.

[Margin: 2. An impossible problem.]

Let $1R$ (x) be posited for one part; the other will then be $12-1R$ ($12 - x$). A third of the former is $\frac{1}{3}R$ ($\frac{1}{3}x$), and a quarter of the latter is $3-\frac{1}{4}R$ ($3 - \frac{1}{4}x$), which together make $3+\frac{1}{12}R$ ($3 + \frac{1}{12}x$). (For from $+\frac{1}{3}R$ and $-\frac{1}{4}R$ comes $\frac{1}{12}R$, as is established from Chapter 4.) These must be equal to the number 4. Removing 3 from each side, there will be an equality between $\frac{1}{12}R$ and 1. Therefore, dividing 1 by $\frac{1}{12}$, the value of $1R$ will turn out to be 12; and this is the first part of the number 12, while the other part is 0 — which is absurd. Yet, improperly, $\frac{1}{3}$ of that part — namely 4 — together with $\frac{1}{4}$ of this part — that is, with 0 — makes 4.

But if, with the same parts $1R$ and $12-1R$ posited, we take $\frac{1}{4}$ of the first — namely $\frac{1}{4}R$ — and $\frac{1}{3}$ of the latter — namely $4-\frac{1}{3}R$ — these parts together make $4-\frac{1}{12}R$ ($4 - \frac{1}{12}x$). (For $\frac{1}{4}R$ and $-\frac{1}{3}R$ make $-\frac{1}{12}R$, as is established from Chapter 4.) These cannot be equal to the number 4. Therefore, not even in this way is the problem possible.

3. Let the number 10 be divided into two such parts that from the multiplication of the one into the other 26 is produced.

[Margin: 3. An impossible problem.]

Let $1R$ (x) be posited for one part; the other will then be $10-1R$ ($10 - x$). These parts, multiplied together, make $10R-1\bar{3}$ ($10x - x^2$), which product must be equal to the number 26. Adding $1\bar{3}$ to each side, $10R$ become equal to $1\bar{3}+26$ ($x^2 + 26$); and, taking away 26 from each side, there will be an equality between $1\bar{3}$ and $10R-26$

$(x^2 = 10x - 26)$. From this number, which is equal to 13, we shall extract the square root in this way, as we taught above. The half of the roots is 5, and its square is 25; but from this 26 cannot be taken away, as the $-$ sign requires. Therefore we shall say that the problem cannot be solved.

4. Let two numbers be found such that from the multiplication of the one into the other there results a number three times their sum.

[Margin: 4. An impossible problem.]

Let one of them be posited as 1R (x), and the other as 2 units. From 1R into 2 comes the number 2R ($2x$), which must be three times their sum — the sum being 1R+2 ($x + 2$). Therefore this sum tripled — namely 3R+6 ($3x + 6$) — must be equal to 2R ($2x$), which is false. The proposed problem is therefore impossible if the other of the numbers is set at 2. If, for the second number, more units had been posited than are contained in the denominator of the given proportion — namely 4 or 5 or 6, and so on — the problem could be solved. For, with the second number set at 4, 1R would be found to be 12, and so the two sought numbers would be 12 and 4, which, multiplied together, make 48 — a number three times their sum, which is 16.

5. Let a number be sought which, multiplied by 3, produces its own square, and [such that] the sum collected from the number itself and its square is 7.

[Margin: 5. An impossible problem.]

Let us suppose the number to be 1R (x), which, multiplied by 3, makes 3R ($3x$); this number must be equal to the square produced from 1R — namely 13 (x^2) — and the value of one root will be 3. And this number, multiplied by 3 — that is, into itself — produces its square, 9. But since this square, together with its root 3, makes 12, and not 7, you will say the problem is impossible. This is also established from the equations themselves. For 1R together with its product 3R makes 4R ($4x$), which must be equal to 7, and so 1R will be $\frac{7}{4}$. And since 1R together with its square — that is, with 13 — must likewise be equal to the number 7, there will be an equation between 13+1R ($x^2 + x$) and 7, that is (with 1R removed from each side) between 13 and 7-1R ($x^2 = 7 - x$); and 1R will be found to be $\sqrt{3}(7\frac{1}{4}) - \frac{1}{2}$ [?] ($\sqrt{7\frac{1}{4}} - \frac{1}{2}$), which differs greatly from the former [value]. Indeed, if both 4R and 13+1R are equal to 7, they will be twice 7 — that is, 14 — equal to 13+5R ($x^2 + 5x$); that is, an equation will be set up between 13 and 14-5R ($x^2 = 14 - 5x$), and 1R will be 2, which also differs greatly from each of the values above. Finally, if both 4R and 13+1R are equal to 7, there will also be an equality between 4R and 13+1R; and, with 1R removed from each side, 3R will be equal to 13; the

value of $1R$, therefore, will be 3. And so, since such varied values of $1R$ are found, the problem cannot be solved — especially since by none of the roots found is the problem satisfied, except that its first part [is satisfied] by the root 3.

6. *Let a number be sought whose square, together with the number 40, is equal to 12 roots of the same square.*

[Margin: 6. An impossible problem.]

Let the number be posited as $1R$ (x). Its square will be $1\mathfrak{z}$ (x^2). Therefore $1\mathfrak{z}+40$ ($x^2 + 40$) is equal to $12R$ ($12x$); that is, $1\mathfrak{z}$ will be equal to $12R-40$ ($x^2 = 12x - 40$). Take the half of the number of roots — namely 6 — and, since from the square of this half — that is, from 36 — the number 40 cannot be subtracted, the problem will be impossible.

7. But now, if it should ever happen that the equation found is between two cossic numbers that are equal and of the same denomination, the problem will be solvable by any number whatever.

[Margin: 7. A problem possible indeed, but trifling.]

[Note on notation: the cossist power-signs are rendered here with the printed tokens — R = *res* (the unknown, x); $\mathfrak{z}$ = *census/zensus* (x^2); $\mathfrak{C}$ = *cubus* (x^3); $\mathfrak{z}\mathfrak{z}$ = *zensizensus* (x^4); $\mathfrak{B}$ = *sursolidum* (x^5); $\mathfrak{z}\mathfrak{C}$ = *zensicubus* (x^6); and so on — with the numeral coefficient written immediately before the sign (so " $12R$ " = twelve *res*). The "second roots" are the letters A, B, C, D, each a distinct relative unknown; a cossic sign after the letter raises that second root ($A\mathfrak{z} = A^2$, $A\mathfrak{C} = A^3$), and a cossic sign before the letter multiplies by a power of the first unknown ($RA = res \times A$).]

As, for instance, if a number is sought which, multiplied by 2 and the product then multiplied into itself, makes a number equal to the one produced from the number multiplied into itself and then by 4.

[Margin: When a problem can be solved by any number.]

Let the number be $1R$ [*res*, the unknown]. From $1R$ multiplied by 2 comes $2R$; and from $2R$ multiplied into itself come $4\mathfrak{z}$ [*census*, x^2]. And since $1R$ multiplied into itself generates $1\mathfrak{z}$, and $1\mathfrak{z}$ multiplied by 4 gives $4\mathfrak{z}$, an equation will be found between $4\mathfrak{z}$ and $4\mathfrak{z}$ — which is trifling and useless. The problem, therefore, can be worked with any number. For example, from 10 multiplied by 2 comes 20, and from 20 multiplied into itself comes 400 — which is exactly what results from 10 multiplied into itself, and then that product, 100, multiplied by 4.

[Margin: Problem 8, possible but trifling.]

8. Again, let the number 10 be divided into two such parts that the number produced from one part multiplied by the whole number is equal to the square of that same part together with the number produced from that same part multiplied by the other.

Let the first part be $1R$, and therefore the other $10 - 1R$. Now since from the first part — that is, from $1R$ multiplied by the whole number 10 — comes $10R$, while that same part multiplied into itself makes 13 , and $1R$ multiplied by $10 - 1R$ makes $10R - 13$; these, added to 13 , give $10R$ (because for 13 to be added to -13 , a subtraction must be performed, as was said under addition). Therefore an equation is found between $10R$ and $10R$, which is likewise trifling; and what is proposed can be done in any number, however it is divided into two parts, as Euclid demonstrated in Book 2, Proposition 3 (*Elements* II.3).

But if the other part, $10 - 1R$, is multiplied by the whole number 10, it makes $100 - 10R$. And if that same part is multiplied into itself, it makes $100 - 20R + 13$. And the two parts, $1R$ and $10 - 1R$, multiplied together make $10R - 13$, which together with $100 - 20R + 13$ make $100 - 10R$ [recte: $100 - 10R$]; and these must be equal to $100 - 10R$ — namely to the product of the second part multiplied by 10. This equation too is trifling.

[Margin: Problem 9, trifling although possible.]

9. Again, let the number 10 be divided into two such parts that the products of the whole number 10 into those parts are together equal to the square of the whole number.

Let one part be $1R$, and therefore the other $10 - 1R$. From the whole number 10 multiplied into $1R$ and into $10 - 1R$ come $10R$ and $100 - 10R$, which numbers together make 100, as is clear from the rule of addition. And since 10 multiplied into itself makes 100 as well, there will be an equation between 100 and 100, which is useless. The problem, therefore, is solved by any number whatever divided into any two parts, as Euclid showed in Book 2, Proposition 2 (*Elements* II.2).

It is clear, then, that from the operation itself one can recognize whether a proposed problem can be solved or not — or indeed whether it is empty and trifling, that is, one that holds for every number.

ON SECOND ROOTS. CHAPTER XV.

[*De secundis radicibus. Caput XV.*]

[Margin: Why second roots were devised.]

Not infrequently two, or three, or even more numbers are met with [?] [opening verb uncertain — reads *-æruntur* after the decorative initial] under an uncertain proportion. For that reason, once 1R has been set down for the first number, it does not seem convenient to set down 1R again for the second, and 1R again for the third, and so on — because, since 1R placed in the first position usually has a value different from a root placed in the second position, and likewise from a root placed in the third, it would easily happen, in working an example, that the roots would be confused with one another unless they were marked by different characters. For this reason "second roots" were devised, which Girolamo Cardano [Cardano] calls *surd quantities*, and Pedro Nunes [Nunes] and others call *simple* or *absolute quantities*, marking them thus: 1q, 2q, etc. — that is, one quantity, two quantities, and so on. We, with others, shall call them second roots, and mark them thus: 1A, that is, 1R of a second [root], distinct from the one placed in the first position; 1B, that is, 1R of a second [root] distinct from the two placed in the first and second positions; and so on. For in this way the algorithm of second roots will be more easily explained.

In notation, when a number has two signs, the number is understood as the number with the earlier sign multiplied into the unit of the latter sign. Thus 1RA means 1R multiplied into 1A; and 3RA means 3R multiplied into 1A. So too 13A3 indicates 13 multiplied into 1A3, and so on.

[Margin: Addition of second roots.]

Addition is carried out by adding the numbers together and appending the same character of the second root — that is, when the second roots are of the same kind. Thus from 3A and 4A come 7A, and from 3B and 4B come 7B, etc. But when the second roots are not of the same kind, the addition is done by the sign of addition, +. Thus from 3A and 4B come 3A + 4B; and likewise from 3R and 4A come 3R + 4A, etc.

[Margin: Subtraction of second roots.]

Subtraction likewise, when the second roots are of the same kind, is done by subtracting number from number and appending the same character of the second root to the remaining number. Thus 3A from 7A leaves 4A, and 3B from 7B leaves 4B, etc. But when the second roots are different, the subtraction is done by the sign of subtraction, -. Thus 3A from 4B leaves 4B - 3A.

[Margin: Multiplication of second roots.]

Multiplications are done as follows. When a number of the first root is to be multiplied by a number of the second root marked only with the letter A, or B, etc., the numbers are multiplied together and the same signs appended. Thus from 2R into 2A come 4RA, that is, 4R multiplied into 1A. Similarly, from 2A into 2R come 4AR, that is, 4A multiplied into 1R. So too from 33 into 4B come 123B, that is, 123 multiplied into 1B, etc. So also from 1R into 1A comes 1RA, that is, 1R multiplied into 1A. And from 13 into 1A3 comes 13A3, that is, 13 multiplied into 1A3.

When an absolute number is multiplied into a number of a second root, a number of the second root results. Thus from 6 into 3C come 18C. And from 7 into 4B come 28B, etc.

When a number of a second root is multiplied into a number of a second root of a different letter, number is multiplied into number and the same letters are appended to the product. Thus from 3A into 9B come 27AB, that is, 27A multiplied into 1B.

When a number of a second root is multiplied into a number of a second root of the same letter, the character 3 is produced, with the same letter prefixed. Thus from 3A into 4A come 12A3.

When a number of a second root is multiplied into itself squarely, or cubically, and so on, the character 3, or C, and so on, is produced, with the same letter prefixed. Thus 1A squared makes 1A3, that is, the 1 square of the second root. Again, from 1RA into 1RA comes 13A3, that is, 13 of the first root multiplied into 13 of the second root. Likewise 3B squared makes 9B3; and so 3B cubed makes 27BC.

When a number of a second root is multiplied into another number of the same second root that also bears a cossic character, the first number is understood to bear as well the cossic sign R. Thus from 1A into 1A3 comes 1AC. And from 3B into 4BC come 12B33.

When a number bearing a cossic sign after the letter of a second root is multiplied into a number that likewise bears a cossic sign after the same letter of the second root, a number is produced with the cossic character that the exponents of the characters yield [i.e., the sum of the exponents], with the same letter of the second root prefixed besides. Thus from 2A3 into 5AC come 10Aß [10 *sursolidum* of A, A⁵]. Likewise from 3B3 into 4BC come 12Bß.

[Margin: Division of second roots.]

But when a number $\mathbb{R}$ is divided by a number of second roots, a fraction results in the quotient. Thus $2\mathbb{R}$ divided by $4\mathbb{A}$ gives $\frac{2\mathbb{R}}{4\mathbb{A}}$ [as a fraction].

But when the number does not have such a root, or the cossic character is not of the same denomination as the root to be extracted, the radical sign is prefixed to the whole number together with its letter and character. Thus the cube root of the number 3A3 is $\sqrt[3]{3A3}$; and the square root of the number 4A $\mathfrak{C}$ is $\sqrt{4A\mathfrak{C}}$.

The aforesaid multiplications and divisions are proved by geometric progressions beginning from 1. For the sake of clarity we will assign the progression of double ratios to the first roots, and the progression of triple ratios to the second roots — which are as follows:

N	R	3	¢	33	ß	3¢	Bß	333	¢¢	3ß	&c
1	2	4	8	16	32	64	128	256	512	1024	.

N	R	3	ē	33	ß	3ē	Bß	333	ēē	3ß	&c
1	3	9	27	81	24	729	218	6561	19683	5904	&c
					3		7			9	.

And so, to let you see it: from 2R into 2A results 4RA, that is, 4R multiplied into 1A. Resolved, 2R makes 4, and 2A makes 6; and 4 into 6 yields 24 — which is exactly what 4RA makes, that is, 8 [= 4R] multiplied into 1A, that is, into 3.

Next, from 2A into 2R we said comes 4AR, that is, 4A multiplied into 1R. And since 2A make 6 and 2R make 4, these become 24, from 6 into 4 — which is exactly what 4A makes, that is, 12, multiplied into 1R, namely into 2.

Moreover, from 1R into 1A is produced 1RA, since 1R is 2 and 1A is 3, so 6 results, from 2 into 3 — which is exactly what comes from 1R, that is, from 2, into 1A, that is, into 3.

So too, from 13 into 1A3 comes 13A3, namely 13 multiplied into 1A3. And since 13 is 4 and 1A3 is 9, 36 results, from 4 into 9 — which is exactly what comes from 13, that is, from 4, into 1A3, that is, into 9.

Again, from 33 into 4B are made 123B, that is, 123 multiplied into 1B. For 33 make 12, and 4B also make 12; and 144 results, from 12 into 12 — which is exactly what comes from 123, that is, from 48, into 1B, that is, into 3.

Likewise, from 6 into 3C come 18C — that is, from 6 into 9 come 54, that is, 18C. So from 7 into 4B are made 28B, namely 84 — which is exactly what comes from 7 into 4B, that is, into 12.

From 3A into 9B are produced 27AB, that is, 27A multiplied into 1B. For 3A make 9 and 9B make 27; and 243 results, from 9 into 27 — which is exactly what comes from 27A, that is, from 81, into 1B, that is, into 3.

Likewise, from 3A into 4A come 12A3, that is, 123 of the second root. For 3A make 9 and 4A make 12; and 108 results, from 9 into 12 — which is exactly what 123 of the second root make.

Moreover, from 1A squared comes 1A3, that is, 13 of the second root. Likewise 3B squared makes 9B3, that is, 93 of the second root. For 3B make 9, and from 9 into 9 come 81 — namely 93 of the second root. Likewise, from 1RA into 1RA comes 13A3, that is, 13 of the first root multiplied into 13 of the second root. For 1RA means 1R of the first [root] multiplied into 1 second root, that is, 2 multiplied into

3, which is 6. And from 6 into 6 comes the number 36 — which is exactly [what continues below].

[...continuing] — which is exactly what comes from 13 of the first root, that is, from 4, into the 1 square of the second root, namely into 9. So 3B cubed makes 27B℄, that is, 27℄ of the second root. Since 3B are 9, and 9 cubed makes 729 — which is exactly what 27℄ of the second root make.

After this, from 1A into 1A3 comes 1A℄, that is, the cube of the second root, namely 27. For 1A is worth 3, and 1A3 of the second root is worth 9; and clearly from 3 into 9 is generated 27.

So too, from 3B into 4B℄ come 12B33, that is, 1233 [*zensizensus*] of the second root. For 3B make 9, and 4B℄ of the second root [make] 108; and 972 results, from 9 into 108 — which is exactly what 1233 of the second root make.

Furthermore, from 2℄ into 4A3 come 8℄A3, that is, 8℄ multiplied into 13 of the second root. For 2℄ are 16, and 4A3 of the second root [are] 36; and 576 results, from 16 into 36 — which is exactly what 8℄ make, that is, 64, multiplied into 13 of the second root, namely into 9.

Likewise, from 1℄ into 1R&A3 comes 133A3, that is, 133 multiplied into 13 of the second root. For 1℄ is 8, and 1R&A3 — that is, 1R multiplied into 1A3 of the second root — is 18; so 144 results, from 8 into 18 — which is exactly what 133 makes, namely 16, multiplied into 13 of the second root, that is, into 9.

Likewise, from 2A3 into 5A℄ come 10Aß. For 2A3 of the second root are 18, and 5A℄ of the second root are 135; and 2430 results, from 18 into 135 — which is exactly what 10ß of the second root make. Likewise, from 3B3 into 4B℄ come 12Bß. For 3B3 of the second root make 27, and 4B℄ of the second root make 108; and 2916 results, from 27 into 108 — and just as much do 12ß of the second root make.

Finally, from 1A℄ into 13A comes 13A33, that is, 13 multiplied into 133 of the second root. For 1A℄ of the second root is 27, and 13 is 4; so 108 results, from 27 into 4; and from 108 into 1 second root — that is, into 3 — come 324. And just as much results from 13 into 133 of the second root, that is, from 4 into 81. And that from 1R&A3 squared there likewise comes 13A33 will become clear thus: the number 1R&A3 means 1R of the first [root], that is, 2, multiplied into 13 of the second root, that is, into 9 — which multiplication makes 18. But from 18 into 18 comes the number 324, which is equivalent to the number 13A33, that is, 13 of the first root multiplied into 133 of the second root, that is, into 81. Similarly,

from $3A^3$ into $4CA$ come $12CA^C$. For $3A^3$ of the second root are 27, and $4CA$ — that is, $4C$ multiplied into $1A$, or into a second root, that is, from 32 into 3 — make 96; and 2592 results, from 27 into 96. And just as much do $12CA^C$ make, that is, $12C$ multiplied into $1C$ of the second root, namely from 96 into 27.

Now we shall prove in the same way that from dividing $8CA^3$ by $4A^3$ the quotient $2C$ is produced. For $8CA^3$ — that is, $8C$ multiplied by $1A^3$, that is, 64 by 9 — make 576; and this number divided by $4A^3$ of the second root, namely by 36, gives the quotient 16, that is, $2C$. Likewise it is clear that from dividing $8CA^3$ by $4C$ the quotient $2A^3$ is generated. For $8CA^3$ — that is, $8C$ multiplied into $1A^3$, namely 64 into 9 — are 576; and this number divided by $4C$, that is, by 32, gives the quotient 18, namely $2A^3$ of the second root.

By the same reasoning, it is also evident that the square root of the number $25A^3$ is $5A$. For $5A$ are 15, and from 15 into 15 comes the number 225, to which 25^3 of the second root are equal. In no other way shall we show that $3A$ is the cube root of the number $27A^C$. For $3A$ are 9, and from 9 cubed come 729, namely $27C$ of the second root. So too the zensizensic [fourth] root of the number $16D^3$ will be $2D$. For $2D$ are 6, and from 6 raised zensizensically [to the fourth power] come 1296, namely 16^3 of the second root.

I have chosen to test all these operations on second roots by geometric progressions, so that their truth might be established for everyone — since perhaps more than one person might doubt them, their rationale being a little more obscure than that of the operations on cossic numbers. The use of second roots, moreover, will be explained in the solving of the riddles [problems] by the rule of algebra.

ON IRRATIONAL NUMBERS, OR SURDS. CHAPTER XVI.

[*De numeris irrationalibus, siue surdis. Caput XVI.*]

[Margin: What irrational numbers, or surds, are.]

Irrational numbers, or surds, are roots of numbers that cannot be expressed by numbers — and hence cannot be "heard" either; for which reason they are called "deaf" [surd] roots, and are marked in this way: $\sqrt{3}$, $\sqrt{C}$, $\sqrt{3^3}$, $\sqrt{B}$, $\sqrt{3C}$, etc., as we explained at the end of chapter 3. Thus $\sqrt{3} 10$ — that is, the square root of the number 10 — is called surd, or irrational, because no number can be given, whether whole or whole-with-fraction, which multiplied into itself squarely produces 10.

As for a whole number, this is evident. For 3 multiplied into itself produces 9, a number smaller than 10; but 4 multiplied into itself generates 16, a number larger than 10. And therefore neither 3 nor 4 — and hence no number smaller than 3 nor larger than 4 — can be the square root of the number 10.

As for a whole number with a fraction, the matter is also clear. For that root would have to lie between 3 and 4. But although between 3 and 4 there are infinitely many fractions which together with 3 make up a whole-with-fraction — such as $3\frac{1}{2}$, $3\frac{1}{4}$, $3\frac{1}{3}$, $3\frac{5}{7}$ [?], $3\frac{8}{11}$ [?], etc. — none of them multiplied into itself squarely can produce 10, a whole number, as we demonstrated in the Lemma at the end of Definition 8, Book 5, of Euclid (*Elements* V, def. 8). So too $\sqrt[3]{10}$ [the cube root of 10] is irrational, and $\sqrt[3]{100}$ [the sixth root of 100], and so on.

These numbers must now be treated, so that the very many problems in which irrational numbers occur can be explained. It is indeed a subject wrapped in no few difficulties, yet one that can be plainly understood, provided only that intense and eager study is applied. But if anyone finds himself less suited to grasping this matter perfectly, let him not despair altogether — for without this knowledge almost countless problems can be worked out by the rule of algebra. Therefore, setting aside for a short time this treatment of irrational numbers, which is contained in 13 chapters, he may turn to chapters 29 and 30, where he will find almost countless riddles, most delightful to know, worked out without any knowledge of irrational numbers. Then, after he has exercised himself for a while in the problems of algebra, he will be able to return to master this treatment of irrational numbers.

Cossist sign legend (as printed; radical order given at first use): $\sqrt{3}$ = square (zensic) root [$\sqrt{3}$ = census/zensus, the square]; $\sqrt[3]{\epsilon}$ = cube root [ϵ = cubus]; $\sqrt[3]{33}$ = fourth root [33 = zensizensus]; $\sqrt[3]{\beta}$ = fifth root [β = surdesolidus, the sursolid]; $\sqrt[3]{3\epsilon}$ = sixth root [3ϵ = zensicubus]; $\sqrt[3]{33\epsilon}$ = twelfth root; $\sqrt[3]{3\epsilon\epsilon}$ = eighteenth root; $\Re$ = res, the cossic unknown; $\sqrt{0}$ = a root of exponent 0, i.e. the absolute number itself. The signs are transcribed with the concatenated power-tokens exactly as they stand in the source.

Chapter XVI [continued]

Before anything else, some writers are accustomed to dispute whether numbers that are called irrational really deserve to be called numbers or not. [Margin: Whether surd roots are numbers or not.] For since they are neither whole numbers nor fractions — as we said a little earlier — and cannot be expressed in numbers that would bear to whole or fractional numbers any fixed ratio, they do

not seem entitled to be called numbers. Yet because they fall under a rule, they must be reckoned to be numbers in some fashion. For $\sqrt{3}10$ multiplied by itself produces 10, just as the square root of 9 yields 9. Likewise $\sqrt{3}3$ multiplied by $\sqrt{3}12$ produces $\sqrt{3}36$, that is, 6 — in the same way that 6 itself arises from 2 times 3, as will be clear from the algorithm of irrational numbers, which we shall treat shortly.

Irrational numbers, moreover, have a certain kinship with fractional numbers. For just as between any two whole numbers infinitely many fractional numbers fall — as appears here, where between 2 and 3 all these fractions fall:

$2\frac{1}{2}$. $2\frac{1}{3}$. $2\frac{2}{3}$. $2\frac{1}{4}$. $2\frac{3}{4}$. $2\frac{1}{5}$. $2\frac{2}{5}$. $2\frac{3}{5}$. $2\frac{4}{5}$. $2\frac{1}{6}$. $2\frac{5}{6}$. &c.

so too between any two whole numbers infinitely many irrational numbers fall. Thus between 2 and 3 all these fall:

$\sqrt{35}$. $\sqrt{36}$. $\sqrt{37}$. $\sqrt{38}$. $\sqrt{39}$. $\sqrt{40}$. $\sqrt{41}$. $\sqrt{42}$. $\sqrt{43}$. $\sqrt{44}$. $\sqrt{45}$. $\sqrt{46}$. $\sqrt{47}$. $\sqrt{48}$.
 $\sqrt{49}$. $\sqrt{50}$. $\sqrt{51}$. $\sqrt{52}$. $\sqrt{53}$. $\sqrt{54}$. $\sqrt{55}$. $\sqrt{56}$. $\sqrt{57}$. $\sqrt{58}$. $\sqrt{59}$. $\sqrt{60}$.
 $\sqrt{61}$. $\sqrt{62}$. $\sqrt{63}$. &c.

For each of those fractions, and each of these surds, is greater than 2 but less than 3, as is evident.

Writers put forward two kinds of surd roots. [Margin: Two kinds of surd roots, simple and composite.] Some are *simple*, such as $\sqrt{3}$ of a number that is not a square, $\sqrt{4}$ of a number that is not a cube, $\sqrt{5}$ of a number that is not a fourth power [zensizensus], $\sqrt{6}$ of a number that is not a fifth power [surdesolidus], and so on for the other roots of numbers whose roots cannot be expressed in numbers. Some call these simple roots *medial numbers*, because through them are found the mean proportionals between two numbers, as we shall explain in Chapter 21. [Margin: What the medial numbers are.] Other surd roots are *composite*, formed by the interposed signs + and −; these too are described by writers under two kinds. Under the first are contained those they call *bound* [Ligatae], namely when we understand a sum collected from several roots, or from one root or more together with some number, or with a cossic number. [Margin: What the bound root is.]

For example, by the number $\sqrt{39}+\sqrt{34}$ we understand 5 units — that is, the sum collected from the root 9 [read: the square root of 9] and the square root of the number 4; and such a number is called a bound root, which is equivalent to the number 5. In the same way a bound root will be $\sqrt{37}+\sqrt{34}+3$, which will be equivalent to this: $\sqrt{37}+5$ (for we signify that 5 is added to $\sqrt{37}$). By the same

method a bound root will be $\sqrt{33+2R}$ [$R = \text{res}$, the cossic unknown], for we denote that $2R$ [2 res] are added to $\sqrt{33}$. Again, a bound root will be $\sqrt{349-\sqrt{27}}$, and likewise $\sqrt{100-\sqrt{39}}$. In the former we understand that from $\sqrt{349}$ — that is, from 7 — has been taken $\sqrt{27}$, that is, 3; and so its value will be 4. In the latter we intend that from $\sqrt{100}$ there be subtracted $\sqrt{39}$, namely 3.

But these bound roots we shall henceforth call *composite* or *diminished* irrational numbers; indeed, for the sake of brevity, we shall in what follows generally call composite [numbers those made by addition, and diminished those made by subtraction].

The other kind of composite roots embraces the surd roots called *universal* [Vniuersales]: namely when the root is understood to be taken of some bound root, or of a composite or diminished number. [Margin: What the universal root is.] For instance, the universal root of the composite number $22+\sqrt{39}$ is 5. For the sense is that to 22 be added the square root of 9, which is 3, and that the root be taken of the sum 25, which is 5. In the same way, if the parts of the same number be inverted so that the number becomes $\sqrt{39}+22$, its universal root will be the same, namely 5. [Margin: The root of the whole composite (or diminished) number.]

These universal roots we shall call simply the roots of composite or diminished irrational numbers; and therefore before such numbers we shall place the radical sign $\sqrt{3}$ or $\sqrt{\text{C}}$. And to signify that the sign refers to the whole composite number, we shall enclose the composite number between two parentheses, in this way: $\sqrt{3}(22+\sqrt{39})$, or, what is the same thing, $\sqrt{3}(\sqrt{39}+22)$. Take these other examples of roots of this kind: $\sqrt{3}(\sqrt{35}+12R)$; likewise $\sqrt{3}(48+12R)$; likewise $\sqrt{3}(\sqrt{39}+\sqrt{34}) + \sqrt{3}(11+\sqrt{325})$. The sense of this last root is that one take the sum made up of $\sqrt{39}$ and $\sqrt{34}$ — namely of 3 and 2, which is 5 — and to this sum add the root of the aggregate of 11 and $\sqrt{325}$, that is, of the aggregate 16, which is 4, so that the whole sum becomes 9, and its root 3.

But if those two composite numbers be inverted, in this way — $\sqrt{3}(11+\sqrt{325}) + \sqrt{3}(\sqrt{39}+\sqrt{34})$ — the sense will change; for the radical sign $\sqrt{3}$ placed at the beginning appears to refer to the whole of what follows it. Thus 11 is taken together with 5, the root of 25, to make 16; to these are added $\sqrt{39}$ and $\sqrt{34}$, namely 5, so that the whole sum becomes 21, and its root $\sqrt{321}$ — which differs greatly from the former, which was 3.

Finally, the root of this number $\sqrt{3}(7+\sqrt{348})$ will be $2+\sqrt{33}$. For this number multiplied by itself produces $7+\sqrt{348}$, as will be clear from what follows.

There is also another kind of root among certain writers, which they call *Distinct* [Distincta], in which the parts are taken separately and not their sum. [Margin: What the distinct root is.] For example, this root $\sqrt[3]{16} + \sqrt[3]{9}$, when the parts are taken by themselves — as 4 and 3, and not joined together as 7 (for so it would be a bound root, such as we called simply a composite irrational number) — is called a Distinct root; and it differs greatly from the bound root. For if it be taken as bound and multiplied by itself, the number 49 will result; but taken as Distinct, the numbers 16 and 9 will result, which together make only 25. But we shall explain in the proper places what the sense of these roots is; for writers speak variously on this matter. You will find more on the same subject in Chapter 24.

On the Reduction of Simple Roots to the Same Denomination

Chapter XVII

For simple surd roots to be capable of being multiplied and divided by one another, it is necessary — if they are of different denomination — to reduce them to the same denomination. [Margin: By what method roots of different denomination are reduced to the same denomination.] This reduction is carried out in nearly the same way as fractions of different denominators are brought to the same denominator. For once the radical signs have been placed beneath the numbers to which they belong, the numbers themselves are to be multiplied *crosswise* into the radical signs, so that new numbers are generated; and then the signs themselves are to be added together — that is, their exponents are likewise to be multiplied together — so that a common sign, or the exponent of the common sign, is produced.

For example, these two roots $\sqrt[3]{5}$ and $\sqrt[4]{4}$ are in this way brought to the same radical sign. Once the numbers and the signs have been set down, as has been said and as the figure indicates, the number 5 is to be multiplied *cubically*, on account of the sign $\sqrt[3]{}$, so that a new number 125 arises; while the number 4 is to be multiplied *zensically* (squared), on account of the sign $\sqrt[4]{}$, so that a new number 16 is generated. Then the sign $\sqrt[3]{}$ is to be added to the sign $\sqrt[4]{}$, so that the sign $\sqrt[3\&4]{}$ arises, whose exponent is of course 6, produced from multiplying 2, the exponent of $\sqrt[3]{}$, into 3, the exponent of $\sqrt[4]{}$, as we taught in Chapter 2. This resulting sign $\sqrt[3\&4]{}$ is to be prefixed to each of the numbers previously produced; so that $\sqrt[3]{5}$ and $\sqrt[4]{4}$ are reduced to $\sqrt[3\&4]{125}$ and $\sqrt[3\&4]{16}$ [$\sqrt[3\&4]{}$ = sixth root, zensicubus]. Once this is done, $\sqrt[3\&4]{125}$ will be equal to $\sqrt[3]{5}$, and $\sqrt[3\&4]{16}$ equal to $\sqrt[4]{4}$. For $\sqrt[3]{5}$ is the $\sqrt[3\&4]{}$ of

the number 125, because $\sqrt[3]{5}$ multiplied into itself squarely makes 5, and 5 multiplied into itself cubically makes 125. By the same reasoning $\sqrt[3]{4}$ is the $\sqrt[3]{\text{C}}$ of the number 16, because $\sqrt[3]{\text{C}}$ multiplied into itself cubically makes 4, and 4 multiplied into itself squarely makes 16 — which is also evident from this: since $\sqrt[3]{\text{C}}$, multiplying $\sqrt[3]{3}$ and 5, produced $\sqrt[3]{3\text{C}}$ and 125 [Margin: *17 Sept.*], the ratio of $\sqrt[3]{3\text{C}}$ to 125 will be the same as that of $\sqrt[3]{3}$ to 5. So too, since $\sqrt[3]{3}$, multiplying $\sqrt[3]{\text{C}}$ and 4, makes $\sqrt[3]{3\text{C}}$ and 16, the ratio of $\sqrt[3]{3\text{C}}$ to 16 will be the same as that of $\sqrt[3]{\text{C}}$ to 4; and therefore $\sqrt[3]{35}$ will be equal to $\sqrt[3]{3\text{C}}125$, and $\sqrt[3]{\text{C}}4$ equal to $\sqrt[3]{3\text{C}}16$.

[Margin: Demonstration of the reduction.] But that roots are correctly reduced to the same denomination by the aforesaid method — that is, in the given example, that $\sqrt[3]{35}$ is $\sqrt[3]{3\text{C}}125$ and that $\sqrt[3]{\text{C}}4$ is equivalent to $\sqrt[3]{3\text{C}}16$ — we shall also show geometrically thus:

Diagram. A cross ("X") schematic for the crosswise reduction of $\sqrt[3]{35}$ and $\sqrt[3]{\text{C}}4$ to a common (sixth-root) denomination. Above the two arms of the X stand the products (with rule-lines beneath); below the arms stand the original numbers and their radical signs; the product of the exponents ($2 \times 3 = 6$) stands at the foot.

- Upper left: 125 (= 5^3 , over a rule-line)
- Upper right: 16 (= 4^2 , over a rule-line)
- Left arm, above: 5
- Right arm, above: 4
- Left arm, below: $\sqrt[3]{3}$
- Right arm, below: $\sqrt[3]{\text{C}}$
- Foot, left: 2 — right: 3
- Foot, centre: 6

Lettered demonstration (first example). [Values assigned to the letters, as set out on the page with the group-headings C = 2, F = 3 and K = 6 below:]

A	B	G	C	D	E	H	F	K
$\sqrt[3]{35}$	5	125	2	$\sqrt[3]{\text{C}}4$	4	16	3	6

Let A be the zensic root of the number B, the exponent of whose root is C. Likewise let D be the cubic root of the number E, whose exponent is F. Let B — the square of this number $\sqrt[3]{35}$ — be multiplied into itself cubically (on account of the exponent F, or the sign C ; for thus the multiplication is made crosswise), so that the number 125 arises, so that B is $\sqrt[3]{\text{C}}$ of the produced number 125, whose exponent is F. Likewise let E — the cube of this number $\sqrt[3]{\text{C}}4$ — be multiplied into itself squarely, on account of the sign 3, or the exponent C, so that the number 16

arises, so that E is $\sqrt[3]{3}$ of the produced number 16, whose exponent is C. Finally let the exponent C be multiplied into the exponent F, and let the number K be made. I say that A is the $\sqrt[3]{3}\mathfrak{C}$ of the number G — namely, denominated from 6 — and that D is the $\sqrt[3]{3}\mathfrak{C}$ of the number H, denominated from the same number K.

For since A is the root of B, denominated from C, there will be between B and unity as many ratios equal to the ratio of A to unity as there are units in C — namely two, as was explained in Chapter 2, and as is here evident: 1. $\sqrt[3]{5}$. 5. Again, since B is the root of G, denominated from F, there will for the same reason be between G and unity as many ratios equal to the ratio of B to unity as there are units in F — namely three, as here too appears: 1. 5. 25. 125. That is, between G and B there will be two ratios equal to the ratio of B to unity, since between G and unity there are three ratios equal to the ratio of B to unity. Since, then, between B and unity there are two ratios equal to the ratio of A to unity, there will be six such ratios between G and unity — as many as there are units in K, a number generated by multiplying the two ratios between B and unity into the three ratios between G and unity, that is, C into F. And therefore G will be a zensicube [zensicubus], whose exponent is 6; that is, A will be the $\sqrt[3]{3}\mathfrak{C}$ of the number G. Not otherwise shall we demonstrate that D is the $\sqrt[3]{3}\mathfrak{C}$ of the number H. And the same reasoning holds for the others, as in this second example of rational numbers.

Lettered demonstration (second example). [Group-headings C = 3, F = 4 and K = 12 below:]

A	B	G	C	D	E	H	F	K
2	8	4096	3	3	81	531441	4	12

A is the $\sqrt[3]{3}\mathfrak{C}$ of the number B, denominated from C. But D is the $\sqrt[3]{3}\mathfrak{C}$ of the number E, denominated from F. From B, multiplied into itself zensizensically [to the fourth power], on account of the exponent F, let G be made, so that B is the $\sqrt[3]{3}\mathfrak{C}$ of the number 4096. Likewise from E, multiplied into itself cubically, on account of the exponent C, let H be made, so that E is the $\sqrt[3]{3}\mathfrak{C}$ of the number H. And finally from C into F let K be made. I say that A is the $\sqrt[3]{3}\mathfrak{C}$ of the number G, denominated from K — that is, from the number 12 — and that D is the $\sqrt[3]{3}\mathfrak{C}$ of the number H, denominated from the same number 12.

For since A is the root of B, denominated from C, there will be between B and unity as many ratios equal to the ratio of A to unity as there are units in C — namely three, as here appears: 1. 2. 4. 8. Again, since B is the root of G, denominated from F, there will be between G and unity as many ratios — each

equal to the ratio of B to unity — as there are units in F, namely four, as here appears: 1. 8. 64. 512. 4096. That is, between G and B there will be three ratios equal to the ratio of B to unity, since between G and unity there are four ratios equal to the ratio of B to unity. Since, then, between B and unity there are three ratios equal to the ratio of A to unity, there will be between G and unity twelve ratios equal to the ratio of A to unity — as many as there are units in the number K, namely 12, a number generated by the multiplication of the three ratios between B and unity into the four ratios between G and unity, that is, of C into F. And therefore G will be a $\sqrt[3]{336}$, whose exponent is 12; that is, A will be the $\sqrt[3]{336}$ of the number G. And by the same reasoning D will be the $\sqrt[3]{336}$ of the number H.

Again, let two numbers be brought back to a common denomination: 6 — that is, $\sqrt[0]{6}$ (since the exponent of an absolute number is 0) — and $\sqrt[3]{3312}$. [Margin: In what manner an absolute number and a root are reduced to the same denomination.] Let 6 be multiplied into itself zensizensically, so that 1296 arises. But 12 is not to be multiplied into $\sqrt[0]{6}$, on account of the exponent 0; nor is 4, the exponent of $\sqrt[3]{33}$, to be multiplied into 0, for the same reason. Therefore 6 and $\sqrt[3]{3312}$ will be reduced to $\sqrt[3]{331296}$ and $\sqrt[3]{3312}$.

Diagram. A cross ("X") schematic for reducing the absolute number 6 and the root $\sqrt[3]{3312}$ to a common denomination. The absolute number, having exponent 0, is written under the sign $\sqrt[0]{6}$; only it is raised (to the fourth power) while the surd $\sqrt[3]{3312}$ is left unchanged.

- Upper left: 1296 (= 6^4 , over a rule-line)
- Upper right: 12 (over a rule-line)
- Left arm, above: 6
- Right arm, above: 12
- Left arm, below: $\sqrt[0]{6}$
- Right arm, below: $\sqrt[3]{33}$
- Foot, centre: 4

And so, when an absolute number and some root are proposed, only the absolute number is to be multiplied into itself, as the character of the root requires. Thus in the preceding example the absolute number 6, multiplied into itself zensizensically, was reduced to $\sqrt[3]{331296}$ and $\sqrt[3]{3312}$.

Chapter XVII [continued]

[Compendious method.] [Margin: A shortcut for the reduction.] Reductions of this kind can sometimes be made more compendiously, by *discarding* the radical

sign — namely when each number has the root that its radical sign indicates. For then that root is extracted, and its radical sign is deleted. For example, if these two roots $\sqrt[3]{27}$ and $\sqrt[4]{25}$ are to be brought to the same denomination, I extract from 25 the zensic root, 5, and prefix to it the remaining sign 3, in this way: $\sqrt[3]{5}$. Then from 27 I extract the cubic root, 3, which I multiply zensically, so that $\sqrt[3]{9}$ arises. The two given roots have therefore been reduced to these: $\sqrt[3]{9}$ and $\sqrt[3]{5}$.

If, on the other hand, the cubic root is first extracted from 27, namely 3, and the sign $\sqrt[3]{}$ is deleted, an absolute number 3 remains. Then, once the zensic root has been extracted from 25 — namely 5 — I prefix to it the remaining sign 3, in this way: $\sqrt[3]{5}$. But since the number found first was the absolute number 3, it is to be multiplied into itself zensically, so that $\sqrt[3]{9}$ arises; and thus again as before.

Without the shortcut, they would be reduced to these two: $\sqrt[3]{3^4 5^3 1441}$ and $\sqrt[3]{3^4 15625}$, which are finally reduced to $\sqrt[3]{9}$ and $\sqrt[3]{5}$ in this way: I extract from 531441 the cubic root, 81, and from this the zensizensic root, 3, to which I prefix no radical sign, since the number had $\sqrt[3]{3^4}$. Then from 15625 I extract the cubic root, 25, and from this the zensic root, 5, to which I prefix the remaining sign 3, in this way: $\sqrt[3]{5}$. But since the number found first was the absolute number 3, let 3 be multiplied into itself zensically, so that $\sqrt[3]{9}$ arises; and thus the reduction will have been made to $\sqrt[3]{9}$ and $\sqrt[3]{5}$, as before.

Diagram. A cross ("X") schematic for the (uncompensious) reduction of $\sqrt[3]{27}$ and $\sqrt[4]{25}$ to a common denomination. The exponents (3 and 4) are cross-multiplied; each number is raised accordingly ($27^4 = 531441$, $25^3 = 15625$).

- Upper left: 531441 (= 27^4 , over a rule-line)
- Upper right: 15625 (= 25^3 , over a rule-line)
- Left arm, above: 27
- Right arm, above: 25
- Left arm, below: $\sqrt[3]{}$
- Right arm, below: $\sqrt[4]{}$

So too, if $\sqrt[3]{1024}$ and $\sqrt[4]{216}$ are to be reduced: I extract from 1024 the surdesolid [fifth] root, 4, and, the character β being deleted, $\sqrt[3]{4}$ remains. Then, having extracted the zensic root from 4, we shall have 2. Likewise from 216 I extract the cubic root, 6, to which I prefix the sign 3. But since we had before the absolute number 2, we shall multiply 2 into itself zensically, so that $\sqrt[3]{4}$ arises. The said roots will therefore be reduced to $\sqrt[3]{4}$ and $\sqrt[3]{6}$. And in this way a shortcut is made by discarding some radical sign.

[By assumption of a sign.] Sometimes a shortcut can also be made by *assuming* some radical sign. For example, if $\sqrt[3]{2}$ and $\sqrt[3]{6}$ are to be reduced, I multiply 2 into itself zensically, on account of the sign 3 (which, besides 3, is found in the other root), and I make 4, to which I prefix 3; and the reduction will have been made to $\sqrt[3]{4}$ and $\sqrt[3]{6}$. Without the shortcut the reduction would be made, as you see here, to $\sqrt[3]{64}$ and $\sqrt[3]{216}$ [$\sqrt[3]{64} = \sqrt[3]{2^6}$ = eighteenth root]. And if from these numbers the cubic roots, 4 and 6, are extracted, and the signs 3 deleted one from each, the reduction will have been made to $\sqrt[3]{4}$ and $\sqrt[3]{6}$, as before. In the same way $\sqrt[3]{2}$ and $\sqrt[3]{8}$ will be reduced to $\sqrt[3]{4}$ and $\sqrt[3]{8}$, if namely 2 be multiplied into itself zensically, and to the product be added the same radical sign 3, with the sign 3 moreover prefixed.

Diagram. A cross ("X") schematic for the (uncompensious) reduction of $\sqrt[3]{2}$ and $\sqrt[3]{6}$, cross-multiplying the exponents (3 and 6) so that each number is raised to the eighteenth-root denomination ($2^6 = 64$, $6^3 = 216$).

- Upper left: 64 (over a rule-line)
- Upper right: 216 (over a rule-line)
- Left arm, above: 2
- Right arm, above: 6
- Left arm, below: $\sqrt[3]{}$
- Right arm, below: $\sqrt[3]{}$

[General rule.] [Margin: An easy rule for reducing certain roots.] Finally, to set still further examples before your eyes: whenever the same character appears in both roots, those roots will readily be reduced to the same denomination, if the number of the smaller denomination be multiplied into itself according to the requirement of the other character (the one found in the larger denomination), and to the product be added that character according to which the multiplication was made. For example, $\sqrt[3]{3}$ and $\sqrt[3]{16}$ are reduced to $\sqrt[3]{9}$ and $\sqrt[3]{16}$, if namely the number 3, of the smaller denomination, be multiplied into itself zensically, on account of the character 3 which, besides the [shared] 3, is found in the larger denomination.

Likewise, $\sqrt[3]{8}$ and $\sqrt[3]{9}$ are reduced to $\sqrt[3]{8}$ and $\sqrt[3]{729}$, if namely the number 9, of the smaller denomination, be multiplied into itself cubically, on account of the character 3 which, besides 3, is contained in the larger denomination.

So too, $\sqrt[3]{20}$ and $\sqrt[3]{10}$ are reduced to $\sqrt[3]{20}$ and $\sqrt[3]{1000000}$, if namely the number 10, of the smaller denomination, be multiplied into itself zensicubically, on account of the character $\sqrt[3]{}$ which, besides 3, is found in the larger

denomination — that is, either first into itself zensically, and the product 100 into itself cubically; or first into itself cubically, and the product 1000 into itself zensically. For by either method the number 1000000 is produced.

In the same way, too, $\sqrt[3]{3}\sqrt[3]{30}$ and $\sqrt[3]{3}\sqrt[3]{15}$ are reduced to $\sqrt[3]{3}\sqrt[3]{30}$ and $\sqrt[3]{3}\sqrt[3]{225}$, if namely the number 15, of the smaller denomination, be multiplied into itself zensically, and to the product 225 there be added, besides the character $\sqrt[3]{}$, the character 3 also; and so for the others.

And therefore the larger character always remains untouched, and only the smaller is changed, by assuming another character besides the one it has — the character, I say, according to whose requirement the number of the smaller denomination was multiplied, and which is found in the larger denomination beyond the character of the smaller denomination.

On the Multiplication and Division of Simple Roots

Chapter XVIII

That the order of doctrine may be preserved, we must treat the multiplication and division of simple roots before their addition and subtraction, because without multiplication the addition and subtraction of roots cannot be carried out. [Margin: Multiplication and division of simple roots, how it is done.] When, therefore, two roots of the same kind are to be multiplied or divided, their numbers are multiplied or divided by one another, and to the resulting number the same radical sign is prefixed.

Thus from the product of $\sqrt[3]{37}$ into $\sqrt[3]{310}$ comes $\sqrt[3]{370}$. From $\sqrt[3]{33}$ into $\sqrt[3]{312}$ comes $\sqrt[3]{336}$, that is, 6. From $\sqrt[3]{32\frac{1}{4}}$ into $\sqrt[3]{38}$ comes $\sqrt[3]{318}$. From $\sqrt[3]{24}$ into $\sqrt[3]{10}$ comes $\sqrt[3]{240}$. From $\sqrt[3]{4}$ into $\sqrt[3]{16}$ comes $\sqrt[3]{64}$, that is, 4.

[Cap. XVIII, continued] — Multiplication and division of surds of different natures

[Notation. Throughout these pages the radical sign is written $\sqrt{}$ followed by a cossist power-sign that names the *degree* of the root: $\sqrt{3}$ = square root (*radix quadrata*, degree 2); $\sqrt[3]{}$ = cube root (*radix cubica*, degree 3); $\sqrt[3]{3}$ = fourth root (*radix zensizensica*, degree 4); $\sqrt[3]{3}$ = fifth or sursolid root (*radix sursolida*, degree

5); $\sqrt[3]{\text{C}}$ = sixth root (*radix zensicubica*, degree 6); $\sqrt[3]{3\text{C}}$ = twelfth root. The radicand is printed after the sign.]

The following worked examples continue the multiplication and division of surds of the same denomination:

- From $\sqrt[3]{9}$ [fourth root of 9] times $\sqrt[3]{8}$ comes $\sqrt[3]{72}$.
- From $\sqrt[5]{4}$ [fifth root of 4] times $\sqrt[5]{8}$ comes $\sqrt[5]{32}$ — that is, 2. [Since $4 \times 8 = 32 = 2^5$.]
- Likewise from the division of $\sqrt[5]{70}$ by $\sqrt[5]{7}$ comes $\sqrt[5]{10}$.
- From $\sqrt[5]{36}$ by $\sqrt[5]{12}$ comes $\sqrt[5]{3}$.
- From $\sqrt[5]{18}$ by $\sqrt[5]{8}$ comes $\sqrt[5]{2\frac{1}{4}}$ — that is, $\sqrt[5]{\frac{9}{4}}$.
- From $\sqrt[3]{240}$ [cube root of 240] by $\sqrt[3]{24}$ comes $\sqrt[3]{10}$.
- From $\sqrt[3]{64}$ by $\sqrt[3]{4}$ comes $\sqrt[3]{16}$.
- From $\sqrt[3]{72}$ by $\sqrt[3]{8}$ comes $\sqrt[3]{9}$.
- From $\sqrt[5]{32}$ by $\sqrt[5]{8}$ comes $\sqrt[5]{4}$.

When, however, two proposed roots are not of the same nature, they must first be reduced to the same denomination; then the multiplication or division is carried out as before.

Thus from $\sqrt[5]{8}$ [i.e. $\sqrt[3]{8}$, square root of 8] multiplied by $1\frac{1}{2}$ comes $\sqrt[3]{18}$. For the number $1\frac{1}{2}$, squared, makes $\sqrt[3]{2\frac{1}{4}}$ — that is, $\sqrt[3]{\frac{9}{4}}$ — and so from $\sqrt[3]{8}$ times $\sqrt[3]{\frac{9}{4}}$ comes $\sqrt[3]{\frac{72}{4}}$, that is, $\sqrt[3]{18}$.

- From $\sqrt[3]{2}$ [cube root of 2] times $\sqrt[3]{6}$ comes $\sqrt[3]{12}$. For those two roots are reduced to these two, $\sqrt[3]{4}$ and $\sqrt[3]{6}$. [$2 \times 6 \rightarrow$ the radicand product route via $\sqrt[3]{4 \cdot 6}$.]
- From $\sqrt[5]{5}$ [fifth root of 5] times $\sqrt[3]{4}$ comes $\sqrt[3]{\text{C} 2000}$. For the two given roots are recalled to these, $\sqrt[3]{\text{C} 125}$ and $\sqrt[3]{\text{C} 16}$. [5 raised to make a sixth root gives $5^3 = 125$; 4 gives $4^3 = 64$... — see note.]

Likewise **for division**: from $\sqrt[3]{\text{C} 18}$ by $1\frac{1}{2}$ comes $\sqrt[3]{\text{C} 8}$. For the number $1\frac{1}{2}$, squared, makes $2\frac{1}{4}$, to which the sign 3 must be prefixed, so that it becomes $\sqrt[3]{3 2\frac{1}{4}}$; and thus $\sqrt[3]{\text{C} 18}$ divided by $\sqrt[3]{3 2\frac{1}{4}}$ — that is, by $\sqrt[3]{\frac{9}{4}}$ — gives the quotient $\sqrt[3]{8}$.

- From $\sqrt[3]{\text{C} 24}$ by $\sqrt[3]{2}$ comes the quotient $\sqrt[3]{\text{C} 6}$. For those two roots are reduced to these, $\sqrt[3]{\text{C} 24}$ and $\sqrt[3]{\text{C} 4}$: namely, the number 2 is squared, and to the product 4 — besides the sign C — the sign 3 is also appended; and so $\sqrt[3]{\text{C} 24}$ divided by $\sqrt[3]{\text{C} 4}$ gives the quotient $\sqrt[3]{\text{C} 6}$.

- From $\sqrt[3]{2000}$ by $\sqrt[3]{5}$ comes the quotient $\sqrt[3]{4}$. For those two roots are reduced to $\sqrt[3]{4000000}$ and $\sqrt[3]{15625}$. For 2000 squared makes 4000000; and 5 raised zensicubically [to the sixth power] makes 15625. To these products the common sign $\sqrt[3]{}$ is prefixed, whose exponent 12 arises from the exponent 6 of the sign $\sqrt[3]{}$ multiplied by the exponent 2 of the sign $\sqrt[3]{}$. And so if you divide $\sqrt[3]{4000000}$ by $\sqrt[3]{15625}$, you will get the quotient $\sqrt[3]{256}$ — that is, $\sqrt[3]{4}$. For the fourth root [$\sqrt[4]{}$] of the number 256 is 4, whose cube root [$\sqrt[3]{}$] is $\sqrt[3]{4}$.
- From $\sqrt[3]{3/4}$ by $\sqrt[3]{2/3}$ comes the quotient $\sqrt[3]{9/8}$ — that is, $\sqrt[3]{1\frac{1}{8}}$. For from the product of the quotient $\sqrt[3]{9/8}$ into the divisor $\sqrt[3]{2/3}$ comes $\sqrt[3]{3/4}$, the number divided. [$3/4 \div 2/3 = 9/8$; and $9/8 \times 2/3 = 3/4$.]

Multiplying a surd into itself squarewise, cubewise, zensizensically, etc.

[Margin: Multiplication of roots into themselves squarewise, cubewise, zensizensically, etc.]

When a surd is multiplied by itself squarewise or cubewise, according to what the requirement of its own radical sign demands, the number itself is to be taken as the product, with the radical sign deleted.

- Thus $\sqrt[3]{4}$ raised to the second power makes 4. For $\sqrt[3]{4}$ times $\sqrt[3]{4}$ makes $\sqrt[3]{16}$, that is, 4.
- $\sqrt[3]{18}$ raised to the second power makes 18 — that is, $\sqrt[3]{324}$, which is 18.
- $\sqrt[4]{4}$ raised to the third power makes 4. For $\sqrt[4]{4}$ times $\sqrt[4]{4}$ makes $\sqrt[4]{16}$, and $\sqrt[4]{4}$ times $\sqrt[4]{16}$ makes $\sqrt[4]{64}$, that is, 4; and the cube root of this product is $\sqrt[4]{4}$.
- Likewise $\sqrt[3]{16}$ raised to the fourth power (zensizensically) makes 16. For $\sqrt[3]{16}$ times $\sqrt[3]{16}$ gives $\sqrt[3]{256}$; and $\sqrt[3]{256}$ times itself gives $\sqrt[3]{65536}$ — namely 16, since the square root of 65536 is 256, and the square root of this is 16.

But when a surd is multiplied into itself according to the requirement of *another* radical sign [i.e. raised to a power lower than its own degree], the radical sign of the multiplied number is to be prefixed to the product.

- Thus from $\sqrt[3]{6}$ cubed comes $\sqrt[3]{216}$; whence the cube root of this product will be $\sqrt[3]{6}$. For $\sqrt[3]{6}$ times itself gives $\sqrt[3]{36}$, and $\sqrt[3]{6}$ times $\sqrt[3]{36}$ gives $\sqrt[3]{216}$.

- From $\sqrt[4]{8}$ raised to the second power comes $\sqrt[4]{64}$ — that is, 4. Whence the square root of the number $\sqrt[4]{64}$ is $\sqrt[4]{8}$, that is, 2.
- From $\sqrt[4]{6}$ raised to the second power comes $\sqrt[4]{36}$. And thus the square root of the number $\sqrt[4]{36}$ is $\sqrt[4]{6}$.
- From $\sqrt[3]{6}$ raised to the second power comes $\sqrt[3]{36}$ — that is, $\sqrt[4]{6}$. For the square root [*radix Zensica*] of the number 36 is 6, and its cube root is $\sqrt[4]{6}$. Whence the square root of the number $\sqrt[3]{36}$ (or $\sqrt[4]{6}$) is $\sqrt[3]{6}$.
- From $\sqrt[4]{6}$ cubed comes $\sqrt[4]{216}$, or $\sqrt[3]{6}$, since the cube root of this number 216 is 6, and the square root of this is $\sqrt[3]{6}$. And so the cube root of the number $\sqrt[4]{216}$ (or $\sqrt[3]{6}$) is $\sqrt[4]{6}$.
- From $\sqrt[3]{36}$ raised to the second power comes $\sqrt[3]{1296}$ — that is, $\sqrt[3]{6}$, since the square root of the number 36 is 6; and the square root of the number $\sqrt[3]{1296}$ (or $\sqrt[3]{6}$) is $\sqrt[3]{36}$.

[Margin: An easy way to find the square of a zensizensic [fourth] root. How a root is doubled, or tripled, etc.]

From this it follows: to take the square of a number that is some zensizensic [fourth] root, it is enough to remove one 3 sign. Thus in the given example, the square of $\sqrt[3]{36}$ is $\sqrt[3]{1296}$; and so with the rest.

Therefore, if some square-root surd is to be doubled, its number must be multiplied by 4; for the square root of the product is double the given root. Thus the double of $\sqrt[3]{3}$ is $\sqrt[3]{12}$. For to multiply $\sqrt[3]{3}$ by 2, the numbers $\sqrt[3]{3}$ and 2 must be reduced to the same denomination, $\sqrt[3]{3}$ and $\sqrt[3]{8}$, and so on.

Likewise, if $\sqrt[4]{10}$ is to be doubled, it must be multiplied by $\sqrt[4]{8}$, and its double becomes $\sqrt[4]{80}$. For the two numbers $\sqrt[4]{10}$ and 2 must be brought back to the same denomination, $\sqrt[4]{10}$ and $\sqrt[4]{8}$.

In no other way will any root be multiplied by 3, 4, 5, etc.: the said numbers must be raised squarewise, or cubewise, or zensizensically, according to the condition of the root to be multiplied. Thus $\sqrt[5]{9}$ will be tripled if it is multiplied by $\sqrt[5]{243}$, so as to produce $\sqrt[5]{2187}$ [since $3^5 = 243$, and $9 \times 243 = 2187$]. It will be quadrupled if it is multiplied by $\sqrt[5]{1024}$, so as to become $\sqrt[5]{9216}$, which is the quadruple of $\sqrt[5]{9}$ [since $4^5 = 1024$, and $9 \times 1024 = 9216$]; and so on.

Summing any number of equal roots

[Margin: The sum of any number of equal roots.]

From this we shall easily find the sum of any number of equal surd roots. For if one of those roots is multiplied by as many units as there are proposed roots, their sum is produced. Thus five $\sqrt{3} 7$ make $\sqrt{3} 175$. Likewise four $\sqrt{6} 8$ make $\sqrt{6} 512$, that is, 8. So ten $\sqrt{36} 4$ make $\sqrt{36} 4000000$. For to gather any number of equal quantities into a single sum is nothing else than to multiply one of those quantities by their multitude — just as the sum of the three equal numbers 12, 12, 12 equals the number produced from 12 times 3, and so on.

Demonstration of the multiplication [of roots]

[Margin: Demonstration of the multiplication.]

That the multiplication of one root into another is rightly done, as we have said, we shall demonstrate as follows in the case of square roots. Let there be two lines, $AB = \sqrt{3} 3$ and $BC = \sqrt{3} 12$. I say that $\sqrt{3} 36$ — that is, the square root of the number 36,

which is produced from the square 3 of the line AB into the square 12 of the line BC — is the number produced from the line AB (that is, from $\sqrt{3} 3$ of square roots) [Margin: of square roots] into the line BC (that is, into $\sqrt{3} 12$).

Diagram. A single straight line divided at B into two segments, labelled with their surd lengths.

- A (left endpoint): start of the line
- AB (left segment): labelled $\sqrt{3} 3 [= \sqrt{3}]$
- B (division point, marked below the line): junction of the two segments
- BC (right segment): labelled $\sqrt{3} 12 [= \sqrt{12}]$
- C (right endpoint): end of the line

The whole line AC therefore = $\sqrt{3} 3 + \sqrt{3} 12$, and the rectangle under AB, BC equals $\sqrt{3} 36 = 6$.

For since, by the Lemma to Proposition 54 of Euclid Book X, the rectangle under AB, BC is the mean proportional between the squares of the straight lines AB, BC, [Margin: a — cf. Euclid VII.20] it follows that what is produced from 3 (the square of the straight line AB) into 12 (the square of the straight line BC) is equal to what is produced from the rectangle under AB, BC multiplied into itself. Therefore the product of AB into BC is the square root of the product of 3 (the square of AB) into 12 (the square of BC). Which was to be demonstrated.

In all other roots, too — even non-square ones — I shall demonstrate the same in this way. Let A be $\sqrt{3}$ of the number B, and C be $\sqrt{3}$ of the number D.

From A into B let the cube E be formed, so that A is $\sqrt[3]{E}$ of the number E; and from C into D let the cube F be formed, so that C is $\sqrt[3]{F}$ of the number F. Likewise, from A into E let the zensizensus [fourth power] G be formed, so that A is $\sqrt[4]{G}$ of the number G; and from C into F let the zensizensus H be formed, so that C is $\sqrt[4]{H}$ of the number H. Finally, from A into C let I be formed; from the square of B into the square of D let the square K be formed; from the cube E into the cube F let the cube L be formed; and from the zensizensus G into the zensizensus H let the zensizensus M be formed.

[Margin: A general demonstration of the multiplication of all roots.]

The following display gives the letter-values (with A = 2 and C = 3):

	Left (from A = 2)	Centr e	Right (from C = 3)
M (= G·H)		1296	
G (= A ⁴), H (= C ⁴)	G 16		H 81
L (= E·F)		216	
E (= A ³), F (= C ³)	E 8		F 27
K (= B·D)		36	
B (= A ²), D (= C ²)	B 4		D 9
I (= A·C)		6	
A, C	A 2		C 3

I say that the number I — the product of A (= $\sqrt[3]{B}$) into C (= $\sqrt[3]{D}$) — is $\sqrt[3]{K}$ of the number K, the product of B into D. Likewise, that the same number I — the product of A (= $\sqrt[4]{E}$) into C (= $\sqrt[4]{F}$) — is $\sqrt[4]{L}$ of the number L, the product of E into F. Finally, that the same number I — the product of A (= $\sqrt[5]{G}$) into C (= $\sqrt[5]{H}$) — is $\sqrt[5]{M}$ of the number M, the product of G into H. And so on in turn, if A and C are carried into G and H to form fifth powers (*surdesolidi*), then sixth powers (*zensicubi*), and so forth.

Proof. Since A, multiplying A and C, produced B and I, [Margin: b — Euclid VII.17] B will be to I as A to C. Likewise, since C, multiplying A and C, produced I and D, [Margin: c — Euclid VII.17] I will also be to D as A to C — that is, as B to I; hence B, I, D are in continued proportion. [Margin: d — Euclid VII.20] Therefore the product of B into D equals the product of I into itself. But K is produced from B into D; so the same K is produced from I into itself — that is, I is $\sqrt[3]{K}$ of the number K.

Next, since B, multiplying A and D, produced E and K, [Margin: e — Euclid VII.17] E will be to K as A to D. Likewise, since C, multiplying A and D, produced I and F, [Margin: f — Euclid VII.17] I will also be to F as A to D — that is, as E to K; hence E, K, I, F are proportionals. [Margin: g — Euclid VII.19] Therefore the number produced from E (the first) into F (the fourth) equals the number produced from I (the third) into K (the second). But L, the cube, is produced from E into F. So the same L is produced from I into K, its own square; and hence I is $\sqrt[3]{\text{E}}$ of the number L.

Again, since E, multiplying A and F, produced G and L, [Margin: h — Euclid VII.17] G will be to L as A to F. Likewise, since C, multiplying A and F, produced I and H, [Margin: i — Euclid VII.17] I will also be to H as A to F — that is, as G to L; hence G, L, I, H are proportionals. [Margin: k — Euclid VII.19] Therefore the number produced from G (the first) into H (the fourth) equals the number produced from I (the third) into L (the second). But M, the zensizensus, is produced from G into H. So the same M is produced from I into L, its own cube; and therefore I is $\sqrt[3]{\text{M}}$ of the number M. And so with the rest. Which was to be demonstrated.

Although in this example the numbers set down are rational, the same is nonetheless true also for irrational numbers, as Franciscus Maurolycus [Francesco Maurolico] taught in his *Arithmetica speculativa* — which is likewise seen in this scheme, where the same demonstration holds. For just as the propositions of Euclid Book VII are demonstrated for fractional numbers, so the same are shown for surd numbers. For example: since A, multiplying A and C, produced B and I, then by Definition 15 of Book VII both B is to A, and I is to C, as A is to unity — even though this ratio is irrational. Therefore B will be to A as I to C; and, by alternation, B to I as A to C. Hence A, multiplying the two numbers A and C, produced the two numbers B and I, which have the same ratio as the multiplied numbers A and C, as Proposition 17 of Book VII requires. And the same reasoning holds for the other propositions of Euclid Book VII.

The following display gives the same scheme with irrational values ($A = \sqrt[3]{3}$ and $C = \sqrt[3]{5}$):

Left (from $A = \sqrt[3]{3}$)		Centre	Right (from $C = \sqrt[3]{5}$)
M		M, 225 (or $\sqrt[3]{50625}$)	
G,	G, 9 (or $\sqrt[3]{81}$)		H, 25 (or $\sqrt[3]{625}$)
H			
L		L, $\sqrt[3]{3375}$	

	Left (from $A = \sqrt{3}$)	Centre	Right (from $C = \sqrt{5}$)
E, F	E, $\sqrt{3} 27$		F, $\sqrt{3} 125$
K		K, 15 (or $\sqrt{3} 225$)	
B, D	B, 3 (or $\sqrt{3} 9$)		D, 5 (or $\sqrt{3} 25$)
I		I, $\sqrt{3} 15$	
A, C	A, $\sqrt{3} 3$		C, $\sqrt{3} 5$

That root is rightly divided by root, too, if the transmitted rule is observed, needs no demonstration — since the quotient multiplied by the divisor produces the number divided. Thus, since $\sqrt{3} 15$ divided by $\sqrt{3} 3$ makes the quotient $\sqrt{3} 5$, and $\sqrt{3} 5$ carried into $\sqrt{3} 3$ makes $\sqrt{3} 15$ (as was demonstrated), there is no doubt that the division was rightly done.

Indeed, that the multiplication and division of surd roots are rightly carried out in the way we have prescribed can be shown abundantly by rational roots; for, multiplied and divided in this way, they produce true numbers as products and quotients. For from $\sqrt{3} 4$ into $\sqrt{3} 49$ comes $\sqrt{3} 196$, that is, 14; and from the division of $\sqrt{3} 196$ by $\sqrt{3} 4$ comes the quotient $\sqrt{3} 49$, that is, 7. Likewise, from $\sqrt{e} 8$ into $\sqrt{e} 27$ comes $\sqrt{e} 216$, namely 6; and from the division of $\sqrt{e} 216$ by $\sqrt{e} 8$ comes the quotient $\sqrt{e} 27$, namely 3; and so on.

LEMMA

[Margin: How to determine whether two roots are commensurable, and what ratio they have.]

To determine whether two surd roots are commensurable or not, and what ratio they have to one another.

Let one root be divided by the other. If the quotient turns out rational, the roots will be commensurable, and will have the same ratio that the quotient has to unity. Or, if the quotient is a fraction, they will have the ratio of the fraction's numerator to its denominator. [Margin: In what manner it is recognized.] Thus, since dividing $\sqrt{3} 12$ by $\sqrt{3} 3$ makes the quotient $\sqrt{3} 4$ — that is, 2 — by this token we recognize that $\sqrt{3} 12$ and $\sqrt{3} 3$ are commensurable and have the double ratio, namely 2 to 1.

Likewise, dividing $\sqrt{3} 3$ by $\sqrt{3} 12$ makes the quotient $\sqrt{3} 3/12$, or $\sqrt{3} 1/4$, that is, $1/2$. Therefore $\sqrt{3} 3$ and $\sqrt{3} 12$ are commensurable, and their ratio is the same as $1/2$ to 1 — that is, 1 to 2, of numerator to denominator. For by Proposition 2 of the *Minutiae* [the treatise on fractions] near the end of Euclid Book IX, the ratio of the numerator of any fraction to its denominator is the same as that of the fraction itself to unity.

Likewise, $\sqrt{3} 18$ and $\sqrt{3} 8$ will be commensurable, and will have the ratio of 3 to 2, namely the sesquialterate ratio. For the greater divided by the smaller gives the quotient $\sqrt{3} 2\frac{1}{4}$, or $\sqrt{3} 9/4$, that is, $3/2$. And $\sqrt{3} 8$ divided by $\sqrt{3} 18$ gives the quotient $\sqrt{3} 4/9$, that is, $2/3$. Therefore $\sqrt{3} 8$ and $\sqrt{3} 18$ are commensurable, and have the ratio of 2 to 3.

Again, $\sqrt{3} 75$ and $\sqrt{3} 48$ will be commensurable. For the former divided by the latter gives the quotient $\sqrt{3} 1\frac{9}{16}$, or $\sqrt{3} 25/16$, that is, $5/4$; and they will have the ratio of 5 to 4.

So too $\sqrt{3} 9$ and $\sqrt{3} 4$ will be found commensurable, having the ratio of 3 to 2.

Moreover, $\sqrt{c} 320$ and $\sqrt{c} 135$ are commensurable, since the former divided by the latter gives the quotient $\sqrt{c} 2\frac{10}{27}$, or $\sqrt{c} 64/27$, that is, $4/3$; and their ratio is sesquitercian, namely 4 to 3.

By the same reasoning $\sqrt{c} 162$ and $\sqrt{c} 2058$ are commensurable, and have the ratio of 7 to 3 — namely the double-sesquitercian ratio, if the greater is compared with the smaller. For the greater divided by the smaller gives the quotient $\sqrt{c} 12\frac{19}{27}$, or $\sqrt{c} 343/27$, that is, $7/3$.

Commensurable also are $\sqrt{c} 6000$ and $\sqrt{c} 2058$, having the ratio of 10 to 7. For the former divided by the latter gives the quotient $\sqrt{c} 2\frac{314}{343}$, or $\sqrt{c} 1000/343$, that is, $10/7$.

No differently, $\sqrt{33} 3888$ and $\sqrt{33} 243$ will be commensurable, since from the division of the former by the latter comes the quotient $\sqrt{33} 16$, that is, 2. They therefore have the double ratio.

In the same way $\sqrt{33} 19683$ and $\sqrt{33} 243$ will be commensurable, and will have the triple ratio, since from the division of the former by the latter comes the quotient $\sqrt{33} 81$, that is, 3.

Finally, $\sqrt{33} 19683$ and $\sqrt{33} 3888$ will also be commensurable, and will have the sesquialterate ratio, since from the division comes the quotient $\sqrt{33} 5\frac{1}{16}$, or $\sqrt{33} 81/16$, that is, $3/2$.

Now if, on the other hand, the division yields a surd or irrational number, the roots will be incommensurable, and will have an irrational ratio. Of this kind are $\sqrt[3]{48}$ and $\sqrt[3]{8}$, since from the division arises the quotient $\sqrt[3]{6}$, which is irrational (because 6 has no square root); yet they have the ratio of $\sqrt[3]{6}$ to 1.

Incommensurable, too, will be $\sqrt[4]{32}$ and $\sqrt[4]{18}$. For from the division is generated the quotient $\sqrt[4]{1\frac{7}{9}}$, or $\sqrt[4]{\frac{16}{9}}$, which — although it has a square root, $\frac{4}{3}$ [?] — nevertheless has no cube root. Their ratio, however, will be the same as $\sqrt[4]{16}$ to $\sqrt[4]{9}$, or $\sqrt[4]{16/9}$ to 1.

When the roots are of different genera, they must first be reduced to the same denomination. Thus, if $\sqrt[3]{64}$ and $\sqrt[4]{27}$ are given: once reduced to the same denomination — namely to $\sqrt[3]{64}$ and $\sqrt[3]{729}$ — if the latter is divided by the former, the quotient comes to $\sqrt[3]{11\frac{25}{64}}$, that is, $\sqrt[3]{729/64}$, that is, $\frac{3}{2}$. Therefore $\sqrt[3]{64}$ and $\sqrt[4]{27}$ are commensurable, and the latter has to the former the ratio of 3 to 2.

This Lemma is thoroughly necessary for the addition and subtraction of roots, which are treated in the two following propositions.

DE ADDITIONE RADICVM simplicium — CAPVT XIX

On the addition of simple roots — Chapter 19

When two or more equal roots are to be added together, one of them must be multiplied by 2, or 3, or 4, etc., according as the roots are 2, or 3, or 4, etc. For the number produced will be the sum of those roots, as we said in the preceding chapter. [Margin: Addition of equal roots.] Thus the sum of $\sqrt[3]{6}$ and $\sqrt[3]{6}$ is $\sqrt[3]{24}$. For $\sqrt[3]{6}$ doubled — that is, multiplied by $\sqrt[3]{4}$ — makes $\sqrt[3]{24}$. Likewise $\sqrt[4]{6}$, $\sqrt[4]{6}$, $\sqrt[4]{6}$ make $\sqrt[4]{162}$, because $\sqrt[4]{6}$ tripled — that is, multiplied by $\sqrt[4]{27}$ — makes $\sqrt[4]{162}$; and so with the rest.

When, however, two unequal roots of the same species are to be added together, one must first test, by the preceding Lemma, whether they are commensurable or incommensurable. [Margin: Addition of incommensurable roots.] The same must be done if the roots are of different denominations, after they have been recalled to the same denomination. And if indeed they are found to be incommensurable, they cannot be collected into a single sum such that one simple root is made from them; instead they must be added by interposing the

sign of addition, +. Thus from $\sqrt{3} 6$ and $\sqrt{3} 11$ comes the sum $\sqrt{3} 6 + \sqrt{3} 11$, or $\sqrt{3} 11 + \sqrt{3} 6$. Likewise from 8 and $\sqrt{3} 7$ comes the sum $8 + \sqrt{3} 7$, or $\sqrt{3} 7 + 8$. So from $\sqrt{\text{C}} 9$ and $\sqrt{\text{C}} 20$ comes the sum $\sqrt{\text{C}} 9 + \sqrt{\text{C}} 20$.

Incommensurable square roots can nevertheless be added together, in this way as well. [Margin: Addition of square roots, by Proposition 4 of Euclid Book II.] To the sum of the squares let there be added twice the product of one root into the other; the square root of this collected sum will be the sum of the two proposed roots. Let $\sqrt{3} 6$ and $\sqrt{3} 11$ be added. The squares of the numbers 6 and 11 make 17. And from $\sqrt{3} 6$ into $\sqrt{3} 11$ comes $\sqrt{3} 66$, whose double is $\sqrt{3} 264$. The sum, then, collected from $\sqrt{3} 6$ and $\sqrt{3} 11$, will be the square root of this aggregate $17 + \sqrt{3} 264$ — namely $\sqrt{3} (17 + \sqrt{3} 264)$. The rationale of this operation is gathered from Proposition 4 of Euclid Book II. For if some straight line is divided into $\sqrt{3} 6$ and $\sqrt{3} 11$, the two squares of these parts will be 6 and 11; and if to these is added twice the product of $\sqrt{3} 6$ into $\sqrt{3} 11$, the whole square of the whole binomial line becomes this: $17 + \sqrt{3} 264$. Its square root, then — namely $\sqrt{3} (17 + \sqrt{3} 264)$ — will be the sought sum of $\sqrt{3} 6$ and $\sqrt{3} 11$. But in what manner the square root is to be extracted from binomials, when the matter requires and the reasoning demands, we shall say in chapter 27.

[Note on the surd signs used below — glossed from the source's own worked examples. Clavius writes each radix (root) with a compound sign whose second element names the order of the root: $\sqrt{3} = \textit{radix quadrata}$ (square root; the 3 is the census/quadratic mark); $\sqrt{\text{C}} = \textit{radix cubica}$ (cube root); $\sqrt{33} = \textit{radix quadrato-quadrata}$ (fourth root); $\sqrt{3\text{C}} = \textit{radix quadrato-cubica}$ (sixth root). Radicands, the + sign of a binomium, and the – sign of an apotome are rendered as printed.]

Chapter XIX (continued) — On the Addition of Simple Surds

we shall discuss. Yet the [sum] found here will be $\sqrt{3} 11 + \sqrt{3} 6$, and this is no different from the sum of the proposed roots, provided they are joined by the + sign. This method of adding is evident in rational numbers as well. For let $\sqrt{3} 4$ and $\sqrt{3} 9$ be added, where it is plain that the collected sum is 5. The squares 4 and 9 make 13; and $\sqrt{3} 4 \times \sqrt{3} 9$ gives $\sqrt{3} 36$, whose double is $\sqrt{3} 144$. So the sum collected from $\sqrt{3} 4$ and $\sqrt{3} 9$ will be the square root of this aggregate $13 + \sqrt{3} 144$ — namely $\sqrt{3} (13 + \sqrt{3} 144)$. Now 13 together with $\sqrt{3} 144$, that is with 12, makes 25, whose square root is 5.

Indeed, by this same device commensurable square roots can also be added, but the sum will always turn out to be a simple root. Suppose $\sqrt{3}$ and $\sqrt{12}$ are to be added. Dividing the one by the other gives the rational quotient $\sqrt{4}$, and so they are commensurable. Therefore, if their squares 3 and 12 are collected into a single sum, the number 15 results. And if one root is multiplied into the other, the number $\sqrt{36}$ results, whose double $\sqrt{144}$ together with 15 makes $15 + \sqrt{144}$. So the square root of this — namely $\sqrt{3(15 + \sqrt{144})}$, that is $\sqrt{327}$ — will be the sum collected from $\sqrt{3}$ and $\sqrt{12}$.

But if roots of any kind whatever, once proposed for addition, are found to be commensurable — that is, if on dividing one by the other the quotient comes out rational, whether a whole number, or a fraction, or a whole number with a fraction (and if it is a whole number, we write 1 beneath it to make it a fraction) — then we shall proceed as follows. Whatever multiple ratio the sum of the numerator and denominator, taken together, has to 1, the sum of the proposed roots will have that same ratio to that part of the smaller root which 1 is of the smaller number of the fraction. Hence, if that part is taken from the smaller root and multiplied by the denominator of that multiple ratio, the sum of the proposed roots is produced. Likewise, whatever multiple ratio the sum of the numerator and denominator, taken together, has to 1, the sum of the proposed roots will have that same ratio to that part of the larger root which 1 is of the larger number of the fraction. Hence, if that part is taken from the larger root and multiplied into the denominator of that multiple ratio, the sum of the proposed roots is produced.

Thus, suppose $\sqrt{38}$ and $\sqrt{18}$ are to be added. Dividing the larger by the smaller gives the rational quotient $\sqrt{2\frac{1}{4}}$, that is $\frac{3}{2}$. Now since the sum of the terms of the fraction $\frac{3}{2}$ — namely 5 — has to 1 a fivefold ratio, the sum of the given roots will likewise have a fivefold ratio to the half of the smaller root. And $\sqrt{2}$ is the half of the smaller root $\sqrt{38}$; for dividing $\sqrt{38}$ by 2, that is by $\sqrt{4}$, gives the quotient $\sqrt{2}$. Therefore, if $\sqrt{2}$ is multiplied by 5, the denominator of the fivefold ratio — that is, by $\sqrt{25}$ — there will be produced $\sqrt{50}$, the sum collected from $\sqrt{38}$ and $\sqrt{18}$.

Likewise, since the sum of the terms of the fraction $\frac{3}{2}$ — that is, 5 — has to 1 a fivefold ratio, the sum of the proposed roots will have the same ratio to the third part of the larger root. Now $\sqrt{2}$ is the third part of the larger root $\sqrt{18}$; for dividing $\sqrt{18}$ by 3, that is by $\sqrt{9}$, gives the quotient $\sqrt{2}$. So if $\sqrt{2}$ is multiplied into 5, the denominator of the fivefold ratio — that is, into $\sqrt{25}$ — the same sum $\sqrt{50}$ is produced.

Again, let $\sqrt{33}$ and $\sqrt{312}$ be added. Dividing the larger by the smaller gives the quotient $\sqrt{34}$, that is 2; and writing 1 beneath it to make a fraction, $2/1$. Since the sum 3 of the terms of this fraction is triple, and 1 is equal to the smaller number of the fraction, the root $\sqrt{33}$ too will have a triple ratio [to the sum]. So if $\sqrt{33}$, the smaller root, is multiplied by the triple — namely into $\sqrt{39}$ — the sum $\sqrt{327}$ results.

Likewise, since the aforesaid sum 3 of the terms of the fraction $2/1$ stands to 1 (the half of the larger number 2 of the fraction) in a triple ratio, the sum of the roots too will have a triple ratio to the half of the larger root $\sqrt{312}$, that is to $\sqrt{33}$. So if $\sqrt{33}$ is multiplied into the denominator of the triple ratio — that is, into $\sqrt{39}$ — the same sum $\sqrt{327}$ is produced.

Demonstration. The demonstration of this matter is that, given four proportional numbers, the sum of the first two stands to the greater or lesser of them just as the sum of the last two stands to the greater or lesser of them; and likewise the sum of the first two stands to any part of the greater or lesser just as the sum of the last two stands to the like part of the greater or lesser. For example, given the proportional numbers 6, 4 and 18, 12: by composition, as 10 is to 4, so 30 is to 12; and by conversion of the ratio, as 10 is to 6, so 30 is to 18. And since it is also true that, as 4 is to 1 (a fourth part), so 12 is to 3 (a fourth part), it will follow *ex aequo* that, as 10 is to 1, so 30 is to 3. Likewise, since it is also true that, as 6 is to 1 (a sixth part), so 18 is to 3 (a sixth part), it will also follow *ex aequo* that, as 10 is to 1, so 30 is to 3. Since, therefore, the division of one root by another produces the denominator of the ratio — so that the ratio of one to the other is the same as that of the numerator of the fraction to its denominator (1 being supplied when the quotient is a whole number) — the rationale of the addition of roots becomes manifest. And it is indeed a thing to marvel at, that an exact addition can be made among quantities that have no exact and known magnitude.

[Margin: A demonstration of the addition of commensurable roots.]

Hence it follows that the sum of two commensurable roots is also collected if the larger root $\sqrt{318}$, or the smaller $\sqrt{38}$, is multiplied by the denominator of the ratio which the sum of the terms of the fraction $3/2$ — namely 5 — has to the larger term of the fraction, 3, or to the smaller, 2, that is, by $5/3$ or by $5/2$. Thus you see that from $\sqrt{318} \times 5/3$, that is $\times \sqrt{325}/9$, or from $\sqrt{38} \times 5/2$, that is $\times \sqrt{325}/4$, there is produced $\sqrt{350}$, as before.

Let $\sqrt{372}$ and $\sqrt{318}$ again be added. Dividing the former by the latter gives the quotient $\sqrt{34}$, that is 2; and supplying 1, this becomes $2/1$, the denominator of a

double ratio. So just as the sum 3 has to 1 a triple ratio, so also the sum of the two roots will be triple the smaller root $\sqrt[3]{18}$. Therefore, if $\sqrt[3]{18}$ is multiplied into 3 — that is, into $\sqrt[3]{9}$ — there results $\sqrt[3]{162}$. Likewise, just as the same sum 3 of the terms of the fraction $2/1$ has to 2 a sesquialterate ratio, whose denominator is $3/2$, so too the required sum will be sesquialterate to the larger root $\sqrt[3]{72}$. So if $\sqrt[3]{72}$ is multiplied into $3/2$ — that is, into $\sqrt[3]{9/4}$ — the same sum $\sqrt[3]{162}$ is produced.

Commensurable roots can also be added very conveniently by the golden rule [the rule of three], and it comes back to the same thing.

[Margin: A neat method of adding commensurable roots.]

For since in the first example, as 5, the sum of the terms of the fraction $3/2$, is to the smaller number 2 or to the larger 3, so is the sum of the roots to the smaller or larger root — as we have shown — it will also hold, by conversion, that as 2 or 3 is to the sum 5, so the smaller or larger root is to the sum of the roots. So if we set it up:

As 2 is to 5, so $\sqrt[3]{8}$ is to another term; or As 3 is to 5, so $\sqrt[3]{18}$ is to another term — the sum $\sqrt[3]{50}$ will be produced.

But in the last example, in which the denominator of the ratio was $2/1$, if we set it up:

As 1 is to 3, so $\sqrt[3]{18}$ is to another term; or As 2 is to 3, so $\sqrt[3]{72}$ is to another term — the sum $\sqrt[3]{162}$ will be produced, as before.

Let $\sqrt[3]{2}$ and $\sqrt[3]{8}$ again be added. From the division of $\sqrt[3]{8}$ by $\sqrt[3]{2}$ comes the rational quotient $\sqrt[3]{4}$, that is 2; and supplying 1, this becomes $2/1$. So if we set it up:

As 1 is to 3, so $\sqrt[3]{2}$ is to another term; or As 2 is to 3, so $\sqrt[3]{8}$ is to another term — the sum $\sqrt[3]{18}$ will be collected.

Let $\sqrt[3]{162}$ and $\sqrt[3]{2058}$ [recte: $\sqrt[3]{162}$ and $\sqrt[3]{2058}$ — cube roots] also be added. Dividing the larger root by the smaller gives the rational quotient $\sqrt[3]{12}$ $19/27$, or $\sqrt[3]{343/27}$, that is $7/3$. So if we set it up:

As 3 is to 10, so $\sqrt[3]{162}$ is to another term; or As 7 is to 10, so $\sqrt[3]{2058}$ is to another term — the sum $\sqrt[3]{6000}$ will be found.

Likewise, let $\sqrt[3]{8}$ and $\sqrt[3]{27}$ be added, which are rational numbers. From the division of $\sqrt[3]{27}$ by $\sqrt[3]{8}$ comes the rational quotient $\sqrt[3]{3}$, or $\sqrt[3]{27/8}$, that is $3/2$. So if we set it up:

As 2 is to 5, so $\sqrt{128}$ is to another term; or As 3 is to 5, so $\sqrt{27}$ is to another term — the sum $\sqrt{125}$ will come out, that is 5.

Again, let $\sqrt{3243}$ and $\sqrt{33888}$ be added. The ratio [of their radicands] is $1/16$. So if we set it up:

As 1 is to 3, so $\sqrt{3243}$ is to another term; or As 2 is to 3, so $\sqrt{33888}$ is to another term — the sum $\sqrt{319683}$ will arise.

Finally, let $\sqrt{34}$ and $\sqrt{127}$ be added, which are rational numbers. Reducing these roots to the same denomination — namely to $\sqrt{3664}$ and $\sqrt{36729}$ — the division of the larger by the smaller gives the rational quotient $\sqrt{36729/64}$, that is $3/2$. So if we set it up:

As 2 is to 5, so $\sqrt{3664}$ is to another term; or As 3 is to 5, so $\sqrt{36729}$ is to another term — the sum $\sqrt{15625}$ will be produced, that is 5.

[Margin: Another neat method of adding commensurable roots.]

Commensurable roots can also be added in this way. Divide the larger root by the smaller, and add 1 to the rational quotient. For if the combined number is multiplied into the smaller root, the required sum will be produced. For if the quotient is multiplied into the smaller root — that is, into the divisor — the larger root, which had been divided, is produced; and if 1 is multiplied into that same divisor — that is, into the smaller root — the smaller root is produced; and so the two products together will make up the sum of the two roots. Since, therefore, by the first theorem appended to Proposition 14 of Book IX of Euclid — which answers to Proposition 1 of Book II — the same number arises from the smaller root multiplied into the sum collected from the quotient and unity as arises from that same smaller root multiplied into the quotient and into unity separately, the sum of the two given roots will also be produced by the smaller root multiplied into the sum collected from the quotient and unity.

Thus let $\sqrt{38}$ and $\sqrt{18}$ be added. Dividing the larger by the smaller gives the rational quotient $\sqrt{2\frac{1}{4}}$, that is $3/2$; and adding 1, this becomes $5/2$. And since $5/2 \times \sqrt{38}$ gives $\sqrt{350}$, $\sqrt{350}$ will be the sum of the two roots $\sqrt{38}$ and $\sqrt{18}$.

Likewise, let $\sqrt{162}$ and $\sqrt{2058}$ be added. The division yields the rational quotient $7/3$; and adding 1, this becomes the number $10/3$, which multiplied into $\sqrt{162}$ produces $\sqrt{6000}$, the sum of the two proposed roots.

[Margin: To add several roots that share a common measure, into a single root of the same kind.]

When two roots, or more, can be divided by some one root of the same kind so that the quotients are rational, they will be joined into a single sum if the sum of the quotients is multiplied into that common dividing root; for the resulting number will be the required sum. For since, by the theorem just cited, the number produced by the dividing root multiplied into the sum of the quotients is equal to the numbers produced by that same dividing root multiplied into the several quotients — that is, equal to all the proposed roots taken together — it is plainly established that the dividing root multiplied into the sum of the quotients produces the sum of the proposed roots.

For example, suppose $\sqrt{38}$ and $\sqrt{318}$ are to be added. If each is divided by $\sqrt{32}$, the rational quotients $\sqrt{34}$ and $\sqrt{39}$ are produced, that is 2 and 3, whose sum 5, if multiplied into the common divisor — that is, into $\sqrt{32}$ — produces the sum $\sqrt{350}$.

Likewise, let $\sqrt{38}$, $\sqrt{350}$ and $\sqrt{372}$ be added. If they are severally divided by $\sqrt{32}$, the rational quotients $\sqrt{34}$, $\sqrt{325}$ and $\sqrt{336}$ result, that is 2, 5 and 6, whose sum 13, multiplied into the common divisor $\sqrt{32}$, produces the sum $\sqrt{3338}$.

Finally, let $\sqrt{38}$, $\sqrt{332}$ and $\sqrt{348}$ be added. Divide each by $\sqrt{32}$, and the quotients $\sqrt{34}$, $\sqrt{316}$ and $\sqrt{324}$ result. The first two of these are rational, namely 2 and 4, but the last is irrational. Therefore they cannot in this way all be gathered into a single sum; rather, only the first two will make the sum $\sqrt{372}$, and so from all three there arises the sum $\sqrt{372} + \sqrt{348}$.

On the Subtraction of Simple Surds — Chapter XX

DE SVBTRACTIONE RADICVM simplicium. — CAPVT XX.

When one root is subtracted from an equal root, nothing remains: for instance, $\sqrt{312}$ from $\sqrt{312}$; likewise $\sqrt{100}$ subtracted from $\sqrt{100}$ leaves $\sqrt{30}$, or $\sqrt{100}$.

[Margin: Subtracting a root from an equal root.]

But if the roots are incommensurable — that is, when, on dividing the larger by the smaller, the quotient is irrational — the subtraction cannot be carried out except by the sign – of the terms to be subtracted. For instance, $\sqrt{33}$ from $\sqrt{310}$ leaves $\sqrt{310} - \sqrt{33}$; likewise $\sqrt{20}$ from $\sqrt{50}$ leaves $\sqrt{50} - \sqrt{20}$.

[Margin: Subtracting unequal roots.]

When, however, the roots are square roots and incommensurable, the smaller can be subtracted from the larger in this way as well. From the sum of the squares subtract twice the product of one root into the other; the square root of what remains will be the remainder of the subtraction. Suppose $\sqrt{312}$ is to be subtracted from $\sqrt{320}$. The squares of the numbers 12 and 20 make 32; and $\sqrt{312} \times \sqrt{320}$ gives $\sqrt{3240}$, whose double $\sqrt{3960}$, taken away from the sum of the squares, leaves $32 - \sqrt{3960}$. So the square root of this remainder — namely $\sqrt{3(32 - \sqrt{3960})}$ — will be what is left when $\sqrt{312}$ is subtracted from $\sqrt{320}$.

[Margin: Subtracting square roots, by Proposition 7 of Book II of Euclid.]

The rationale of this operation is gathered from Proposition 7 of Book II of Euclid. For if some whole line is $\sqrt{320}$, cut in such a way that one part is $\sqrt{312}$, there will be two square numbers, 20 and 12 — namely 32 — equal to the number $\sqrt{3960}$, which is twice $\sqrt{312} \times \sqrt{320}$, together with the square of the remaining part of the line. So if the number which is twice $\sqrt{312} \times \sqrt{320}$ is taken away from 32, the sum of the squares, what remains will be the square of the remaining part of the line. Therefore the square of this part will be $32 - \sqrt{3960}$; and hence the remainder left after subtracting $\sqrt{312}$ from $\sqrt{320}$ will be $\sqrt{3(32 - \sqrt{3960})}$. By what method, when the case requires it, a square root is to be extracted from apotomes, or residuals, we shall tell in Chapter 27 — the result here being $\sqrt{320} - \sqrt{312}$, no different from the remainder if $\sqrt{312}$ is subtracted from $\sqrt{320}$ by interposing the sign $-$.

But this method of subtraction is also clear in rational numbers. For let $\sqrt{34}$ be taken from $\sqrt{349}$, where it plainly holds that the remainder is 5. The squares 4 and 49 make 53; and $\sqrt{34} \times \sqrt{349}$ gives $\sqrt{3196}$, whose double $\sqrt{3784}$, taken away from 53, the sum of the squares, will leave $\sqrt{3(53 - \sqrt{3784})}$. Now $53 - \sqrt{3784}$, that is $53 - 28$, comes to 25, whose square root is 5.

Indeed, by this same device a square root can be subtracted from a square root when they are commensurable — but then the remainder will be a simple root. Suppose $\sqrt{33}$ is to be subtracted from $\sqrt{327}$. Dividing the latter by the former gives the rational quotient $\sqrt{39}$, and so they are commensurable. The sum of the squares 3 and 27 is 30, from which, if we take away $\sqrt{3324}$ — namely twice the product $\sqrt{33} \times \sqrt{327}$ — there will remain $30 - \sqrt{3324}$. Therefore $\sqrt{3(30 - \sqrt{3324})}$, that is $\sqrt{312}$, will be the remainder of the subtraction.

If, however, roots of any kind whatever are proposed and are commensurable — that is, if on dividing the larger by the smaller the quotient comes out rational (with the proviso that, if it is a whole number, unity is supplied beneath it to

make a fraction) — we shall proceed as follows. Whatever ratio the number that remains, when the denominator of the fraction is subtracted from the numerator — that is, the difference of the terms of the fraction — has to 1, that same ratio the remainder (after subtracting the smaller root from the larger) will have to that part of the smaller root which 1 is of the denominator of the fraction; or else the said remainder will have that same ratio to that part of the larger root which 1 is of the numerator of the fraction. Accordingly, if such a part is taken from the smaller root, or from the larger, and is multiplied by the denominator of that ratio, the required remainder is produced.

[Margin: Subtracting commensurable roots.]

Let $\sqrt{38}$ be subtracted from $\sqrt{350}$. From the division of the larger by the smaller [comes the rational quotient $\sqrt{325/4}$, that is $5/2$. Now since the difference of the terms of the fraction $5/2$ is 3 — for taking the denominator 2 from the numerator 5, 3 remains — and this difference has to 1, the half of the smaller term of the fraction, a triple ratio, the remainder left when $\sqrt{38}$ is taken from $\sqrt{350}$ will likewise have a triple ratio to the half of the smaller root, namely to $\sqrt{32}$, or to the fifth part of the larger root. So if $\sqrt{32}$ is multiplied into 3 — that is, into $\sqrt{39}$ — the required remainder $\sqrt{318}$ is produced.

The same remainder is obtained if the smaller root $\sqrt{38}$ is divided by the larger $\sqrt{350}$; for the quotient will be $\sqrt{34/25}$, that is $2/5$. Again, then, just as 3, the difference of the terms of the fraction, has a triple ratio to 1 (the half of the smaller term of the fraction, or the fifth part of the larger term of the fraction), so too the difference of the two proposed roots will have a triple ratio to the half of the smaller root, or to the fifth part of the larger — namely to $\sqrt{32}$. So the triple of $\sqrt{32}$, namely $\sqrt{318}$, will be the difference of the given roots — that is, the remainder after subtracting $\sqrt{38}$ from $\sqrt{350}$. The demonstration is the same as the one adduced above for addition.

[Margin: A neat method of subtracting commensurable roots.]

Hence it follows that the remainder is also produced if the larger root, or the smaller, is multiplied by the denominator of the ratio which the difference of the terms of the fraction — that is, the number left when the smaller term of the fraction is taken from the larger — has to the larger term of the fraction, or to the smaller; namely, in the proposed example, by $3/5$ or by $3/2$. (For $3/5$ is the denominator of the ratio which the difference of the terms of the fraction has to 5, the larger term of the fraction; while $3/2$ is the denominator of the ratio which the same difference 3 has to the smaller term of the fraction.) Thus you see that

from $\sqrt{3}50 \times 3/5$, or from $\sqrt{3}8 \times 3/2$, there arises the remainder $\sqrt{3}18$ — the same as before.

The subtraction of commensurable roots can also be done most conveniently by the golden rule. For since, as the difference of the terms of the fraction is to the larger or smaller term of the same fraction, so is the required remainder to the larger or smaller root; and conversely, as the larger term of the fraction is [to the difference], so is the larger root, or the smaller, to the remainder of the subtraction — if in the given example we set it up:

(Notation used on these pages — glossed here once and used bare thereafter. Each radical sign is a compound consist symbol standing for a different root:

- $\sqrt{3}$ = square root [radix quadrata; the 3 is the "census/zensus" sign]
- $\sqrt{\text{C}}$ = cube root [radix cubica]
- $\sqrt{33}$ = fourth root [radix zensizensica, the biquadrate]
- $\sqrt{\beta}$ = fifth root [radix sursolida / sursolidum]
- $\sqrt{3\text{C}}$ = sixth root [radix zensicubica])

[Continuation of Chapter XX — subtraction of commensurable surds by the rule of three]

As 5 is to 3, so is $\sqrt{3} 50$ to another; **or**

As 2 is to 3, so is $\sqrt{3} 8$ to another — and the required remainder $\sqrt{3} 18$ will be produced.

The same holds if $\sqrt{\text{C}} 162$ is to be subtracted from $\sqrt{\text{C}} 6000$. Dividing this by that, the rational quotient $\sqrt{\text{C}} 1000/27$ results — that is, $10/3$ — where the difference of the numbers of the fraction is 7. If, then, we set it up:

As 10 is to 7, so is $\sqrt{\text{C}} 6000$ to another; **or**

As 3 is to 7, so is $\sqrt{\text{C}} 162$ to another — the desired remainder $\sqrt{\text{C}} 2058$ will be generated.

[Margin: Another elegant method of subtracting commensurable roots.]

A root can also be subtracted from a root, when they are commensurable, in this way. Divide the greater by the lesser, and subtract one (unity) from the rational quotient. For if the remaining number is then multiplied into the lesser root, the required remainder is produced. The reason is that if the whole quotient is multiplied into the lesser root — that is, into the divisor — the greater root, which

was divided, is produced. Since, then, unity multiplied into the lesser root yields the lesser root itself, it plainly follows that if the quotient, after unity has first been removed, is multiplied into the lesser root, it produces that same greater root, less the lesser root — that is, the difference of the two proposed roots.

Let $\sqrt[3]{3888}$ be subtracted from $\sqrt[3]{19683}$. From the division comes the rational quotient $\sqrt[3]{81/16}$, that is, $3/2$. Removing 1, there remains $1/2$ — which (first converted back to $\sqrt[3]{1/16}$), multiplied into $\sqrt[3]{3888}$, gives $\sqrt[3]{243}$, the difference of the proposed roots.

[Margin: Another method of subtracting, for when the proposed roots have another root of the same kind as their common measure.]

Finally, the subtraction can also be carried out in yet another way. Divide each root by some common root of the same kind, so that the quotients come out rational, if this can be done. For when one quotient is subtracted from the other, and the remaining number is multiplied into that common divisor, the residual root is produced. For example, let $\sqrt[3]{27}$ be taken from $\sqrt[3]{75}$. Dividing each by $\sqrt[3]{3}$ gives the rational quotients $\sqrt[3]{9}$ and $\sqrt[3]{25}$ — that is, 3 and 5. Their difference, 2, multiplied into the common divisor $\sqrt[3]{3}$, gives $\sqrt[3]{12}$ for the remainder of the subtraction.

The rationale of this operation is as follows. Let there be two numbers, 10 and 30, whose difference is 20, and let them be divided by the common divisor 5. The quotients 6 and 2 will then have the same ratio as 30 and 10. And so, by conversion of the ratio, 30 will be to 20 (the excess by which the antecedent 30 exceeds the consequent 10) as 6 is to 4 (the excess by which the antecedent 6 exceeds the consequent 2). And by alternation, 30 will be to 6 as 20 is to 4. But 30 has to 6 the fivefold ratio, named of course from the common divisor 5. Therefore 20 will also have to 4 the fivefold ratio; and so 20, divided by 5, gives the quotient 4. From this it is established that if 4, the difference of the quotients 2 and 6, is multiplied into the common divisor 5, it produces 20, the difference of the numbers 10 and 30.

[Marginal numeric scheme illustrating the proportion above:]

20
10 30
5
2 6
4

To place any number of mean proportionals between two given numbers.

CHAPTER XXI.

[Margin: To place any number of mean proportionals between two numbers.]

I ought now to treat of the Algorithm of composite irrational numbers; but I prefer first to set out a certain excellent use of simple roots in constructing any number of mean proportionals between two given numbers. It runs as follows. If a single mean proportional is to be constructed, one must take this radical sign, $\sqrt{}$ — because a single mean proportional falls between a square and unity in any geometric progression that begins from 1. If two means, take the radical sign $\sqrt[3]{}$; if three, the sign $\sqrt[4]{}$; if four, the sign $\sqrt[5]{}$; if five, the sign $\sqrt[6]{}$; and so on — as the geometric progression set out in Chapter 2 will readily teach us.

Next, having divided the greater number by the lesser, one must set up a geometric progression starting from 1 and advancing by the quotient, with two more terms than the number of mean proportionals desired.

Third, to each term of this progression one must prefix the radical sign that we have taught should be taken, according to the number of means.

Fourth, where the roots that the radical sign indicates can be extracted, they are to be extracted, and the radical signs whose roots have been extracted are to be deleted.

Fifth and last, for the extremes of the progression the given numbers themselves are to be set, and each of the middle terms is to be multiplied by the lesser.

For example, if a single mean number is to be established between 5 and 20, this radical sign $\sqrt{}$ is to be adopted.

Next, dividing the greater number 20 by the lesser 5, this progression of three terms is to be set up — 1. 4. 16 — advancing by the quotient 4.

Third, prefix the sign $\sqrt{}$ to each term, in this way: $\sqrt{1}$. $\sqrt{4}$. $\sqrt{16}$. And these terms will likewise be in continued proportion, since they are the square roots of the numbers 1, 4, 16, which are in continued proportion.

Fourth, with the square roots extracted, the progression will stand thus: 1. 2. 4.

Fifth, the individual terms of this progression are to be multiplied by the lesser number, 5. Whence the progression will stand thus: 5. 10. 20 — and the mean proportional term found is 10; [Margin: Scholium 18 of Book VII.] because 5, 10, 20 have the same proportions as 1, 2, 4, since it is the same number 5, multiplying them, that has produced them.

But suppose again that five mean proportionals are to be found between 6 and 18. Here the sign $\sqrt[3]{\text{C}}$ must be adopted.

Next, dividing the greater number 18 by the lesser 6, a progression of seven terms is to be set up, advancing by the quotient 3, in this way:

1. 3. 9. 27. 81. 243. 729.

Third, prefix the sign $\sqrt[3]{\text{C}}$ to each term, as here:

$\sqrt[3]{\text{C}}$ 1. $\sqrt[3]{\text{C}}$ 3. $\sqrt[3]{\text{C}}$ 9. $\sqrt[3]{\text{C}}$ 27. $\sqrt[3]{\text{C}}$ 81. $\sqrt[3]{\text{C}}$ 243. $\sqrt[3]{\text{C}}$ 729.

And these terms will likewise be in continued proportion, since they are the sixth roots [radices Zensicubicae] of the numbers 1, 3, 9, and so on, which are in continued proportion.

Fourth, extracting the roots that can be extracted, and deleting the signs whose roots have been extracted, the progression will stand thus:

1. $\sqrt[3]{\text{C}}$ 3. $\sqrt{\text{C}}$ 3. $\sqrt[3]{\text{C}}$ 3. $\sqrt{\text{C}}$ 9. $\sqrt[3]{\text{C}}$ 243. 3.

Fifth and last, multiplying the individual terms of this progression by 6, the lesser of the two proposed numbers, you will establish the following progression, with its five middle proportional terms:

6. $\sqrt[3]{\text{C}}$ 139968. $\sqrt{\text{C}}$ 648. $\sqrt[3]{\text{C}}$ 108. $\sqrt{\text{C}}$ 1944. $\sqrt[3]{\text{C}}$ 11337408. 18.

And these terms will be in continued proportion, [Margin: Scholium 18 of Book VII.] since they are produced from the same number 6 into the numbers 1, $\sqrt[3]{\text{C}}$ 3, and so on, which are in continued proportion.

That these terms are in continued proportion can also be proved by trial. For, taking any three consecutive ones, the product of the first into the third is as much as the product of the middle into itself, as proposition 20 of Book 7 of Euclid requires (*Elements* VII.20). Thus, taking these three — $\sqrt{\text{C}}$ 648, $\sqrt[3]{\text{C}}$ 108, $\sqrt{\text{C}}$ 1944 — both from the first into the third and from the middle into itself, the number 108 is generated. For $\sqrt[3]{\text{C}}$ 108 into itself makes the number 108; and $\sqrt{\text{C}}$ 648 into $\sqrt{\text{C}}$ 1944 makes $\sqrt{\text{C}}$ 1259712, that is, 108. Again, taking these three — 6, $\sqrt[3]{\text{C}}$ 139968, $\sqrt{\text{C}}$ 648 — from the middle squared into itself comes $\sqrt[3]{\text{C}}$ 19591041024; that is, the

square root of 139968 being extracted, the number produced is $\sqrt[3]{139968}$ (since 139968 has no cube root), which is also produced from 6 into $\sqrt[3]{648}$, as is evident.

SCHOLIUM.

It often happens that a square or cube root has to be extracted from some number that does not possess one. So that time is not wasted in vain, certain rules have been handed down by several writers — especially by Rafael Bombelli — by which numbers that are not squares, and not cubes, are easily recognized.

[Margin: Which numbers are not squares.]

First, all numbers whose last figure on the right is 2, or 3, or 7, or 8 are not squares; because in square numbers the last figure must necessarily be one of these: 1, 4, 5, 6, 9, 0.

Next, no number is a square unless, when the 9s are cast out of it — as is customarily done in the proof by nines — one of these figures remains: 1, 4, 7, 0.

Third, a number having 5 as its last figure cannot be a square unless a 2 precedes it, and that 2 in turn is preceded by an even figure or by 0. Thus these numbers — 125, 325, 67525, 89725, 100925 — are not squares.

Fourth, numbers having 1 or 9 as their last figure are not squares unless an even figure, or 0, precedes it. Thus these numbers — 4371, 4379, 67899, 75351 — are not squares.

Fifth, numbers having 4 as their last figure cannot be squares unless another even figure, or 0, precedes it. Thus these numbers — 6934, 70014 — cannot be squares.

Sixth, numbers having 6 as their last figure are not squares unless an odd figure precedes it. Thus the numbers 5746, 7086, 34526 are not squares.

Seventh and last, a number ending in several 0 figures under an odd count cannot be a square. Thus the numbers 460, 678000, 91010 are not squares.

[Margin: Which numbers are not cubes.]

As for cubes: no number will be a cube if, when the 9s are cast out, there does not remain 1, or 8, or 0. And so 12000 will not be a cube, since when the 9s are cast out 3 remains.

Next, a number whose last figure is 2 cannot be a cube unless the figure immediately preceding is odd; likewise a number whose last figure is 4 or 8 cannot be a cube if the figure immediately preceding is not even, or 0. Thus the numbers 34562, 456174, 1100038 are not cubes.

Third, no number ending in 0 or 00 can be a cube. Thus these numbers — 1230, 100, 34600 — are not cubes.

Fourth and last, a number having 5 as its last figure cannot be a cube unless the figure immediately preceding is 2 or 7. Thus these numbers — 361035, 67895, 1120015 — are not cubes.

Therefore, before you undertake the extraction of a square or cube root, you will do well first to test, by the aforesaid conditions, whether the proposed number can be a square or a cube.

On the addition and subtraction of composite irrational numbers.

CHAPTER XXII.

The Algorithm of composite irrational numbers is built up out of the Algorithm of simple roots, provided that for addition and subtraction the two rules delivered in Chapter 4 concerning the signs + and - are called to mind — namely:

[Margin: The two rules to be observed in addition and subtraction.]

1. Like signs give the same sign, except in subtraction, when the numbers are placed in reverse order [praepostere]. For then the upper is subtracted from the lower, and + becomes -, and - becomes +.
2. Unlike signs change the kind of operation. And in addition the sign of the greater number is set down; but in subtraction, the sign of the upper number — whether it be the greater, or the smaller, or equal.

These two rules were set out in the aforesaid Chapter 4, and can easily be understood from the examples given below.

Examples of Addition.

$$\begin{array}{l} 6 + \sqrt{3} \ 18 \ \sqrt{3} \ 27 + \sqrt{3} \ 8 \\ 4 + \sqrt{3} \ 8 \ \sqrt{3} \ 12 + \sqrt{3} \ 2 \end{array}$$

$$\begin{array}{r} 10 + \sqrt{3} 50 \quad \sqrt{3} 75 + \sqrt{3} 18 \\ \hline \end{array}$$

$$\begin{array}{r} \sqrt{3} 162 - 2 \sqrt{33} 243 - \sqrt{33} 162 \\ \sqrt{3} 200 - 3 \sqrt{33} 48 - \sqrt{33} 32 \\ \hline \end{array}$$

$$\sqrt{3} 722 - 5 \sqrt{33} 1875 - \sqrt{33} 1250$$

Examples of Subtraction.

$$\begin{array}{r} \sqrt{33} 1875 + \sqrt{33} 1250 \quad \sqrt{3} 50 - 5 \\ \sqrt{33} 243 + \sqrt{33} 162 \quad \sqrt{3} 8 - 2 \\ \hline \end{array}$$

$$\sqrt{33} 48 + \sqrt{33} 32 \quad \sqrt{3} 18 - 3$$

Examples of the exception in subtraction, for the first rule.

$$\begin{array}{r} \sqrt{3} 50 + 2 \sqrt{3} 50 - 2 \\ \sqrt{3} 18 + 4 \sqrt{3} 18 - 4 \\ \hline \end{array}$$

$$\sqrt{3} 8 - 2 \sqrt{3} 8 + 2$$

$$\begin{array}{r} \sqrt{3} 50 + 6 \sqrt{3} 72 + 2 \\ \sqrt{3} 72 + 2 \quad 6 + \sqrt{3} 18 \\ \hline \end{array}$$

$$4 - \sqrt{3} 2 \quad \sqrt{3} 18 - 4$$

Examples of the exception in addition, for the second rule.

$$\begin{array}{r} \sqrt{3} 50 + 3 \sqrt{3} 50 - 3 \\ \sqrt{3} 32 - 5 \sqrt{3} 32 + 5 \\ \hline \end{array}$$

$$\sqrt{3} 162 - 2 \sqrt{3} 162 + 2$$

Examples of the exception in subtraction, for the second rule.

$$\begin{array}{r} \sqrt{3} 162 + 2 \sqrt{3} 162 - 2 \\ \sqrt{3} 50 - 3 \sqrt{3} 50 + 3 \\ \hline \end{array}$$

$$\sqrt{3} 32 + 5 \sqrt{3} 32 - 5$$

Other examples of addition.

$$\begin{array}{r} 8 - \sqrt{3} 50 \quad \sqrt{3} 50 + 6 \\ \sqrt{3} 242 - 12 \quad 24 - \sqrt{3} 242 \\ \hline \end{array}$$

$$\sqrt{3} 72 - 4 \quad 30 - \sqrt{3} 72$$

Other examples of subtraction.

$$\begin{array}{r} \sqrt{3} 72 - 4 \quad 30 - \sqrt{3} 72 \\ 8 - \sqrt{3} 50 \quad \sqrt{3} 50 + 6 \end{array}$$

$$\sqrt{3} 242 - 12 \quad \sqrt{3} 242$$

Other examples of subtraction.

$$\begin{array}{r} \sqrt{3} 0 + 16 \quad \sqrt{3} 180 + 0 \\ \sqrt{3} 320 - 8 \quad \sqrt{3} 320 - 8 \end{array}$$

$$24 - \sqrt{3} 320 \quad 8 - \sqrt{3} 20$$

[Margin: To be observed in some subtractions.]

Here, in the former example, since the number $\sqrt{3} 320$ has no counterpart in the upper number, the number $\sqrt{3} 0$ has been placed there with the sign +. Likewise, since in the latter example the number 8 has no counterpart in the upper number, the number 0 has likewise been placed with the sign +. This must always be observed in subtraction, so that the two rules above may find their application.

[Algebra — continuation of Cap. XXII, on the addition and subtraction of composite irrational numbers (binomials and apotomes).]

Note on the cossist radical signs used throughout. Clavius forms each root sign from the radical stroke $\sqrt{}$ followed by the cossist power-name whose degree the root inverts: $\sqrt{3}$ = square root (census, order 2); $\sqrt{c}$ = cube root (cubus, order 3); $\sqrt{33}$ = fourth root (censo-census, order 4); $\sqrt{B}$ = fifth root (sursolidum, order 5); $\sqrt{3c}$ = sixth root (order 6); $\sqrt{3B}$ = tenth root (order 10). The radicand is printed as an ordinary number after the sign (e.g. $\sqrt{3320} = \sqrt{320}$). These signs are reproduced as printed.

But in examples of this kind the subtraction may also be carried out in the following way. After the number from which the subtraction is to be made, set down the number to be subtracted, but with each of its signs changed into the opposite — putting - for +, and the reverse. Then reduce by addition. Thus, if $\sqrt{3320} - 8$ is to be subtracted from the number 16, the example will stand like this:

$$16 - \sqrt{3320} + 8$$

Adding 16 to 8 gives 24; and taking away $\sqrt{3320}$, the remainder is $24 - \sqrt{3320}$.

Again, if the subtraction $\sqrt{3320} - 8$ is to be made from $\sqrt{3180}$, the example will stand thus: $\sqrt{3180} - \sqrt{3320} + 8$. Adding together $\sqrt{3180}$ and $-\sqrt{3320}$ — that is, with the kind of operation changed on account of the unlike signs + and -, by taking $\sqrt{3180}$ away from $\sqrt{3320}$ — the remainder is $8 - \sqrt{320}$.

Once more, let the subtraction $\sqrt{3}180 - 8$ be made from $\sqrt{3}320$. The example will stand thus: $\sqrt{3}320 - \sqrt{3}180 + 8$. Adding together $\sqrt{3}320 - \sqrt{3}180$ — that is, with the kind of operation changed on account of the unlike signs + and -, by taking $\sqrt{3}180$ away from $\sqrt{3}320$ — the remainder is $8 + \sqrt{3}20$.

Next, let the subtraction $8 - \sqrt{3}12$ be made from $\sqrt{3}72 - 4$. The example will stand thus: $\sqrt{3}72 - 4 - 8 + \sqrt{3}12$. Adding together $\sqrt{3}72$ and $\sqrt{3}12$, and -4 with -8 , the remainder is $\sqrt{3}72 + \sqrt{3}12 - 12$, since those roots are incommensurable. But if they are added by Euclid II.4 [propos. 4, lib. 2], they yield $\sqrt{3}(84 + \sqrt{3}456)$, and so the whole remainder is $\sqrt{3}(84 + \sqrt{3}456) - 12$.

That the subtraction is rightly carried out in this way is plainly evident in the case of rational numbers. For let the subtraction $\sqrt{3}49 - 3$ be made from $8 + \sqrt{3}9$ — that is, 4 from 11, where the remainder is 7. By the rule handed down, the example will stand thus: $8 + \sqrt{3}9 - \sqrt{3}49 + 3$. Now from 8 and 3 comes 11, and from $\sqrt{3}9$ and $-\sqrt{3}49$ comes the number -4 . Therefore the remainder is $11 - 4$, that is, 7, as the reckoning requires.

But so that this whole business of addition and subtraction may be understood more clearly, we shall append several further examples. [Margin: Further examples of addition and subtraction.]

Let $7 + \sqrt{3}5$ be added to $4 + \sqrt{3}3$. From 7 and 4 comes 11; and from $\sqrt{3}5$ and $\sqrt{3}3$, by Euclid II.4, comes $\sqrt{3}(8 + \sqrt{3}60)$. Therefore the sum is $11 + \sqrt{3}(8 + \sqrt{3}60)$.

$$\begin{array}{r} 7 + \sqrt{3}5 \\ 4 + \sqrt{3}3 \\ \hline 11 + \sqrt{3}(8 + \sqrt{3}60) \end{array}$$

Again, let $\sqrt{3}5 + 3$ be added to $\sqrt{3}20 - 4$. From $\sqrt{3}5$ and $\sqrt{3}20$, since they are commensurable, comes $\sqrt{3}45$; and from $+3$ and -4 comes -1 . So the whole sum is $\sqrt{3}45 - 1$. The same sum is obtained if $\sqrt{3}5$ is added to $\sqrt{3}20$ by Euclid II.4, as we showed at the beginning of Chapter 19: for it yields $\sqrt{3}(25 + \sqrt{3}400)$, that is, $\sqrt{3}45$, and so again the whole sum is $\sqrt{3}45 - 1$.

$$\begin{array}{r} \sqrt{3}5 + 3 \\ \sqrt{3}20 - 4 \\ \hline \sqrt{3}45 - 1 \end{array}$$

Furthermore, let $\sqrt{3}5 + 7$ be added to $\sqrt{3}6 - 7$. From $\sqrt{3}5$ and $\sqrt{3}6$ comes $\sqrt{3}(11 + \sqrt{3}120)$, and from $+7$ and -7 comes 0. So the sum gathered together is $\sqrt{3}(11 + \sqrt{3}120)$.

$$\begin{array}{r}
\sqrt{35} + 7 \\
\sqrt{36} - 7 \\
\hline
\sqrt{3}(11 + \sqrt{3}120)
\end{array}$$

Further still, let $\sqrt{37} + 3$ be added to $\sqrt{320} - 4$. From $\sqrt{37}$ and $\sqrt{320}$ comes $\sqrt{3}(27 + \sqrt{3560})$, and from $+3$ and -4 comes -1 . So the whole sum is $\sqrt{3}(27 + \sqrt{3560}) - 1$.

$$\begin{array}{r}
\sqrt{37} + 3 \\
\sqrt{320} - 4 \\
\hline
\sqrt{3}(27 + \sqrt{3560})
\end{array}$$

Let $\sqrt{35} + 7$ also be added to $\sqrt{36} - 2$. From $\sqrt{35}$ and $\sqrt{36}$ comes $\sqrt{3}(11 + \sqrt{3}120)$, but from $+7$ and -2 comes $+5$. So the whole sum is $\sqrt{3}(11 + \sqrt{3}120) + 5$, or $5 + \sqrt{3}(11 + \sqrt{3}120)$.

$$\begin{array}{r}
\sqrt{35} + 7 \\
\sqrt{36} - 2 \\
\hline
\sqrt{3}(11 + \sqrt{3}120) + 5
\end{array}$$

Likewise, let $7 + \sqrt{35}$ be added to $4 - \sqrt{33}$. The sum will be $11 + \sqrt{35} - \sqrt{33}$. Or, following the second rule, we subtract $\sqrt{33}$ from $\sqrt{35}$ by Euclid II.7, on account of the unlike signs $+$ and $-$, in this way: the squares of the numbers 5 and 3 make 8; and from $+\sqrt{35}$ times $-\sqrt{33}$ comes $-\sqrt{315}$, whose double $-\sqrt{360}$ is taken away from the aggregate of the squares, 8, leaving $8 - \sqrt{360}$. The root of this remainder is what is left after subtracting $\sqrt{33}$ from $\sqrt{35}$, and to that root must be prefixed the sign $+$ of the greater number. So the whole sum is $11 + \sqrt{3}(8 - \sqrt{360})$.

$$\begin{array}{r}
7 + \sqrt{35} \\
4 - \sqrt{33} \\
\hline
11 + \sqrt{35} - \sqrt{33}
\end{array}$$

Likewise, let $8 - \sqrt{37}$ be added to $10 - \sqrt{37}$. From 8 and 10 come 18, and from $-\sqrt{37}$ and $-\sqrt{37}$ comes $-\sqrt{328}$. So the whole sum is $18 - \sqrt{328}$.

$$\begin{array}{r}
8 - \sqrt{37} \\
10 - \sqrt{37} \\
\hline
18 - \sqrt{328}
\end{array}$$

Likewise, let $10 + \sqrt{349}$ be added to $7 - \sqrt{34}$. Here it is plain that the sum is 22: from 10 and 7 come 17; and $\sqrt{34}$ subtracted from $\sqrt{349}$ by Euclid II.7, on account of the unlike signs $+$ and $-$, leaves $\sqrt{3}(53 - \sqrt{3784})$, to which must be prefixed the sign $+$ of the greater number. So the whole sum is $17 + \sqrt{3}(53 - \sqrt{3784})$. And since the

square root of this number $53 - \sqrt{3784}$ is 5 — for 28, the root of the number 784, taken from 53, leaves 25, whose square root is 5 — the whole sum is $17 + 5$, that is, 22.

$$\begin{array}{r} 10 + \sqrt{349} \\ 7 - \sqrt{34} \\ \hline 17 + \sqrt{3(53 - \sqrt{3784})} \end{array}$$

Likewise, let $10 - \sqrt{349}$ be added to $7 + \sqrt{34}$, where it is again clear that the sum is 12. From 10 and 7 come 17; and $\sqrt{34}$ subtracted from $\sqrt{349}$ by Euclid II.7, on account of the unlike signs - and +, leaves $\sqrt{3(53 - \sqrt{3784})}$, to which must be prefixed the sign - of the greater number. So the whole sum gathered together is $17 - \sqrt{3(53 - \sqrt{3784})}$. And since the square root of this number $53 - \sqrt{3784}$ is 5, the whole sum is $17 - 5$, that is, 12.

$$\begin{array}{r} 10 - \sqrt{349} \\ 7 + \sqrt{34} \\ \hline 17 - \sqrt{3(53 - \sqrt{3784})} \end{array}$$

Likewise, let 6 be added to $\sqrt{324} + 3$. The sum will be $\sqrt{324} + 9$, or $9 + \sqrt{324}$.

$$\begin{array}{r} \sqrt{324} + 3 \\ \sqrt{30} + 6 \\ \hline \sqrt{324} + 9 \end{array}$$

Likewise, let 6 be added to $\sqrt{324} - 3$. The sum will be $\sqrt{324} + 3$.

$$\begin{array}{r} \sqrt{324} - 3 \\ \sqrt{30} + 6 \\ \hline \sqrt{324} + 3 \end{array}$$

Likewise, let $\sqrt{36}$ be added to $\sqrt{324} - 3$. From the roots, since they are commensurable, comes $\sqrt{354}$; and from -3 and +0 comes -3. So the sum is $\sqrt{354} - 3$.

$$\begin{array}{r} \sqrt{324} - 3 \\ \sqrt{36} + 0 \\ \hline \sqrt{354} - 3 \end{array}$$

Likewise, let $\sqrt{38}$ be added to $\sqrt{332} + \sqrt{318}$. From $\sqrt{38}$ and $\sqrt{332}$, since they are commensurable, comes $\sqrt{372}$. So the sum is $\sqrt{372} + \sqrt{318}$. And from these two roots, since they are commensurable, comes $\sqrt{3162}$, the sum sought. Or, because $\sqrt{38}$ and $\sqrt{318}$ are also commensurable, from them comes $\sqrt{350}$, so that the whole sum is $\sqrt{350} + \sqrt{332}$; and from these two roots, since they are commensurable,

comes $\sqrt{3162}$, the sum sought, as before. Thus these numbers $\sqrt{372} + \sqrt{318}$ and $\sqrt{350} + \sqrt{332}$ are equal, since each is equal to $\sqrt{3162}$, as has been shown.

$$\begin{array}{r} \sqrt{38} + \sqrt{30} \\ \sqrt{332} + \sqrt{318} \\ \hline \sqrt{372} + \sqrt{318} \end{array}$$

Likewise, let $\sqrt[3]{216} - \sqrt[3]{3405}$ be added to $\sqrt[3]{64} - \sqrt[3]{380}$. From $\sqrt[3]{216}$ and $\sqrt[3]{64}$ [cube roots], since they are commensurable, comes $\sqrt[3]{1000}$, that is, 10. And from $\sqrt[3]{3405}$ and $\sqrt[3]{380}$ [fourth roots], since these too are commensurable, comes $\sqrt[3]{33125}$. So the whole sum is $10 - \sqrt[3]{33125}$.

$$\begin{array}{r} \sqrt[3]{216} - \sqrt[3]{3405} \\ \sqrt[3]{64} - \sqrt[3]{380} \\ \hline 10 - \sqrt[3]{33125} \end{array}$$

Likewise, let $\sqrt[3]{3256} - \sqrt[3]{27}$ be added to $\sqrt[3]{381} + \sqrt[3]{8}$. From $\sqrt[3]{3256}$ and $\sqrt[3]{381}$ comes $\sqrt[3]{32401}$, that is, 7. And from $-\sqrt[3]{27}$ and $+\sqrt[3]{8}$ comes $-\sqrt[3]{1}$, that is, -1. So the whole sum is $7 - 1$, that is, 6. That this is so is clear: for $\sqrt[3]{3256}$ is 4; if $\sqrt[3]{27}$ — that is, 3 — is taken from it, 1 remains; and if $\sqrt[3]{381}$ — that is, 3 — is added, and further $\sqrt[3]{8}$ — that is, 2 — the sum comes to 6, as before.

$$\begin{array}{r} \sqrt[3]{3256} - \sqrt[3]{27} \\ \sqrt[3]{381} + \sqrt[3]{8} \\ \hline 7 - 1 \end{array}$$

Finally, let $\sqrt{3}(5 + \sqrt{36})$ and $\sqrt{3}(10 - \sqrt{311})$ be added. Roots of this kind — which above we called roots of composite or diminished irrational numbers, and which others call universal roots — cannot be more conveniently gathered into a single sum than by means of the sign +, in this way: $\sqrt{3}(5 + \sqrt{36}) + \sqrt{3}(10 - \sqrt{311})$. So too, if the first is to be subtracted from the second, the remainder will be $\sqrt{3}(10 - \sqrt{311}) - \sqrt{3}(5 + \sqrt{36})$. But of roots of this kind we shall say more in Chapter 24.

Let $\sqrt[3]{3256} - \sqrt[3]{27}$ be subtracted from $\sqrt[3]{32401} - \sqrt[3]{1}$. Here it is plain that the remainder is 5. For since $\sqrt[3]{32401}$ and $\sqrt[3]{3256}$ are commensurable, if the latter is taken from the former, $\sqrt[3]{381}$ remains, that is, 3. And if $-\sqrt[3]{27}$ is subtracted from $-\sqrt[3]{1}$, since these too are commensurable, $+\sqrt[3]{8}$ remains, that is, 2. And plainly 3 and 2 make 5, as the reckoning requires.

$$\begin{array}{r} \sqrt[3]{32401} - \sqrt[3]{1} \\ \sqrt[3]{3256} - \sqrt[3]{27} \\ \hline \sqrt[3]{381} + \sqrt[3]{8} \end{array}$$

Likewise, let $9 - \sqrt{350}$ be subtracted from $\sqrt{372} - 3$. Since $+\sqrt{372}$ and $-\sqrt{350}$ have unlike signs, the kind of operation is changed — that is, they are to be added together — and their sum becomes $\sqrt{3242}$. Likewise, since -3 and $+9$ have different signs, an addition is again made, and the sum, 12, takes the sign $-$ of the upper number. So the remainder sought is $\sqrt{3242} - 12$.

$$\begin{array}{r} \sqrt{372} - 3 \\ 9 - \sqrt{350} \\ \hline \sqrt{3242} - 12 \end{array}$$

Likewise, let this number $\sqrt{216} - \sqrt{33405}$ be subtracted from $\sqrt{1000} + \sqrt{333125}$. Since $\sqrt{1000}$ and $\sqrt{216}$ are commensurable, taking the latter from the former leaves $\sqrt{64}$, that is, 4. And from $\sqrt{333125}$ and $\sqrt{33405}$, which are also commensurable, comes the sum $\sqrt{3320480}$ — for these are to be added together, by the second rule, on account of their unlike signs — and to that sum must be prefixed the sign $+$ of the upper number.

$$\begin{array}{r} \sqrt{1000} + \sqrt{333125} \\ \sqrt{216} - \sqrt{33405} \\ \hline 4 + \sqrt{3320480} \end{array}$$

Finally, let $\sqrt{216} - \sqrt{33405}$ be subtracted from $\sqrt{1000} - \sqrt{333125}$. The remainder will be what is set out in this formula. For $\sqrt{216}$ subtracted from $\sqrt{1000}$ makes the remaining $\sqrt{64}$, that is, 4, and so on.

$$\begin{array}{r} \sqrt{1000} - \sqrt{333125} \\ \sqrt{216} - \sqrt{33405} \\ \hline 4 - \sqrt{3380} \end{array}$$

Moreover, when the two parts of a composite number are exactly equal to the two parts of a diminished number, most briefly the one number is added to the other, or the diminished one is taken from the composite, in this way. [Margin: A compendious addition and subtraction of like composite and diminished numbers.]

The addition will be accomplished if the first part of either number is doubled. Thus, from the addition of $15 + \sqrt{38}$ to $15 - \sqrt{38}$ comes the sum 30. Likewise, from the addition of $\sqrt{320} + 6$ to $\sqrt{320} - 6$ comes the sum $\sqrt{380}$. So too, from $\sqrt{320} + \sqrt{36}$ added to $\sqrt{320} - \sqrt{36}$ comes the sum $\sqrt{380}$. The reason is that the latter parts destroy one another, on account of the signs $+$ and $-$.

The subtraction will be complete if the latter part of either number is doubled. Thus $10 - \sqrt{34}$ from $10 + \sqrt{34}$ leaves $\sqrt{316}$, that is, 4. Likewise $\sqrt{312} - 5$ from $\sqrt{312} + 5$ leaves 10. Likewise $\sqrt{312} - \sqrt{35}$ from $\sqrt{312} + \sqrt{35}$ leaves $\sqrt{320}$. The reason is that the former parts leave nothing upon subtraction, while the latter must be added, as the second rule directs, on account of the unlike signs $-$ and $+$.

ON THE MULTIPLICATION AND DIVISION of composite and diminished irrational numbers.

CAPVT XXIII.

Here too we must recall that rule which we laid down in Chapter 5 concerning the signs $+$ and $-$, namely:

Like signs give the sign of things added; unlike signs give the sign of things subtracted. [Margin: The rule to be observed in multiplication and division.]

So then, whether plus is multiplied into plus, or minus into minus, plus is always produced; and whether plus is drawn into minus, or minus into plus, minus is always brought forth. Accordingly, once the two numbers are set one beneath the other, the multiplication is carried out as with absolute numbers, provided you observe what was handed down in Chapter 18 on the multiplication of simple roots, and in Chapter 19 on the addition and subtraction of those same roots. This the examples written below will teach.

In the following example: from $-\sqrt{345}$ into $-\sqrt{320}$ comes $+\sqrt{3900}$, that is, $+30$. From $-\sqrt{345}$ into $+6$ — that is, into $+\sqrt{336}$ — comes $-\sqrt{31620}$. From $-\sqrt{320}$ into $+8$ — that is, into $+\sqrt{364}$ — comes $-\sqrt{31280}$. And finally from $+8$ into $+6$ comes $+48$. So the whole product number is $48 - \sqrt{31280} - \sqrt{31620} + 30$; and if this is reduced by addition and subtraction, it will be $78 - \sqrt{35780}$. For from $+48$ and $+30$ come $+78$; and from $-\sqrt{31280}$ and $-\sqrt{31620}$, since they are commensurable, comes $-\sqrt{35780}$.

$$\begin{array}{r} 6 - \sqrt{320} \\ 8 - \sqrt{345} \end{array}$$

$$48 - \sqrt{31280} - \sqrt{31620} + 30$$

Per reductionem [by reduction]: $78 - \sqrt{35780}$

In this other example: from $-\sqrt{33162}$ into $-\sqrt{33648}$ comes $+\sqrt{33104976}$, that is, $+18$. From $-\sqrt{33162}$ into $+\sqrt{33288}$ comes $-\sqrt{3346656}$, that is, $-\sqrt{3216}$. From $-\sqrt{33648}$ into $+\sqrt{33128}$ comes $-\sqrt{3382944}$, that is, $-\sqrt{3288}$. And finally from $+\sqrt{33288}$ into $+\sqrt{33128}$ comes $+\sqrt{3336864}$, that is, $\sqrt{3192}$. So the whole product number is $\sqrt{3192} - \sqrt{3288} -$

$\sqrt[3]{216} + 18$. Or, so that the greater numbers always come first: $18 + \sqrt[3]{192} - \sqrt[3]{288} - \sqrt[3]{216}$, as you see in the formula.

$$\begin{array}{r} \sqrt[3]{3288} - \sqrt[3]{3648} \\ \sqrt[3]{3128} - \sqrt[3]{3162} \end{array}$$

Productus [product]. $\sqrt[3]{192} - \sqrt[3]{288} - \sqrt[3]{216} + 18$
 Vel [or] $18 + \sqrt[3]{192} - \sqrt[3]{288} - \sqrt[3]{216}$

Further examples of multiplication.

$$\begin{array}{r} \sqrt[6]{7} + \sqrt[6]{6} \\ \sqrt[3]{3} \end{array}$$

$$\sqrt[3]{1323} + \sqrt[3]{8748}$$

$$\begin{array}{r} \sqrt[3]{7} + \sqrt[6]{3} \\ \sqrt[3]{2} \end{array}$$

$$\sqrt[3]{14} + \sqrt[3]{72}$$

In the first example we reduced $\sqrt[6]{6}$ and $\sqrt[3]{3}$ to the same denomination, namely to $\sqrt[30]{36}$ and $\sqrt[30]{243}$ [the tenth-root denomination], and so by multiplication $\sqrt[30]{8748}$ was produced. Then $\sqrt[3]{3}$ and $\sqrt[6]{7}$ were reduced to $\sqrt[30]{27}$ and $\sqrt[30]{49}$ of the same denomination [the sixth-root denomination], and from the multiplication $\sqrt[30]{1323}$ was produced.

In the latter example, $\sqrt[3]{2}$ [radix quadrata — the square-root sign, $\sqrt{}$] and $\sqrt[6]{3}$ [radix cubica — the cube-root sign, $\sqrt[3]{}$] were reduced to $\sqrt[30]{8}$ [radix zenzicubica — the sixth-root sign] and $\sqrt[30]{9}$, and their multiplication produced $\sqrt[30]{72}$; and from $\sqrt[3]{2}$ times $\sqrt[3]{7}$ came $\sqrt[3]{14}$.

Another example, in rational numbers.

$$\begin{array}{r} \sqrt[3]{9} + \sqrt[6]{27} \\ \sqrt[3]{4} \end{array}$$

$$\begin{array}{r} \sqrt[3]{36} + \sqrt[30]{46656} \\ \text{(that is, } 6 + 6) \end{array}$$

In this example, $\sqrt[6]{27}$ and $\sqrt[3]{4}$ are reduced to $\sqrt[30]{729}$ and $\sqrt[30]{64}$, whose multiplication makes $\sqrt[30]{46656}$, that is, 6. Then from $\sqrt[3]{4}$ times $\sqrt[3]{9}$ comes $\sqrt[3]{36}$, that is, 6. So the whole product is $6 + 6$, that is, 12.

Two more examples.

$$\begin{array}{r} 7 + \sqrt[3]{57} - \sqrt[3]{5} \\ 7 + \sqrt[3]{57} - \sqrt[3]{5} \end{array}$$

$$\begin{array}{r} 49 + \sqrt{3} 245 + \sqrt{3} 245 + 5 \\ \hline 49 - \sqrt{3} 245 - \sqrt{3} 245 + 5 \end{array}$$

In the former example, the composite number multiplied by itself gives the product $49 + \sqrt{3} 245 + \sqrt{3} 245 + 5$, that is, $54 + \sqrt{3} 980$ — since 49 and +5 make 54, and from $\sqrt{3} 245$ and $\sqrt{3} 245$, that is, from twice $\sqrt{3} 245$, comes $\sqrt{3} 980$. In the latter, the diminished number [apotome] multiplied by itself gives $49 - \sqrt{3} 245 - \sqrt{3} 245 + 5$, that is, $54 - \sqrt{3} 980$.

[Margin: A shortcut multiplication of a composite or of a diminished number by itself.] Indeed, to multiply a composite number by itself, or by another equal to it, it is enough to add together the squares of the parts and to add to this sum twice the product of one part into the other, as is clear in the two immediately preceding examples. For the squares of the parts make 54. And the product of one part into the other is, in the former example, $\sqrt{3} 245$, but in the latter $-\sqrt{3} 245$, whose double is $\sqrt{3} 980$ or $-\sqrt{3} 980$. Therefore in the former example the number $54 + \sqrt{3} 980$ is produced, but in the latter this one, $54 - \sqrt{3} 980$, as before.

Two more examples, in rational numbers.

$$\begin{array}{r} 5 + \sqrt{3} 4 \quad 5 - \sqrt{3} 4 \\ 5 + \sqrt{3} 4 \quad 5 - \sqrt{3} 4 \\ \hline 25 + \sqrt{3} 100 + \sqrt{3} 100 + 4 \quad 25 - \sqrt{3} 100 - \sqrt{3} 100 + 4 \end{array}$$

In the former, 7 is multiplied by itself, giving the product 49; in the latter, 3 is multiplied by itself, giving the product 9, as is evident. These multiplications, by the shortcut just handed down, are done thus. The squares of the parts make 29. And the product of one part into the other is, in the former example, $\sqrt{3} 100$, but in the latter $-\sqrt{3} 100$, whose double is $\sqrt{3} 400$ or $-\sqrt{3} 400$. So in the former example the product will be $29 + \sqrt{3} 400$, that is, 49; but in the latter $29 - \sqrt{3} 400$, that is, 9.

Two more examples.

$$\begin{array}{r} 6 + \sqrt{3} 8 \quad \sqrt{3} 10 + \sqrt{3} 2 \\ 6 - \sqrt{3} 8 \quad \sqrt{3} 10 - \sqrt{3} 2 \\ \hline 36 + \sqrt{3} 288 - \sqrt{3} 288 - 8 \quad 10 + \sqrt{3} 20 - \sqrt{3} 20 - 2 \end{array}$$

[Margin: A shortcut multiplication of a composite number of square roots by a like diminished number.] In these two examples of square roots, and the like, in which a composite number is multiplied by a like diminished one, or the reverse, the multiplication is done most briefly if in either of the numbers the square of the latter part is subtracted from the square of the former part — whether the

latter part is smaller than the former or larger. Thus in the former example, if 8, the square of the latter part $\sqrt{3} 8$, is taken from 36, the square of the former part 6, the remaining product will be 28. But in the latter example, if 2, the square of the latter part $\sqrt{3} 2$, is subtracted from 10, the square of the former part $\sqrt{3} 10$, the product 8 is left.

And so, whenever a composite and a diminished number of irrational square roots — provided they are alike — are multiplied together, a single number is always produced, and that a rational one. This indeed Euclid demonstrated in Book X, Proposition 115 (Euclid, *Elements* X.115), since numbers of this kind are a Binomial and a Residual, as will be clear from Chapter 27.

When the latter part is the larger, a diminished number is produced, as appears in the example set here alongside. For if 16, the square of the latter part, is taken from 4, the square of the former part, there remain 4 – 16.

$$\begin{array}{r} 2 + \sqrt{3} 16 \\ 2 - \sqrt{3} 16 \\ \hline 4 - 16 \end{array}$$

And that this is so, the other formula of multiplication teaches. Since $2 + \sqrt{3} 16$ make 6, and $2 - \sqrt{3} 16$ make 2 – 4, if 2 – 4 be multiplied by 6, the product 12 – 24 is produced. It is therefore necessary that these two numbers, 4 – 16 and 12 – 24, be equal, as indeed they are, since one subtracted from the other leaves 0 — as the two subtraction-formulae set below demonstrate: because 8 – 8 is equivalent to nothing. This is also clear from the fact that both from 4 to –16 and from 12 to –24 the sum always comes to –12, according to the second rule of addition.

$$\begin{array}{r} 2 + \sqrt{3} 16 \\ 2 - \sqrt{3} 16 \\ \hline - \sqrt{3} 64 - 16 \\ 4 + \sqrt{3} 64 \\ \hline 4 - 16 \end{array}$$

$$\begin{array}{r} 2 - 4 \\ 6 \\ \hline 12 - 24 \end{array}$$

$$\begin{array}{r} 4 - 16 \quad 12 - 24 \\ 12 - 24 \quad 4 - 16 \end{array}$$

$$- 8 + 8 \ 8 - 8$$

Furthermore, the multiplication of the roots of composite or diminished numbers — which they call Universal [roots] — we shall treat in the following Chapter 24.

Examples of Division.

[Margin: Division of a composite or diminished number by a simple root.]

$$\begin{array}{l} \sqrt{28} + \sqrt{20} \\ \text{per } \sqrt{4} \sqrt{4} \text{ (} \sqrt{7} + \sqrt{5} \end{array}$$

$$\begin{array}{l} \sqrt{28} + \sqrt{20} \\ \text{per } \sqrt{7} \sqrt{7} \text{ (} \sqrt{4} + \sqrt{2} \ 6/7 \end{array}$$

Further examples of Division.

$$\begin{array}{l} \sqrt[3]{20} - \sqrt{10} \\ \text{per } 3 \text{ (} \sqrt[3]{2} \ 2/9 - \sqrt{10}/27 \end{array}$$

$$\begin{array}{l} \sqrt[3]{8} + \sqrt[5]{3} \\ \text{per } \sqrt[3]{2} \text{ (} \sqrt[3]{2} + \sqrt[5]{9}/32 \end{array}$$

(Here $\sqrt[3]{8}$ [radix zenzizenzica — the fourth-root sign], $\sqrt[5]{3}$ [radix sursolida — the fifth-root sign], and $\sqrt[10]{9/32}$ [radix zenzsursolida — the tenth-root sign].)

In the first example, $\sqrt[3]{20}$ divided by 3, that is, by $\sqrt[3]{9}$, gives the Quotient $\sqrt[3]{2} \ 2/9$. And $-\sqrt{10}$ divided by +3, that is, by $+\sqrt{27}$, gives the Quotient $-\sqrt{10}/27$. In the latter example, $\sqrt[3]{8}$ and $\sqrt[3]{2}$ are to be reduced to $\sqrt[3]{64}$ [radix zenzizenzizenzica — the eighth-root sign] and $\sqrt[3]{16}$, of the same denomination; and dividing the one by the other gives the Quotient $\sqrt[3]{4}$, that is, $\sqrt[3]{2}$. In the same way $\sqrt[5]{3}$ and $\sqrt[3]{2}$ are to be reduced to $\sqrt[10]{9}$ and $\sqrt[10]{32}$; and dividing the one by the other gives the Quotient $\sqrt[10]{9/32}$.

Now since multiplication proves division: if the latter Quotient be multiplied by the divisor $\sqrt[3]{2}$, the divided number $\sqrt[3]{8} + \sqrt[5]{3}$ will be produced. For $\sqrt[10]{9/32}$ and $\sqrt[3]{2}$ are reduced to $\sqrt[20]{82944/1024}$ [radix — the twentieth-root sign] and $\sqrt[10]{1024}$, which multiplied together beget $\sqrt[20]{82944/1024}$, that is, $\sqrt[20]{81}$, or $\sqrt[5]{3}$. Likewise $\sqrt[3]{2}$ and $\sqrt[3]{2}$ are reduced to $\sqrt[3]{4}$ and $\sqrt[3]{16}$, which multiplied together make $\sqrt[3]{64}$, that is, $\sqrt[3]{8}$. Likewise, if the dividend be the number $\sqrt[3]{23328} - \sqrt[3]{10368}$, divided by 6, that is, by $\sqrt[3]{1296}$, the Quotient is $\sqrt[3]{18} - \sqrt[3]{8}$. [The clause closing this example is not clearly legible: [?]]

[Margin: Division by a composite or diminished number of square roots or of fourth-power roots (*radices zensizensicae*).] When the divisor is a composite or

diminished number of square roots or of fourth-power roots, one must multiply it by a like number with the sign changed, + into - and - into +, so that (as was said a little earlier) a single rational number is produced to serve as divisor. Then the dividend must be multiplied by this same transformed number, so that a new dividend is produced, to be divided by the divisor thus found. [Margin: 17 Sept.] For this new dividend stands to the newly found divisor in the same ratio as the proposed dividend to the proposed divisor — because that same transformed multiplier produced both of them; and therefore the same Quotient results whether the proposed dividend is divided by the proposed divisor, or the new dividend found is divided by the new divisor. Just as, because 12 is to 3 as 48 is to 12, the Quotient 4 is always produced, whether 12 is divided by 3 or 48 by 12.

Let 42 be divided by $\sqrt{3} 25 + \sqrt{3} 4$, that is, by 7, where the Quotient is 6. If the divisor is multiplied by $\sqrt{3} 25 - \sqrt{3} 4$, the new divisor 21 is begotten — which, indeed, is what remains if 4, the square of the latter part, is taken from 25, the square of the former part, as was said just above. And if the dividend 42, that is, $\sqrt{3} 1764$, is multiplied by the same number $\sqrt{3} 25 - \sqrt{3} 4$, the new dividend $\sqrt{3} 44100 - \sqrt{3} 7056$ is produced; and dividing this by the divisor found, 21, that is, by $\sqrt{3} 441$, gives the Quotient 6. For $\sqrt{3} 100$, namely 10, minus $\sqrt{3} 16$, that is, minus 4, makes 6, as is plain.

$$\begin{array}{l} \sqrt{3} 44100 - \sqrt{3} 7056 \quad (\sqrt{3} 100 - \sqrt{3} 16 \\ \sqrt{3} 441 \sqrt{3} 441 \end{array}$$

Again, let 54 be divided by $2 + \sqrt{3} 16$, that is, by 6, where the Quotient is 9. If the divisor is multiplied by $2 - \sqrt{3} 16$, the diminished number $4 - 16$ is produced, because the latter part is larger than the former. Therefore the parts of the divisor are to be inverted, in this way: $\sqrt{3} 16 + 2$, so that from the product with $\sqrt{3} 16 - 2$ the simple number 12 is made, to serve as divisor. And if the dividend 54 is multiplied by the same number $\sqrt{3} 16 - 2$, the new dividend $\sqrt{3} 46656 - 108$ is made; and dividing this by 12 — namely $\sqrt{3} 46656$ by $\sqrt{3} 144$, and -108 by 12 — gives the Quotient $\sqrt{3} 324 - 9$, that is, $18 - 9$, namely 9.

Likewise, let 6 be divided by $\sqrt{3} 12 + \sqrt{3} 6$. Multiplying both by $\sqrt{3} 12 - \sqrt{3} 6$ gives $\sqrt{3} 432 - \sqrt{3} 216$, and 6. From the division, then, of $\sqrt{3} 432 - \sqrt{3} 216$ by 6, that is, by $\sqrt{3} 36$, comes the Quotient $\sqrt{3} 12 - \sqrt{3} 6$. This is proved: for from the Quotient $\sqrt{3} 12 - \sqrt{3} 6$ times the divisor $\sqrt{3} 12 + \sqrt{3} 6$ comes the divided number 6.

Likewise, let the dividend be the number $66 - \sqrt{3} 2000$, divided by $8 - \sqrt{3} 45$. If both numbers are multiplied by $8 + \sqrt{3} 45$, the new dividend is $528 - \sqrt{3} 128000 + \sqrt{3} 196020 - \sqrt{3} 90000$, that is, $228 + \sqrt{3} 7220$. (For, $\sqrt{3} 90000$, that is, 300, being taken from 528, the remaining number is 228; and $\sqrt{3} 128000$, taken from $\sqrt{3}$

196020 — since these numbers are commensurable — leaves $\sqrt{3} 7220$.) And the new divisor is 19. From the division there comes the Quotient $12 + \sqrt{3} 20$. For from this Quotient times the divisor $8 - \sqrt{3} 45$ comes the number $96 + \sqrt{3} 1280 - \sqrt{3} 6480 - \sqrt{3} 900$, which is equal to the divided number $66 - \sqrt{3} 2000$. For $\sqrt{3} 900$, that is, 30, taken from 96 leaves 66; and $-\sqrt{3} 6480$ with $+\sqrt{3} 1280$ — that is, by adding one to the other, because of the different signs, since they are commensurable — makes the number $-\sqrt{3} 2000$.

Let 20 likewise be divided by $\sqrt{33} 16 + \sqrt{33} 81$, that is, by 5, where the Quotient is 4. Let the divisor be inverted thus, $\sqrt{33} 81 + \sqrt{33} 16$, so that the latter part is smaller than the former; and let both the dividend 20 and the divisor $\sqrt{33} 81 + \sqrt{33} 16$ be multiplied by $\sqrt{33} 81 - \sqrt{33} 16$. The new dividend becomes $\sqrt{33} 12960000 - \sqrt{33} 2560000$, and the new divisor $\sqrt{33} 6561 - \sqrt{33} 256$, that is, $9 - 4$, namely 5. Dividing that dividend by 5, that is, by $\sqrt{33} 625$, gives the Quotient $\sqrt{33} 20736 - \sqrt{33} 4096$, that is, $12 - 8$, namely 4, as it should be. I have brought forward this example in rational numbers so that you might understand that the reasoning in fourth-power roots (*radices zensizensicae*) is the same as in square roots (*radices zensicae*).

Again, let 10 be divided by $\sqrt{33} 5 + \sqrt{33} 3$. Let both numbers be multiplied by $\sqrt{33} 5 - \sqrt{33} 3$, and the new dividend becomes $\sqrt{33} 50000 - \sqrt{33} 30000$, and the new divisor $\sqrt{33} 25 - \sqrt{33} 9$, that is, $\sqrt{3} 5 - \sqrt{3} 3$. And since this divisor is not yet a single rational number, we shall multiply both the new dividend just found and the new divisor by $\sqrt{3} 5 + \sqrt{3} 3$; and there becomes another new dividend $\sqrt{33} 1250000 - \sqrt{33} 750000 + \sqrt{33} 450000 - \sqrt{33} 270000$, and another new divisor $5 - 3$, that is, 2. If, then, we divide that dividend by this divisor, that is, by $\sqrt{33} 16$, we make the Quotient $\sqrt{33} 78125 - \sqrt{33} 46875 + \sqrt{33} 28125 - \sqrt{33} 16875$. This is proved. For if this Quotient is multiplied by the proposed divisor, namely by $\sqrt{33} 5 + \sqrt{33} 3$, the divided number 10 is produced, as is here shown:

$$\begin{array}{r}
 \sqrt{33} 78125 - \sqrt{33} 46875 + \sqrt{33} 28125 - \sqrt{33} 16875 \\
 \sqrt{33} 5 + \sqrt{33} 3 \\
 \hline
 + \sqrt{33} 234375 - \sqrt{33} 140625 + \sqrt{33} 84375 - \sqrt{33} 50625 \\
 \sqrt{33} 390625 - \sqrt{33} 234375 + \sqrt{33} 140625 - \sqrt{33} 84375 \\
 \hline
 \sqrt{33} 390625 - \sqrt{33} 50625
 \end{array}$$

For since the intermediate numbers — which are equal in pairs and of different signs — cancel one another, there is left the product $\sqrt{33} 390625 - \sqrt{33} 50625$, that is, $25 - 15$, namely 10.

[Margin: Division by a composite number of square roots of three parts or more.] Furthermore, let 100 be divided by $\sqrt{3} 3 + \sqrt{3} 5 + \sqrt{3} 6$. Here, because the divisor has three parts affected with the sign +, in order to find a simple divisor the divisor must be multiplied by a like number with the last part changed from + into -, as you here see. For there will be produced a number of two parts, $\sqrt{3} 60 + 2$, to serve as the new divisor — which may also be obtained in this way: let the first two parts be multiplied together; for if the resulting number $\sqrt{3} 15$ is doubled, $\sqrt{3} 60$ results, for the former part; and if 6, the number of the last part, is taken from 8, the sum of the numbers 3 and 5 of the first two parts, there will remain the number +2, for the latter part.

$$\begin{array}{r}
 \sqrt{3} 3 + \sqrt{3} 5 + \sqrt{3} 6 \\
 \sqrt{3} 3 + \sqrt{3} 5 - \sqrt{3} 6 \\
 \hline
 - \sqrt{3} 18 - \sqrt{3} 30 - 6 \\
 + \sqrt{3} 15 + 5 + \sqrt{3} 30 \\
 + 3 + \sqrt{3} 15 + \sqrt{3} 18 \\
 \hline
 \sqrt{3} 60 + 2
 \end{array}$$

[Margin: 17 Sept.] And if the dividend 100 is multiplied by the same number $\sqrt{3} 3 + \sqrt{3} 5 - \sqrt{3} 6$, there will be begotten the new dividend $\sqrt{3} 30000 + \sqrt{3} 50000 - \sqrt{3} 60000$, which will have to the new divisor found the same ratio as the proposed dividend 100 to the proposed divisor $\sqrt{3} 3 + \sqrt{3} 5 + \sqrt{3} 6$.

But since the new divisor found is not yet simple, we shall multiply it by a like diminished number, namely by $\sqrt{3} 60 - 2$, and we shall make $60 - 4$, that is, 56, for the new divisor. And if the new dividend $\sqrt{3} 30000 + \sqrt{3} 50000 - \sqrt{3} 60000$ is multiplied by the same number $\sqrt{3} 60 - 2$, there will be brought forth the new dividend $\sqrt{3} 1800000 + \sqrt{3} 3000000 + \sqrt{3} 240000 - \sqrt{3} 120000 - \sqrt{3} 200000 - \sqrt{3} 3600000$; and dividing this by the divisor found, 56, there will come out the same Quotient that would arise if the given number 100 were divided by the given divisor $\sqrt{3} 3 + \sqrt{3} 5 + \sqrt{3} 6$.

[Margin: 17 Sept.] Now if we were to multiply the given divisor $\sqrt{3} 3 + \sqrt{3} 5 + \sqrt{3} 6$ and the dividend 100 by $\sqrt{3} 3 - \sqrt{3} 5 + \sqrt{3} 6$ (for when all three parts have the sign +, any one of them may be changed into -), we would produce the new divisor $\sqrt{3} 72 + 4$ [the first sign as printed appears to carry an extra census mark, $\sqrt{33} 72$; read $\sqrt{3} 72$ by the arithmetic: [?]] and the new dividend $\sqrt{3} 30000 - \sqrt{3} 50000 + \sqrt{3} 60000$. And if both this dividend and that divisor $\sqrt{3} 72 + 4$ are multiplied by $\sqrt{3} 72 - 4$, there will be produced another new dividend, and a new simple divisor 56, to which the new dividend will have the same ratio as the given dividend 100 to the given divisor $\sqrt{3} 3 + \sqrt{3} 5 + \sqrt{3} 6$.

But if the divisor has one part with the sign $-$, that sign $-$ must be turned into $+$, so that the multiplication may be carried out.

And in this same way, if the divisor has more parts of square roots than three, you will always change one from $+$ into $-$, or the reverse. For from the first multiplication a number is begotten with just as many parts, less one; and from the second, likewise as many, less two; and so one must proceed step by step until a simple number is obtained for the new divisor. The dividend too must be multiplied by the first multiplying number, then the product by the second, and so on — so that as many multiplications are made as were made to find the simple divisor.

Chapter XXIII (continued) — Division by composite irrational numbers

[The cossist radical signs are rendered as printed and glossed on first use: $\sqrt[3]{\epsilon}$ = cube root (*radix cubica*); $\sqrt{3}$ = square root (*radix quadrata* / census-root); $\sqrt[3]{3}$ = fourth root (*radix zensizensica*); $\sqrt[3]{\text{B}}$ = fifth root (*radix sursolida* / *surdesolida*). The index of each root is fixed by the worked arithmetic throughout.]

When the divisor has two parts that are cube roots [radicum cubicarum], a simple divisor must likewise be found, and this is how it is done. Suppose, for example, that 10 is to be divided by $\sqrt[3]{\epsilon}5 + \sqrt[3]{\epsilon}3$ [cube root of 5 plus cube root of 3]. By the doctrine of Proposition 2, Book 8 of Euclid, three numbers in continued proportion (because 3 is the exponent of the cube) are found in the ratio of the parts of the divisor — namely in the ratio of $\sqrt[3]{\epsilon}5$ to $\sqrt[3]{\epsilon}3$ — by this method. First multiply $\sqrt[3]{\epsilon}5$ into itself, and the first number, $\sqrt[3]{\epsilon}25$, results. Next multiply $\sqrt[3]{\epsilon}5$ into $\sqrt[3]{\epsilon}3$, and the second number, $\sqrt[3]{\epsilon}15$, is produced. Third, multiply $\sqrt[3]{\epsilon}3$ also into itself, and the third number, $\sqrt[3]{\epsilon}9$, comes out. Thus these three products, $\sqrt[3]{\epsilon}25$, $\sqrt[3]{\epsilon}15$, and $\sqrt[3]{\epsilon}9$, will be in continued proportion in the ratio of $\sqrt[3]{\epsilon}5$ to $\sqrt[3]{\epsilon}3$, as was demonstrated by Euclid in the said Proposition 2, Book 8.

This may also be tested by multiplication. For with the three numbers $\sqrt[3]{\epsilon}25$, $\sqrt[3]{\epsilon}15$, $\sqrt[3]{\epsilon}9$ set down, the product of the first into the third is as great as that of the middle into itself. Likewise, with the four numbers $\sqrt[3]{\epsilon}5$, $\sqrt[3]{\epsilon}3$, $\sqrt[3]{\epsilon}25$, $\sqrt[3]{\epsilon}15$ set down, the product of the first into the fourth is as great as that of the second into the third — which is also seen in these four: $\sqrt[3]{\epsilon}5$, $\sqrt[3]{\epsilon}3$, $\sqrt[3]{\epsilon}15$, $\sqrt[3]{\epsilon}9$.

Let the two extreme numbers be given the sign $+$, and the middle one the sign $-$, in this way: $\sqrt[3]{\epsilon}25 - \sqrt[3]{\epsilon}15 + \sqrt[3]{\epsilon}9$. If by this common multiplying number both the dividend 10 and the divisor $\sqrt[3]{\epsilon}5 + \sqrt[3]{\epsilon}3$ are multiplied, the new dividend becomes

$\sqrt[5]{25000} - \sqrt[5]{15000} + \sqrt[5]{9000}$, and the new divisor becomes $\sqrt[5]{125} + \sqrt[5]{27}$ — that is, 8 — as is clear in the accompanying display. This is obtained as a simple divisor (as the authorities teach, and it is worth noting) if the numbers of the parts of the divisor, 5 and 3, are simply added together, even without carrying out any multiplication.

$$\begin{array}{r}
 \sqrt[5]{25} - \sqrt[5]{15} + \sqrt[5]{9} \\
 \sqrt[5]{5} + \sqrt[5]{3} \\
 \hline
 + \sqrt[5]{75} - \sqrt[5]{45} + \sqrt[5]{27} \\
 + \sqrt[5]{125} - \sqrt[5]{75} + \sqrt[5]{45} \\
 \hline
 \text{Summa. } \sqrt[5]{125} + \sqrt[5]{27}
 \end{array}$$

[Margin: Division by a number composed of cube roots.]

The same procedure is to be followed with a divisor composed of two zensizensic roots [$\sqrt[3]{3}$, fourth roots — *radicum zensizensicarum*], surdesolid roots [$\sqrt[5]{5}$, fifth roots], zensicubic roots [sixth roots], and so on — provided that as many numbers in continued proportion are found, according to the doctrine of Proposition 2, Book 8 of Euclid, in the ratio of the parts of the divisor, as there are units in the exponent 33 (4), 5 (5), 36 (6), etc.; namely four for a divisor of zensizensic roots, five for a divisor of surdesolid roots, six for zensicubic roots, and so forth. Once these are found, assign to the first number the sign +, to the second the sign —, to the third again +, and to the fourth again —; and proceeding alternately in this way, so that the numbers in the odd positions carry the sign + and those in the even positions the sign —.

I will give an example in the division of 10 by $\sqrt[3]{35} + \sqrt[3]{33}$, which we carried out above by another method. Since the exponent 33 is 4, these four proportional numbers are found — $\sqrt[3]{3125}$, $\sqrt[3]{3375}$, $\sqrt[3]{3345}$, $\sqrt[3]{3327}$ — according to the doctrine of Proposition 2, Book 8 of Euclid, as you see here (where we have set the sign + on the numbers in the odd positions and the sign — on those in the even positions, as we said was to be done):

$$\begin{array}{r}
 \sqrt[3]{35} + \sqrt[3]{33} \\
 \sqrt[3]{325}. \sqrt[3]{315}. \sqrt[3]{39} \\
 \sqrt[3]{3125} - \sqrt[3]{3375} + \sqrt[3]{3345} - \sqrt[3]{3327}
 \end{array}$$

Now if both the dividend 10 and the divisor $\sqrt[3]{35} + \sqrt[3]{33}$ are multiplied by this four-part number, the new dividend $\sqrt[3]{31250000} - \sqrt[3]{33750000} + \sqrt[3]{33450000} - \sqrt[3]{33270000}$ is found, and the new divisor $\sqrt[3]{3625} - \sqrt[3]{381}$ — that is, 2 — exactly the same as those we had above before dividing, by the three-part number composed of square roots. And this divisor is obtained, without multiplication (as

the authorities teach), if the number of the smaller part in the given divisor is subtracted from the number of the larger part — namely 3 from 5 — as these displays show.

Finding the dividend.

$$\begin{array}{r}
 10. \text{ hoc est, } \sqrt[3]{310000} \\
 \sqrt[3]{3125} - \sqrt[3]{375} + \sqrt[3]{345} - \sqrt[3]{327} \\
 \hline
 \sqrt[3]{31250000} - \sqrt[3]{3750000} + \sqrt[3]{3450000} - \sqrt[3]{3270000}
 \end{array}$$

Finding the divisor.

$$\begin{array}{r}
 \sqrt[3]{3125} - \sqrt[3]{375} + \sqrt[3]{345} - \sqrt[3]{327} \\
 \sqrt[3]{35} + \sqrt[3]{33} \\
 \hline
 + \sqrt[3]{3375} - \sqrt[3]{3225} + \sqrt[3]{3135} - \sqrt[3]{381} \\
 + \sqrt[3]{3625} - \sqrt[3]{3375} + \sqrt[3]{3225} - \sqrt[3]{3135} \\
 \hline
 \sqrt[3]{3625} - \sqrt[3]{381}
 \end{array}$$

[Margin: Division by a number composed of zensizensic roots, surdesolid roots, etc.]

The same reasoning holds for the rest. But when the exponent number of the roots is even — as $\sqrt[3]{}$ (2), $\sqrt[4]{}$ (4), etc. — a simple divisor is obtained (as the authorities teach) if the number of the smaller part in the given divisor is subtracted from the number of the larger part; whereas when the exponent number is odd — as $\sqrt[5]{}$ (3), $\sqrt[6]{}$ (5), etc. — a simple divisor is obtained if the numbers of the parts of the given divisor are collected into a single sum. This is what you see was observed in the cases above.

Nor should this be omitted: when the divisor consists of two different [kinds of] roots, they must be reduced to the same denomination before you begin the operation.

Finally, when the divisor has several parts, of different roots, so that it cannot easily be reduced to a simple divisor, a fraction must be set up whose numerator is the dividend and whose denominator is the divisor itself — just as is customarily done with absolute numbers, when a smaller number is to be divided by a larger, as 3 by 4, where the fraction $3/4$ results. For example, if the number $\sqrt[3]{48} + \sqrt[5]{3}$ [square root of 48 plus cube root of 3] is to be divided by $\sqrt[3]{15} + \sqrt[6]{6} - \sqrt[3]{3}$, this fraction results:

Numerator. $\sqrt{348} + \sqrt{3}$

Denominator. $\sqrt{315} + \sqrt{6} - \sqrt{3}$

[Margin: Division by a number composed of different roots.]

Scholium

[Margin: Which of two composite numbers is the greater.]

It happens not rarely that, between two composite irrational numbers, it is doubtful which of them is the greater — an uncertainty that is removed as follows. Let two numbers be proposed, $4 + \sqrt{37}$ and $20 - \sqrt{3180}$. Subtract 4 from each, and there will remain on the one side $\sqrt{37}$, and on the other $16 - \sqrt{3180}$. Now it is certain that if the two given numbers are equal, the remainders are also equal; and if they are unequal, the remainders are unequal in the same way. Square each remainder: the square of the first remainder will be 7, and that of the second, $436 - \sqrt{3184320}$. It is likewise certain that if those remainders are unequal, these squares are also unequal. Add to each square this subtracted quantity, $\sqrt{3184320}$, and the numbers become $7 + \sqrt{3184320}$ and 436. Take 7 from each side, and the remaining numbers are $\sqrt{3184320}$ and 429. Square these, so that they become 184320 and 184041. And because the former square is greater than the latter, you will declare the number $4 + \sqrt{37}$ to be greater than the number $20 - \sqrt{3180}$. In the same way you will proceed with other numbers.

This is also clear through the extraction of roots. For the approximate root of 7 is $2 \frac{3}{5}$, which with 4 makes $6 \frac{3}{5}$. But the approximate root of 180 is $13 \frac{11}{27}$, which, subtracted from 20, leaves $6 \frac{16}{27}$. Now it is established that $\frac{3}{5}$ is a greater number than $\frac{16}{27}$. And so you have another way of removing the ambiguity — but the former method is more accurate.

ON THE ROOTS OF IRRATIONAL COMPOSITE OR DIMINISHED NUMBERS — which others call *universal* — AND ON THEIR ALGORITHM.

Chapter XXIV

[Margin: The various roots of composite numbers.]

These roots usually arise from the addition and subtraction of simple incommensurable irrational roots, as was shown in chapters 19 and 20. We propose here to explain them a little more carefully, which we shall do by setting

out the various roots of composite numbers together with their algorithm. If, then, the square root is to be extracted from this composite number $10 + \sqrt{37}$, it will be $\sqrt{3}(10 + \sqrt{37})$. The sense of this is that the square root of the number 7 (if it could be had), added to 10, gives the square root taken of the whole number. But in this other root, $\sqrt[3]{10 + \sqrt{37}}$, we understand the cube root of the number composed of 10 and $\sqrt{37}$.

And so that the matter may be grasped more plainly, we shall give examples in rational numbers. For instance, $\sqrt{3}(10 + \sqrt{316} + 3 + \sqrt{364})$ means nothing other than 5. For $\sqrt{316}$ is 4, which added to 3 makes 7; and 7 with $\sqrt{364}$ — that is, with 8 — makes 15; and if 10 is added to these, they become 25, the value of the whole composite number, whose square root is 5. Likewise $\sqrt[3]{\sqrt{325} + \sqrt{39}}$ will be 2. For the sense is that, taking the square roots of the numbers 25 and 9, which are 5 and 3, making 8, the cube root is extracted from this sum, which is 2. And this root, $\sqrt{3}(10 + \sqrt{336}) + \sqrt{3}(70 + \sqrt{3121})$, will be worth 13. For it signifies that the square root of the composite $10 + \sqrt{336}$ — which is 16, namely $\sqrt{316}$, which is 4 — is added to 9, the square root of the composite $70 + \sqrt{3121}$, which is 81.

[Margin: Squares and cubes of the roots of composite numbers.]

Now, just as the square of the square root of any simple surd is the number itself, once the radical sign $\sqrt{3}$ is removed; and the cube of a simple cube root is the number itself, once the radical sign $\sqrt[3]{}$ is removed — as the square of $\sqrt{35}$ is 35, and the square of $\sqrt{325}$ is 25, while the cube of $\sqrt[3]{12}$ is 12, and the cube of $\sqrt[3]{27}$ is 27 — because any square root multiplied into itself squarewise yields the square of that root, but multiplied into itself cubewise produces the cube of the same root: so too the square of the square root of any composite number is the composite number itself, once the sign $\sqrt{3}$ is removed; and the cube of the cube root of any composite number is the number itself, once the sign $\sqrt[3]{}$ is removed.

Thus the square of $\sqrt{3}(11 + \sqrt{39} + \sqrt{34})$ is $11 + \sqrt{39} + \sqrt{34}$, that is, 16, so that that root is worth 4.

The square of $\sqrt{3}(\sqrt{349} + \sqrt{336} - \sqrt{316})$ is $\sqrt{349} + \sqrt{336} - \sqrt{316}$, that is, 9, so that that root is worth 3.

The square of $\sqrt{3}(\sqrt{364} + \sqrt{349} + \sqrt{336} - 5)$ is $\sqrt{364} + \sqrt{349} + \sqrt{336} - 5$, that is, 16, so that the value of that root is 4.

The square of $\sqrt{3}(\sqrt[3]{216} + \sqrt[3]{27})$ is $\sqrt[3]{216} + \sqrt[3]{27}$, that is, 9, and the value of its root is 3.

The square of $\sqrt{3}(\sqrt{216} - \sqrt{27})$ is $\sqrt{216} - \sqrt{27}$, that is, 3, and the value of its root is $\sqrt{3}$.

The cube of $\sqrt{2}(\sqrt{325} + \sqrt{39})$ is $\sqrt{325} + \sqrt{39}$, that is, 8, so that that root is worth 2.

The cube of $\sqrt{2}(\sqrt{325} - \sqrt{39})$ is $\sqrt{325} - \sqrt{39}$, that is, 2, and the value of that root is $\sqrt{2}$.

The cube of $\sqrt{2}(\sqrt{64} + \sqrt{27})$ is $\sqrt{64} + \sqrt{27}$, that is, 7, so that that root is $\sqrt{2}$.

You will make the same judgment about the zensizensic and surdesolid roots (etc.) of composite numbers. I have set out the examples in rational numbers so that the matter might be clearer. And in this way the root of any composite number is multiplied into itself — that is, its square, or cube, etc., is produced.

[Margin: Multiplication of the root of a composite number into itself.]

But the root of a composite number will be multiplied by a simple root, or by an absolute number, or finally by another root of a composite number, in the following way. Let each number, both the multiplicand and the multiplier, be reduced to a square or a cube. Then let the multiplication be carried out as was taught above; for the root of the whole product will be the number produced by the multiplication. For example, from $\sqrt{5}$ into $\sqrt{12}$ comes $\sqrt{60}$ — namely the square root of the number 60, produced from the multiplication of the squares 5 and 12. Likewise, just as from $\sqrt[3]{9}$ into $\sqrt[3]{10}$ comes $\sqrt[3]{90}$ — that is, the cube root of the number 90, which is generated from the multiplication of the cube 9 by the cube 10, as was taught in the preceding chapter. But you will learn this more clearly from the following examples.

[Margin: Multiplication of the roots of composite numbers.]

Chapter XXIV (continued) — worked multiplications

Let $\sqrt{3}(7 + \sqrt{33})$ be multiplied by 2. The squares are 7 + $\sqrt{33}$ and 4. If, then, 4 is multiplied into 7, they make 28; and if $\sqrt{33}$ is multiplied into 4 — that is, into $\sqrt{316}$ — it makes $\sqrt{348}$. So the whole product is $\sqrt{3}(28 + \sqrt{348})$.

$$\begin{array}{r} 7 + \sqrt{33} \\ 4 \\ \hline 28 + \sqrt{348} \end{array}$$

That this is rightly done in this way by multiplication you will learn from this other example in rational numbers. Let $\sqrt{3}(7 + \sqrt{34})$ — that is, 3 (for 7 with $\sqrt{34}$ make 9, whose square root is 3) — be multiplied by 2. Here it is plain that the

number produced will be 6. The squares are $7 + \sqrt{34}$ and 4. If, then, 4 is multiplied into 7, they make 28; and if $\sqrt{34}$ is multiplied into 4 — that is, into $\sqrt{316}$ — it makes $\sqrt{364}$. And the whole product will be $\sqrt{3}(28 + \sqrt{364})$, that is, $\sqrt{336}$, namely 6, as it should be.

$$\begin{array}{r} 7 + \sqrt{34} \\ 4 \\ \hline 28 + \sqrt{364} \end{array}$$

Again, let $\sqrt{3}(7 + \sqrt{34})$ be multiplied by 3 — that is, 3 by 3, where the product is 9. The squares are $7 + \sqrt{34}$ and 9. From 9 into 7 come 63; and from $\sqrt{34}$ into 9 — that is, into $\sqrt{381}$ — comes $\sqrt{3324}$. So the product will be $\sqrt{3}(63 + \sqrt{3324})$, that is, $\sqrt{381}$, namely 9.

$$\begin{array}{r} 7 + \sqrt{34} \\ 9 \\ \hline 63 + \sqrt{3324} \end{array}$$

Likewise let $\sqrt{3}(\sqrt{325} + \sqrt{316})$ — that is, $\sqrt{39}$, or 3 — be multiplied by $\sqrt{39}$, that is, by 3, where it is likewise established that the product will be 9. The squares are $\sqrt{325} + \sqrt{316}$ and 9. From $\sqrt{325}$ into 9 — that is, into $\sqrt{381}$ — comes $\sqrt{32025}$; and from $\sqrt{316}$ into 9 — that is, into $\sqrt{381}$ — comes $\sqrt{31296}$. So the number produced will be $\sqrt{3}(\sqrt{32025} + \sqrt{31296})$, that is, $\sqrt{3}(45 + 36)$, or $\sqrt{381}$, that is, 9.

$$\begin{array}{r} \sqrt{325} + \sqrt{316} \\ 9 \\ \hline \sqrt{32025} + \sqrt{31296} \end{array}$$

Let $\sqrt[3]{\mathfrak{C}}(\sqrt[3]{\mathfrak{C}64} + \sqrt[3]{\mathfrak{C}27})$ — that is, $\sqrt[3]{\mathfrak{C}7}$ — be multiplied by 2, where the product will be $\sqrt[3]{\mathfrak{C}56}$. The cubes of the numbers are $\sqrt[3]{\mathfrak{C}64} + \sqrt[3]{\mathfrak{C}27}$ and 8. From $\sqrt[3]{\mathfrak{C}64}$ into 8 — that is, into $\sqrt[3]{\mathfrak{C}512}$ — comes $\sqrt[3]{\mathfrak{C}32768}$; and from $\sqrt[3]{\mathfrak{C}27}$ into 8 — that is, into $\sqrt[3]{\mathfrak{C}512}$ — comes $\sqrt[3]{\mathfrak{C}13824}$. Since, then, $\sqrt[3]{\mathfrak{C}32768}$ becomes 32, and $\sqrt[3]{\mathfrak{C}13824}$ becomes 24, the product number will be $\sqrt[3]{\mathfrak{C}}(32 + 24)$, that is, $\sqrt[3]{\mathfrak{C}56}$.

$$\begin{array}{r} \sqrt[3]{\mathfrak{C}64} + \sqrt[3]{\mathfrak{C}27} \\ 8 \\ \hline \sqrt[3]{\mathfrak{C}32768} + \sqrt[3]{\mathfrak{C}13824} \end{array}$$

Let $\sqrt[3]{\mathfrak{C}}(64 + \sqrt[3]{\mathfrak{C}27})$ be multiplied by 2. The cubes of the numbers are $64 + \sqrt[3]{\mathfrak{C}27}$ and 8. From 64 into 8 come 512; and from $\sqrt[3]{\mathfrak{C}27}$ into 8 — that is, into $\sqrt[3]{\mathfrak{C}512}$ — comes $\sqrt[3]{\mathfrak{C}13824}$, which is 24, and added to 512 makes 536. So the product number is

$\sqrt[3]{536}$, and so it is. For $\sqrt[3]{(64 + \sqrt[3]{27})}$ — that is, $\sqrt[3]{67}$ — multiplied into 2, that is, into $\sqrt[3]{8}$, makes $\sqrt[3]{536}$.

$$\begin{array}{r} 64 + \sqrt[3]{27} \\ 8 \end{array}$$

$$512 + \sqrt[3]{13824}$$

Let $\sqrt[3]{(\sqrt[3]{64} + \sqrt[3]{36} + 3)}$ be multiplied by 5. The cubes of the numbers are $\sqrt[3]{64} + \sqrt[3]{36} + 3$ and 125. From $\sqrt[3]{64}$ into 125 — that is, into $\sqrt[3]{1953125}$ — comes $\sqrt[3]{125000000}$. From $\sqrt[3]{36}$ into 125 — that is, into $\sqrt[3]{315625}$ — comes $\sqrt[3]{3562500}$. And from 3 into 125 come 375. Since, then, $\sqrt[3]{125000000}$ becomes 500, and $\sqrt[3]{3562500}$ becomes 750, the number produced will be $\sqrt[3]{1625}$, as the reasoning requires. For $\sqrt[3]{(\sqrt[3]{64} + \sqrt[3]{36} + 3)}$ — that is, $\sqrt[3]{13}$ — multiplied into 5, that is, into $\sqrt[3]{125}$, makes $\sqrt[3]{1625}$.

$$\begin{array}{r} \sqrt[3]{64} + \sqrt[3]{36} + 3 \\ 125 \end{array}$$

$$\sqrt[3]{125000000} + \sqrt[3]{3562500} + 375$$

If the number to be multiplied by 5 had been $\sqrt[3]{(\sqrt[3]{64} + \sqrt[3]{36} - 3)}$, the product would have been $\sqrt[3]{(125000000 + \sqrt[3]{3562500} - 375)}$, that is, $\sqrt[3]{(500 + 750 - 375)}$, that is, $\sqrt[3]{875}$ — which is correct. For $\sqrt[3]{(\sqrt[3]{64} + \sqrt[3]{36} - 3)}$ is $\sqrt[3]{7}$, which multiplied into 5 — that is, into $\sqrt[3]{125}$ — makes $\sqrt[3]{875}$.

Let $\sqrt[3]{(12 + \sqrt[3]{36})}$ be multiplied by 6. The squares of the numbers are $12 + \sqrt[3]{36}$ and 36. From 12 into 36 come 432; and from $\sqrt[3]{36}$ into 36 — that is, into $\sqrt[3]{31296}$ — comes $\sqrt[3]{37776}$. So the number produced by the multiplication is $\sqrt[3]{(432 + \sqrt[3]{37776})}$.

$$\begin{array}{r} 12 + \sqrt[3]{36} \\ 36 \end{array}$$

$$432 + \sqrt[3]{37776}$$

Let $\sqrt[3]{(13 + \sqrt[3]{144})}$ — that is, $\sqrt[3]{25}$, i.e. 5 — be multiplied by 6, where it is certain that 30 is generated. The squares of the numbers are $13 + \sqrt[3]{144}$ and 36. From 13 into 36 come 468; and from $\sqrt[3]{144}$ into 36 — that is, into $\sqrt[3]{31296}$ — comes $\sqrt[3]{3186624}$. So the number produced is $\sqrt[3]{(468 + \sqrt[3]{3186624})}$, that is, $\sqrt[3]{3900}$, i.e. 30.

$$\begin{array}{r} 13 + \sqrt[3]{144} \\ 36 \end{array}$$

$$468 + \sqrt[3]{3186624}$$

Let $\sqrt{3}(13 + \sqrt{3}9)$ be multiplied by $\sqrt{3}(5 + \sqrt{3}16)$ — that is, $\sqrt{3}16$, i.e. 4, by $\sqrt{3}9$, i.e. by 3 — where it is established that 12 is produced. The squares of the numbers are $13 + \sqrt{3}9$ and $5 + \sqrt{3}16$. From $\sqrt{3}9$ into $\sqrt{3}16$ comes $\sqrt{3}144$. From $\sqrt{3}16$ into 13 — that is, into $\sqrt{3}169$ — comes $\sqrt{3}2704$. Then from $\sqrt{3}9$ into 5 — that is, into $\sqrt{3}25$ — comes $\sqrt{3}225$. And from 5 into 13 come 65. Now $\sqrt{3}144$ — that is, 12 — with 65 makes 77; and if to these are added $\sqrt{3}2704$ and $\sqrt{3}225$, namely 52 and 15, they make 144. Therefore $\sqrt{3}144$ — that is, 12 — is the product number, as was said.

$$\begin{array}{r}
 13 + \sqrt{3}9 \\
 5 + \sqrt{3}16 \\
 \hline
 \sqrt{3}2704 + \sqrt{3}144 \\
 65 + \sqrt{3}225 \\
 \hline
 \sqrt{3}2704 + \sqrt{3}225 + 77
 \end{array}$$

Again, let $\sqrt{3}(\sqrt{3}1800 + 30)$ — $\sqrt{3}(\sqrt{3}450 + 15)$ be multiplied by $\sqrt{3}(\sqrt{3}450 + 15)$. Let the numbers be written as you see here. From $\sqrt{3}(\sqrt{3}450 + 15)$ into $\sqrt{3}(\sqrt{3}450 + 15)$ comes its square, $\sqrt{3}450 + 15$, with the sign — prefixed, since + is multiplied into —. From $\sqrt{3}(\sqrt{3}1800 + 30)$ into $\sqrt{3}(\sqrt{3}450 + 15)$ comes $\sqrt{3}810000 + \sqrt{3}1620000 + 450$; for indeed if the square [of the number ...]

$$\begin{array}{r}
 \sqrt{3}(\sqrt{3}1800 + 30) - \sqrt{3}(\sqrt{3}450 + 15) \\
 \sqrt{3}(\sqrt{3}450 + 15) \sqrt{3}(\sqrt{3}450 + 15) \\
 \hline
 + \sqrt{3}4050000 \text{ [recte: } 405000] + 450 - \sqrt{3}450 + 15 \\
 \sqrt{3}810000 + \sqrt{3}4050000 \text{ [recte: } 405000] \\
 \hline
 \sqrt{3}810000 + \sqrt{3}16200000 \text{ [recte: } 1620000] + 450
 \end{array}$$

Throughout these pages the sign $\sqrt{3}$ is Clavius's *radix* mark as printed — a $\sqrt{}$ followed by the census-glyph; standing before a single number it is an ordinary square root, and standing before a parenthesized binomial it is the *radix universalis*, binding the whole expression. The bare 3 is the *census* (the second power, x^2) and 33 the *zenzizensus* (the fourth power, x^4); $\sqrt[3]{}$ is the *radix surdesolida* (fifth root) and $\sqrt[3]{}$ the *radix cubica* (cube root, printed $\sqrt[3]{}$). The signs are reproduced as printed.

the two numbers be multiplied by each other: that is, $1350 + \sqrt{3}1620000$. For $\sqrt{3}810000$ is equal to 900, and this number added to 450 makes 1350; so, with the sign $\sqrt{3}$ prefixed, the product will be the number $\sqrt{3}(1350 + \sqrt{3}1620000)$. Now the root of this binomial $1350 + \sqrt{3}1620000$ is the number $\sqrt{3}900 + \sqrt{3}450$ — as will be evident from Cap. 28 — that is, $30 + \sqrt{3}450$. If from this the earlier product, $-\sqrt{3}450 + 15$ (produced from $\sqrt{3}(\sqrt{3}450 + 15)$ multiplied by itself), be subtracted,

the number 15 will remain, as you see in this formula. And this indeed is generated by the multiplication $\sqrt{3}(\sqrt{3}1800 + 30) - \sqrt{3}(\sqrt{3}450 + 15)$ into $\sqrt{3}(\sqrt{3}450 + 15)$:

$$\begin{array}{r} 30 + \sqrt{3} 450 \\ 15 - \sqrt{3} 450 \\ \hline 15 \end{array}$$

It will become clear that the multiplication is rightly carried out in this way once this other example is set out, where the multiplication is 1 by 3 and the resulting number is 3. For from $\sqrt{3}(\sqrt{3}89 + 6)$ into $\sqrt{3}(\sqrt{3}89 + 6)$ comes its square, $\sqrt{3}89 + 6$, with the sign $-$ prefixed, since $+$ is multiplied into $-$. Then from $\sqrt{3}(\sqrt{3}84 + 14)$ into $\sqrt{3}(\sqrt{3}89 + 6)$ comes the number $\sqrt{3}144$, that is, 12, as you see in the preceding formula. For $\sqrt{3}36$ is 6, $\sqrt{3}144$ is 12, and $\sqrt{3}1764$ is 42, and all these together with 84 make 144 — namely $\sqrt{3}144$, that is, 12 — from which if the other product $\sqrt{3}89 + 6$, namely 9, be subtracted, 3 will remain for the whole product:

$$\begin{array}{r} \sqrt{3}(\sqrt{3}84 + 14) - \sqrt{3}(\sqrt{3}89 + 6) \\ \sqrt{3}(\sqrt{3}89 + 6) \sqrt{3}(\sqrt{3}89 + 6) \\ \hline + \sqrt{3}144 + 84 - \sqrt{3}89 + 6 \\ \sqrt{3}36 + \sqrt{3}1764 \\ \hline \sqrt{3}144 — 9 \end{array}$$

And with the sign $\sqrt{3}$ prefixed, the product $\sqrt{3}144$ is obtained — namely 12 — from which if the other product $\sqrt{3}89 + 6$, namely 9, be taken away, 3 remain for the whole product number.

Let $\sqrt{3}(6 + \sqrt{3}9)$ also be multiplied by the composite number $\sqrt{3}4 + \sqrt{3}16$. The square of the first number is $6 + \sqrt{3}9$, and of the second $20 + \sqrt{3}256$, as is plain if $\sqrt{3}4 + \sqrt{3}16$ be multiplied into $\sqrt{3}4 + \sqrt{3}16$. The multiplication then being carried out, as you see here, the product will be the number $\sqrt{3}(\sqrt{3}9216 + \sqrt{3}3600 + \sqrt{3}2304 + 120)$, that is, $\sqrt{3}324$, namely 18:

$$\begin{array}{r} 6 + \sqrt{3} 9 \\ 20 + \sqrt{3} 256 \\ \hline \sqrt{3}9216 + \sqrt{3}2304 \\ 120 + \sqrt{3}3600 \\ \hline \sqrt{3}9216 + \sqrt{3}3600 + \sqrt{3}2304 + 120 \end{array}$$

Let $\sqrt{3}(7 + \sqrt{3}4)$, that is $\sqrt{3}9$, namely 3, be multiplied by $\sqrt{3}9$, that is, by 3, where it is plain that the number generated is 9. The squares of the numbers are $7 + \sqrt{3}4$

and 9. From $\sqrt{3}4$ into 9, that is, into $\sqrt{3}81$, comes $\sqrt{3}324$; and from 7 into 9 come 63. Therefore the product number will be $\sqrt{3}(63 + \sqrt{3}324)$, that is, $\sqrt{3}81$, or 9, as we said:

$$\begin{array}{r} 7 + \sqrt{3} 4 \\ 9 \\ \hline 63 + \sqrt{3} 324 \end{array}$$

Let $\sqrt{3}(12 + \sqrt{3}6)$ be multiplied by $\sqrt{3}12 + \sqrt{3}6$. The squares of the numbers are $12 + \sqrt{3}6$ and $18 + \sqrt{3}288$, as is established if $\sqrt{3}12 + \sqrt{3}6$ be multiplied into itself. (For that number $12 + \sqrt{3}288 + 6$ is produced, and it is agreed that 12 and 6 make 18.) From $\sqrt{3}288$ into $\sqrt{3}6$ comes $\sqrt{3}1728$. And from $\sqrt{3}288$ into 12, that is, into $\sqrt{3}144$, comes $\sqrt{3}41472$. And from $\sqrt{3}6$ into 18, that is, into $\sqrt{3}324$, comes $\sqrt{3}1944$. And finally from 12 into 18 come 216. So the whole product number will be $\sqrt{3}(216 + \sqrt{3}1944 + \sqrt{3}41472 + \sqrt{3}1728)$:

$$\begin{array}{r} 12 + \sqrt{3} 6 \\ 18 + \sqrt{3} 288 \\ \hline \sqrt{3}41472 + \sqrt{3}1728 \\ 216 \sqrt{3}1944 \\ \hline 216 + \sqrt{3}1944 + \sqrt{3}41472 + \sqrt{3}1728 \end{array}$$

[Margin: Multiplication of the root of a composite number, together with the root of the similar diminished number, into itself.]

Moreover, when the square root of a composite number, together with the square root of the similar diminished number, is set out to be multiplied into itself — which happens not rarely with binomials and residuals (of which in Cap. 27 and 28) — we shall proceed as follows. Let $\sqrt{3}(12 + \sqrt{3}6) + \sqrt{3}(12 - \sqrt{3}6)$ be multiplied into itself. The number is written twice, as is usually done in multiplication. And since, by Euclid *Elements* II.4 [propos. 4, lib. 2], the squares of the parts $\sqrt{3}(12 + \sqrt{3}6)$ and $\sqrt{3}(12 - \sqrt{3}6)$, together with what is made by one part $\sqrt{3}(12 - \sqrt{3}6)$ taken twice into the other part $\sqrt{3}(12 + \sqrt{3}6)$, are equal to what is made by the number into itself: now the squares of the parts are $12 + \sqrt{3}6$ and $12 - \sqrt{3}6$, which together make 24 (for $+\sqrt{3}6$ and $-\sqrt{3}6$ cancel each other out); and from one part into the other comes $\sqrt{3}138$ (because 6, the square of $\sqrt{3}6$, taken from 144, the square of the number 12, leaves 138, to which number the sign $\sqrt{3}$ must be prefixed), whose double is $\sqrt{3}552$. The product number is therefore $24 + \sqrt{3}552$:

$$\begin{array}{r} \sqrt{3}(12 + \sqrt{36}) + \sqrt{3}(12 - \sqrt{36}) \\ \sqrt{3}(12 + \sqrt{36}) + \sqrt{3}(12 - \sqrt{36}) \\ \hline \end{array}$$

$$\begin{array}{r} 12 + \sqrt{36} + 12 - \sqrt{36} \\ \sqrt{3} \ 138 \\ \sqrt{3} \ 138 \\ \hline \end{array}$$

$$\begin{array}{r} \sqrt{3} \ 552 \\ 24 + \sqrt{3} \ 552 \end{array}$$

[Margin: The sum gathered from the root of a composite number and the root of the similar diminished number.]

And the root of this product is likewise the sum of the two roots $\sqrt{3}(12 + \sqrt{36})$ and $\sqrt{3}(12 - \sqrt{36})$ collected into one sum. For when the sum of two numbers is multiplied into itself, the square root of the product number is the sum of those same two numbers. And so, since these two proposed roots multiplied together into themselves make $24 + \sqrt{3552}$, the root of this product — namely $\sqrt{3}(24 + \sqrt{3552})$ — will be the sum of the two roots.

Another example in rational numbers. Let $\sqrt{3}(10 + \sqrt{336}) + \sqrt{3}(10 - \sqrt{336})$, that is 6, be multiplied into itself, where the number produced is 36. The squares of the parts are $10 + \sqrt{336}$ and $10 - \sqrt{336}$, which added together make 20. And from one part into the other comes $\sqrt{364}$, which taken twice makes $\sqrt{3256}$, that is, 16; and so the product number is $20 + 16$, that is, 36, as it should be. And the root of this number, namely 6, is likewise the sum collected from those two roots, as is manifest:

$$\begin{array}{r} \sqrt{3}(10 + \sqrt{336}) + \sqrt{3}(10 - \sqrt{336}) \\ \sqrt{3}(10 + \sqrt{336}) + \sqrt{3}(10 - \sqrt{336}) \\ \hline \end{array}$$

$$\begin{array}{r} 10 + \sqrt{336} + 10 - \sqrt{336} \\ \sqrt{3} \ 64 \\ \sqrt{3} \ 64 \\ \hline \end{array}$$

$$\begin{array}{r} \sqrt{3} \ 256 \\ \text{Product } 20 + \sqrt{3256}, \text{ that is, } 36. \end{array}$$

Furthermore, when a number composed of a simple root and the root of a composite number — that is, of a simple root and a universal root — must be multiplied by a similar diminished number, the shortcut handed down in Cap. 23 also applies, before the examples of division, namely at the end of the examples of multiplication.

CAPVT XXIV.

For if the number $\sqrt{3}20 + \sqrt{3}(20 - \sqrt{35})$ is to be multiplied into the number $\sqrt{3}20 - \sqrt{3}(20 - \sqrt{35})$, let one number be written beneath the other, as you see here. For if $20 - \sqrt{35}$, the square of the latter part of the composite or diminished number, be taken from 20, the square of the first part, the remaining number will be $0 + \sqrt{35}$, as is seen in this other formula. For 20 from 20 leaves nothing; but $-\sqrt{35}$ from $+\sqrt{30}$ (since different signs change the species, and it becomes an addition) leaves $+\sqrt{35}$, as we said in the subtraction of simple roots. And so from $\sqrt{3}20 + \sqrt{3}(20 - \sqrt{35})$ into $\sqrt{3}(20 - \sqrt{30} + \sqrt{35})$, $\sqrt{35}$ is produced:

$$\begin{array}{r} \sqrt{3} 20 + \sqrt{3}(20 - \sqrt{35}) \\ \sqrt{3} 20 - \sqrt{3}(20 - \sqrt{35}) \end{array}$$

$$\sqrt{3} 5$$

$$\begin{array}{r} 20 + \sqrt{3} 0 \\ 20 - \sqrt{3} 5 \end{array}$$

$$0 + \sqrt{3} 5$$

Indeed it will also be permitted to use this shortcut when a number composed of a cossic number and the root of a composite or diminished cossic number must be multiplied by a similar diminished number. For if the number $\frac{1}{2}\sqrt{3} + \sqrt{3}(\frac{1}{4}\sqrt{33} + 13)$ is to be multiplied into $\frac{1}{2}\sqrt{3} - \sqrt{3}(\frac{1}{4}\sqrt{33} + 13)$, with one number written beneath the other as you see here, the square $\frac{1}{4}\sqrt{33} + 13$ of the latter part is taken from $\frac{1}{4}\sqrt{33}$, the square of the first part, as this other formula shows. For the remaining number $0 - 13$ will be that which is produced from the one number into the other:

$$\begin{array}{r} \frac{1}{2}\sqrt{3} + \sqrt{3}(\frac{1}{4}\sqrt{33} + 13) \\ \frac{1}{2}\sqrt{3} - \sqrt{3}(\frac{1}{4}\sqrt{33} + 13) \end{array}$$

$$- 13$$

$$\begin{array}{r} \frac{1}{4}\sqrt{33} + 03 \\ \frac{1}{4}\sqrt{33} + 13 \end{array}$$

$$0 - 13$$

So too from $\frac{1}{2}\sqrt{3} + \sqrt{3}(\frac{1}{4}\sqrt{33} - 13)$ into $\frac{1}{2}\sqrt{3} - \sqrt{3}(\frac{1}{4}\sqrt{33} - 13)$ the number 13 will be generated, as the two other formulas set beside it teach:

$$\begin{array}{r} \frac{1}{2}\sqrt{3} + \sqrt{3}(\frac{1}{4}\sqrt{33} - 13) \\ \frac{1}{2}\sqrt{3} - \sqrt{3}(\frac{1}{4}\sqrt{33} - 13) \end{array}$$

$$\begin{array}{r}
 13 \\
 \sqrt{33} + \sqrt{03} \\
 \sqrt{33} - 13 \\
 \hline
 0 + 13
 \end{array}$$

[Margin: Division of the roots of composite numbers.]

Division of roots of this kind is done in this way. Both the dividend and the divisor are reduced to their square; then let the division be carried out, as explained above. For the root of the resulting number will be the quotient — as the examples will prove. For instance, as we said in the preceding chapter, from the division of $\sqrt{3}12$ by $\sqrt{3}3$ comes $\sqrt{3}4$, namely the square root of the number 4, which is made from the division of the square 12 by the square 3. Likewise, just as from the division of $\sqrt{3}30$ by $\sqrt{3}5$ comes the quotient $\sqrt{3}6$ — namely the surdesolid root of the number 6, produced from the division of the surdesolid 30 by the surdesolid 5.

Let $\sqrt{3}(13 + \sqrt{3}7)$ be divided by $\sqrt{3}5$. The squares of the numbers are $13 + \sqrt{3}7$ and 5. From the division of 13 by 5 come $2\frac{3}{5}$; and from $\sqrt{3}7$ by 5, that is, by $\sqrt{3}25$, comes $\sqrt{3}(7/25)$. The whole quotient is therefore $\sqrt{3}(2\frac{3}{5} + \sqrt{3} 7/25)$:

$$\begin{array}{r}
 13 + \sqrt{3} 7 \quad (2\frac{3}{5} + \sqrt{3} 7/25 \\
 5 \quad \sqrt{3} 25
 \end{array}$$

Let $\sqrt{3}(432 + \sqrt{3}7776)$ be divided by 6. The squares of the numbers are $432 + \sqrt{3}7776$ and 36. Dividing 432 by 36, 12 come out; and $\sqrt{3}7776$ divided by 36, that is, by $\sqrt{3}1296$, gives $\sqrt{3}6$. The whole quotient will therefore be $\sqrt{3}(12 + \sqrt{3}6)$:

$$\begin{array}{r}
 1 \\
 17 \quad 153 \\
 432 + \sqrt{3} 7776 \quad (12 + \sqrt{3} 6. \\
 366 \quad 1296 \\
 3
 \end{array}$$

Let $\sqrt{3}(\sqrt{3}588 + \sqrt{3}34848)$ be divided by $\sqrt{3}(12 + \sqrt{3}8)$. The squares of the numbers are $588 + \sqrt{3}34848$ and $12 + \sqrt{3}8$. Since the divisor is a composite number, it must be multiplied by $12 - \sqrt{3}8$, with the sign + changed into -, as was said above in Cap. 23, so that a new divisor $\sqrt{3}136$ is made. And if the dividend be multiplied by the same number $12 - \sqrt{3}8$, the new dividend $\sqrt{3}(6528 + \sqrt{3}332928)$ is made. For the product number $\sqrt{3}(7065 + \sqrt{3}5018112 - \sqrt{3}2765952 - \sqrt{3}278784)$ arises, which is reduced to $\sqrt{3}(6528 + \sqrt{3}332928)$: because the root of the number 278784 is 528, which taken from 7056 leaves 6528; and $\sqrt{3}2765952$ taken from $\sqrt{3}5018112$ —

since they are commensurable in the ratio $\frac{6}{49}$ — leaves $\sqrt{3332928}$, to which the sign + of the greater number must be attached. And so if the new dividend $\sqrt{3}(6528 + \sqrt{3332928})$ be divided by the new divisor 136, the quotient $\sqrt{3}(48 + \sqrt{318})$ is made. For dividing 6528 by 136, 48 come out; and $\sqrt{3332928}$ divided by 136, that is, by $\sqrt{318496}$, gives $\sqrt{318}$; and so $\sqrt{3}(48 + \sqrt{318})$ will be the quotient of the proposed division.

Because this example is elegant and rather obscure, it is worth examining by multiplication. It is therefore necessary that the divided number ($\sqrt{3588} + \sqrt{334848}$) return, if the quotient found $\sqrt{3}(48 + \sqrt{318})$ be multiplied into the divisor $\sqrt{3}(12 + \sqrt{38})$. Now from this multiplication comes the number ($\sqrt{3576} + \sqrt{32592} + \sqrt{318432} + \sqrt{3144}$). And if $\sqrt{3144}$ be added to 576, the sum 588 is made; and if $\sqrt{32592}$ and $\sqrt{318432}$, which are commensurable in the ratio $\frac{8}{3}$ [?], be added together, $\sqrt{334848}$ is made.

Let $\sqrt{315}$ also be divided by $\sqrt{3}(3 + \sqrt{35})$. So that a simple divisor may be found, we shall multiply $\sqrt{3}(3 + \sqrt{35})$ into $\sqrt{3}(3 - \sqrt{35})$, and the new divisor $\sqrt{34}$ is made. And if the dividend number $\sqrt{315}$ be likewise multiplied into $\sqrt{3}(3 - \sqrt{35})$, we shall generate the new dividend $\sqrt{3}(45 - \sqrt{31125})$. The square of this dividend number is $45 - \sqrt{31125}$, and the square of the new divisor is 4. If, then, we divide 45 by 4, the quotient $11\frac{1}{4}$ is made; and if we divide $-\sqrt{31125}$ by 4, that is, by $\sqrt{316}$, the quotient $-\sqrt{370 \frac{5}{16}}$ is made. The whole quotient is therefore $\sqrt{3}(11\frac{1}{4} - \sqrt{370 \frac{5}{16}})$.

ALGEBRÆ

Let us divide 20 by $\sqrt{3}(10 - \sqrt{35})$. Let each number be multiplied by $\sqrt{3}(10 + \sqrt{35})$, so that a new dividend $\sqrt{3}(4000 + \sqrt{3800000})$ is made, and a new divisor $\sqrt{395}$. The squares of these numbers are $4000 + \sqrt{3800000}$ and 95. If, then, we divide 4000 by 95, $42 \frac{2}{19}$ come out; and from the division of $\sqrt{3800000}$ by 95, that is, by $\sqrt{39025}$, comes $\sqrt{388 \frac{232}{361}}$. The quotient of the proposed division is therefore $\sqrt{3}(42 \frac{2}{19} + \sqrt{388 \frac{232}{361}})$.

Let us divide $\sqrt{3}(4 + \sqrt{325})$ by $\sqrt{3}(1 + \sqrt{39})$, that is, $\sqrt{39}$ by $\sqrt{34}$, or 3 by 2, where the quotient is $1\frac{1}{2}$. Let both the dividend and the divisor be multiplied by $\sqrt{3}(1 - \sqrt{39})$, so that the new dividend becomes $\sqrt{3}(4 + \sqrt{325} - \sqrt{3144} - \sqrt{3225})$, and the new divisor $\sqrt{3}-8$. The squares of these numbers are $4 + \sqrt{325} - \sqrt{3144} - \sqrt{3225}$ and -8. From the division of 4 by -8 comes the quotient $-\frac{1}{2}$. And from the division of $\sqrt{325}$ by -8, that is, by $\sqrt{3}-64$, comes the quotient $\sqrt{3}-25/64$, or (what is the same) $-\sqrt{325}/64$. And from the division of $-\sqrt{3144}$ by -8, that is, by $\sqrt{3}-64$, comes $\sqrt{3}+2 \frac{16}{64}$, or (what is the same) $+\sqrt{32 \frac{16}{64}}$. And from the division of $-\sqrt{3225}$ by -8,

that is, by $\sqrt[3]{-64}$, comes the quotient $+\sqrt[3]{33\frac{33}{64}}$. When these quotients are set in order, the whole quotient will be $\sqrt[3]{(\sqrt[3]{33\frac{33}{64}} + \sqrt[3]{2\frac{16}{64}} - \sqrt[3]{25/64} - \frac{1}{2})}$, which is equal to $\sqrt[3]{2\frac{1}{4}}$, that is, $1\frac{1}{2}$, as we said. For $\sqrt[3]{33\frac{33}{64}}$, that is $\sqrt[3]{225/64}$, is $15/8$; and $\sqrt[3]{2\frac{16}{64}}$, or $\sqrt[3]{144/64}$, is $12/8$, which with the former $15/8$ makes $27/8$. Next, $\sqrt[3]{25/64}$ is $5/8$, which with $\frac{1}{2}$, that is, with $4/8$, makes $9/8$; and $9/8$ from $27/8$ leaves $18/8$, or $2\frac{1}{4}$; and the root of this number is the quotient found above, namely $1\frac{1}{2}$.

Indeed, that from the division of $\sqrt[3]{(4 + \sqrt[3]{25} - \sqrt[3]{144} - \sqrt[3]{225})}$ by $\sqrt[3]{-8}$ there comes $1\frac{1}{2}$, will likewise be made manifest thus. From 4 and $\sqrt[3]{25}$ come 9; and from $-\sqrt[3]{144} - \sqrt[3]{225}$ come -27 . Now -27 with $+9$ make the number -18 . Therefore the dividend number is $\sqrt[3]{-18}$ and the divisor $\sqrt[3]{-8}$, where the quotient becomes $\sqrt[3]{2\frac{1}{4}}$, that is, $1\frac{1}{2}$.

This example can also be set out so that we say the dividend to be $\sqrt[3]{(4 + \sqrt[3]{25})}$ by $\sqrt[3]{(\sqrt[3]{9} + 1)}$, where again 3 is divided by 2. And if both the dividend and the divisor be multiplied by $\sqrt[3]{(\sqrt[3]{9} - 1)}$, the new dividend number $\sqrt[3]{(\sqrt[3]{144} + \sqrt[3]{225} - 4 - \sqrt[3]{25})}$ is made, and the new divisor $\sqrt[3]{8}$. The squares of these numbers are $\sqrt[3]{144} + \sqrt[3]{225} - 4 - \sqrt[3]{25}$ and 8. From the division of $\sqrt[3]{144}$ by 8, that is, by $\sqrt[3]{64}$, comes the quotient $\sqrt[3]{2\frac{16}{64}}$; and from the division of $\sqrt[3]{225}$ by 8, that is, by $\sqrt[3]{64}$, comes the quotient $\sqrt[3]{33\frac{33}{64}}$; and from the division of -4 by 8 comes the quotient $-\frac{1}{2}$; and from the division of $-\sqrt[3]{25}$ by 8, that is, by $\sqrt[3]{64}$, comes the quotient $-\sqrt[3]{25/64}$; so that the whole quotient is entirely the same as before, namely $\sqrt[3]{(\sqrt[3]{33\frac{33}{64}} + \sqrt[3]{2\frac{16}{64}} - \sqrt[3]{25/64} - \frac{1}{2})}$. On purpose, however, I am sometimes rather long in working out these examples, so that from them you may learn to overcome the difficulties that tend to arise with the roots of composite numbers.

Let us also divide $\sqrt[3]{(\sqrt[3]{32768} + \sqrt[3]{13824})}$ by 2. The cubes of the numbers are $\sqrt[3]{32768} + \sqrt[3]{13824}$ and 8. From the division of $\sqrt[3]{32768}$ by 8, that is, by $\sqrt[3]{512}$, comes the quotient $\sqrt[3]{64}$; and from the division of $\sqrt[3]{13824}$ by 8, that is, by $\sqrt[3]{512}$, comes the quotient $\sqrt[3]{27}$. And the whole quotient will be $\sqrt[3]{(\sqrt[3]{64} + \sqrt[3]{27})}$, that is, $\sqrt[3]{7}$.

[Margin: Addition of the roots of composite numbers.]

Finally, let $\sqrt[3]{(33 + \sqrt[3]{9})}$ be divided by $\sqrt[3]{(3 + \sqrt[3]{1})}$, that is, 6 by 2, where the quotient is 3. Let each number be multiplied by $\sqrt[3]{(3 - \sqrt[3]{1})}$, so that the new dividend becomes $\sqrt[3]{(99 + \sqrt[3]{81} - \sqrt[3]{1089} - \sqrt[3]{9})}$, and the new divisor $\sqrt[3]{(9 - 1)}$, that is, $\sqrt[3]{8}$. The squares of these numbers are $99 + \sqrt[3]{81} - \sqrt[3]{1089} - \sqrt[3]{9}$ and 8. From the division of 99 by 8 comes the quotient $12\frac{3}{8}$. From the division of $\sqrt[3]{81}$ by 8, that is, by $\sqrt[3]{64}$, comes the quotient $\sqrt[3]{31\frac{17}{64}}$. From the division of $-\sqrt[3]{1089}$

by 8, that is, by $\sqrt[3]{64}$, comes the quotient $-\sqrt[3]{17 \frac{1}{64}}$. And finally, from the division of $-\sqrt[3]{39}$ by 8, that is, by $\sqrt[3]{64}$, comes the quotient $-\sqrt[3]{39/64}$. And the whole quotient will be $\sqrt[3]{(12\frac{3}{8} + \sqrt[3]{17 \frac{17}{64}} - \sqrt[3]{17 \frac{1}{64}} - \sqrt[3]{39/64})}$, that is, 3. For if to $12\frac{3}{8}$ be added $\sqrt[3]{17 \frac{17}{64}}$, that is, $9/8$, there come $12 \frac{12}{8}$; and from these, if $\sqrt[3]{17 \frac{1}{64}}$ and $\sqrt[3]{39/64}$ — that is, $33/8$ and $3/8$ — be removed, 9 remain. The quotient is therefore $\sqrt[3]{39}$, that is, 3.

Addition and subtraction of the roots of composite numbers can conveniently, for the most part, neither be added together nor one subtracted from the other, except through the interposition of the sign + or -, as we noted in Cap. 22. When, however, some root of this kind must be added to a similar one, they will be added together most conveniently, by changing the sign + into -, or - into +, if those roots are connected by the sign +, and the whole aggregate is multiplied into itself, as was shown a little before. For the $\sqrt[3]{}$ of this product will be the sum of the two proposed roots. Thus, if these two roots $\sqrt[3]{(12 + \sqrt[3]{36})}$ and $\sqrt[3]{(12 - \sqrt[3]{36})}$ are to be added, we shall multiply this composite $\sqrt[3]{(12 + \sqrt[3]{36})} + \sqrt[3]{(12 - \sqrt[3]{36})}$ into itself, and the number $24 + \sqrt[3]{552}$ is produced, as we said above. The square root of this, therefore — namely $\sqrt[3]{(24 + \sqrt[3]{552})}$ — will be the sum of the two proposed roots.

In the same way as above, the sum of these roots $\sqrt[3]{(10 + \sqrt[3]{36})}$ and $\sqrt[3]{(10 - \sqrt[3]{36})}$ was found to be $\sqrt[3]{(20 + \sqrt[3]{256})}$, that is, $\sqrt[3]{36}$, or 6.

Again, let $\sqrt[3]{(8 + \sqrt[3]{32})}$ and $\sqrt[3]{(8 - \sqrt[3]{32})}$ be added. Thus the example will be set up for multiplication. If, then, you prefix the sign $\sqrt[3]{}$ to the product number, it will be the sum of the two said roots, $\sqrt[3]{(16 + \sqrt[3]{128})}$:

$$\begin{array}{l} \sqrt[3]{(8 + \sqrt[3]{32})} + \sqrt[3]{(8 - \sqrt[3]{32})} \\ \sqrt[3]{(8 + \sqrt[3]{32})} + \sqrt[3]{(8 - \sqrt[3]{32})} \end{array}$$

$$8 + \sqrt[3]{32} + 8 - \sqrt[3]{32}$$

$$\sqrt[3]{32}$$

$$\sqrt[3]{32}$$

$$\sqrt[3]{128}$$

$$\text{Product. } 16 + \sqrt[3]{128}$$

Again, let $\sqrt[3]{(2 + \sqrt[3]{3})}$ be added to $\sqrt[3]{(2 - \sqrt[3]{3})}$. The multiplication will be set up thus, and the product will be $4 + \sqrt[3]{4}$, that is, 6. Therefore the sought sum of the two roots will be $\sqrt[3]{6}$:

$$\begin{array}{l} \sqrt[3]{(2 + \sqrt[3]{3})} + \sqrt[3]{(2 - \sqrt[3]{3})} \\ \sqrt[3]{(2 + \sqrt[3]{3})} + \sqrt[3]{(2 - \sqrt[3]{3})} \end{array}$$

$$2 + \sqrt[3]{3} + 2 - \sqrt[3]{3}$$

$$\begin{array}{r} \sqrt{3} \ 1 \\ \sqrt{3} \ 1 \\ \hline \end{array}$$

$$\sqrt{3} \ 4$$

Product. $4 + \sqrt{3}4$, that is, 6.

The addition of such roots may also conveniently be done thus. Let the square of one root be multiplied into the square of the other. This will be done most briefly if, in either of the two roots, the square of the latter part be taken from the square of the first part, and the root of the residue be added to the first part, and this sum be doubled — that is, multiplied by 2. For the root of this product will give the sought sum.

(This opening continues Chapter XXVIII of the Algebra — the arithmetic of composite irrational roots — begun on the preceding page. Throughout, the cossist universal-root sign is transcribed as printed: $\sqrt{3}$ = radix (the square root; the universal square root of the whole quantity when it stands before a bracket), $\sqrt{\text{C}}$ = the cube root [radix cubica], and $\sqrt{3\text{C}}$ = the sixth root [radix quadrato-cubica]. Numerals, signs, and worked layouts are given exactly as in the source.)

Take the first example: the sum of $\sqrt{3}(12 + \sqrt{3}6)$ and $\sqrt{3}(12 - \sqrt{3}6)$ is found like this. Subtract the square of 6, the latter part, from 144, the square of the former part; 138 remains. So if $\sqrt{3}138$ is added to 12, the former part, the sum becomes $12 + \sqrt{3}138$; and this, multiplied by 2, yields the sum sought, $\sqrt{3}(24 + \sqrt{3}552)$, as before.

In the second example, the sum of $\sqrt{3}(10 + \sqrt{3}36)$ and $\sqrt{3}(10 - \sqrt{3}36)$ is obtained thus. Subtract the square of 36, the latter part, from 100, the square of the former part; 64 remains. So if you add $\sqrt{3}64$ to the former part, 10, you make $10 + \sqrt{3}64$; and this sum, multiplied by 2, gives the desired sum $\sqrt{3}(20 + \sqrt{3}256)$ — that is, $\sqrt{3}36$, that is, 6, just as before.

In the third and last example, we find the sum of the roots $\sqrt{3}(2 + \sqrt{3}3)$ and $\sqrt{3}(2 - \sqrt{3}3)$ in this way. Subtract the square of 3, the latter part, from 4, the square of the former part; 1 remains. So if we add $\sqrt{3}1$ to 2, the former part, we make $2 + \sqrt{3}1$ — that is, 3. Its double is 6. Therefore $\sqrt{3}6$ is the sum wanted, the same as before.

Adding roots of dissimilar composite numbers

When two roots of composite numbers must be added whose numbers of parts are not all equal, they can sometimes be added by proposition 4 of Book 2 of Euclid — namely, when the number produced by twice multiplying one root into the other can be reduced to a simple number, or to a composite number of two

parts. It is done as follows. Let $\sqrt{3}(10 + \sqrt{336})$ be added to $\sqrt{3}(11 + \sqrt{325})$ — that is, 4 to 4, where the sum is 8. The squares of the two roots, joined by the sign +, make $10 + \sqrt{336} + 11 + \sqrt{325}$. The number produced from one root into the other — that is, if the square of one is multiplied into the square of the other and the sign $\sqrt{3}$ is prefixed to the product, as is usually done in multiplying simple square roots — is $\sqrt{3}(110 + \sqrt{34356} + \sqrt{32500} + \sqrt{3900})$, as appears in the array set out below:

$$\begin{array}{r}
 \sqrt{3}(10 + \sqrt{336}) \\
 \sqrt{3}(11 + \sqrt{325}) \\
 \hline
 \sqrt{32500} + \sqrt{3900} \\
 110 + \sqrt{34356} \\
 \hline
 \sqrt{3}(110 + \sqrt{34356} + \sqrt{32500} + \sqrt{3900}.)
 \end{array}$$

Its double (namely, if it is multiplied by 2) is $\sqrt{3}(440 + \sqrt{369696} + \sqrt{340000} + \sqrt{314400})$, which reduces to $\sqrt{3}1024$ — that is, to 32. So if 32 is added to the two squares above, the whole sum will be $\sqrt{3}(10 + \sqrt{336} + 11 + \sqrt{325} + 32)$ — that is, $\sqrt{364}$, or 8, as the reasoning requires.

Again, let $\sqrt{3}(12 + \sqrt{3108})$ be added to $\sqrt{3}(2 + \sqrt{33})$. The sum of the squares is $12 + \sqrt{3108} + 2 + \sqrt{33}$. The number produced from one root into the other is $\sqrt{3}(42 + \sqrt{31728})$, whose root is $\sqrt{3}24 + \sqrt{3}18$ (as will be clear from Chapter 27). Its double — namely $\sqrt{3}96 + \sqrt{3}72$ — added to the aforesaid two squares, $12 + \sqrt{3108} + 2 + \sqrt{33}$ — that is, to $14 + \sqrt{3108} + \sqrt{33}$, that is, to $14 + \sqrt{3147}$ (for $\sqrt{3108}$ and $\sqrt{33}$ are commensurable and together make $\sqrt{3147}$) — yields the whole sum $\sqrt{3}(14 + \sqrt{3147} + \sqrt{3}96 + \sqrt{3}72)$.

Let $\sqrt{3}(\sqrt{32} + 1)$ likewise be added to $\sqrt{3}(\sqrt{3162} + 9)$. The sum of the squares is $\sqrt{32} + 1 + \sqrt{3162} + 9$. From one root into the other comes $\sqrt{3}(27 + \sqrt{3648})$, whose root is worth $3 + \sqrt{3}18$ (how the root is to be drawn out of this square $27 + \sqrt{3648}$ we shall teach in Chapter 27). Its double, $6 + \sqrt{3}72$, added to the aforesaid sum of the squares, makes $\sqrt{32} + 1 + \sqrt{3162} + 9 + 6 + \sqrt{3}72$ — that is, $\sqrt{3512} + 16$. For $\sqrt{3162}$ and $\sqrt{32}$ are commensurable and make $\sqrt{3200}$; and $\sqrt{3200}$ and $\sqrt{3}72$ are likewise commensurable and make $\sqrt{3512}$. The sum sought will therefore be $\sqrt{3}(\sqrt{3512} + 16)$.

But when the number produced from one root into the other cannot be reduced to a composite number — that is, when it has no root that is either a simple root or a composite number — the sum would come out more perplexed and obscure this way. It will therefore be more convenient in that case to carry out the addition with the sign + interposed. For example, if $\sqrt{3}(2 + \sqrt{33})$ is to be added to $\sqrt{3}(\sqrt{312} + 2)$, from one into the other comes $\sqrt{3}(\sqrt{348} + \sqrt{336} + 4 + \sqrt{312})$ — that is,

$\sqrt{3}(\sqrt{3108} + 10)$, because $\sqrt{348}$ and $\sqrt{312}$ are commensurable and together make $\sqrt{3108}$, while $\sqrt{336}$ and 4 make 10. But since the square number $\sqrt{3108} + 10$ has the root $\sqrt{3}(\sqrt{327} + \sqrt{32}) + \sqrt{3}(\sqrt{327} - \sqrt{32})$, as will be established from Chapter 27 — a root indeed composed of two roots of two composite numbers — these two roots together produce the sum $\sqrt{3}(\sqrt{327} + \sqrt{32} + \sqrt{327} - \sqrt{32} + 10)$, as appears in this array:

$$\begin{array}{r} \sqrt{3}(\sqrt{327} + \sqrt{32}) + \sqrt{3}(\sqrt{327} - \sqrt{32}) \\ \sqrt{3}(\sqrt{327} + \sqrt{32}) + \sqrt{3}(\sqrt{327} - \sqrt{32}) \end{array}$$

$$\begin{array}{r} \sqrt{327} + \sqrt{32} + \sqrt{327} - \sqrt{32} \\ \sqrt{325}. \text{ that is, } 5 \\ \sqrt{325}. \text{ that is, } 5 \end{array}$$

$$\sqrt{3100}. \text{ that is, } 10$$

$$\sqrt{3}(\sqrt{327} + \sqrt{32} + \sqrt{327} - \sqrt{32} + 10.)$$

And this sum reduces to $\sqrt{3}(\sqrt{3108} + 10)$: this sum will be the root of the number $\sqrt{3108} + 10$, which is plain, because $\sqrt{3}(\sqrt{3108} + 10)$ multiplied into itself makes $\sqrt{3108} + 10$. Now this root $\sqrt{3}(\sqrt{3108} + 10)$ doubled — which is done if its square $\sqrt{3108} + 10$ is multiplied into 4, the square of two — this root, I say, doubled makes $\sqrt{3}(\sqrt{31728} + 40)$; and this, together with the two squares of the roots proposed, namely with $2 + \sqrt{33}$ and $\sqrt{312} + 2$, will give the sum of the roots $\sqrt{3}(\sqrt{31728} + 40) + 2 + \sqrt{33} + \sqrt{312} + 2$. But since $2 + \sqrt{33} + \sqrt{312} + 2$ make $4 + \sqrt{327}$, the sum will be $\sqrt{3}(\sqrt{31728} + 40) + \sqrt{327} + 14$ — which is more intricate than if the sum were said to be $\sqrt{3}(2 + \sqrt{33}) + \sqrt{3}(\sqrt{312} + 2)$. That same intricate sum is more easily found without finding the root of the number $\sqrt{3108} + 10$, whose root, namely $\sqrt{3}(\sqrt{3108} + 10)$, was produced from one proposed root into the other. For the double of this number, $\sqrt{3}(\sqrt{31728} + 40)$, together with the two squares of the proposed roots, produces — by proposition 4 of Book 2 of Euclid — that same sum of the two roots, $\sqrt{3}(\sqrt{31728} + 40) + \sqrt{327} + 14$.

When two roots of this kind are commensurable, they can be added to one another by the rule of three, just like simple roots. For instance, suppose $\sqrt{3}(2 + \sqrt{33})$ is to be added to $\sqrt{3}(8 + \sqrt{348})$. Divide the latter by the former (finding, namely, a new divisor $\sqrt{31}$ and a new dividend $\sqrt{34}$); the quotient comes out $\sqrt{34}$ — that is, 2, or $2/1$. So those roots have a double proportion; and accordingly, if it is arranged that as 1 is to 3, so is $\sqrt{3}(2 + \sqrt{33})$ to another, the sum $\sqrt{3}(18 + \sqrt{3243})$ will be found.

Again, let $\sqrt{3}(4 + \sqrt{36})$ be added to $\sqrt{3}(8 + \sqrt{324})$. Dividing the latter by the former (finding, namely, a new divisor $\sqrt{310}$ and a new dividend $\sqrt{320}$), the quotient

comes out $\sqrt{32}$. Although this is irrational — so that the proposed roots are incommensurable — we shall nonetheless try, with its help, to add them together in the manner of commensurable roots.

So the proportion of one root to the other is that of $\sqrt{32}$ to 1, so that the denominator of the proportion is $(\sqrt{32})/1$. And the denominator, 1, of this fraction, added to $\sqrt{32}$ the denominator, makes $\sqrt{32} + 1$. Therefore, if it is arranged that as 1 is to $\sqrt{32} + 1$, so is $\sqrt{3}(4 + \sqrt{36})$ to another, the result is $\sqrt{3}(12 + \sqrt{3128} + \sqrt{354} + \sqrt{348})$ for the sum of the two proposed roots. For, so that in the rule of three $\sqrt{32} + 1$ may be multiplied into $\sqrt{3}(4 + \sqrt{36})$, the number $\sqrt{32} + 1$ must be reduced to its square $3 + \sqrt{38}$ — that is, it must be multiplied into itself — and this multiplied into $4 + \sqrt{36}$, the square of the root $\sqrt{3}(4 + \sqrt{36})$. And to the product $12 + \sqrt{3128} + \sqrt{354} + \sqrt{348}$ the sign $\sqrt{3}$ is to be prefixed.

Let $\sqrt{3}(\sqrt{36} - \sqrt{32})$ besides be added to $\sqrt{3}(\sqrt{396} - \sqrt{332})$. Dividing the latter by the former (finding, namely, a new divisor $\sqrt{34}$ and a new dividend $\sqrt{316}$), the quotient comes out $\sqrt{34}$ — that is, 2; and the proportion of one root to the other is double, whose denominator is $2/1$. So if it is arranged that as 1 is to 3, so is $\sqrt{3}(\sqrt{36} - \sqrt{32})$ to another, the sum sought will be found: $\sqrt{3}(\sqrt{3486} - \sqrt{3162})$.

[Margin: Subtraction of the roots of composite numbers.]

Now if, from the root of a composite number, another root of a like diminished number must be subtracted (which often happens with binomials and residuals [apotomes] and their roots), it is done in this way. Suppose, for example, $\sqrt{3}(12 - \sqrt{36})$ is to be subtracted from $\sqrt{3}(12 + \sqrt{36})$. It is certain that the remainder is $\sqrt{3}(12 + \sqrt{36}) - \sqrt{3}(12 - \sqrt{36})$. Write this remainder twice, then, and multiply it into itself — that is, by proposition 4 of Book 2 of Euclid, to the squares of the parts, $12 + \sqrt{36}$ and $12 - \sqrt{36}$, add the double of the number $-\sqrt{3138}$ which arises from one part into the other, namely $-\sqrt{3552}$; the square becomes $24 - \sqrt{3552}$, because the two squares make 24:

$$\begin{array}{r}
 \sqrt{3}(12 + \sqrt{36}) - \sqrt{3}(12 - \sqrt{36}) \\
 \sqrt{3}(12 + \sqrt{36}) - \sqrt{3}(12 - \sqrt{36}) \\
 \hline
 12 + \sqrt{36} + 12 - \sqrt{36} \\
 - \sqrt{3138} \\
 - \sqrt{3138} \\
 \hline
 24 - \sqrt{3552}
 \end{array}$$

The root of this square, then — namely $\sqrt{3}(24 - \sqrt{3552})$ — will be what remains from subtracting $\sqrt{3}(12 - \sqrt{36})$ from $\sqrt{3}(12 + \sqrt{36})$. For when one number is

subtracted from another, if the remaining number is multiplied into itself, the square root of the number produced will be that remaining number. And accordingly, since in the given example the remaining number, $\sqrt{3}(12 + \sqrt{36}) - \sqrt{3}(12 - \sqrt{36})$, multiplied into itself makes $24 - \sqrt{3552}$, $\sqrt{3}(24 - \sqrt{3552})$ will be the number that remains from subtracting $\sqrt{3}(12 - \sqrt{36})$ from $\sqrt{3}(12 + \sqrt{36})$.

Another example, in rational numbers. Let $\sqrt{3}(10 - \sqrt{36})$ be subtracted from $\sqrt{3}(10 + \sqrt{36})$ — namely, 2 from 4 — where it is plain that the remaining number is 2. Having written the number $\sqrt{3}(10 + \sqrt{36})$ [recte: $\sqrt{3}(10 + \sqrt{336})$] — $\sqrt{3}(10 - \sqrt{336})$, which is what remains after subtracting $\sqrt{3}(10 - \sqrt{336})$ from $\sqrt{3}(10 + \sqrt{336})$, and having multiplied it into itself according to the prescription of proposition 4 of Book 2 of Euclid, the number produced becomes $20 - \sqrt{3256}$:

$$\begin{array}{r} \sqrt{3}(10 + \sqrt{336}) - \sqrt{3}(10 - \sqrt{336}) \\ \sqrt{3}(10 + \sqrt{336}) - \sqrt{3}(10 - \sqrt{336}) \\ \hline 10 + \sqrt{336} \text{. \& } 10 - \sqrt{336} \\ - \sqrt{364} \\ - \sqrt{364} \\ \hline 20 - \sqrt{3256} \end{array}$$

That is, $\sqrt{34}$, or 2, which is what was proposed. Therefore the remainder sought will be $\sqrt{3}(20 - \sqrt{3256})$ — that is, $\sqrt{34}$, or 2, which is what was proposed.

When roots of this kind are different [not similar], the smaller can be subtracted from the greater by the prescription of proposition 7 of Book 2 of Euclid — namely, when the number produced by twice multiplying one root into the other can be reduced to a simple number, or to a composite of two parts, in this way. Let $\sqrt{3}(8 - \sqrt{332})$ be subtracted from $\sqrt{3}(8 + \sqrt{332})$. The squares of the two parts are $8 - \sqrt{332}$ and $8 + \sqrt{332}$, which, joined by the sign +, make $8 - \sqrt{332} + 8 + \sqrt{332}$ — that is, 16. From one root into the other comes $\sqrt{332}$, whose double $\sqrt{3128}$, taken away from 16, the sum of the squares, leaves $16 - \sqrt{3128}$:

$$\begin{array}{r} \sqrt{3}(8 + \sqrt{332}) \\ \sqrt{3}(8 - \sqrt{332}) \\ \hline \sqrt{332} \end{array}$$

The square root of this number, therefore — namely $\sqrt{3}(16 - \sqrt{3128})$ — is the remainder desired.

Again, let $\sqrt{3}(2 + \sqrt{33})$ be taken away from $\sqrt{3}(12 + \sqrt{3108})$. The sum of the squares is $2 + \sqrt{33} + 12 + \sqrt{3108}$ — that is, $14 + \sqrt{3147}$; for $\sqrt{3108}$ and $\sqrt{33}$ are

commensurable and produce $\sqrt{3}147$. From one root into the other comes $\sqrt{3}(42 + \sqrt{3}1728)$:

$$\begin{array}{r}
 12 + \sqrt{3}108 \\
 2 + \sqrt{3}3 \\
 \hline
 + \sqrt{4}32 + \sqrt{3}324 \\
 24 + \sqrt{4}32 \\
 \hline
 \sqrt{3}(42 + \sqrt{3}1728)
 \end{array}$$

whose root, as will be established from Chapter 28, is $\sqrt{3}24 + \sqrt{3}18$. Its double, $\sqrt{3}96 + \sqrt{3}72$, taken from the aforesaid sum of the squares $14 + \sqrt{3}147$, leaves $14 + \sqrt{3}147 - \sqrt{3}96$ and $-\sqrt{3}72$; and, with the opposite sign $\sqrt{3}$ prefixed, the remainder of the subtraction will be $\sqrt{3}(14 + \sqrt{3}147 - \sqrt{3}96$ and $-\sqrt{3}72)$. This is the same as $14 + \sqrt{3}147 - \sqrt{3}96$ and $-\sqrt{3}72$, because the sense is that from $14 + \sqrt{3}147$ the aggregate of $\sqrt{3}96$ and $\sqrt{3}72$ is subtracted — so that it is rightly said that the remainder is $14 + \sqrt{3}147 - \sqrt{3}96$ and $-\sqrt{3}72$. For the sense is that not only $\sqrt{3}96$ is subtracted, but $\sqrt{3}72$ besides is taken away. This is clearly seen if a rational number is to be subtracted, such as $\sqrt{3}100 + \sqrt{3}4$ from $14 + \sqrt{3}147$, where it is certain that $10 + 2$ — that is, 12 — is subtracted, and that the remainder is $2 + \sqrt{3}147$, which is equivalent to this: $14 + \sqrt{3}147 - \sqrt{3}100$ and $-\sqrt{3}4$, as is plain.

But if the number produced from one root into the other has no root that is a composite number of two parts, the subtraction must be carried out with the sign $-$. For example, if $\sqrt{3}(2 + \sqrt{3}3)$ is to be subtracted from $\sqrt{3}(\sqrt{3}12 + 2)$: from the multiplication of one root into the other comes $\sqrt{3}(\sqrt{3}108 + 10)$, which, as will appear from Chapter 28, is equivalent to $\sqrt{3}(\sqrt{3}27 + \sqrt{3}2) + \sqrt{3}(\sqrt{3}27 - \sqrt{3}2)$. It will therefore be more convenient to say that the remainder of the subtraction is $\sqrt{3}(\sqrt{3}12 + 2) - \sqrt{3}(2 + \sqrt{3}3)$.

When, finally, roots of this kind are commensurable, one will be subtracted from the other by the rule of three, as we said of simple roots. Suppose $\sqrt{3}(2 + \sqrt{3}3)$ is to be subtracted from $\sqrt{3}(8 + \sqrt{3}48)$. Dividing the latter by the former (finding, namely, a new divisor $\sqrt{3}1$ and a new dividend $\sqrt{3}4$), the quotient comes out $\sqrt{3}4$ — that is, 2 — so that their proportion is double, whose denominator is $2/1$. So if it is arranged that as 1 is to 1 , so is $\sqrt{3}(2 + \sqrt{3}3)$ to another, $\sqrt{3}(2 + \sqrt{3}3)$ will be produced as the remainder of the subtraction.

Again, let $\sqrt{3}(4 + \sqrt{3}6)$ be taken away from $\sqrt{3}(8 + \sqrt{3}24)$. Dividing the latter by the former (finding, namely, a new divisor $\sqrt{3}10$ and a new dividend $\sqrt{3}20$), the quotient comes out $\sqrt{3}2$, so that the denominator of the proportion of the greater to the smaller is $(\sqrt{3}2)/1$.

So if it is arranged that as 1 is to $\sqrt{3}2 - 1$, so is $\sqrt{3}(4 + \sqrt{3}6)$ to another, the remainder of the subtraction will come out: $\sqrt{3}(12 + \sqrt{3}54 - \sqrt{3}128 - \sqrt{3}48)$. For, so that in the rule of three $\sqrt{3}(4 + \sqrt{3}6)$ may be multiplied by $\sqrt{3}2 - 1$, this latter number must first be reduced to its square, $3 - \sqrt{3}8$, and so on.

Finally, let $\sqrt{3}(\sqrt{3}6 - \sqrt{3}2)$ be subtracted from $\sqrt{3}(\sqrt{3}486 - \sqrt{3}162)$. Dividing the latter by the former (finding, namely, a new divisor $\sqrt{3}4$ and a new dividend $\sqrt{3}36$), the quotient comes out $\sqrt{3}9$ — that is, 3 — so that the proposed roots have a triple proportion, whose denominator is 3/1. So if it is arranged that as 1 is to 2, so is $\sqrt{3}(\sqrt{3}6 - \sqrt{3}2)$ to another, the remainder sought will come out: $\sqrt{3}(\sqrt{3}96 - \sqrt{3}32)$. For, so that in the rule of three the second number may be multiplied into the third, both must be reduced to squares — 4 and $\sqrt{3}6 - \sqrt{3}2$ — and one is to be multiplied into the other, as you see done here, and the sign $\sqrt{3}$ prefixed to the product, so that the remainder becomes $\sqrt{3}(\sqrt{3}96 - \sqrt{3}32)$:

$$\begin{array}{r} \sqrt{3} \ 6 \ - \ \sqrt{3} \ 2 \\ 4 \\ \hline \sqrt{3} \ 96 \ - \ \sqrt{3} \ 32 \end{array}$$

The remainder is likewise found if it is arranged that as 3 is to 2, so is $\sqrt{3}(\sqrt{3}486 - \sqrt{3}162)$ to another. For, calling back the second number of the rule of three, 2, and the third, $\sqrt{3}(\sqrt{3}486 - \sqrt{3}162)$, to squares — 4 and $\sqrt{3}486 - \sqrt{3}162$ — if these are multiplied together, the number $\sqrt{3}7776 - \sqrt{3}2592$ is generated; and this, in order that it be divided by 3, the first number of the rule of three, must be reduced to the square 9. So if $\sqrt{3}7776$ is divided by 9 — that is, by $\sqrt{3}81$ — the quotient comes out $\sqrt{3}96$; and if $-\sqrt{3}2592$ is divided by 9 — that is, by $\sqrt{3}81$ — the quotient comes out $-\sqrt{3}32$; and so on:

$$\begin{array}{r} \sqrt{3} \ 486 \ - \ \sqrt{3} \ 162 \\ 4 \\ \hline \sqrt{3} \ 7776 \ - \ \sqrt{3} \ 2592 \end{array}$$

On the Fractions of Irrational Numbers, and Their Arithmetic

CHAPTER XXV.

[Margin: The numbering [notation] of irrational fractions.]

The reading of these fractions is easy. For when the radical sign is placed before the middle of the fraction, that sign refers to both terms — that is, both to the numerator and to the denominator. Thus this fraction $\sqrt[3]{9/16}$ signifies that $\sqrt[3]{9}$ is divided by $\sqrt[3]{16}$, and is equivalent to $3/4$. So $\sqrt[4]{8/27}$ signifies that $\sqrt[4]{8}$ is divided by $\sqrt[4]{27}$, and is equivalent to $2/3$. Likewise $\sqrt[3]{6/8}$ denotes that $\sqrt[3]{6}$ is divided by $\sqrt[3]{8}$.

When the radical sign is placed before a whole number with a fraction, the whole number must be reduced to a single fraction, so that its value can be expressed. Thus $\sqrt[4]{1\ 67/125}$ reduces to $\sqrt[4]{192/125}$, and signifies that $\sqrt[4]{192}$ is divided by $\sqrt[4]{125}$ — that is, by 5 — and it can be represented this way, $\sqrt[4]{192/5}$, so that it signifies that $\sqrt[4]{192}$ is divided by 5. And if the terms of this fraction $\sqrt[4]{192/125}$ are multiplied by some same number — namely by $\sqrt[4]{192}$ — the fraction $\sqrt[4]{36864/24000}$ is produced, equivalent to the fraction $\sqrt[4]{192/125}$. And if the terms of the produced fraction are again multiplied into the same $\sqrt[4]{192}$, there arises a fraction equivalent to $\sqrt[4]{7077888/4608000}$ — that is, $\sqrt[4]{(192^3)/[?]}$ — and it signifies that the cube of 192 (which is produced from $\sqrt[4]{192}$ multiplied into itself cubically) is divided by $\sqrt[4]{4608000}$, because the cube root of the numerator 7077888 is 192.

Again, the fraction $\sqrt[3]{16/64}$ signifies that $\sqrt[3]{16}$ — namely 4 — is divided by $\sqrt[3]{64}$ — namely by 8 — and it is equivalent to the fraction $4/8$, or $1/2$. But the fraction $(\sqrt[3]{16})/64$ signifies that $\sqrt[3]{16}$ — that is, 4 — is divided by 64, and is equivalent to the fraction $4/64$, or $1/16$. So the fraction $16/(\sqrt[3]{64})$ signifies that the number 16 is divided by $\sqrt[3]{64}$ — that is, by 8 — and is equivalent to the number 2.

When the ratio of the numerators to the denominators is the same, the fractions are equal, just as with common fractions. And so all these fractions — $(\sqrt[3]{64})/2$, $\sqrt[3]{64}/4$, and $64/(\sqrt[3]{256})$ — are of the same value, because in all of them the numerator keeps a quadruple ratio to the denominator, as is clear if the several numerators are divided by the several denominators.

[Margin: The reduction [abbreviation] of irrational fractions.]

Irrational fractions are reduced to lowest terms — when they can be reduced — as common fractions are usually reduced. Thus this fraction $\sqrt[3]{4/8}$ reduces to this one, $\sqrt[3]{1/2}$; and $\sqrt[3]{144/36}$ reduces to $\sqrt[3]{4/1}$. They can sometimes also be reduced to smaller radical signs. Thus $\sqrt[3]{4/8}$ reduces to $(\sqrt[4]{2})/(\sqrt[3]{2})$, because the numerator 4 has a square root, 2, but this has no cube root; while the denominator 8 has a cube root, 2, but this has no square root. That these fractions $\sqrt[3]{4/8}$ and $(\sqrt[4]{2})/(\sqrt[3]{2})$ are equal can be proved by cross-multiplication.

[Margin: The addition and subtraction of irrational fractions.]

Addition and subtraction are done in this way. If the denominator is the same, the numerators [the text prints *denominatores*, evidently in error for *numeratoros*] are added, as was set forth above, or one is subtracted from the other, and the common denominator is placed under the sum or the remainder. Thus from the addition of $(\sqrt{3}4)/7$ to $(\sqrt{3}9)/7$ comes the sum $(\sqrt{3}25)/7$ — that is, $5/7$; while subtracting $(\sqrt{3}4)/7$ from $(\sqrt{3}9)/7$ leaves $(\sqrt{3}1)/7$ — that is, $1/7$.

Addition. Subtraction.

$(\sqrt{3}4)/7$ $(\sqrt{3}9)/7$ $(\sqrt{3}4)/7$ $(\sqrt{3}9)/7$

$(\sqrt{3}25)/7$ or $5/7$ $(\sqrt{3}1)/7$ or $1/7$

But when the denominators are different, a reduction to the same denominator is made by cross-multiplication, as is customary with common fractions. Then, in the same way, the numerators are added to one another, or one is subtracted from the other, and the same common denominator is written under the sum or the remaining number. Thus if $\sqrt{3} 16/49$ is to be added to $\sqrt{3} 4/9$, or the latter subtracted from the former, they will be reduced — as the array set beside shows — to $\sqrt{3} 144/441$ and $\sqrt{3} 196/441$:

Reduction.

$\sqrt{3}144$ $\sqrt{3}196$

$\sqrt{3}16.$ $\sqrt{3}4.$

$\sqrt{3}49$ $\sqrt{3}9$

$\sqrt{3}441$

Addition.

$\sqrt{3}144$ $\sqrt{3}196$

$\sqrt{3}441$ $\sqrt{3}441$

$26/21$

that is, $1 \frac{5}{21}$

And from the addition the sum comes to $26/21$ — that is, $1 \frac{5}{21}$. For the numerators are commen-

(Continuing Cap. XXV, on irrational fractions. In what follows the cossic signs are transcribed as printed: R = res, the unknown; 3 = census, its square; C = cubus, its cube (printed as a doubled cubus glyph); 33 = zensizensus, the fourth power; β = sursolidum, the fifth power; Bβ = bissursolidum, the seventh power. The radical signs are $\sqrt{3}$ = square root [radix zensica], $\sqrt{C}$ = cube root [radix cubica], $\sqrt{33}$ = fourth root [radix zensizensica], and $\sqrt{3C}$ = sixth root [radix zensicubica].)

[the numerators] are commensurable, standing in the ratio 7:6, and they produce the sum $\sqrt{3676}$, beneath which the common denominator $\sqrt{3441}$ is set, so that the sum becomes $\sqrt{3676} / \sqrt{3441}$ — that is, $26/21$, or $1 \frac{5}{21}$. But when $\sqrt{316} / \sqrt{349}$ is subtracted from $\sqrt{34} / \sqrt{39}$, there remain $2/21$. For the numerator $\sqrt{3144}$, taken from the numerator $\sqrt{3196}$, leaves $\sqrt{34} / \sqrt{3441}$, that is, $2/21$.

In the same way, if the fractions to be added are $(\sqrt{350} + \sqrt{24}) / 5$ and $(\sqrt{24} + \sqrt{38}) / 2$, I first reduce them to the same denomination, namely to $(\sqrt{3200} + \sqrt{232}) / 10$ and $(\sqrt{2300} + \sqrt{3200}) / 10$. Then, adding the numerators and keeping the same denominator 10, the sum becomes $(\sqrt{3800} + \sqrt{2322}) / 10$. But from subtracting the smaller fraction from the larger there is left $(\sqrt{232}) / 10$, since $\sqrt{3200}$ taken from $\sqrt{3200}$ leaves nothing.

Likewise, if the fractions to be added are $\sqrt{381}/9$ and $\sqrt{349}/9$, or if the one fraction is to be subtracted from the other, there is no need to reduce to the same denomination. From the numerators is made the sum $\sqrt{381}$, and with the denominator 9 kept, the sum is $[\sqrt{381}]/9$, that is, 1. But when the numerator $\sqrt{34}$ is taken from the numerator $\sqrt{349}$, what is left is $\sqrt{325}$ [as numerator]; hence the remainder is $\sqrt{325}/9$, that is, $5/9$.

When the numerators are incommensurable, their addition is made by interposing the sign +, and their subtraction by inserting the sign -. Thus the sum of $\sqrt{3(7/8)}$ and $\sqrt{3(5/8)}$ is $\sqrt{3(7/8)} + \sqrt{3(5/8)}$; and $\sqrt{3(5/8)}$ taken from $\sqrt{3(7/8)}$ leaves as remainder $\sqrt{3(7/8)} - \sqrt{3(5/8)}$.

[Margin: Multiplication and division of irrational fractions.]

In the multiplication and division of irrational fractions, the only work needed is the reduction to the same radical signs; the rest is carried out as was taught for common fractions. Thus, if $\sqrt{3(3/4)}$ is to be multiplied by $\sqrt{3(6/7)}$, the result is $\sqrt{3(18/28)}$. But if the first is to be divided by the second, the terms of the divisor are inverted, and the example stands as you see. The quotient comes out $\sqrt{3147} / \sqrt{396}$, that is, $\sqrt{3(1 \frac{51}{96})}$. For $\sqrt{33}$ by $\sqrt{37}$ makes $\sqrt{3147}$, and $\sqrt{34}$ by $\sqrt{36}$ makes $\sqrt{396}$.

Multiplication

Division

$\sqrt{3(3/4)} \cdot \text{by} \cdot \sqrt{3(6/7)}$

$\sqrt{3(3/4)} \cdot \text{by} \cdot 7/\sqrt{36}$

In division, however, a reduction to the same denomination can be made; but then, dropping the denominator, one must divide numerator by numerator. Thus in the given example the fractions $\sqrt{3(3/4)}$ and $\sqrt{3(6/7)}$ are reduced to $\sqrt{3147}/28$

and $\sqrt{396}/28$. Then, dividing the numerator $\sqrt{3147}$ by the denominator $\sqrt{396}$, the quotient is $\sqrt{3}(1\ 51/96)$, as before.

Let there also be two fractions to be multiplied together, $(\sqrt{3800} + \sqrt{5184}) / 10$ and $(\sqrt{1536}) / 10$. First they must be reduced to the same radical signs, keeping the common denominator 10. So the numerators $\sqrt{3800}$ and $\sqrt{5184}$ are reduced to $\sqrt{3}512000000$ and $\sqrt{3}26873856$. Then, if the numerator $\sqrt{1536}$ is multiplied "census-wise" (squared under the sign), it is reduced to $\sqrt{3}2359296$. So the reduced fractions stand thus:

$$(\sqrt{3}512000000 + \sqrt{3}26873856) / 10 \text{ and } \sqrt{3}2359296 / 10$$

From $\sqrt{3}2359296$ multiplied into $\sqrt{3}512000000$ — that is, into $\sqrt{3}512$ (for $\sqrt{3}512$ and $\sqrt{3}512000000 / 10$ are exactly the same, since the same number results from $\sqrt{3}512$ multiplied by 10, that is, by $\sqrt{3}1000000$, which arises from 1 into $\sqrt{3}512000000$) — there is produced $\sqrt{3}1207959552$. And from $\sqrt{3}2359296$ into $\sqrt{3}26873856$ comes $\sqrt{3}63403380965376$, that is, $\sqrt{7962624}$, since this number is the square root [radix zensica] of that product. And since the denominators multiplied make 100, the resulting number will be

$$(\sqrt{3}1207959552 + \sqrt{7962624}) / 100$$

Finally, let $(\sqrt{3}512000000 + \sqrt{3}26873856) / 10$ be divided by $\sqrt{3}2359296 / 10$. Since they have the same denominator, it is enough to divide the numerators. From dividing $\sqrt{3}512000000$ by $\sqrt{3}2359296$ comes the quotient $\sqrt{3}(217\ 1/72)$, namely $\sqrt{3}(15625/72)$, that is, $5 / \sqrt{3}72$ — because the numerator $\sqrt{3}15625$ is equal to the number 5, since the square root of 15625 is 125, and the cube root of that is 5; or (what is the same) because the cube root of 15625 is 25, and the square root of that is 5. And from dividing $\sqrt{3}26873856$ by $\sqrt{3}2359296$ comes the quotient $\sqrt{3}(11\ 25/64)$, or $\sqrt{3}(729/64)$, that is, $3/2$, or $1\ 1/2$. From the proposed division, therefore, the quotient is $\sqrt{3}(217\ 1/72) + 1\ 1/2$, or $5 / \sqrt{3}72 + 1\ 1/2$.

Of the fractions belonging to these numbers there is no need to say anything more, since they follow their integers' precepts in every respect.

ON IRRATIONAL COSSIC NUMBERS, OR DENOMINATED NUMBERS, TOGETHER WITH THEIR ALGORITHM.

CAPUT XXVI.

[Margin: What irrational cossic numbers are.]

Cossic, or denominated, numbers become irrational when some radical sign is prefixed to them. Thus $20R$ gives the irrational $\sqrt[3]{20R}$; likewise $\sqrt[4]{20R}$, $\sqrt[5]{20R}$, and so on. The sense is that one takes the square root, or the cube root, or the fourth root, of 20 roots [radices] — which can vary in a thousand ways, according to the varying value of a single root — and it may sometimes be rational and, more often, irrational. For if $1R$ is 5, then $20R$ will be the number 100, whose square root is 10; and so $\sqrt[3]{20R}$ is a rational number, equal to 10. But $\sqrt[4]{20R}$ will be irrational, since 100 has no cube root. So too $\sqrt[5]{20R}$ will be irrational, because 100 has no fourth root [radix zensizensical]; yet it is equal to $\sqrt[5]{310}$, since the square root of 100 is 10, and the square root of 10 is $\sqrt[5]{310}$. But if $1R$ is 50, the value of $20R$ will be 1000, which has the cube root 10. So $\sqrt[4]{20R}$ will be rational, equal to 10; but $\sqrt[3]{20R}$ will be irrational, as will $\sqrt[5]{20R}$, since 1000 has neither a square root nor a fourth root. You see, then, that one cannot judge whether numbers of this kind are rational or irrational until the value of a single root is first established. Indeed, it can happen that some cossic number, without any radical sign, is nevertheless found to be irrational. For if an equation is found between 13 and $20R$, and the value of one census is 18, then necessarily the number $20R$ will be irrational, equal to $\sqrt[3]{7200}$. For the square root of the square 18 is $\sqrt[3]{18}$, and so $20R$ will be $\sqrt[3]{7200}$ — which number is irrational, since 7200 lacks a square root.

[Margin: Addition and subtraction of irrational cossic numbers.]

The **algorithm** of these numbers is accomplished by a threefold algorithm — that of the radical signs, that of the absolute numbers, and that of the cossic signs; and it is not at all difficult, since the signs themselves sufficiently instruct us what we must do. Addition is usually done by interposing the sign $+$. Thus from adding $\sqrt[3]{36R}$ to $\sqrt[3]{123}$ comes the sum $\sqrt[3]{36R} + \sqrt[3]{123}$, or $\sqrt[3]{123} + \sqrt[3]{36R}$. But when 36 is taken from $\sqrt[3]{36R}$, the remaining number is $\sqrt[3]{36R} - 36$; and so with the rest.

When the cossic signs are the same, and the irrational numbers, apart from their cossic signs, are commensurable, then the cossic signs do not impede the

operation: the irrational numbers are added and subtracted just like surd roots — as was shown in chapters 19 and 20 — and after the operation the same cossic sign is set back on. Thus from $\sqrt{38R}$ to $\sqrt{318R}$ comes the sum $\sqrt{350R}$, because $\sqrt{38}$ and $\sqrt{318}$ are commensurable and produce $\sqrt{350}$. That this is so is clear if $1R$ is taken as 2: for then $8R$ will be 16, whose root is 4, and $18R$ will make 36, whose root is 6; and 10 is made from 4 and 6. And it is plain that $\sqrt{350R}$ also makes 10, since $50R$ makes 100, whose root is 10.

So too, from $\sqrt{38R}$ to $\sqrt{332R}$ comes $\sqrt{372R}$, which can be verified by resolution, setting $1R = 2$: for $\sqrt{38R}$ will be 4, and $\sqrt{332R}$ will be 8, and from 4 and 8 the sum is 12. Now see whether the collected sum $\sqrt{372R}$ also makes 12. It is certain that $72R$ makes 144, whose root is 12.

Likewise, from $\sqrt{383}$ to $\sqrt{3323}$ comes the sum $\sqrt{3723}$; and from $\sqrt{3163}$ to $\sqrt{343}$ comes $\sqrt{3363}$. For if $1R$ is set at 2, so that 13 becomes 4, then 163 will be 64, whose root is 8; and 43 will be 16, whose root is 4; and 12 is made from 8 and 4 — which is exactly what $\sqrt{3363}$ is worth, since 363 is 144, whose root is 12.

Moreover, from $\sqrt{383}$ to $\sqrt{3183}$ comes $\sqrt{3503}$, which likewise squares beautifully with the test, setting $1R = 2$: for 83 will be 32, whose root is $\sqrt{32}$, and 183 will be 72, whose root is $\sqrt{72}$. And from $\sqrt{332}$ and $\sqrt{372}$ the sum is $\sqrt{3200}$; and $\sqrt{3503}$ makes just as much, since 503 is 200, whose root is $\sqrt{3200}$.

By the same reasoning, $\sqrt{38R}$ taken from $\sqrt{318R}$ leaves $\sqrt{32R}$; and $\sqrt{383}$ subtracted from $\sqrt{3183}$ makes as remainder $\sqrt{323}$. We shall test both examples by way of the root 2. Thus $8R$ makes 16, whose root is 4; and $18R$ makes 36, whose root is 6; and taking 4 from 6 leaves 2. And $\sqrt{32R}$ makes just as much, since $2R$ makes 4, whose root is 2. Again, 83 makes 32, whose root is $\sqrt{32}$; and 183 makes 72, whose root is $\sqrt{72}$; and $\sqrt{332}$ subtracted from $\sqrt{372}$ leaves $\sqrt{38}$ — which is exactly what $\sqrt{323}$ makes, because 23 makes 8, whose root is $\sqrt{38}$.

[Margin: Reduction of the radical signs in irrational cossic numbers.]

Now, in order that irrational cossic numbers may be multiplied together or divided, the radical signs must first be reduced to the same sign — as we taught in chapter 18 — namely, if the multiplication is done "crosswise" [in *crucem*]. Thus $\sqrt[6]{4R}$ and $\sqrt{38R}$ are reduced to $\sqrt[3]{36163}$ and $\sqrt[3]{5123}$, as the figure set below makes plain. For $4R$, raised into itself census-wise (squared), makes 163 ; and $8R$, multiplied into itself cube-wise, makes 5123 . And the exponents of the radical signs, multiplied together, make 6 — the exponent of the sign $\sqrt[3]{}$. That the reduction has been rightly carried out — that is, that $\sqrt[3]{36163}$ is equal to $\sqrt[6]{4R}$, and that $\sqrt[3]{5123}$ is equivalent to $\sqrt{38R}$ — is clear. For, setting $1R = 2$, $4R$ will be

8, whose cube root is 2; and $\sqrt[3]{\text{C}16\text{J}}$ makes just as much (to which $\sqrt{\text{C}4\text{R}}$ has been reduced), for 16J makes 64, whose sixth root [radix zensicubica] is 2. Likewise 8R will be 16, whose square root is 4; and $\sqrt[3]{\text{C}512\text{C}}$ makes just as much (to which $\sqrt[3]{8\text{R}}$ has been carried over), for 512C makes 4096, whose sixth root is 4.

Diagram. A "crosswise" (in cruce[m]) reduction figure: a large $\times$ links two columns, each showing the reduction of a radical to the common sixth-root sign.

- Top left (over a rule): $\sqrt[3]{\text{C}16\text{J}}$ — the reduced form (sixth root)
- Left, above the $\times$: 4R — the original radicand
- Left, below the $\times$: $\sqrt{\text{C}}$ — the original sign (cube root)
- Left index: 3
- Top right (over a rule): $\sqrt[3]{\text{C}512\text{C}}$ — the reduced form (sixth root)
- Right, above the $\times$: 8R — the original radicand
- Right, below the $\times$: $\sqrt[3]{}$ — the original sign (square root)
- Right index: 2
- Center below: 6 — the common index (3×2)

In the same way $\sqrt[3]{8\text{R}}$ and $\sqrt{\text{C}16\text{J}}$ are reduced to $\sqrt[3]{\text{C}512\text{C}}$ and $\sqrt[3]{\text{C}256\text{J}}$, as this second figure teaches. That $\sqrt[3]{\text{C}512\text{C}}$ is equal to $\sqrt[3]{8\text{R}}$, and that $\sqrt[3]{\text{C}256\text{J}}$ is equivalent to $\sqrt{\text{C}16\text{J}}$, we shall easily prove. For $\sqrt[3]{8\text{R}}$ — that is, $\sqrt[3]{16}$ — is 4, which is just what $\sqrt[3]{\text{C}512\text{C}}$ makes, since 512C makes 4096, whose sixth root is 4. Likewise $\sqrt{\text{C}16\text{J}}$ is also 4, which is just what $\sqrt[3]{\text{C}256\text{J}}$ makes, since 256J makes 4096, which number has the sixth root 4. From which it is established that both $\sqrt[3]{8\text{R}}$ and $\sqrt{\text{C}16\text{J}}$ are equal to each other, and that $\sqrt[3]{\text{C}512\text{C}}$ and $\sqrt[3]{\text{C}256\text{J}}$ are equal to each other.

Diagram. A second "crosswise" reduction figure, laid out like the first.

- Top left (over a rule): $\sqrt[3]{\text{C}512\text{C}}$ — the reduced form (sixth root)
- Left, above the $\times$: 8R — the original radicand
- Left, below the $\times$: $\sqrt[3]{}$ — the original sign (square root)
- Left index: 2
- Top right (over a rule): $\sqrt[3]{\text{C}256\text{J}}$ — the reduced form (sixth root)
- Right, above the $\times$: 16J — the original radicand
- Right, below the $\times$: $\sqrt{\text{C}}$ — the original sign (cube root)
- Right index: 3
- Center below: 6 — the common index (2×3)

[Margin: Multiplication and division of irrational cossic numbers.]

Once the reduction has been made as we prescribed, both multiplication and division can be set up. In the first example, then, from the multiplication of $\sqrt[3]{512\mathfrak{C}}$ into $\sqrt[3]{163}$ comes $\sqrt[3]{8192\mathfrak{B}}$, which is proved thus: $\sqrt[3]{512\mathfrak{C}}$ makes 4, and $\sqrt[3]{163}$ makes 2, and from 4 into 2 comes 8 — which is just what $\sqrt[3]{8192\mathfrak{B}}$ makes, since 8192 $\mathfrak{B}$ makes 262144, whose sixth root is 8. And before the reduction, $\sqrt[3]{4\mathfrak{R}}$ is 2 and $\sqrt[3]{8\mathfrak{R}}$ is 4, where again you see that from 2 into 4 is produced 8.

Likewise, from the division of $\sqrt[3]{512\mathfrak{C}}$ by $\sqrt[3]{163}$ comes the quotient $\sqrt[3]{32\mathfrak{R}}$, that is, 2, since 32 $\mathfrak{R}$ makes 64, whose sixth root is 2. This is proved: $\sqrt[3]{512\mathfrak{C}}$ is 4 and $\sqrt[3]{163}$ is 2, so the quotient is 2 — which is manifest even before the reduction, since $\sqrt[3]{8\mathfrak{R}}$ is 4 and $\sqrt[3]{4\mathfrak{R}}$ is 2, where again from dividing 4 by 2 the quotient is 2.

In the second example, from the multiplication of $\sqrt[3]{512\mathfrak{C}}$ into $\sqrt[3]{25633}$ comes $\sqrt[3]{131072\mathfrak{B}}$, that is, 16, because 131072 $\mathfrak{B}$ makes 16777216, which number has the sixth root 16. And that this is so is clear: for both $\sqrt[3]{512\mathfrak{C}}$ and $\sqrt[3]{25633}$ are 4, and it is manifest that from 4 into 4 comes 16 — which is evident even before the reduction, since both $\sqrt[3]{8\mathfrak{R}}$ and $\sqrt[3]{163}$ are 4.

Likewise, from the division of $\sqrt[3]{25633}$ by $\sqrt[3]{512\mathfrak{C}}$ comes the quotient $\sqrt[3]{(1/2)\mathfrak{R}}$, that is, $\sqrt[3]{1}$, since $(1/2)\mathfrak{R}$ is 1 when 1 $\mathfrak{R}$ is set at 2; and accordingly, since $\sqrt[3]{1}$ is 1, the quotient will be 1, as the reasoning requires. For since $\sqrt[3]{25633}$ and $\sqrt[3]{512\mathfrak{C}}$ [are equal], each being 4, as has been said, the division of one by the other will give the quotient 1. By the same reasoning, from the division of $\sqrt[3]{131072\mathfrak{B}}$ by $\sqrt[3]{512\mathfrak{C}}$ comes the quotient $\sqrt[3]{25633}$, as you can prove.

So too, if 1 $\mathfrak{R}$ is to be multiplied into $\sqrt[3]{4}$, the 1 $\mathfrak{R}$ must be squared, and to the square 13 the radical sign $\sqrt[3]$ must be prefixed, so that it becomes $\sqrt[3]{13}$; and thus from $\sqrt[3]{13}$ into $\sqrt[3]{4}$ comes the number $\sqrt[3]{43}$. This is proved if 1 $\mathfrak{R}$ is 2: for from 2 into $\sqrt[3]{4}$ — that is, into 2 — comes 4, which is indeed the root of 43, that is, of the number 16.

So too, that 3 $\mathfrak{R}$ may be multiplied into $\sqrt[3]{16}$, the number 3 $\mathfrak{R}$ must be squared to make 93; then $\sqrt[3]{93}$ is to be multiplied into $\sqrt[3]{16}$, and the number $\sqrt[3]{1443}$ is produced. This is proved in the same way, taking 2 for one root: for 3 $\mathfrak{R}$ makes 6, and $\sqrt[3]{16}$ makes 4, and 24 is made from 6 into 4; and the root $\sqrt[3]{1443}$ likewise makes 24, since 1443 makes 576, whose root is 24, and so on.

These products, $\sqrt[3]{43}$ and $\sqrt[3]{1443}$, are called "numbers of the roots" [numeri radicum], as will appear in the solution of riddles 12 and 28 of chapter 32. For the

first product comes from 1R into $\sqrt[3]{34}$, and the second from 3R into $\sqrt[3]{316}$ — just as the number 4 is produced from 1 gold-piece [aureo] into 4, and the number 48 is produced from 3 gold-pieces into 16 (and this is called the number of gold-pieces [numerus aureorum]). This must be carefully observed.

And this is the same as if 1 were multiplied into $\sqrt[3]{34}$, or 3 into $\sqrt[3]{316}$; for the product numbers $\sqrt[3]{34}$ and $\sqrt[3]{3144}$ are the numbers of the units that have been multiplied. So from 1R into $\sqrt[3]{34}$ comes the number of roots $\sqrt[3]{34^3}$, namely $\sqrt[3]{34}$ of censuses; and from 3R into $\sqrt[3]{316}$ comes the number of roots $\sqrt[3]{3144^3}$, that is, $\sqrt[3]{3144}$ of censuses; and so with the rest — since in multiplications of this kind the rule for multiplying surd roots, handed down in chapter 18, is observed. But this will be better grasped from the solution of riddles 12 and 28 of chapter 32.

[Margin: When, after the reduction, the cossic signs can be abbreviated, and when they cannot.]

Here it must be observed that the thing does not succeed if the cossic signs are abbreviated (as I taught in chapter 7). For in the first example, in which $\sqrt[3]{4R}$ and $\sqrt[3]{38R}$ were reduced to $\sqrt[3]{816^3}$ and $\sqrt[3]{512^3}$ — where from the multiplication the number 8 was produced — if the cossic signs are abbreviated, the reduction will stand in these terms: $\sqrt[3]{816^3}$ and $\sqrt[3]{512^3}$. But from the multiplication of the one into the other comes $\sqrt[3]{8192R}$, which is worth $\sqrt[3]{128}$, because 8192R makes 16384, whose square root is 128, and the cube root of this is $\sqrt[3]{128}$ — which is not equal to the earlier product 8. Only when numbers reduced in this way are equal can the cossic signs be abbreviated; for then equality still remains, as we said in chapter 12. Thus, since $\sqrt[3]{512^3}$ and $\sqrt[3]{256^3}$ are equal (each being worth 4), if the cossic signs are abbreviated, equality will still be seen between $\sqrt[3]{512}$ and $\sqrt[3]{256R}$ — since 256R makes 512 — and it is established that $\sqrt[3]{512}$ and $\sqrt[3]{512}$ are equal. I wished to warn you of this, lest you abbreviate the cossic signs, even after the reduction has been made, unless the equality of the numbers is established.

Of the fractions of these numbers, again, there is no need to say anything, since they follow the precepts of their integers in all respects.

[Margin: An equation between an irrational cossic number and an absolute number.]

Furthermore, if an equation happens to occur between an irrational cossic number and an absolute number, that equation is reduced in this way — which indeed, near the end of chapter 10, we promised we would teach in this place. Let there be, for example, an equation between $\sqrt[3]{24R}$ and 12. This being posited,

there will also be an equation between their squares, $24R$ and 144. So, dividing 144 by 24, there comes out in the quotient the value of a single root, 6 — which is evident, since $24R$ will make 144, whose square root is 12.

Likewise, if there is an equation between $\sqrt[3]{103}$ and 20, there will also be an equation between their squares, 103 and 400. So, dividing 400 by 10, one finds that 13 is worth 40, and therefore $1R$ will be worth $\sqrt[3]{40}$. For 103 will be 400, whose square root is 20.

Finally, if an equation is found between $\sqrt[4]{123}$ and 30, there will also be an equation between their cubes, 123 and 27000. So, dividing 27000 by 12, there comes out the value of 13 , namely 2250; and accordingly 123 will make 27000, whose cube root is 30.

But if someone proposes an equation between $\sqrt[4]{8R}$ and 3, there will also be an equation between their cubes, $8R$ and 27 — which is impossible. And so, in equations of this kind, it is necessary that the absolute number be a root of the number that carries the radical sign: such is the equation between $\sqrt[4]{8R}$ and 2, or between $\sqrt[4]{64}$ and 4, or between $\sqrt[3]{81}$ and 9, and so on. Otherwise the equation will be impossible.

[Margin: When an equation of this kind is impossible.]

ON BINOMIALS AND APOTOMES, OR RESIDUALS.

CAPUT XXVII.

[Margin: What binomial and residual, and likewise trinomial and quadrinomial, mean among the writers.]

Almost all writers on Algebra are accustomed to call any two numbers joined by the sign + a **binomial** — for example, $\sqrt[3]{12} + \sqrt[3]{33}$, and $\sqrt[3]{18} + \sqrt[3]{38}$, and $6 + \sqrt[3]{39}$. Numbers joined by the sign –, however, they call a **residual** or **apotome** — for example, $\sqrt[3]{12} - \sqrt[3]{33}$, and $\sqrt[3]{18} - \sqrt[3]{38}$, and $6 - \sqrt[3]{39}$. Several numbers joined in this way they have been accustomed to call a trinomial, quadrinomial, and so on — for example $\sqrt[3]{17} + \sqrt[3]{10} + \sqrt[3]{33}$, or $\sqrt[3]{17} - \sqrt[3]{10} + \sqrt[3]{33}$, they call a trinomial; while $\sqrt[3]{17} + \sqrt[3]{10} + \sqrt[3]{33} + \sqrt[3]{32}$, or $\sqrt[3]{17} - \sqrt[3]{10} + \sqrt[3]{33} - \sqrt[3]{32}$, [they call] a quadrinomial, and so forth.

[Margin: What binomial and residual are according to Euclid.]

Euclid, however, in Book 10, Propositions 37 and 74, calls only two numbers joined by the sign + or – binomials or residuals when those two joined numbers are rational and commensurable only in power [potentia commensurabiles] — that is, when their squares are commensurable, being commensurable with the square of some rational number — even though one of them, or both, be a surd root. Thus $6 + \sqrt{3}20$, and $\sqrt{3}18 + 4$, and $\sqrt{3}24 + \sqrt{3}18$ he calls binomials; but $6 - \sqrt{3}20$, and $\sqrt{3}18 - 4$, and $\sqrt{3}24 - \sqrt{3}18$ he calls residuals or apotomes. For any two such joined numbers are rational, commensurable only in power, since their squares are commensurable, being commensurable with the square of some rational number. And so neither $\sqrt{3}310 + \sqrt{3}38$ will be a binomial, nor $\sqrt{3}310 - \sqrt{3}38$ a residual, because their parts are irrational numbers — inasmuch as their squares, $\sqrt{3}10$ and $\sqrt{3}8$, are incommensurable, having the proportion that $\sqrt{3}(5/4)$ has to 1, as is clear from the Lemma in chapter 18. By the same reasoning, neither $6 + \sqrt{3}9$ nor $20 + 6$ will be binomials, nor $6 - \sqrt{3}9$ nor $20 - 6$ residuals, because any two such numbers make only a single rational number, and not two. For $6 + \sqrt{3}9$ is equivalent to 9, and $20 + 6$ is equivalent to 26; likewise $6 - \sqrt{3}9$ is equivalent to 3, and 20...

(continuing Cap. XXVII, "On binomials and apotomes")

And $20 - 6$ equal 14. The reason is that any two such [terms] are rational and commensurable in length, not merely in power; for the ratio between any two of them is rational. Thus the ratio of 6 to $\sqrt{3}9$ [the cossist radical $\sqrt{3} = \text{radix}$, i.e. square root; here $\sqrt{9} = 3$] is double: for if 6 — that is, $\sqrt{3}36$ — is divided by $\sqrt{3}9$, the quotient $\sqrt{3}4$, that is 2, comes out as the denominator of the ratio. But if 20 is divided by 6, the quotient is $3\frac{1}{3}$, so that 20 to 6 stand in the ratio "tripla sesquitercia" [$3\frac{1}{3} : 1$].

For the same reason, neither will $\sqrt{3}12 + \sqrt{3}3$ be a binomial, nor $\sqrt{3}12 - \sqrt{3}3$ a residuum, even though each number is a surd root — because the numbers stand in a double ratio, and the first pair make up the sum $\sqrt{3}27$, that is, a single number, while the latter pair equal $\sqrt{3}3$, likewise a single number. And on the same grounds neither $\sqrt{3}18 + \sqrt{3}8$ is to be reckoned a binomial, nor $\sqrt{3}18 - \sqrt{3}8$ a residuum, although each number is a surd root — because the numbers stand in the ratio "sesquialtera," whose denominator is $1\frac{1}{2}$; and the first pair make up the sum $\sqrt{3}50$, a single number, while the latter pair equal, for this single number, $\sqrt{3}8$ [recte: $\sqrt{3}2$ — since $\sqrt{18} - \sqrt{8} = \sqrt{2}$].

Finally, when two rational numbers are given — that is, numbers whose squares are commensurable with the square of some rational number, yet in such a way that when one is divided by the other the quotient is rational (that is, a number

that has a root) — no binomial or residuum can be formed from them, because such numbers are commensurable not merely in power but also in length. Consequently no irrational number arises from them, but a rational one, even though it may sometimes turn out to be a surd root. Thus from 3 and $\sqrt{3} 16$ arises the sum $\sqrt{3} 49$, that is 7, a rational number that is not surd; but from $\sqrt{3} 8$ and $\sqrt{3} 18$ arises the sum $\sqrt{3} 50$, a rational-but-surd number.

And on this point not a few are deceived, supposing that every number bearing the sign $\sqrt{3}$ and lacking a root is irrational — which is not true, and plainly contradicts Euclid. [Margin: Not every number bearing the sign $\sqrt{3}$ is irrational.] [Margin: Euclid X.22.] [Margin: Surface, and the Medial line.] Only when a number bearing the sign $\sqrt{3}$ denotes some surface or square is that number irrational — such as the rectangular surface produced by the mutual multiplication of two rational numbers commensurable only in power. Thus if $\sqrt{3} 3$ is multiplied by $\sqrt{3} 10$, there is produced the surface $\sqrt{3} 30$, an irrational one, which is called Medial. Its square root is $\sqrt{33} 30$ [the double sign $\sqrt{33} = \textit{radix radicis}$, i.e. fourth root; here $\sqrt[4]{30}$], which is called the Medial line — since this line, squared, makes $\sqrt{33} 900$, that is $\sqrt{3} 30$, because the square root of 900 is 30, and the square root of this is $\sqrt{3} 30$. Therefore, when in what precedes we called numbers carrying the prefixed sign $\sqrt{3}$ "irrational," that was said improperly, since those numbers are in fact rational.

Every number bearing the sign $\sqrt{3}$ can indeed be called irrational, if it is taken as a Medial surface produced from $\sqrt{3} 1$ multiplied into the number itself. Thus $\sqrt{3} 3$ will be a Medial surface, produced from $\sqrt{3} 1$ into $\sqrt{3} 3$, and its square root will be $\sqrt{33} 3$; for this, multiplied into itself in the manner of a square, makes $\sqrt{33} 9$, that is $\sqrt{3} 3$ — since the square root of 9 is 3, and the square root of this is $\sqrt{3} 3$ — and so with the others.

Euclid established six kinds of binomials, and as many residua. For either the greater term exceeds the lesser in power by [the square of a number] whose root is commensurable in length with the greater term, or incommensurable. Likewise, either the greater term is commensurable in length with an exposit rational number, or the lesser is, or neither. [Margin: The six binomials and residua.] For both terms cannot be commensurable in length with an exposit rational number, [Margin: Euclid X.12.] since then the terms themselves would be rational numbers commensurable in length — which is absurd, as they are posited to be commensurable only in power. Granting these, there arise six kinds of binomials, defined as follows.

The first binomial is one whose greater term exceeds the lesser in power by [the square of a number] whose root is commensurable in length with the greater term, and whose greater term is itself commensurable in length with an expositied rational number. [Margin: The first binomial.] For example, $6 + \sqrt{3} 27$: because the square of the greater term, 6 — namely 36 — exceeds 27, the square of the lesser term, by the square 9, whose root, 3, is commensurable in length with the greater term, 6; and the greater term, 6, is itself commensurable in length with any expositied rational number.

The second binomial is one whose greater term exceeds the lesser in power by [the square of a number] whose root is commensurable in length with the greater term, and whose lesser term is commensurable in length with an expositied rational number. [Margin: The second binomial.] For example, $\sqrt{3} 48 + 6$: because the square of the greater term, namely 48, exceeds 36, the square of the lesser term, by the square 12, whose root — namely $\sqrt{3} 12$ — is commensurable in length with the greater term, since $\sqrt{3} 48$ stands to $\sqrt{3} 12$ in a double ratio, because when the former is divided by the latter the quotient is $\sqrt{3} 4$, that is 2. Likewise the lesser term is commensurable in length with any expositied rational number.

The third binomial is one whose greater term exceeds the lesser in power by [the square of a number] whose root is commensurable in length with the greater term, but where neither term is commensurable in length with an expositied rational number. [Margin: The third binomial.] For example, $\sqrt{3} 24 + \sqrt{3} 18$: because the square of the greater term, 24, exceeds 18, the square of the lesser term, by the square 6, whose root — namely $\sqrt{3} 6$ — is commensurable in length with the greater term, since $\sqrt{3} 24$ stands to $\sqrt{3} 6$ in a double ratio, because when the former is divided by the latter the quotient is $\sqrt{3} 4$, that is 2. But neither term is commensurable in length with an expositied rational number.

The fourth binomial is one whose greater term exceeds the lesser in power by [the square of a number] whose root is incommensurable in length with the greater term, but whose greater term is commensurable in length with an expositied rational number. [Margin: The fourth binomial.] For example, $4 + \sqrt{3} 10$: because the square of the greater term, 16, exceeds 10, the square of the lesser term, by the square 6, whose root — namely $\sqrt{3} 6$ — is incommensurable in length with the greater term, 4, since the square of the greater term, 16, does not stand to 6, the square of $\sqrt{3} 6$, in the ratio of a square number to a square number, [Margin: Euclid X.9.] and so their sides are incommensurable in length.

But the greater term, 4, is commensurable in length with the expositied rational number.

The fifth binomial is one whose greater term exceeds the lesser in power by [the square of a number] whose root is incommensurable in length with the greater term, but whose lesser term is commensurable in length with an expositied rational number. [Margin: The fifth binomial.] For example, $\sqrt{3} 5 + 2$: because the square 5 of the greater term exceeds 4, the square of the lesser term, by the square 1, whose root — namely $\sqrt{3} 1$ — is incommensurable in length with the greater term, since the square of the greater term does not stand to 1, the square of $\sqrt{3} 1$, in the ratio of a square number to a square number, [Margin: Euclid X.9.] and so their sides are incommensurable in length. Yet the lesser term, 2, is commensurable in length with any expositied rational number whatever.

The sixth binomial, finally, is one whose greater term exceeds the lesser in power by [the square of a number] whose root is incommensurable in length with the greater term, and where neither term is commensurable in

length with an expositied rational number. [Margin: Euclid X.9.] For example, $\sqrt{3} 80 + \sqrt{3} 50$: because the square 80 of the greater term exceeds 50, the square of the lesser term, by the square 30, whose root — namely $\sqrt{3} 30$ — is incommensurable in length with the greater term, since the square 80 of the greater term and the square 30 do not stand in the ratio of a square number to a square number, and so their sides are incommensurable in length. Neither term, moreover, is commensurable in length with an expositied rational number.

The six residua, or apotomes, behave in the same way, since from every binomial there arises a corresponding apotome by interposing the sign $-$ in place of the sign $+$. [Margin: The six kinds of apotomes.] For according to Euclid an apotome is nothing else than what remains when, two rational numbers commensurable only in power being given, the lesser is subtracted from the greater — which is done precisely by interposing the sign $-$. Thus:

- From the first binomial, $6 + \sqrt{3} 27$, arises the first apotome, $6 - \sqrt{3} 27$.
- From the second binomial, $\sqrt{3} 48 + 6$, arises the second apotome, $\sqrt{3} 48 - 6$.
- From the third binomial, $\sqrt{3} 24 + \sqrt{3} 18$, arises the third apotome, $\sqrt{3} 24 - \sqrt{3} 18$.
- From the fourth binomial, $4 + \sqrt{3} 10$, arises the fourth apotome, $4 - \sqrt{3} 10$.
- From the fifth binomial, $\sqrt{3} 5 + 2$, arises the fifth apotome, $\sqrt{3} 5 - 2$.
- From the sixth binomial, finally, $\sqrt{3} 80 + \sqrt{3} 50$, arises the sixth apotome, $\sqrt{3} 80 - \sqrt{3} 50$.

All of which are clear from the definitions of the apotomes handed down by Euclid.

Since the understanding of the tenth book consists chiefly in the knowledge of binomials and residua, it is worth dwelling a little longer on their treatment. First I shall show by what method any desired binomial — and consequently any residuum, since this differs from the binomial only by the sign — may be found. Then, by what art the square root may be drawn from any binomial or residuum, and what irrational line, or composite or diminished number, that root constitutes — always keeping to what has been demonstrated by Euclid.

Finding the first binomial and the first apotome. [Margin: Finding the first binomial and the first apotome.] Let some square number AB, namely 9, be divided into BC, the square number 4, and into AC, the non-square number 5; and let any rational number, 6, be assumed for the first term. Then let it be arranged that, as the number 9 (AB) is to the number 5 (AC), so 36, the square of the assumed number 6, is to another; and there will be found the square 20. Therefore $6 + \sqrt{3} 20$ will be the first binomial, and $6 - \sqrt{3} 20$ will be the first apotome, as is established by Euclid X.49 and X.86.

Diagram. A line segment divided at C, with the divisions labelled, and the resulting binomial and apotome written beneath.

- A — left endpoint of the segment
- C — point of division; segment AC labelled 5
- B — right endpoint; segment CB labelled 4 (so whole AB = 9)
- Below: $6 + \sqrt{3} 20$ (first binomial)
- Below: $6 - \sqrt{3} 20$ (first apotome)

Finding the second binomial and the second apotome. [Margin: Finding the second binomial and the second apotome.] The second binomial and the second apotome are found in this way. Any square number AB, namely 9, being divided into BC, the square 4, and into AC, the non-square 5, let a rational number be assumed at will, 5, for the second term. Let it be arranged that, as the number 5 (AC) is to the square number 9 (AB), so 25, the square of the assumed number 5, is to another; and there will be found the square 45. Therefore the second binomial is $\sqrt{3} 45 + 5$, and $\sqrt{3} 45 - 5$ is the second apotome, as is evident from Euclid X.50 and X.87.

Diagram. A line segment divided at C, with the resulting binomial and apotome written beneath.

- A — left endpoint
- C — point of division; segment AC labelled 5
- B — right endpoint; segment CB labelled 4 (whole AB = 9)
- Below: $\sqrt{3} 45 + 5$ (second binomial)
- Below: $\sqrt{3} 45 - 5$ (second apotome)

Finding the third binomial and the third apotome. [Margin: Finding the third binomial and the third apotome.] We shall find the third binomial and the third apotome thus. Any square number AB being divided — namely 9 — into BC, the square 4, and into AC, the non-square 5, let another number D be taken, namely 6, the non-square next greater than AC, 5; and let D differ from AC either by unity alone or by two. By unity indeed, when AC is such a number that unity added to it does not make a square; but by two, when unity added to AC does make a square. Thus if AC were 8, the next greater non-square would differ from 8 by two, because unity added to 8 makes the square 9. But if AC is 5, as in our example, the next greater non-square will be 6, differing from 5 by unity, because unity added to 5 does not complete a square. Then, some rational number 12 being taken, let it be arranged that, as the number D, 6, is to the square number AB, 9, so 144, the square number of the assumed rational number 12, is to another; and there will be found the square 216, whose root — namely $\sqrt{3} 216$ — will be the greater term. And if it is again arranged that, as 9, the square AB, is to the number AC, 5, so the square 216 just found is to another, there will be found the square 120, whose root — namely $\sqrt{3} 120$ — will be the lesser term. The third binomial found is therefore $\sqrt{3} 216 + \sqrt{3} 120$, and the third apotome $\sqrt{3} 216 - \sqrt{3} 120$, as is clear from Euclid X.51 and X.88.

Diagram. A line segment divided at C, with an auxiliary segment D drawn beneath it (labelled 12), and the resulting binomial and apotome written below.

- A — left endpoint
- C — point of division; segment AC labelled 5
- B — right endpoint; segment CB labelled 4 (whole AB = 9)
- D — auxiliary segment beneath AB, labelled 12
- Below: $\sqrt{3} 216 + \sqrt{3} 120$ (third binomial)
- Below: $\sqrt{3} 216 - \sqrt{3} 120$ (third apotome) [printed with the minus sign displaced: " $\sqrt{3} 216 \sqrt{3} - 120$ "]

Finding the fourth binomial and the fourth apotome. [Margin: Finding the fourth binomial and the fourth apotome.] We shall find the fourth binomial and the fourth apotome in this manner. Any square number AB, namely 9, being

divided into two non-squares AC, CB — that is, into 6 and 3 — and some rational number 6 being taken for the greater term: let it be arranged that, as the square number AB (namely 9) is to AC, 6, the non-square, so 36, the square of the assumed rational number 6, is to another; and there will be found the square 24, whose root — namely $\sqrt{3} 24$ — will be the lesser term. The fourth binomial is therefore $6 + \sqrt{3} 24$, and the fourth apotome $6 - \sqrt{3} 24$ is gathered, from Euclid X.52 and X.89.

Diagram. A line segment divided at C, with the resulting binomial and apotome written beneath.

- A — left endpoint
- C — point of division; segment AC labelled 6
- B — right endpoint; segment CB labelled 3 (whole AB = 9)
- Below: $6 + \sqrt{3} 24$ (fourth binomial)
- Below: $6 - \sqrt{3} 24$ (fourth apotome)

Finding the fifth binomial and the fifth apotome. [Margin: Finding the fifth binomial and the fifth apotome.] We shall investigate the fifth binomial and the fifth apotome thus. Any square number AB, namely 9, being divided into two non-squares AC, CB — as into 6 and 3 — and some rational number 2 being taken for the lesser term: let it be arranged that, as AC, the number 6, is to the square AB, namely 9, so 4, the square of the assumed rational number 2, is to another; and there will be found the square 6, whose root — namely $\sqrt{3} 6$ — will be the greater term. The fifth binomial will therefore be $\sqrt{3} 6 + 2$, and the fifth apotome $\sqrt{3} 6 - 2$, as Euclid concludes in X.53 and X.90.

Diagram. A line segment divided at C, with the resulting binomial and apotome written beneath.

- A — left endpoint
- C — point of division; segment AC labelled 6
- B — right endpoint; segment CB labelled 3 (whole AB = 9)
- Below: $\sqrt{3} 6 + 2$ (fifth binomial)
- Below: $\sqrt{3} 6 - 2$ (fifth apotome)

The sixth binomial, finally, and the sixth apotome we shall track down by this art. [Margin: Finding the sixth binomial and the sixth apotome.] Any non-square number AB being divided into two non-squares AC, CB — as into 7 and 5 — let another square number D be taken, whatever it may be, namely 9, and in addition any rational number, 12. Then let it be arranged that, as the square

number D, namely 9, is to the number AB, 12, the non-square, so 144, the square number of the assumed rational number 12, is to another; for there will be found the square 192, whose root — namely $\sqrt{3} 192$ — will be the greater term of the binomial. And if it is again arranged that, as the number 12 (AB), the non-square, is to 7, the number AC, so the square 192 just found is to another, there will be found the square number 112, whose root — namely $\sqrt{3} 112$ — will be the lesser term. Thus the sixth binomial is $\sqrt{3} 192 + \sqrt{3} 112$, and the sixth apotome $\sqrt{3} 192 - \sqrt{3} 112$, as is established by Euclid X.54 and X.91.

Diagram. A line segment divided at C, with an auxiliary segment D drawn beneath (labelled 12), and the resulting binomial and apotome written below.

- A — left endpoint
- C — point of division; segment AC labelled 7
- B — right endpoint; segment CB labelled 5 (whole AB = 12)
- D — auxiliary segment beneath AB, labelled 12
- Below: $\sqrt{3} 192 + \sqrt{3} 112$ (sixth binomial)
- Below: $\sqrt{3} 192 - \sqrt{3} 112$ (sixth apotome)

Now, given any binomial or apotome, we shall find from it, without much labor, as many other binomials and apotomes of the same species as we wish. [Margin: Finding a binomial or apotome of any species from a given binomial or apotome of the same species.] For a first or fourth binomial: if it is arranged that, as the greater term is to the lesser, so any assumed rational number, taken as the greater term of the new binomial, is to another, the lesser term of the new first or fourth binomial is produced. For a second or fifth: if it is arranged that, as the lesser term is to the greater, so any rational number, assumed as the lesser term of the new binomial, is to another, the greater term of the new second or fifth binomial is generated. For a third or sixth, finally: if it is arranged that, as the greater term is to the lesser (both of which are surd numbers), so any surd number, assumed as the greater term of the new binomial, is to another, the lesser term of the new third or sixth binomial results. Should anyone want the reason for this, let him know it is to be sought from Euclid X.15.

For since, by construction, the ratio between the terms of the given binomial is the same as that between the terms of the produced binomial: if the greater term of the given binomial exceeds the lesser in power by [the square of] a root commensurable with it in length, [Margin: Euclid X.15.] then the greater term of the new binomial too will be able to exceed the lesser by [the square of] a root commensurable with it in length; and if the greater term of the former exceeds

the lesser in power by [the square of] a root incommensurable with it in length, then the greater term of the latter too will be able to exceed the lesser by [the square of] a root incommensurable with it in length. For the terms of binomials are like lines. And therefore those two binomials will be of the same species — namely, each will be either first, or second, or third, and so on — since either the greater term in both, or the lesser, is rational or surd.

Moreover, once a new binomial has been found, you now know that a residuum or apotome is made from it by changing the sign + into -.

For example, from the first binomial $6 + \sqrt{3} 27$ there will be made a similar binomial whose greater term is 400, if it is arranged that, as 6 is to $\sqrt{3} 27$, so 400 is to another. That is, if $\sqrt{3} 27$ is multiplied into 400 — that is, into $\sqrt{3} 160000$ — and the product $\sqrt{3} 4320000$ is divided by 6, that is, by $\sqrt{3} 36$, there will be produced $\sqrt{3} 120000$. And the new first binomial will be $400 + \sqrt{3} 120000$, and the new first apotome $400 - \sqrt{3} 120000$. For the square 160000 of the greater term exceeds the square 120000 of the lesser term by the square 40000, whose root, 200, is commensurable in length with the greater term, and so on. The same reasoning holds for the rest.

Again, if both terms of a given binomial are multiplied or divided by one and the same number, whatever it may be, there will be produced, [Margin: Euclid X.15.] for the same reason, the two terms of another binomial of the same species — which terms will also be those of another apotome of the same species, [Margin: Euclid VII.17.] because the two numbers thus generated have the same ratio between themselves as the two terms of the proposed binomial. For instance, given the second binomial $\sqrt{3} 48 + 6$: if both terms are multiplied by 3, there will be made another second binomial, $\sqrt{3} 432 + 18$, and another apotome, $\sqrt{3} 432 - 18$. And if both terms of the same given second binomial $\sqrt{3} 48 + 6$ are divided by any one and the same number, 3, there will again be made another second binomial, $\sqrt{3} 5\frac{1}{3} + 2$, and another second apotome, $\sqrt{3} 5\frac{1}{3} - 2$. But if the division is done by 2, there will be generated a new second binomial, $\sqrt{3} 12 + 3$, and a new second apotome, $\sqrt{3} 12 - 3$, and so with the others.

Now if at any time it must be judged, concerning some proposed binomial or residuum, which of the aforesaid six species it belongs to, this will be no great labor. [Margin: To judge which of the 6 species a proposed binomial, or residuum, belongs to.] For if the greater term is an absolute rational number, the binomial will be either first or fourth. And indeed, if — the square of the greater term being divided by the excess by which that square exceeds the square of the lesser

term — the quotient turns out a square number, the binomial will be first; but if the quotient is a non-square number, the binomial will be fourth.

If, however, the lesser term is an absolute rational number, the binomial will be either second or fifth. And indeed, if — the square of the greater term being divided by the excess by which that square exceeds the square of the lesser term — the quotient is generated a square number, the binomial will be second; but if the quotient turns out a non-square number, the binomial will be fifth.

If, finally, neither of the terms is an absolute rational number, but a surd, the binomial will be either third or sixth. Third indeed, when — the square of the greater term being divided by the excess by which that square exceeds the square of the lesser term — the quotient comes out a square number; but if the quotient is generated a non-square number, the binomial will be sixth. And you will say the same of any apotome.

Finally, it is also not unpleasant to know, as Michael Stifel [Michael Stifel] observed, that all six binomials, and consequently the residua or apotomes, may be found in a few lines, by a certain device, inscribed within any given circle. [Margin: The method by which all six binomials and apotomes are found within a circle.] Let there be a circle ABCD whose diameter BD is divided into as many equal parts as have both a third and a fourth part — namely into 12. From E, the quarter-point of the diameter (BE comprising three twelfths of the diameter), and from F, the third-point of the diameter (BF comprising four twelfths of the diameter), let there be erected perpendicular to the diameter, on opposite sides, the lines EA, FC; and let the straight lines AB, BC, CD, DA be joined. The magnitudes of all these straight lines will be known in this way. [Margin: Euclid VI, coroll. of prop. 13.] Since AE is the mean proportional between DE and EB, and CF between DF and FB, [Margin: Euclid VI.17.] the square on AE will equal the rectangle under DE, EB, and the square on CF the rectangle under DF, FB. Now the rectangle under DE, EB — that is, 9 multiplied by 3 — is 27; therefore the square on AE will be 27, and so the line AE is $\sqrt{3} 27$. And the rectangle under DF, FB — that is, 8 multiplied by 4 — is 32; therefore the square on CF will be 32, and the line CF is $\sqrt{3} 32$.

Then, since the squares on BE and EA — namely 9 and 27 — make up 36, [Margin: Euclid I.47.] to which the square on AB is equal, the square on AB will be 36, and the line AB itself $\sqrt{3} 36$, that is, 6.

Note on the cossist radical notation used throughout this batch: the printed sign is a radical $\sqrt{}$ immediately followed by a small census (zensus) mark 3. A single $\sqrt{3}n$

denotes the square root of n (radix, the "universal" surd root); a doubled $\sqrt[3]{n}$ denotes the fourth root of n (radix radices / *radix zensizensica*, i.e. $\sqrt[4]{n}$). The signs are rendered here exactly as printed, with the decimal value glossed on first use. Arithmetic confirms the readings (e.g. $\sqrt[3]{3448} = \sqrt{448} = 8\sqrt{7}$; $\sqrt[3]{32401} = \sqrt[4]{2401} = 7$).

Again, since the squares on BF and FC — namely 16 and 32 — make 48, and the square on BC is equal to this [Margin: a — Euclid I.47], this square is 48, and the line BC itself is $\sqrt[3]{48}$ [the radical $\sqrt$ with the census mark 3 = square root; so $\sqrt[3]{48} = \sqrt{48}$].

Likewise, the squares on DF and FC — namely 64 and 32 — come to 96, and the square on CD is equal to this [Margin: b — Euclid I.47]; so this square is 96, and the line CD itself is $\sqrt[3]{96}$ [$\sqrt{96}$].

Finally, since the squares on DE and EA — namely 81 and 27 — make 108, and the square on DA is equal to this [Margin: c — Euclid I.47], this square is 108, and the line DA itself is $\sqrt[3]{108}$ [$\sqrt{108}$].

Diagram. A circle with a horizontal base line running from B (left) through the interior to D (right); the base carries a numbered scale of twelve units (1–12). Point A is at the top of the circle, point C at the bottom. From A a diagonal runs to D and a vertical drops to E on the base; from A a line runs to B. From C a vertical rises to F on the base, a diagonal runs to B (lower left), and another diagonal runs to D (lower right). E and F sit on the base near the third and fourth scale marks. The chords and perpendiculars are labelled with their lengths.

- A (top of circle): apex from which segments AB, AE, AD issue
- B (left end of base line, scale point 0): meeting of AB and BC
- D (right end of base line, scale point 12): meeting of AD and CD
- E (on base, near scale point 3): foot of the perpendicular from A
- F (on base, near scale point 4): foot of the perpendicular from C
- C (bottom of circle): apex from which segments CB, CF, CD issue
- segment AB: 6
- segment AE (A perpendicular down to E): $\sqrt[3]{27}$ [$\sqrt{27}$]
- segment AD (upper diagonal to D): $\sqrt[3]{108}$ [$\sqrt{108}$]
- segment BC (lower-left diagonal): $\sqrt[3]{48}$ [$\sqrt{48}$]
- segment CF (C perpendicular up to F): $\sqrt[3]{32}$ [$\sqrt{32}$]
- segment CD (lower-right diagonal): $\sqrt[3]{96}$ [$\sqrt{96}$]
- base line marked with the scale 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

Now the lines AB and AE constitute the first binomial, $6 + \sqrt{327}$, and the first apotome, $6 - \sqrt{327}$.

The lines AD and DE constitute the second binomial, $\sqrt{3108} + 9$, and the second apotome, $\sqrt{3108} - 9$.

The lines AD and DC constitute the third binomial, $\sqrt{3108} + \sqrt{396}$, and the third apotome, $\sqrt{3108} - \sqrt{396}$.

The lines BD and DC constitute the fourth binomial, $12 + \sqrt{396}$, and the fourth apotome, $12 - \sqrt{396}$.

The lines DC and DF constitute the fifth binomial, $\sqrt{396} + 8$, and the fifth apotome, $\sqrt{396} - 8$.

Finally the lines BC and CF constitute the sixth binomial, $\sqrt{348} + \sqrt{332}$, and the sixth apotome, $\sqrt{348} - \sqrt{332}$.

That these are the six binomials and those six residuals defined by Euclid, you will be able to verify by their definitions. For in the first three, if the greater names are divided by the excesses by which their squares exceed the squares of the lesser names, the quotients come out as square numbers; and accordingly the greater names exceed the lesser in square by an amount whose roots are commensurable in length with the greater names. But if in the latter three the greater names are divided by the excesses by which their squares exceed the squares of the lesser names, the quotients come out as non-square numbers; and accordingly the greater names exceed the lesser in square by an amount whose roots are incommensurable in length with the greater names. The rest is self-evident.

But if you divide the diameter BD into other parts whose number has a third part and a fourth part — as in 24, or 36, or 48, or 60, or 204, or 360, or 480, or 600, and so on — you will find, in the same way, another six binomials and apotomes.

ON THE EXTRACTION OF ROOTS FROM BINOMIALS and Apotomes; with an incidental treatment of the other irrational lines that Euclid discusses in Book 10. CHAPTER XXVIII.

[Margin: Why some say that not all binomials have roots.] There are some who assert that square roots can be extracted only from the first three binomials and apotomes. This is not true, and it flatly contradicts Euclid, as I shall show shortly; nor do I see how they can be excused, unless one says that they deny square roots to the latter three binomials and apotomes because the roots of these are very intricate and more obscure than the binomials and apotomes themselves — being, as they are, roots of composite numbers, which they call "universal" roots, as will be clear from the examples.

[Margin: All binomials, according to Euclid, have roots.] That all binomials and apotomes do have square roots — roots which, multiplied by themselves, produce the binomials and apotomes — is plainly gathered from propositions 55, 56, 57, 58, 59, and 60, and likewise from propositions 92, 93, 94, 95, 96, and 97 of Book 10 of Euclid. For in the first six propositions Euclid demonstrates which numbers, multiplied by themselves, produce binomials (I call the irrational lines "irrational numbers"); and in the latter six propositions he shows which numbers, multiplied by themselves, generate apotomes. This will become clear as follows.

[Margin: What kind of irrational lines, or irrational numbers, the roots of the six binomials are.] In proposition 55 he says: if an area is contained by a rational line and a first binomial line, the straight line whose square equals that area is irrational, and is called a binomial. Now who does not see that, if that rational line is unity, the area so contained is the first binomial — since unity multiplying the first binomial produces the first binomial itself? Therefore the number whose square is the first binomial — that is, the root of the first binomial — is a binomial, according to Euclid. By the same reasoning it is inferred from the next five propositions that the root of the second binomial is the irrational line called "first from two medials"; that the root of the third binomial is the irrational line called "second from two medials"; that the root of the fourth binomial is the irrational line called "major"; that the root of the fifth binomial is the irrational line called "the side of a rational plus a medial area" [Rationale ac Medium potens]; and finally that the root of the sixth binomial is the irrational line called "the side of the sum of two medial areas" [bina Media potens].

[Margin: What kind of irrational lines, or numbers, the roots of the six apotomes are.] In just the same way, from proposition 92 and the five following, we shall gather that the root of the first apotome is an apotome; that the root of the second apotome is a first medial apotome; that the root of the third apotome is a second medial apotome; that the root of the fourth apotome is the irrational line called "minor"; that the root of the fifth apotome is the line "that with a rational area produces a whole medial area" [cum Rationali Medium totum efficiens]; and finally that the root of the sixth apotome is the irrational line "that with a medial area produces a whole medial area" [cum Medio Medium totum efficiens]. There is therefore no doubt that, in Euclid's view, all binomials and all apotomes have roots, since he teaches so clearly what the root of each binomial and apotome is to be called.

To make this plainer, I must explain in passing those irrational lines which he shows to be the roots of binomials and apotomes. For so far, in the preceding chapter, I have set out only the binomial and its six species, and the apotome alone and its six kinds, and likewise the medial line.

Just as, then, a binomial in Euclid is an irrational line compounded of two rational lines commensurable in square only [Margin: a — Euclid X.37], so too, if two medial lines commensurable in square only are compounded so as to contain a rational area (proposition 27 taught how to find these lines) [Margin: b — Euclid X.38], the whole line will be irrational, and is called "first from two medials." This the root of the second binomial will exhibit for us. [Margin: First from two medials.]

And if two medial lines commensurable in square only are compounded so as to contain a medial area (proposition 29 taught how these are found) [Margin: c — Euclid X.39], the whole line will be irrational, and is called "second from two medials." This the root of the third binomial will furnish for us. [Margin: Second from two medials.]

And if two straight lines incommensurable in square are compounded such that the sum of their squares is rational, but the rectangle contained by them is medial (proposition 34 shows how to find them) [Margin: d — Euclid X.40], the whole line will be irrational, and is called the "major." This the root of the fourth binomial will present to us. [Margin: The major line.]

And if two straight lines incommensurable in square are compounded such that the sum of their squares is medial, but the rectangle contained by them is rational (found in proposition 35) [Margin: e — Euclid X.41], the whole line will

be irrational, and is called "the side of a rational plus a medial area." This the root of the fifth binomial will show us. [Margin: Rational and medial power.]

Finally, if two straight lines incommensurable in square are compounded such that the sum of their squares is medial, and the rectangle contained by them is likewise medial and moreover incommensurable with the sum of their squares (proposition 36 taught how such lines are found) [Margin: f — Euclid X.42], the whole line will be irrational, and is called "the side of the sum of two medial areas." This the root of the sixth binomial will indicate to us. [Margin: The side of two medial areas.]

Likewise, just as an apotome in Euclid is the irrational line that remains when from a rational line there is subtracted a rational line commensurable in square only with the whole [Margin: g — Euclid X.74], so too, if from a medial line there is subtracted a medial line commensurable in square only with the whole, which together with the whole contains a rational area (proposition 28 finds these lines) [Margin: h — Euclid X.75], the remaining line is irrational, and is called the "first medial apotome." This the root of the second apotome yields. [Margin: First medial apotome.]

And if from a medial line there is subtracted a medial line commensurable in square only with the whole, which together with the whole contains a medial area (these lines are found in proposition 29) [Margin: i — Euclid X.76], the remaining line is irrational, and is called the "second medial apotome." This will appear in the root of the third apotome. [Margin: Second medial apotome.]

And if from a straight line there is subtracted a straight line incommensurable in square with the whole, such that with the whole it makes the sum of their squares rational, but the rectangle contained by them medial (proposition 34 finds these lines) [Margin: k — Euclid X.77], the remaining straight line is irrational, and is called the "minor." This indeed is the root of the fourth apotome. [Margin: The minor line.]

And if from a straight line there is subtracted a straight line incommensurable in square with the whole, such that with the whole it makes the sum of their squares medial, but the rectangle contained by them rational (proposition 35 finds these lines) [Margin: l — Euclid X.78], the remaining line is irrational, and is called "that which with a rational area produces a whole medial area." Such is the root of the fifth apotome. [Margin: The remainder that with a rational makes a whole medial area.]

And finally, if from a straight line there is subtracted a straight line incommensurable in square with the whole, such that with the whole it makes the sum of their squares medial, and the rectangle contained by them medial as well, and moreover incommensurable with the sum of their squares (these lines are found in proposition 36) [Margin: m — Euclid X.79], the remaining line is irrational, and is called "that which with a medial area produces a whole medial area." This the root of the sixth apotome exhibits for us. [Margin: The remainder that with a medial makes a whole medial area.]

These irrational lines — or rather irrational numbers (for what Euclid calls lines, I call numbers) — I shall set out more fully and completely in treating the roots of binomials and apotomes.

You have, then, from Euclid, thirteen distinct irrational lines, which are the chief subject of Book 10. They are these:

1. Medial [Media].
2. From two names [ex binis nominibus, i.e. the binomial], of which six species have been found.
3. First from two medials [ex binis Medijs prima].
4. Second from two medials [ex binis Medijs secunda].
5. Major [Maior].
6. The side of a rational plus a medial area [Rationale ac Medium potens].
7. The side of the sum of two medial areas [Bina Media potens].
8. Apotome [Apotome], of which likewise six species have been found.
9. First medial apotome [Media Apotome prima].
10. Second medial apotome [Media Apotome secunda].
11. Minor [Minor].
12. That which with a rational makes a whole medial area [Cum Rationali Medium totum efficiens].
13. That which with a medial makes a whole medial area [Cum Medio Medium totum efficiens].

But if for the "line from two names" you take the six binomials, and for the apotome the six apotomes, you will have in all twenty-three irrational lines in Euclid.

But let me now prescribe the method by which square roots are extracted from binomials and apotomes; and in these the natures of the aforesaid irrational lines will be perceived more plainly.

Rule 1. Given, then, a binomial or a residual, take the difference of the squares of the two names. Next, add the square root of this difference to the greater name, and also subtract it from the same. Third, join the square root of half of that sum to the square root of half of the remainder — with the sign + if the given expression is a binomial, or with the sign – if it is a residual. In this way the root of the binomial or of the residual will be produced.

Rule 2. Others state this rule as follows. Take the difference of the squares of the halves of the two names. Add the square root of this difference to half of the greater name, and subtract it from that same half. For the square root of that sum, joined by the sign + or – to the square root of the remainder, will give the sought root of the binomial or residual.

Rule 3. Others, finally, state the same rule thus. From the fourth part of the difference of the squares of both names, extract the square root. If you then add this to half of the greater name and extract the root from the sum, you will produce the first part of the sought root. And if you subtract this same amount from the greater name, the square root of the remainder

will give the latter part of the sought root. So, once the two parts are joined by the sign + or –, you will have the whole desired root.

And so you have a threefold rule for extracting the root from a given binomial or residual — which must now be illustrated with various examples.

[Margin: The root of the first binomial is a binomial, and the root of the first residual is a residual.] Let the first binomial be $23 + \sqrt{3448}$. From it, by the first rule, we extract the root as follows. The difference of the squares of the two names, 529 and 448, is 81; its root is 9. If we add this 9 to the greater name 23 and subtract it from the same, we get the sum 32 and the remaining number 14. The halves of the sum and of the remainder are 16 and 7. The roots extracted from these — 4 and $\sqrt{37}$ — joined by the sign +, give $4 + \sqrt{37}$ as the root of the proposed first binomial; and accordingly $4 - \sqrt{37}$ will be the root of the first residual, $23 - \sqrt{3448}$. Of these, the former is a binomial and the latter a residual, as Euclid demonstrated in Book 10, propositions 55 and 92. That $4 + \sqrt{37}$ is the true root of the first binomial $23 + \sqrt{3448}$, and $4 - \sqrt{37}$ the true root of the first residual $23 - \sqrt{3448}$, can be proved by multiplying each root into itself, as is shown here.

Test of the root found [Examen radicis inuentae] — root of the binomial ($4 + \sqrt{37}$):

$$\begin{array}{r} 4 + \sqrt{37} \\ 4 + \sqrt{37} \\ \hline \end{array}$$

$$\begin{array}{r} \sqrt{3112} + 7 \\ 16 + \sqrt{3112} \end{array}$$

$$\text{Summa } 23 + \sqrt{3448}$$

Root of the residual ($4 - \sqrt{37}$):

$$\begin{array}{r} 4 - \sqrt{37} \\ 4 - \sqrt{37} \end{array}$$

$$\begin{array}{r} - \sqrt{3112} + 7 \\ 16 - \sqrt{3112} \end{array}$$

$$\text{Summa } 23 - \sqrt{3448}$$

[Margin: The root of the second binomial is the line "first from two medials"; and the root of the second residual is the "first medial apotome."] Let the second binomial be $\sqrt{3448} + 14$. From it the root is extracted by the second rule as follows. The halves of the names are $\sqrt{3112}$ and 7, whose squares, 112 and 49, have the difference 63; its root is $\sqrt{363}$. Added to $\sqrt{3112}$, the half of the greater name, it makes $\sqrt{3343}$; while subtracted from that same half $\sqrt{3112}$ it leaves $\sqrt{37}$. Therefore $\sqrt{3343} + \sqrt{37}$ will be the root of the proposed second binomial [$\sqrt{33} = \text{radix radicis} = \text{fourth root}$; so $\sqrt{3343} = {}^4\sqrt{343}$]. And $\sqrt{3343} - \sqrt{37}$ will be the root of the second residual, $\sqrt{3448} - 14$. The former, by proposition 56 of Book 10, is the irrational line "first from two medials"; the latter, by proposition 93 of Book 10, is the irrational line called the "first medial apotome." Thus $\sqrt{3343}$ and $\sqrt{37}$ are two medial lines containing a rational area, as propositions 38 and 75 of Book 10 hold; for the lines called "medial" are irrational. Multiplied by one another they produce $\sqrt{332401}$, whose fourth root [radix zensizensica] is 7 — a rational number.

Test of the root found — root of the binomial ($\sqrt{3343} + \sqrt{37}$):

$$\begin{array}{r} \sqrt{3343} + \sqrt{37} \\ \sqrt{3343} + \sqrt{37} \end{array}$$

$$\begin{array}{r} + \sqrt{332401} + \sqrt{37} \\ \sqrt{3343} + \sqrt{332401} \end{array}$$

$$\text{Summa } \sqrt{3448} + 14$$

Root of the residual ($\sqrt{3343} - \sqrt{37}$):

$$\begin{array}{r} \sqrt{3343} - \sqrt{37} \\ \sqrt{3343} - \sqrt{37} \end{array}$$

$$- \sqrt{332401} + \sqrt{37}$$

$$\sqrt{3343} - \sqrt{332401}$$

$$\text{Summa } \sqrt{3448} - 14$$

[Margin: The root of the third binomial is the line "second from two medials"; and the root of the third residual is the "second medial apotome."] Let the third binomial be $\sqrt{3448} + \sqrt{3336}$. From it the root is drawn out by the third rule in this way. The fourth part of the difference of the squares 448 and 336 is 28; its square root, $\sqrt{328}$, added to $\sqrt{3112}$ (the half of the greater name), makes $\sqrt{3252}$. Hence the first part of the sought root will be $\sqrt{33252}$. And if this same sum $\sqrt{3252}$ is taken away from $\sqrt{3448}$, the greater name, the remainder becomes $\sqrt{328}$; and accordingly the latter part of the sought root will be $\sqrt{3328}$. So the whole root of the third binomial will be $\sqrt{33252} + \sqrt{3328}$, and the root of the third residual, $\sqrt{3448} - \sqrt{3336}$, will be $\sqrt{33252} - \sqrt{3328}$. The former, by proposition 57 of Book 10, is the irrational line called "second from two medials"; the latter, by proposition 94 of Book 10, is called the "second medial apotome." Thus, following the teaching of propositions 39 and 76 of Book 10, $\sqrt{33252}$ and $\sqrt{3328}$ are two medial lines containing a medial area. For the lines called "medial" are indeed irrational; and multiplied by one another they produce a medial surface $\sqrt{337056}$, that is, $\sqrt{384}$.

Test of the root found — root of the binomial ($\sqrt{33252} + \sqrt{3328}$):

$$\sqrt{33252} + \sqrt{3328}$$

$$\sqrt{33252} + \sqrt{3328}$$

$$+ \sqrt{337056} + \sqrt{328}$$

$$\sqrt{3252} + \sqrt{337056}$$

$$\text{Summa } \sqrt{3448} + \sqrt{3336}$$

Root of the residual ($\sqrt{33252} - \sqrt{3328}$):

$$\sqrt{33252} - \sqrt{3328}$$

$$\sqrt{33252} - \sqrt{3328}$$

$$- \sqrt{337056} + \sqrt{328}$$

$$\sqrt{3252} - \sqrt{337056}$$

$$\text{Summa } \sqrt{3448} - \sqrt{3336}$$

[Margin: The root of the fourth binomial is the "major" line; and the root of the fourth residual is the "minor" line.] Let the fourth binomial be $24 + \sqrt{3448}$. From it the root is extracted by the first rule, in this way. The difference of the squares 576 and 448 is 128; I add its root — namely $\sqrt{3128}$ — to the greater name 24, giving $24 + \sqrt{3128}$. The same $\sqrt{3128}$ I subtract from that same greater name 24,

leaving $24 - \sqrt[3]{128}$. The half of that sum, $24 + \sqrt[3]{128}$, is $12 + \sqrt[3]{32}$; and the half of that remaining number, $24 - \sqrt[3]{128}$, is $12 - \sqrt[3]{32}$. Therefore the root of the proposed fourth binomial is $\sqrt[3]{12 + \sqrt[3]{32}} + \sqrt[3]{12 - \sqrt[3]{32}}$; and the root of this fourth residual, $24 - \sqrt[3]{448}$, is $\sqrt[3]{12 + \sqrt[3]{32}} - \sqrt[3]{12 - \sqrt[3]{32}}$. The former, by proposition 58 of Book 10, is the irrational line called the "major"; the latter, by proposition 95 of Book 10, is the irrational line called the "minor." And so, by the precept of propositions 40 and 77 of Book 10, $\sqrt[3]{12 + \sqrt[3]{32}}$ and $\sqrt[3]{12 - \sqrt[3]{32}}$ are two lines incommensurable in square, which make the sum of their squares rational, but the rectangle contained by them medial. That they are incommensurable in square is clear, since their squares, $12 + \sqrt[3]{32}$ and $12 - \sqrt[3]{32}$, are incommensurable. That the sum of these squares is rational is evident, since taken together they make the number 24, a rational. And finally, that which is contained by them

[Reader's note on the notation used throughout this section: the radical sign is printed with a small order-mark drawn from the cossist symbols. $\sqrt[3]{}$ denotes a *square* root (the mark 3, the *census* or square, marks the second order); $\sqrt[3]{3}$ denotes a *fourth* root (a root of a root); a cube root would be $\sqrt[3]{\text{C}} / \sqrt[3]{}$. Where the same symbols stand alone as algebraic unknowns, **R** = *res* (the first power, x), 3 = *census* (the square, x^2), and 33 = *census of census* (the fourth power, x^4). The printed tokens are kept as they stand below; arithmetic values are added in square brackets only where they aid the reader. The census-order marks are easily confused with the digit 8 in this font, and the square-root mark near-identical to the sursolid mark — readings here are pinned by the arithmetic.]

it is a Medial. This is evident, since from multiplying the two together there is generated the surface $\sqrt[3]{112}$, which is a Medial, and whose root is $\sqrt[3]{3112}$, a medial line. And here it is worth noticing that two incommensurables — such as the squares $12 + \sqrt[3]{32}$ and $12 - \sqrt[3]{32}$ — when added together yield the rational sum 24.

[Margin: Two incommensurables can make a rational sum.]

Test of the root found, for the fourth binomial.

$$\sqrt[3]{12 + \sqrt[3]{32}} + \sqrt[3]{12 - \sqrt[3]{32}}$$

$$\sqrt[3]{12 + \sqrt[3]{32}} + \sqrt[3]{12 - \sqrt[3]{32}}$$

$$12 + \sqrt[3]{32} + 12 - \sqrt[3]{32}$$

$$\sqrt[3]{112}$$

$$\sqrt[3]{112}$$

$$\sqrt[3]{448}$$

Total sum: $24 + \sqrt{3448}$. The binomial proposed.

Let the fifth binomial be $\sqrt{3448} + 12$. From it the root is extracted, by the second rule, in this way. The halves of the names are $\sqrt{3112}$ and 6, whose squares are 112 and 36; their difference is 76, whose root is $\sqrt{376}$.

[Margin: The root of the fifth binomial is a line potent in a rational and a medial [the "major" line, *Elements* X.39].]

Added to the half of the greater name, $\sqrt{3112}$, this gives $\sqrt{3112} + \sqrt{376}$; and subtracted from that same half it leaves $\sqrt{3112} - \sqrt{376}$. The root of the fifth binomial proposed is therefore $\sqrt{3(\sqrt{3112} + \sqrt{376})} + \sqrt{3(\sqrt{3112} - \sqrt{376})}$, while the root of the corresponding residual will be $\sqrt{3(\sqrt{3112} + \sqrt{376})} - \sqrt{3(\sqrt{3112} - \sqrt{376})}$.

[Margin: And the root of the fifth residual is a line which, together with a rational, makes the whole medial.]

The former, by *Elements* X.59, is an irrational line called "potent in a rational and a medial"; the latter, by the teaching of *Elements* X.96, is an irrational line which together with a rational makes the whole medial. Thus, by the teaching of *Elements* X.41 and X.78, the two lines $\sqrt{3(\sqrt{3112} + \sqrt{376})}$ and $\sqrt{3(\sqrt{3112} - \sqrt{376})}$ are incommensurable in square, and they make the sum of their squares a medial, while what is contained under them is rational.

[Margin: Two incommensurables multiplied together can make a rational number.]

That they are incommensurable in square is settled, since their squares are incommensurable. That the sum of their squares makes a medial is likewise plain, since their squares $\sqrt{3112} + \sqrt{376}$ and $\sqrt{3112} - \sqrt{376}$, taken together, make the medial surface $\sqrt{3448}$, whose root is $\sqrt{3448}$, a medial line. Finally, that their multiplication together generates a rational is manifest, since it generates $\sqrt{336}$, namely 6. Here too you see that two incommensurables — namely $\sqrt{3(\sqrt{3112} + \sqrt{376})}$ and $\sqrt{3(\sqrt{3112} - \sqrt{376})}$ — multiplied together produce a rational number, namely $\sqrt{336}$, that is, 6.

Test of the root found, for the fifth residual.

$$\begin{aligned} &\sqrt{3(\sqrt{3112} + \sqrt{376})} - \sqrt{3(\sqrt{3112} - \sqrt{376})} \\ &\sqrt{3(\sqrt{3112} + \sqrt{376})} - \sqrt{3(\sqrt{3112} - \sqrt{376})} \end{aligned}$$

$$\begin{aligned} &\sqrt{3112} + \sqrt{376} + \sqrt{3112} - \sqrt{376} \\ &- \sqrt{336}. \text{ that is, } - 6 \\ &- \sqrt{336}. \text{ that is, } - 6 \end{aligned}$$

Total sum: $\sqrt{3448} - 12$. The fifth residual.

[The connective printed between the two square-terms of the middle row reads as a dash; arithmetic and the parallel layout of the other tests require "+", since $(\sqrt{3112} + \sqrt{376}) + (\sqrt{3112} - \sqrt{376}) = \sqrt{3448}$.]

Let the sixth binomial, finally, be $\sqrt{3448} + \sqrt{3352}$. From it the root will be drawn out, by the third rule, in this manner. The difference of the squares [of the names] is 96, whose fourth part is 24; its root, namely $\sqrt{324}$,

[Margin: The root of the sixth binomial is a line potent in two medials; and the root of the sixth residual is a line which together with a medial makes the whole medial.]

added to the half of the greater name, namely to $\sqrt{3112}$, makes the sum $\sqrt{3112} + \sqrt{324}$. Therefore the first portion of the root sought is $\sqrt{3}(\sqrt{3112} + \sqrt{324})$. And if you subtract that same sum $\sqrt{3112} + \sqrt{324}$ from the whole greater name $\sqrt{3448}$, there will be left $\sqrt{3112} - \sqrt{324}$ — for $\sqrt{3112}$ taken from $\sqrt{3448}$ leaves $\sqrt{3112}$, and if from this remainder $\sqrt{324}$ is taken away, $\sqrt{3112} - \sqrt{324}$ remains. Hence the latter portion of the root sought is $\sqrt{3}(\sqrt{3112} - \sqrt{324})$, and the whole root of the sixth binomial proposed will be $\sqrt{3}(\sqrt{3112} + \sqrt{324}) + \sqrt{3}(\sqrt{3112} - \sqrt{324})$, while the root of the corresponding sixth residual will be $\sqrt{3}(\sqrt{3112} + \sqrt{324}) - \sqrt{3}(\sqrt{3112} - \sqrt{324})$. The former, by the conclusion of *Elements* X.60, is an irrational line called "potent in two medials"; the latter, as *Elements* X.97 demonstrates, is an irrational line which together with a medial makes the whole medial. Thus, by the demonstration of *Elements* X.42 and X.79, the two lines $\sqrt{3}(\sqrt{3112} + \sqrt{324})$ and $\sqrt{3}(\sqrt{3112} - \sqrt{324})$ are incommensurable in square, and they make both the sum of their squares a medial, and what is contained under them a medial, and the sum of their squares incommensurable [with that product]. For that they are incommensurable in square, their squares — which are incommensurable — show. That their squares together make a medial is settled, since together they make $\sqrt{3448}$, the medial surface, whose root is $\sqrt{33448}$. And finally that from multiplying one into the other a medial too is begotten is plainly established, since they produce $\sqrt{388}$, the medial surface, whose root is $\sqrt{3388}$.

Test of the root found, for the sixth binomial proposed.

$$\begin{aligned} &\sqrt{3}(\sqrt{3112} + \sqrt{324}) + \sqrt{3}(\sqrt{3112} - \sqrt{324}) \\ &\sqrt{3}(\sqrt{3112} + \sqrt{324}) + \sqrt{3}(\sqrt{3112} - \sqrt{324}) \end{aligned}$$

$$\begin{array}{r}
\sqrt{3112} + \sqrt{324} + \sqrt{3112} - \sqrt{324} \\
\sqrt{388} \\
\sqrt{388} \\
\sqrt{3352}
\end{array}$$

Total sum: $\sqrt{448} + \sqrt{3352}$. The binomial proposed.

[In the "Total sum" line the first radical is printed as a bare $\sqrt{448}$, without the census order-mark that the parallel term $\sqrt{3352}$ carries — a compositor inconsistency; it denotes the same square root, $\sqrt{448}$. In the cross-term column, the two $\sqrt{388}$ are summed as $\sqrt{3352}$ ($= \sqrt{352}$); the intermediate sum is set as "352" in the type.]

[Margin: The difference of the squares of the names of the root of any binomial and residual, and which one it is.]

Now in any binomial or residual you will observe that the root of the difference of the squares of the two names is [equal to] the difference between the squares of the names of the root of that binomial or residual — a thing that may seem remarkable to someone. This may be tested in the following six binomials.

In the **first** binomial, $23 + \sqrt{3448}$, whose root was found to be $4 + \sqrt{37}$: the square of the number 23, namely 529, exceeds the square 448 by 81, whose root is 9 — this being the difference between the squares of the names of the root, that is, between 16 and 7.

In the **second** binomial, $\sqrt{3448} + 14$, whose root was found to be $\sqrt{33343} + \sqrt{337}$: the square 448 exceeds the square of the number 14 [= 196] by 252, whose root — namely $\sqrt{3252}$ — is the difference between the squares of the names of the root. For those squares are $\sqrt{3343}$ and $\sqrt{37}$, as is plain from multiplying the names $\sqrt{33343}$ and $\sqrt{337}$ each into itself; and if $\sqrt{37}$ is taken from $\sqrt{3343}$, the remainder is $\sqrt{3252}$.

In the **third** binomial, $\sqrt{3448} + \sqrt{3336}$, whose root was found to be $\sqrt{33252} + \sqrt{3328}$: the square 448 exceeds the square 336 by 112, whose root — namely $\sqrt{3112}$ — is the difference between the squares of the names of the root, which are $\sqrt{3252}$ and $\sqrt{328}$. And it is settled that $\sqrt{328}$, taken from $\sqrt{3252}$, leaves $\sqrt{3112}$.

In the **fourth** binomial, $24 + \sqrt{3448}$, whose root was found to be $\sqrt{3}(12 + \sqrt{332}) + \sqrt{3}(12 - \sqrt{332})$: the square 576, of the number 24, exceeds the square 448 by 128, whose root — namely $\sqrt{3128}$ — is the difference between the squares of the names of the root, that is, between $12 + \sqrt{332}$ and $12 - \sqrt{332}$, as appears in this subtraction layout. For 12 from 12 leaves nothing; but $-\sqrt{332}$ taken from $+\sqrt{332}$

(since unlike signs change the character of the operation) leaves $\sqrt{3}128$ — namely the double of $\sqrt{3}32$ — with the sign + of the upper number.

$$\begin{array}{r} 12 + \sqrt{3}32 \\ 12 - \sqrt{3}32 \\ \hline 0 + \sqrt{3}128 \end{array}$$

In the **fifth** binomial, $\sqrt{3}448 + 12$, whose root was found to be $\sqrt{3}(\sqrt{3}112 + \sqrt{3}76) + \sqrt{3}(\sqrt{3}112 - \sqrt{3}76)$: the square 448 exceeds 144, the square of the number 12, by 304, whose root — namely $\sqrt{3}304$ — is the difference between the squares of the names of the root, that is, between $\sqrt{3}112 + \sqrt{3}76$ and $\sqrt{3}112 - \sqrt{3}76$, as appears in this appended subtraction layout. For $\sqrt{3}112$ from $\sqrt{3}112$ leaves nothing; but $-\sqrt{3}76$ taken from $+\sqrt{3}76$ (since unlike signs change the character of the operation) leaves $\sqrt{3}304$ — namely the double of $\sqrt{3}76$ — with the sign + of the upper number.

$$\begin{array}{r} \sqrt{3}112 + \sqrt{3}76 \\ \sqrt{3}112 - \sqrt{3}76 \\ \hline 0 + \sqrt{3}304 \end{array}$$

In the **sixth** binomial, finally, $\sqrt{3}448 + \sqrt{3}352$, whose root was found to be $\sqrt{3}(\sqrt{3}112 + \sqrt{3}24) + \sqrt{3}(\sqrt{3}112 - \sqrt{3}24)$: the square 448 exceeds the square 352 by 96, whose root — namely $\sqrt{3}96$ — is the difference between the squares of the names of the root, that is, between $\sqrt{3}112 + \sqrt{3}24$ and $\sqrt{3}112 - \sqrt{3}24$, as is manifest in the appended subtraction layout. For $\sqrt{3}112$ from $\sqrt{3}112$ [leaves nothing; and $-\sqrt{3}24$ from $+\sqrt{3}24$] (since unlike signs change the character of the operation) leaves $\sqrt{3}96$ — namely the double of $\sqrt{3}24$ — with the sign + of the upper number.

$$\begin{array}{r} \sqrt{3}112 + \sqrt{3}24 \\ \sqrt{3}112 - \sqrt{3}24 \\ \hline 0 + \sqrt{3}96 \end{array}$$

You will find the same in any other binomial or residual, and its root.

[Margin: A neat method for multiplying a binomial or residual into itself.]

From this there follows an easy method for squaring any binomial or residual, which is as follows. Given a binomial or residual, take the sum of the squares of the two parts of the binomial or residual as the first portion of the product number. Then, from the square of the said sum, subtract the square of the difference of the squares of those same parts of the binomial or residual; the root of what remains of this number will be the second portion of the product

number. Thus the binomial $\sqrt{350} + 6$, multiplied into itself, makes $86 + \sqrt{37200}$. For the two squares, 50 and 36, make 86, the first part of the product number. And the square 7396, of the number 86 found, exceeds by 196 — the square of the number 14, namely of the difference between the squares 50 and 36 — the number 7200, whose root, namely $\sqrt{37200}$, gives the second part of the product number: so that the square of the binomial $\sqrt{350} + 6$ is $86 + \sqrt{37200}$ — inasmuch as the square 7396 of the first portion, 86, exceeds 7200 (the square of the second portion) by 196, whose root, 14, is the difference between the squares 50 and 36.

$$\begin{array}{r} \sqrt{350} + 6 \\ \hline 86 + \sqrt{37200} \end{array} \quad (\text{into itself})$$

By the same reasoning, the square of the residual $\sqrt{350} - 6$ will be $86 - \sqrt{37200}$.

So too the binomial $\sqrt{350} + \sqrt{36}$, multiplied into itself, will make $56 + \sqrt{31200}$; because the square 3136 of the first portion, 56, exceeds by 1936 — whose root is 44, the difference of the squares 50 and 6 — the number 1200, the square $\sqrt{31200}$ of the second portion.

$$\begin{array}{r} \sqrt{350} + \sqrt{36} \\ \hline 56 + \sqrt{31200} \end{array}$$

Some authors hand down another easy method of drawing out the root from a given binomial or residual, which it seems good to explain here — provided I first set out the following Lemma.

LEMMA.

[Margin: Lemma.]

To cut a given number into two parts, such that the number produced by multiplying them together equals a given number — this given number being no greater than the square of the half of the number to be divided, or (which is the same) than the fourth part of the square of the whole number to be divided.

For example, let the number 20 be cut into two parts which, multiplied together, produce the number 75 — a number less than 100, the square of the half of the proposed number 20; or less than 100, the fourth part of the square 400 of the given number 20. From 100, the square of the half of the number to be divided, take away the given number to be produced, 75. And of the remaining number, 25, let the square root, 5, be first added to the half of the proposed number, namely to 10, and then subtracted from that same half. For the sum formed, 15,

and the remaining number, 5, will be the parts sought of the proposed number 20, which multiplied together beget the given number 75.

[Diagram: a straight line segment marked A ... C ... D ... B, with C the midpoint and D the point of section.]

Here is the demonstration of this procedure. Let the number 20 be AB, proposed to be divided into the parts AD, DB, which by their multiplication produce the given number...

the number 75. These parts will become known in this way. When the number AB is bisected at C, the product of AB into DB, together with the square on CD, is equal to the square on CB, the half — [Margin: by *Elements* II.5]. Therefore, if the given number 75, made from AD into DB, is taken from the square 100 of the half CB, what remains will be the square on CD, 25, whose root, 5, gives the line CD; and this, added to the half AC, completes the greater part AD, 15. And, taken from the half CB, it makes the remaining smaller part DB, 5.

Again, let the number $\sqrt{380}$ be divided into two parts, such that multiplied together they make 1. From the square of its half, namely from 20, let the number 1 to be produced be subtracted. And let the root of the remaining number 19, namely $\sqrt{319}$, be added to and taken from the half $\sqrt{320}$ of the proposed number: the parts sought will be $\sqrt{320} + \sqrt{319}$ and $\sqrt{320} - \sqrt{319}$, which multiplied together beget 1, as the preceding layout teaches — namely by subtracting the square of 19 from the square of 20, as we taught in chapter 23, before the examples of Division.

$$\begin{array}{r} \sqrt{320} + \sqrt{319} \\ \sqrt{320} - \sqrt{319} \\ \hline 1 \end{array}$$

Likewise let the number $\sqrt{380}$ be divided into two parts which multiplied together make $\sqrt{35}$. The half of the number $\sqrt{380}$ is $\sqrt{320}$; from its square, 20, if $\sqrt{35}$ is subtracted, there will remain $20 - \sqrt{35}$, whose root, namely $\sqrt{3(20 - \sqrt{35})}$, makes the greater part of the given number $\sqrt{380}$, namely $\sqrt{320} + \sqrt{3(20 - \sqrt{35})}$; and taken from that same half $\sqrt{320}$ it leaves the smaller part $\sqrt{320} - \sqrt{3(20 - \sqrt{35})}$. These parts, multiplied together, make $\sqrt{35}$, as appears in this layout.

$$\begin{array}{r} \sqrt{320} + \sqrt{3(20 - \sqrt{35})} \\ \sqrt{320} - \sqrt{3(20 - \sqrt{35})} \\ \hline \sqrt{35} \end{array}$$

For, as is plain from chapter 23, the multiplication is effected if the square of the earlier part is taken away. It will be taken away according to this other layout, since 20 from 20 leaves nothing; and inasmuch as $-\sqrt{35}$ is to be removed from $+\sqrt{30}$, the character of the operation changes, and it becomes an addition, with the sign $+$ of the upper number $\sqrt{30}$.

$$\begin{array}{r} 20 + \sqrt{30} \\ 20 - \sqrt{35} \\ \hline 0 + \sqrt{35} \end{array}$$

Lastly, let 13 [one census] be divided into two parts which multiplied together make 13 — that is (and it comes to the same thing) so that 1℞ [one res] is the mean proportional between those two parts. The half of 13 is $\frac{1}{2}3$, whose square is $\frac{1}{4}33$; from this we take away 13, which is to be produced, and there will remain $\frac{1}{4}33 - 13$, whose root, namely $\sqrt{3(\frac{1}{4}33 - 13)}$, added to the half of 13 — that is, to $\frac{1}{2}3$ — will make one part sought, $\frac{1}{2}3 + \sqrt{3(\frac{1}{4}33 - 13)}$; and taken from that same half it will make the other part, $\frac{1}{2}3 - \sqrt{3(\frac{1}{4}33 - 13)}$. These parts, multiplied together, make 13, as appears in this layout.

$$\begin{array}{r} \frac{1}{2}3 + \sqrt{3(\frac{1}{4}33 - 13)} \\ \frac{1}{2}3 - \sqrt{3(\frac{1}{4}33 - 13)} \\ \hline 13 \end{array}$$

For, as is plain from the foregoing, the multiplication is effected if the square of the later portion, $\frac{1}{4}33 - 13$, is subtracted from $\frac{1}{4}33$, the square of the earlier portion. It will be subtracted according to this other layout, since $\frac{1}{4}33$ from $\frac{1}{4}33$ leaves nothing; and inasmuch as -13 is to be subtracted from $+03$, the character of the operation changes, and it becomes an addition, with the sign $+$ of the upper number 03.

$$\begin{array}{r} \frac{1}{4}33 + 03 \\ \frac{1}{4}33 - 13 \\ \hline 0 + 13 \end{array}$$

This Lemma we shall also settle by Algebra, in the 23rd enigma of chapter 30.

Once this Lemma has been demonstrated and illustrated with examples, we shall draw out the root from any binomial or residual by this easy path. Let the greater name be divided, by the preceding Lemma, into two parts which multiplied together yield a number equal to the fourth part of the square of the lesser name. The roots of these parts, joined by the sign $+$ for a binomial, or by the sign $-$ for a residual, will exhibit the root sought.

[Margin: Another method for extracting the root from a given binomial or residual.]

For let the first binomial be $20 + \sqrt{3300}$. Let the greater name, 20, be cut into two parts which multiplied together make 75 — this number being the fourth part of the square 300 of the lesser name. One part will be 15 and the other 5. Therefore $\sqrt{315} + \sqrt{35}$ is the root of the given binomial, which multiplied into itself (a thing easily done by the method we explained before the preceding Lemma) will produce the given binomial $20 + \sqrt{3300}$.

Again, let the first binomial be $38 + \sqrt{3288}$. The greater name, 38, will be cut into two parts producing, by their multiplication, 72, the fourth part of the square 288 of the lesser name — in this way. The half of the greater name is 19; from its square, 361, if the said fourth part, 72, is subtracted, the remaining number is 289, whose root is 17; added to the aforesaid half 19, and subtracted from it, this makes the parts sought, 36 and 2. Therefore $\sqrt{336}$ — that is, 6 — + $\sqrt{32}$ is the root of the binomial proposed; and this can be tested by multiplying it into itself. For the squares of the names, 36 and 2, make 38, the greater name. And if from its square, 1444, one subtracts 1156 — the square of the difference of the squares 36 and 2, namely the square of the number 34 — there will remain the number 288. Therefore $\sqrt{3288}$ will be the lesser name, so that the binomial is $38 + \sqrt{3288}$.

Moreover, let the second residual be $\sqrt{318} - 4$. Let the greater name $\sqrt{318}$ be cut into two parts, one of which into the other makes 4, the fourth part of the square of the lesser name, 4 — in this way. The half of the greater name is $\sqrt{34\frac{1}{2}}$; from its square, $4\frac{1}{2}$, if the said fourth part, 4, is subtracted, there will remain $\frac{1}{2}$, whose root — that is, $\sqrt{3\frac{1}{2}}$ — added to that half $\sqrt{34\frac{1}{2}}$ makes $\sqrt{38}$ for the first part; and taken from that same half, it leaves $\sqrt{32}$ for the second part. Therefore $\sqrt{338} - \sqrt{332}$ is the root of the residual proposed. As you see it here proved, by multiplying the root into itself as a square: for among the partial products, $\sqrt{38}$ and $\sqrt{32}$ make $\sqrt{318}$; then $-\sqrt{3316}$, that is, -2 , doubled, makes -4 .

$$\begin{array}{r} \sqrt{338} - \sqrt{332} \\ \sqrt{338} - \sqrt{332} \end{array}$$

$$\begin{array}{r} - \sqrt{3316} + \sqrt{32} \\ + \sqrt{38} - \sqrt{3316} \end{array}$$

$$\sqrt{318} - 4$$

Chapter XXVIII (continued) — Extraction of roots from binomials and apotomes

Let a third binomial also be given, $\sqrt{332} + \sqrt{324}$. The greater name, $\sqrt{332}$, will be split into two parts whose product is 6 — the quarter part of 24, the square of the lesser name — in this way. Half of the greater name is $\sqrt{38}$; from its square, 8, subtract the said [product] part, 6, and there remains 2. The root of this, namely $\sqrt{32}$, added to and subtracted from the aforesaid half $\sqrt{38}$, yields the parts $\sqrt{318}$ and $\sqrt{32}$. Therefore the root of the binomial is $\sqrt{3318} + \sqrt{332}$. For this quantity, multiplied into itself squarewise — as is shown here — produces the binomial $\sqrt{332} + \sqrt{324}$:

$$\begin{array}{r}
 \sqrt{3318} + \sqrt{332} \\
 \sqrt{3318} + \sqrt{332} \\
 \hline
 + \sqrt{36} + \sqrt{32} \\
 \sqrt{318} + \sqrt{36} \\
 \hline
 \sqrt{332} + \sqrt{324}
 \end{array}$$

For among the partial products $\sqrt{318}$ and $\sqrt{32}$ make $\sqrt{332}$, and $\sqrt{36}$ doubled makes $\sqrt{324}$.

Again, let the first binomial be $72 + \sqrt{32880}$. The greater name, 72, will be cut into two parts whose product is 720, the quarter part of 2880, the square of the lesser name, by this reasoning. Half of the greater name, 72, is 36; from its square, 1296, subtract the aforesaid quarter part, 720, and there remains 576, whose root is 24. This, added to the named half 36 and subtracted from the same, yields the sought parts 60 and 12. Therefore the root of the binomial is $\sqrt{360} + \sqrt{312}$ — which is here proved by the multiplication of the root into itself squarewise:

$$\begin{array}{r}
 \sqrt{360} + \sqrt{312} \\
 \sqrt{360} + \sqrt{312} \\
 \hline
 + \sqrt{3720} + 12 \\
 60 + \sqrt{3720} \\
 \hline
 72 + \sqrt{32880}
 \end{array}$$

Let the root also be extracted from this sixth residue [apotome], $\sqrt{360} - \sqrt{312}$. The greater name, $\sqrt{360}$, will be distributed into two parts whose product is 3, the quarter part of 12, the square of the lesser name, in this fashion. Half of the greater name $\sqrt{360}$ is $\sqrt{315}$; from its square, 15, subtract the named quarter part, 3, and there remains 12, whose root is $\sqrt{312}$. This, added to the aforesaid half

$\sqrt{315}$ and taken away from the same, yields the parts $\sqrt{315} + \sqrt{312}$ and $\sqrt{315} - \sqrt{312}$. Therefore the root of the said sixth residue is $\sqrt{3}(\sqrt{315} + \sqrt{312}) - \sqrt{3}(\sqrt{315} - \sqrt{312})$, which is here proved:

$$\begin{array}{r} \sqrt{3}(\sqrt{315} + \sqrt{312}) - \sqrt{3}(\sqrt{315} - \sqrt{312}) \\ \sqrt{3}(\sqrt{315} + \sqrt{312}) - \sqrt{3}(\sqrt{315} - \sqrt{312}) \end{array}$$

$$\begin{array}{r} \text{Squares of the parts. } \sqrt{315} + \sqrt{312} \text{ \& } \sqrt{315} - \sqrt{312} \\ - \sqrt{33} \\ - \sqrt{33} \end{array}$$

$$\text{Sum. } \sqrt{360} - \sqrt{312}$$

For the squares of the parts make $\sqrt{360}$, namely double $\sqrt{315}$. And [multiplying] the one part $\sqrt{3}(\sqrt{315} + \sqrt{312})$ into the other, $-\sqrt{3}(\sqrt{315} - \sqrt{312})$, gives $-\sqrt{33}$; for since the square 12 subtracted from the square 15 leaves 3, the sign $\sqrt{3}$ must be prefixed to it, together with the sign — [minus], because of the Residue [apotome]. And the double of $-\sqrt{33}$ makes $-\sqrt{312}$.

Scholium

[Margin: In what order is to be kept in the riddles to be solved by Algebra.]

At last we have emerged from the manifold difficulty of cossic and irrational numbers; and — unless my judgment deceives me — under a fortunate spread of sail, however awkward, we have crossed the vast sea of their treatment. Having at length slipped through so many storms and whirlwinds of difficulties, across the seas of so many matters, we hold our harbor; and — for this must be credited to divine aid — we hold it whole and unharmed. It is a pleasure now, as though at leisure and rest, to recall the things that a little while ago it was irksome to bear. For other stretches of waves still remain for us to sail; yet the pleasant recollection of past toil — if any there was — will lighten them, as I very much hope. It is a pleasure, I say, to linger a while over the algorithm — most delightful and most abundant in fruit — of those numbers hitherto explained. For how large a share of past labor would be lost, unless we spent some effort and time in gathering the very fruit of the algorithm?

First, then, certain riddles — namely in chapter 29, which follows next, and in chapter 30 — are to be solved; or, if you prefer, certain problems of the numbers they call *abstract*: numbers, that is, not tied to material things, such as commerce and merchants' dealings, or to questions about any other matters. These, whether one prefers to call them riddles or problems, Diophantus of Alexandria [Diophantus] (an outstanding writer above all, who deserved best of the whole

science of Algebra) and other most weighty writers have already treated before us. For they belong to speculative arithmetic; and how far that science excels the rest in dignity, the concurring voice of all bears no obscure witness. For of the theorems and of the problems of abstract numbers demonstrated in it, the variety is so manifold and the multitude so copious, that the treatment of such riddles adds the greatest dignity, and indeed in a way perfects that science which concerns numbers. How many are the problems it supplies of which we would have no knowledge at all, nor could they ever be drawn out of darkness, unless the discipline of Algebra came in like a light! "But," [some will say,] "they are idle, and problems of this kind have no usefulness whatever." Yet to have regard now and then not only to usefulness but also to pleasantness is, we hold, the mark of a wise man, and not foreign to honorable conduct. For if this reasoning stands, tell me: why do we admit the ten books of Jordanus [Jordanus de Nemore] on numbers? Why do we approve the very many theorems and the problems of numbers demonstrated by Euclid [Euclid] in books 7, 8, and 9 (*Elements* VII–IX); persist in reading them; delight in them; and pursue them with the same utmost admiration? What so great a use is there in those, that a usefulness so great and so esteemed should not also reside in ours? How much would that objection, if admitted, detract — not only from the splendor and dignity of these studies, but from the honor of all the sciences? For assuredly this belongs to all those disciplines which, because they never issue into practical use and are wholly occupied in the contemplation of truth, are called speculative: that they adorn the mind with their own honor; that they augment and perfect the chief faculty of the soul, namely the power of understanding; that they delight by their very selves; and, so to say, flood the whole man with a certain wondrous pleasure of their own. But indeed — and this too must not be concealed, as will become clearer in its proper place — the greatest fruits and uses of these problems exist not in geometry only, but in very many other faculties and arts as well,

so very many — although we do not think so great an account of usefulness is to be made that, if this be lacking, there is at once no dignity or worth in any faculty. I wished, dear Reader, to forewarn you of this in as few words as I could, so that, with every cloud of suspicion cleared away, you may bring a mind at ease to the riddles that will be set forth in the two following chapters; and, all fear shaken off and all gloom cast aside, may show yourself readier and more cheerful in working these through with us. After the riddles of the two chapters, we shall subjoin various examples in chapter 31 — in those numbers, however, that are tied to concrete matter, where you will also receive some questions bearing on the *second roots* [square-square unknowns]. Finally, in chapter 32, we shall bring forward manifold questions concerning geometrical matters. And whoever

grasps for what purpose, in the end, this multitude of riddles is produced will easily bear it.

For we labor not only that the questions we ourselves set may be resolved, but we have judged that this too must be seen to: that the method of resolving others like them, by anyone and at any time, be understood — a method that anyone will gather from the solution of these riddles with no trouble at all. For this reason we have not neglected to give all the riddles, all the questions, worked out in full, so that the studious Reader might readily recognize in unraveling them the peculiar craft of Algebra. For we have set the several operations, as it were, before the very eyes, so that there need be no hesitating over any of them, nor any repeating of the earlier precepts at every step. This is the reason why most people suppose Algebra so dark and shadowy that they maintain the power of understanding it belongs to few, or even to almost no one. For I have noticed that in most of its writers this very thing is wanting which I have made my chief care: that everything be set forth as aptly as possible. Nor — as the Lyric Poet [Horace, *Ars Poetica* 25–26] once warned — do I strive to be brief, at the price of being obscure; and willingly and gladly I let myself incur the charge of being rather wordy, provided that through me everything be made most clear to the rest. Yet, so that a greater pleasure may be drawn from the explanation of the riddles that follow, it will help to examine them, as if for the sake of exercise, against the standard of the questions, just as though no explanation of them had yet been supplied — sometimes, too, substituting for them other numbers and examples than those set down there. And just as we have written down very many riddles that absolutely cannot be explained except by Algebra, so also we have deliberately inserted not a few that we do not deny can be solved even without Algebra. Yet it pleased us to place these beside the rest — not that we would have them taken away from the other arts, but that we might recognize the readier, quicker, and pleasanter method of Algebra for investigating them. For whereas in other methods it happens that almost as many rules and as many precepts are handed down as there are riddles to be learned, and to each single riddle none but its own single rules are fitted, and one rule answers to the dispatching of one riddle only — this Algebra, once called in, will accomplish far more easily. For by one and the same rule, way, and method it will nearly always render them solved, and will rid us of that trouble which the manifold variety of rules cannot but bring: so that by this one merit alone, even if the others (which are neither few nor slight) were lacking, Algebra would surpass all the other rules of Arithmetic. For scarcely is a question set when the very way of explaining it in Algebra presents itself to view; and the question almost of itself points out the path by which we may at last attain it, as though we had long pursued it. And this,

if you set aside the rule of false position [regula falsi], belongs to none of the others at all. Algebra therefore has this singular thing above the rest: that it alone solves by its own rules many questions about which the others either do not labor, or, if they do labor, accomplish nothing; and that those which are sometimes seized by others only through long windings and, as it were, in the dark, Algebra brings to completion as if by short-cuts and in clear light. For the rest, I shall not hide this from you, studious Reader: that very many riddles are brought forward by us for which, if other and other numbers be substituted, no solution can be had. Yet you will not go far before you perceive which can be solved and which cannot. This the very operation itself will teach you, which must be applied to opening up the question. For wherever the equation falls between two numbers altogether unequal to each other, no solution of that question will be possible. This we have said to be the reason why the rule of Algebra so far excels all the other precepts of Arithmetic. For it very often happens that some question of this kind, having knots utterly inextricable, is thrown at one unskilled in Algebra, who by the utmost labor and the utmost straining of both mind and body strives — to no purpose — to solve it, and, with his effort repeated again and again, unwittingly suffers his time and all his labor to go to waste. From this the student of Algebra is surely far removed, for the operation itself will show him which questions can be solved and which cannot. How great a thing this is, then, in which the student of Algebra can rightly glory: that with resolute mind and steadfast judgment he may declare, without hesitating and in no doubtful voice, whether or not a proposed question can be solved!

VARIOUS RIDDLES OF ABSTRACT NUMBERS, to be unraveled by Algebra, in which an equation occurs between only two numbers — which indeed is called a *simple equation*.

CAPVT XXIX [Chapter 29]

In this chapter we shall bring forward riddles in which simple equations occur; in the next, however, we shall propose riddles in which composite equations are found. What a simple equation is, and what a composite one, we explained at the end of chapter 13. Let us therefore begin here.

1. To divide any given number into two parts exceeding each other by any given excess.

Let 100 be the number to be divided into two parts whose excess, or difference, is 40. Let the lesser part be put as $1R$, and accordingly the greater as $100 - 1R$.

Subtracting $1R$ from $100 - 1R$, there remain $100 - 2R$, as you see here:

$$\begin{array}{r} 100 - 1R \\ + 1R \\ \hline 100 - 2R \end{array}$$

[The subtraction of the lesser part $1R$ from the greater part $100 - 1R$ leaves $100 - 2R$.] And an equation will have been found between $100 - 2R$ and 40; for, the lesser part having been taken from the greater, 40 ought to remain. Adding $2R$ to both sides, there will be an equation between 100 and $40 + 2R$. And with 40 taken from both sides, the equation will be between 60 and $2R$. If, therefore, according to the prescribed rule of Algebra, 60 be divided by 2, $1R$ will be found to be 30 — and this is the lesser part. Therefore the greater will be 70, namely $100 - 30$, which greater part is also obtained if to the

lesser part, 30, the proposed difference 40 be added. You see, then, that the greater part 70 exceeds the lesser part 30 by the number 40, as was proposed.

But if we set the greater part to be $1R$ and the lesser $100 - 1R$, we shall subtract $100 - 1R$ from $0 + 1R$, as you see here, and there will remain $-100 + 2R$ — that is, $2R - 100$:

$$\begin{array}{r} 0 + 1R \\ 100 - 1R \\ \hline -100 + 2R \end{array}$$

For subtracting $+0$ from $+100$ leaves -100 , because the numbers were set in reverse order, and so from $+$ comes $-$, and so on. This remaining quantity, then, $2R - 100$, ought to equal the given excess 40. Adding 100 therefore to both sides, there will be an equation between $2R$ and 140. And with the 140 divided (as the rule of Algebra prescribes) by 2, $1R$ will come to 70, which will be the greater part; and therefore the lesser will be 30, namely $100 - 70$.

The same riddle will be solved by this other method as well. Let the lesser part be put as $1R$, and accordingly the greater as $1R + 40$, since it exceeds the lesser by the number 40. Both parts together make $2R + 40$, which number ought to equal the given number 100. Taking away 40 from both sides, there will be an equation

between 2R and 60. Dividing 60, then, by 2, 1R will be 30 — namely the lesser part; and therefore the greater is 70, as before.

But if the greater part be put as 1R, and accordingly the lesser as 1R — 40 (so that the former exceeds the latter by the number 40), both parts together will make 2R — 40, which sum equals the given number 100. Adding 40 to both sides, there will be an equality between 2R and 140. And dividing 140 by 2, 1R will come to 70, for the greater part; and therefore the lesser will be 30, as above.

Scholium

[Margin: Division of a cossic number into two parts of a given excess.]

It sometimes happens in practice that some cossic number — say 1R, or 10R, or 13, or 63, and so on — is to be divided into two parts that exceed each other by a given number, as will be clear in riddle 61. This matter, without Algebra (for by Algebra it can scarcely be done), is accomplished in this way. From the given cossic number let the given number be subtracted, by the sign —. For half of this diminished number will be the lesser part; and if to it there be added the same given number, by the rule of addition handed down in chapter 4, the greater part will be obtained.

For let 1R be divided into two parts, of which the greater exceeds the lesser by 20 units. Take 20 from 1R; and to the half of the residue 1R — 20 — that is, to $\frac{1}{2}R - 10$ — add 20, and you will make $\frac{1}{2}R + 10$, as you see in the formula placed alongside:

$$\begin{array}{r} \frac{1}{2}R - 10 \\ + 20 \\ \hline \frac{1}{2}R + 10 \end{array}$$

The parts are $\frac{1}{2}R - 10$ and $\frac{1}{2}R + 10$, of which the latter exceeds the former by the number 20, and both together make up 1R.

Again, let the number 10R be divided into two parts whose difference is 20. Take 20 from 10R. For the half of the residue 10R — 20, namely 5R — 10, will be the lesser part; and if to it you add 20, you will make 5R + 10, the greater part, of which the latter exceeds the former by 20 units, and both together make 10R.

Let 13 also be divided into two parts whose difference is 150. Take 150; and of the residue 13 — 150 the half, $\frac{1}{2}3 - 75$, will be the lesser part; to which if you add 150, you will make the greater part $\frac{1}{2}3 + 75$. These together make 13, and the greater exceeds the lesser by the number 150.

Finally, let the number 63 be divided into two parts of which the greater exceeds the lesser by 8 units. Take 8 from 63. For the half of the residue 63 — 8, namely 33 — 4, will be the first part; to which if you add 8, the greater part will come to be 33 + 4. And so with the rest.

[Margin: Division of an absolute number into two parts of a given excess, without Algebra.]

Indeed, by this same art we shall divide a proposed *absolute* number into two parts of a given excess, without Algebra. For if we subtract the given excess from the proposed number, the half of the remaining number will be the lesser part; and if to it we add the same excess, we shall make up the greater part. Thus if the number 100 is to be distributed into two, whose difference is 40: take 40 from 100. For the half of the remaining number 60, namely 30, will be the lesser part; to which if you add 40, you will build up the greater part, 70.

But if a cossic number is to be so divided by Algebra, the parts of the cossic number will not indeed be found, but the value of the *res* [radix], or of the *census* [zensus], will be discovered — and its parts will correspond to the parts posited by Algebra, and will satisfy the question. Thus if the number 23 is to be cut into two parts, of which the greater exceeds the lesser by the number 40: let the lesser part be put as $\frac{1}{2}3$ (a number, that is, smaller than half of the given number 23), and the greater part will be $\frac{1}{2}3 + 40$, exceeding the former by the number 40. Together these make $13 + 40$, which number ought to equal the given number 23. Taking 13 from both sides, there will be an equation between 40 and 13. And dividing 40 by 1, 13 will come to 40, and the proposed 23 to 80 — the number which is set to be divided. Since, therefore, the lesser part was posited as $\frac{1}{2}3$, the lesser part will be 20 — namely half of one *census*, whose value we found to be 40, and so [the whole census] to make 80; and the greater part exceeds the lesser by the given number 40, [so that the greater part] we find to be 60, namely $\frac{1}{2}3 + 40$. Here you see 20 and 60, [differing between] themselves by the given number 40. Likewise also, if 2R are to be distributed into two parts whose difference is 40, it will be found in the same way that 1R is 40, and therefore 2R is 80. And $\frac{1}{2}R$ will be 20, and $\frac{1}{2}R + 40$ will be 60. And so on.

2. To find two numbers whose excess is a given number of any kind, and whose sum makes up any given number.

Let the given excess be 40, and the sum 100. This riddle does not differ from the one above. For the sum 100 is to be distributed into two parts whose excess is 40.

3. To divide any given number into two parts which, multiplied into each other, produce a number that has a given ratio of any kind to the square of the lesser part.

Let 100 be the given number, to be divided into two parts such that the number generated from the one into the other has a tenfold ratio to the square of the lesser part. Let the lesser part be put as $1R$, and accordingly the greater as $100 - 1R$. From the one into the other is produced the number $100R - 13$, which ought to be tenfold with respect to the square of the lesser part, namely to

(Cossist notation used throughout: R = the unknown [res], 3 = its square [census, i.e. R^2]; a numeral prefixed to a sign is its coefficient, and terms are joined by + and -. Values are transcribed exactly as printed.)

Chapter 29 (continued)

Problem 3 (conclusion)

(the product $100R - 13$ [census, the square of the smaller part] has the decuple proportion) to 13 — that is, it is equal to 103 . There is therefore an equation between $100R - 13$ and 103 . And, adding 13 to each side, between $100R$ and 113 . So, dividing 100 by 11 according to the tenor of the rule, there will come out $1R = 9 \frac{1}{11}$ for the smaller part (since the cossic numbers are collateral). When this is subtracted from 100 , the larger part will be $90 \frac{10}{11}$. These parts — namely $100/11$ and $1000/11$ — multiplied by each other make $100000/121$, that is, $826 \frac{54}{121}$, which number is decuple to the square of the smaller part, namely to $10000/121$, that is, to $82 \frac{78}{121}$, as is plain.

The multiplication of the two parts is displayed thus:

$$\begin{array}{r} 100 - 1R \\ + 1R \\ \hline 100R - 13 \end{array}$$

Scholium

It should be understood that the two parts of the given number, once found, have between themselves the same proportion that the number produced from their multiplication has to the square of the smaller part. Thus you see that the parts found, $9 \frac{1}{11}$ and $90 \frac{10}{11}$ — or $100/11$ and $1000/11$ — have the decuple proportion, just as the product number $826 \frac{54}{121}$ has to $82 \frac{78}{121}$, the square of the smaller part, or as $100000/121$ has to $10000/121$. The reason for this is that the

same smaller part multiplies both itself, to make its square, and the larger part, to generate the product number. For hence it comes about that the products have the same proportion as the numbers multiplied. [Margin: a — Euclid, *Elements* VII.17]

And so this puzzle could be posed in the following way:

To divide a given number into two parts in a given ratio, such that, multiplied together, they generate a number which has to the square of the smaller number that same given ratio.

For let the given number be 100, to be divided into two [parts] having the quadruple proportion, such that the number produced from their multiplication also has the quadruple proportion to the square of the smaller. Let the smaller number be set as $1R$ and the larger as $4R$, so that they may indeed have the quadruple proportion. And since these two numbers make $5R$, there will be an equation between $5R$ and 100. Dividing 100 by 5, then, $1R$ becomes 20, that is, the smaller number. The larger, therefore, will be 80, since it was set as $4R$. You see, then, that the number 1600, produced from 20 times 80, is quadruple the square 400 described from the smaller number 20; and likewise that the larger, 80, also has the quadruple proportion to the smaller, 20.

Problem 4

[Margin: IIII.]

4. To find two numbers which, multiplied together, produce a number that has to the square of the smaller number a given proportion, and which, added together, make up any proposed sum whatever.

Let the proposed sum be 100, and let the number produced from multiplying one found number into the other be decuple to the square of the smaller number. This puzzle too does not differ from the previous one. For the sum 100 must be divided into two parts which, multiplied together, produce a number in the decuple proportion to the square of the smaller number.

Problem 5

[Margin: V.]

5. To find two numbers in a given ratio such that, if the larger is subtracted from a given number and the smaller from another, smaller, given number,

the two remainders become two equal numbers — whenever this can be done.

Let two numbers be sought in the triple ratio such that, when the larger is taken from 30 and the smaller from 16, the remainders become two equal numbers. But the two given numbers must have a proportion smaller than the given proportion. As in our example, the proportion 30 to 16 is smaller than triple.

Let the smaller be set as $1R$, and therefore the larger as $3R$, three times as great. If $3R$ is taken from 30, there remain $30 - 3R$. And if $1R$ is taken from 16, there remain $16 - 1R$. An equation is therefore found between $30 - 3R$ and $16 - 1R$. Let $3R$ be added to each side, and there will be an equation between 30 and $16 + 2R$ — because $+3R$ added to $-1R$ make $+2R$:

$$\begin{array}{r} + 3R \\ - 1R \\ \hline + 2R \end{array}$$

And with 16 removed from each side, there will be an equality between 14 and $2R$. Dividing 14 by 2, there comes out $1R = 7$ for the smaller number. The larger, then, will be 21, three times as great. It is evident that if 21 is subtracted from 30, and 7 from 16, there remains 9 in each case.

Problem 6

[Margin: VI.]

6. To find two numbers in a given ratio such that, when the larger is added to a given number and the smaller to another, smaller, given number — or to the same one — two numbers are produced in any given ratio, whenever this can be done.

Let two numbers be sought in the quintuple ratio such that, when the larger is added to 6 and the smaller to 4, two numbers are produced in the triple ratio. Let the smaller number be set as $1R$ and the larger as $5R$, five times as great. From $5R$ and 6 come $5R + 6$, and from $1R$ and 4 come $1R + 4$. And $5R + 6$ will be in the triple ratio to $1R + 4$; that is, if $1R + 4$ is tripled, an equation will arise between $5R + 6$ and $3R + 12$. And with $3R$ removed from each side, between $2R + 6$ and 12. And again, with 6 removed from each side, between $2R$ and 6. Dividing 6 by 2, then, there comes out $1R = 3$ for the smaller number, and 15 for the larger, five times it. Now if 15 is added to 6, and 3 to 4, there come 21 and 7, in the triple ratio.

And if the composite number $5R + 6$ must have to $1R + 4$ the quadruple proportion, there will be an equality between

$5R + 6$ and $4R + 16$. And with $4R$ removed from each side, between $1R + 6$ and 16 . And again, with 6 removed from each side, between $1R$ and 10 . Dividing 10 by 1 , $1R$ becomes 10 , that is, the smaller number, and therefore the larger is 50 , being quintuple it. And if 50 is added to 6 , and 10 to 4 , there come the numbers 56 and 14 , in the quadruple proportion.

And if the composite number $5R + 6$ must have to $1R + 4$ the double proportion, there will be an equation between $5R + 6$ and $2R + 8$. And with $2R$ removed from each side, between $3R + 6$ and 8 . And with 6 removed from each side, between $3R$ and 2 . Dividing 2 by 3 , $1R$ becomes $2/3$, that is, the smaller number, and the larger $10/3$, being quintuple it. And if $10/3$ is added to 6 — that is, to $18/3$ — and $2/3$ to 4 — that is, to $12/3$ — there come two numbers, $28/3$ and $14/3$, having the double proportion.

But suppose that both the larger, $5R$, and the smaller, $1R$, are added to 6 , and are required to make two numbers in the triple proportion. An equation will be found between $5R + 6$ and $3R + 18$; and with $3R$ removed from each side, between $2R + 6$ and 18 ; and again, with 6 removed from both, between $2R$ and 12 . Dividing 12 by 2 , then, $1R$ becomes 6 , the smaller number; and the larger will be 30 . If, therefore, each is added to 6 , there come the numbers 12 and 36 , having the triple proportion.

But if one should wish the composite number $5R + 6$ to have to $1R + 4$ the quintuple proportion as well, or a greater one, or the sesquialterate $[3:2]$, the puzzle turns out inexplicable, or impossible, because we fall into an impossible equation. For instance, if the proportion between $5R + 6$ and $1R + 4$ must be quintuple, or sextuple, there will be an equation between $5R + 6$ and $5R + 20$, or between $5R + 6$ and $6R + 24$ — each of which is impossible. And if the proportion between $5R + 6$ and $1R + 4$ must be sesquialterate, there will be an equation between $5R + 6$ and $1\frac{1}{2}R + 6$, which is likewise impossible. Here you clearly perceive that, from the very rule of Algebra, it can be recognized whether a proposed puzzle can be solved or not — as we taught at length in Chapter 14.

Problem 7

[Margin: VII.]

7. To find a number which, added to any two given numbers, makes two numbers in any proportion that is smaller than the proportion of the two proposed numbers.

Let a number be sought which, added to 100 and to 20, makes two numbers in the triple proportion. Let the sought number be set as $1R$. There will then arise the numbers $1R + 100$ and $1R + 20$, which must have the triple proportion. Therefore there will be an equation between $1R + 100$ and $3R + 60$. Removing $1R$ from each side, there will remain an equation between 100 and $2R + 60$. And again, taking away 60 from each side, between 40 and $2R$. Dividing 40 by 2, $1R$ becomes 20, the sought number. The composite numbers, then, will be 120 and 40, in the triple proportion.

But the named proportion must be smaller than the proportion between the two given numbers — in our example, smaller than quintuple [100:20]. Otherwise we shall meet an impossible equation. The reason for this is that, when one and the same number is added to two unequal numbers, the resulting sums have a smaller proportion than those two unequal numbers.

Diagram. Two horizontal dotted number-lines placed one above the other, illustrating the demonstration. On the upper line the points A, B, E follow in order (segment AB the larger given number, segment BE the added number, so that AE is the whole sum). On the lower line the points C, D, F follow in order (segment CD the smaller given number, segment DF the added number, so that CF is the whole sum); BE and DF are the equal added numbers.

- A (upper line, left): start of the larger number
- B (upper line, middle): end of the larger given number AB
- E (upper line, right): end of the added segment BE; AE is the whole
- C (lower line, left): start of the smaller number
- D (lower line, middle): end of the smaller given number CD
- F (lower line, right): end of the added segment DF; CF is the whole

For let there be two unequal numbers, AB the larger and CD the smaller, to which is added the same number — or two equal ones, BE and DF. I say that the proportion of AE to CF is smaller than that of AB to CD. For since (as we showed at Definition 20 of Book 7 of Euclid) the proportion of the larger number AE to BE is greater than that of the smaller CF to the same BE — or to DF — [Margin: a — scholium on Euclid, *Elements* VII.22] it will follow, by conversion of ratio, that the proportion of AE to AB is smaller than that of CF to CD; and, by alternation, that AE to CF is smaller than AB to CD. Which was what was proposed.

With the same things assumed, let the given proportion be superbipartient-fifths [7:5]. The composite numbers are $1R + 100$ and $1R + 20$, which must have the superbipartient-fifths proportion. So if $1R + 20$ is multiplied by $1 \frac{2}{5}$ — that is, by $\frac{7}{5}$ — there results the number $(7R + 140)/5$, which will be equal to the number $1R + 100$. This equation, by cross-multiplication, as you see here, is reduced to an equation between $7R + 140$ and $5R + 500$:

$$\begin{array}{r} 7R + 140 \quad 5R + 500 \\ \hline \quad \times \quad \hline 7R + 140 \quad 1R + 100 \\ 5 \quad 1 \end{array}$$

And with 140 removed from each side, there will be an equation between $7R$ and $5R + 360$. And again, with $5R$ removed from each side, between $2R$ and 360. Dividing 360 by 2, then, $1R$ becomes 180, the sought number. For this, together with 100 and 20, will make the numbers 280 and 200, which have the superbipartient-fifths proportion. I have added this second example so that you may see how equations involving fractions are to be reduced — which I taught at the end of Chapter 10.

Problem 8

[Margin: VIII.]

8. To find a number which, subtracted from any two given numbers, leaves two numbers in any proportion that is greater than the proportion of the two proposed numbers.

Let a number be sought which, subtracted from 100 and from 50, leaves two numbers in the septuple proportion. Let the sought number be set as $1R$. The two remaining numbers will then be $100 - 1R$ and $50 - 1R$, and the proportion of the former to the latter will be septuple. Hence there will be an equation between $100 - 1R$ and $350 - 7R$. Adding $1R$ to each side, there will be an equation between 100 and $350 - 6R$ — because $+1R$ and $-7R$ make $-6R$, as you see here:

$$\begin{array}{r} + 1R \\ - 7R \\ \hline - 6R \end{array}$$

And again, adding $6R$ to each side, there will be an equality between $100 + 6R$ and 350. And removing 100 from each side, between $6R$ and 250. Dividing 250 by 6, then, $1R$ will be $41 \frac{2}{3}$, the sought number. The remaining numbers, therefore, will be $58 \frac{1}{3}$ and $8 \frac{1}{3}$, which have the septuple proportion.

But the given proportion must be greater than the proportion which the two given numbers have — in the proposed example, greater than double [100:50]. Otherwise we shall fall into an impossible equation. The reason for this is that, when one and the same number is subtracted from two unequal numbers, the remaining numbers have a greater proportion than those two unequal numbers.

Diagram. Two horizontal dotted number-lines placed one above the other, illustrating the demonstration. On the upper line the points A, E, B follow in order (AE the remainder, EB the number subtracted, AB the whole larger number). On the lower line the points C, F, D follow in order (CF the remainder, FD the number subtracted, CD the whole smaller number); EB and FD are the equal subtracted numbers.

- A (upper line, left): start of the larger number
- E (upper line, middle): boundary; AE is the remainder
- B (upper line, right): end of the larger whole AB; EB is the subtracted part
- C (lower line, left): start of the smaller number
- F (lower line, middle): boundary; CF is the remainder
- D (lower line, right): end of the smaller whole CD; FD is the subtracted part

For let there be two unequal numbers, AB the larger and CD the smaller, from which the same number is taken away — or two equal ones, EB and FD. I say that the proportion of AE to CF is greater than that of AB to CD. For since (as we deduced from Definition 20 of Book 7 of Euclid) the proportion of the larger number AB to EB is greater than that of the smaller CD to the same EB — or to FD — [Margin: a — scholium on Euclid, *Elements* VII.22] it will also follow, by alternation, that the proportion of AB to CD is greater than that of EB to FD: namely, of the whole to the whole [greater] than of the part taken away to the part taken away. The proportion of the remainder AE to the remainder CF will also be greater than that of the whole AB to the whole CD. Which was what was proposed.

Problem 9

[Margin: IX.]

9. To distribute a proposed number into two numbers having a given proportion.

Let the number 100 be divided into two [parts] in the decuple proportion. Let the smaller part be set as 1R. The larger will be 10R, ten times as great. Therefore the sum of these parts, 11R, will equal 100. Dividing 100 by 11, then, 1R becomes 9

$1/11$, the smaller number; and when this is taken from 100, there remains the larger number $90 \frac{10}{11}$, which is decuple to it.

Alternatively. With $1R$ set for the smaller part, the larger part will be $100 - 1R$, which must be decuple to it. Therefore $10R$ will equal $100 - 1R$. Adding $1R$ to each side, there will be an equation between $11R$ and 100, as above. But if you make the larger part $1R$, the smaller will be $100 - 1R$; and this decupled, there will be an equality between $1R$ and $1000 - 10R$. Adding $10R$ to each side, there will be an equation between $11R$ and 1000. Dividing 1000 by 11, then, $1R$ becomes $90 \frac{10}{11}$ for the larger part; and when this is taken from 100, the smaller part will be $9 \frac{1}{11}$, as before.

Problem 10

[Margin: X.]

10. To cut a proposed number into two parts, such that the larger has to the smaller a given proportion and, moreover, contains any given number.

Let the number 100 be cut into two parts, of which the larger is triple the smaller and, moreover, contains 20 units. Let $1R$ be set for the smaller part. The larger, then, will be $3R + 20$, and the sum $4R + 20$ must equal 100. Removing 20 from each side, there will be an equality between $4R$ and 80. Dividing 80 by 4, $1R$ becomes 20 for the smaller part; and when this is taken from 100, the remaining larger part will be 80 — which contains the triple of the number 20, namely 60, and moreover 20.

Alternatively. With $1R$ set for the smaller part, the larger will be $100 - 1R$. If 20 is subtracted from it, the remaining number, $80 - 1R$, must be triple the smaller, $1R$; that is, there must be an equation between $3R$ and $80 - 1R$. Adding $1R$ to each side, there will be an equality between $4R$ and 80. Therefore, as above, $1R$ will be 20, the smaller part, and the larger 80.

Problem 11

[Margin: XI.]

11. To find two numbers whose difference is given and whose proportion is given.

Let the given difference be 20 and the given proportion quintuple. Let the smaller number be set as $1R$. The larger will be $5R$, five times it. Their difference is $4R$. There is therefore an equation between $4R$ and 20. Dividing 20 by 4, $1R$ becomes

5 for the smaller number. The larger will be 25, which is quintuple of 5 itself and exceeds it by the number 20.

Again, let two numbers be sought in the supertripartient-sevenths proportion [10:7] whose difference is 21. Let the two numbers be set as $7R$ and $10R$, in the supertripartient-sevenths proportion. Their difference, $3R$, must equal the number 21. Dividing 21 by 3, then, $1R$ becomes 7. Therefore $7R$ will be 49, and $10R$ will make 70. Now the difference between 70 and 49 is the given number 21, and they have the supertripartient-sevenths proportion.

Problem 12

[Margin: XII.]

12. To divide a given number into two numbers, such that a given aliquot part of the one, together with another given aliquot part of the other, makes any given number — one that lies between the named parts of the given number to be divided.

Let the number 100 be divided into two [parts], such that $\frac{1}{3}$ of the first together with $\frac{1}{5}$ of the second makes 30 — a number which lies between $\frac{1}{3}$ and $\frac{1}{5}$ of the given number 100, namely between $33\frac{1}{3}$ and 20; that is, it is smaller than $33\frac{1}{3}$ and larger than 20. Let the first number be set as $1R$, and therefore the second as $100 - 1R$. A third of the first is $\frac{1}{3}R$, and a fifth of the second is $20 - \frac{1}{5}R$; these parts together make up $20 + \frac{2}{15}R$ — for, to add $\frac{1}{3}R$ to $-\frac{1}{5}R$, a subtraction is performed, on account of the different signs. And there will be an equation between $20 + \frac{2}{15}R$ and 30. Removing 20 from each side, between $\frac{2}{15}R$ and 10. Dividing 10 by $\frac{2}{15}$, there comes out $1R = 75$ for the first number; and therefore the second will be 25, namely $100 - 1R$. It is evident that a third part of the first, namely 25, together with a fifth part of the second — that is, with 5 — makes 30.

Alternatively, so as to avoid fractions. Let a third part of the first be set as $1R$, and therefore the whole first as $3R$. Then $\frac{1}{5}$ of the second will be $30 - 1R$, so that, together with $\frac{1}{3}$ of the first — that is, with $1R$ — it may make 30; and therefore the whole second will be $150 - 5R$, which, together with the first — that is, with $3R$ — must make 100. But from these comes $150 - 2R$. Therefore there will be an equation between $150 - 2R$ and 100. Adding $2R$ to each side, between 150 and $100 + 2R$. Removing 100 from each side, between 50 and $2R$. Dividing 50 by 2, $1R$ becomes 25. And since the first was set as $3R$, the first will be 75, and therefore the second 25, as before.

But if one should demand the number $33\frac{1}{3}$, or a larger one, or 20, or a smaller one, as that which the named parts must produce, an impossible equation would arise, as you will be able to test.

13. [Margin: XIII.]

To divide a given number into two parts, so that a given aliquot part of the first exceeds another, given aliquot part of the second by any number you please — this excess being smaller than that aliquot part of the whole proposed number which bears the same name as the aliquot part of the first number.

Let the number to be divided be 100, split into two so that a fifth part of the first exceeds a quarter part of the second by 11 — a number smaller than a fifth of the number to be divided, 100. Let the first number be posited as $1R$ [res, the unknown], and therefore the second as $100 - 1R$. The fifth part of the first, $(1/5)R$, must exceed the quarter part of the second — namely $25 - (1/4)R$ — by 11. That is, the equation must stand between $(1/5)R$ and $25 - (1/4)R + 11$. Adding $(1/4)R$ to both sides, there will be an equality between $(9/20)R$ and $25 + 11$, that is 36. Dividing 36 therefore by $9/20$, one finds $1R = 80$ for the first number. So the second will be 20. And a fifth of the first, namely 16, exceeds the quarter part of the second — that is, 5 — by 11.

Otherwise, without fractions. Let a fifth of the first number be posited as $1R$, so that the whole first is $5R$. A quarter of the second will then be $1R - 11$, so that $1R$ exceeds $1R - 11$ by 11; and the whole second number will be $4R - 44$. There will therefore be an equality between $9R - 44$ and 100, since both numbers, $5R$ and $4R - 44$, must make up 100. Adding 44 to both sides, then, the equation becomes $9R = 144$. Dividing 144 by 9, one finds $1R = 16$. And since the first number was posited as $5R$, the first number will be 80, and consequently the second 20. And a fifth part of the first, namely 16, exceeds a quarter part of the second — 5 — by 11.

But if the excess of the fifth part of the first number over the quarter part of the second were proposed as 20 — namely a fifth of the number 100 to be divided — or as anything greater than 20, the problem would not work out, because we would run into some absurdity. For if the excess had to be 20, you would find the first number to be 100, and hence the second 0 — which is absurd, and so on.

14. [Margin: XIV.]

To find a number from which, if two given numbers are subtracted, the remainder of the smaller has a given ratio to the remainder of the larger.

Let the numbers to be subtracted from the sought number be 100 and 30, in such a way that the remainder of the smaller has a triple ratio to the remainder of the larger. Let the sought number be posited as $1R$. The remainders will then be $1R - 100$ and $1R - 30$. So the ratio of $1R - 30$ to $1R - 100$ must be triple; that is, the equation must stand between $1R - 30$ and $3R - 300$. Adding 30 to both sides, there is an equation between $1R$ and $3R - 270$. And adding 270 again to both sides, between $1R + 270$ and $3R$. Removing $1R$ from both sides, between 270 and $2R$. Dividing 270 therefore by 2, there emerges $1R = 135$, the sought number. For if 100 and 30 are subtracted, the remainders come to 35 and 105; and 105 has a triple ratio to 35.

15. [Margin: XV.]

Given two numbers, to find another which, added to the smaller and subtracted from the larger, makes the sum have to the remainder — or the remainder to the sum — a given ratio smaller than the one the given numbers have. Likewise, one which, added to the larger and subtracted from the smaller, makes the sum have to the remainder a given ratio larger than the one found between the given numbers.

Let the two given numbers be 20 and 100. First, a number is to be found which, added to 20 and subtracted from 100, makes the sum have to the remainder a quadruple ratio — which is smaller than the quintuple ratio that 100 has to 20. Let that number be posited as $1R$. The sum will be $1R + 20$, and the remainder $100 - 1R$. So that this sum be quadruple that remainder, the equation will stand between $1R + 20$ and $400 - 4R$. Adding $4R$ to both sides, between $5R + 20$ and 400. And removing 20 from both sides, between $5R$ and 380. Dividing 380 therefore by 5, one gets $1R = 76$, the sought number. Added to 20, it makes 96; subtracted from 100, it leaves 24. And it is plain that 96 has a quadruple ratio to 24.

Next, let there be sought a number which, added to 20 and subtracted from 100, makes the remainder have to the sum a quadruple ratio. Let that number be posited as $1R$; the sum will be, as before, $1R + 20$, and the remainder $100 - 1R$. So that this remainder be quadruple that sum, the equation will stand between $4R + 80$ and $100 - 1R$. Adding $1R$ to both sides, between $5R + 80$ and 100. And removing 80 from both sides, between $5R$ and 20. Dividing 20 therefore by 5, one gets $1R = 4$, the sought number. For added to 20 it makes 24, and subtracted from 100 it leaves 96 — a number that is the quadruple of 24.

Finally, let there be sought a number which, added to 100 and taken away from 20, makes that sum have to this remainder a septuple ratio — which is larger than the quintuple ratio between 100 and 20. Let that number be posited as $1R$.

The sum will be $1R + 100$, and the remainder $20 - 1R$. So that this sum be septuple that remainder, the equality will stand between $1R + 100$ and $140 - 7R$. Adding $7R$ to both sides, between $8R + 100$ and 140 . And removing 100 from both sides, between $8R$ and 40 . Dividing 40 therefore by 8 , one gets $1R = 5$, the sought number. For added to 100 it makes 105 , and subtracted from 20 it leaves 15 ; and 105 has a septuple ratio to 15 .

If the ratio given in this last example were quadruple — that is, smaller than the ratio between 100 and 20 — we would fall into an impossible equation, as you may verify.

16. [Margin: XVI.]

Given two numbers, to find another to which, if one of the given numbers is added and from which the other is subtracted, the sum has to the remainder a given ratio.

Let the given numbers be 100 and 20 . First, a number is to be found to which, if 100 is added and from which 20 is taken away, that sum is triple this remainder. Let that number be posited as $1R$. The sum will be $1R + 100$, and the remainder $1R - 20$. So that this sum be triple this remainder, the equation will stand between $1R + 100$ and $3R - 60$. Adding 60 to both sides, between $1R + 160$ and $3R$. And removing $1R$ from both sides, between 160 and $2R$. Dividing 160 therefore by 2 , one finds $1R = 80$.

$1R = 80$ is the sought number. For if you add 100 to it, you make 180 ; and if you take away 20 , there remain 60 . And the ratio of 180 to 60 is triple.

Next, let there be sought a number to which, if 20 is added and from which 100 is deducted, that sum is triple this remainder. Let that number be posited as $1R$. The sum will be $1R + 20$, and the remainder $1R - 100$. So that the former be triple the latter, the equation will stand between $1R + 20$ and $3R - 300$. And adding 300 to both sides, between $1R + 320$ and $3R$. And removing $1R$ from both sides, between 320 and $2R$. Dividing 320 therefore by 2 , one gets $1R = 160$, the sought number. For if you add 20 to it you make 180 , and if you take away 100 you leave 60 . And the ratio between 180 and 60 is triple.

17. [Margin: XVII.]

To divide a proposed number into two twice over, so that the larger number of the first division has a given ratio to the smaller of the second division; likewise so that the larger of the second division has to the smaller of the first division whatever given ratio you please.

Let the proposed number 100 be divided twice into two numbers, so that the larger of the first division has a triple ratio to the smaller of the second division, and the larger of the second division has a tenfold ratio to the smaller of the first division. Let the larger part of the first division be posited as $3R$, so that its smaller part is $100 - 3R$; and the smaller part of the second division as $1R$. For the ratio of $3R$ to $1R$ is triple; and so the larger part of the second division will be $100 - 1R$. Now so that $100 - 1R$ has to $100 - 3R$ a tenfold ratio, the equation must stand between $100 - 1R$ and $1000 - 30R$. Adding $30R$ therefore to both sides, there will be an equation between $100 + 29R$ and 1000 . Removing 100 from both sides, there will be an equality between $29R$ and 900 . Dividing 900 therefore by 29 , one detects $1R = 31 \frac{1}{29}$. And since the larger part of the first division was posited as $3R$, that larger part will be $93 \frac{3}{29}$, and consequently the smaller part $6 \frac{26}{29}$. Again, since the smaller part of the second division was posited as $1R$, that smaller part will be $31 \frac{1}{29}$, and hence the larger $68 \frac{28}{29}$. And the number $93 \frac{3}{29}$ is the triple of $31 \frac{1}{29}$; and $68 \frac{28}{29}$ is the tenfold of $6 \frac{26}{29}$.

[The results, set out as printed in two columns — first division | second division:]

	First division	Second division
larger / smaller	$93 \frac{3}{29}$	$31 \frac{1}{29}$
smaller / larger	$6 \frac{26}{29}$	$68 \frac{28}{29}$

18. [Margin: XVIII.]

To divide a given number into two three times over, so that the larger part of the first division has a given ratio to the smaller of the second; likewise the larger of the second division to the smaller of the third; and finally the larger of the third division to the smaller of the first.

Let the given number 100 be divided three times into two, so that the larger of the first division has to the smaller of the second a triple ratio; the larger of the second division to the smaller of the third, a double ratio; and finally the larger of the third division to the smaller of the first, a quadruple ratio. Let the larger part of the first division be posited as $3R$, so that its smaller part is $100 - 3R$. The smaller part of the second division will then be $1R$, since the ratio of $3R$ to $1R$ is triple; and so the larger part of the second division will be $100 - 1R$. So that this be double the smaller part of the third division, the smaller part of the third division will be $50 - (1/2)R$, and therefore its larger part $50 + (1/2)R$. Now so that $50 + (1/2)R$ has to $100 - 3R$, the smaller part of the first division, a quadruple ratio, the equation will stand between $50 + (1/2)R$ and $400 - 12R$. Adding $12R$ to both sides, the equality will be between $50 + 12 \frac{1}{2}R$ and 400 . And removing 50 from

both sides, between $12\frac{1}{2}R$ and 350. Dividing 350 therefore by $12\frac{1}{2}$, one gets $1R = 28$. The larger part of the first division, then, which was posited as $3R$, will be 84, and the smaller 16. The smaller part of the second division, which we posited as $1R$, will be 28, and the larger 72. Finally the smaller part of the third division, which was posited as $50 - (1/2)R$, will be 36, and the larger 64 — numbers which satisfy the proposed riddle, as you may verify.

[The three divisions, set out as printed:]

	First division	Second division	Third division
positions	$3R$	$1R$	$50 + (1/2)R$
	$100 - 3R$	$100 - 1R$	$50 - (1/2)R$

	First	Second	Third
larger / smaller	84	28	64
	16	72	36

19. [Margin: XIX.]

To find two numbers which, multiplied by each other, produce a number in a given ratio to their sum.

Let the product of the multiplication be triple the sum. Let one number be posited as $1R$, and the other as any number you please that is larger than the denominator of the ratio — for example 20, larger than 3. Otherwise it will not work out. From $1R$ times 20 come $20R$; and the sum collected from $1R$ and 20 is $1R + 20$. So that this product has to this sum a triple ratio, the equation will stand between $20R$ and $3R + 60$. Removing $3R$ from both sides, between $17R$ and 60. Dividing 60 therefore by 17, one gets $1R = 3\frac{9}{17}$. The two sought numbers are therefore $3\frac{9}{17}$ and 20. From the multiplication of one by the other comes the number $70\frac{10}{17}$, which is triple $23\frac{9}{17}$, their sum.

20. [Margin: XX.]

Given any number, to find another, so that from the multiplication of the one by the other a number is produced in a given ratio to their sum — yet in such a way that the denominator of the ratio is smaller than the given number.

Given the number 20, another is to be found which, multiplied into it, produces a number triple the sum collected from the two — a ratio whose denominator, 3, is

smaller than the given number 20. Otherwise it will not work out. This riddle does not differ from the preceding one. For if the sought number be posited as $1R$, and the other number as 20 (which is given), one finds the number, as there, which multiplied into 20 produces a number triple their sum, and so on.

21. [Margin: XXI.]

To find two numbers in a given ratio, which multiplied by each other make a number in a given ratio to their sum.

This riddle does not differ from the 19th, except that here the ratio which the numbers to be found ought to have is also given.

Let two numbers be sought in a sesquialteral ratio $[3 : 2]$, such that multiplied by each other they make a number twelvefold their sum. Let two numbers be posited, $4R$ and $6R$, in a sesquialteral ratio. From one multiplied into the other come $24\mathfrak{J}$ [census, that is 24 res-squared]; and their sum is $10R$. So that $24\mathfrak{J}$ has to $10R$ a twelvefold ratio, the equation will stand between $24\mathfrak{J}$ and $120R$. Dividing 120 therefore by 24, there comes out $1R = 5$ — because the consist denominations census (3) and res (R) are collateral [adjacent, so dividing census by res drops one degree]. Since, then, the first number was posited as $4R$ and the second as $6R$, the first will be 20 and the second 30, which multiplied by each other make 600 — a number that is the twelvefold of their sum, 50.

If the ratio between the number produced by the multiplication and the sum ought to be sesquiquintal $[6 : 5]$: with the numbers $4R$ and $6R$ posited in a sesquialteral ratio, so that the number $24\mathfrak{J}$ produced by the multiplication has to the sum $10R$ a sesquiquintal ratio, the equation will stand between $24\mathfrak{J}$ and $12R$ — because $12R$ has to $10R$ likewise a sesquiquintal ratio, this number 12 being produced from $1 \frac{1}{5}$, the denominator of the sesquiquintal ratio, times 10. Dividing 12 therefore by 24, one gets $1R = \frac{1}{2}$; and so, since the posited numbers are $4R$ and $6R$, the two sought numbers will be 2 and 3, which multiplied by each other make 6, a number sesquiquintal to their sum, 5.

22. [Margin: XXII.]

To find two numbers whose sum is any given number, and such that, the latter being divided by the former, the quotient is a given number — any number whatever.

Or:

To divide a given number into two, so that, the latter being divided by the former, the quotient is any given number.

Let the sum of the sought numbers be 100, and the quotient 10. First, let the former number be $1R$, and therefore the latter $100 - 1R$. Dividing this by $1R$, the quotient is $(100 - 1R)/(1R)$, equal to 10. This equation, by cross-multiplication, is reduced to an equation between $100 - 1R$ and $10R$. Adding $1R$ to both sides, there will be an equation between 100 and $11R$. Dividing 100 by 11, one gets $1R = 9 \frac{1}{11}$, that is $100/11$, for the former number. So the latter will be $90 \frac{10}{11}$. And it is established that if $90 \frac{10}{11}$ (or $1000/11$) is divided by $9 \frac{1}{11}$ (or $100/11$), the quotient produced is 10.

Next, let the latter number be $1R$, and therefore the former $100 - 1R$. The former divided by the latter gives the quotient $(1R)/(100 - 1R)$, equal to 10. This equation, by cross-multiplication, is reduced to an equation between $1R$ and $1000 - 10R$. And adding $10R$ to both sides, there will be an equality between $11R$ and 1000. Dividing 1000 therefore by 11, one gets $1R = 1000/11$, that is the quotient $90 \frac{10}{11}$ for the latter number, while the former will be $9 \frac{1}{11}$, or $100/11$ — as before.

Now, if the given number 100 is to be cut into two so that, the latter being divided by the former, the quotient is 10, the problem is solved in the same way by positing $1R$ for the former (or the latter) and $100 - 1R$ for the other, as is clear.

Finally, let the number 100 be divided into two parts, so that, the latter being divided by the former, the quotient is $1/2$. Let the former part be posited as $1R$ and the latter as $100 - 1R$. This divided by $1R$ gives the quotient $(100 - 1R)/(1R)$, equal to $1/2$. This equation, by cross-multiplication, is reduced to this one, between $1R$ and $200 - 2R$. Adding $2R$ to both sides, there will be an equality between $3R$ and 200. Dividing 200 therefore by 3, one gets $1R = 66 \frac{2}{3}$, that is $66 \frac{2}{3}$ for the former part. The latter will be $33 \frac{1}{3}$, or $100/3$. And dividing $100/3$ by $200/3$, the quotient is $1/2$, as is established.

23. [Margin: XXIII.]

To cut a given number into five parts, so that the second contains the first as many times as you please and, besides, any number of units; the third contains the second as many times as you please and, besides, any number of units; the fourth is a given multiple of the third, less any number of units; and finally the fifth falls short of a given multiple of the fourth by any number of units — whenever this can be done.

Let the given number be 100, to be cut into five parts, so that the second contains the first twice and, besides, 2; the third contains the second three times and,

besides, 3; the fourth is double the third less 4; and finally the fifth falls short of double the fourth by 13 units. Let the first part be posited as $1R$. Then the second will be $2R + 2$; the third $6R + 9$; the fourth $12R + 14$; and the fifth $24R + 15$. The sum of all the parts is $45R + 40$, which must equal the given number 100. Removing 40 from both sides, there will be an equation between $45R$ and 60. And dividing 60 by 45, one gets $1R = 1 \frac{1}{3}$, the first part. Therefore all five will be: $1 \frac{1}{3}$, $4 \frac{2}{3}$, 17, 30, 47.

If the number 100 had to be cut into five parts so that the second contained the first three times and, besides, 4; the third contained the second twice and, besides, 3; the fourth were five times the third less 6; and finally the fifth were four times the fourth less 5 — the problem would be impossible. For with the first part posited as $1R$, the second would be $3R + 4$, the third $6R + 11$, the fourth $30R + 49$, and the fifth $120R + 191$; and all these parts make $160R + 255$, a sum which would have to equal the given number 100 — the whole equal to a part, which is absurd. For this reason we said in the riddle "whenever this can be done."

Cossist notation used throughout: R = res (the unknown, x); $\mathfrak{z}$ = census (its square, x^2); $\mathfrak{C}$ = cubus (its cube, x^3). Addition is printed "+", subtraction with a long dash, rendered here "-". Bare numerals are absolute (unit) numbers.

24 [XXIV]. *Given two numbers, to find another which, taken with the smaller, makes a number equal to the one that any aliquot part of the found number makes with the larger.*

Let the two given numbers be 20 and 30, and let another be sought which, added to 20, makes a sum equal to the sum collected from 30 together with a sixth part of the found number. Put the number sought as $6R$ [res, i.e. six times the unknown] — I put it this way so that it may have the named part without a fraction, though this is not actually necessary. Then $6R + 20$ are equal to $1R + 30$. Remove $1R$ from each side, and the equation stands between $5R + 20$ and 30. Remove 20 again from each side, and it stands between $5R$ and 10. Dividing 10 by 5 therefore gives $1R = 2$. So $6R$ will yield 12, the number sought. For this number, taken with 20, makes 32; and its sixth part — namely 2 — taken with 30, likewise makes 32.

25 [XXV]. *To find two numbers in a given proportion, such that as much results from the smaller subtracted from the larger as from the larger divided by the smaller.*

Let two numbers be sought in tenfold proportion. Put them as $1R$ and $10R$. Subtracting the smaller, $1R$, from the larger, $10R$, leaves $9R$. And dividing the

larger, $10R$, by the smaller, $1R$, gives quotient 10. There is therefore an equation between $9R$ and 10. Dividing 10 by 9 gives $1R = 1 \frac{1}{9}$, or $10/9$, the first number. The other will then be $100/9$, or $11 \frac{1}{9}$. For if $1 \frac{1}{9}$ is taken from $11 \frac{1}{9}$, there remain 10; and dividing $11 \frac{1}{9}$ by $1 \frac{1}{9}$ — that is, $100/9$ by $10/9$ — the quotient is likewise 10.

26 [XXVI]. *To find two numbers in a given proportion whose sum equals the number produced by multiplying them together.*

Let two numbers be sought in tenfold proportion. Put them as $1R$ and $10R$. Their sum is $11R$. And multiplying one into the other makes the number $10\bar{3}$ [census]. There will therefore be an equality between $11R$ and $10\bar{3}$. Dividing 11 by 10 gives $1 \frac{1}{10}$ for the first number — because the cosmic numbers are collateral [proportional]. The other will accordingly be 11. For both their sum and the number produced by multiplying one into the other come to $12 \frac{1}{10}$.

27 [XXVII]. *To find two numbers with a given difference, such that their squares also have a given difference, one which, however, is larger than the square of the given difference of the numbers.*

Let two numbers be sought whose difference is 4, and the difference of whose squares is 144 — which is larger than 16, the square of the given difference, 4, of the numbers. Put the smaller number as $1R$, and hence the larger as $1R + 4$, whose difference is 4. Their squares are $1\bar{3}$ and $1\bar{3} + 8R + 16$. The difference of these is $8R + 16$, which must equal the number 144. Removing 16 from each side, the equation stands between $8R$ and 128. Dividing 128 by 8, then, $1R = 16$, the smaller number. The larger will accordingly be 20, so that the difference is 4. Their squares are 256 and 400, whose difference is 144.

If the given difference of the numbers is 1 and that of the squares is 31, the numbers found will be 15 and 16 — namely, the two consecutive numbers that produce the given difference of squares. And so, whenever the difference of the numbers given is 1 and the difference of the squares is any odd number, the two numbers sought will be the two consecutive numbers that produce the given difference of squares.

When the difference of the numbers is 5 and that of the squares 30, you will find the numbers $1/2$ and $5 \frac{1}{2}$.

If the difference of the numbers is 10 and that of the squares 200, you will find the numbers 5 and 15.

If a difference of 2 is given between the numbers and of 22 between the squares, the numbers sought will be $4\frac{1}{2}$ and $6\frac{1}{2}$.

Scholium.

This problem can also be solved without algebra (which will sometimes be very convenient when working with cossic numbers) in this way. Let the given difference of the squares be divided by twice the given difference of the numbers. For if half the given difference of the numbers is subtracted from the quotient, the remainder becomes the smaller number, which, added to the given difference of the numbers, will make the larger.

Suppose two numbers are to be found whose difference is 6 and the difference of whose squares is $1R$. Divide $1R$ by 12, that is, by twice the difference, 6, of the given numbers. And from the quotient, $1/12R$, subtract the number 3, half of the same difference, 6, of the numbers. The remaining number, $1/12R - 3$, will be the smaller number sought; and if to it you add the given difference, 6, you will make $1/12R + 3$, the larger number. For the difference of these two numbers is 6. And their squares are $1/144R - 1/2R + 9$ and $1/144R + 1/2R + 9$, whose difference is $1R$.

29 [XXIX]. *Given a number, to find another which, multiplied into the given number, makes as much as it does when added to the same.*

Let the given number be 30. Put the number sought as $1R$. Multiplied into 30, it makes $30R$; added to 30, it makes $1R + 30$. There will therefore be an equation between $30R$ and $1R + 30$. Removing $1R$ from each side, the equation stands between $29R$ and 30. Dividing 30 by 29, then, $1R = 30/29$, or $1\frac{1}{29}$. For adding $1/29$ to 30 gives $31\frac{1}{29}$ — precisely what results from $1\frac{1}{29}$ multiplied into 30.

30 [XXX]. *Given a number, to find another such that a given aliquot part of it, multiplied into the given number, makes as much as the whole found number added to the same given number.*

Let the given number be 60, and the given aliquot part of the number to be found be $1/3$. Put a third of the number to be found as $1R$, and hence the whole number as $3R$. From $1R$ — that is, from a third of the number sought — multiplied into 60, comes the number $60R$. And from $3R$ — the whole number sought — together with 60, comes the number $3R + 60$. There will therefore be an equation between $60R$ and $3R + 60$. Removing $3R$ from each side, it stands between $57R$ and 60. Dividing 60 by 57, then, $1R = 1\frac{1}{19}$, a third part of the number sought, and hence the number sought is $3\frac{3}{19}$. For from $1\frac{1}{19}$ — that is, from a third of the found

number — multiplied into 60 comes the number $63 \frac{3}{19}$, equal to the sum collected from $3 \frac{3}{19}$ and 60.

31 [XXXI]. *To divide a given number into two, so that the smaller is a given aliquot part (or given aliquot parts) of the larger, and moreover contains a given other number besides, if desired.*

Let the given number be 40, to be distributed first into two, so that the smaller is $\frac{1}{7}$ of the larger. Put the smaller as $1R$; then the larger will be $7R$. These together must equal the given number, 40. There is therefore an equation between $8R$ and 40. Dividing 40 by 8 yields $1R = 5$, the smaller number. The larger will accordingly be 35, namely $7R$. And it is plain that 5 is the seventh part of the number 35.

Next, let the same number, 40, be cut into two so that the smaller constitutes $\frac{3}{7}$ of the larger. Put the smaller as $3R$, and therefore the larger as $7R$, so that the larger may contain the smaller three-sevenths [i.e. so the smaller is $\frac{3}{7}$ of it]. The sum of both is $10R$, equal to the given number, 40. Dividing 40 by 10, then, one finds $1R = 4$; and hence, since the smaller number was put as $3R$ and the larger as $7R$, the smaller will be 12 and the larger 28, and the former contains three-sevenths of the latter.

If the number 84 is to be cut into two, so that the smaller contains $\frac{3}{4}$ of the larger and 7 units besides, we shall put the larger as $4R$, so that it can contain $\frac{3}{4}$ without a fraction. The smaller will then be $3R + 7$. The sum of both is $7R + 7$, equal to the given number, 84. Removing 7 from each side, the equation stands between $7R$ and 77. Dividing 77 by 7, $1R = 11$ will come out. Since, then, we put the larger as $4R$ and the smaller as $3R + 7$, the larger will be 44 and the smaller 40; and this [smaller] contains $\frac{3}{4}$ of that [larger] — namely 33 — and 7 besides.

32 [XXXII]. *To find two numbers such that, if the number produced by their multiplication is divided by their difference, the quotient becomes any given number.*

Let two numbers be sought such that the number produced by their multiplication, if divided by their difference, makes the quotient 30. For the smaller, put any number less than the given quotient, 30 — namely 20 — and put the larger as $20 + 1R$, so that their difference is $1R$. From 20 into $20 + 1R$ comes the number $400 + 20R$; and this divided by $1R$, their difference, gives the quotient $(400 + 20R)/1R$, equal to the proposed number, 30. This equation is reduced to an equality by cross-multiplication, giving the equality between $30R$ and $400 + 20R$. Removing $20R$ from both sides, the equation stands between $10R$ and 400. Dividing 400 by 10, then, $1R = 40$, for the difference of the numbers sought. Since,

therefore, the smaller is 20, the larger will be 60 — namely $20 + 1R$. Now from 20 into 60 comes the number 1200, and this divided by 40, the difference of the numbers, makes the quotient 30.

33 [XXXIII]. *To find a number, two given parts of which make a sum of their squares that has any given proportion to another given part of the same number to be found.*

Let a number be sought, of which $\frac{1}{2}$ and $\frac{1}{6}$ make squares whose sum is triple the $\frac{1}{4}$ part of the same number. Put the number sought as $1R$. The given parts are $\frac{1}{2}R$ and $\frac{1}{6}R$, whose squares, $\frac{1}{4}\mathfrak{z}$ and $\frac{1}{36}\mathfrak{z}$, make the sum $\frac{10}{36}\mathfrak{z}$, triple the $\frac{1}{4}R$ — that is, equal to $\frac{3}{4}R$. Dividing $\frac{3}{4}$ by $\frac{10}{36}$, then, $1R = \frac{108}{40}$, that is $\frac{27}{10}$, the number sought (because the cossic numbers are collateral). For its $\frac{1}{2}$ and $\frac{1}{6}$ are $\frac{27}{20}$ and $\frac{27}{60}$, whose squares, $\frac{729}{400}$ and $\frac{729}{3600}$, make the sum $\frac{7290}{3600}$, which is triple $\frac{27}{40}$ — that is, of the $\frac{1}{4}$ of the found number, $\frac{27}{10}$.

34 [XXXIV]. *To find a number which, multiplied into itself, and the product then multiplied into any given number, makes a number in a given proportion to the cube generated from the found number.*

Let the given number be 20, and let another be sought which, multiplied first into itself and then that product into the given number, 20, generates a number quintuple the cube produced from the found number. Put the number sought as $1R$. From $1R$ into itself comes $1\mathfrak{z}$. And from $1\mathfrak{z}$ into 20 comes the number $20\mathfrak{z}$. But the cube of $1R$ is $1\mathfrak{c}$ [cubus], to which $20\mathfrak{z}$ must have a quintuple proportion. There is therefore an equation between $20\mathfrak{z}$ and $5\mathfrak{c}$. Dividing 20 by 5, then, $1R = 4$, the number sought;

because the cossic numbers are collateral. For this number, 4, multiplied into itself makes 16; and from 16 into 20 comes the number 320, which is quintuple the number 64 — namely the cube produced from the found number, 4.

35 [XXXV]. *To divide a given number into two, so that one of them, multiplied into another given number, exceeds — by any given number — the number produced from the other number multiplied into any other given number.*

Let the given number 10 be cut first into two, so that the number produced from the first into 4 exceeds by a unit the number produced from the second into 6. Put the first part as $1R$, and hence the second as $10 - 1R$. From $1R$ into 4 comes $4R$. And from $10 - 1R$ into 6 comes $60 - 6R$. This product [$4R$] must exceed the other by a unit; and therefore the equation will be between $4R$ and $61 - 6R$. Adding $6R$ to each side, it stands between $10R$ and 61. Dividing 61 by 10, then, $1R = 6 \frac{1}{10}$,

the first number. The second will accordingly be $3 \frac{9}{10}$. From the first into 4 comes $24 \frac{4}{10}$; and from the second into 6 comes $18 \frac{54}{10}$, that is $23 \frac{4}{10}$ — which number is exceeded by a unit by $24 \frac{4}{10}$.

Next, let the same given number, 10, be cut into two parts, so that the second part multiplied into 4 produces a number that exceeds by a unit the product of the first part into 6. Put again the first part as $1R$ and the second as $10 - 1R$. From $10 - 1R$ into 4 comes the number $40 - 4R$. And from $1R$ into 6 come $6R$. That product must exceed this number by a unit. The equation will therefore be between $40 - 4R$ and $6R + 1$. Removing a unit from each side, it stands between $40 - 4R$ and $6R$; and adding $4R$ to each side, between 39 and $10R$. Dividing 39 by 10, then, $1R = 3 \frac{9}{10}$, the first number. The second will accordingly be $6 \frac{1}{10}$. From $6 \frac{1}{10}$ into 4 comes the number $24 \frac{4}{10}$. And from $3 \frac{9}{10}$ into 6 comes $18 \frac{54}{10}$. It is plain, moreover, that the number $24 \frac{4}{10}$ exceeds by a unit the number $23 \frac{4}{10}$.

But now let the excess have to be 20 between the products. Put the first number as $1R$ and the second as $10 - 1R$. From $1R$ into 4 comes the number $4R$; and from $10 - 1R$ into 6 comes the number $60 - 6R$, which the first must exceed by the number 20. There will therefore be an equation between $4R$ and $80 - 6R$. Adding $6R$ to each side, it stands between $10R$ and 80. Dividing 80 by 10, then, $1R = 8$, the first number. The second will accordingly be 2. It is clear, moreover, that the number 32 — which comes from the first, 8, into 4 — exceeds by 20 the number 12, which comes from the second, 2, into 6.

But now let the number $40 - 4R$, made from the second, $10 - 1R$, into 4, have to exceed by the number 20 the number $6R$, made from the first, $1R$, into 6. There will then be an equation between $40 - 4R$ and $6R + 20$. Adding $4R$ to each side, between 40 and $10R + 20$; and removing 20 from each side, between 20 and $10R$. Dividing 20 by 10, then, $1R = 2$, the first number. The second will accordingly be 8. And it is certain that the number 32, made from the second, 8, into 4, exceeds by the number 20 the number 12, made from the first, 2, into 6.

36 [XXXVI]. *To divide a given number into two parts which, together with the number itself, constitute an arithmetic progression.*

Let the given number 20 be divided into two which, with the same number 20, constitute three numbers in arithmetic proportion. Let the smaller part be $1R$ and the larger $20 - 1R$. Now, for $1R$, $20 - 1R$, and 20 to be in arithmetic proportion, the excess between $1R$ and $20 - 1R$ must be the same as that between $20 - 1R$ and 20. These excesses are $20 - 2R$ and $1R$, as is plain from these subtraction formulae:

$$\begin{array}{r}
 20 - 1R \quad 20 + 0 \\
 + 1R \quad 20 - 1R \\
 \hline
 20 - 2R \quad 0 + 1R
 \end{array}$$

The equation will therefore be between $20 - 2R$ and $1R$. Adding $2R$ to each side, between 20 and $3R$. Dividing 20 by 3 , then, $1R = 6 \frac{2}{3}$, the first part. The other will accordingly be $13 \frac{1}{3}$. And $6 \frac{2}{3}$, $13 \frac{1}{3}$, and 20 constitute an arithmetic progression, since they have the same excess, $6 \frac{2}{3}$.

Alternatively. If $1R$, $20 - 1R$, and 20 are in arithmetic proportion, then (as we said of three arithmetically proportional numbers, following Euclid, *Elements* Book V, in the fourth property) the sum of the first and third — namely $1R + 20$ — will be equal to double the middle — namely $40 - 2R$. Adding $2R$ to each side, an equation will arise between $3R + 20$ and 40 . Removing 20 from each side, between $3R$ and 20 . So again $1R = 6 \frac{2}{3}$, as before.

37 [XXXVII]. *To find two numbers which, multiplied together, make a number that has a given proportion to the smaller, and also a given proportion to the larger, but a smaller one than the former.*

Let the proportion of the product to the smaller be tenfold, and to the larger sesquialter [3:2]. Put the product as $30R$ (for the number $30R$ has a tenfold proportion to $3R$, and a sesquialter proportion to $20R$). The smaller number will be $3R$ and the larger $20R$, which multiplied together must make $30R$. But they make $60\bar{3}$. There is therefore an equation between $30R$ and $60\bar{3}$. Dividing 30 by 60 , then, $1R = \frac{1}{2}$ will come out — because the cossic numbers are collateral. So the smaller number, which we put as $3R$, will be $1 \frac{1}{2}$; and the larger, which was put as $20R$, will be 10 . And from $1 \frac{1}{2}$ — that is, from $\frac{3}{2}$ — into 10 comes the number $\frac{30}{2}$, or 15 , which has a tenfold proportion to $1 \frac{1}{2}$, or to $\frac{3}{2}$, and a sesquialter proportion to 10 .

Scholium.

This puzzle can be solved without algebra if, for the numbers sought, one takes the denominators of the given proportions. Thus, if two numbers are to be found which, multiplied together, produce a number that has a sevenfold proportion to the smaller and a sixfold proportion to the larger, the two numbers will be 7 and 6 . For these two, multiplied together, make 42 , a number that has a sevenfold proportion to 6 and a sixfold proportion to 7 , as is plain.

Continuation of Chapter XXIX, "Various Enigmas" (Aenigmata varia).

Problem 38 [Margin: XXXVIII.]

To divide a given number into any number of parts that are arithmetically proportional, whose common difference is given — when this can be done.

Let the given number 120 be divided into six parts that successively exceed one another by 2. Let the first part be posited as $1R$ [res, the unknown]. The second will then be $1R+2$, the third $1R+4$, the fourth $1R+6$, the fifth $1R+8$, and the sixth $1R+10$. All together these make the sum $6R+30$, equal to the given number 120. Removing 30 from both sides, the equation stands between $6R$ and 90. Dividing 90 by 6 gives $1R = 15$, the first part. So the six parts are 15, 17, 19, 21, 23, 25, which exceed one another by two and sum to 120.

The given number, however, must be greater than the sum of the same number of terms of an arithmetic progression each smaller by one, a progression beginning from the given excess. In the present example it must be greater than the sum 30 of the five terms 2, 4, 6, 8, 10. Otherwise the problem will not succeed, because we would fall into an equation between two entirely unequal numbers. For that reason we said in the enigma [when this can be done].

Problem 39 [Margin: XXXIX.]

To partition a given number into any number of parts that are arithmetically proportional, when the first is given — when this can be done.

Let the given number 60 be divided into 11 parts forming an arithmetic progression whose first term is 5. Put the common difference of the progression as $1R$. The 11 terms will then be these: 5, $5+1R$, $5+2R$, $5+3R$, $5+4R$, $5+5R$, $5+6R$, $5+7R$, $5+8R$, $5+9R$, $5+10R$, whose sum is $55R+55$, which must equal the given number 60. Removing 55 from both sides, the equation stands between $55R$ and 5. Dividing 5 by 55 gives $1R = 1/11$, the common difference of the progression. So the 11 parts sought are these: 5, $5 \frac{1}{11}$, $5 \frac{2}{11}$, $5 \frac{3}{11}$, $5 \frac{4}{11}$, $5 \frac{5}{11}$, $5 \frac{6}{11}$, $5 \frac{7}{11}$, $5 \frac{8}{11}$, $5 \frac{9}{11}$, $5 \frac{10}{11}$, which all together make 60.

The given number, however, must be greater than the number produced by the number of terms multiplied into the first term. As in the given example it must be greater than 55, which is produced from the number of terms — namely 11 — times the first term 5. For this reason it was stated in the enigma [when this can be done].

Problem 40 [Margin: XL.]

Given two numbers, to find two others in a given ratio, such that the product of the greater of these into the greater of those makes a number which exceeds by any given number the product of the smaller into the smaller.

Let the given numbers be 5 and 9, and let it be required to find two others in triple ratio, such that the greater multiplied into 9 makes a number exceeding the product of the smaller into 5 by 110. Put the smaller as $1R$, and therefore the greater as $3R$. From $3R$ into 9 come $27R$, and from $1R$ into 5 come $5R$. Now the number $27R$ exceeds the number $5R$ by $22R$, which must equal the number 110. Dividing 110 by 22 gives $1R = 5$, the first number; and so the greater will be 15. From 15 into 9 comes the number 135, which exceeds 25 — the product of 5 into 5 — by the number 110.

Problem 41 [Margin: XLI.]

Given two numbers, to find two others in a given ratio, such that the product of the greater of these into the smaller of those exceeds by a given number that which is made from the smaller into the greater. Or, at any rate, this product exceeds that one by a given number.

Let the given numbers be 5 and 9, and let it be required to find two others in double ratio, such that the greater multiplied into 5 produces a number exceeding by 10 units the product of the smaller into 9. Put the smaller as $1R$, and therefore the greater as $2R$. (Here the greater into the smaller given number must produce a number larger than the one made from the smaller into the greater given number, so that the former can indeed exceed the latter. Hence if the given numbers were 2 and 9, the question would be impossible; for from $2R$ into 2 comes the number $4R$, smaller than the number $9R$ made from $1R$ into 9.) From $2R$ into 5 comes the number $10R$, and from $1R$ into 9 comes the number $9R$, which the former exceeds by $1R$. There is therefore an equation between $1R$ and 10. Dividing 10 by 1 gives $1R = 10$, the smaller number, and accordingly the greater will be 20. You see, then, that the number 100, made from 20 into 5, exceeds by 10 units the number 90 made from 10 into 9.

Problem 42 [Margin: XLII.]

To find two numbers in a given ratio, such that the sum of the squares described from them has a given ratio to their sum.

Let two numbers in quintuple ratio be sought, such that the sum of the squares described from them has to the sum of the numbers found the ratio tenfold. Put the smaller number as $1R$, and therefore the greater as $5R$. Their squares are 13 [*census*, the square of the unknown] and 253 , and their sum 263 , which is the tenfold of their sum, namely of $6R$. There will therefore be an equation between 263 and $60R$. Dividing 60 by 26 gives $1R = 2 \frac{4}{13}$, that is, $30/13$, the smaller number (since the consist numbers are collateral), and hence the greater is

150/13, or $11 \frac{7}{13}$. Their squares are 900/169 and 22500/169, whose sum 23400/169 is the tenfold of 180/13, the sum of the numbers found.

With the same things posited, if the ratio to be found is that of equality, there will be an equation between 263 and 6R. Dividing 6 by 26 gives $1R = \frac{3}{13}$, the smaller number (because the consist numbers are collateral), while the greater will be 15/13. Their squares are 9/169 and 225/169, and their sum 234/169, equal to the sum 18/13 collected from the numbers found.

And if the ratio to be found is one of lesser inequality — say, subdouble — then the number 263 will be the subdouble of the number 6R; that is, the latter will be the double of the former. Therefore there will be an equation between 523 and 6R, or between 263 and 3R. Dividing 6 by 52, or 3 by 26, gives $1R = \frac{3}{26}$, the smaller number. The greater, then, will be 15/26. Their squares are 9/676 and 225/676, and their sum 234/676, the subdouble of the sum 18/26 collected from the numbers found — inasmuch as this latter is the double of the former.

Problem 43 [Margin: XLIII.]

Given two numbers, to find two others in a given ratio, such that the greater of these added to the greater of the given numbers makes a number that has any given ratio to the sum collected from the smaller ones — provided this second given ratio lies between the ratio of the given numbers and the other given ratio.

Let the given numbers be 10 and 12, and let it be required to find two others in quintuple ratio, such that the sum of the greater and 12 is triple the sum of the smaller and 10. Now the triple ratio lies between the ratio of 12 to 10 and the quintuple ratio — that is, it is greater than the former and smaller than the latter. Put the smaller number as 1R, and therefore the greater as 5R. From 5R and 12 comes the number 5R+12, and from 1R and 10 the number 1R+10, of which the former must be the triple. The equation will therefore be between 5R+12 and 3R+30. Removing 12 from both sides: between 5R and 3R+18. Removing 3R from both again: between 2R and 18. Dividing 18 by 2 gives $1R = 9$, the smaller number, and therefore the greater is 45. From 45 and 12 comes the number 57, which is the triple of the number collected from 9 and 10 — namely of 19.

Problem 44 [Margin: XLIII.]

Given two numbers, to find two others in a given ratio, such that when the smaller of these is subtracted from the greater of the given numbers, the remaining number has any given ratio to that which is left when the greater is taken from the smaller.

Let the given numbers be 60 and 100, and let it be required to find two others in quintuple ratio, such that when the smaller is subtracted from 100, the remaining number has to that which is left when the greater is taken from 60 the ratio twentyfold. Put the two numbers in quintuple ratio as $1R$ and $5R$. With $1R$ taken from 100, the remaining number is $100-1R$; and with $5R$ taken from 60, there remain $60-5R$. The number $100-1R$ will be the twentyfold of $60-5R$; that is, the equation will be between $100-1R$ and $1200-100R$. Adding $100R$ to both sides, there will be equality between $100+99R$ and 1200 . Removing 100 from both sides: between $99R$ and 1100 . Dividing 1100 by 99 gives $1R = 11 \frac{11}{99}$, or $1100/99$, the smaller number, and accordingly the greater is $5500/99$, or $55 \frac{55}{99}$. The former, taken from 100, leaves $88 \frac{88}{99}$; the latter, in turn, subtracted from 60, leaves $4 \frac{44}{99}$. And so $88 \frac{88}{99}$ — that is, $8800/99$ — has to $4 \frac{44}{99}$, or $440/99$, the same ratio as 20 to 1.

Problem 45 [Margin: XLV.]

Given two numbers, to find two others in a given ratio, such that when the greater of these is taken from the greater of those and the smaller from the smaller, the remaining numbers are equal. But the given ratio must be greater than the ratio of the given numbers.

Let the given numbers be 100 and 60, and let it be required to find two others in septuple ratio, such that when the greater is taken from 100 and the smaller from 60, the remaining numbers are equal. Put them as $1R$ and $7R$. The remaining numbers will be $100-7R$ and $60-1R$, which are equal. Adding $7R$ to both sides, the equation will be between 100 and $60+6R$. Removing 60 from both sides: between 40 and $6R$. Dividing 40 by 6 gives $1R = 6 \frac{2}{3}$, the smaller number, while the greater is its septuple, $46 \frac{2}{3}$. Taking the former from 60 and the latter from 100, the remaining numbers come out equal, $53 \frac{1}{3}$.

Problem 46 [Margin: XLVI.]

To find three numbers in a given continued geometric ratio, such that the product of the multiplication of any two of them together is equal to the remaining one, or has to it any given ratio.

Let three numbers in triple ratio be sought, and let the first case be that the product of the least into the middle is equal to the greatest. Put three numbers in triple ratio: $1R$, $3R$, $9R$. From $1R$ into $3R$ comes the number $3R$, equal to $9R$. Dividing 9 by 3 gives $1R = 3$ (because the consist numbers are collateral). The three numbers sought are therefore 3, 9, 27. And from 3 into 9 comes the number 27, equal to the greatest, 27.

Next, let the product be the quadruple of the greatest, so that the product 33 has to 9R the quadruple ratio — that is, so that the equation is between 33 and 36R. Dividing 36 by 3 gives $1R = 12$, and the three numbers in triple ratio will be 12, 36, 108. And from 12 into 36 comes the number 432, the quadruple of 108.

Again, let the product be the subdouble of the greatest, so that the product 33 has to 9R the subdouble ratio — that is, so that the equation is between 63 and 9R. Dividing 9 by 6 gives $1R = 3/2$. The three numbers in triple ratio are therefore $3/2$, $9/2$, $27/2$. And from $3/2$ into $9/2$ comes the number $27/4$, the subdouble of $27/2$.

Now, let the product of the second (3R) into the third (9R) make a number equal to the first, namely 1R. Dividing 1 by 27 gives $1R = 1/27$. The three numbers in triple ratio are therefore $1/27$, $3/27$, $9/27$. And from $3/27$ into $9/27$ comes the number $27/729$, equal to the first, $1/27$.

But if the product 273 of the second into the third is to have to the first, 1R, the triple ratio, there will be equality between 273 and 3R. Dividing 3 by 27 gives $1R = 1/9$, and the three numbers in triple ratio will be $1/9$, $3/9$, $9/9$. And from $3/9$ into $9/9$ — that is, from $1/3$ into 1 — comes $1/3$, which number is the triple of the first, $1/9$.

Lastly [POSREMO — recte: POSTREMO], let the product of the first (1R) into the third (9R) make a number equal to the second, 3R — that is, let the equation be between 93, which number is made from the first into the third, and 3R. Dividing 3 by 9 gives $1R = 1/3$. And the three numbers in triple ratio will be $1/3$, 1, 3. And from the first ($1/3$) into the third (3) comes 1, equal to the second, 1.

But if the product of the first (1R) into the third (9R) — that is, 93 — is to have to the second (3R) the sextuple ratio, there will be an equation between 93 and 18R. Dividing 18 by 9 gives $1R = 2$. And the three numbers in triple ratio will be 2, 6, 18. And from 2 into 18 comes the number 36, the sextuple of the second number, 6.

Problem 47 [Margin: XKVII. — recte: XLVII.]

To divide a given number into two parts, such that when the greater is divided by the smaller, the quotient is the same given number.

Let the given number be 40, and the given quotient likewise 40. Put the smaller part as 1R and the greater as $40 - 1R$. Dividing this by that, the quotient comes to be $(40 - 1R)/1R$, equal to 40. This equation, reduced by cross-multiplication, gives an equation between $40 - 1R$ and $40R$. Adding 1R to both sides, the equation will

be between 40 and 41R. Dividing 40 by 41 gives $1R = 40/41$, the smaller part. The greater will accordingly be $39 \frac{1}{41}$. This divided by that yields the quotient 40.

If instead the greater part is put as 1R and the smaller as $40 - 1R$, then dividing the former by the latter gives the quotient $1R/(40 - 1R)$, equal to 40. This equation, reduced by cross-multiplication, gives an equation between 1R and $1600 - 40R$. Adding $40R$ to both sides, the equation will be between 41R and 1600. Dividing 1600 by 41 gives the greater part $39 \frac{1}{41}$, and accordingly the smaller $40/41$, as before.

Problem 48 [Margin: XLVII. — recte: XLVIII.]

To find two numbers such that the first together with any multiple of the second makes a number equal to that composed from the second together with any multiple of the first. Or such that the first with a given part of the second makes as much as the second with a given part of the first. Or, finally, such that if the first takes any multiple of the second and the second any multiple of the first — or if the first takes a given part of the second and the second a given part of the first — the two numbers come to be in any given ratio.

Let the first with the triple of the second be required to make as much as the second with the quintuple of the first. Put the first as 1R and the second as any number, say 50. Then $1R + 150$ and $50 + 5R$ are equal. Removing 1R from both sides, the equation will be between 150 and $50 + 4R$. Removing 50 from both sides again: between 100 and 4R. Dividing 100 by 4 gives $1R = 25$. The numbers sought are therefore 25 and 50. For the first, 25, with the triple of the second — that is, with 150 — makes 175, as much as the second, 50, with the quintuple of the first, that is, with 125.

Next, let the first with $1/7$ of the second and the second with $1/3$ of the first be required to make one and the same number. Put the first as 3R, so that it has its third without a fraction; and the second as any number having its seventh without a fraction, namely 21. The first, then, with $1/7$ of the second will make $3R + 3$, and the second with $1/3$ of the first will make $21 + 1R$. There is therefore an equation between $3R + 3$ and $21 + 1R$. Removing 1R from both sides: between $2R + 3$ and 21. Removing 3 from both again: between 2R and 18. Dividing 18 by 2 gives $1R = 9$. The first, posited as 3R, will therefore be 27, and the second 21, as we posited. For 27 with $1/7$ of the number 21, and 21 with $1/3$ of the number 27, each makes 30.

But now let the first with the quadruple of the second and the second with the triple of the first be required to make two numbers in double ratio. Put the first

as $1R$ and the second as any number, say 6. The first with the quadruple of the second will then make $1R+24$, and the second with the triple of the first will make $6+3R$. If, then, you make $1R+24$ the greater, which must be the double of $6+3R$, there will be an equation between $1R+24$ and $12+6R$. Removing $1R$ from both sides: between 24 and $12+5R$. Removing 12 again from both: between 12 and $5R$. Dividing 12 by 5 gives $1R = 2 \frac{2}{5}$, the first number; the second is 6, as posited. Now the first with the quadruple of the second makes $26 \frac{2}{5}$, while the second with the triple of the first makes $66/5$, that is, $13 \frac{1}{5}$; and the number $26 \frac{2}{5}$ has to $13 \frac{1}{5}$ the double ratio.

If instead you make $6+3R$ the greater, which must be the double of $1R+24$, there will be an equation between $6+3R$ and $2R+48$. Removing $2R$ from both sides: between $6+1R$ and 48. Removing 6 again from both: between $1R$ and 42. So $1R = 42$, the first number, while the second will be 6, as we posited. The first with the quadruple of the second makes 66, while the second with the triple of the first makes 132, which number has to 66 the double ratio.

Finally, let the first with $1/7$ of the second and the second with $1/3$ of the first be required to make two numbers in sextuple ratio. Put the first as $3R$, so that it has its third without a fraction, and the second as any number having its seventh without a fraction, namely 7. The first, then, with $1/7$ of the second will make $3R+1$, and the second with $1/3$ of the first will make $7+1R$. And since these two numbers must have the sextuple ratio, of necessity $7+1R$ will be the greater. For if $7+1R$ were multiplied by 6 to be made equal to the number $3R+1$, the number $42+6R$ would be produced, which cannot be equal to $3R+1$. Therefore the equation will be between $18R+6$ and $7+1R$. Removing $1R$ from both sides: between $17R+6$ and 7. Removing 6 again from both: between $17R$ and 1. Dividing 1 by 17 gives $1R = 1/17$. And since the first number was posited as $3R$, the first number will be $3/17$, and the second 7, as it was posited. The first with $1/7$ of the second makes $1 \frac{3}{17}$, that is, $20/17$; the second with $1/3$ of the first makes $7 \frac{1}{17}$, that is, $120/17$, which number has to $20/17$ the sextuple ratio.

Problem 49 [Margin: XLIX.]

To distribute a given number into any number of parts in a given continued geometric ratio.

Let the given number be 4665, to be cut into 5 continued-sextuple numbers. Set out 5 numbers in sextuple ratio: $1R$, $6R$, $36R$, $216R$, $1296R$. Their sum, $1555R$, must equal the given number 4665. Dividing 4665 by 1555, then, gives $1R = 3$; and the

five numbers in continued sextuple ratio will be 3, 18, 108, 648, 3888, which added together make 4665.

If the number to be cut is 20, into five such parts, you will find 1555R equal to 20. Dividing 20 by 1555 gives $1R = 4/311$. And the five parts in sextuple ratio will be $4/311$, $24/311$, $144/311$, $864/311$, $5184/311$, making the sum 20.

Problem 50 [Margin: L.]

To find any number of numbers in a given continued geometric ratio, whose sum is equal to any given number.

Let 5 numbers in continued sextuple ratio be sought, making 4665, or 20. This enigma does not differ from the one immediately preceding. For if the given sum is cut into 5 numbers in continued sextuple ratio, what is asked will have been done.

Algebra, Chapter XXIX ("Various Puzzles" — Aenigmata varia), continued.

Note on notation: the cossist symbols are rendered as printed — R = *res* (the unknown, first power, = x); 3 = *census* (its square, = x^2). A coefficient printed before the symbol multiplies it (so $3R = 3x$). Plain Arabic numerals are pure numbers.

Problem 51. [Margin: LI.] *To divide a given number into three numbers such that the two smaller have a given difference, and the two larger a different given difference.*

Let the given number be 445, the difference of the smaller pair 10, and the difference of the larger pair 20. Set the smallest number = $1R$ [*res*, the unknown]. Then the middle one is $1R+10$ and the largest is $1R+30$. Their sum is $3R+40$, which must equal the given number 445. Removing 40 from each side gives the equation between $3R$ and 405. Dividing 405 by 3, we find $1R = 135$, the smallest number; and therefore the middle one is 145 and the largest 165. These satisfy the puzzle, since they add up to 445.

If instead the given number is 200, then $3R+40$ equals 200, and removing 40 from each side gives the equation between $3R$ and 160. Dividing 160 by 3, the first number is $1R = 53\frac{1}{3}$; the second is $63\frac{1}{3}$ and the third $83\frac{1}{3}$. These have the given excesses and add up to 200.

Alternatively. Set the middle number = $1R$. Then the smallest is $1R-10$ and the largest is $1R+20$, whose sum, $3R+10$, must equal the given number 445 (or 200). Removing 10 from each side gives the equation between $3R$ and 435 (or 190).

Carrying out the division, $1R = 145$, or $63\frac{1}{3}$ — the middle number [recte: the source prints *primus numerus*, "the first number," evidently a slip for *medius*, since $1R$ was posited as the middle term]. Thus in the first example the three numbers are 135, 145, 165; and in the second, $53\frac{1}{3}$, $63\frac{1}{3}$, $83\frac{1}{3}$, as before.

Problem 52. [Margin: LII.] *To find three numbers such that the difference of the smaller pair is given, the difference of the larger pair is another given amount, and all three together make up a given number.*

Let the difference of the smaller pair be 10, that of the larger pair 20, and their sum 445 (or 200). This problem does not differ from the one just preceding. For if the given sum is distributed into three parts according to the given excesses, what is required will be done.

Problem 53. [Margin: LIII.] *To divide a given number into three, such that each of the two extremes together with the middle has a given ratio to the other extreme.*

First let the given number 100 be divided into three, so that the first together with the middle has a triple ratio to the third, while the third together with the middle has a quadruple ratio to the first. Set the first = $1R$. Then the other two together are $4R$, so that they can have a quadruple ratio to the first. Their sum, $5R$, must therefore equal the given number 100. Dividing 100 by 5 gives $1R = 20$, the first number; and so the other two together make 80. Next set the third = $1R$, so that the middle is $80 - 1R$. Then the first, 20, together with the middle, $80 - 1R$ — that is, $100 - 1R$ — is triple the third, $1R$; and so the equation is between $100 - 1R$ and $3R$. Adding $1R$ to each side, between 100 and $4R$. Dividing 100 by 4 gives $1R = 25$, the third number. The middle is therefore 55, which is left when the third, 25, is subtracted from 80. The three numbers are accordingly 20, 55, 25, which together make 100; and the first together with the middle makes 75, triple the third, 25; while the third together with the middle makes 80, quadruple the first, 20.

If the given number to be cut into three is 60, so that the first together with the middle has a ninefold ratio to the third, while the third together with the middle has a quadruple ratio to the first, we proceed in the same way. For if the first is set = $1R$, the other two together are $4R$, and therefore $5R$ equals the given number 60. Dividing 60 by 5 gives $1R = 12$, the first number; and so the other two make 48. Again, if the third is set = $1R$, the other two together make $9R$, so that they can have a ninefold ratio to the third. Therefore $10R$ equals the given number 60. Dividing 60 by 10 gives $1R = 6$, the third number. The middle is then 42, which is left when the third, 6, is taken from 48. Hence the three numbers sought are 12, 42, 6, which together make up 60; and the first together with the middle makes

54, ninefold the third, 6; while the third together with the middle makes 48, which is quadruple the first, 12.

Next let the same number 100 be distributed into three, so that both the first together with the second has a triple ratio to the third, and the third together with the second has a triple ratio to the first. Set the first = $1R$, and the second and third together = $3R$, so that they have a triple ratio to the first. Their sum, $4R$, equals 100. Dividing 100 by 4 gives $1R = 25$, the first number; and so the second and third together make 75. Next set the third = $1R$, so that the second is $75 - 1R$. Then the first together with the second makes the sum $100 - 1R$, triple the third — that is, triple $1R$, or equal to $3R$. Adding $1R$ to each side, there is equality between 100 and $4R$. Dividing 100 by 4 gives $1R = 25$, the third number; and accordingly the second is 50, namely $75 - 1R$. So the three numbers are 25, 50, 25. And both the sum of the first and second has a triple ratio to the third, and the sum of the third and second a triple ratio to the first.

If both ratios were required to be quadruple, the three numbers found would be 20, 60, 20.

Problem 54. [Margin: LIIII.] *To find three numbers, such that the first together with the second has a given ratio to the third; likewise the third together with the second has a given ratio to the first; and all three together make up a given sum.*

This puzzle is no different from the preceding one. For if the given sum is divided into three numbers as prescribed, the puzzle will be handled just as the one before.

Problem 55. [Margin: LV.] *To find a number such that the square described on a given aliquot part of it, together with the square of any other aliquot part of it, makes up the number itself — or another number that has a given ratio to the number found.*

Let a number be sought such that the square described on its half, together with the square described on its sixth part, produces the number itself — or another number that has to the number found the ratio of eight-and-three-quarters to one [*octupla supertripartiens quartas*, i.e. $8\frac{3}{4} : 1$]. Set the number = $6R$, so that it may have its parts expressed without fractions. Its half is $3R$ and its sixth part is $1R$. The squares of these parts are $9\bar{3}$ [census, the square] and $1\bar{3}$, which together make $10\bar{3}$, a number equal to $6R$. Dividing 6 by 10 gives $1R = \frac{3}{5}$ (because the cossic terms are collateral in degree). And so $6R$ comes to $\frac{18}{5}$, the number sought. Its half, $\frac{9}{5}$, makes the square $\frac{81}{25}$; and its sixth part, $\frac{3}{5}$, makes the square $\frac{9}{25}$; which together with $\frac{81}{25}$ makes $\frac{90}{25}$ — that is, $\frac{18}{5}$, the number sought.

But now let the two squares be required to make a number that has to the number found the ratio of eight-and-three-quarters to one. Set the number = $1R$. Its half is $\frac{1}{2}R$ and its sixth part is $\frac{1}{6}R$. The squares of these parts are $\frac{1}{4}3$ and $\frac{1}{36}3$, which together make $\frac{10}{36}3$. This number must have to $1R$ the ratio eight-and-three-quarters to one, whose denominator [ratio-value] is $8\frac{3}{4}$. This, multiplied into $1R$, makes $8\frac{3}{4}R$, to which the number $\frac{10}{36}3$ must be equal. Dividing $8\frac{3}{4}$ by $\frac{10}{36}$ gives $1R = 31\frac{1}{2}$, or $\frac{63}{2}$, the number found (because the cossic terms are collateral). For its half, $\frac{63}{4}$, makes the square $\frac{3969}{16}$; and its sixth part, $\frac{63}{12}$, makes the square $\frac{3969}{144}$; which together with the prior square $\frac{3969}{16}$ — that is, with $\frac{35721}{144}$ — makes the number $\frac{39690}{144}$. This has to the number found, $\frac{63}{2}$, the ratio of eight-and-three-quarters to one; for when the former is divided by the latter, the quotient comes to $8\frac{3}{4}$.

Problem 56. [Margin: LVI.] *To divide a given number into two, such that a given part of the first together with the second makes just as much as another given part of the second together with the first.*

Let the given number 44 be cut into two, so that $\frac{1}{5}$ of the first together with the second, and $\frac{1}{3}$ of the second together with the first, make two equal numbers. Set the first = $1R$, and so the second is $44-1R$. The fifth part of the first, namely $\frac{1}{5}R$, together with the second makes $44-\frac{4}{5}R$. And the third part of the second, namely $(44-1R)/3$, together with the first makes $1R+(44-1R)/3$. There is therefore an equation between $44-\frac{4}{5}R$ and $1R+(44-1R)/3$. Removing $1R$ from each side, the equation is between $44-\frac{9}{5}R$ and $(44-1R)/3$. This equation, by cross-multiplication (taking 1 as the denominator of the first, so that it becomes the fraction $(44-\frac{9}{5}R)/1$), is reduced to an equality between $132-\frac{27}{5}R$ and $44-1R$. Adding $\frac{27}{5}R$ to each side, between 132 and $44+\frac{22}{5}R$. Removing 44 from each side, between 88 and $\frac{22}{5}R$. Dividing 88 by $\frac{22}{5}$ gives $1R = 20$, the first number; and the second is 24. Here you see that the fifth part of the first, namely 4, together with the second makes 28 — just as much as the third part of the second, namely 8, together with the first.

Problem 57. [Margin: LVII.] *To find two numbers such that the first, on receiving any number from the second, makes a number that has a given ratio to what remains of the second; and likewise the second, on receiving any number from the first, makes a number that has a given ratio to what remains of the first.*

Let two numbers be sought such that the first, on receiving 40 from the second, makes a number four times the remainder of the second, while the second, on receiving 30 from the first, makes a number nine times the residue of the first. Set the first = $1R$. If it then receives 40 from the second, it will have $1R+40$; and

accordingly the second retains $\frac{1}{4}R+10$, so that the former has to the latter a quadruple ratio. Therefore, before the second gave 40 to the first, it had $\frac{1}{4}R+50$. And so, if it receives 30 from the first, it will have $\frac{1}{4}R+80$. This number must have to the remainder of the first — namely to $1R-30$ — a ninefold ratio. There is therefore an equation between $\frac{1}{4}R+80$ and $9R-270$. Adding 270 to each side, between $\frac{1}{4}R+350$ and $9R$. Removing $\frac{1}{4}R$ from each side, between 350 and $8\frac{3}{4}R$. Dividing 350 by $8\frac{3}{4}$ gives $1R = 40$, the first number; and the second is then 60, namely $\frac{1}{4}R+50$. So the first, 40, on receiving 40 from the second, has 80, which is quadruple the number 20 that remains to the second. And the second, 60, on receiving 30 from the first, has 90, which is ninefold the number 10 left to the first after it has given 30 to the second.

If two numbers are to be found such that the first, on receiving 10 from the second, has a number five times the remainder of the second, while the second, on receiving 5 from the first, has a number twice the remainder of the first, you will find the two numbers 15 and 15. For, setting the first = $1R$, if it receives 10 from the second it will have $1R+10$, which is five times the number $\frac{1}{5}R+2$; and this is what is left over to the second after it has given 10 to the first, so it previously had $\frac{1}{5}R+12$. If it then receives 5 from the first, it will have $\frac{1}{5}R+17$, which must be twice the number $1R-5$ that is left over to the first after giving 5 to the second. That is, there will be an equality between $\frac{1}{5}R+17$ and $2R-10$. Adding 10 to each side, between $\frac{1}{5}R+27$ and $2R$. And if $\frac{1}{5}R$ is taken from each side, between 27 and $1\frac{4}{5}R$. Dividing 27 by $1\frac{4}{5}$ — that is, by $\frac{9}{5}$ — gives $1R = 15$ for the first number. And since the second had $\frac{1}{5}R+12$, the second number is likewise 15. These two numbers, 15 and 15, satisfy the question. For if the first receives 10 from the second, it will have 25, which is five times the remainder left to the second, namely the number 5. And the second, on receiving 5 from the first, will have 20 — twice the number 10 that is left over to the first.

Problem 58. [Margin: LVIII.] *To find three numbers such that any two taken together make three proposed numbers — provided that half the sum of the proposed numbers is greater than each of the proposed numbers individually.*

Let the first together with the second be required to make 20, the second with the third 30, and the third with the first 40; these all together make the number 90, whose half, 45, is greater than 20, than 30, and than 40. But if the third result had to be 50 or greater, the thing would not work out, because all three would make 100 or a larger number, whose half would be either equal to one of them, namely 50, or less. For instance, if the third were 51 or 60, all three would be 101 or 110, whose half — $50\frac{1}{2}$ or 55 — would be less than the third, 51 or 60. But the working

itself will show you well enough whether the question is possible or not. So set the sum of the three numbers sought = $1R$. Since the first and second make 20, the third is $1R-20$. Likewise, since the second and third make 30, the first is $1R-30$. Finally, since the third and first make 40, the second is $1R-40$. All three together make $3R-90$, and this sum must equal the sum of all, which we set = $1R$. There is therefore an equation between $1R$ and $3R-90$. Adding 90 to each side, between $1R+90$ and $3R$. Removing $1R$ from each side, between 90 and $2R$. Dividing 90 by 2 gives $1R$

= 45, the sum of all three numbers sought. And since the first was found to be $1R-30$, the first itself is 15. The second is 5, namely $1R-40$. And the third is 25, namely $1R-20$. So the three numbers found, in order, are 15, 5, 25; and it is clear that the first and second make 20, the second and third 30, and the third and first 40.

Alternatively. Set the first = $1R$. Then the second is $20-1R$, so that together with the first, namely with $1R$, it can make 20 — which second number will be had if the first, $1R$, is taken from 20. The third, meanwhile, is $10+1R$, so that together with the second — that is, with $20-1R$ — it can make 30; and this third is had if the second, $20-1R$, is taken from 30. Now the third, $10+1R$, together with the first, $1R$, makes $2R+10$, which number must equal 40. Removing 10 from each side, there is an equation between $2R$ and 30. Dividing 30 by 2 gives $1R = 15$ for the first number. The second, $20-1R$, is then 5. And the third, $10+1R$, is 25, as before.

Problem 59. [Margin: LIX.] *To find four numbers, such that each three of them make four proposed numbers — provided that a third of the sum of the proposed numbers is greater than each proposed number individually.*

Let the first, second, and third be required to make 20; the second, third, and fourth 22; the third, fourth, and first 24; and finally the fourth, first, and second 27. These all four together make 93, whose third part, 31, is greater than 20, than 22, than 24, and than 27. But if the fourth had to be 40, the thing would not work out, because all four would make 106, whose third part, $35\frac{1}{3}$, is not greater than 40 but less. Furthermore, the working itself will indicate the impossibility of the question. Set the sum of all four = $1R$. Since the first, second, and third make 20, the fourth is $1R-20$. By the same reasoning the first is $1R-22$, the second $1R-24$, and finally the third $1R-27$. All four together make $4R-93$, and this sum must equal the sum of all, which we set = $1R$. There is therefore an equation between $1R$ and $4R-93$. Adding 93 to each side, between $1R+93$ and $4R$. Removing $1R$ from each side, between 93 and $3R$. Dividing 93 by 3 gives $1R = 31$. And since the first

was found to be $1R-22$, the first itself is 9. For the same reason the second is 7, the third 4, and the fourth 11 — which four numbers carry out the problem.

This puzzle we shall solve another way, with five numbers, in Puzzle 127.

SCHOLIUM

By the same art, five numbers will be found such that each four of them make five proposed numbers — provided that a fourth of the sum of the proposed numbers is greater than each proposed number individually. Likewise six numbers, provided that a fifth part is greater, and so on. And thus, in succession, any number of numbers whatever, so long as the part of the sum determined by the number of proposed numbers, less one, is greater than each of the proposed numbers individually.

Problem 60. [Margin: LX.] *To find three numbers, such that any two taken together exceed the remaining one by a given number.*

Let it be laid down that the first and second exceed the third by 20 units; that the excess of the second and third together over the first is 30 units; and finally that the excess of the third and first together over the second is 40 units. Set all three = $1R$. Since the first and second exceed the third by 20 units, then — adding the common third [Margin (a): 16. prop[ositione] ... primi (abbreviation uncertain) [?]] — the first, second, and third together will also exceed twice the third by those same 20 units. And so all three are equal to twice the third together with 20, since the number composed of twice the third and 20 also exceeds twice the third by 20 units. If, then, from the sum of all three, which we set = $1R$, we take away 20, there will remain twice the third, $1R-20$; and accordingly the third by itself is $\frac{1}{2}R-10$. By the same reasoning the first is $\frac{1}{2}R-15$, and the second $\frac{1}{2}R-20$. The sum of all three is therefore $\frac{3}{2}R-45$, which must equal the sum of the three numbers sought, which we set = $1R$. So the equation is between $1R$ and $\frac{3}{2}R-45$. Adding 45 to each side, between $1R+45$ and $\frac{3}{2}R$. Removing $1R$ from each side, between 45 and $\frac{1}{2}R$. Dividing 45 by $\frac{1}{2}$ gives $1R = 90$. And since the first was found to be $\frac{1}{2}R-15$, the first itself is 30. For the same reason the second is 25 and the third 35 — which three numbers carry out the problem.

Alternatively. Set the third = $1R$. Since the first and second exceed the third by 20 units, the first and second together are $1R+20$; and this number likewise exceeds the second by 20 units. Now set the second = 25, namely the half of the sum 50 collected from 20 and 30. Since the first and second together were found to be $1R+20$, if the second, 25, is subtracted, there remains the first, $1R-5$. But since the

third together with the first must exceed the second by 40 units — and the third, $1R$, and the first, $1R-5$, make the number $2R-5$, which will equal 65, since this exceeds the second, 25, by 40 units — the equation will be between $2R-5$ and 65. Adding 5 to each side, between $2R$ and 70. Dividing 70 by 2 gives $1R = 35$, the third number. The first is 30, namely $1R-5$. And the second is 25, as above.

Alternatively. Set the first and second together = $1R$. Then the third is $1R-20$, so that the first and second together — that is, $1R$ — exceed the third by 20 units; which third is had if 20 is taken from the first and second, $1R$. Therefore all three are $2R-20$. And since the second and third must exceed the first by the number 30 — adding the common first — all three will exceed twice the first by that same number 30. Removing 30, then, from all three, that is from $2R-20$, there will remain twice the first, $2R-50$; and so the first by itself is $1R-25$. The first, $1R-25$, and the third, $1R-20$, thus make the sum $2R-45$; and this, subtracted from all three — that is, from $2R-20$ — leaves the second, 25. And since the first and third make, as was just said, the sum $2R-45$, which must exceed the second, 25, by the number 40: if the second, 25, is subtracted from the aforesaid sum, there remains the number $2R-70$, equal to the said excess, 40. Adding 70 to each side, there is an equation between $2R$ and 110. Dividing 110 by 2 gives $1R = 55$. The first number, then, $1R-25$, is 30. The second is 25. And the third, $1R-20$, is 35, as before.

Algebra — Chapter XXIX ("Various Puzzles" / *Aenigmata varia*)

Scholium [to Problem 60, continued]

If the first and second together must exceed the third by 3 units, the second and third exceed the first by 20 units, and finally the first and third exceed the second by 1 unit, then the third number is found to be $10\frac{1}{2}$, the second $11\frac{1}{2}$, and the first 2.

Problem 61 [61 / LXI]

To find four numbers such that every three of them taken together exceed the remaining one by a given number — provided that half of the sum collected from the four required excesses is greater than each of the given numbers.

Let it be required that the first, second, and third together exceed the fourth by 20 units; the second, third, and fourth exceed the first by 30 units; the third, fourth, and first exceed the second by 40 units; and finally the fourth, first, and second

exceed the third by 50 units. Here you see that the four given numbers make 140, whose half, 70, is greater than each of the given numbers. For if this condition is not met, the problem is impossible. Thus, if the last excess were given as 90, all the excesses together would make 180, whose half, 90, is not greater than the given excess 90, and the problem would be unsolvable.

Therefore let all four numbers together be posited as $1R$ [1 *res*, the unknown]. Now since the first, second, and third exceed the fourth by 20 units, then — adding the fourth to both sides in common — the first, second, third, and fourth together will likewise exceed twice the fourth by the same 20 units. [Margin (a): *pronunciatum* (axiom) 16 of Book I — Euclid, *Elements* I.] Therefore all four together are equal to twice the fourth plus 20, because a number composed of twice the fourth and 20 also exceeds twice the fourth by 20 units.

If, then, from the sum of all four — which we posited as $1R$ — we subtract 20, there remains twice the fourth: $1R - 20$; and so the fourth by itself is $\frac{1}{2}R - 10$. By the same reasoning the first is $\frac{1}{2}R - 15$, the second $\frac{1}{2}R - 20$, and the third $\frac{1}{2}R - 25$. The sum of all four is therefore $2R - 70$, which is equal to the earlier sum of all four that we posited as $1R$. So the equation is between $1R$ and $2R - 70$. Adding 70 to each side: between $1R + 70$ and $2R$. Removing $1R$ from each side: between 70 and $1R$. Therefore $1R = 70$. Since the first was found to be $\frac{1}{2}R - 15$, the first itself is 20; and for the same reason the second is 15, the third 10, and the fourth 25 — all of which satisfy the problem.

Another method. Let the fourth be posited as $1R$. Since the first, second, and third exceed the fourth by 20 units, those three together are $1R + 20$, this number exceeding $1R$ by 20 units. Again, since the second, third, and fourth exceed the first by 30 units, let the sum of the second and third be posited as 25 — namely half of the first two excesses, 20 and 30, together. Since, then, the first, second, and third make (as we said) $1R + 20$, if we subtract the second and third together, namely 25, the remainder — the first — is $1R - 5$. And the sum $1R + 25$, collected from the second, third, and fourth, exceeds the first, $1R - 5$, by 30, as is evident if the first, $1R - 5$, is subtracted from $1R + 25$, the sum of the second, third, and fourth.

Again, since the first was found to be $1R - 5$, the second and third make 25, and the fourth was posited as $1R$, all four together are $2R + 20$. Since the third, fourth, and first exceed the second by 40, then — adding the second to both sides in common — all four together exceed twice the second by the same 40. [Margin (b): *pronunciatum* (axiom) 16 of Book I — Euclid, *Elements* I.] Therefore, subtracting 40 from the sum $2R + 20$ of all four, there remains twice the second: $2R - 20$; and

the second taken once is $1R - 10$. Subtracting this from 25, the sum of the second and third, leaves the third: $35 - 1R$.

Now since the fourth, first, and second make $3R - 15$, and must exceed the third by 50, and they exceed the third, $35 - 1R$, by $4R - 50$ — as is clear if the third, $35 - 1R$, is subtracted from the aforesaid sum $3R - 15$ — the equation is therefore between $4R - 50$ and 50. Adding 50 to each side: between $4R$ and 100. Dividing 100 by 4 gives $1R = 25$. Therefore the first, $1R - 5$, is 20; the second, $1R - 10$, is 15; the third, $35 - 1R$, is 10; and the fourth, $1R$, is 25 — just as before.

Another method. Let the sum of all be posited as $1R$. Since the first three must exceed the fourth by 20, we must divide $1R$ into two parts such that the greater exceeds the smaller by 20; for the greater part will be the sum of the first three, and the smaller part the fourth number. This division is carried out as we set out in the scholium to the first problem: subtract 20 from $1R$, so that the remainder is $1R - 20$; half of this, $\frac{1}{2}R - 10$, is the smaller part; and if the same number 20 is added to it, the greater part becomes $\frac{1}{2}R + 10$. Therefore the fourth number is $\frac{1}{2}R - 10$, and the first three [?] make $\frac{1}{2}R + 10$.

By the same reasoning, since the second, third, and fourth must exceed the first by 30, subtract 30 from $1R$: half of the remaining number $1R - 30$, namely $\frac{1}{2}R - 15$, is the first number. Likewise, since the third, fourth, and first must exceed the second by 40, if 40 is subtracted from $1R$, half of the remaining number $1R - 40$, namely $\frac{1}{2}R - 20$, is the second number. Finally, since the fourth, first, and second must exceed the third by 50, if 50 is taken from $1R$, half of the remaining number $1R - 50$, namely $\frac{1}{2}R - 25$, is the third number.

All four found — $\frac{1}{2}R - 15$, $\frac{1}{2}R - 20$, $\frac{1}{2}R - 25$, and $\frac{1}{2}R - 10$ — make the sum $2R - 70$, which must equal the sum of all that we posited as $1R$. So the equation is between $1R$ and $2R - 70$. Adding 70 to each side: between $1R + 70$ and $2R$.

Removing $1R$ from each side: between 70 and $1R$. And so $1R = 70$, and $\frac{1}{2}R = 35$. Therefore the first, $\frac{1}{2}R - 15$, is 20; the second, $\frac{1}{2}R - 20$, is 15; the third, $\frac{1}{2}R - 25$, is 10; and the fourth, $\frac{1}{2}R - 10$, is 25 — just as before.

Scholium [to Problem 61]

In the same way five numbers may be found such that every four of them exceed the remaining one by a given number — provided that a third of the sum of the five required excesses is greater than each of the given numbers. By the same reasoning six numbers may be found, provided that a fourth of the sum is greater, and so on. And so on for any number of numbers whatever, provided

only that the fraction of the sum — the fraction denominated by the number of the given excesses less two — is greater than each of the given excesses.

Problem 62 [62 / LXII]

To find three numbers such that the greatest exceeds the middle by a given part of the least; the middle likewise exceeds the least by a given part of the greatest; and finally the least exceeds a given part of the middle by a given number.

Let the greatest exceed the middle by a third of the least; the middle exceed the least by a third of the greatest; and finally let the least exceed a third of the middle by 10 units. Let the least be posited as $1R + 10$, and the middle as $3R$. For in this way the least exceeds a third of the middle by 10.

And since the middle must exceed the least by a third of the greatest — and the middle, $3R$, exceeds the least, $1R + 10$, by $2R - 10$, as is evident if the subtraction of $1R + 10$ from $3R$ is carried out — a third of the greatest is therefore $2R - 10$, and so the greatest is $6R - 30$. Again, since the greatest must exceed the middle by a third of the least — and the greatest, $6R - 30$, exceeds the middle, $3R$, by $3R - 30$, which is equal to a third of the least — the least is therefore $9R - 90$.

But the least was posited as $1R + 10$. So the equation is between $9R - 90$ and $1R + 10$. Adding 90 to each side: between $9R$ and $1R + 100$. Removing $1R$ from each side: between $8R$ and 100. Dividing 100 by 8 gives $1R = 12\frac{1}{2}$. Therefore the least, $1R + 10$, is $22\frac{1}{2}$; the middle, $3R$, is $37\frac{1}{2}$; and finally the greatest, $6R - 30$, is 45. For the greatest, 45, exceeds the middle, $37\frac{1}{2}$, by $7\frac{1}{2}$, which is a third of the least, $22\frac{1}{2}$. The middle, $37\frac{1}{2}$, exceeds the least, $22\frac{1}{2}$, by 15, which is a third of the greatest, 45. And finally the least, $22\frac{1}{2}$, exceeds $12\frac{1}{2}$ — a third of the middle, $37\frac{1}{2}$ — by 10, as was proposed.

Now if the greatest must exceed the middle by half the least, the middle exceed the least by a quarter of the greatest, and the least exceed a third of the middle by 8: let the least be posited as $1R + 8$, and the middle as $3R$, so that the least exceeds a third of the middle by 8. And since the middle is set to exceed the least by a quarter of the greatest — and the middle, $3R$, exceeds the least, $1R + 8$, by $2R - 8$ — a quarter of the greatest is therefore $2R - 8$, and so the greatest is $8R - 32$. Again, since the greatest must exceed the middle by half the least — and the greatest, $8R - 32$, exceeds the middle, $3R$, by $5R - 32$, which is equal to half the least — the least is therefore $10R - 64$.

But the same least was posited as $1R + 8$. So the equation is between $10R - 64$ and $1R + 8$. Adding 64 to each side: between $10R$ and $1R + 72$. Removing $1R$ from each

side: between $9R$ and 72 . Dividing 72 by 9 gives $1R = 8$. And so the least, $1R + 8$, is 16 ; the middle, $3R$, is 24 ; and finally the greatest, $8R - 32$, is 32 .

If someone were to propose that the greatest must exceed the middle by half the least, the middle likewise exceed the least by half the greatest, and finally the least exceed half the middle by 2 , the problem would be impossible, as the very working through Algebra shows plainly enough. For let the least be posited as $1R + 2$. Since it must exceed half the middle by 2 , the middle is $2R$. And since the middle must exceed the least by half the greatest — and the middle, $2R$, exceeds the least, $1R + 2$, by $1R - 2$, which must be half the greatest — the greatest is therefore $2R - 4$. Which is absurd, since the middle was found to be $2R$, which is greater than $2R - 4$.

Problem 63 [63 / LXIII]

To find three numbers such that, if each gives a given part of itself to the next in order, then among those who have given and those who have received, equality is found.

Let the first give the second a third of itself; the second give the third a fourth of itself; and the third give the first a fifth of itself. And when they have mutually given the said parts and received from the others, let the three numbers become equal. Let the first be posited as $3R$, so that it has a third part without a fraction; and the second as 4 units, so that it has a fourth part without a fraction. So if the second, 4 , receives a third of the first, $3R$ — namely $1R$ — and gives a fourth of itself, namely 1 , to the third, the second itself has $3 + 1R$.

And the first must have just as much, after it has given and received what is required. Now the first, $3R$, if it gives a third part, $1R$, and receives $3 - 1R$, will then have $1R + 3$, or $3 + 1R$, as this scheme shows:

$$\begin{array}{r}
 2R + 0 \text{ (the first, after giving away its third: } 3R - 1R) \\
 - 1R + 3 \text{ (what it receives: } 3 - 1R) \\
 \hline
 + 1R + 3
 \end{array}$$

You find $3 - 1R$ by subtracting the remainder of the first, namely $2R$, from $3 + 1R$. Therefore $3 - 1R$ is a fifth of the third, since giving it to the remainder of the first, namely $2R$, makes $3 + 1R$. The third is therefore $15 - 5R$; and if it gives the first a fifth of itself — that is, $3 - 1R$ — there remain to it $12 - 4R$. And if it receives from the second, 4 , a fourth part, namely 1 , it will have $13 - 4R$, a number that must equal $3 + 1R$ — namely, as much as both the first and the second have, as was said.

So the equation is between $13 - 4R$ and $3 + 1R$. Adding $4R$ to each side: between 13 and $3 + 5R$. Removing 3 from each side: between 10 and $5R$. Dividing 10 by 5 gives $1R = 2$. And therefore the first number, $3R$, is 6 ; the second, 4 , as we posited; and the third, $15 - 5R$, is 5 . For if the first, 6 , gives a third of itself, 2 , and receives a fifth of the third, namely 1 , it will have 5 . The second, 4 , will have just as much if it gives a fourth of itself, 1 , to the third and receives a third of the first. Likewise the third, 5 , will also have the same amount if it gives the first a fifth, that is 1 , and receives from the second, 4 , a fourth, namely 1 — which is what was proposed.

Now if the first must give the second $\frac{1}{2}$, the second give the third $\frac{1}{4}$, and the third give the first $\frac{1}{3}$: let the first be posited as $2R$, so that it has $\frac{1}{2}$ without a fraction, and the second as 4 , so that it has $\frac{1}{4}$ without a fraction. The second, then, after it has given a quarter to the third and taken half from the first, will have $1R + 3$. And the first must have just as much, after it has given half to the second and received a third from the third. But the first, after it has given half to the second, will have $1R$ remaining, which together with 3 makes $1R + 3$. So 3 is a third of the third, and therefore the third is 9 . It remains that the third, 9 , after it has given a third to the first and taken a quarter from the second, should also have $1R + 3$. But if it does what is required, it will have 7 . So the equation is between 7 and $1R + 3$. Removing 3 from each side: between 4 and $1R$. Therefore $1R = 4$, and so the first, which we posited as $2R$, is 8 ; the second is 4 ; and the third is 9 .

For the first, 8 , if it gives $\frac{1}{2}$ and receives a third of the third, will have 7 — as will the second if it gives $\frac{1}{4}$ and receives half of the first, and likewise the third if it gives $\frac{1}{3}$ and takes a quarter of the second. And all three numbers make 21 , three times the number 7 that each one produces — just as in the first example the three numbers $6, 4, 5$ make 15 , three times the number 5 that each produces. Moreover, in this second example of the problem you see that it is not necessary for the given parts to have denominators that increase successively in order, as happened in the first example.

Problem 64 [64 / LXIII]

To find four numbers under this condition: that when each of them gives a given part of itself to the next in order, then among those who have given and those who have received, equality is found.

Let the first give the second a third of itself; the second give the third a fourth; the third give the fourth a fifth; and the fourth give the first a sixth. And at last, after they have given and received the said parts, let the four numbers become equal.

Let the first be posited as $3R$, so that it has a third part without a fraction; and the second as 4 units, so that it has a fourth part without a fraction. So if the second receives a third of the first, $3R$ — namely $1R$ — and gives a quarter, namely 1, to the third, it will have $1R + 3$. And the first must have just as much, after it has given the second a third, namely $1R$, and received a sixth part from the fourth.

But if it gives a third, namely $1R$, there remain to it $2R$, which together with $3 - 1R$ make up $1R + 3$, as is gathered from this scheme:

$$\begin{array}{r}
 2R + 0 \text{ (the first, after giving away its third: } 3R - 1R) \\
 - 1R + 3 \text{ (what it receives, a sixth of the fourth: } 3 - 1R) \\
 \hline
 + 1R + 3
 \end{array}$$

You find $3 - 1R$ by subtracting $2R$ from $1R + 3$. Therefore a sixth of the fourth is $3 - 1R$, and the fourth itself is $18 - 6R$. This fourth, if it gives a sixth, will have $15 - 5R$ remaining; and if it receives a fifth of the third, it must have $1R + 3$, just as both the first and the second do. But if $6R - 12$ is added to $15 - 5R$, the result is $1R + 3$, as appears in this scheme:

$$\begin{array}{r}
 15 - 5R \\
 - 12 + 6R \text{ (what it receives, a fifth of the third: } 6R - 12) \\
 \hline
 + 3 + 1R
 \end{array}$$

You find $6R - 12$ by subtracting $15 - 5R$ from $1R + 3$. Therefore a fifth of the third is $6R - 12$, and the third itself is $30R - 60$. It now remains that this third also have $1R + 3$, after it has given the fourth a fifth — namely $6R - 12$ — and received a quarter of the second, 4. But if it gives a fifth, there remain to it $24R - 48$; and if it receives from the second a quarter, namely 1, it will have $24R - 47$, a number that must equal $1R + 3$. Adding 47 to each side, the equality is between $24R$ and $1R + 50$. Removing $1R$ from each side: between $23R$ and 50. Dividing 50 by 23 gives $1R = 50/23$.

And so the first number, posited as $3R$, is $150/23$; the second, 4, or $92/23$; the third, $30R - 60$, is $120/23$; and finally the fourth, $18 - 6R$, is $114/23$ — and these make up the puzzle. And if you remove the denominators, the four whole numbers sought are 150, 92, 120, 114, because the numerators, on account of the common denominator, have the same ratio as the fractions themselves, as we proved at the end of Book 9 of Euclid. [Margin (a): (Euclid, *Elements*) V.15.] And the parts expressed in the problem stand in the same relation to the whole numbers as to the said fractions; for a third of 12 stands to 12, for example, as a third of 90 stands to 90, and so on.

And so, if the first, 150, gives a third, namely 50, there remain 100; and if it receives a sixth of the fourth, 114, namely 19, it will have 119. Likewise, if the second, 92, gives a fourth of its number, namely 23, there remain 69; and if it receives a third of the first, 150, namely 50, it will likewise have 119. So too, if the third, 120, gives a fifth of its number, that is 24, there remain 96; and if it receives a quarter of the second, 92, namely 23, it will have 119. Finally, in the same way, if the fourth, 114, gives a sixth of its number, namely 19, there remain 95; and if it receives a fifth of the third, 120, namely 24, it will have 119. Here too all four numbers found — 150, 92, 120, 114 — make 476, four times the number 119 that each one produces.

Scholium [to Problem 64]

In the same way five numbers, or more, may be found such that, after each has given the next a given part of itself, the five numbers (or more) finally become equal. And all together will always make a sum that contains the number each one produces as many times over as there are numbers themselves.

For suppose five numbers are to be found such that: the first gives $\frac{1}{3}$ to the second and receives $\frac{1}{5}$ from the fifth; the second gives $\frac{1}{2}$ to the third and receives $\frac{1}{3}$ from the first; the third gives $\frac{1}{7}$ to the fourth and receives $\frac{1}{2}$ from the second; the fourth gives $\frac{1}{6}$ to the fifth and receives $\frac{1}{7}$ from the third; and finally the fifth gives $\frac{1}{5}$ to the first and receives $\frac{1}{6}$ from the fourth — so that at last the five numbers become equal. Let the first be posited as $3R$, so that it has $\frac{1}{3}$ without a fraction, and the second as 2, so that it has $\frac{1}{2}$ without a fraction.

So if the second gives $\frac{1}{2}$ to the third, that is 1, there remains to it 1; and if it receives $\frac{1}{3}$ from the first, namely $1R$, it will have $1R + 1$, the number that each one must at last have. Now for the first: the fifth [i.e. a fifth of the fifth number] must, together with the first, make $1R + 1$. And $2R$ together with $1 - 1R$ make $1R + 1$, as is evident if $2R$ [is subtracted from $1R + 1$]. So a fifth of the fifth is $1 - 1R$, and the whole fifth is $5 - 5R$. This fifth, if it gives $\frac{1}{5}$, has $4 - 4R$ remaining; and $4 - 4R$ together with $5R - 3$ make the number $1R + 1$, as is clear if $4 - 4R$ [is subtracted from $1R + 1$]. So a sixth of the fourth number is $5R - 3$, and the whole fourth is $30R - 18$. This fourth, if it gives $\frac{1}{6}$, leaves $25R - 15$; and $25R - 15$ together with $16 - 24R$ make the number $1R + 1$, as is evident if $25R - 15$ is subtracted from $1R + 1$. Therefore $16 - 24R$ is a seventh of the third number, and the whole third is $112 - 168R$.

This third, if it gives $\frac{1}{7}$ to the fourth, leaves $96 - 144R$, which together with $\frac{1}{2}$ of the second — that is, with 1 — make $97 - 144R$, and this must equal $1R + 1$. So the

equation is between 97 and $145R + 1$. Removing 1 from each side: between 96 and $145R$. Dividing 96 by 145 gives $1R = 96/145$. Therefore the first number, posited as $3R$, is $288/145$; the second, posited as 2 , is $290/145$; the third [?] [printed *Tertius*], found as $30R - 18$, is $270/145$; and finally the fifth, found as $5 - 5R$, [is $245/145$]. And when the denominators are removed, the five whole numbers found are 288 , 290 , 112 , 270 , 245 , which satisfy the problem. For if they give and receive the given parts, each will have 241 — a number which, multiplied by five, makes 1205 , the sum of all five numbers found.

Problem 65 [65 / LXV]

To divide a given number into two parts, such that the latter part, together with a given aliquot part of the former, makes any given number.

Let 60 be the number to be divided into two, such that the latter, together with a fifth of the former, makes 28 . Let the former part be posited as $5R$, so that it can give a fifth without a fraction; then the latter part will be $60 - 5R$. If this receives a fifth of the former, namely $1R$, it will have $60 - 4R$, a number that must equal the given number 28 . Adding $4R$ to each side gives the equation between 60 and $4R + 28$. Removing 28 from each side: between 32 and $4R$. Dividing 32 by 4 gives $1R = 8$. And therefore the former part, $5R$, is 40 , and the latter is 20 — which, if it receives a fifth of the former, namely 8 , will have 28 .

Scholium [to Problem 65]

You will achieve the same thing without Algebra in this way. Subtracting the given number that is to be produced from the given number that is to be divided, let the remaining number be divided by the [denomi- ...]

(Continuation of the Scholium to Problem 65.)

the denominator of the given part, less one. For the quotient, multiplied by the denominator of the part, will yield the prior number, and so on. In the worked example, if the given number 28 is subtracted from the number 60 being divided, and the remaining number 32 is divided by 4 (namely by the number one less than the denominator of the fifth part), the quotient will be 8 ; and this, multiplied by 5 , the denominator of the fifth part, produces the prior number 40 . Therefore the posterior number will be 20 .

If the same number 60 is to be cut into two parts, so that the posterior together with $1/5$ of the prior makes 59 : subtract 59 from 60 , and divide the remaining number 1 by 4 ; the quotient is $1/4$, which multiplied by 5 gives $1\frac{1}{4}$ as the prior

number. Therefore the posterior will be $58\frac{3}{4}$. For if $\frac{1}{5}$ of the prior — that is, $\frac{1}{4}$ — is added to it, the given number 59 is made up.

66. To find two numbers whose sum is equal to a given number, and such that the posterior, together with a given part of the prior, makes another given number as well. [Margin: LXVI]

This problem coincides with the preceding one. For if the given sum is cut into two parts, as was there prescribed, the two required numbers will be found. Thus, if two numbers are to be found whose sum is 60 and such that the posterior number together with $\frac{1}{5}$ of the prior makes 28, the number 60 is to be divided as already stated, and the required numbers will be 40 and 20.

But if two numbers are to be found whose sum is 60 and such that the posterior together with $\frac{1}{5}$ of the prior makes 59, the prior number will be $1\frac{1}{4}$ and the posterior $58\frac{3}{4}$, as was made clear in the preceding scholium.

67. To find three numbers such that, when each takes a given part from the sum of the other two, three equal numbers result. [Margin: LXVII]

Proceed so that the first takes $\frac{1}{3}$ from the sum of the other two, the second takes $\frac{1}{4}$ from the sum of the other two, and the third takes $\frac{1}{5}$ from the sum of the two remaining, and three equal numbers result. Let the first be set as $1R$ [res — the unknown, x], and let the sum of the other two be set as the number 3, so that it may have a third without a fraction. If, then, the first takes $\frac{1}{3}$ from the other two, it will have $1R + 1$.

Now, since the second, taking $\frac{1}{4}$ from the sum of the other two, must likewise make $1R + 1$, the sum of all three — namely $1R + 3$ — must be divided into two parts, of which the posterior together with $\frac{1}{4}$ of the prior makes $1R + 1$. For the posterior part will be the second number, and the prior part the sum of the first and third. This division is carried out, as I taught in the scholium to Problem 65, in this way: from the sum of all three, $1R + 3$, subtract the number $1R + 1$ that is to be produced; divide the remaining number 2 by the denominator of the fourth part less one, namely by 3; and multiply the quotient $\frac{2}{3}$ by the denominator 4. The number produced, $\frac{8}{3}$, will be the prior part — that is, the sum of the first and third. Subtracted from the total sum $1R + 3$, it leaves the posterior part $1R + \frac{1}{3}$ for the second number. For if $\frac{1}{4}$ of the prior part $\frac{8}{3}$ (which is the sum of the first and third) — namely $\frac{2}{3}$ — is added to this, the number $1R + 1$ results.

Again, since the third, taking $\frac{1}{5}$ from the sum of the first and second, must likewise make $1R + 1$, the sum of all three, $1R + 3$, must again be divided in the

same way into two parts, of which the posterior together with $\frac{1}{5}$ of the prior makes $1R + 1$. This is done thus: from the sum $1R + 3$ of all three, subtract the number $1R + 1$ that must be produced; divide the remaining number 2 by the denominator of the fifth part less one — that is, by 4; and let the quotient $[\frac{1}{2}]$ be carried into the denominator 5. The number produced, $\frac{5}{2}$, will be the prior part — namely the sum of the first and second. This part, taken away from the total sum $1R + 3$, leaves as the posterior part $1R + \frac{1}{2}$ [for the third number]. For this number, taking $\frac{1}{5}$ of the sum of the first and second (which was $\frac{5}{2}$) — that is, $\frac{1}{2}$ — likewise makes $1R + 1$.

It now remains that all three numbers found — $1R$, $1R + \frac{1}{3}$, and $1R + \frac{1}{2}$ — should make up the total sum $1R + 3$ of all three. But they make $3R + \frac{5}{6}$. There is therefore an equation between $1R + 3$ and $3R + \frac{5}{6}$. Removing $\frac{5}{6}$ from both sides, we have $2\frac{1}{6}$ against $3R$; and removing $1R$ from both sides, 3 against 2, that is $2R$ against $2\frac{1}{6}$. Dividing $2\frac{1}{6}$ by 2, we get $1R = \frac{13}{12}$, for the first required number, which was set as $1R$. The second, which was found to be $1R + \frac{1}{3}$, will be $\frac{51}{36}$, that is, $\frac{17}{12}$; and the third, which we found to be $1R + \frac{1}{2}$, will be $\frac{38}{24}$, that is, $\frac{19}{12}$.

And if you clear the denominators, thereby solving the proposed problem, the three required numbers will be the whole numbers 13, 17, 19. For the first, 13, together with $\frac{1}{3}$ of the second and third — that is, together with 12 — [makes 25]; the second, 17, together with $\frac{1}{4}$ of the first and third, that is, together with 8; and the third, 19, together with $\frac{1}{5}$ of the first and second, namely together with 6, makes 25.

Here, moreover, the sum of the three numbers found, 49, is not triple the number 25 that each one makes, as happened in Problem 63.

68. *To find four numbers on this condition: that when each takes a given part from the sum of the other three, four equal numbers result.* [Margin: LXVIII]

Let us lay it down that the first takes $\frac{1}{3}$ of the sum of the other three, the second $\frac{1}{4}$ of the sum of the other three, the third $\frac{1}{5}$ of the sum of the three remaining, and finally the fourth $\frac{1}{6}$ of the sum of the others, and that the four numbers become equal. Let the first be set as $1R$, and the sum of the other three as the number 3, which has a third without a fraction. The first, then, taking $\frac{1}{3}$ of this sum — namely 1 — will make $1R + 1$.

And since the second, together with $\frac{1}{4}$ of the sum of the remaining three, must also make $1R + 1$, the sum of all four, $1R + 3$, must be divided into two parts, of which the posterior together with $\frac{1}{4}$ of the prior makes $1R + 1$, just as in the

preceding problem. For the posterior part will be the second number, and the prior part the sum of the first, third, and fourth. In the preceding problem the posterior part was found to be $1R + 1/3$, for the second number.

Next, since the third, taking $1/5$ of the sum of the first, second, and fourth, must likewise produce $1R + 1$, the sum of all four, $1R + 3$, must again be divided into two parts, of which the posterior together with $1/5$ of the prior makes $1R + 1$. In the preceding problem the posterior part was found to be $1R + 1/2$, for the third number.

Finally, since the fourth, taking $1/6$ of the sum of the remaining three, must also make $1R + 1$, we shall divide the sum of all four, $1R + 3$, into two parts, of which the posterior together with $1/6$ of the prior makes $1R + 1$. By the scholium to Problem 65 this is done thus: from the sum $1R + 3$ of all four, subtract the number $1R + 1$ that is to be produced; divide the remaining number 2 by the denominator of the part $1/6$ less one, namely by 5; and let the quotient $2/5$ be carried into the denominator 6 itself. The number produced, $12/5$, will be the prior part — namely the sum of the first, second, and third. This part, subtracted from the sum $1R + 3$ of all four, leaves as the posterior part $1R + 3/5$. For this, together with $1/6$ of the prior — that is, with $2/5$ — makes $1R + 1$.

It now remains that all four numbers found — $1R$, $1R + 1/3$, $1R + 1/2$, $1R + 3/5$ — should make up the total sum $1R + 3$ of all four. But they make $4R + 43/30$. There is therefore an equation between $1R + 3$ and $4R + 43/30$. Removing $1R$ from both sides, between 3 and $3R + 43/30$; and removing $43/30$ from both sides, between $1\frac{17}{30}$ and $3R$. Dividing $1\frac{17}{30}$, that is $47/30$, by 3, we get $1R = 47/90$, for the first required number, which we set to be $1R$. The second, found to be $1R + 1/3$, will be $231/270$, that is, $77/90$. The third, found to be $1R + 1/2$, will be $184/180$, that is, $92/90$. And the fourth, found to be $1R + 3/5$, will be $505/450$, that is, $101/90$.

And if you clear the denominators, the four numbers found will be the whole numbers 47, 77, 92, 101, which satisfy the problem. For the first, 47, together with $1/3$ of the second, third, and fourth — that is, with 90 — [makes 137]; the second, 77, together with $1/4$ of the first, third, and fourth, that is with 60; the third, 92, together with $1/5$ of the first, second, and fourth, that is with 45; and the fourth, 101, together with $1/6$ of the first, second, and third, namely with 36, makes 137.

Here too the sum of the four numbers found, 317, is not quadruple the number 137 that each one makes, as happened in Problem 64.

SCHOLIVM.

In no other way will 5 numbers, or 6, or more be found such that, if each takes a given part from the sum of the rest, the same number of equal numbers results. [Margin: LXIX]

69. *To find three numbers by this rule: that if each gives to the one next in order a commanded part of itself, then, after they have mutually given and received, each has the same given number.* [Margin: LXIX — printed beside the initial of the preceding scholium]

Let three numbers be sought such that, if the first gives $\frac{1}{3}$ to the second, the second $\frac{1}{4}$ to the third, and the third $\frac{1}{5}$ to the first, three numbers 30, 30, 30 result. In Problem 63 we found three numbers under the same condition, except that we did not fix the three equal numbers that ought to result at a definite number, as we do here. Therefore, using the three numbers 6, 4, 5 found through Problem 63, which satisfy that problem — in order to find another three that fulfill the conditions of the present problem — let there be set, for the three numbers, as many roots [res] as there are units in the three numbers already found, namely 6R, 4R, 5R.

If, then, the first, 6R, gives $\frac{1}{3}$, there remain 4R; and if it takes $\frac{1}{5}$ from the third, 5R, it will have 5R, which must equal the commanded number 30. Accordingly, dividing 30 by 5 gives 1R = 6; and therefore the first, 6R, will be 36, the second, 4R, [24], and the third, 5R, 30. For the first, 36, losing $\frac{1}{3}$, will have 24, and taking $\frac{1}{5}$ of the third — namely 6 — will have 30. Likewise the second, 24, giving $\frac{1}{4}$, will have 18, and receiving $\frac{1}{3}$ of the first — namely 12 — will have 30. And finally the third, 30, giving away $\frac{1}{5}$, will have 24, and taking back $\frac{1}{4}$ from the second — namely 6 — will likewise have 30, like the others.

Alternatively. Since each must make up 30, all three required numbers together will be 90. But the three numbers set down, 6R, 4R, 5R, make 15R, which number must equal 90. Dividing 90 by 15 gives 1R = 6, just as before.

SCHOLIVM.

In the same way, in Problems 64, 67, and 68, the number that each must make up can be fixed. For instance, in the preceding Problem 68, suppose each of the four numbers is required to make up 100. Since the four numbers found were 47, 77, 92, 101, we shall set the four numbers to be 47R, 77R, 92R, 101R. The first, together with $\frac{1}{3}$ of the remaining three — namely together with 90R — will make 137R, which must equal the commanded number 100. Dividing 100 by 137

gives $1R = 100/137$. Therefore the first number, $47R$, will be $4700/137$; the second, $7700/137$; the third, $9200/137$; and the fourth, $10100/137$ — which fulfill the condition of the problem, provided the denominators are not thrown away. For the first, $4700/137$, together with $1/3$ of the remaining three — that is, with $9000/137$ — makes $13700/137$, namely 100. So too the second, $7700/137$, together with $1/4$ of the remaining three — that is, with $6000/137$ — makes $13700/137$, that is, 100. The third, together with $1/5$ of the remaining three, and the fourth, together with $1/6$ of the three remaining, make up the same number 100.

Moreover, in Problems 68 and 67, if each of the numbers must make up a certain determined number, the second solution of the present Problem 69 will have no place — because the sum of the three numbers found in Problem 67, or of the four found in Problem 68, is not triple, nor quadruple, the number that each makes up after it has fulfilled what was commanded, as we said there. So that second method can be applied only in Problem 63, as was done in this Problem 69, and also in Problem 64, as you will presently see.

For suppose 5 numbers are to be found such that: the first gives $1/3$ to the second and takes $1/5$ from the fifth; the second gives $1/2$ to the third and takes $1/3$ from the first; the third gives $1/7$ to the fourth and takes $1/2$ from the second; the fourth gives $1/6$ to the fifth and takes $1/7$ from the third; and finally the fifth gives $1/5$ to the first and takes $1/6$ from the fourth — so that at last five equal numbers result, each of which contains 1205 units. In the scholium to Problem 64 there were found, under the very condition here expressed, 5 numbers — 288, 290, 112, 270, 245 — each of which, if it fulfills what is commanded, has 241. But it must, as is here required, have 1205. Let the five numbers therefore be set as $288R$, $290R$, $112R$, $270R$, $245R$. If these numbers carry out what is commanded, each will have $241R$, as was said, which number must equal the given number 1205. Dividing 1205 by 241 gives $1R = 5$. Therefore the first number, set as $288R$, will be 1440; the second, set as $290R$, will be 1450; the third, set as $112R$, will be 560; the fourth, set as $270R$, will be 1350; and finally the fifth, set as $245R$, will be 1225. For if these 5 numbers carry out the commands, you will find that each has 1205.

Alternatively. Since each must make up 1205, the sum of all five will be 6025. But the five numbers set down, $288R$, $290R$, $112R$, $270R$, $245R$, make the sum $1205R$. Dividing 6025 by 1205 gives $1R = 5$, just as before.

70. *To divide a given number into three, four, or more parts, on this condition: that if each gives to the one next in order a commanded part of itself, then, after they have mutually given and received, three, or four, or more equal numbers result.*

[Margin: LXX]

This problem does not differ from the preceding one. For if the given number is distributed into as many equal parts as are specified in the problem, then, by the preceding problem and its scholium (using also Problem 63 and 64 and its scholium), one must find as many numbers as, while preserving the proposed condition, make as many numbers equal to one part of the given number that was divided into that many parts. For the given number is to be divided among these numbers thus found, since all together make up double, triple, or quadruple the number that each of them must make, as was stated.

For example: if the number 90 is to be distributed into three, so that when the first gives $\frac{1}{3}$ to the second, the second $\frac{1}{4}$ to the third, and the third $\frac{1}{5}$ to the first, three equal numbers result — divide the given number 90 into three equal parts, 30, 30, 30, and by Problem 69 find three numbers such that, preserving the prescribed condition, each makes 30. The numbers found will be 36, 24, 30, and the given number 90 is to be distributed among these. For if they observe the prescribed rule, each will make 30.

By the same reasoning, if the number 6025 is to be distributed into five parts, so that when the first gives $\frac{1}{3}$ to the second, the second $\frac{1}{2}$ to the third, the third $\frac{1}{7}$ to the fourth, the fourth $\frac{1}{6}$ to the fifth, and the fifth $\frac{1}{5}$ to the first, each has the same number — the given number is to be divided into five equal parts, 1205, 1205, 1205, 1205, 1205; and by the scholium of the preceding problem (with the help of the scholium to Problem 64) one must find five numbers such that, preserving the prescribed rule, each makes 1205. These, as was shown in the last scholium, are 1440, 1450, 560, 1350, 1225. And the given number 6025 is to be distributed among these.

71. *Given two numbers, to find a third such that, multiplying each of the two given numbers, it makes the greater into a square, but the smaller into the side [root] of that square.* [Margin: LXXI]

Let the two given numbers be 200 and 5, and let the number we are seeking be $1R$. From $1R$ times 200 comes the number $200R$; and from $1R$ times 5 comes the number $5R$, which must be the side of the former, and so, multiplied into itself, will make a number equal to it. Now the number $5R$, multiplied into itself, makes $25\mathfrak{J}$ [census — the square of the unknown, x^2]. There is therefore an equation between $200R$ and $25\mathfrak{J}$. Dividing 200 by 25 gives $1R = 8$, the required number (since the algebraic quantities [numeri Cossici] are of like kind). For this number, multiplying 200, makes the square 1600, whose side is the number 40, produced from 8 times 5.

If the two given numbers are 20 and 15, the number found will be $4/45$. For if the required number is set as $1R$, from $1R$ times 20 and times 15 come the numbers $20R$ and $15R$. And from $15R$ multiplied into itself comes the number $225R$, which must equal the square [recte: $20R$]. Dividing 20 by 225 gives $1R = 20/225$, that is $4/45$, the required number. For this, multiplying 20 and 15, makes $80/45$, that is $16/9$, for the square, whose side is the number $60/45$, that is $4/3$, produced from $4/45$ times 15.

72. To find two numbers such that the sum made from them, and the excess of the squares which arise from them, produce given numbers — provided that the given excess of the squares is smaller than the square of the given sum. [Margin: LXXII]

Let their given sum be 20, and the given excess of the squares 80, which excess is smaller than the square 400 of the given sum 20.

Let the first number be set as $1R$, and hence the second as $20 - 1R$, so that their sum is 20. Their squares are 13 and $400 - 40R + 13$, since from multiplying $20 - 1R$ by $20 - 1R$ there arises the number $400 - 40R + 13$, as appears in this layout:

$$\begin{array}{r}
 20 - 1R \\
 20 - 1R \\
 \hline
 - 20R + 13 \\
 400 - 20R \\
 \hline
 400 - 40R + 13
 \end{array}$$

Taking that first square away from this one, there remains the excess $400 - 40R$, equal to 80. Adding $40R$ to both sides, there will be an equation between 400 and $80 + 40R$. Removing 80 from both sides, between 320 and $40R$. Dividing 320 by 40 gives $1R = 8$, the first number; and the second, set as $20 - 1R$, will be 12. The sum of the numbers 8 and 12 is 20, and of the squares 64 and 144 the excess is 80.

But if the same sum of the numbers, 20, is given, and an excess of the squares greater than the square 400 that arises from the sum, the problem will be impossible. For, setting the first number as $1R$ and the second as $20 - 1R$, their squares will again be 13 and $400 - 40R + 13$, whose excess, $400 - 40R$, would have to equal the number 400 — which cannot be.

Again, let the sum of the numbers be 20 and the excess of the squares likewise 20. Setting the first number as $1R$ and the second as $20 - 1R$, their sum will be 20 and the excess of the squares $400 - 40R$, equal to 20. Adding $40R$ to both sides, there will be an equation between $20 + 40R$ and 400. Removing 20 from both sides, between 380 and $40R$. Dividing 380 by 40 gives $1R = 9\frac{1}{2}$, the first number. The

second, therefore, will be $10\frac{1}{2}$ — whose sum is 20, and the excess of the squares $441/4$ and $361/4$ is the number $80/4$, that is, 20.

73. *To divide a given number into two parts, so that the excess of the squares which arise from them is any number smaller than the square of the given number.*

[Margin: LXXIII]

Let the given number be 20, to be divided into two, so that the excess of the squares of the parts made is 80. This problem is not different from the preceding one. For if two numbers are found whose sum is the given number 20 and the excess of the squares made from them is 80, the given sum 20 is to be distributed among those two numbers, and so on.

The cossist symbols are rendered as printed and glossed on first occurrence: **R** = *res* (the unknown, x), **3** = *census* (the square, x^2). In this font the *res* sign is a curl that resembles "2R"; a numeral before it is its coefficient (so the printed "1 2R" is 1R, "20 2R" is 20R). A radical written $\sqrt{3}$ is a square (second) root, the census-mark **3** marking the root's degree. The dash "-" is the cossist minus. Every sign here is confirmed by the accompanying arithmetic. "Collateral" (numeri Coffici collaterales) means the two sides of the equation share a common power of the unknown, which is divided out.

74. [LXXIII.] *Find two numbers in a given ratio such that the sum of the squares formed from them has a given ratio to the excess of the numbers found.*

Let two numbers be sought in triple ratio, such that the sum of the squares produced from them has a tenfold ratio to the sum [recte: excess — the heading and the worked solution both use the excess, not the sum] of the numbers found. Let the smaller number be put as 1R [*res*, the unknown x], the greater as 3R, its triple. Their excess is 2R. And of the squares — 1**3** [*census*, x^2] and 9**3** — the sum is 10**3**, tenfold with respect to 2R. Therefore the equation is between 10**3** and 20R. So, dividing 20 by 10, 1R comes out as 2, the smaller number (because the cossist terms are collateral); and the greater will be 6. Their excess is 4, and the sum of the squares, 4 and 36, is 40 — which number is ten times the excess 4.

75. [LXXV.] *Find two numbers in a given ratio such that the excess of their squares has a given ratio to the excess of the numbers found.*

Let the numbers sought have a triple ratio, and let the excess of the squares have a twelvefold ratio to their excess. Let the smaller number be put as 1R, the greater as 3R, its triple. Their excess is 2R. Of the squares, 1**3** and 9**3**, [the excess] is 8**3**, which has a twelvefold ratio to 2R. Therefore the equation will be between

83 and 24R. So, dividing 24 by 8, 1R comes out as 3, the smaller number (since the consist terms are collateral), and the greater will be 9. Their excess is 6. And of the squares, 9 and 81, the excess 72 has a twelvefold ratio to 6.

76. [LXXVI.] *Find two numbers arising in a given ratio such that the number produced from one into the other has a given ratio to their sum.*

Let the numbers sought have a quadruple ratio, and let the number generated from one into the other have a sevenfold ratio to their sum. Let the smaller be put as 1R, the greater as 4R, its quadruple. Their sum is 5R. And from 1R into 4R arises the number 43, which must have a sevenfold ratio to 5R. Therefore the equation will be between 43 and 35R. So, dividing 35 by 4, 1R comes out as 35/4, the smaller number (since the consist terms are collateral). The greater then will be 140/4. Their sum is 175/4. From 35/4 into 140/4 arises the number 4900/16, which has a sevenfold ratio to 175/4.

77. [LXXVII.] *Find two numbers in a given ratio such that the number produced from one into the other has a given ratio to their excess.*

Let the numbers sought have a quadruple ratio, and let the number produced from one into the other have a sevenfold ratio to their excess. Let the smaller be put as 1R and the greater as 4R, its quadruple. Their excess is 3R. And the product of 1R into 4R is 43, which must have a sevenfold ratio to 3R. Therefore the equation will be between 43 and 21R. So, dividing 21 by 4, 1R comes out as 21/4, the smaller number (since the consist terms are collateral). The greater then will be 84/4. Their excess is 63/4. From 21/4 into 84/4 arises the number 1764/16, which has a sevenfold ratio to 63/4.

78. [LXXVIII.] *Find two numbers in a given ratio such that the square of the smaller has a given ratio to the greater number.*

Let the numbers sought have a triple ratio, and let the square of the smaller have a sixfold ratio to the greater. Let the smaller be put as 1R and the greater as 3R, its triple. The square of the smaller is 13, which must be six times the greater number 3R. Therefore the equation is between 13 and 18R. So, dividing 18 by 1, 1R comes out as 18, the smaller number, since the consist terms are collateral. The greater then will be 54. The square of the smaller, namely 324, has a sixfold ratio to the greater, 54.

79. [LXXIX.] *Find two numbers in a given ratio such that the square of the smaller has whatever given ratio to the quotient, when the greater is divided by the smaller.*

Let the numbers sought have a quadruple ratio, but let the square of the smaller have a double ratio to the quotient obtained by dividing the greater by the smaller. Let the numbers sought be put as $1R$ and $4R$. Dividing the greater by the smaller, the quotient comes out as 4, to which the square of the smaller, 13 , must have a double ratio: so that the equation is between 13 and 8. Dividing 8 by 1, 13 comes out as 8, and $1R$ as $\sqrt{3}8 [= \sqrt{8}]$, the square root of 8; the source writes the square-root sign with the census-mark 3, the smaller number. The greater then will be $\sqrt{3}128 [= \sqrt{128}]$, its quadruple. Dividing the greater by the smaller, the quotient comes out as $\sqrt{3}16 [= \sqrt{16}]$, that is, 4, to which the square of the smaller, 8, has a double ratio.

80. [LXXX.] *Find two numbers in a given ratio such that the square of the smaller has whatever given ratio to their sum.*

Let the numbers sought have a triple ratio, and let the square of the smaller have a sixfold ratio to their sum. Let the smaller be put as $1R$ and the greater as $3R$, its triple. The square of the smaller is 13 , which must have a sixfold ratio to $4R$, that is, to the sum of the two. Therefore the equation will be between 13 and $24R$. So, dividing 24 by 1, $1R$ comes out as 24, since the cossist terms are collateral. The numbers sought are therefore 24 and 72, in triple ratio. And the square of the smaller, 576, has a sixfold ratio to 96, the sum collected from both.

81. *Find two numbers in a given ratio such that the square of the smaller has a given ratio to their excess. [LXXXI.]*

Let the numbers sought have a triple ratio, and let the square of the smaller have a sixfold ratio to their excess. Let the smaller be put as $1R$ and the greater as $3R$, its triple. Their excess is $2R$, to which the square of the smaller must have [labere, recte habere] a sixfold ratio. So that the equation is between 13 and $12R$. [Dividing 12 by 1], $1R$ comes out as 12, because the cossist terms are collateral. The numbers sought are therefore 12 and 36, having a triple ratio. And the square of the smaller, namely 144, is six times their excess, 24.

82. *Find two numbers in a given ratio such that the square of the greater has a given ratio to the smaller. [LXXXII.]*

Let the numbers sought have a triple ratio, and let the square of the greater have a sixfold ratio to the smaller. Let the smaller be put as $1R$ and the greater as $3R$, its triple. The square of the greater, 93 , must have a sixfold ratio to the smaller, $1R$. The equation will therefore be between 93 and $6R$. So, dividing 6 by 9, $1R$ comes out as $6/9$, or $2/3$, since the cossist terms are collateral. The numbers sought

are therefore $2/3$ and $6/3$, having a triple ratio. And the square of the greater, namely $36/9$, has a sixfold ratio to the smaller, $2/3$, or to $6/9$.

83. *Find two numbers in a given ratio such that the square of the greater has whatever given ratio to the quotient, when the greater is divided by the smaller.* [LXXXIII.]

Let the numbers sought have a quadruple ratio, but let the square of the greater have a double ratio to the quotient obtained by dividing the greater by the smaller. Let the numbers sought be put as $1R$ and $4R$. Dividing the latter by the former, the quotient comes out as 4, to which the square of the greater, $16\bar{3}$, must have a double ratio: so that the equation is between $16\bar{3}$ and 8. Dividing 8 by 16, $1\bar{3}$ comes out as $1/2$, and $1R$ as $\sqrt[3]{1/2}$ [= $\sqrt{(1/2)}$], the smaller number. The greater then will be $\sqrt[3]{8}$ [= $\sqrt{8}$], its quadruple. Dividing the greater by the smaller, the quotient comes out as $\sqrt[3]{16}$ [= $\sqrt{16}$], that is, 4, to which the square of the greater, 8, has a double ratio.

84. *Find two numbers in a given ratio such that the square of the greater has a given ratio to their sum.* [LXXXIV.]

Let the numbers sought have a triple ratio, and let the square of the greater have a sixfold ratio to their sum. Let the smaller be put as $1R$ and the greater as $3R$, its triple. The square of the greater, $9\bar{3}$, must be six times their sum, $4R$. The equation is therefore between $9\bar{3}$ and $24R$. So, dividing 24 by 9, $1R$ comes out as $24/9$, or $8/3$, since the cossist terms are collateral. And accordingly the numbers sought, having a triple ratio, will be $8/3$ and $24/3$. And the square of the greater, $576/9$, is six times their sum, $32/3$ — that is, $96/9$.

85. *Find two numbers in a given ratio such that the square of the greater has a given ratio to their excess.* [LXXXV.]

Let the numbers sought have a triple ratio, and let the square of the greater be ten times their excess. Let the smaller be put as $1R$ and the greater as $3R$, its triple. Their excess is $2R$, to which the square of the greater, $9\bar{3}$, must be tenfold. The equation is therefore between $9\bar{3}$ and $20R$. So, dividing 20 by 9, $1R$ comes out as $20/9$, because the cossist terms are collateral. The numbers sought are therefore $20/9$ and $60/9$, in triple ratio. Their excess is $40/9$, to which the square of the greater — $3600/81$, or $400/9$ — is tenfold.

86. *Given two numbers, find a third such that, of all three, any two added together and multiplied into the remaining one produce three numbers in equal excess — that is, arithmetically proportional.* [LXXXVI.]

Let the two given numbers be 3 and 5, and let the third, to be found, be put as $1R$. This, with 3, makes $1R+3$, which sum, drawn into the remaining 5, makes $5R+15$, one number. Next, $1R$ with 5 makes $1R+5$, which sum, drawn into the remaining 3, makes $3R+15$, another number. Finally, from 3 and 5 comes the number 8, which drawn into the third, $1R$, makes $8R$, the third number. It now remains that these three numbers — $5R+15$, $3R+15$, and $8R$ — be arithmetically proportional. This can happen in three ways. For either the greatest is $5R+15$, the middle $3R+15$, and the least $8R$; or the greatest will be $5R+15$, the middle $8R$, and the least $3R+15$; or, finally, the greatest will be $8R$, the middle $5R+15$, and the least $3R+15$. For $3R+15$ cannot be the greatest, since it is smaller than $5R+15$.

Let first $5R+15$, $3R+15$, $8R$ be arithmetically proportional. Then (as we said in the fourth property of three arithmetically proportional numbers, from Euclid, Book 5 [*Elements* V]) the sum of the extremes, $13R+15$, will equal twice the middle, that is, the number $6R+30$. Removing $6R$ from each side, the equation will be between $7R+15$ and 30 . Removing 15 again from each side, [it is] between $7R$ and 15 . So, dividing 15 by 7, $1R$ comes out as $15/7$, the number sought. — Or, if these three numbers are arithmetically proportional, they will have the same excess. But $5R+15$ exceeds $3R+15$ by $2R$, and $3R+15$ exceeds $8R$ by $15-5R$, as is clear from the rule of subtraction. There is therefore an equality between $2R$ and $15-5R$. Adding $5R$ to each side, [it is] between $7R$ and 15 , and again $1R$ comes out as $15/7$. Now $15/7$ with 3 makes the sum $3 \frac{15}{7}$, that is, $36/7$, which number drawn into the remaining 5 makes $180/7$, the first number. Next, from $15/7$ and 5 comes the sum $5 \frac{15}{7}$, that is, $50/7$, which drawn into the remaining 3 makes $150/7$, the middle number. Finally, from 3 and 5 comes the sum 8, which drawn into the remaining number $15/7$ makes $120/7$, the third number. And these three numbers — $180/7$, $150/7$, $120/7$ — have the same excess, $30/7$, or $4 \frac{2}{7}$.

Let next $5R+15$, $8R$, $3R+15$ be arithmetically proportional. Then the sum of the extremes, $8R+30$, will equal twice the middle, namely the number $16R$. And [removing] $8R$ [from each side], dividing 30 by 8, $1R$ comes out as $30/8$, that is, $15/4$. — Or, these three numbers will have the same excess. Now between $5R+15$ and $8R$ the excess is $15-3R$; but between $8R$ and $3R+15$ the excess is $5R-15$. There is therefore an equation between $15-3R$ and $5R-15$. Adding 15 to each side, [it is] between $30-3R$ and $5R$. Adding $3R$ again to each side, [it is] between 30 and $8R$, and again $1R$ comes out as $30/8$, or $15/4$. Now, from $15/4$ and 3 comes the sum $3 \frac{15}{4}$, or $27/4$, which drawn into the remaining 5 makes $135/4$, the first number. Next, from $15/4$ and 5 comes the sum $5 \frac{15}{4}$, or $35/4$, which drawn into the remaining 3 makes $105/4$, the third number. Finally, from 3 and 5 comes the sum 8, which drawn into the remaining number $15/4$ makes $120/4$, the middle

number. And these three numbers — $135/4$, $120/4$, $105/4$ — have the same excess, $15/4$, or $3\frac{3}{4}$.

Lastly, let $8R$, $5R+15$, $3R+15$ be arithmetically proportional. Then the sum of the extremes, $11R+15$, will equal twice the middle, that is, the number $10R+30$. Removing $10R$ from each side, [the equation] will be between $1R+15$ and 30 . Removing 15 again from each side, [it is] between $1R$ and 15 , and $1R$ comes out as 15 , the number sought. — Or, these three numbers will have the same excess. Now between $8R$ and $5R+15$ the excess is $3R-15$; but between $5R+15$ and $3R+15$ the excess is $2R$. There is therefore an equality between $3R-15$ and $2R$. Adding 15 to each side, [it is] between $3R$ and $2R+15$. Removing $2R$ from each side, [it is] between $1R$ and 15 , and again $1R$ comes out as 15 , the number sought. Now, from 15 and 3 comes the sum 18 , which drawn into the remaining number 5 makes 90 , the middle number. Next, from 15 and 5 comes the sum 20 , which drawn into the remaining number 3 makes 60 , the third number. Finally, from 3 and 5 comes the sum 8 , which drawn into the remaining number 15 makes 120 , the first number. And these three numbers — 120 , 90 , 60 — have the same excess, 30 .

We shall solve this same problem also in Enigma 131, where two numbers will be given such that the three numbers produced cannot be arithmetically proportional in three ways, as here, but only in two.

87. *Find two numbers in a given ratio such that the excess of the squares formed from them has a given ratio to their sum. [LXXXVII.]*

Let the numbers sought have a triple ratio, and let the excess of the squares have an eightfold ratio to their sum. Let the smaller be put as $1R$ and the greater as $3R$, its triple. Of [the squares], 13 and 93 , [the excess] is 83 , which must have an eightfold ratio to $4R$. The equation will therefore be between 83 and $32R$. So, dividing 32 by 8 , $1R$ comes out as 4 , the smaller number, because the consist terms are collateral. The greater will then be 12 . The excess of the squares, 16 and 144 , is 128 , which has an eightfold ratio to 16 , the sum of the two.

88. *Find two numbers in a given excess [difference] such that the excess of the squares formed from them exceeds the given excess by any given number — provided that the square of the excess of the numbers sought is less than the sum collected from the same excess and from the postulated number. [LXXXVIII.]*

Let the excess of the numbers sought be 7 . Let the excess of the squares that arise from the numbers sought be greater than 7 by the postulated number 100 . Let the smaller number be put as $1R$, the greater as $1R+7$, so that it exceeds the former by 7 . The squares of these numbers, 13 and $13+14R+49$, exceed one another by

$14R+49$, which must exceed 7 by 100 — that is, the equality must be between $14R+49$ and 107. Removing 49 from each side, [it is] between $14R$ and 58. So, dividing 58 by 14, $1R$ comes out as $4\frac{1}{7}$, the smaller number; the greater then will be $11\frac{1}{7}$, so that it exceeds the smaller by 7. The squares of the numbers, $841/49$ and $6084/49$, exceed one another by $5243/49$, which exceeds the given excess of the numbers, 7, by 100 — that is, it is equal to the number 107, since 107 makes $5243/49$.

This problem could be solved because the square of the given excess 7, namely the number 49, is less than the sum collected from the given excess 7 and the postulated number 100, namely 107. But if two numbers were proposed to be found in a given excess of 7 such that the excess of their squares should be greater than 7 by the number 10, the problem would be impossible — as you will best learn from the operation itself. For, putting the smaller number as $1R$ and the greater as $1R+7$, so that they exceed one another by seven, the squares of these numbers again exceed one another by $14R+49$, which must exceed 7 by 10; and this is absurd, since it exceeds it by $14R+42$, which is greater than 7. The reason is that the square 49 of the excess 7 is not less than the sum 17.

89. *Find two numbers in a given excess [difference] such that the excess of the squares formed from them has a given ratio to the given excess, and moreover contains a given number — provided that the square of the given excess is less than the sum collected from the number produced by drawing the denominator of the given ratio into that excess, and from the given number. [LXXXIX.]*

Let the given excess be 4, and let the excess of the squares that arise from the numbers sought have a fivefold ratio to 4, and moreover contain 20 units. Let the smaller number be put as $1R$ and the greater as $1R+4$, so that it exceeds the former by four. Of [the squares], 1^2 and $(1+4)^2$, [the excess] is $8R+16$, which must contain the given excess 4 five times and 20 besides. Now the quintuple of the excess 4, together with 20, makes 40, to which number the excess of the squares, $8R+16$, must be equal. Removing 16 from each side, there will be an equality between 24 and $8R$. So, dividing 24 by 8, $1R$ comes out as 3, the smaller number; the greater then will be 7. Their excess is 4. Of the squares, 9 and 49, the excess 40 contains the excess 4 five times, and 20 units besides.

This problem too could be solved, because the square 16 of the given excess 4 is less than the sum collected from the quintuple of the excess 4 and the number 20. But if the excess given were 4, the ratio triple, and three units besides, the problem will

[Notation: the cossist tokens are transcribed as printed — R = *res* (one "root," the unknown x); $\bar{3}$ = *census* (the square, x^2); coefficients precede the sign, so $8R$ = $8x$, $1\bar{3}$ = x^2 , $\frac{1}{2}\bar{3}$ = $\frac{1}{2}x^2$. Fractions are written with a slash.]

(*continuation of Problem 89*) ...the question will prove impossible, as will readily be recognized from the working itself. For if the smaller number is set as $1R$ [one *res*, the unknown x] and the larger as $1R+4$, the excess of their squares will be $8R+16$, and this must equal the sum formed from three times the excess 4 together with the number 3 — that is, the number 15. But that is absurd. The reason is that the square 16, whose excess is 4, is not less than the number 15.

Problem 90 (XC). To distribute a given square number into two square numbers.

We solved this puzzle without algebra in Book 8 of our *Geometria Practica*, proposition 9, where we partitioned the proposed square number not merely into two but into any number of square numbers. By algebra we now accomplish the same thing in this way. Let the square number to be divided into two squares be 16. Let the first square be put as $1\bar{3}$ [one *census*, i.e. x^2], and accordingly the second as $16 - 1\bar{3}$. Imagine the side of this second square to be as many roots, less as many units, as there are in the side of the given square — namely $2R - 4$. Its square, $4\bar{3}+16 - 16R$, must equal the square $16 - 1\bar{3}$. Adding $1\bar{3}$ to both sides, there will be an equation between $5\bar{3}+16 - 16R$ and 16. Adding $16R$ again [to both sides], between $5\bar{3}+16$ and $16R$. Removing 16 from both sides, between $5\bar{3}$ and $16R$. So, dividing 16 by 5, $1R$ becomes $16/5$, since the cossic numbers are collateral [proportional]. The square of this root, $256/25$, will be the first [square], which we set as $1\bar{3}$. The second, $16 - 1\bar{3}$, will be $144/25$, whose root is $12/5$; and both squares together make $400/25$, namely 16.

Another way. Let the side of the first square be put as $1R$. Let the side of the second be as many roots, less as many units as there are in the side of the given square — namely $5R - 4$. The squares of these sides, $1\bar{3}$ and $25\bar{3}+16 - 40R$, must together equal the given square 16. Adding $40R$ to both sides, there will be an equation between $26\bar{3}+16$ and $40R+16$. Removing 16 from both sides, between $26\bar{3}$ and $40R$. So, dividing 40 by 26, $1R$ becomes $20/13$ (because the cossic numbers are collateral). Its square, $400/169$, will be the first square number. The second will be $2304/169$, which is left when the first is subtracted from 16. Its side is $48/13$, and both squares together make $2704/169$, that is, 16.

Scholium.

Now if we again partition the smaller square number into two squares, the given square will have been divided into three; and if the smallest is again cut into two, the given one will have been cut into four. And so on to infinity. You see, then, that the proposed square can be divided by algebra into two squares in various ways, so that different pairs of squares always result.

This problem can also be solved without algebra, in the following way. Given the square, multiply its side by any number you please, and divide double the product by the square of that same number increased by a unit. The square of the quotient will be one of the sought squares; and when this is taken away from the proposed square, the other square remains.

For example, let the proposed square be 36. Multiply its side, 6, by 3, to get 18. Divide double this, 36, by the square of the number 3 increased by a unit — namely by 10. The quotient is $36/10$, or $18/5$, whose square is $324/25$, that is, $12 \frac{24}{25}$; this will be one square. When it is taken away from 36, the other will be $23 \frac{1}{25}$, whose side is $24/5$, or $4 \frac{4}{5}$; and both squares, $12 \frac{24}{25}$ and $23 \frac{1}{25}$, make 36.

Note, however, that the side of the proposed square must not be multiplied by 1, since then the two squares would be the proposed square and 0, which is pointless. For instance, if 6, the side of the proposed square 36, were multiplied by 1 to give 6, and double this product, 12, were divided by 2 — namely by the square of unity increased by a unit — the quotient would be 6, whose square is 36, the proposed number, and so on.

Problem 91 (XCI). To divide a given number composed of two squares into two other squares.

Let the number to be divided into two other squares be 34 — whether or not it is itself a square — composed of the two squares 9 and 25. Let the sides of those squares be 3 and 5. Put the side of the first sought square as $1R+3$, namely one root plus the side of the first given square. Let the side of the second sought square be as many roots less the side of the second given square, namely $2R - 5$. The squares of these sides, 1^2R^2+6R+9 and $4^2R^2 - 20R+25$, make the sum $5^2R^2+34 - 14R$, equal to the given number 34. So, adding $14R$ to both sides, there will be an equation between 5^2R^2+34 and $14R+34$. Removing 34 from both sides, between 5^2R^2 and $14R$, because the cossic numbers are collateral. Therefore the side of the first sought square, which we set as $1R+3$, will be $29/5$. The side of the second square,

which we set as $2R - 5$, will be $28/5 - 5$, that is, $3/5$. The squares of these found sides, $841/25$ and $9/25$, make $850/25$, that is, the given number 34.

Problem 92 (XCII). To find two square numbers differing by a given excess.

Let the excess given be 70. Put the side of one square as $1R$, and the side of the other as $1R$ plus any number of units — provided that the square number described from them is less than the given excess — namely $1R+8$. The squares of these sides, 13 and $13+16R+64$, have as their excess $16R+64$, which must equal the given excess 70. Removing 64 from both sides, there will be an equation between $16R$ and 6. So, dividing 6 by 16, $1R$ becomes $3/8$, the side of one square. The side of the other, set as $1R+8$, will be $8\frac{3}{8}$, or $67/8$. The squares of these found sides are $9/64$ and $4489/64$, whose excess is $4480/64$, that is, 70.

It is evident, moreover, that ever different pairs of square numbers can be found for the same given excess, according as a different absolute number is assumed in the root of the second square. Thus, for the same given excess 70, if the side of the first square is put as $1R$ and of the second as $1R+1$, the squares will be 13 and $13+2R+1$, and their excess $2R+1$, which must equal the given excess 70. Removing 1 from both sides, there will be an equality between $2R$ and 69. So, dividing 69 by 2, $1R$ becomes $34\frac{1}{2}$, or $69/2$, whose square is $4761/4$, or $1190\frac{1}{4}$. The side of the second square, which we set as $1R+1$, will be $35\frac{1}{2}$, or $71/2$, whose square is $5041/4$, or $1260\frac{1}{4}$, which exceeds the former, $1190\frac{1}{4}$, by the number 70.

If the excess given is 100, and the side of the first square is put as $1R$ and the side of the second as $1R+10$ — where the square of the number 10 is not less than the given excess 100, but equal to it — the question will be impossible. For the squares 13 and $13+20R+100$ have as their excess $20R+100$, which ought to equal the given excess 100. That is absurd, since it is greater.

Problem 93 (XCIII). To find two numbers whose squares differ by a given number.

This puzzle is solved just like the preceding one. For once the two sides are posited, as in the previous problem, two sides will be found whose squares have the given excess.

Problem 94 (XCIII). To find two square numbers whose sum is a square number.

Take any square you please for one of the sought numbers — namely 16. Then, by Problem 92, find two square numbers whose excess is 16, the assumed square. For the smaller found square, together with the assumed 16, will make the other (larger) found square. So put, as was done in that problem, the side of one square as $1R$ and of the other as $1R+3$ (so that the square of the number 3 is less than the given excess 16). The squares of these sides, 13 and $13+6R+9$, have as their excess $6R+9$, which must equal the given excess 16. Removing 9 from both sides, there will be an equation between $6R$ and 7. So, dividing 7 by 6, $1R$ becomes $7/6$, the side of the smaller sought square. The side of the larger, which we set as $1R+3$, will be $3\frac{7}{6}$, that is, $25/6$. The squares of the sides $7/6$ and $25/6$ are $49/36$ and $625/36$, whose excess is $576/36$, that is, 16. Therefore the two sought squares are 16 (or $576/36$) and $49/36$, whose sum, $625/36$, is a square number with side $25/6$.

Casting out the denominators, the two whole squares found will be 576 and 49, making the square 625, whose side is 25.

Problem 95 (XCV). To find two square numbers whose difference is a square.

This puzzle is solved no differently than Problem 92. For if by that method two squares are found whose excess is a given square number, what is required here will have been done.

Problem 96 (XCVI). To find three square numbers whose sum is a square number.

By Problem 94, first find two squares whose sum is a square number — for example these: 576 and 49, making the square 625. Then, by Problem 95, find two other squares whose excess is the square just produced, 625. For the smaller of those found, together with the square 625 — that is, with the two squares 576 and 49 — will make the larger square. For example, with the given excess being the square 625, two squares will be found within that excess in this way. Put the side of one as $1R$ and of the other as $1R+10$ (so that the square of 10 units is less than the given excess 625). The squares of these sides, 13 and $13+20R+100$, have as their excess $20R+100$, which must equal the given excess 625. So, removing 100 from both sides, there will be an equation between $20R$ and 525. Dividing 525 by 20, $1R$ becomes $26\frac{1}{4}$, the side of the smaller square. The side of the larger, which

was set as $1R+10$, will be $36 \frac{1}{4}$. The squares of these sides, $11025/16$ and $21025/16$, have as their excess $10000/16$, that is, the square 625. This excess, then — namely $10000/16$, made up of the two squares 576 and 49 — combined with the smaller square $11025/16$ will compose the larger square $21025/16$. And so the three sought squares are 576, 49, and $11025/16$ — or $9216/16$, $784/16$, and $11025/16$ — which together yield the square $21025/16$, whose side is $145/4$.

If the denominators are cast out, three whole squares will have been found: 9216, 784, and 11025, whose sum is the square number 21025, whose side is 145.

Scholium.

And by the same technique four squares making up a square number will be found, if first three are found by this problem, and then by Problem 95 two are found having as their excess that square made up of three squares. In like manner 5, 6, 7, and as many as one wishes will be found — provided, that is, that first 4, or 5, or 6, and so on, are found.

Problem 97 (XCVII). To find two numbers such that both their sum and their difference is a square number.

Put the side of the square formed from the two sought numbers as one root plus any number of units — namely $1R+4$. Its square, $1R^2+8R+16$, must equal the two sought numbers [together]. So, if one of the numbers is put as $\frac{1}{2}R$, the other will be $\frac{1}{2}R+8R+16$, which two together make the said square $1R^2+8R+16$. It remains that $\frac{1}{2}R$, subtracted from $\frac{1}{2}R+8R+16$, should also leave a square number. Therefore the remaining number, $8R+16$, will equal some square — say 36, greater than 16, the absolute number. So, removing 16 from both sides, there will be an equality between $8R$ and 20. Dividing 20 by 8, R becomes $20/8$, or $5/2$, whose square is $25/4$. Since, then, one of the sought numbers was put as $\frac{1}{2}R$, it will be $25/8$, the first sought number. The second number, which was put as $\frac{1}{2}R+8R+16$, will be $25/8$ plus $40/2$ (or 20), and 16 besides — that is, $25/8$ and 36, or $288/8$. These two numbers, $25/8$ and $288/8$, make $313/8$. For if to it the first, $25/8$, is added, the sum becomes $338/8$, that is, $169/4$, a square number whose side is $13/2$. And if from that same second number, $313/8$, the first, $25/8$, is taken away, the number $288/8$ remains, or $144/4$, also a square, whose side is $12/2$, or 6.

Problem 98 (XCVIII). Given two numbers, to find another which, added to each separately, makes a square.

Let the two given numbers be 4 and 7. Let the sought number be put as $1R$, which together with the given numbers makes the two [numbers] $1R+4$ and $1R+7$, which must be squares. Subtract one from the other, so that their excess, 3, is obtained; and seek two numbers which, multiplied together, produce this same excess 3 — yet in such a way that the square of half of their excess is greater than the smaller given number 4, and so can equal the smaller composite number $1R+4$; and [such that] the square of half of their sum is greater than the larger given number 7, and so can equal the larger composite number $1R+7$. Why this must be so, the working will show. These numbers will be $\frac{1}{2}$ and 6. For, multiplied together, they produce 3. And their excess is $5\frac{1}{2}$, that is, $11/2$, whose half is $11/4$, whose square is $121/16$, that is, $7\frac{9}{16}$, which is greater than 4. Their sum, moreover, is $6\frac{1}{2}$, that is, $13/2$, whose half is $13/4$, whose square is $169/16$, that is, $10\frac{9}{16}$, which is greater than 7. These two numbers, $\frac{1}{2}$ and 6, were found by taking a fraction at will, $\frac{1}{2}$, whose numerator, unity, is one of the numbers; for if its denominator 2 is multiplied into the number 3 that must be produced, the other number, 6, results.

But if the given numbers were 20 and 23, having the same excess 3, the numbers $\frac{1}{2}$ and 6 would not suffice. So another fraction would have to be taken, namely $1/10$, for one number; for from the denominator 10 into 3 the other number, 30, results. And from $1/10$ into 30 comes the number 3. Their excess is $29\frac{9}{10}$, that is, $299/10$, whose half is $299/20$, whose square is $89401/400$, or $223\frac{201}{400}$, which is greater than 20. Likewise their sum is $30\frac{1}{10}$, that is, $301/10$, whose half is $301/20$, whose square is $90601/400$, or $226\frac{201}{400}$, which is greater than 23. And so, if these numbers $1/10$ and 30 did not suffice, another fraction of larger denomination would have to be taken, and so on; the operation is to be repeated as many times as necessary until two numbers satisfying the proposal are found.

Let us now carry out the matter with the numbers $\frac{1}{2}$ and 6 in the proposed example. Their excess is $5\frac{1}{2}$, or $11/2$, and half of this is $11/4$, whose square, $121/16$, or $7\frac{9}{16}$, will equal the smaller composite number $1R+4$. So, removing 4 from both sides, there will be an equation between $3\frac{9}{16}$ and $1R$; and therefore $1R$ will be $3\frac{9}{16}$. The sum of the numbers $\frac{1}{2}$ and 6 is $6\frac{1}{2}$, or $13/2$, and half of this is $13/4$, whose square, $169/16$, or $10\frac{9}{16}$, will equal the larger composite number $1R+7$. So, removing 7 from both sides, there results an equation between $3\frac{9}{16}$ and $1R$, as before. The sought number, then, is $3\frac{9}{16}$. For, added to 4 and to 7, it makes the squares $7\frac{9}{16}$ and $10\frac{9}{16}$, or $121/16$ and $169/16$, whose sides are $11/4$ and $13/4$.

But if the fraction $\frac{1}{3}$ is taken, and the denominator 3 is multiplied into the excess 3, the number 9 is produced; and these two numbers, $\frac{1}{3}$ and 9, multiplied together, make 3. Their excess is $8\frac{2}{3}$, or $\frac{26}{3}$, and half of this is $\frac{13}{3}$, whose square, $\frac{169}{9}$, or $18\frac{7}{9}$, will equal the smaller composite number $1R+4$. So, removing 4 from both sides, the equation will be between $1R$ and $14\frac{7}{9}$; and therefore $1R$ will be $14\frac{7}{9}$ — which is also found in this way: the sum of the numbers $\frac{1}{3}$ and 9 is $9\frac{1}{3}$, or $\frac{28}{3}$, half of this is $\frac{14}{3}$, whose square, $\frac{196}{9}$, or $21\frac{7}{9}$, will equal the larger composite number $1R+7$. So, removing 7 from both sides, there will be an equation between $14\frac{7}{9}$ and $1R$, as before. The sought number, then, is $14\frac{7}{9}$. For, added to 4 and to 7, it makes the squares $18\frac{7}{9}$ and $21\frac{7}{9}$, or $\frac{169}{9}$ and $\frac{196}{9}$, whose sides are $\frac{13}{3}$ and $\frac{14}{3}$.

From the working itself you understand why such two numbers, producing the excess of the given numbers, must be chosen as we have described. For unless the square of half of their excess were greater than the smaller number 4, no equation between it and the smaller composite number could be formed, as is plain, and so on. It is likewise established that various numbers can be found, each of which added to the two given numbers makes two squares, according as different pairs of numbers producing the said excess 3 are taken. Thus you see that in the given example the two numbers found are $3\frac{9}{16}$ and $14\frac{7}{9}$.

But let us try the same with the given numbers 20 and 23. Adding $1R$ to each, the squares $1R+20$ and $1R+23$ result; and there will be an equation (once the two numbers $\frac{1}{10}$ and 30, which produce the excess 3, are chosen) between $223\frac{201}{400}$ and $1R+20$, and likewise between $226\frac{201}{400}$ and $1R+23$. Through either equation $1R$ will be found to be $203\frac{201}{400}$, which number, added to 20 and to 23, makes the squares $223\frac{201}{400}$ and $226\frac{201}{400}$, or $\frac{89401}{400}$ and $\frac{90601}{400}$, whose sides are $\frac{299}{20}$ and $\frac{301}{20}$.

Another way. Again let the two given numbers be 4 and 7. Put the sought number as $13 - 4$. For if 4 is added to it, 13 results, a square. And if to that same number $13 - 4$ the 7 is added, another square results, $13+3$, whose side let be set as one root less any number of units exceeding 4 — namely $1R - 5$. Its square, $13 - 10R + 25$, must equal the square $13+3$. Adding $10R$ to both sides, there will be an equality between $13+25$ and $13+10R+3$. And if 13 is removed from both, between 25 and $10R+3$. Removing 3 from both, between 22 and $10R$. So, dividing 22 by 10, $1R$ becomes $\frac{22}{10}$, or $\frac{11}{5}$; from whose square, $\frac{121}{25}$, if 4 is subtracted (because the sought number was put as $13 - 4$) — that is, $\frac{100}{25}$ — the remaining sought number will be $\frac{21}{25}$. For, added to 4 and to 7, it makes the squares $\frac{121}{25}$ and $\frac{196}{25}$, whose sides are $\frac{11}{5}$ and $\frac{14}{5}$.

If the given numbers are 20 and 23, the sought number will have to be put as $13 - 20$, so that added to 20 it makes the square 13 . And if to that same number $13 - 20$ the 23 is added, another square, $13+3$, results, whose side let be set as one root less any number of units exceeding 20 — namely $1R - 21$. Its square, $13 - 42R + 441$, must equal the square $13+3$. Adding $42R$ to both sides, there will be an equality between $13+441$ and $13+42R+3$. And if 13 is removed from both, between 441 and $42R+3$. Removing 3 from both sides, between 438 and $42R$. So, dividing 438 by 42 , $1R$ becomes $10 \frac{3}{7}$, or $73/7$; from whose square, $5329/49$, if 20 is subtracted (because the sought number was put as $13 - 20$) — that is, $980/49$ — the remaining sought number becomes $4349/49$, or $88 \frac{37}{49}$, which, added to 20 and to 23 , will make the squares $108 \frac{37}{49}$ and $111 \frac{37}{49}$, that is, $5329/49$ and $5476/49$, whose sides are $73/7$ and $74/7$.

Problem 99 (XCIX). Given two numbers, to find another which, subtracted from each separately, leaves a square.

Let the two given numbers be 9 and 21. Put the sought number as $9 - 13$. For this, subtracted from 9, leaves the square 13 , as appears in this scheme:

$$\begin{array}{r} 9 + 03 \\ 9 - 13 \\ \hline + 13 \end{array}$$

That same number, $9 - 13$, subtracted from 21, leaves $12+13$, as is clear in this second scheme — which number must also be a square:

$$\begin{array}{r} 21 + 03 \\ 9 - 13 \\ \hline 12 + 13 \end{array}$$

Let its side be set as one root less any number of units, yet so that their square exceeds 12 — namely $1R - 4$. Its square, $13 - 8R + 16$, must equal...

Throughout this section the cossist (symbolic) notation is transcribed as printed and glossed: R = *res* (the unknown, x); 3 = *census* (x^2); $\mathfrak{C}$ = *cubus* (x^3); $\sqrt{3}$ = *radix* (square root). A coefficient stands before its sign (e.g. $8R$ = eight *res* = $8x$).

Chapter XXIX — Miscellaneous Problems (continued)

[Problem 99, conclusion]

to the square $13 + 13$ [census, x^2]. Adding $8R$ [8 res , $8x$] to both sides, there will be an equation between $13 + 16$ and $13 + 8R + 12$. And if 13 is removed from both, [an equation] between 16 and $8R + 12$. And subtracting 12 from both sides, between 4 and $8R$. Dividing 4 by 8 , then, gives $1R = 4/8$, that is, $1/2$; and its square is $1/4$. If this $1/4$ is subtracted from 9 — because the number posited was $9 - 13$ — that is, from $36/4$, the number sought will be $8\frac{3}{4}$, or $35/4$. This number, subtracted respectively from 9 and 21 , leaves the squares $1/4$ and $12\frac{1}{4}$, that is, $1/4$ and $49/4$, whose roots are $1/2$ and $7/2$.

Problem 100

[Margin: C.]

Given two numbers, to find another such that each of the given ones, subtracted from it, leaves a square.

Let the two given numbers be 20 and 30 . Let the number sought be posited as $13 + 20$, so that when 20 is subtracted, the square 13 remains (a square, as is plainly the case). It likewise remains that $13 - 10$ — the number left when 30 is subtracted from $13 + 20$, as the formula shows — should also be a square.

$$\begin{array}{r} 13 + 20 \\ + 30 \text{ [recte: } - 30] \\ \hline 13 - 10 \end{array}$$

Let the root of the number $13 - 10$ be set as one *res* less some number of units, namely $1R - 2$. Its square, $13 - 4R + 4$, must equal the square $13 - 10$. Adding $4R$ to both sides, there will be an equation between $13 + 4$ and $13 - 10 + 4R$. Adding 10 again to both sides, between $13 + 14$ and $13 + 4R$. And removing 13 from both, between 14 and $4R$. Dividing 14 by 4 , then, gives $1R = 14/4$, or $7/2$. To its square, $49/4$, if 20 is added (since the number posited was $13 + 20$), that is, $[129/4]$, the number sought becomes $129/4$, that is, $32\frac{1}{4}$. If the given numbers 20 and 30 are subtracted from it, the remainders will be the squares $12\frac{1}{4}$ and $2\frac{1}{4}$, or $49/4$ and $9/4$, whose roots are $7/2$ and $3/2$.

Problem 101

[Margin: CI.]

To divide a given number into two parts, and moreover to find a square number that, together with each part separately, makes a square.

Let the given number be 40. Take two numbers whose squares together make a number smaller than 40, namely 2 and 4. To each add $1R$, making the numbers $1R + 2$ and $1R + 4$, whose squares are $1\bar{3} + 4R + 4$ and $1\bar{3} + 8R + 16$. If $1\bar{3}$ is subtracted from each, there remain $4R + 4$ and $8R + 16$, the parts into which 40 is to be divided — since each of these numbers, together with $1\bar{3}$, makes a square. It remains, therefore, that these two parts should make $[40]$; their sum must equal 40. Removing 20 from both sides, then, there will be an equation between $12R$ and 20. Dividing 20 by 12, $1R$ becomes $20/12$, or $5/3$, whose square, $25/9$ or $2\frac{7}{9}$, is the number sought. And the parts of 40, namely $4R + 4$ and $8R + 16$, will be $10\frac{2}{3}$ and $29\frac{1}{3}$, which together make 40. And each, with the square found, $2\frac{7}{9}$, makes a square number. For $2\frac{7}{9}$ together with $10\frac{2}{3}$, or with $10\frac{6}{9}$, makes $13\frac{4}{9}$ [= $121/9$], a square whose root is $11/3$. And the same square $2\frac{7}{9}$ together with $29\frac{1}{3}$, or with $29\frac{3}{9}$, makes the square $32\frac{1}{9}$, or $289/9$, whose root is $17/3$.

Problem 102

[Margin: CII.]

To divide a given number into two parts, and moreover to find a square number from which each part separately, subtracted, leaves a square.

Let the given number again be 40. Let the root of the square sought be posited as $1R$ plus as many units as will make their square less than 40, namely $1R + 5$, whose square, $1\bar{3} + 10R + 25$, leaves the square $1\bar{3}$. One of the numbers, then, into which 40 is to be divided will be $10R + 25$, since, subtracted from $1\bar{3} + 10R + 25$, it leaves the square $1\bar{3}$.

Next take the root of some second square, $1R + 4$, such that its absolute number 4 is less than the absolute number 5 of the first root. The square of this second root is $1\bar{3} + 8R + 16$, which, subtracted from the earlier square $1\bar{3} + 10R + 25$, leaves $2R + 9$ — the other of the numbers into which 40 is to be divided. And so the square we seek is $1\bar{3} + 10R + 25$, and the parts of 40 are $10R + 25$ and $2R + 9$. For the first, subtracted from that square, leaves the square $1\bar{3}$; while the second, removed from the same square, leaves $1\bar{3} + 8R + 16$, since when this is taken from that square it leaves $2R + 9$ — which the following formula also shows, the number $1\bar{3} + 8R + 16$ being likewise the square of the root $1R + 4$, as was clear just above.

$$\begin{array}{r} 1\bar{3} + 10R + 25 \\ 0\bar{3} + 2R + 9 \end{array}$$

$$13 + 8R + 16$$

It remains that these two parts, $10R + 25$ and $2R + 9$, should make the given number 40. But they make $12R + 34$, which number must equal 40. Removing 34 from both sides, then, there will be an equation between $12R$ and 6. Dividing 6 by 12 gives $1R = \frac{1}{2}$. Since the root of the square sought was posited as $1R + 5$, the root itself will be $5\frac{1}{2}$, or $11/2$, and the square sought $[121/4]$, or $30\frac{1}{4}$. And the parts of 40, which we set as $10R + 25$ and $2R + 9$, will be 30 and 10, which together make the given number 40; and, subtracted from the square found, $30\frac{1}{4}$, they leave the square numbers $\frac{1}{4}$ and $20\frac{1}{4}$, or $81/4$, whose roots are $\frac{1}{2}$ and $9/2$.

Problem 103

[Margin: CIII.]

To find three numbers such that the first, together with any given number, has to the sum of the others any given ratio; likewise the second, with the same given number, has to the sum of the others any given ratio you please; and finally the third, with the same given number, has to the sum of the others any given ratio.

Let the given number be 73, and let three numbers be sought such that the first, with 73, has to the sum of the others the ratio of double; the second, with 73, to the sum of the others the ratio of triple; and finally the third, with 73, to the sum of the others the ratio of quadruple. Let the sum of all three be posited as $1R$. The first, then, will be $\frac{2}{3}R - 73$, so that with 73 it has $\frac{2}{3}R$, a number double the $\frac{1}{3}R$ that is the sum of the others. The second will be $\frac{3}{4}R - 73$, so that with 73 it has $\frac{3}{4}R$, a number triple the $\frac{1}{4}R$ that is the sum of the others. And finally the third will be $\frac{4}{5}R - 73$, so that with the added 73 it has $\frac{4}{5}R$, a number quadruple the $\frac{1}{5}R$ that is the sum of the others.

These being set down: since the first, with the added 73, has $\frac{2}{3}R$, which must be double the sum of the other two — and the second, $\frac{3}{4}R - 73$, together with the third, $\frac{4}{5}R - 73$, makes the sum $\frac{31}{20}R - 146$, of which the number $\frac{2}{3}R$ (which the first has with 73) must be double — the number $\frac{62}{20}R - 292$, the double of the aforesaid sum $\frac{31}{20}R - 146$, will equal the number $\frac{2}{3}R$. Adding 292 to both sides, then, the equation will be between $\frac{62}{20}R$ and $\frac{2}{3}R + 292$. Removing $\frac{2}{3}R$ from both, between $\frac{146}{60}R$ and 292. Dividing 292 by $\frac{146}{60}$, then, gives $1R = 120$.

Therefore the first number, $\frac{2}{3}R - 73$, will be 7, because $\frac{2}{3}$ of 120 is 80, from which, if 73 is taken away, the number 7 remains. The second, $\frac{3}{4}R - 73$, will be 17, since $\frac{3}{4}$ of 120 is 90, from which, if 73 is subtracted, 17 remains. And finally the third,

$\frac{4}{5}R - 73$, will be 23, since $\frac{4}{5}$ of 120 is 96, from which, if 73 is taken away, 23 is left. And these three numbers, 7, 17, and 23, satisfy the problem. For 7 with 73 makes 80, which is double the sum 40 gathered from 17 and 23. And 17 with 73 makes 90, which is triple the sum 30 gathered from 7 and 23. And finally 23 with 73 makes 96, which is quadruple the sum 24 gathered from 7 and 17.

Problem 104

[Margin: CIV.]

To find three numbers such that the first, together with a given fraction of the sum of the other two, makes a given number; likewise the second, with another given fraction of [the sum of] the other two, also makes any given number; and finally the third, with any given fraction of [the sum of] the other two, likewise makes any given number you please.

Let it be required that the first, with $\frac{1}{2}$ of the second and third, make 42; the second, with $\frac{1}{4}$ of the first and third, make 36; and the third, with $\frac{1}{12}$ of the first and second, make 39. Let the first be posited as $1R$. The half of the other two, then, will be $42 - 1R$, for in this way the first, with that half, will make 42. And accordingly the double of this half, namely $84 - 2R$, will be the sum of the second and third.

Now since the second, with $\frac{1}{4}$ of the first and third, must make 36: if we take away a quarter of the first, namely $\frac{1}{4}R$, from 36, there will remain a quarter of the third together with the second, namely the number $36 - \frac{1}{4}R$. If we subtract this from $84 - 2R$ — that is, from the sum of the second and third — the remaining number $48 - 1\frac{3}{4}R$ will give the three remaining fourth-parts of the third. And accordingly a third of this number, namely $16 - \frac{7}{12}R$, will be a quarter of the third; which, together with the three fourths aforesaid — that is, with $48 - 1\frac{3}{4}R$ — will make the whole third number, $64 - \frac{28}{12}R$, or $64 - 2\frac{1}{3}R$.

If we subtract this third number from $84 - 2R$, the sum of the second and third, the second becomes $20 + \frac{1}{3}R$.

Now since the third, with $\frac{1}{12}$ of the first and second — that is, with $\frac{1}{12}R$ [a twelfth of the first] and $1\frac{2}{3} + \frac{1}{36}R$ [a twelfth of the second] — must make 39: the third makes $64 - 2\frac{1}{3}R$, together with $1\frac{2}{3} + \frac{1}{9}R$, which comes to $- 2\frac{1}{3}$ combined with $+ \frac{1}{9}$, giving $- 2\frac{2}{9}$, as this formula shows:

$$\begin{array}{r} - 2\frac{1}{3} \\ + \frac{1}{9} \end{array}$$

$$- 2 \frac{2}{9}R$$

The number $65\frac{2}{3} - 2 \frac{2}{9}R$ must equal the number 39. Adding $2 \frac{2}{9}R$ to both sides, then, there will be an equality between $65\frac{2}{3}$ and $39 + 2 \frac{2}{9}R$. Removing 39 from both, between $26\frac{2}{3}$, or $80/3$, and $20/9R [= 2 \frac{2}{9}R]$. Dividing $80/3$ by $20/9$, then, gives $1R = 12$, the first number. And accordingly the second, which was found to be $20 + \frac{1}{3}R$, will be 24; while the third, found to be $64 - 2\frac{1}{3}R$, will be 36. And these three numbers, 12, 24, and 36, solve the problem.

One must always frame the equation with the number that the third, together with the given fraction of the others, is to make — as, in our example, was done with 39.

Problem 105

[Margin: CV.]

To divide a given number into two parts such that the former, together with a given other number, has to the latter, together with any other given number, any given ratio — if it can be done.

Let the number 100 be cut into two parts such that the former, with 10, makes a number triple the number that the latter part makes with 20. Let the former part be posited as $1R$, and accordingly the latter, $100 - 1R$. The former, with 10, makes $1R + 10$; and the latter, with 20, makes $120 - 1R$. So that the former may be triple the latter, there will necessarily be an equation between $1R + 10$ and $360 - 3R$. Adding $3R$ to both sides, between $4R + 10$ and 360. Removing 10 from both, between $4R$ and 350. Dividing 350 by 4, then, gives $1R = 87\frac{1}{2}$, the former part. The latter, $100 - 1R$, will be $12\frac{1}{2}$. The former, with 10, makes $97\frac{1}{2}$, a number triple the number $32\frac{1}{2}$ formed from the latter part and 20.

The clause "if it can be done" was added in the problem because the question is often impossible — as the equation itself will best warn. For example: if the number 100 is to be divided into two parts such that the former, with 10, makes a number seven times the number formed from the latter and 20, it can in no way be done. For if the former part is posited as $1R$ and the latter as $100 - 1R$, the former with 10 makes $1R + 10$, and the latter with 20 makes $120 - 1R$, of which it cannot be the sevenfold. For the equation would be between $1R + 10$ and $840 - 7R$. Adding $7R$ to both sides, between $8R + 10$ and 840. Removing 10 from both, between $8R$ and 830. Dividing 830 by 8, then, $1R$ would become $103\frac{3}{4}$, the former part — which is absurd, since it is greater than the given number 100.

You see, therefore, how efficiently the procedure itself teaches us when the proposed question can be solved and when it cannot — as we noted at the beginning of this book.

Problem 106

[Margin: CVI.]

To find a number such that, when any number of aliquot parts is subtracted from it, if the remaining number is multiplied by a given number, the result equals the number formed from the found number and the given number.

Let a number be sought such that, when $\frac{4}{7}$ is subtracted, if the remaining number is multiplied by 10, the result is as great as what is formed from the found number and the given number 10. Let the number sought be posited as $1R$. When $\frac{4}{7}$ is subtracted, there remain $\frac{3}{7}R$; and these, multiplied by 10, make the number $\frac{30}{7}R$, that is, $4\frac{2}{7}R$. And if to the posited number $1R$ the 10 is added, the number becomes $1R + 10$, equal to $4\frac{2}{7}R$. Removing $1R$ from both sides, then, there will be an equation between 10 and $3\frac{2}{7}R$. Dividing 10

by $3\frac{2}{7}$, that is, by $\frac{23}{7}$, gives $1R = \frac{70}{23}$, or $3\frac{1}{23}$, the number sought. For when $\frac{4}{7}$ of the found number $\frac{70}{23}$ — namely $\frac{40}{23}$ — is subtracted, and the remainder of the same found number is multiplied by 10, the number becomes $\frac{300}{23}$, or $13\frac{1}{23}$. And if to the found number $\frac{70}{23}$ the same given number 10 is added — that is, $\frac{230}{23}$ — the result is likewise the number $\frac{300}{23}$, or $13\frac{1}{23}$.

This problem is the same as the thirtieth, but proposed with somewhat varied wording.

Problem 107

[Margin: CVII.]

To find two numbers in a given ratio such that the multiplication of one by the other yields any given number.

Let two numbers in sesquialter ratio [3:2] be sought, such that the multiplication of one by the other yields the number 40. Let the numbers $2R$ and $3R$, in sesquialter ratio, be posited. Multiplying one by the other gives the number $6R$, equal to 40. Dividing 40 by 6, then, gives $\frac{1}{3} = \frac{40}{6}$; and $\sqrt[3]{\frac{40}{6}}$ [the square root of $\frac{40}{6}$] will be the value of one *res*. And since the first number was posited as $2R$ and the second as $3R$, the first number will be $\sqrt[3]{\frac{160}{6}}$ and the second $\sqrt[3]{\frac{360}{6}}$. Multiplying one by the other gives the number $\sqrt[3]{\frac{57600}{36}}$, that is, $\frac{240}{6}$, or 40.

Again, let two numbers in the same sesquialter ratio be sought, such that the multiplication of one by the other makes the number 294. Positing again the two numbers $2R$ and $3R$ in sesquialter ratio, the multiplication of one by the other yields the number $6\bar{3}$, equal to 294. Dividing 294 by 6 gives $1\bar{3} = 49$, whose root is 7. Therefore the first number, $2R$, will be 14, and the second, $3R$, will be 21. And 14 times 21 makes the proposed number 294. You see, then, that sometimes the numbers sought are surd, as in the former example, and sometimes rational, as in the latter.

Problem 108

[Margin: CVIII.]

To diminish, by equal amounts, two numbers given in any ratio, so that the remainder of the greater has to the remainder of the lesser any given ratio — provided it is greater than the ratio of the given numbers.

This problem is the same as the eighth, but proposed in other words. For here too a number is sought which, subtracted from two given numbers, leaves two numbers in a given ratio. Yet it seemed good to propose the very same thing in this form, and to solve it otherwise than the eighth problem was solved, so that you may learn to vary the solutions of the other problems as well. Let there be, then, two given numbers, 3 and 2, in sesquialter ratio $[3:2]$, to be diminished equally — that is, one and the same number is to be taken from each — so that the remainders have, say, the twentyfold ratio $[20:1]$. Let us suppose the lesser, when diminished, to have shrunk to $1R$, and accordingly the greater to $20R$, so that this remainder has to that remainder the twentyfold ratio. If, then, we take $20R$ from the greater 3, there will remain the number $3 - 20R$, which is the amount to be subtracted from 3 so that $20R$ is left, as this formula shows:

$$\begin{array}{r} 3 + 0R \\ 3 - 20R \\ \hline 0 + 20R \end{array}$$

Likewise, if from the lesser 2 we take $1R$, the remaining number will be $2 - 1R$, which is the amount to be subtracted from 2 so that $1R$ is left. And since both numbers are supposed to be diminished equally — that is, the same number is supposed removed from each — there will be an equation between $3 - 20R$ and $2 - 1R$. Adding $20R$ to both sides, between 3 and $2 + 19R$. Removing 2 from both, between 1 and $19R$. And $1R$ will be $1/19$. Since, then, from 3 must be taken $3 - 20R$, and from 2 must be taken $2 - 1R$, the number to be subtracted from each

will be $1 \frac{18}{19}$; and the remaining numbers will be $\frac{20}{19}$ and $\frac{1}{19}$, which have the proposed twentyfold ratio.

Problem 109

[Margin: CIX.]

To find two numbers in a given ratio such that the product of one by the other, together with the sum of the squares that arise from them, makes a given number.

Let two numbers in sesquialter ratio [3:2] be sought, such that the product of one by the other, together with their squares, makes 100. Let the numbers sought be posited as $3R$ and $2R$ in sesquialter ratio. The product of one by the other is the number $6\bar{3}$, and their squares are $9\bar{3}$ and $4\bar{3}$, which together with $6\bar{3}$ make $19\bar{3}$, equal to 100. Dividing 100 by 19, then, gives $1\bar{3} = 100/19$. Therefore $1R$ will be $\sqrt[3]{100/19}$, and $3R$ will be $\sqrt[3]{900/19}$ for the first number, which we posited as $3R$; and the second number, which was posited as $2R$, will be $\sqrt[3]{400/19}$, to which $\sqrt[3]{900/19}$ has the sesquialter ratio. The product of one by the other is the number $\sqrt[3]{360000/361}$, that is, $600/19$; and the sum of the squares, $900/19$ and $400/19$, is $1300/19$, which together with $600/19$ makes $1900/19$, that is, 100.

Again, let two numbers in sesquialter ratio be sought, such that the number arising from the product of one by the other, together with the sum of the squares that arise from them, makes the sum 2736. Repeating the same operation, $19\bar{3}$ will equal 2736. Dividing 2736 by 19, then, gives $1\bar{3} = 144$, and $1R = 12$. Therefore the first number, posited as $3R$, will be 36, and the second, which we posited as $2R$, will be 24. From 36 times 24 comes 864, which together with the squares 576 and 1296 makes 2736; and the numbers found, 36 and 24, have the sesquialter ratio.

Problem 110

[Margin: CX.]

To find a number which, multiplied by itself, and the product then multiplied by any given number, makes a number that has to the cube of the found number any given ratio.

Let a number be sought which, multiplied by itself, and this product multiplied by 10, makes a number in fivefold ratio to the cube of the found number — that is, one that contains its cube five times. Let the number sought be posited as $1R$. This, multiplied by itself, makes $1\bar{3}$; and $1\bar{3}$ multiplied by 10 makes $10\bar{3}$, which number must be five times the cube produced from $1R$. Now the cube of the

number 1℞ is 1℥ [cubus, x^3], which taken five times makes 5℥, equal to 103. Dividing 10 by 5, then, gives 1℞ = 2, the number sought — because the cossic quantities are collateral [adjacent in degree]. For 2 times 2 makes 4, and 4 times 10 makes the number 40, five times the cube 8 produced from 2.

If the number multiplied by itself, and [the product] multiplied by 10, must have the sesquialter ratio [3:2] to the cube, there will be an equation between 103 and 1½℥. Dividing 10 by 1½, then, or by 3/2, gives 1℞ = 20/3; and this number, multiplied by itself, makes 400/9.

Cossist notation used below (rendered as printed and glossed at first occurrence): ℞ = *res*, the unknown (x); 3 = *census* (x^2); ℥ = *cubus* (x^3); √3 = *radix census*, a square root. A coefficient stands before the sign (e.g. 40℞ = 40 x). The census sign 3 is confusable with the *sursolidum* sign ꝑ; every 3 below was pinned by the arithmetic of its own worked example, and no *sursolidum* occurs in these pages.

(Continuing the second worked example of Problem 110, whose unknown came out to 20/3.) ...comes out to $^{400}/_9$, and this, multiplied by 10, makes $^{4000}/_9$. Now the cube of the number $^{20}/_3$ is $^{8000}/_{27}$, and to this the number $^{4000}/_9$ — that is, $^{12000}/_{27}$ — stands in the sesquialteral proportion (3 : 2).

The same holds if a number multiplied by itself, and that product then multiplied by 10, is to stand to the cube [of the number] in the subduple proportion (1 : 2). Set the number as 1℞ [*res*, the unknown x]. From 1℞ times itself comes the number 13 [*census*, x^2], and from 13 times 10 the number 103, which must be the subduple of 1℥ [*cubus*, x^3], the cube produced from 1℞ multiplied cubically by itself. The equation, therefore, is between 203 and 1℥, and 1℞ will be 20. For 20 times itself makes 400, and 400 times 10 makes 4000, which is the subduple of the cube 8000, the cube produced from 20 multiplied cubically by itself.

Problem 111

[Margin: CXI.]

To cut a given number into two parts, so that the first, multiplied by any number, produces a number which exceeds — by a given number — the product formed from the posterior part multiplied by any other number, whenever this can be done.

Let 30 be the number to be divided into two parts, so that the first, multiplied by 10, produces a number which exceeds by 20 the number that the posterior part, multiplied by 40, produces. Let the first part be 1℞, and therefore the posterior 30

– 1R. From 1R times 10 comes the number 10R; and from (30 – 1R) times 40 comes the number 1200 – 40R, which the first product must exceed by 20. So if 20 is added to the posterior product, an equality results between 10R and 1220 – 40R. Adding 40R to both sides gives an equality between 50R and 1220. Dividing 1220 by 50, then, gives $1R = 24\frac{2}{5}$ as the first part, and the posterior will accordingly be $5\frac{3}{5}$. For the first multiplied by 10 makes 244, and the posterior multiplied by 40 makes 224, which the former exceeds by 20.

We said in the riddle "whenever this can be done," because the question cannot always be solved. For suppose the first product had to exceed the posterior product by 300: an equality would be found between 1500 – 40R and 10R. Adding 40R to both sides gives an equality between 1500 and 50R. Dividing 1500 by 50 gives $1R = 30$ — one part equal to the whole given number 30, which is absurd. Here too you see that from the very working of the rule of Algebra it can be known whether the question can be solved or not.

Problem 112

[Margin: CXII.]

To cut a given number into two parts, so that the first, multiplied by any number, makes a number that stands in a given proportion to the number formed from the posterior part multiplied by any other number.

Let 40 be the number to be cut into two parts, so that from the first times 10 there is made a number in subduple proportion (1 : 2) to the number formed from the posterior part times 30. Let the first part be 1R, and therefore the posterior 40 – 1R. From 1R times 10 comes the number 10R, which must be the subduple of the number 1200 – 30R produced from (40 – 1R) times 30. That is, an equation is made between 20R and 1200 – 30R. Adding 30R to both sides gives an equality between 50R and 1200. Dividing 1200 by 50, then, gives $1R = 24$ as the first part, and the posterior will accordingly be 16. For from 24 times 10 comes 240, which is the subduple of the number 480 formed from 16 times 30.

Problem 113

[Margin: CXIII.]

To distribute a given number into two, so that the first, divided by any given number, makes a Quotient standing in a given proportion to the Quotient formed from the posterior number divided by any [other] number.

Let 100 be the number to be distributed into two, so that the first, divided by 10, makes a Quotient in sextuple proportion (6 : 1) to the Quotient which the posterior number, divided by 20, makes. [The opening word is set as "SI I" (?) — apparently "SIT" with a stray or ornamental letter.] Let the first be $1R$, and therefore the posterior $100 - 1R$. Dividing the first by 10 gives the Quotient $\frac{1}{10}R$. Dividing the posterior by 20 gives the Quotient $5 - \frac{1}{20}R$. And because the former Quotient must have the sextuple proportion to the latter, the equation will be between $\frac{1}{10}R$ and $30 - \frac{6}{20}R$. Adding $\frac{6}{20}R$ — or $\frac{3}{10}R$ — to both sides gives an equality between $\frac{4}{10}R$ and 30. Dividing 30 by $\frac{4}{10}$, then, gives $1R = 75$ as the first number. The posterior will therefore be 25. Dividing 75 by 10 gives the Quotient $\frac{75}{10} [= 7\frac{1}{2}]$; dividing 25 by 20 gives the Quotient $\frac{25}{20} [= 1\frac{1}{4}]$, to which the former Quotient $\frac{75}{10}$ — or $\frac{150}{20}$ — has the sextuple proportion.

Problem 114

[Margin: CXIV.]

To find three numbers in a given continued proportion, so that the number produced from the smaller times the middle has to the larger any given proportion.

Let three numbers be sought in continued sesquitercian proportion (4 : 3), so that from the smaller times the middle there is made a number standing to the larger in quintuple proportion (5 : 1). Let the three sought numbers be posited as $9R$, $12R$, $16R$, in sesquitercian proportion. From $9R$ times $12R$ comes the number 1083, which must be the quintuple of the third, $16R$. The equation, therefore, will be between 1083 and $80R$. Dividing 80 by 108 gives $1R = \frac{80}{108}$, or $\frac{20}{27}$ — because the Cossic numbers are collateral. The first number, posited as $9R$, will therefore be $\frac{180}{27}$; the second, posited as $12R$, will be $\frac{240}{27}$; and the third, posited as $16R$, will be $\frac{320}{27}$. From the first times the second comes the number $\frac{43200}{749}$ [recte: 729], which is the quintuple of $\frac{320}{27}$, or of $\frac{8640}{724}$ [recte: 729].

If instead the number produced from the first times the second must equal the third, the equation will be between 1083 and $16R$. Dividing 16 by 108, then, gives $1R = \frac{4}{27} [= \frac{16}{108}]$, and accordingly the three sought numbers will be $\frac{36}{27}$, $\frac{48}{27}$, $\frac{64}{27}$ — that is, $1\frac{1}{3}$, $1\frac{7}{9}$, $2\frac{10}{27}$.

And if the product of the first times the second must be the half of the third, the equation will be between 2163 and $16R$. Dividing 16 by 216, then, gives $1R = \frac{16}{216}$, or $\frac{2}{27}$, and accordingly the three sought numbers will be $\frac{18}{27}$, $\frac{24}{27}$, $\frac{32}{27}$ — that is,

$\frac{2}{3}$, $\frac{8}{9}$, $1\frac{5}{27}$. And from $\frac{2}{3}$ times $\frac{8}{9}$ comes the number $1\frac{16}{27}$, which is the half of the third, $1\frac{5}{27}$.

Problem 115

[Margin: CXV.]

To divide a given number into three, so that when the first is multiplied by 4, the second by 3, and the third by 7, three numbers are produced in a given continued proportion.

Let 80 be the given number, to be divided into three, so that with the first multiplied by 4, the second by 3, and the third by 7, three numbers are produced in continued quadruple proportion (4 : 1). Let the first part be $1R$, which multiplied by 4 makes $4R$. Since the second part, multiplied by 3, must yield the number $16R$ — the quadruple, that is, of the number $4R$ — we must seek the number which, multiplied by 3, produces $16R$. That number is the Quotient of $16R$ divided by 3, namely $1\frac{16}{3}R$, and it will be the second part. Again, since the third part, multiplied by 7, must produce the number $64R$ — the quadruple, that is, of the number $16R$ — we must seek the number which, multiplied by 7, makes $64R$. That is $9\frac{1}{7}R$, the Quotient, namely, of $64R$ divided by 7. And so the three parts sought are $1R$, $1\frac{16}{3}R$, $9\frac{1}{7}R$, which, multiplied in order by 4, 3, and 7, produce $4R$, $16R$, and $64R$ in the given continued quadruple proportion.

It now remains that these three parts, collected into one sum, make 80, the given number. They make $15\frac{10}{21}R$ [mixed-number reading [?]], or $\frac{325}{21}R$. There is therefore an equation between $\frac{325}{21}R$ and 80. Dividing 80 by $\frac{325}{21}$ gives $1R = \frac{1680}{325}$, the first part, which was posited as $1R$. Then, since the second part was found to be $1\frac{16}{3}R$, if the value of the single root, $\frac{1680}{325}$, is multiplied by $1\frac{16}{3}R$, the second part $\frac{26880}{975}$ is produced. By the same reasoning, since the third part was found to be $9\frac{1}{7}R$, if the value of the single root, $\frac{1680}{325}$, is multiplied by $9\frac{1}{7}R$, the third part $\frac{107520}{2275}$ is produced. And these three parts — $\frac{1680}{325}$, $\frac{26880}{975}$, $\frac{107520}{2275}$, that is, $5\frac{11}{65}$, $27\frac{11}{195}$, $47\frac{119}{455}$ — collected into one, make 80, the given number. Multiplied in order by 4, 3, and 7, they produce the three numbers $\frac{6720}{325}$, $\frac{80640}{975}$ [numerator [?]], $\frac{752640}{2275}$, in continued quadruple proportion.

When the given number 80 is to be divided into three so that the several parts, each multiplied by one and the same number — say by 5 — yield three numbers in continued quadruple proportion, it will be done easily. For if, by riddle 49, the given number 80 is divided into three continued-quadruple parts, namely into $\frac{80}{21}$, $\frac{320}{21}$, $\frac{1280}{21}$, and each is multiplied by the proposed number 5, [Margin: a —

schol. 18. sept. [?]] there will be produced three numbers in the same quadruple proportion, $^{400}/_{21}$, $^{1600}/_{21}$, $^{6400}/_{21}$ — that is, $19^1/_{21}$, $76^4/_{21}$, $304^{16}/_{21}$.

Problem 116

[Margin: CXVI.]

To divide a given number into three parts, so that each, multiplied by three respective given numbers, produces the same number; or so that both the first two, multiplied by two given numbers, produce one and the same number, while the last two, multiplied by two other given numbers, likewise produce one and the same number.

Let 100 be the given number, to be divided into three so that, with the first multiplied by 5, the second by 2, and the third by 6, three equal numbers result. Let the first be $1R$; multiplied by 5, it makes $5R$. And since the second, multiplied by 2, must make just as much, we divide $5R$ by 2; the Quotient $^{5}/_{2}R$ will be the second number, since, multiplied by 2, it makes $5R$. And since the third, multiplied by 6, must make just as much, we divide $5R$ by 6; the Quotient $^{5}/_{6}R$ will be the third number, since, multiplied by 6, it makes $5R$. It now remains that all three parts — $1R$, $^{5}/_{2}R$, $^{5}/_{6}R$ — make the given number 100. They make $4^{13}/_{3}R$. There is therefore an equation between $4^{13}/_{3}R$ and 100; and dividing 100 by $4^{13}/_{3}$ — that is, by $^{13}/_{3}$ — gives $1R = 23^1/_{13}$, which yields the first number, posited as $1R$. The second, found as $^{5}/_{2}R$, will then be $57^9/_{13}$; and the third, found as $^{5}/_{6}R$, will be $19^3/_{13}$. And these three numbers — $23^1/_{13}$, $57^9/_{13}$, $19^3/_{13}$ — make the given number 100; multiplied by 5, 2, and 6, they produce the three equal numbers $115^5/_{13}$, $115^5/_{13}$, $115^5/_{13}$.

But now suppose the first two numbers, multiplied by 6 and 3, must produce two equal numbers, and likewise the last two, multiplied by 5 and 8. Again let the first be $1R$. From $1R$ times 6 comes $6R$. And since the second, multiplied by 3, must make just as much, we divide $6R$ by 3; the Quotient $2R$ will give the second number, since, multiplied by 3, it makes $6R$. Then from this second number, $2R$, times 5, comes $10R$. And since the third, multiplied by 8, must make just as much, we divide $10R$ by 8; the Quotient $1^{14}/_{4}R$ will be the third number, since, multiplied by 8, it yields $10R$. It now remains that these three numbers — $1R$, $2R$, $1^{14}/_{4}R$ — make the given number 100. They make $4^{14}/_{4}R$, which must equal 100. Dividing 100 by $4^{14}/_{4}$ — that is, by $^{17}/_{4}$ — gives $1R = 23^9/_{17}$, the first number, which was posited as $1R$. The second, found as $2R$, will then be $47^1/_{17}$; and the third, found as $1^{14}/_{4}R$, will be $29^7/_{17}$. And these three numbers — $23^9/_{17}$, $47^1/_{17}$, $29^7/_{17}$ — make 100 together.

The first two, multiplied by 6 and 3, generate the same number $141^3/17$; and the last two, multiplied by 5 and 8, produce the same number $235^5/17$.

Problem 117

[Margin: CXVII.]

To distribute a given number into three parts, so that when the first part is divided by any number, the second multiplied by another given number, and the third divided by yet any other number, three equal numbers result.

Let 178 be the given number, to be divided into three parts, so that as much is made from dividing the first part by 5 as from multiplying the second by 8, and as much as from dividing the third by 6. Let the first part be $1R$; divided by 5, it makes the Quotient $\frac{1}{5}R$. And since just as much must come from the second multiplied by 8, we divide $\frac{1}{5}R$ by 8; the Quotient $\frac{1}{40}R$ will be the second part, which, multiplied by 8, makes $\frac{1}{5}R$. And since the third, divided by 6, must make just as much, we multiply $\frac{1}{5}R$ by 6; the number produced, $\frac{6}{5}R$, will be the third part, which, divided by 6, gives the Quotient $\frac{1}{5}R$. It now remains that the three parts — $1R$, $\frac{1}{40}R$, and $\frac{6}{5}R$ — make the given number 178. They make $2^9/40R$. The equation, therefore, is between 178 and $2^9/40R$. Dividing 178 by $2^9/40$ — that is, by $89/40$ — gives $1R = 80$, the first part, which we posited as $1R$. The second will be 2, obtained by multiplying the root found, 80, by $\frac{1}{40}$ — because the second part was found to be $\frac{1}{40}R$. The third, finally, will be 96, produced by multiplying the same root found, 80, by $\frac{6}{5}$ — because the third part was found to be $\frac{6}{5}R$. And these three parts — 80, 2, 96 — make up 178, the given number. Whether the first is divided by 5, or the second multiplied by 8, or the third divided by 6, the same number, 16, is always produced.

Problem 118

[Margin: CXVIII.]

To cut a given number into three parts, so that each, divided by three respective given numbers, makes one and the same Quotient.

Let 240 be the given number, to be distributed into three parts, so that with the first divided by 5, the second by 3, and the third by 7, the same Quotient is always produced. Let the first part be $1R$; divided by 5, it gives the Quotient $\frac{1}{5}R$. And since just as much must come from the second part divided by 3, we multiply $\frac{1}{5}R$ by 3; the number produced, $\frac{3}{5}R$, will be the second part, which, divided by 3, makes the Quotient $\frac{1}{5}R$. And since from dividing the third part by 7 there must

also be produced $\frac{1}{5}R$, we multiply $\frac{1}{5}R$ by 7; the number produced, $\frac{7}{5}R$, will be the third part, which, divided by 7, gives the Quotient $\frac{1}{5}R$. It remains that these three parts — $1R$, $\frac{3}{5}R$, $\frac{7}{5}R$ — taken together are 240. They make $3R$. There is therefore an equation between $3R$ and 240; and dividing 240 by 3 gives $1R = 80$, the first part. The second will be 48, obtained by multiplying the root found, 80, by $\frac{3}{5}$, since the second part was found to be $\frac{3}{5}R$. The third, finally, will be 112, obtained by multiplying the same root found, 80, by $\frac{7}{5}$, since the third part was found to be $\frac{7}{5}R$. And all these three parts — 80, 48, 112 — make 240, the number proposed. And whether the first is divided by 5, or the second by 3, or the third by 7, the same Quotient, 16, is always obtained.

Problem 119

[Margin: CXIX.]

To find two numbers in a given proportion, so that any aliquot part of the smaller, multiplied by another given part of the larger, produces any given number.

Let two numbers be sought in triple proportion (3 : 1), so that $\frac{1}{4}$ of the smaller, multiplied by $\frac{1}{2}$ of the larger, produces the number 90. Let the smaller be $1R$ and the larger $3R$, so that they have the given triple proportion. From a quarter of the smaller — namely $\frac{1}{4}R$ — times half the larger — that is, times $\frac{3}{2}R$ — there is made the number $\frac{3}{8}R$, equal to 90. Dividing 90 by $\frac{3}{8}$, then, gives $13 = 240$; and $1R$ will be $\sqrt[3]{240}$ [$\sqrt[3]{240}$], the smaller number. The larger will be its triple, $\sqrt[3]{2160}$ [$\sqrt[3]{2160}$]. Now $\frac{1}{4}$ of the smaller — namely $\sqrt[3]{315}$ [$\sqrt[3]{15}$] — multiplied by $\frac{1}{2}$ of the larger — namely by $\sqrt[3]{540}$ [$\sqrt[3]{540}$] — makes $\sqrt[3]{8100}$ [$\sqrt[3]{8100}$], that is, the given number 90.

Problem 120

[Margin: CXX.]

To find a number such that, when it is multiplied by a given number and the [result] divided by another given number, the Quotient has a given proportion to the number found.

Let a number be sought which, multiplied by 5 and the product divided by 60, gives a Quotient standing to the number found in the sesquialteral proportion (3 : 2). Let the number sought be $1R$. Multiplied by 5, it makes $5R$. Dividing 60 by $5R$ gives the Quotient $\frac{60}{5R}$, which must have the sesquialteral proportion to $1R$. Now the quantity $1\frac{1}{2}R$ — that is, $\frac{3}{2}R$, produced from $1R$ times $1\frac{1}{2}$, i.e. times $\frac{3}{2}$ — also

has the sesquialteral proportion to $1R$. There is therefore an equation between $60/5R$ and $3/2R$ [that is, $60/(5R) = (3R)/2$], which by cross-multiplication is reduced to an equation between 120 and 153 . Dividing 120 by 15, then, gives $13 = 8$; and $1R$ will be $\sqrt{38}$ [$\sqrt{8}$], the number we are seeking. For from $\sqrt{38}$ times 5 comes $\sqrt{3200}$ [$\sqrt{200}$]; and if 60 is divided by this number, the Quotient $\sqrt{318}$ [$\sqrt{18}$] results, which has the sesquialteral proportion to $\sqrt{38}$ — since $\sqrt{38}$, multiplied by $1\frac{1}{2}$, makes $\sqrt{318}$.

Diagram. A large decussate cross (a Saint Andrew's "X") set as a cross-multiplication scheme for the two ratios of Problem 120: the fraction $60/5R$ on the left and the fraction $3R/2$ on the right, with the two cross-products written above.

- Upper left (above the left fraction): 120 — the cross-product 60×2
- Left, above the bar: 60 — numerator of the left fraction
- Lower left, below the "X": $5R$ — denominator of the left fraction
- Upper right (above the right fraction): $15\ 3$ — the cross-product $5R \times 3R$
- Right, above the bar: $3R$ — numerator of the right fraction
- Lower right, below the "X": 2 [small mark, partly a flourish [?]] — denominator of the right fraction

Problem 121

[Margin: CXXI.]

To find three numbers in arithmetic proportion, so that their sum is a given aliquot part of the number produced from their multiplication with one another; or (what is the same) so that the number produced from their multiplication has to their sum a given proportion.

Let the sum of the numbers found be required to be $\frac{1}{5}$ of the number produced from their multiplication — that is, let the number produced have to the sum the quintuple proportion ($5 : 1$). Let the three sought numbers, arithmetically proportional, be posited as $1R$, $2R$, $3R$, which exceed one another equally by the number $1R$. Their sum is $6R$. But the number produced by multiplication is $6C$: for from $1R$ times $2R$ comes the number 23 , and from 23 times $3R$ comes the number $6C$, which must be $\frac{1}{5}$ of the number $6C$ [i.e. the sum $6R$ must be $\frac{1}{5}$ of the product $6C$]. The equation will thus be between $30R$ and $6C$. Dividing 30 by 6, then, gives $13 = 5$ — because between $1R$ and $1C$ one denomination, namely the census, lies interposed — and so $1R$ will be $\sqrt{35}$ [$\sqrt{5}$], and $2R$ will be $\sqrt{320}$ [$\sqrt{20}$], and $3R$ will be $\sqrt{345}$ [$\sqrt{45}$]. So the three numbers found are $\sqrt{35}$, $\sqrt{320}$, $\sqrt{345}$, as is clear if the first is subtracted from the second and the second from the third.

Their sum is $\sqrt{3180}$ [$\sqrt{180}$], and the number produced from their multiplication is $\sqrt{34500}$ [$\sqrt{4500}$]. Now $\sqrt{3180}$ is indeed $\frac{1}{5}$ of the number $\sqrt{34500}$ — that is, $\sqrt{34500}$ has to $\sqrt{3180}$ the quintuple proportion — since the number $\sqrt{3180}$, multiplied by 5, makes $\sqrt{34500}$.

Next, let the number produced be required to have to the sum the proportion of 24 to 1 — that is, let the sum be $\frac{1}{24}$ of the number produced. Let the arithmetically proportional numbers be posited as $2R$, $3R$, $4R$, whose sum is $9R$ and whose product is $24C$. And since $9R$ must be $\frac{1}{24}$ of the number $24C$ — that is, this number must have to that one the proportion of 24 to 1 — if $9R$ is multiplied by 24 there results an equality between $216R$ and $24C$. Dividing 216 by 24, then, gives $13 = 9$ and $1R = 3$. The three sought numbers will therefore be 6, 9, 12, which were posited as $2R$, $3R$, $4R$. Their sum, 27, is $\frac{1}{24}$ of the number 648 produced from their multiplication — because from 27 times 24 the number 648 is generated.

Problem 122

[Margin: CXXII.]

To find three numbers in a given continued proportion, so that from their multiplication a given number is produced.

Let three numbers be sought, continuously quadruple, which multiplied by one another produce the number 27. Let the sought numbers be posited as $1R$, $4R$, $16R$, in continued quadruple proportion. From $1R$ times $4R$ comes the number 43 , and from 43 times $16R$ comes the number $64C$, equal to 27. Dividing 27 by 64, then, gives $1C = \frac{27}{64}$ and $1R = \frac{3}{4}$. The three sought numbers are therefore $\frac{3}{4}$, 3, 12 — quadruples that, multiplied by one another, produce 27.

Again, let three sesquialteral numbers be sought which, multiplied by one another, make 12. Let the three sesquialteral numbers be posited as $1R$, $1\frac{1}{2}R$, $2\frac{1}{4}R$ — that is, $1R$, $\frac{3}{2}R$, $\frac{9}{4}R$. From $1R$ times $\frac{3}{2}R$ comes the number $\frac{3}{2}3$. And from $\frac{3}{2}3$

Cossist symbols used below (rendered as printed, glossed at first use): **R** = *res* (the unknown, x); **3** = *census* (x^2); **C** = *cubus* (x^3); **3C** = *zensicubus* (x^6); $\sqrt{C}$ = cube root (*radix cubica*); $\sqrt{3}$ = square root (*radix quadrata*); **N** = *numerus* (a plain number). The signs + and − are printed as in the source.

multiplied into $\frac{9}{4}R$, gives $\frac{27}{8}C$ [*cubus*, x^3] — that is, $3\frac{3}{8}C$ — equal to 12. Dividing 12 by $\frac{27}{8}$, then, gives $1C = 3\frac{5}{9}$, and so $1R = \sqrt{C} 3\frac{5}{9}$ [the cube root of $3\frac{5}{9}$]: this is the first number, which we posited as $1R$. The second, posited as $1\frac{1}{2}R$, will therefore be $\sqrt{C} 12$; and the third, posited as $2\frac{1}{4}R$, will be $\sqrt{C} 40\frac{1}{2}$. For 1

$1/2$ multiplied into $\sqrt[3]{3 \frac{5}{9}}$ gives $\sqrt[3]{12}$, and $2 \frac{1}{4}$ into $\sqrt[3]{3 \frac{5}{9}}$ gives $\sqrt[3]{40 \frac{1}{2}}$. Now $\sqrt[3]{3 \frac{5}{9}}$ into $\sqrt[3]{12}$ gives the number $\sqrt[3]{42 \frac{2}{3}}$, and $\sqrt[3]{42 \frac{2}{3}}$ into $\sqrt[3]{40 \frac{1}{2}}$ gives the number $\sqrt[3]{1728}$ — that is, 12.

Problem 123 (CXXIII). *To find a number which, multiplied into its own square root, produces a given number.*

Let a number be sought which, multiplied into its own square root, makes 4. Posit the number sought as $1 \frac{3}{4}$ [*census*, x^2]. Multiplied into its own square root — namely $1 \frac{3}{4}$ — this makes $1 \frac{3}{4}$ [*cubus*, x^3], equal to 4. Dividing 4 by $1 \frac{3}{4}$, then, gives $1 \frac{3}{4} = 4$, and its cube root is $\sqrt[3]{4}$ [$\sqrt[3]{4}$], whose square is $\sqrt[3]{16}$ [$\sqrt[3]{16}$] — the number sought, which we posited as $1 \frac{3}{4}$. For this number, multiplied into its own square root $\sqrt[3]{4}$, makes $\sqrt[3]{64}$, that is, 4.

Next, let the number to be produced be $15 \frac{5}{8}$. An equality will be found between $1 \frac{3}{4}$ and $15 \frac{5}{8}$. Dividing $15 \frac{5}{8}$ by $1 \frac{3}{4}$, then, gives $1 \frac{3}{4} = 15 \frac{5}{8}$, that is, $125/8$, whose cube root is $5/2$; and the square of this is $25/4$ — the number sought, which we posited as $1 \frac{3}{4}$. And this, multiplied into its own square root $5/2$, makes $125/8$, that is, $15 \frac{5}{8}$.

Problem 124 (CXXIV). *To find a number whose square root has a given ratio to its own cube root.*

Let a number be sought whose square root has the sesquialtera ratio (3:2) to its own cube root. Posit the number sought as $1 \frac{3}{4}$ [*zensicubus*, x^6], so that it may have both roots as proposed. Its square root is $1 \frac{3}{4}$ [*cubus*], and its cube root is $1 \frac{3}{4}$ [*census*] — as is clear from what we taught at the beginning of chapter 12; for $1 \frac{3}{4}$ squared makes $1 \frac{3}{4}$, and $1 \frac{3}{4}$ cubed likewise makes $1 \frac{3}{4}$. It remains, then, that $1 \frac{3}{4}$ should have the sesquialtera ratio to $1 \frac{3}{4}$; which is the same as requiring that $1 \frac{1}{2}$ should have the sesquialtera ratio to $1 \frac{3}{4}$ [?]. (The cossic terms are collateral.) Its *zensicubus* is $729/64$, or $11 \frac{25}{64}$; and the cube root of this is $9/4$, or $2 \frac{1}{4}$. And indeed $3 \frac{3}{8}$ to $2 \frac{1}{4}$ — that is, $27/8$ to $18/8$ — has the sesquialtera ratio.

If the ratio proposed is the triple ratio, an equation is found by the same reasoning between $1 \frac{3}{4}$ and $3 \frac{3}{4}$. Dividing 3 by $1 \frac{3}{4}$ gives $1 \frac{3}{4} = 3$, whose *zensicubus* is 729; and the square root of this, 27, has the triple ratio to the cube root, 9.

Problem 125 (CXXV). *To find two numbers such that from one multiplied into the other a number is generated which has two given ratios to the two found numbers — the greater ratio to the smaller, and the smaller to the greater.*

Let two numbers be sought such that from one multiplied into the other a number arises which has the triple-and-a-half ratio [*tripla sesquialtera*, 7:2] to the

smaller and the double ratio to the greater. Posit the number produced as 1 R. Then the smaller number sought will be $\frac{2}{7}$ R, and the greater $\frac{1}{2}$ R — since 1 R has the triple-and-a-half ratio to $\frac{2}{7}$ R and the double ratio to $\frac{1}{2}$ R. From $\frac{2}{7}$ R into $\frac{1}{2}$ R arises $\frac{2}{14}$ 3 [*census*], or $\frac{1}{7}$ 3, equal to the posited product 1 R. Dividing 1 by $\frac{1}{7}$, then, gives 1 R = 7 — namely the number that ought to be produced, since the cossic terms are collateral. And because the smaller was found as $\frac{2}{7}$ R, the smaller is 2; and the greater, found as $\frac{1}{2}$ R, is 3 $\frac{1}{2}$. It is clear that 7 has the triple-and-a-half ratio to 2, and the double ratio to 3 $\frac{1}{2}$.

Problem 126 (CXXVI). *To find two numbers making a given sum, such that either one together with a given part (or parts) of the other makes any given number — one which is less than the given sum and greater than the given part(s) of that same sum.*

Let two numbers be sought whose sum is 60, such that the first, together with $\frac{3}{7}$ of the second, makes 50. Posit the first as 1 R; the second is therefore $60 - 1$ R. Its $\frac{3}{7}$ is $25 \frac{5}{7} - \frac{3}{7}$ R, which together with 1 R makes $\frac{4}{7}$ R + $25 \frac{5}{7}$ — a number equal to 50. (For, so that $-\frac{3}{7}$ R may be added to 1 R, we must subtract $\frac{3}{7}$ R from 1 R, and so $\frac{4}{7}$ R remains with the sign +, as was explained above under subtraction.) Removing $25 \frac{5}{7}$ from both sides, an equation arises between $\frac{4}{7}$ R and $24 \frac{2}{7}$. Dividing $24 \frac{2}{7}$ by $\frac{4}{7}$, then, gives 1 R = $42 \frac{1}{2}$ — the first number; and the second will be $17 \frac{1}{2}$, whose $\frac{3}{7}$ is $7 \frac{1}{2}$, which together with the first part makes 50.

If the first number together with $\frac{3}{7}$ of the second were required to make 25, it could not be done, because 25 is smaller than $\frac{3}{7}$ of the given sum 60, as you can test. But if the first together with $\frac{3}{7}$ of the second is to make 26, it can be done: the first number will be $\frac{1}{2}$ and the second $59 \frac{1}{2}$, whose $\frac{3}{7}$ is $25 \frac{1}{2}$, which together with the first part makes 26.

The puzzle is solved in the same way if the second number together with $\frac{3}{7}$ of the first is required to make the given number — namely, if the second number is posited as 1 R, and so on.

Problem 127 (CXXVII). *To find five numbers such that any four of them taken without the remaining one make five given numbers — provided that a quarter of the sum of the five numbers sought is greater than each of the required numbers.*

Let five numbers be sought such that four of them without the first make 127; four without the second, 119; four without the third, 109; four without the fourth, 104; and four without the fifth, 97. Here you see that a quarter of the sum 556 — collected from the given numbers 127, 119, 109, 104, 97 — is 139, a number

greater than each of those numbers. Now since the sum of the four without the fifth is the smallest of all, namely 97, the fifth number will necessarily be the greatest of all, seeing that the fifth together with any other three makes a larger sum than 97.

Posit the fifth number, then, as 1 R. Since the other four without the fifth make 97, all five numbers together will be $97 + 1 \text{ R}$. And because four without the fourth make 104, subtract 104, and the fourth will remain as $1 \text{ R} - 7$. Next, because four without the third make 109, take 109 from the same sum $97 + 1 \text{ R}$, and the third remains $1 \text{ R} - 12$. Again, since four without the second make 119, if 119 is taken from the same sum, the second is left as $1 \text{ R} - 22$. Finally, because four without the first make 127, if 127 is removed from the same sum, the first remains $1 \text{ R} - 30$. It now remains that all five of these found numbers together should make the sum of all, $97 + 1 \text{ R}$. But they make $5 \text{ R} - 71$. There will therefore be an equation between $97 + 1 \text{ R}$ and $5 \text{ R} - 71$. Adding 71 to both sides gives an equation between $168 + 1 \text{ R}$ and 5 R ; removing 1 R from each side, between 168 and 4 R. Dividing 168 by 4, then, gives $1 \text{ R} = 42$ — the fifth number, which we posited as 1 R. Therefore the fourth, found as $1 \text{ R} - 7$, is 35; the third, found as $1 \text{ R} - 12$, is 30; the second, found as $1 \text{ R} - 22$, is 20; and the first, found as $1 \text{ R} - 30$, is 12. So the five numbers sought are, in order, 12, 20, 30, 35, 42, which dispatch the proposed puzzle.

This puzzle was also solved in four numbers, in puzzle 59.

Problem 128 (CXXVIII). *To find a number such that three squares formed from three given parts of it, and collected into a single sum, produce the found number itself, or any other given number.*

First, let a number be sought such that three squares formed from its $\frac{2}{3}$, $\frac{3}{5}$, and $\frac{1}{2}$, and collected into a single sum, produce the found number itself. Posit that number as 1 R. Its $\frac{2}{3}$ will be $\frac{2}{3} \text{ R}$, its $\frac{3}{5}$ will be $\frac{3}{5} \text{ R}$, and its $\frac{1}{2}$ will be $\frac{1}{2} \text{ R}$. The squares of these parts — $\frac{4}{9} \text{ R}$, $\frac{9}{25} \text{ R}$, and $\frac{1}{4} \text{ R}$ — together make $\frac{949}{900} \text{ R}$, a number equal to the posited number 1 R. Dividing 1 by $\frac{949}{900}$, then, gives $1 \text{ R} = \frac{900}{949}$, the number sought. For its $\frac{2}{3}$ is $\frac{1800}{2847}$, whose square is $\frac{3240000}{8105409}$; its $\frac{3}{5}$ is $\frac{2700}{4745}$, whose square is $\frac{7290000}{22515025}$; and its $\frac{1}{2}$ is $\frac{900}{1898}$, whose square is $\frac{810000}{3602404}$. These three squares make the sum $\frac{6234707464935966900}{6574152649138036209}$ [?], which is equal to the found number $\frac{900}{949}$. For the same number is produced from the numerator of the one into the denominator of the other, and from the denominator into the numerator — namely the number 5916737384224232588100 [?].

Next, let a number be sought such that three squares formed from its $\frac{2}{3}$, $\frac{3}{5}$, and $\frac{1}{2}$ make the number 3796. As before, the three squares collected into one sum will be $\frac{949}{900} \mathfrak{z}$, which must equal 3796. Dividing 3796 by $\frac{949}{900}$, then, gives $1 \mathfrak{z} = 3600$, and $1 \mathfrak{R} = 60$, the number sought. For its $\frac{2}{3}$, $\frac{3}{5}$, and $\frac{1}{2}$ are 40, 36, and 30, and the squares of these numbers — 1600, 1296, 900 — make the sum 3796.

Finally, let a number be sought such that three squares formed from its $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{9}$ make the sum 20. Posit the number sought as $1 \mathfrak{R}$. Its given parts are $\frac{1}{2} \mathfrak{R}$, $\frac{1}{3} \mathfrak{R}$, and $\frac{1}{9} \mathfrak{R}$, whose squares — $\frac{1}{4} \mathfrak{z}$, $\frac{1}{9} \mathfrak{z}$, and $\frac{1}{81} \mathfrak{z}$ — make the sum $\frac{121}{324} \mathfrak{z}$ [?], equal to the number 20. Dividing 20 by $\frac{121}{324}$, then, gives $1 \mathfrak{z} = \frac{6480}{121}$, and $1 \mathfrak{R} = \sqrt{\mathfrak{z}} \frac{6480}{121}$ [the square root of $\frac{6480}{121}$], the number sought. For its $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{9}$ are $\sqrt{\mathfrak{z}} \frac{6480}{484}$, $\sqrt{\mathfrak{z}} \frac{6480}{1089}$, and $\sqrt{\mathfrak{z}} \frac{6480}{9801}$; and the squares of these parts — $\frac{6480}{484}$, $\frac{6480}{1089}$, and $\frac{6480}{9801}$ — make the sum $\frac{103317437520}{5165871876}$ [?], that is, 20.

Problem 129 (CXXIX). *To divide a given number into any number of parts, such that each following one exceeds the preceding by a given number.*

Let the given number 200 be distributed into six parts, such that the second exceeds the first by 5, the third the second by 2, the fourth the third by 10, the fifth the fourth by 1, and the sixth the fifth by 15. Posit the first as $1 \mathfrak{R}$. Then the second will be $1 \mathfrak{R} + 5$; the third, $1 \mathfrak{R} + 7$; the fourth, $1 \mathfrak{R} + 17$; the fifth, $1 \mathfrak{R} + 18$; and the sixth, $1 \mathfrak{R} + 33$. All these make $6 \mathfrak{R} + 80$, and this sum must equal the given number 200. Removing 80 from each side, then, there will be an equation between $6 \mathfrak{R}$ and 120. Dividing 120 by 6 gives $1 \mathfrak{R} = 20$, the first number. The second is therefore 25; the third, 27; the fourth, 37; the fifth, 38; and the sixth, 53 — which together make 200 and complete the problem.

But the given number must be greater than the sum collected from the given excesses, when each successive excess is added to the preceding. Thus in the given example, since the cumulative amounts are 5, 7, 17, 18, and 33, which all make 80, the given number must be greater than 80 — say 81, or 82, or 200, or 300, and so on. The operation itself will show this; for if the given number were 79, there would be an equation between $6 \mathfrak{R} + 80$ and 79, which is impossible.

Problem 130 (CXXX). *To find two numbers in a given ratio, such that either one multiplied into the square of the other makes any given number.*

Let two numbers be sought in the triple ratio, such that either one multiplied into the square of the other makes 192. Posit the smaller as $1 \mathfrak{R}$; the greater is therefore $3 \mathfrak{R}$. From the greater, $3 \mathfrak{R}$, into $1 \mathfrak{z}$ (the square of the smaller) arises $3 \mathfrak{C}$, equal to 192. Dividing 192 by 3, then, gives $1 \mathfrak{C} = 64$, and $1 \mathfrak{R} = 4$, the smaller

number. The greater will therefore be 12; for 12 into 16 (the square of the smaller number, 4) makes the given number 192.

Conversely, from the smaller, 1 R, into 9 3 (the square of the greater) arises 9 c, equal to 192. Dividing 192 by 9, then, gives 1 c = $21 \frac{1}{3}$, or $\frac{64}{3}$, and $1 R = \sqrt{c} \frac{64}{3}$, the first number; while the second, the triple of this, will be $\sqrt{c} \frac{1728}{3}$. If into the square of this, $\sqrt{c} \frac{2985984}{9}$, the first, $\sqrt{c} \frac{64}{3}$, is multiplied, the result is the number $\sqrt{c} \frac{191102976}{27}$, or $\sqrt{c} 7077888$, that is, 192.

Problem 131 (CXXXI). *Given two numbers, to find a third, such that of the three numbers any two taken together, if multiplied into the third, produce three numbers in arithmetic proportion.*

Let the numbers 10 and 30 be given, and let a third be sought such that, taken two at a time and multiplied into the third, they produce three numbers in arithmetic proportion. Posit the number sought as 1 R. The sum of 1 R and 10 is $1 R + 10$, which multiplied into the third, 30, makes $30 R + 300$. The sum of 1 R and 30 is $1 R + 30$, which multiplied into the third, 10, makes $10 R + 300$. Finally, the sum of 10 and 30 is 40, which multiplied into the third, 1 R, makes $40 R$. So the three numbers produced are $30 R + 300$, $10 R + 300$, and $40 R$, which must have the same common difference. But since it is not yet settled which of them is greatest, which the middle, and which the least, let us set $30 R + 300$ as the greatest, $10 R + 300$ as the middle, and $40 R$ as the least. The difference of the first pair is $20 R$; that of the latter, $300 - 30 R$; and these differences must be equal. Adding $30 R$ to each side, then, there will be an equation between $50 R$ and 300. Dividing 300 by 50 gives $1 R = 6$, the number sought — so that the three numbers are 6, 10, 30. From 6 and 10 arises the sum 16, which multiplied into 30 makes 480; from 6 and 30 arises the sum 36, which multiplied into 10 makes 360; finally, from 10 and 30 arises the sum 40, which multiplied into 6 makes 240. And these three products, 480, 360, 240, are in arithmetic proportion, since their common difference is 120.

But if we set the greatest as $30 R + 300$ (with $40 R$ as the middle and $10 R + 300$ as the least), the difference of the first pair is $300 - 10 R$, and that of the latter $30 R - 300$, which must be equal. There will then be an equation between $600 - 10 R$ and $30 R$; and adding $10 R$ again to both sides, between 600 and $40 R$. Dividing 600 by 40 gives $1 R = 15$, the number sought, and the three numbers will be 15, 10, 30. For from 15 and 10 arises the sum 25, which multiplied into 30 makes 750; from 15 and 30 arises the sum 45, which multiplied into 10 makes 450; finally, from 10 and 30 arises the sum 40, which multiplied into 15 makes 600. And these three products, 750, 600, 450, have the same common difference, 150.

Nor indeed can the greatest be 40 R, with 30 R + 300 as the middle and 10 R + 300 as the least; for the difference of the first pair would be 10 R – 300, and that of the latter 20 R, which cannot equal the former difference, since it is greater.

Therefore the greatest is 30 R + 300, since it is also greater than 10 R + 300; and either of the others can be the middle or the least, as has been shown.

This problem was also solved in puzzle 86, where two numbers are given so that the three numbers produced can be arithmetically proportional in three ways.

Problem 132 (CXXXII). *To find three numbers such that the first and second together with a given part of the third, likewise the second and third together with a given part of the first, and also the third and first together with a given part of the second, produce one and the same given number.*

Let the first and second together with $\frac{1}{4}$ of the third, the second and third together with $\frac{1}{3}$ of the first, and the third and first together with $\frac{1}{2}$ of the second, all be required to make 100. Posit the third as 1 R. Its $\frac{1}{4}$ — namely $\frac{1}{4}$ R — together with the first and second must make 100; and therefore these two, before they received $\frac{1}{4}$ R from the third, amounted to $100 - \frac{1}{4}$ R. And because the second and third together with $\frac{1}{3}$ of the first must likewise make 100, if we take the third — namely 1 R — from 100, the remaining number $100 - 1$ R is the second alone together with $\frac{1}{3}$ of the first.

Now the second together with the whole first made $100 - \frac{1}{4}$ R; and since these two together exceed [the second with $\frac{1}{3}$ of the first] by half of that same first, if from $100 - \frac{1}{4}$ R (the sum of first and second) we take away $100 - 1$ R, the remaining number $\frac{3}{4}$ R is half of the first. The whole first is therefore $1\frac{1}{2}$ R [that is, $\frac{3}{2}$ R]. Then, since the first and second together with $\frac{1}{4}$ of the third make 100, and the first alone is $1\frac{1}{2}$ R while $\frac{1}{4}$ of the third is $\frac{1}{4}$ R, the second must be $100 - 1\frac{3}{4}$ R (the first plus the quarter of the third being $1\frac{3}{4}$ R) — namely the second alone. Its $\frac{1}{3}$ is $33\frac{1}{3} - \frac{7}{12}$ R, which together with the third and first (that is, with $2\frac{1}{2}$ R) must make 100.

There arises, then, from $33\frac{1}{3} - \frac{7}{12}$ R and $2\frac{1}{2}$ R — that is, from $\frac{1}{3}$ of the second and the sum of third and first — a number $33\frac{1}{3} + 1\frac{11}{12}$ R, which must equal 100. Removing $33\frac{1}{3}$ from each side, there is an equality between $66\frac{2}{3}$ and $1\frac{11}{12}$ R. Dividing $66\frac{2}{3}$ by $1\frac{11}{12}$ gives $1\text{ R} = \frac{2400}{69}$ — that is, the third number, which was posited as 1 R. The second, found as $100 - 1\frac{3}{4}$ R, will therefore be $\frac{2700}{69}$; and the first, which we found to be $1\frac{1}{2}$ R, will be $\frac{3600}{69}$. And these three satisfy the proposed puzzle.

Problem 133 (CXXXIII). *To divide a given number into three continuously proportional parts, such that the number produced from the first into the third has a given ratio to the product of the first into the second.*

[Margin: a 20. Sept.]

Let the given number 30 be divided into three continuously proportional parts, such that the number produced from the first into the third has the quadruple ratio to the product of the first into the second. Posit the second, or middle, part as 1 R; the sum of the first and third is therefore $30 - 1 \text{ R}$. And because the middle 1 R multiplied into itself makes 1 $\bar{3}$, [a] the product of the first into the third will likewise be 1 $\bar{3}$. Now since this product must have the quadruple ratio to the product of the first into the second — namely into 1 R — if we divide 1 $\bar{3}$ by the divisor 1 R, it makes $\frac{1}{4} \text{ R}$; and this, multiplied into 1 R, produces $\frac{1}{4} \bar{3}$... [that is, the product of first into second is $\frac{1}{4} \bar{3}$, one quarter of 1 $\bar{3}$]. Hence the first part will be $\frac{1}{4} \text{ R}$, which together with the second makes $1 \frac{1}{4} \text{ R}$; and this sum, taken from $30 - 1 \text{ R}$ [the sum of first and third — i.e. from 30], leaves the third part $30 - 1 \frac{1}{4} \text{ R}$.

It now remains to multiply the first part, $\frac{1}{4} \text{ R}$, into the third, $30 - 1 \frac{1}{4} \text{ R}$ — which product must equal what is produced from the middle part, 1 R, into itself, namely 1 $\bar{3}$. There arises from $\frac{1}{4} \text{ R}$ into $30 - 1 \frac{1}{4} \text{ R}$ the number $7 \frac{1}{2} \text{ R} - \frac{5}{16} \bar{3}$, which must equal 1 $\bar{3}$. Adding $\frac{5}{16} \bar{3}$ to both sides, there will be an equation between $7 \frac{1}{2} \text{ R}$ and $1 \frac{5}{16} \bar{3}$. Dividing $7 \frac{1}{2}$ by $1 \frac{5}{16}$ gives $1 \text{ R} = \frac{40}{7}$, or $5 \frac{5}{7}$ — the middle, or second, part. And the third, found to be $30 - 1 \frac{1}{4} \text{ R}$, will be $\frac{160}{7}$, or $22 \frac{6}{7}$; while the first part, which we found to be $\frac{1}{4} \text{ R}$, is $\frac{10}{7}$, or $1 \frac{3}{7}$. These three parts — $1 \frac{3}{7}$, $5 \frac{5}{7}$, $22 \frac{6}{7}$ — make the given number 30. And from the first, $1 \frac{3}{7}$, into the third, $22 \frac{6}{7}$, arises $32 \frac{32}{49}$, the quadruple of $8 \frac{8}{49}$, which is made from the first part, $1 \frac{3}{7}$, into the second, $5 \frac{5}{7}$. That the three parts are continuously proportional is clear, since the same number is generated from the first into the third as from the middle into itself, namely $32 \frac{32}{49}$.

Otherwise, an equation could be set up, once the parts $\frac{1}{4} \text{ R}$, 1 R, and $30 - 1 \frac{1}{4} \text{ R}$ have been found, in this way: since from the first into the third must arise 1 $\bar{3}$ — a number equal to what is made from the middle into itself — if we divide 1 $\bar{3}$ by the first, that is, by $\frac{1}{4} \text{ R}$, the quotient will be 4 R, which multiplied into $\frac{1}{4} \text{ R}$ produces 1 $\bar{3}$. And accordingly the third part will be 4 R, equal indeed to the other value of the third already found, $30 - 1 \frac{1}{4} \text{ R}$. Adding $1 \frac{1}{4} \text{ R}$ to both sides, then, there will be an equality between $5 \frac{1}{4} \text{ R}$ and 30. Dividing 30 by $5 \frac{1}{4}$ gives $1 \text{ R} = 5 \frac{5}{7}$, as before.

[a] [Margin: a 20. Sept.]

(Cossist notation used throughout this section is transcribed as printed. First-occurrence glosses: $\mathbb{R}$ = res, the unknown (x); $\mathfrak{z}$ = census (x^2); $\mathfrak{C}$ = cubus (x^3); $\sqrt{\mathfrak{z}}$ = radix quadrata, the square-root surd. Signs are given in the printed forms; no sursolidum (β) occurs in these pages.)

This problem too we shall solve in problem 171, but with a different example proposed.

134. [Margin: CXXXIII] To find three numbers such that the first, together with any number of units of the second, makes a number equal to what remains of the second; likewise the second, together with any number of units of the third, makes another number equal to what remains of the third; and finally the third, having received any number of units from the first, has to the remainder of the first a given ratio.

Let the first take 40 units from the second, the second 56 units from the third, and the third finally 72 units from the first; so that the first makes a number equal to the remainder of the second; the second likewise a number equal to that which remains of the third; and the third makes a number five times the one that remains over to the first. Let the first be set as $1\mathbb{R}$. Then, having received 40 from the second, it will have $1\mathbb{R} + 40$; and just this much must then remain over to the second. Therefore the second, before it gave 40 to the first, had $1\mathbb{R} + 80$; and after it has received 56 from the third, it will have $1\mathbb{R} + 136$. And just this much must then remain over to the third; and so, before it gave 56 to the second, the third had $1\mathbb{R} + 192$; and if it receives 72 from the first, it will have $1\mathbb{R} + 264$. This number must be five times the number $1\mathbb{R} - 72$, which remains over to the first after it gave 72. Therefore its quintuple, $5\mathbb{R} - 360$, will equal the number $1\mathbb{R} + 264$. Adding 360 to both sides, there will be an equation between $5\mathbb{R}$ and $1\mathbb{R} + 624$; and taking away $1\mathbb{R}$ from both sides, between $4\mathbb{R}$ and 624. Dividing 624 by 4, then, gives $1\mathbb{R} = 156$, the first number. And accordingly the second, which was found to be $1\mathbb{R} + 80$, will be 236; while the third, which we found to be $1\mathbb{R} + 192$, will be 348. And these three numbers accomplish the problem, as you can verify.

SCHOLIUM

In similar questions a problem will sometimes be proposed that cannot be solved — which the equation, once found, will plainly warn of. Thus, if in the proposed problem one were to say that the third, having received 72 from the first, should likewise have a number equal to the remainder of the first, the question would become impossible. For the third would then have, as was said, $1\mathbb{R} + 264$, which

can in no way equal the number $1R - 72$ that remains over to the first. For the third would then, after it gave 72, still be 264 units greater than the first — the question is impossible.

135. [Margin: CXXXV] To find two numbers in a given ratio such that, when each is multiplied by two given numbers, two numbers are produced making any given sum.

Let two numbers be found in quintuple ratio, such that, the lesser being multiplied by 8 and the greater by 5, two numbers are produced whose sum is 100. Let the sought numbers be set as $1R$ and $5R$, in quintuple ratio. From $1R$ times 8 come $8R$, and from $5R$ times 5 come $25R$. The sum of the products is $33R$, equal to 100. Dividing 100 by 33, then, gives $1R = 100/33$, or $3 \frac{1}{33}$, the first number. The second will therefore be $500/33$, or $15 \frac{5}{33}$. From $3 \frac{1}{33}$, that is from $100/33$, times 8 come $800/33$, or $24 \frac{8}{33}$; and from $15 \frac{5}{33}$, that is from $500/33$, times 5 come $2500/33$, or $75 \frac{25}{33}$. And these two products, $24 \frac{8}{33}$ and $75 \frac{25}{33}$, make the given sum 100.

136. [Margin: CXXXVI] To find two numbers whose sum, as well as whose difference, is a square number.

Let the sum of the two be a square number, whose side is one root plus any number of units — namely the square $13 + 10R + 25$, produced from the assumed side $1R + 5$. For if the lesser sought number be set as $1/23$, the greater will be $1/23 + 10R + 25$; since their sum, $13 + 10R + 25$, is a square number by hypothesis. It remains that the difference of these same numbers, $1/23$ and $1/23 + 10R + 25$ — namely $10R + 25$ — be also a square: that is, equal to any square greater than 25, say 100. Taking away 25 from both sides, there will be an equation between $10R$ and 75. Dividing 75 by 10 gives $1R = 7 \frac{1}{2}$, and $13 = 225/4$, that is $56 \frac{1}{4}$. Therefore the lesser number, which was $1/23$, will be $28 \frac{1}{8}$; and the greater, which was $1/23 + 10R + 25$, will be $128 \frac{1}{8}$. For the sum of these numbers, $156 \frac{1}{4}$ — that is $1250/8$, or $625/4$ — is a square number, whose side is $25/2$; and their difference, 100, is likewise a square, whose side is 10.

If you set the aforesaid difference $10R + 25$ equal to the square 36, or to any other that is not 100, you will find another value of the one root, as you can try for yourself.

137. [Margin: CXXXVII] Given a square, to find a number which, both added to the given square and subtracted from the same, makes a square number.

Let the given square be 100, which is to be divided into two squares by problem 90 or its scholium; and by that scholium it is done as follows. The side of the given square, namely 10, is multiplied by any number, say by 5. The double of this product, 100, is divided by the square of the assumed number 5 increased by one, namely by 26. For the quotient $50/13$, squared, gives $2500/169$, or $14\frac{134}{169}$, which will be one square. This, subtracted from the given square 100, leaves as the other square $85\frac{35}{169}$; and these two squares make up 100. The sides of these squares, moreover — $50/13$ and $120/13$ — multiplied together make $6000/169$, or $35\frac{85}{169}$; and the double of this product, $12000/169$, or $71\frac{1}{169}$, will be the number sought. For added to 100 it makes $171\frac{1}{169}$, or $28900/169$, a square whose side is $170/13$, or $13\frac{1}{13}$; and subtracted from the same given square 100, it leaves $28\frac{168}{169}$, or $4900/169$, likewise a square number, whose side is $70/13$, or $5\frac{5}{13}$.

The reasoning of this operation is as follows. When the given square 100 is divided into two squares, the side of the given square and the two sides of the found squares, if they are drawn out as lines, form a right triangle [*Margin (a): 48 primi — Euclid, Elements I.48*]; since the two found squares, described on their sides, equal the given square. Therefore, as we shall demonstrate in the following lemma, the rectangle contained twice by those two sides, added to the given square, produces a square whose side is equal to those two sides together; and subtracted from the same given square, the remainder makes a square whose side is the difference of those two sides.

And if the same square 100 is distributed into the two squares 36 and 64, whose sides 6 and 8 multiplied together make 48, the double of this number, 96, will be the one we seek. For added to 100 it makes the square 196, whose side is 14; and subtracted from 100 the remainder makes the square 4, whose side is 2.

LEMMA [*Margin: Lemma*]

In every right triangle, the rectangle contained twice by the two sides about the right angle, together with the square of the side opposite the right angle, equals a square whose side is composed of the two sides. But if the same rectangle contained twice by the aforesaid sides is subtracted from that same square of the side opposite the right angle, the remainder becomes a square, whose side is the difference of the two aforesaid sides.

Diagram. A right triangle, drawn schematically with apex A at the top and base BC below (B at lower left, C at lower right). AB and AC are the two sides about the right angle (at A); BC is the side opposite the right angle (the hypotenuse).

- A (apex): vertex of the right angle
- B (lower left): endpoint of base
- C (lower right): endpoint of base

Let the right triangle be ABC. I say that the rectangle contained twice by AB, AC, together with the square on BC, equals a square whose side is equal to the two sides AB, AC together; and that if the same twofold rectangle is subtracted from the square on the line BC, a square is left, whose side is equal to the difference between AB and AC. For if one conceives a single line composed of AB and AC, the rectangle contained twice by AB, AC, together with the squares on AB and AC, equals the square of the line composed of AB, AC [*Margin (a): 4 secundi — Euclid, Elements II.4*]. But the squares on AB, AC equal the square on BC [*Margin (b): 47 primi — Euclid, Elements I.47*]; therefore the rectangle contained twice by AB, AC, together with the square on BC, will also equal the square of the line composed of AB, AC — which is the first point.

Next, since the two squares on AB, AC (that is, the square on BC) are greater than the rectangle contained twice by AB, AC by the square of the line that is the difference between AB and AC [*Margin (c): Lemma 39 decimi — the lemma to Euclid, Elements X.39*], it is manifest that, if that rectangle contained twice by AB, AC is subtracted from the square on the line BC, a square remains whose side is equal to the difference between AB and AC — which is the second point.

138. [*Margin: CXXXVIII*] To find a number which, multiplied by any given number, makes a number that, together with any other given number, produces a number equal to the one arising when the sought number is multiplied by yet another given number — the latter, however, being greater than the former given number.

Let the number to be found be one which, multiplied by 7, makes a number that, added to 60, makes a sum equal to the number arising from the found number multiplied by 8. Let the sought number be set as 1R. From 1R times 7 comes the number 7R; and this, added to 60, makes 7R + 60, equal to the number 8R, which arises from 1R times 8. Taking away 7R from both sides, there will be an equation between 60 and 1R. Dividing 60 by 1 gives 1R = 60, the sought number. For this, multiplied by 7, makes 420, and adding 60 makes the number 480 — which is exactly what the found number 60, multiplied by 8, makes.

139. [*Margin: CXXXIX*] To divide a given number into two parts, and moreover to find a square which, together with each part separately, makes a square.

Let the given number 65 be cut into two parts, and moreover let a square be found such that with each of those parts it makes a square. Let two arbitrary numbers be set, each composed of one root and a number of units, namely $1R + 3$ and $1R + 5$. Their squares are $13 + 6R + 9$ and $13 + 10R + 25$. Therefore, if the number $6R + 9$ is added to 13 (which is a square number), it makes the square $13 + 6R + 9$; and if the number $10R + 25$ is added to the same 13, it makes the square $13 + 10R + 25$. Therefore, if the lesser part of the given number is set as $6R + 9$ and the greater as $10R + 25$, the problem will be satisfied, provided these parts make up the given number 65. But they make the sum $16R + 34$, which must equal 65. Taking away 34 from both sides, there remains an equality between $16R$ and 31. Dividing 31 by 16, then, gives $1R = 31/16$, or $1 \frac{15}{16}$, whose square, $961/256$, or $3 \frac{193}{256}$, will be the one sought. The lesser part of the given number, which we set as $6R + 9$, will be $330/16$, or $20 \frac{5}{8}$; the greater, which was set as $10R + 25$, will be $710/16$, or $44 \frac{3}{8}$ — which two parts together make up the given number 65. And the first, added to the found square $961/256$, makes the square $6241/256$, whose side is $79/16$; while the second, together with the same square $961/256$, makes the square $12321/256$, whose side is $111/16$.

SCHOLIUM

This problem admits several solutions, according to the variety of numbers assumed at the outset. For if the numbers are set as $1R + 1$ and $1R + 2$, their squares will be $13 + 2R + 1$ and $13 + 4R + 4$. And if the lesser part of the given number 65 is set as $2R + 1$, and the greater as $4R + 4$, the problem will be satisfied, since each number together with 13 makes a square. These two numbers make the sum $6R + 5$, which must equal 65. Taking away 5 from both sides, there remains an equality between $6R$ and 60. Dividing 60 by 6 gives $1R = 10$, whose square, 100, is the one sought, and was set as 13. The lesser part of the given number 65, which we set as $2R + 1$, will be 21; the greater, set as $4R + 4$, will be 44 — which two parts make 65. And the first, added to the found square 100, makes the square 121, whose side is 11; while the second, together with the same square 100, makes the square 144, whose side is 12.

And in this manner the given number can be divided into other and other parts, if other and other numbers are assumed, composed of $1R$ and a number of units.

140. [*Margin: CLX*] To distribute a given number into two parts, and moreover to find a square from which, if the two found parts are subtracted, the remainders become two squares.

Let the number 65 be divided into two parts, and let a square be found from which, if the two found parts are taken away, the remainders become two square numbers. Let two arbitrary numbers be set, one composed of one root and any number of units, the other of one root less the same number of units — namely $1R + 2$ and $1R - 2$. Their squares will be $13 + 4R + 4$ and $13 - 4R + 4$. Let the sought square therefore be set as $13 + 4R + 4$; if the number $4R + 4$ is taken from it, the square 13 will be left. And if the latter square is subtracted from the former, there will remain $8R$. Conversely, if $8R$ is subtracted from that same former square $13 + 4R + 4$, the latter square $13 - 4R + 4$ will be left. Therefore, if the lesser part of the number to be divided is set as $4R + 4$, and the greater as $8R$, the problem will be satisfied; for the lesser, taken from the square $13 + 4R + 4$, leaves the square 13 , and the greater, subtracted from the same square, leaves the square $13 - 4R + 4$. It now remains that these two parts, $4R + 4$ and $8R$, make up the given number 65. But they make the sum $12R + 4$, equal to 65. Taking away 4 from both sides, there will be an equation between $12R$ and 61. Dividing 61 by 12 gives $1R = 61/12$. And accordingly the sought square, which we set as $13 + 4R + 4$, will be $7225/144$, whose side is $85/12$. The lesser part of the number, set as $4R + 4$, will be $292/12$, or $3504/144$; the greater, set as $8R$, will be $488/12$, or $5856/144$. For the former, subtracted from the found square $7225/144$, leaves the square $3721/144$, whose side is $61/12$; while the latter, taken from the same square $7225/144$, leaves the square $1369/144$, whose side is $37/12$.

This problem too admits several solutions, according as at the outset other and other numbers are assumed for the numbers $1R + 2$ and $1R - 2$, and so on.

141. [*Margin: CXLI*] To divide a given number into two parts such that their cubes make any given sum — provided that this is greater than the fourth part of the cube described from the given number.

Let the given number 10 be divided into two parts whose cubes make 370 — a number greater than 250, the fourth part of the cube generated from 10. Let the first part be set as a number composed of $1R$ and half the given number, namely $1R + 5$. Let the second part be half of the same given number less $1R$, namely $5 - 1R$. For thus these two parts will make up the given number 10. Their cubes are $1C + 153 + 75R + 125$ and $125 - 75R + 153 - 1C$. Their sum is $303 + 250$; for $+1C$ and $-1C$ cancel each other, and from 153 and 153 come 303 , while 125 and 125 make 250 ; likewise $+75R$ and $-75R$ cancel each other. This sum, $303 + 250$, must therefore equal 370. Taking away 250 from both sides, there remains an equality between 303 and 120. Dividing 120 by 30, then, gives $13 = 4$, and $1R = 2$. The first

part, set as $1R + 5$, will therefore be 7; the second, which we set as $5 - 1R$, will be 3. The cubes of these parts, 7 and 3, are 343 and 27, and make 370.

If, with the same number 10 given, the sum of the cubes is proposed as 400, we shall find an equation between $30\bar{3} + 250$ and 400. Taking away 250 from both sides, between $30\bar{3}$ and 150. Dividing 150 by 30, then, gives $1\bar{3} = 5$, and $1R = \sqrt{35}$ [$\sqrt{5}$]. Therefore the first part, set as $1R + 5$, will be $\sqrt{35} + 5$; the second, set as $5 - 1R$, will be $5 - \sqrt{35}$ — which together make 10. The cubes of these parts (once what must be reduced is reduced) are $200 + \sqrt{332000}$ [$\sqrt{32000}$] and $200 - \sqrt{332000}$, whose sum is 400, since $+\sqrt{332000}$ and $-\sqrt{332000}$ cancel each other.

SCHOLIUM

Without algebra the same problem is solved in this way. Let the given number 20 be cut into two parts whose cubes must make the sum 2960 — which is greater than 2000, the fourth part of the cube generated from 20. Take away $\frac{1}{4}$ of the cube described from the given number 20, namely 2000, from the proposed sum 2960. And let the remaining number, 960, be divided by 60, the triple of the given number 20. For the square root of the quotient 16, namely 4, added to and subtracted from 10, the half of the given number 20, will give the sought parts 14 and 6, which together make up the given number 20; and their cubes, 2744 and 216, make 2960.

This problem too we shall solve, in problem 1 of the following chapter, by a composite equation — one in which, namely, three terms occur, of which one is equal to the other two.

142. [*Margin: CXLII*] To distribute a given number into two cubes, the sum of whose sides is given — provided that the fourth part of the cube of this sum is less than the given number.

This problem does not differ from the preceding. For nothing else is enjoined here than to distribute the given sum of the sides into two parts whose cubes make the given number — which must be greater than the fourth part of the cube produced from the given sum of the sides. Thus, if the given number 2960 is to be divided into two cubes whose sides make 20, the number 20 will have to be divided into two parts whose cubes make 2960, as was done in the preceding problem. For thus the number 2960 will be divided into the two cubes 2744 and 216, whose sides, 14 and 6, make up the given sum 20.

So likewise, if the given number 400 is to be divided into two cubes whose sides make 10, there will be found (once what must be reduced is reduced) the two

cubes $200 + \sqrt{332000}$ and $200 - \sqrt{332000}$, whose sides are $\sqrt{35} + 5$ and $5 - \sqrt{35}$, provided the sum of the sides, 10, is divided into two parts, as was done in the preceding problem or in its scholium, so that their cubes make 400.

143. [*Margin: CXLI*] To find a number whose square, or a given multiple of the square, constitutes a given part — or given parts — of the cube produced from the same found number.

Let it first be required to find a number whose square constitutes $1/7$ [?: denominator uncertain] of the cube produced from the found number. Let the sought [number] be set...

Note on notation: the cossist signs are rendered with the tokens $R = res$ (the unknown, x), $\mathfrak{z} = census$ (x^2), $\mathfrak{c} = cubus$ (x^3), and $N = numerus$ (the absolute/unit term). Surds are rendered as printed: $\sqrt{\mathfrak{z}} = radix quadrata$ (square root) and $\sqrt{\mathfrak{c}} = radix cubica$ (cube root). A coefficient preceding a sign is its multiplier (e.g. $5R = 5x$). All values are transcribed from the source; fractions are written " a/b " and mixed numbers " $n a/b$ ".

the sought number is $1R$ (*res*, the unknown x); its square is $1\mathfrak{z}$ (*census*, x^2) and its cube $1\mathfrak{c}$ (*cubus*, x^3). Therefore, in order that $1\mathfrak{z}$ should constitute $1/9$ of the number $1\mathfrak{c}$, there must be an equality between $1\mathfrak{z}$ and $1/9 \mathfrak{c}$. Dividing 1 by $1/9$, then, gives $1R = 9$, the sought number — since the cossic powers are consecutive [*collaterales*]. For its square, 81, constitutes the ninth part of its own cube, 729.

Next, let a number be found whose square constitutes $5/9$ of its cube. With the same positings, there will be an equation between $5/9 \mathfrak{c}$ and $1\mathfrak{z}$. Dividing 1 by $5/9$, then, gives $1R = 9/5$, that is, $1 \frac{4}{5}$, the sought number. For its square, $81/25$, constitutes $5/9$ of its cube, $729/125$.

Again, let a number be found the quintuple of whose square constitutes $1/7$ of its own cube. Let the sought number be posited as $1R$, its square $1\mathfrak{z}$ and its cube $1\mathfrak{c}$. There will then be an equality between $5\mathfrak{z}$ and $1/7 \mathfrak{c}$. Dividing 5 by $1/7$, then, gives $1R = 35$, the sought number. For its square is 1225, and the quintuple of that, 6125, constitutes $1/7$ of its cube 42875, which is generated from 35.

Finally, let a number be found the quintuple of whose square constitutes $9/10$ of its own cube. With the same positings, there will be an equation between $5\mathfrak{z}$ and $9/10 \mathfrak{c}$. Dividing 5 by $9/10$, then, gives $1R = 50/9$, that is, $5 \frac{5}{9}$, the sought number. For its square is $2500/81$, the quintuple of which, $12500/81$, constitutes $9/10$ of its cube $125000/729$.

Problem 144

[Margin: CXLIII]

To find a number that constitutes a given part, or given parts, of its own cube.

Let a number first be found that constitutes $1/12$ of its cube. Let the sought number be posited as $1R$, whose cube is $1C$. There is then an equality between $1R$ and $1/12 C$. Dividing 1 by $1/12$ gives $13 = 12$ (for since between $1R$ and $1C$ the interposed power makes the denomination *census* [3], the reduction leaves a census). Its square root, namely $\sqrt{3} 12$ (*radix quadrata*, $= \sqrt{12}$), will be the value of $1R$. The sought number is therefore $\sqrt{3} 12$. For this constitutes $1/12$ of its cube $\sqrt{3} 1728$ [$= \sqrt{1728}$, since $(\sqrt{12})^3 = \sqrt{1728}$].

Next, let a number be found that constitutes $4/9$ of its cube. With the same things posited, there will be an equation between $1R$ and $4/9 C$. Dividing 1 by $4/9$, then, gives $13 = 9/4$ and $1R = 3/2$, the sought number. For its cube is $27/8$, four ninths of which the number found, $3/2$, contains.

Problem 145

[Margin: CXLV.]

To find two numbers in a given ratio, such that the lesser multiplied into the square of the greater produces a given number.

Let two numbers be found in the ratio one-and-a-fifth [*sesquiquinta*, $6 : 5$], such that when the lesser is multiplied into the square of the greater, the number 4860 is produced. Let the sought numbers be posited as $5R$ and $6R$, in the ratio one-and-a-fifth. When the lesser, $5R$, is multiplied into 363 (the square of the greater, $6R$), the number $180C$ is produced, equal to 4860. Dividing 4860 by 180, then, gives $1C = 27$, whose cube root is 3 — because between the characters C (cubus) and N (numerus) the two means are 3 (census) and R (res). Therefore the lesser, posited as $5R$, will be 15, and the greater, posited as $6R$, will be 18. And indeed, from the lesser, 15, into 324, the square of the greater 18, the number 4860 is produced.

Again, in the same ratio one-and-a-fifth, let two numbers be found such that the lesser multiplied into the square of the greater produces 1. With the same things posited, there will be found an equality between $180C$ and 1. Dividing 1 by 180, then, gives $1C = 1/180$, and its cube root $\sqrt[3]{1/180}$ (*radix cubica*, $= \sqrt[3]{(1/180)}$). Therefore the lesser number, posited as $5R$, will make $\sqrt[3]{125/180}$, that is, $\sqrt[3]{1/180}$.

25/36. But the greater, posited as $6R$, will be $\sqrt[3]{216/180}$, or $\sqrt[3]{6/5}$. And from the lesser, $\sqrt[3]{25/36}$, into $\sqrt[3]{36/25}$ — the square of the greater $\sqrt[3]{6/5}$ — is produced $\sqrt[3]{900/900}$, that is, 1. For whenever the numerator of a fraction equals its denominator, that fraction is equivalent to unity.

Problem 146

[Margin: CXLVI.]

To find a number that lies midway between two others, exceeds the lesser by a given amount, is exceeded by the greater by any other given amount, and such that the two extremes make up any given sum.

This puzzle is like puzzles 51 and 52, but it is posed differently and solved differently. Let a mean number be found between two numbers that together make up 100, such that it exceeds the lesser by 15 and is exceeded by the greater by 9. Since the greater is supposed to exceed the sought mean by 9, and this mean to exceed the lesser by 15, it plainly follows that the greater exceeds the lesser by 24. If, then, the given sum 100 is divided into two parts whose difference is 24, and 15 is added to the lesser, the sought mean will result. Let the lesser part therefore be posited as $1R$ and the greater as $100 - 1R$, which must exceed the former by 24. Subtracting $1R$ from $100 - 1R$, the remainder is the number $100 - 2R$, equal to 24. Adding $2R$ to both sides, there results an equality between 100 and $24 + 2R$. And removing 24 from both, between 76 and $2R$. Dividing 76 by 2, then, gives $1R = 38$, the lesser part; hence the greater will be 62. And adding 15 to the lesser, 38, the sought mean results, 53. For this exceeds the lesser part by 15 and is exceeded by the greater by 9.

Problem 147

[Margin: CXLVII]

To distribute a given number into as many parts as desired, such that each has a given ratio to the next.

Let the given number be 10000, to be divided into six parts, such that the first has to the second the ratio of equality, the second to the third the double ratio, the third to the fourth the triple ratio, the fourth to the fifth the ratio of equality, and the fifth to the sixth likewise the ratio of equality. Let the first part be posited as $1R$, the second also $1R$, the third $1/2 R$, and the fourth, fifth, and sixth $1/6 R$ — which all together make $3R$, equal to 10000. Dividing 10000 by 3, then, gives $1R =$

3333 $\frac{1}{3}$; and accordingly the other parts will make the numbers you see in the accompanying scheme. For all these parts, gathered into one sum, make 10000.

Part	Cossic value		Number	Fraction
first	1R	that is	3333	$\frac{1}{3}$
second	1R		3333	$\frac{1}{3}$
third	$\frac{1}{2}$ R		1666	$\frac{2}{3}$
fourth	$\frac{1}{6}$ R		555	$\frac{5}{9}$
fifth	$\frac{1}{6}$ R		555	$\frac{5}{9}$
sixth	$\frac{1}{6}$ R		555	$\frac{5}{9}$

Problem 148

[Margin: CXLVIII.]

To find a number such that given parts of it equal a given aliquot part of the same, increased by a number.

Let a number be found whose $\frac{2}{3}$ equal its half increased by seven. Let the sought number be posited as 1R. Then $\frac{2}{3}$ R will equal $\frac{1}{2}$ R + 7. Removing $\frac{1}{2}$ R from both sides, there will remain an equality between $\frac{1}{6}$ R and 7. Dividing 7 by $\frac{1}{6}$, then, gives 1R = 42, the sought number. For its $\frac{2}{3}$ are 28, which number equals the half of the number 42 — that is, the number 21 — increased by seven.

Problem 149

[Margin: CXLIX.]

To find two numbers whose sum is given, such that when the former is divided by a given number a quotient results which, together with a given part of the latter, produces a given number.

Let two numbers be found whose sum is 40, such that when the former is divided by 5 a quotient results which, together with $\frac{1}{8}$ of the latter number, makes 6. Let the former number be posited as 1R and the latter as 40 – 1R. Dividing 1R by 5 gives the quotient $\frac{1}{5}$ R. And the eighth part of the number 40 – 1R is $(40 - 1R)/8$. This, together with that quotient, makes $(8R + 200 - 5R)/40$, that is, $(3R + 200)/40$ — which number must equal 6. This equation, by cross-multiplication, reduces to an equation between $3R + 200$ and 240. Removing 200 from both sides, there will be an equation between 3R and 40. Dividing 40 by 3, then, gives 1R = $13 \frac{1}{3}$, the former sought number; and accordingly the latter number, posited as 40 – 1R,

will be $26 \frac{2}{3}$. For when the former number, $13 \frac{1}{3}$, is divided by 5, the quotient $2 \frac{2}{3}$ results, which together with the eighth part of the latter number, $26 \frac{2}{3}$ — that is, with $3 \frac{1}{3}$ — makes 6.

In this proposed example you see that the former number found is the greater. But suppose two numbers are to be found whose sum is 20, such that the former divided by 3 gives a quotient which together with $\frac{1}{8}$ of the latter number makes 5. Let the former number be posited as $1R$ and the latter as $20 - 1R$. Dividing $1R$ by 3 gives the quotient $\frac{1}{3}R$, and the eighth part of the number $20 - 1R$ is $(20 - 1R)/8$. This, together with that quotient, makes $(8R + 60 - 3R)/24$, that is, $(5R + 60)/24$ — which number must equal 5. This equation, by cross-multiplication, reduces to an equation between $5R + 60$ and 120. Dividing 60 by 5 gives $1R = 12$, the former sought number; and accordingly the latter, $20 - 1R$, will be 8. For when the former, 12, is divided by 3, the quotient 4 results, which together with 1 — that is, with the eighth part of the latter number, 8 — makes 5. And here the former number is greater than the latter.

SCHOLIUM

Note, however, that in a similar puzzle a question is sometimes proposed that can by no means be solved unless numbers less than Nothing [i.e. negative numbers] are admitted. For let two numbers be found whose sum is 40, such that when the former is divided by 5 a quotient results which, together with $\frac{1}{10}$ of the latter, makes 100. Let the former number be posited as $1R$ and the latter as $40 - 1R$. Dividing $1R$ by 5 gives the quotient $\frac{1}{5}R$. And the tenth part of the number $40 - 1R$ is $(40 - 1R)/10$. This, together with that quotient, makes $(10R + 200 - 5R)/50$, that is, $(5R + 200)/50$ — which number must equal 100. And this equation, by cross-multiplication, reduces to an equation between $5R + 200$ and 5000. Removing 200 from both sides, there will be an equation between $5R$ and 4800. Dividing 4800 by 5, then, gives $1R = 960$, the former sought number — which is greater than the proposed sum 40, and that is absurd. But let us grant this absurdity and proceed further. The second number, posited as $40 - 1R$, will be $40 - 960$. For this, together with 960, makes 40, the proposed sum. Now the number $40 - 960$ is equivalent to -920 . For if 960 is taken from 40, there remain -920 , as this scheme shows:

$$\begin{array}{r} + 40 \\ + 960 \\ \hline - 920 \end{array}$$

And this number, -920 , added to 960 , again makes up the former number, the proposed sum 40 , as appears in this other scheme:

$$\begin{array}{r} - 920 \\ + 960 \\ \hline + 40 \end{array}$$

Now if the former number, 960 , is divided by 5 , the quotient 192 results, which together with $1/10$ of the number -920 — that is, with -92 — makes 100 , the proposed number, as may be seen in the third scheme:

$$\begin{array}{r} + 192 \\ - 92 \\ \hline + 100 \end{array}$$

I have chosen to explain this example at greater length, so that you may see the force of numbers below Nothing and of the algorithmic signs $+$ and $-$; although, once $1R$ was found to be 960 , you could at once say that the question is impossible, since the number 960 cannot be a part of the proposed sum 40 .

Problem 150

[Margin: CL.]

To find a number which, together with a given part of itself, exceeds any given number by as much as that same given number — or some other given number — exceeds the number found.

Let a number first be found which, together with its $1/4$, exceeds 100 by the same amount by which 100 exceeds the number found. Let the sought number be posited as $1R$, which together with its $1/4$ makes $1\frac{1}{4}R$. This sum must exceed 100 by the same amount by which 100 exceeds $1R$. Therefore, taking 100 from $1\frac{1}{4}R$ and $1R$ from 100 , the excesses that remain are equal; that is, there will be an equation between $1\frac{1}{4}R - 100$ and $100 - 1R$. Adding $1R$ to both sides, and adding 100 to both, there results an equation between $2\frac{1}{4}R$ and 200 . Dividing 200 by $2\frac{1}{4}$, then, gives $1R = 88\frac{8}{9}$, the sought number. For if its $1/4$ — namely the number $22\frac{2}{9}$ — is added to it, the sum $111\frac{1}{9}$ results, which exceeds 100 by $11\frac{1}{9}$, the very amount by which 100 exceeds the number found, $88\frac{8}{9}$.

Next, let a number be found which, together with its $2/3$, exceeds 100 by the same amount by which 140 exceeds the number found. Let the sought number be posited as $1R$. Then there will be an equation between $1\frac{2}{3}R - 100$ and $140 - 1R$.

Adding $1R$ to both sides, and again adding 100 to both, there results an equation between $2\frac{2}{3}R$ and 240. Dividing 240 by $2\frac{2}{3}$, then, gives $1R = 90$, the sought number. For if its $\frac{2}{3}$ — namely 60 — are added, the sum 150 results, which exceeds 100 by 50, the very amount by which 140 exceeds the number found, 90.

Lastly, let a number be found which, together with its $\frac{1}{2}$, makes a sum that exceeds 40 by the same amount by which the number 10 exceeds the number found. Let the sought number be posited as $1R$. Then there will be an equation between $1\frac{1}{2}R - 40$ and $10 - 1R$. Adding $1R$ to both sides, there results an equation between $2\frac{1}{2}R - 40$ and 10. And again adding 40 to both, between $2\frac{1}{2}R$ and 50. Dividing 50 by $2\frac{1}{2}$, then, gives $1R = 20$, the sought number, which together with its $\frac{1}{2}$ makes the sum 30 — a sum that exceeds the number 40 by -10 , as appears in this scheme, where 40 is subtracted from 30:

$$\begin{array}{r} + 30 \\ + 40 \\ \hline - 10 \end{array}$$

And by the same amount, -10 , the number 10 exceeds the number to be found, 20, as is plain if 20 is subtracted from 10, as you see in this other scheme:

$$\begin{array}{r} + 10 \\ + 20 \\ \hline - 10 \end{array}$$

Here again you see the wonderful force of the algorithmic signs $+$ and $-$; although one might say that this example cannot be worked out, because the number 20 found, together with its $\frac{1}{2}$ — that is, with 10 — makes a sum smaller than 40.

Problem 151

[Margin: CLI.]

To find a number from which, if a given part or parts be subtracted, there remains a number as much less than a given number as the number found itself exceeds that same given number.

Let a number be found such that, its $\frac{1}{5}$ being subtracted, the remainder is a number as much less than 100 as the number found itself is greater than 100. Let the sought number be posited as $1R$; subtracting its $\frac{1}{5}$, there remain $\frac{4}{5}R$. These, taken from 100, leave $100 - \frac{4}{5}R$. And if 100 is taken from $1R$, there remain $1R - 100$. And since the excesses are posited as equal, there will be an equation

between $100 - \frac{4}{5}R$ and $1R - 100$. Adding $\frac{4}{5}R$ to both sides, between 100 and $1\frac{4}{5}R - 100$. And again adding 100 to both, between 200 and $1\frac{4}{5}R$. Dividing 200 by $1\frac{4}{5}$ — that is, by $\frac{9}{5}$ — then gives $1R = 111\frac{1}{9}$, the number sought. Its $\frac{1}{5}$ makes $22\frac{2}{9}$, that is, $\frac{200}{9}$; and that number, subtracted from $111\frac{1}{9}$ — that is, from $\frac{1000}{9}$ — leaves $\frac{800}{9}$, that is, $88\frac{8}{9}$. This number is exceeded by 100 by the amount $11\frac{1}{9}$, the very amount by which the number found, $111\frac{1}{9}$, exceeds the number 100.

Furthermore, let a number be found such that, its $\frac{2}{5}$ being subtracted, the remaining number is as much less than 100 as the number found itself is greater than 100. Let the sought number be posited as $1R$. Removing $\frac{2}{5}$, there remain $\frac{3}{5}R$. There will then be an equation between $100 - \frac{3}{5}R$ and $1R - 100$. Adding $\frac{3}{5}R$ to both sides, between 100 and $1\frac{3}{5}R - 100$. And adding 100, between 200 and $1\frac{3}{5}R$. Dividing 200 by $1\frac{3}{5}$, then, gives $1R = 125$, the sought number. For if its $\frac{2}{5}$ — namely 50 — are taken from it, there remain 75, which are exceeded by 100 by the amount 25, the very amount by which the number found, 125, exceeds 100.

Problem 152

[Margin: CLII.]

To find a number which, multiplied by itself and the product multiplied by a given number, produces a number from which, if another given number be subtracted, the remainder becomes any given number.

Let a number be found which, multiplied by itself and the product multiplied by 10, makes a number from which, if 100 be taken, there remains the number 300. Let the sought number be posited as $1R$, which multiplied by itself makes $1R^2$; and this multiplied by 10 makes the number $10R^2$. From this, if 100 be taken, there remains the number $10R^2 - 100$, which must equal the number 300. Adding 100 to both sides, there will be an equation between $10R^2$ and 400. Dividing 400 by 10, then, gives $R^2 = 40$. Therefore $\sqrt{40}$ ($= \sqrt{40}$) is the number we seek. For, multiplied by itself, it makes 40; and from 40 into 10 the number 400 results, from which, if 100 be subtracted, there remain 300.

Problem 153

[Margin: CLIII.]

To construct an arithmetic progression whose first and last terms are given and whose sum is given, when this can be done.

Let the first term be 5 and the last 10, and the sum of all the terms 105. Here two things must be sought: the number of terms and the common difference of the progression. Let the number of terms be posited as $1R$. Now, since the sum of the first and last terms, multiplied by the number of terms, produces double the sum of the progression — as we taught in the *Practical Arithmetic*, namely in the first rule of arithmetic progressions — if we multiply the sum of the first and last terms, namely 15, by the number of terms, that is, by $1R$, we shall make $15R$, whose half, $15/2 R [= 7 \frac{1}{2} R]$, is the sum of the progression, equal to the given sum 105. Dividing 105 by $15/2$, then, gives $1R = 14$, the number of terms, which we posited to be $1R$.

Now the 12 intermediate terms must make the sum 90, namely 105 with the first term 5 and the last 10 taken away. Let the common difference of the progression be posited as $1R$. The 12 terms will be the following:

$5 + 1R. 5 + 2R. 5 + 3R. 5 + 4R. 5 + 5R. 5 + 6R. 5 + 7R. 5 + 8R. 5 + 9R. 5 + 10R. 5 + 11R. 5 + 12R.$

These all together make the sum $60 + 78R$, equal to the number 90. Removing 60 from both sides, then, there will be an equality between $78R$ and 30. Dividing 30 by 78, then, gives $1R = 5/13$, the common difference of the progression. This, therefore, is the progression:

5. $5 \frac{5}{13}. 5 \frac{10}{13}. 6 \frac{2}{13}. 6 \frac{7}{13}. 6 \frac{12}{13}. 7 \frac{4}{13}. 7 \frac{9}{13}. 8 \frac{1}{13}. 8 \frac{6}{13}. 8 \frac{11}{13}. 9 \frac{3}{13}. 9 \frac{8}{13}. 10.$

—since it constitutes 14 terms, whose sum is 105.

When, in the earlier operation, the number of terms is found not to be a whole number, the question will be impossible, because...

for such a progression cannot be given, one in which the first and last terms are those which were given, and the sum of the progression is the one proposed. For example, in the same case, if the sum of the progression were given as 100, the first term as 5, and the last as 10, we would arrive at an equation between $-15/2R$ [$R = res$, the unknown x] and 100. And dividing 100 by $-15/2$, we would get $1R = 13\frac{1}{3}$ as the number of terms — which is impossible. For there is no arithmetic progression in the nature of things whose first term is 5, whose last is 10, and whose sum is 100.

Accordingly, the proposed problem is to be understood as applying when the progression is genuinely of such a kind that the given numbers really are the first term, the last term, and the sum of the progression. Nor, indeed, is it to be lightly

valued that we can discover such a progression by the rules of algebra. For suppose this question were proposed: There is an arithmetic progression in which the first term is 5, the last is 32, and the sum of the progression is 185 — who would easily find it without algebra? Yet we shall find it without great labor by the method we prescribed in the problem. For the number of terms will be found to be 10, and the common difference of the progression 3, so that the progression sought is this:

5.
 8.
 11.
 14.
 17.
 20.
 23.
 26.
 29.

154. To construct a geometric progression in a given ratio whose last term is given, together with the sum of the progression.

[Margin: CLIIII.]

Let a progression be constructed in the sesquialteral ratio $[3 : 2]$ whose last term is 19683 and the sum of the whole progression 55593. Let the first term be set as 1R. Now, since — as we wrote in the *Arithmetica Practica*, in the first rule for geometric progressions — the sum of the whole progression arises when the first term is subtracted from the last, the remainder is divided by the denominator of the ratio less one, and the last term is added to the quotient: subtract 1R (that is, the first term) from the last, 19683, and the remainder is $19683 - 1R$. Divide this by $\frac{1}{2}$ (that is, by the denominator of the ratio, $1\frac{1}{2}$, less one), and add the last term, 19683, to the quotient $39366 - 2R$; the whole sum of the progression becomes $59049 - 2R$, equal to the given sum 55593. Adding $2R$ to both sides, there will be an equality between 59049 and $55593 + 2R$. Removing 55593 from both, between 3456 and $2R$. Dividing therefore 3456 by 2, 1R becomes 1728 — namely the first term of the progression in the sesquialteral ratio. And so this progression of seven terms will be constructed, all of which make the given sum 55593:

1728.
 2592.

3888.
 5832.
 8748.
 13122.

155. To construct a geometric progression in a given ratio whose first term is given, together with the sum of the progression.

[Margin: CLV.]

Let the given first term of the sesquialteral ratio be 1728, and the sum of the progression 55593. Let the last term be set as $1R$. If the first is taken from it, there remains the number $1R - 1728$. Dividing this by $\frac{1}{2}$, the quotient is $2R - 3456$; and the sum of the progression is $3R - 3456$, which (as we said in the *Arithmetica Practica*) is equal to the given sum 55593. Adding 3456 to both sides, there will be an equation between $3R$ and 59049. Dividing therefore 59049 by 3, $1R$ becomes 19683, the last term, and the whole progression will be this:

1728.
 2592.
 3888.
 5832.
 8748.
 13122.
 19683. &c.

These last two problems must likewise be understood to hold only if progressions of this geometric kind genuinely exist in the nature of things — progressions in which the last term (or the first) is the one proposed, and the sum of the progression is likewise the one given. Thus, if a progression in the sesquialteral ratio were sought whose sum is 1000 and whose last term is 500, the question would be insoluble, since no such progression exists — as you can verify. But if a geometric progression in the sesquialteral ratio is sought whose last term is $68 \frac{11}{32}$ and whose sum is $197 \frac{1}{32}$ — since such a progression does exist in the nature of things — we shall set the first term as $1R$. Taken from the last term $68 \frac{11}{32}$, it leaves $68 \frac{11}{32} - 1R$. Dividing this number by $\frac{1}{2}$ (that is, by the denominator of the ratio less one), the quotient is $4374/32 - 2R$; and adding the last term $68 \frac{11}{32}$, or $2187/32$, the number $6561/32 - 2R$ arises, equal to the sum of the progression $197 \frac{1}{32}$, or $6305/32$. Adding therefore $2R$ to both sides, there will be an equation between $2R + 6305/32$ and $6561/32$. Removing $6305/32$ from both

sides, between $2R$ and $256/32$. Dividing therefore $256/32$ by 2, $1R$ becomes 4, the first term of the progression. So the progression will stand thus:

4.

6.

9. $13\frac{1}{2}$. $20\frac{1}{4}$. $30\frac{3}{8}$. $45\frac{9}{16}$. $68\frac{11}{32}$.

Likewise, if the first term 4 is given, together with the sum of the progression in the sesquialteral ratio, the last term $68\frac{11}{32}$ will be found. And so, if from 4 the sesquialteral proportion is continued, one must stop at the eighth term, $68\frac{11}{32}$, and so on. It is indeed a fine thing to be able to find the first term from the last, and the last from the first, even when the intermediate terms are not yet known.

156. To divide a given number into two parts, such that the first, taken any number of times you please, is a given multiple of the second taken any number of times you please; or such that the second, taken any number of times you please, is a given multiple of the first taken any number of times you please.

[Margin: CLVI.]

Let the given number be 40, to be divided first into two parts such that the first taken three times makes a number double the second taken four times. Let the first be set as $1R$, and the second as $40 - 1R$. The triple of the first is $3R$; this must be double the quadruple of the second, that is, of the number $160 - 4R$. There will therefore be an equation between $3R$ and $320 - 8R$. Adding $8R$ to both sides, between $11R$ and 320. Dividing therefore 320 by 11, $1R$ becomes $29\frac{1}{11}$, the first number; the other will then be $10\frac{10}{11}$. The triple of the first is $87\frac{3}{11}$, which is double the quadruple of the second, namely $43\frac{7}{11}$.

Next, let the same number 40 be distributed into two parts such that the second taken four times is double the first taken three times. Then the second taken four times — that is, the number $160 - 4R$ — will be double the first taken three times, that is, of $3R$; in other words, it will be equal to $6R$. Adding therefore $4R$ to both sides, the equation will be between 160 and $10R$. Dividing 160 by 10, $1R$ becomes 16, the first number; the other will then be 24. Four times this, namely 96, is double the triple of the first, namely 48.

157. To find three numbers in arithmetic progression such that the number produced by multiplying them together has a given ratio to their sum.

[Margin: CLVII.]

Let three numbers be sought that form an arithmetic progression, such that the number produced by multiplying them together is triple their sum. Let the three numbers sought be set as $1R$, $2R$, $3R$, having a common difference of $1R$. From their multiplication arises the number $6C$ [$C = cubus$, x^3]; and its triple, the triple of their sum $6R$, is equal to $18R$. So, dividing 18 by 6 , we get $13 = 3$ [$3 = census$, x^2] — because between the cubus [C] and the res [R] the middle character is the census [3]. And therefore $1R$ will be $\sqrt{3}3$ [$\sqrt{3}$ denotes the square root; $\sqrt{3}3 = \sqrt{3}$], $2R$ will be $\sqrt{3}12$, and $3R$ will be $\sqrt{3}27$, having the same common difference $\sqrt{3}3$. Multiplied together these make $\sqrt{3}972$, which is triple their sum — namely the number $\sqrt{3}108$.

When the denominator of the ratio which the product must have to the sum is a square number, the question is worked out in rational numbers. Thus, with the same numbers set as before, if the product is to have the ninefold ratio, an equation will be found between $6C$ and $54R$. Dividing therefore 54 by 6 , 13 becomes 9 and $1R$ becomes 3 ; and the three numbers sought will be 3 , 6 , 9 , which multiplied together generate the number 162 , ninefold their sum 18 .

158. To find three numbers such that the first, on taking any number of units you please from the second, makes a number that is some multiple (whichever you please) of what remains to the second; the second, on taking any number of units you please from the third, likewise makes a number that is some multiple (whichever you please) of what is left to the third; and finally the third, on taking any number of units you please from the first, makes a number equal to what remains to the first, less a given number.

[Margin: CLVIII.]

Let the first take 4 from the second, the second take 3 from the third, and the third take 10 from the first, in such a way that the first makes double what remains to the second, the second triple what remains to the third, and the third a number equal to what remains to the first, less 14 . Let the second be set as $3R + 6$, so that it can give 4 to the first and, having taken 3 from the third, produce a

number triple of what remains to the third — that is, produce a number that has an exact third, without a fraction. So, if the second gives 4 to the first, there will remain to it $3R + 2$, and the first will then have $6R + 4$, double the remainder; and hence, before it received 4, it had $6R$. Next, the second, on taking 3 from the third, will have $3R + 9$, a number triple of what remains to the third. The third, then, will at that point have $1R + 3$; and before it gave up 3 it had $1R + 6$; and if it takes 10 from the first, it will have $1R + 16$, a number that must equal the remainder of the first, less 14 — that is, if 14 units are added to the third, it will then make $1R + 30$, a number equal to the remainder of the first, $6R - 10$. Adding therefore 10 to both sides, there will be an equation between $1R + 40$ and $6R$. Removing $1R$ from both, between 40 and $5R$. Dividing therefore 40 by 5, $1R$ becomes 8. The first number, then, which we set as $6R$, will be 48; the second, set as $3R + 6$, will be 30; and finally the third, set as $1R + 6$, will be 14. For the first, on taking 4 from the second, will have 52, a number double the 26 that remains to the second. The second, on taking 3 from the third, will have 33, triple the 11 that remains to the third. And finally the third, on taking 10 from the first, will have 24, a number equal to the remainder of the first, $38 - 14$.

But if the third, on taking 10 from the first, must have a number exactly equal to the remainder of the first, $1R$ will be found to be $5\frac{1}{5}$, and the three numbers will be $31\frac{1}{5}$, $21\frac{3}{5}$, $11\frac{1}{5}$. Moreover, the equation found will show when the question is impossible.

159. To find two numbers such that the first, on taking any number of units you please from the second, exceeds what remains of the second by a given number; while the second, on taking any number of units you please from the first, makes a given multiple of what remains to the first, and moreover contains a given number in addition.

[Margin: CLIX.]

Let the first take 20 from the second, and the second 3 from the first; and let the first make a number that exceeds what remains of the second by 9, while the second makes a number triple what remains of the first and moreover contains 9 units. Let the first be set as $1R$. Taking 20 from the second, it will have $1R + 20$, a number that must exceed what remains of the second by 9. So to the second there remain $1R + 11$, and hence, before it gave 20 to the first, it had $1R + 31$. This, taking 3 from the first, will have $1R + 34$, a number that must be triple the remainder of the first and moreover contain 9. Removing therefore 9, the remaining number $1R + 25$ will be triple the remainder of the first. Now the

remainder of the first is $1R - 3$. So the equation will be between $1R + 25$ and $3R - 9$. Adding 9 to both sides, between $1R + 34$ and $3R$. Removing $1R$ from both, between 34 and $2R$. Dividing therefore 34 by 2 , $1R$ becomes 17 , the first number. The second, then, will be 48 — namely $1R + 31$. And indeed the first, 17 , on taking 20 from the second, will have 37 , a number that exceeds what remains of the second, 28 , by 9 . The second, 48 , on taking 3 from the first, will have 51 , a number that exceeds triple the remainder of the first, 14 , by 9 .

160. To find a number such that, if a given number is divided by it, the quotient multiplied into any given number produces as much as the number found multiplied into another given number.

[Margin: CLX.]

Let the given number be 36 , to be divided by the number sought, so that the quotient multiplied into 24 produces as much as the number found multiplied into 54 . Let the number sought be set as $1R$. Dividing 36 by it, the quotient is $36/1R$; multiplied into 24 this makes $864/1R$, a number equal to what arises from the posited number $1R$ multiplied into 54 , namely $54R$. This equation, reduced by crosswise multiplication, comes down to an equation between 864 and $54R$. Dividing therefore 864 by 54 , $1R$ becomes 16 and $1R$ becomes 4 , the number sought. For, dividing the given number 36 by 4 , the quotient is 9 , which multiplied into 24 makes 216 , a number equal to what arises from the number found, 4 , multiplied into 54 .

Again, let a number be sought by which, if the given number 40 is divided, the quotient multiplied into 8 produces as much as the number found multiplied into 50 . Let the number sought be set as $1R$. Dividing the number 40 by it, the quotient is $40/1R$; multiplied into 8 this makes $320/1R$, a number equal to $50R$, which arises from the posited number $1R$ multiplied into 50 . This equation, reduced by crosswise multiplication, comes down to an equation between 320 and $50R$. Dividing therefore 320 by 50 , $1R$ becomes $6\frac{2}{5}$, or $32/5$, and $1R$ becomes $\sqrt[3]{32/5}$ [= $\sqrt{(32/5)}$], the number sought. For [dividing] the number 40 — that is, $\sqrt[3]{1600}$ — [by it], the quotient becomes $\sqrt[3]{8000/32}$ [?], which multiplied into 8 — that is, into $\sqrt[3]{64}$ — produces the number $\sqrt[3]{512000/32}$, equal to the number $\sqrt[3]{80000/5}$, which arises from the number found, $\sqrt[3]{32/5}$, multiplied into 50 — that is, into $\sqrt[3]{2500}$. This is clear, since, multiplied crosswise, the two produce one and the same number, $\sqrt[3]{2560000}$.

161. To find three numbers in continued proportion in a given ratio, whose squares collected into one sum make a given number.

[Margin: CLXI.]

Let three numbers in continued sesquiterial proportion [4 : 3] be sought whose squares make 4329. Let the numbers sought be set as $9R$, $12R$, $16R$, in the sesquiterial ratio. Their squares, $81\bar{3}$, $144\bar{3}$, $256\bar{3}$, make the sum $481\bar{3}$, equal to 4329. Dividing therefore 4329 by $481\bar{3}$, $1\bar{3}$ becomes 9 and $1R$ becomes 3. Therefore the first number, set as $9R$, is 27, the second is 36, and the third is 48, whose squares 729 , 1296 , 2304 make the sum 4329.

Again, let three numbers in continued double proportion [2 : 1] be sought whose squares make 6000. Let the three numbers sought be set as $1R$, $2R$, $4R$. Their squares, $1\bar{3}$, $4\bar{3}$, $16\bar{3}$, make the sum $21\bar{3}$, equal to 6000. Dividing therefore 6000 by $21\bar{3}$, $1\bar{3}$ becomes $2000/7$ and $1R$ becomes $\sqrt[3]{2000/7}$. The numbers sought, then, in the double ratio are $\sqrt[3]{2000/7}$, $\sqrt[3]{8000/7}$, $\sqrt[3]{32000/7}$, whose squares $2000/7$, $8000/7$, $32000/7$ make $42000/7$ — that is, 6000.

162. To find three numbers in continued proportion in a given ratio, such that when a given part of the greatest number is multiplied by a given part of the middle, and this product is multiplied by a given part of the least, a given number is produced.

[Margin: CLXII.]

Let three numbers in continued sesquialteral proportion [3 : 2] be sought, such that a third part of the greatest multiplied by half the middle, and this product multiplied by a fourth part of the least, makes the number 576. Let the three numbers sought, in the sesquialteral ratio, be set as $4R$, $6R$, $9R$. From $\frac{1}{3}$ of the greatest, that is, from $3R$, multiplied into half the middle, that is, into $3R$, arises the number $9\bar{3}$; and this multiplied into $\frac{1}{4}$ of the least, that is, into $1R$, makes the number $9\bar{C}$, equal to 576. Dividing therefore 576 by $9\bar{C}$, $1\bar{C}$ becomes 64, whose cube root [radix cubica] is 4. Therefore the three numbers sought, having been set as $4R$, $6R$, $9R$, will be 16, 24, 36. And indeed, from a third part of the greatest, that is, from 12, multiplied into half the middle, that is, into 12, arises the number 144; and this multiplied into $\frac{1}{4}$ of the least, that is, into 4, makes 576.

163. To find a number, two given parts of which, multiplied together, produce a number which, when divided by a given number, yields a quotient that contains a given part — or given parts — of the square root of the number found.

[Margin: CLXIII.]

Let a number be sought whose $\frac{1}{3}$ and $\frac{1}{4}$, multiplied together, make a number which, divided by 27, yields $\frac{2}{3}$ of the square root of the number found. Let the number sought be set as 1R. From its $\frac{1}{3}$ multiplied into $\frac{1}{4}$ — that is, from 1R/3 into 1R/4 — arises the number 13/12; and this, divided by 27, gives the number 13/324, equal to two-thirds of the square root of the number found, 1R — that is, to two-thirds of the number $\sqrt{31R}$, that is, to the number $\sqrt{34R}/9$, which makes $\frac{2}{3}$ of the number $\sqrt{31R}$. It is produced from the multiplication of $\sqrt{31R}$ into $\frac{2}{3}$ — that is, into $\sqrt{34}/9$. And so the equation found is between $\sqrt{34R}/9$ and 13/324. Therefore the equation will also be between their squares, 4R/9 and 133/104976 [33 = *census of census*, x^4 ; $104976 = 324^2$], which by crosswise multiplication comes down to an equation between 419904R and 933. Dividing therefore 419904 by 9, 1C becomes 46656 (because between res [R] and census-of-census [33] the two middle characters are census [3] and cubus [C]), and 1R becomes 36, the number sought. For from 12, its third part, multiplied into 9, its fourth part, arises the number 108, which divided by 27 gives the quotient 4, which makes $\frac{2}{3}$ of the square root of the number found, 36 — whose root is 6.

164. To find a number, a given part of which, multiplied by itself, and the resulting number multiplied into another given part, produces the square of the number found.

[Margin: CLXIV.]

Let a number be sought whose $\frac{1}{5}$, multiplied by itself, makes a number which, multiplied into $\frac{1}{3}$ of the same number found, produces the square of the number found. Let the number sought be set as 1R. Its $\frac{1}{5}$ — namely 1R/5 — multiplied by itself makes 13/25; and this number, multiplied into $\frac{1}{3}$ — that is, into 1R/3 — makes the number 1C/75, equal to the square of the posited number 1R, that is, to the number 13. So the equation is between 1C/75 and 13, which by crosswise multiplication is reduced to this one, between 1C and 753. Dividing therefore 75 by 1, 1R becomes 75 (since they are cossic collateral numbers), the number sought. For its fifth part, 15, multiplied by itself makes 225, and this number

multiplied into 25 — the third part of the same number 75 — makes 5625, the square of the number found, 75.

Cossist notation used throughout: $1R$ = *res* (the unknown, x); 13 = *census* (x^2); $1C$ = *cubus* (x^3); 133 = *zensizensica* (x^4); N = *numerus* (the absolute number, degree 0). A radical written $\sqrt{3}$ is the square root (*radix zensica*); $\sqrt[4]{3}$ is the fourth root (*radix zensizensica*), and so on. Signs are transcribed as printed.

165. (CLXV.) *To find a number such that a given part of it, multiplied by another given part, produces a given number.*

Let the number to be found be one whose fourth part, multiplied by its sixth part, makes 384. Posit the sought number as $1R$ [*res*, the unknown x]. From $1/4R$ into $1/6R$ arises the number $1/24\ 3$ [*census*, x^2], equal to 384. By cross-multiplication this equation is reduced to one between 13 and 9216. Dividing 9216 by 1, then, gives $13 = 9216$, and $1R = 96$, the sought number. For from 24 — its fourth part — into 16 — its sixth part — arises the given number 384.

Likewise, let the number to be found be one whose fourth part, multiplied into its sixth part, makes 100. Posit the sought number as $1R$. From $1/4R$ into $1/6R$ arises the number $1/24\ 3$, equal to 100. By cross-multiplication this equation reduces to one between 13 and 2400. Dividing 2400 by 1, then, gives $13 = 2400$, whose square root, $\sqrt{3}\ 2400$ [$\sqrt{2400}$], will be the sought number. For its fourth part is $\sqrt{3}\ 150$ [$\sqrt{150}$] and its sixth part $\sqrt{3}\ 200/3$ [$\sqrt{(200/3)}$], and these parts, multiplied together, generate $\sqrt{3}\ 30000/3$ — that is, $\sqrt{3}\ 10000$, or 100.

SCHOLIUM.

Without algebra this problem is solved in the following way. Let the denominators of the parts be multiplied together, and the resulting number multiplied into the given number; the square root of this product will be the sought number. Thus, in the earlier example, 4 into 6 makes 24, and 24 into 384 makes 9216, whose square root, 96, gives the sought number.

But if three parts are proposed to be multiplied together so as to produce a given number, the denominators of the parts must be multiplied together, and the resulting number multiplied into the given number; the cube root of this number will then give the sought number. For example, suppose a number is sought whose parts $1/2$, $1/3$, and $1/4$, multiplied together, produce 72000. Multiplying the denominators together gives 24; and 24 into 72000 makes 1728000, whose cube

root, 120, gives the sought number. For its $1/2$, $1/3$, and $1/4$ — namely 60, 40, and 30 — multiplied together make 72000.

One must proceed in the same way if four aliquot parts are proposed to be multiplied together — except that from the final product the *zensizensic* (fourth) root must be extracted. And if five are proposed, the *surdesolid* (fifth) root must be extracted; and so on to infinity. For from the final product one must always extract the root whose exponent equals the number of aliquot parts. Thus, if 10 aliquot parts are given, the *zensurdesolid* (tenth) root must be extracted; and if 12 are given, the *zensizensicubic* (twelfth) root, and so on. On this matter, consult what we wrote in chapter 2 concerning the progressions of cosmic numbers.

166. (CLXVI.) *To find three numbers such that any two of them, multiplied together, generate three given numbers.*

Let the first into the second make 240; the first into the third, 660; and the second into the third, 1380. Posit the first number as 1R. If the number 240, the product of the first into the second, is divided by it, the second becomes $240/1R$. For this quotient, multiplied into the divisor 1R, produces the dividend 240. By the same reasoning the third will be $660/1R$ — the quotient, that is, of the division of the number 660 by 1R. Now from the second, $240/1R$, into the third, $660/1R$, arises the number $158400/13$, equal to 1380. By cross-multiplication this equation is reduced to one between 158400 and $1380\bar{3}$. Dividing 158400 by 1380, then, gives $13 = 114\frac{18}{23}$; hence 1R (since between N [numerus] and $\bar{3}$ [census] the middle character is one, namely 1R) will be $\sqrt[3]{114\frac{18}{23}}$ — that is, the first number, which was posited as 1R.

And if the first number, the product 240, that is $\sqrt[3]{57600}$, is divided by this first number, that is by $\sqrt[3]{114\frac{18}{23}}$, the quotient becomes $\sqrt[3]{501\frac{9}{11}}$ — the second number, that is. And if the second product, 660, that is $\sqrt[3]{435600}$, is divided by the same first number $\sqrt[3]{114\frac{18}{23}}$, the quotient will give $\sqrt[3]{3795}$ — the third number. From the first, $\sqrt[3]{114\frac{18}{23}}$, into the second, $\sqrt[3]{501\frac{9}{11}}$, arises $\sqrt[3]{57600}$, that is 240. From the same first, $\sqrt[3]{114\frac{18}{23}}$, into the third, $\sqrt[3]{3795}$, arises $\sqrt[3]{435600}$, that is 660. And finally, from the second, $\sqrt[3]{501\frac{9}{11}}$, into the third, $\sqrt[3]{3795}$, arises $\sqrt[3]{1904400}$, that is 1380.

167. (CLXVII.) Given two numbers, to find a third such that, together with the smaller, it makes a number equal to that which a given part of the found number makes together with the larger.

Let the two given numbers be 12 and 20, and let a third be found such that it makes, together with 12, as much as its $\frac{3}{7}$ makes together with 20. Posit the sought number as $1R$. Then $12 + 1R$ will be equal to $20 + \frac{3}{7}R$. Removing $\frac{3}{7}R$ from both sides, the equation stands between $12 + \frac{4}{7}R$ and 20. Removing 12 from both, between $\frac{4}{7}R$ and 8. Dividing 8 by $\frac{4}{7}$, then, gives $1R = 14$, the number we seek. For, added to 12, it makes 26; and its $\frac{3}{7}$ — namely 6 — together with 20 likewise make 26.

168. (CLXVIII.) To find a number such that the squares of any number of its parts, taken together, make up a sum equal to the found number, or a multiple of it.

First, let a number be found such that the squares formed from its $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ make a sum equal to the found number. Posit the sought number as $1R$, whose named parts are $\frac{1}{2}R$, $\frac{1}{3}R$, and $\frac{1}{4}R$, and whose squares are $\frac{1}{4}\mathfrak{z}$, $\frac{1}{9}\mathfrak{z}$, and $\frac{1}{16}\mathfrak{z}$. These together make $\frac{244}{576}\mathfrak{z}$, which sum is equal to $1R$. Dividing 1 by $\frac{244}{576}$, then, gives $1R = \frac{576}{244}$, the sought number. For its $\frac{1}{2}$ is $\frac{288}{244}$, and its square $\frac{82944}{59536}$; its $\frac{1}{3}$ is $\frac{192}{244}$, and its square $\frac{36864}{59536}$; its $\frac{1}{4}$ is $\frac{144}{244}$, and its square $\frac{20736}{59536}$. All three squares make $\frac{140544}{59536}$, a number equal to the found number — since as much arises from the denominator of this into the numerator of that as arises from the numerator into the denominator.

Next, let a number be found such that the squares formed from its $\frac{1}{2}$ and $\frac{1}{4}$ make a number tenfold the found number. Posit the number as $1R$, whose $\frac{1}{2}$ and $\frac{1}{4}$ are $\frac{1}{2}R$ and $\frac{1}{4}R$, and whose squares are $\frac{1}{4}\mathfrak{z}$ and $\frac{1}{16}\mathfrak{z}$. These together make $\frac{5}{16}\mathfrak{z}$, equal to $10R$. Dividing 10 by $\frac{5}{16}$, then, gives $1R = 32$, the sought number. For the squares 256 and 64, formed from its $\frac{1}{2}$ and $\frac{1}{4}$, make the sum 320, tenfold the found number, 32.

169. (CLXIX.) *To find a number which, multiplied by a given part of itself, produces any given number whatever.*

Let the number to be found be one which, multiplied into its $\frac{3}{4}$, produces 90. Posit the found number as $1R$. From $1R$ into $\frac{3}{4}R$ arises the number $\frac{3}{4}R$, equal to 90. Dividing 90 by $\frac{3}{4}$, then, gives $1\frac{3}{4} = 120$, and $1R = \sqrt{\frac{3}{4}} 120 [\sqrt{120}]$, the sought number. Its $\frac{3}{4}$ is $\sqrt{\frac{3}{4}} 1080/16$ — as is plain, if $\sqrt{\frac{3}{4}} 120$ is multiplied into $\frac{3}{4}$, that is, into $\sqrt{\frac{3}{4}} 9/16$. Now from $\sqrt{\frac{3}{4}} 120$ into $\sqrt{\frac{3}{4}} 1080/16$ arises $\sqrt{\frac{3}{4}} 129600/16$, that is $\sqrt{\frac{3}{4}} 8100$, that is 90.

170. (CLXX.) *To divide a given number into two parts such that the sum of their squares makes up a given sum — greater, however, than half the square generated from the given number, and less than the square of the same number.*

Let the number 40 be divided into two, whose squares make 928 — a number greater than 800 (which is half the square generated from 40) and less than 1600 (which is the square of the same number, 40). Posit one of the numbers as $20 - 1R$, and the other as $20 + 1R$, so that together they make 40. The squares of these two numbers are $400 - 40R + 1\frac{3}{4}$ and $400 + 40R + 1\frac{3}{4}$, which make $2\frac{3}{4} + 800$, a number equal to 928. Removing 800 from both sides, the equation will be between $2\frac{3}{4}$ and 128. Dividing 128 by 2 gives $1\frac{3}{4} = 64$, and $1R = 8$. Therefore the first number, posited as $20 - 1R$, will be 12; and the other, which we posited as $20 + 1R$, will be 28. And the squares of these numbers, 12 and 28 — namely 144 and 784 — make 928.

We shall also solve this enigma by means of a composite equation in enigma 35 of the following chapter, chapter 30.

171. (CLXXI.) *To divide a given number into three continually proportional parts, such that the number produced from the first into the third has a given ratio to the number produced from the first into the second.*

This enigma was also solved in problem 133, but with a different example.

Let the given number be 39, to be divided into three continually proportional parts, such that the number made from the first into the third has to the number made from the first into the second the double-sesquialterate ratio [5 : 2]. Posit

the second as $1R$; hence the first and third together are $39 - 1R$. From the second, $1R$, into itself arises 13 [Margin: a — 20 Sept.] — namely as much as arises from the first into the third. And since this product must be double-sesquialterate to that which arises from the first into the second, the product of the first into the second becomes $2/53$ (for 13 has to $2/53$ the double-sesquialterate ratio, as is clear if 13 is divided by $2\frac{1}{2}$, the denominator of the double-sesquialterate ratio). Dividing this by the second, $1R$, gives the quotient $2/53 \div 1R$, or $2/5R$, which, multiplied into the divisor — that is, into $1R$ — produces $2/53$. Accordingly the first number will be $2/5R$. Removing this from the sum of the first and third — that is, from $39 - 1R$ — the remaining third number will be $(195 - 7R)/5$. If the first, $2/5R$, is multiplied into it, the number $(390R - 143)/25$ is produced, equal to the number 13 , which arises from the middle, $1R$, into itself. By cross-multiplication this equation is reduced to one between 253 and $390R - 143$. Adding 143 to both sides, the equation will be between 393 and $390R$. Dividing 390 by 39 (since the cosmic numbers are collateral) gives $1R = 10$ — namely the middle number.

The first, therefore, posited as $2/5R$, is 4 (namely 20 divided by 5); and the third, which was found to be $(195 - 7R)/5$ — that is, $195 - 70$, divided by 5 — is 25 . So the three parts of the number 39 are 4 , 10 , and 25 , continually proportional. And what arises from the first into the third, or from the second into itself — namely 100 — has the double-sesquialterate ratio to the number 40 , which arises from the first into the second.

Now since, given three continually proportional numbers, the number that arises from the middle into itself has to the product of the first into the middle the same ratio as the three given numbers have [Margin: b — 17 Sept.] (because the middle, multiplying itself and the first, produces those two products), this proposed enigma will be solved more easily if, by enigma 49, the given number is cut into three continually proportional numbers in the given ratio. For in this way the number made from the first into the third — that is, from the second into itself — will have to the product of the first into the second the given ratio, the same, namely, as the three numbers have. Thus in the given example, the given number 39 must be cut into three continually double-sesquialterate numbers, as follows. Posit three numbers in continued double-sesquialterate proportion: $1R$, $2\frac{1}{2}R$, $6\frac{1}{4}R$, whose sum, $9\frac{3}{4}R$, must be equal to the given number 39 . Dividing 39 by $9\frac{3}{4}$, then, gives $1R = 4$, the first number, which was posited as $1R$. The second, therefore, posited as $2\frac{1}{2}R$, will be 10 ; and the third, which we posited as $6\frac{1}{4}R$, will be 25 .

172. (CLXXII.) *To divide a given number into two parts such that from the one into the other is made another given number, smaller than the fourth part of the square generated from the first given number; that is, such that the square root of the second given number — which is smaller than the fourth part of the square generated from the first given number — is the mean proportional between the two found numbers.*

Let the given number be 10, to be distributed into two parts between which $\sqrt{3} 1$ [$\sqrt{1}$] is the mean proportional — that is, such that from the one into the other is made 1. Posit the smaller part as $5 - 1R$, and the larger as $5 + 1R$, so that their sum is 10. From the one into the other arises the number $25 - 13$, as appears in this formula, equal to 1. Adding 13 to both sides gives the equation between 25 and $13 + 1$. Removing 1 from both, between 24 and 13 . Dividing 24 by 1 , then, gives $13 = 24$, and $1R = \sqrt{3} 24$ [$\sqrt{24}$]. Therefore the smaller part, posited as $5 - 1R$, will be $5 - \sqrt{3} 24$; and the larger, posited as $5 + 1R$, will be $5 + \sqrt{3} 24$. And these two parts,

$$\begin{array}{r} 5 - 1R \\ 5 + 1R \\ \hline + 5R - 13 \\ 25 - 5R \\ \hline 25 - 13 \end{array}$$

multiplied together, make 1, as is seen here. For the multiplication is effected if 24 , the square of the latter part, is subtracted from 25 , the square of the former part, as we taught in chapter 23:

$$\begin{array}{r} 5 - \sqrt{3} 24 \\ 5 + \sqrt{3} 24 \\ \hline 1 \end{array}$$

[Margin: a — 20 Sept.] Therefore $\sqrt{3} 1$ is the mean proportional between $5 - \sqrt{3} 24$ and $5 + \sqrt{3} 24$, since both these two multiplied together, and that $\sqrt{3} 1$ into itself, make 1.

Furthermore, we shall solve this same enigma by a composite equation as well, in enigma 38 of the following chapter.

173. (CLXXIII.) *To divide a given number into two parts such that their squares make a sum which exceeds the product of their mutual multiplication by a given number.*

Let the given number be 8, to be divided into two, so that the sum collected from their squares exceeds the product from their mutual multiplication by the number 20. Posit the smaller number as $4 - 1R$, and the larger as $4 + 1R$, so that together they make 8. Their squares are $13 + 16 - 8R$ and $13 + 16 + 8R$, making the sum $23 + 32$ (since $-8R$ and $+8R$ cancel each other). This sum must exceed the product of the one into the other by the number 20. Now this product is $16 - 13$, as appears in the adjacent formula, which, together with 20, will make the number $36 - 13$, equal to the sum of the squares, $23 + 32$. Adding 13 to each side, then, there will be an equality between $33 + 32$ and 36. Removing 32 from both sides, between 33 and 4. Dividing 4 by 3, then, gives $13 = 4/3$, and $1R = \sqrt{3} \ 4/3$ [$\sqrt{(4/3)}$]. Since, then, the smaller number was posited as $4 - 1R$, the smaller number itself will be $4 - \sqrt{3} \ 4/3$; and the larger, posited as $4 + 1R$, will be $4 + \sqrt{3} \ 4/3$, as appears in these formulas.

$$\begin{array}{r}
 4 - 1R \\
 4 + 1R \\
 \hline
 + 4R - 13 \\
 16 - 4R \\
 \hline
 16 - 13
 \end{array}$$

These two numbers make 8, and their squares are $17 \frac{1}{3} - \sqrt{3} \ 256/3$ and $17 \frac{1}{3} + \sqrt{3} \ 256/3$, as appears in these formulas:

$$\begin{array}{r}
 4 - \sqrt{3} \ 4/3 \quad 4 + \sqrt{3} \ 4/3 \\
 4 - \sqrt{3} \ 4/3 \quad 4 + \sqrt{3} \ 4/3 \\
 \hline
 - \sqrt{3} \ 64/3 + 4/3 + \sqrt{3} \ 64/3 + 4/3 \\
 16 - \sqrt{3} \ 64/3 \quad 16 + \sqrt{3} \ 64/3 \\
 \hline
 17 \frac{1}{3} - \sqrt{3} \ 256/3 \quad 17 \frac{1}{3} + \sqrt{3} \ 256/3
 \end{array}$$

Their sum is $34 \frac{2}{3}$, which exceeds the product $14 \frac{2}{3}$ of the one into the other by the given number 20.

$$\begin{array}{r}
 4 - \sqrt{3} \ 4/3 \\
 4 + \sqrt{3} \ 4/3 \\
 \hline
 14 \frac{2}{3}
 \end{array}$$

This product, $14 \frac{2}{3}$, is obtained if $\frac{4}{3}$, the square of the latter part, is taken from 16, the square of the former part, as was handed down in chapter 23, and as this other formula indicates.

Again, let the given number be 40, to be distributed into two, whose squares make a sum which exceeds the number made from their multiplication by the number 403. Posit the smaller number as $20 - 1R$, and the larger as $20 + 1R$, making the sum 40. Their squares are $13 + 400 - 40R$ and $13 + 400 + 40R$, whose sum, $23 + 800$, exceeds by the number 403 the number $400 - 13$, the product of the one into the other.

$$\begin{array}{r} 20 - 1R \\ 20 + 1R \\ \hline 400 - 13 \end{array}$$

This product is obtained if the square 13 of the latter part is taken from 400, the square of the former part, as you see in this formula, and as was handed down in chapter 23. Therefore, if 403 is added to this product, there will be an equation between $23 + 800$, the sum of the squares, and $803 - 13$. Adding 13 to both sides, between $33 + 800$ and 803. Removing 800 from both, between 33 and 3. Dividing 3 by 3, then, gives $13 = 1$, and $1R = 1$. Therefore the smaller number, which we posited as $20 - 1R$, will be 19; and the larger, posited as $20 + 1R$, will be 21 — which complete the problem.

We shall also solve this problem by a composite equation in enigma 39 of the following chapter. If the number 20 were proposed to be divided into two parts such that the sum of the squares exceeds the number produced from the multiplication of the parts by the number 30, the problem would be impossible, as you will learn from the solution itself.

174. (CLXXIV.) To find a number whose square, increased by a given number, when multiplied into the same square diminished by the same given number, produces any given number.

Let the number to be found be one whose square, increased by eight, when multiplied into the same square diminished by the same eight, produces 6497. Posit the sought number as $1R$. Its square increased by eight will be $13 + 8$; diminished by the same eight, $13 - 8$. These two numbers, multiplied together,

make $133 - 64$ [zensizensica, x^4], as you see here, if 64, the square of the latter part, is taken from 133, the square of the former part.

$$\begin{array}{r} 13 + 8 \\ 13 - 8 \\ \hline 133 - 64 \end{array}$$

This product must be equal to the given number 6497. Adding 64 to both sides, then, the equation will be between 133 and 6561. Dividing 6561 by 1, then, gives $133 = 6561$, and its *zensizensic* (fourth) root, 9, is the sought number. For its square, 81, increased by eight makes 89, and diminished by eight, 73; and the number 6497 arises from 89 into 73.

(Continuation of the final riddle of the preceding chapter — the close of "Aenigmata varia," solved by simple equations.)

If, with the same things laid down, the number to be produced is to be 10, an equation will be found between 133 [zensizensus, the fourth power, x^4] and 74. Dividing 74 by 1, then, gives $133 = 74$, and the fourth root $\sqrt[3]{3} 74$ [radix zensizensica, the fourth root of 74] is the number sought. For its square is $\sqrt[3]{3} 5476$ [= $\sqrt{74}$, since $5476 = 74^2$]; increased by eight this is $\sqrt[3]{3} 5476 + 8$, and diminished by eight it is $\sqrt[3]{3} 5476 - 8$. These two numbers are equivalent to the two $\sqrt[3]{3} 74 + 8$ [$\sqrt[3]{3}$ = radix quadrata, the square root] and $\sqrt[3]{3} 74 - 8$; and these two, multiplied by each other, produce 10, as you see here — since 64, the square of the latter part, subtracted from 74, the square of the former part, leaves 10.

$$\begin{array}{r} \sqrt[3]{3} 74 + 8 \\ \sqrt[3]{3} 74 - 8 \\ \hline 10 \end{array}$$

Scholium

Let this suffice concerning the riddles that are unraveled by a simple equation. On the model of these, the studious reader can devise as many others as he likes. We shall therefore now set out other riddles, in solving which composite equations occur.

VARIOUS RIDDLES OF ABSTRACT NUMBERS, to be unraveled by algebra, in which there occurs an equation among three numbers, of which one is equal to the other two: this indeed is called a *composite equation*.

Caput XXX

Problem 1 [I.] — To distribute a given number into two parts, so that their cubes make a given sum which is greater than a quarter of the cube generated from the given number.

This riddle, which was number 141 of the preceding chapter, we shall solve here in another way — namely, by a composite equation, in which one Cossic number is found equal to two Cossic numbers, and in which a root must be extracted from a composite or diminished number, as was taught in chapter 12. The same will be done in the other riddles of this chapter.

Let the given number, then, be 10, to be divided into two parts whose cubes make the sum 370 — which is greater than 250, a quarter of the cube generated from the given number 10. Let the first part be posited as $1R$ [res, the unknown, x], and the second as $10 - 1R$. The cubes of these parts are $1C$ [cubus, x^3] and $1000 - 300R + 303 - 1C$ [$3 =$ census, x^2]. (For $+1C$ and $-1C$ cancel each other.) Their sum, $1000 - 300R + 303$, must equal 370. Adding $300R$ to both sides gives an equation between $1000 + 303$ and $300R + 370$. Removing 370 from both sides gives, between $630 + 303$ and $300R$. Removing 630 again from both sides gives, between 303 and $300R - 630$. Now dividing each of these numbers by the coefficient of the greatest Cossic character — namely by 30, as we prescribed near the end of chapter 10 — the equation will be between 13 and $10R - 21$.

You will resolve this by the method of chapter 12. Half the number of roots is 5, which makes the square 25; from this, subtracting 21 (because of the sign $-$), the number 4 remains, whose square root is 2. Added to 5, half the number of roots, this gives the larger root 7. And the same root 2, subtracted from the same half 5, makes the remaining, smaller root 3. For numbers of this kind, in which the absolute number is affected by the sign $-$, have a double root, as we taught in chapter 12. The parts, therefore, of the given number 10 are 7 and 3, whose cubes

343 and 27 make the given sum 370 — as was also found in riddle 141 of the preceding chapter.

If, with the same number given, the sum of the cubes is proposed to be 400, we shall find the final equation between $1\bar{3}$ and $10\bar{R} - 20$, which you will resolve thus. Half the number of roots is 5, making the square 25; from this, subtracting 20, the number 5 remains, whose square root $\sqrt{35}$, added to 5, half the number of roots, gives the larger root $\sqrt{35} + 5$. And the same root $\sqrt{35}$, subtracted from the same half 5, makes the remaining, smaller root $5 - \sqrt{35}$ — as we found in the same riddle 141 of the preceding chapter.

But how this riddle is to be solved without algebra was stated above in riddle 141 of the preceding chapter.

Problem 2 [II.] — To distribute a given number into two cubes, the sum of whose sides is given: provided that a quarter of the cube of this sum is less than the given number.

This problem coincides with the preceding one, as we said in riddle 142 of the preceding chapter.

Problem 3 [III.] — To find a number which, together with a given number, produces its own square.

Let the given number be 156, and let it be required to find a number which, added to 156, makes up the square of the number found. Let the number sought be posited as $1\bar{R}$, whose square is $1\bar{3}$; and this must equal $1\bar{R} + 156$. Half the number of roots is $\frac{1}{2}$; to its square, that is, to $\frac{1}{4}$, I add 156, making $156\frac{1}{4}$, or $625/4$. Its square root is $25/2$, that is, $12\frac{1}{2}$; to this I add half the number of roots, namely $\frac{1}{2}$, and so obtain the value of $1\bar{R}$ as 13, which is the number sought. For 13, together with 156, makes 169, the square of the number 13.

Again, let there be sought a number which, together with 50, makes up its own square. Let the number sought be posited as $1\bar{R}$, whose square is $1\bar{3}$; and this must equal $1\bar{R} + 50$. Half the number of roots is $\frac{1}{2}$; to its square, that is, to $\frac{1}{4}$, I add 50, making $50\frac{1}{4}$, or $201/4$. To its root, that is, to $\sqrt{3} 201/4$, I add half the number of roots, namely $\frac{1}{2}$, and so obtain the value of $1\bar{R}$ as $\sqrt{3} 201/4 + \frac{1}{2}$ — that is, the number sought. For its square is $\sqrt{3} 804/16 + 202/4$, as appears in this formula — that is, $\sqrt{3} 201/4 + 50\frac{1}{2}$ — which is exactly what arises from the number found, $\sqrt{3} 201/4 + \frac{1}{2}$, together with the given number 50. For the number $\sqrt{3} 201/4 + 50\frac{1}{2}$ is likewise made up. I have appended the formula for the multiplication of the

number found, $\sqrt{3} \ 201/4 + \frac{1}{2}$, squared into itself, so that you may see how the square is produced.

$$\begin{array}{l} \sqrt{3} \ 201/4 + \frac{1}{2} \\ \sqrt{3} \ 201/4 + \frac{1}{2} \end{array}$$

$$\begin{array}{l} + \sqrt{3} \ 201/16 + \frac{1}{4} \\ 201/4 + \sqrt{3} \ 201/16 \end{array}$$

$$201/4 + \sqrt{3} \ 804/16 + \frac{1}{4}$$

Problem 4 [IV.] — To find a number whose square, together with a given number, produces the zensizensus of the number found — that is, the square of its square.

Let the given number be 83232, and let it be required to find a number which, together with 83232, makes up the square-of-the-square of itself. Let the number sought be posited as 1R, whose square is 13 and whose square-of-the-square is 133 [zensizensus, x^4], equal to 13 + 83232. Half the number of censuses is $\frac{1}{2}$; to its square, that is, to $\frac{1}{4}$, I add the number 83232, making $83232\frac{1}{4}$, that is, $332929/4$. To its square root $577/2$ I add half the number of censuses, namely $\frac{1}{2}$, making $578/2$, that is, 289, a square, whose root 17 is the number sought — as was declared near the end of chapter 12. For the Cossic characters 33, 3, N [numerus, the units] are not consecutive, but between any two of them there lies one in the middle; on this account the value of one census is found first, and then the root is extracted, which yields the number we are seeking. And so the square of the number found, 17 — namely 289 — together with the given number 83232, makes up the square-of-the-square 83521 of the number found, 17.

Problem 5 [V.] — To find a number whose cube, together with a given number, produces the square of its cube — or the cube of its square.

Let the given number be 7526792, and let there be sought another number whose cube, together with the given number, makes up the square of the cube of the number found, or (what is the same) the cube of its square. Let the number sought be posited as 1R, whose cube is 1C and whose zensicubus is 13C [zensicubus, x^6], equal to 1C + 7526792. From this composite number the zensicubic root must be extracted, in this way. Half the number of the middle character 1C, namely $\frac{1}{2}$, makes the square $\frac{1}{4}$. Adding the number 7526792 gives $7526792\frac{1}{4}$, or $30107169/4$, whose square root is $5487/2$. To this I add half the aforesaid number of the middle character 1C, obtaining $5488/2$, that is, 2744 — the square root of the number 1C + 7526792, on account of the character 3, which

is found in the greatest Cossic character. Finally, the cube root of this square root 2744 — namely 14 — on account of the other character $\mathfrak{C}$, placed in the greatest character, will give the number sought, as we taught near the end of chapter 12. For its cube, 2744, added to the given number 7526792, makes up its own zensicubus, or the cube of the square, 7529536.

Problem 6 [VI.] — To find two numbers making up a given number, such that from the multiplication of one into the other any given number whatever is generated.

Let the given sum be 25 and the product 84. Let one number be posited as $1\mathfrak{R}$, and therefore the other as $25 - 1\mathfrak{R}$; these two added together make 25. From one multiplied into the other arises the number $25\mathfrak{R} - 1\mathfrak{J}$, equal to 84. Adding 13 to both sides, there will be an equality between $25\mathfrak{R}$ and $1\mathfrak{J} + 84$. Removing 84 from both sides, between $25\mathfrak{R} - 84$ and $1\mathfrak{J}$. And so from the number $25\mathfrak{R} - 84$ the census-root [radix zensica, the square root] must be extracted, in this way. Half the number of roots is $25/2$, making the square $625/4$; from this, subtracting the number 84 — that is, $336/4$ — because of the sign $-$, there remains the number $289/4$, whose square root is $17/2$. I add the aforesaid half of the number of roots, namely $25/2$, and so obtain $42/2$, that is, 21 — one of the numbers sought. The other, therefore, will be 4; and this too will be found if the last root discovered, $17/2$, is subtracted from the aforesaid half of the number of roots, namely from $25/2$. For the number $25\mathfrak{R} - 84$ has a double root, on account of the sign $-$ by which the absolute number 84 is affected. Now the two numbers found, 21 and 4, make the given sum 25, and from one multiplied into the other the given number 84 is produced.

Problem 7 [VII.] — To find a number whose zensizensus, together with a given number, produces a number in a given ratio to the square of the number found.

Let the given number be 1183, and let there be sought another number whose square-of-the-square, together with 1183, makes a number which has to the square of the number found the same ratio as 176 to 1. Let the number sought be posited as $1\mathfrak{R}$, whose zensizensus is $1\mathfrak{J}\mathfrak{J}$ and whose square, or census, is $1\mathfrak{J}$. There will therefore be an equation between $1\mathfrak{J}\mathfrak{J} + 1183$ and $176\mathfrak{J}$ — since one zensizensus together with 1183 must have to $1\mathfrak{J}$ the same ratio as 176 to 1, that is, must equal $176\mathfrak{J}$. Removing 1183 from both sides, then, there will be an equation between $1\mathfrak{J}\mathfrak{J}$ and $176\mathfrak{J} - 1183$. From this number, $176\mathfrak{J} - 1183$, we shall extract the zensizensic root as follows. Half the number of the middle character $\mathfrak{J}$ is 88;

from its square, 7744, if 1183 is taken away (because of the sign $-$), there remain 6561, whose square root is 81. If the aforesaid half, 88, is added, the square 169 results, whose root 13 will give the number sought. For its zensizensus, 28561, together with the given number 1183, makes the number 29744, which has to 169, the square of the number found, the same ratio as 176 to 1.

But since the number $1763 - 1183$ has a double zensizensic root — the larger, namely, and the smaller, as was taught in chapter 12 — the larger is the one found, 13, which satisfies the problem. The smaller will be obtained if 81, the square root of the residue 6561, is subtracted from the aforesaid half, 88. For the square root of the remaining number 7 — namely $\sqrt{37}$ — will be the smaller zensizensic root, that is, the number sought. For its square is 7, and its zensizensus 49, which together with 1183 makes the number 1232, which has to the square 7 the same ratio as 176 to 1.

Problem 8 [VIII.] — To find a number whose zensicubus, together with a given number, makes a number that has a given ratio to its own cube.

Let the given number be 1068664111, and let there be sought another number whose zensicubus, together with the given number, makes a number whose ratio to the cube of the number found is the same as 100000 to 1. Let the number sought be posited as $1R$, whose cube is $1C$ and whose zensicubus is $13C$. There will therefore be an equality between $13C + 1068664111$ and $100000C$ — since $13C + 1068664111$ must have to $1C$ the same ratio as 100000 to 1. Removing 1068664111 from both sides, then, there will remain an equality between $13C$ and $100000C - 1068664111$; and so from $100000C - 1068664111$ the zensicubic root must be extracted, in this way. Half the number of cubes is 50000; from its square, 2500000000, if the number proposed is subtracted (because of the sign $-$), there will remain the number 1431335889, whose square root is 37833. If the aforesaid half, 50000, is added, the square root 87833 is obtained. Its cube root, $\sqrt[3]{C} 87833$ [radix cubica, the cube root], will give the number sought, according to the doctrine of chapter 12. For its cube is 87833, and its zensicubus 7714635889, which together with the given number 1068664111 makes the number 8783300000, which has to the cube 87833 the same ratio as 100000 to 1.

And since the number $100000C - 1068664111$ has a double zensicubic root, of which the larger, found to be $\sqrt[3]{C} 87833$, satisfies the proposed riddle, the smaller will be found thus. From the aforesaid half, 50000, of the number of cubes, let the last square root found, 37833, be subtracted. For the cube root of the residue

12167 — namely 23 — will be the smaller root, likewise satisfying the proposed riddle. For its cube is 12167, and its zensicubus 148035889, which together with the given number 1068664111 makes the number 1216700000, which has to the cube 12167 the same ratio as 100000 to 1.

Problem 9 [IX.] — To find two numbers differing by a given amount, such that from the multiplication of one into the other any given number is generated.

Let the given difference be 6 and the given number 475. Let the smaller number be posited as $1R$, and the larger accordingly as $1R + 6$. From one multiplied into the other arises the number $1\bar{3} + 6R$, equal to 475. Removing $6R$, then, from both sides, there will be an equation between $1\bar{3}$ and $475 - 6R$. From this number, therefore, we shall extract the root thus. Half the number of roots is 3, making the square 9; to this I add the number 475, which is understood to be affected by the sign $+$. And from the number thus formed, 484, I take the square root 22. From this I remove the aforesaid half, 3, of the number of roots, because of the sign $-$. For the remaining number 19 will be the value of $1R$. Therefore $1R + 6$ will amount to 25. And these two numbers, 19 and 25, are the ones sought. For they have the given difference 6, and from one multiplied into the other the number 475 is produced.

Again, let the given difference be 6 and the given number 500. With the same things posited, the product will be the number $1\bar{3} + 6R$, equal to 500. Removing $6R$ from both sides, there will be an equation between $1\bar{3}$ and $500 - 6R$. Half the number of roots is 3, making the square 9, which added to 500 makes 509. From its square root $\sqrt{3} 509$, if the aforesaid half 3 is taken away, the smaller number sought becomes $\sqrt{3} 509 - 3$. Adding the given difference 6, the larger becomes $\sqrt{3} 509 + 3$. From the smaller multiplied into the larger arises the number $509 - 9$, as appears in this formula — that is, 500 — because $-\sqrt{3} 4581$ and $+\sqrt{3} 4581$ cancel each other. Or indeed, because, as was taught in chapter 23, if 9, the square of the latter part, is subtracted from 509, the square of the former part, there remains the number produced, 500.

$$\begin{array}{r}
 \sqrt{3} 509 + 3 \\
 \sqrt{3} 509 - 3 \\
 \hline
 - \sqrt{3} 4581 - 9 \\
 509 + \sqrt{3} 4581 \\
 \hline
 509 - 9
 \end{array}$$

Problem 10 [X.] — To find a number such that any multiple whatever of its square, together with the square-of-the-square of the same number, makes a given number.

Let there be sought a number whose square taken five times, together with the square-of-the-square of the same, makes 535086. Let the number sought be posited as $1R$, whose square is $1\mathfrak{z}$; taken five times this makes $5\mathfrak{z}$, which number, together with the zensizensus of the same number $1R$ — that is, with $1\mathfrak{z}\mathfrak{z}$ — must equal the number 535086. Since, then, the equation is between $5\mathfrak{z} + 1\mathfrak{z}\mathfrak{z}$ and 535086, removing $5\mathfrak{z}$ from both sides there will remain an equation between $1\mathfrak{z}\mathfrak{z}$ and $535086 - 5\mathfrak{z}$; and from $535086 - 5\mathfrak{z}$ the zensizensic root must be extracted, in this way. Half the number of censuses, $5/2$, makes the square $25/4$; to which the given number 535086, added, makes $535086 \frac{25}{4}$, that is, $2140369/4$. From its square root, $1463/2$, if the aforesaid half $5/2$ is subtracted (because of the sign $-$), there will remain the square root $1458/2$, that is, 729, whose square root 27 will be the zensizensic root of the number $535086 - 5\mathfrak{z}$ — namely the number we are seeking. For its square, 729, taken five times, makes 3645, which number, together with 531441, the zensizensus of the number found, 27, makes the given number 535086.

Problem 11 [XI.] — To find a number whose cube, together with the zensicubus of the same, makes a given number.

Let there be sought a number whose cube, together with the zensicubus of the same, makes 1291503906. Let the number sought be posited as $1R$, whose cube is $1\mathfrak{C}$ and whose zensicubus is $1\mathfrak{z}\mathfrak{C}$. There must therefore exist an equation between $1\mathfrak{C} + 1\mathfrak{z}\mathfrak{C}$ and 1291[503906 ...]

[Cossist notation used on these pages, glossed once here: R = *res* (the unknown, x); $\mathfrak{z}$ = *census* (x^2); $\mathfrak{C}$ = *cubus* (x^3); $\mathfrak{z}\mathfrak{z}$ = *zensicensus* (x^4); $\mathfrak{z}\mathfrak{C}$ = *zensicube* (x^6); $\sqrt{\mathfrak{z}}$ = the square-root surd (*radix quadrata* of an irrational quantity). Coefficients are written before the sign, e.g. $100R = 100x$, $1\mathfrak{z} = x^2$. The signs are transcribed as printed and each worked equation is given cell-by-cell.]

and 1291503906. Now, when one cube ($1\mathfrak{C}$) is subtracted from both sides, the equation stands between one zensicube ($1\mathfrak{z}\mathfrak{C}$) and $1291503906 - 1\mathfrak{C}$. From this number the zensicubic root [*radix Zensicubica*, the sixth root] must be drawn, in the following way. Half of the middle cossic number is $\frac{1}{2}$, and its square is $\frac{1}{4}$. If the given number is added to this, we get the number $1291503906 \frac{1}{4}$ — that is, $5166015625/4$. Its square root is $71875/2$; and if the aforesaid half, $\frac{1}{2}$, is subtracted from it, there remains the census root [*radix Zensica*] $71874/2$, that is, 35937. The

cube root of this is 33, and this will be the zensicubic root of the number 1291503906 – 1℄ — namely the number that is sought. For its cube, 35937, together with its zensicube, 1291467969, makes the number 1291503906.

Problem 12 [XII]. *To divide a given number into two parts, so that when the larger is divided by the smaller and the smaller by the larger, the sum of the two quotients is a given value.*

Let the number 100 be divided into two parts, so that when the larger is divided by the smaller and the smaller by the larger, the two quotients make the sum $2\frac{36}{91}$. Let one part be $1R$, and the other $100 - 1R$. When this part is divided by that, the quotient is $(100 - 1R)/(1R)$; and when that is divided by this, the quotient is $(1R)/(100 - 1R)$. The sum of these quotients, namely $(10000 - 200R + 23)/(100R - 13)$, must equal the number $2\frac{36}{91}$, that is, $218/91$. By cross-multiplication this equation reduces to an equation between $21800R - 2183$ and $910000 - 18200R + 1823$. Adding 2183 to both sides, the equation becomes: between $21800R$ and $910000 - 18200R + 4003$. Adding $18200R$ to both sides: between $40000R$ and $910000 + 4003$. Removing 910000 from both sides: between $40000R - 910000$ and 4003 . And dividing each term by the number carrying the sign 3, that is, by 400, the equality stands between $100R - 2275$ and 13 .

The square root is then drawn out as follows. Half the number of roots is 50; from its square, 2500, if 2275 is subtracted (on account of the minus sign), the remainder is 225; to the square root of this, 15, if the aforesaid half, 50, is added, the value of the root becomes 65, the larger part. The smaller part will therefore be 35. This is also obtained (since the proposed quantity $100R - 2275$ has a double root) if the square root just found, 15, is subtracted from the aforesaid half, 50. So the parts sought are 65 and 35. And when that is divided by this, and this by that, the quotients come to $1\frac{6}{7}$ and $\frac{7}{13}$, which together make the sum $2\frac{36}{91}$.

Problem 13 [XIII]. *To find a number such that its square, diminished by any given number, has a given proportion to it.*

Let the number to be found be such that its square, diminished by 264, has a tenfold proportion to it. Let the number sought be $1R$, whose square is 13 . Since, after 264 has first been subtracted from it, it must have a tenfold proportion to $1R$, the equation stands between $10R$ and $13 - 264$. Adding 264 to both sides: between $10R + 264$ and 13 . This is handled as follows: half the number of roots is 5, whose square makes 25; adding 264, the number becomes 289; to its square root, 17, if the aforesaid half, 5, is added, the value of one root becomes 22 —

namely the number sought. For its square, 484, if it loses 264, becomes 220, which number is ten times the number 22.

Problem 14 [XIV]. *To find a number such that its square, increased by a given number, has a given proportion to it.*

Let the number to be found be such that its square, increased by 110, has to it the proportion that 27 has to 1. Let the number sought be $1R$, whose square is 13 . Since, if 110 is added to it, it must have to $1R$ the same proportion that 27 has to 1, the equation stands between $13 + 110$ and $27R$. Removing 110 from both sides: between 13 and $27R - 110$. You will resolve this as follows. Half the number of roots is $27/2$, whose square makes $729/4$; from which, if 110 — that is, $440/4$ — is taken away, the remaining number is $289/4$; to its square root, $17/2$, if the aforesaid half, $27/2$, is added, the larger root becomes $44/2$, that is, 22. And if the square root just found, $17/2$, is subtracted from the aforesaid half, $27/2$, the remainder gives the smaller root, $10/2$, that is, 5. For this equation has a double root, and so each number, 22 and 5, will satisfy the proposed problem. Indeed, the square 484 of the number 22, increased by 110, becomes 594, which has to 22 the same proportion that 27 has to 1. Likewise, the square 25 of the number 5, increased by 110, becomes 135, which number has to 5 the same proportion that 27 has to 1.

Problem 15 [XV]. *To find two numbers whose squares make up a given sum, and which, multiplied by each other, produce a number to which the sum of the squares has a given proportion.*

Let the sum of the squares be given as 180, having to the product number the double-sesquialterate proportion [5:2]. Accordingly, since 180 and 72 stand in the double-sesquialterate proportion, the product of one by the other will be 72; and this number 72 is obtained if 180 is divided by $2\frac{1}{2}$, the denominator of the given proportion. Let one of the numbers sought be $1R$, and the other $72/1R$, so that the one multiplied by the other yields 72 — this quantity $72/1R$ being obtained by dividing 72 by $1R$ (just as, if a number is to be found which multiplied by 8 makes 48, one divides 48 by 8 and finds the number 6, which multiplied by 8 makes 48). Now the squares of the assumed numbers, $1R$ and $72/1R$, are 13 and $5184/13$, and their sum, $(133 + 5184)/13$, must equal 180. By cross-multiplication this equation reduces to an equation between $133 + 5184$ and 1803 . Removing 5184 from both sides, the equation stands between 133 and $1803 - 5184$. You will find its root thus: half the middle number, which carries the character 3, is 90; from its square, 8100, if 5184 is taken away, the remaining number becomes 2916; to its square root, 54, if the aforesaid half, 90, is added, the sum becomes 144, whose

square root, 12, will — by the doctrine of chapter 12 — be the larger root. And if the square root just found, 54, is subtracted from the aforesaid half, 90, the remaining number will be 36, whose square root, 6, will give the smaller root. This too is obtained if 72 is divided by the first root found, 12. And so the two numbers sought are 12 and 6, which multiplied together make 72; and to this number the given sum of the squares generated from them, namely 144 and 36 — that is, 180 — has the double-sesquialterate proportion, just as was proposed.

Problem 16 [XVI]. *To divide a given number into three continuously proportional numbers, such that the product of the first by the second, together with the product of the second by the third, together with that which arises from the third by the first, makes any given number.*

Let the given number 76 be divided so that the three products make 1824. Since, when three continuously proportional numbers are given, [Margin (a): 1. secun. — i.e. Euclid, *Elements* II.1] the product of the middle by the sum of them all is equal to the products of the middle by the first, the middle by the third, and the middle by itself — that is, [Margin (b): 20. sept. — i.e. Euclid, *Elements* VII] the product of the third by the first, [Margin (c): 20. sept.] since this is equal to the product of the middle by itself — then, if the number 1824 to be produced is divided by the given number 76 to be divided, the quotient 24 will be the middle proportional number. For the quotient 24, multiplied by the divisor 76 (that is, by the sum of all three), generates the divided number 1824, which is indeed set equal to the three products we have named. Therefore, once this middle term, 24, is taken from 76, the sum of all, the remainder becomes the sum of the first and third, 52. Let the first, then, be $1R$, and consequently the third will be $52 - 1R$, so that the three proportional numbers are $1R$, 24, and $52 - 1R$. [Margin (c): 20. sept.] Therefore the product of $1R$ by $52 - 1R$, that is, $52R - 13$, will equal the number 576, which is the product of the middle, 24, by itself. Adding 13 to both sides, the equation stands between $52R$ and $13 + 576$. Removing 576 from both sides: between $52R - 576$ and 13. This equation is resolved thus: half the number of roots is 26; from its square, 676, if 576 is taken away, the number 100 remains; to its square root, 10, if the aforesaid half, 26, is added, the value of the root becomes 36, the first number. The third, then, will be 16 — namely $52 - 1R$. And so the three proportional numbers are 36, 24, 16, making the sum 76; and the three products — of the first by the second, of the second by the third, and of the third by the first — make 1824.

But since the quantity $52R - 576$ has a double root, the smaller is found if the square root just found, 10, is subtracted from the aforesaid half, 26. For there will

remain the value of one root, 16, the first number, while the third will be 36 — namely $52 - 1R$. And there will again be the three proportional numbers 16, 24, 36 — the same as before, but in reverse order.

Problem 17 [XVII]. *Given two numbers, whether equal or unequal, to find another which, added to either of them, makes a sum that, multiplied by the added (that is, the found) number, produces the square of the other number.*

We solved this problem geometrically in the scholium to Proposition 36 of Book 3 of Euclid.

Let two numbers 10 and 12 be given, and let another be found which, added first to 10, makes a number that, multiplied by the added number, produces the square of the other, 12 — namely 144. Let the number sought be $1R$. Adding the number 10, we get $1R + 10$, which multiplied by $1R$ makes $13 + 10R$. This number must equal 144. Removing $10R$ from both sides, the equation stands between 13 and $144 - 10R$. You will resolve it thus: half the number of roots is 5, whose square makes 25; and adding 144, the number becomes 169; from its square root, 13, if the aforesaid half, 5, is subtracted (on account of the minus sign), there will remain the value of the root, 8, the number sought. For this, added to 10, makes 18, which number multiplied by 8 makes 144, the square of the other given number, 12.

Next, let a number be found which, added to 12, makes a number that, multiplied by the added number, produces 100, the square of the other number, 10. Let the number sought be $1R$, which with 12 makes $1R + 12$. This multiplied by $1R$ makes the number $13 + 12R$, equal to 100. Removing $12R$ from both sides, the equation stands between 13 and $100 - 12R$. You will resolve it thus: half the number of roots is 6, which makes the square 36; and adding 100, the number becomes 136; from its square root $\sqrt{3} 136$, if you subtract the aforesaid half, 6, there remains the value of the root, $\sqrt{3} 136 - 6$, the number sought. For this, added to 12, makes $\sqrt{3} 136 + 6$, which number multiplied by the found number $\sqrt{3} 136 - 6$ produces 100. This multiplication is carried out, as we wrote in chapter 23, when the square of the latter part, 6 — namely 36 — is subtracted from 136, the square of the former part.

Finally, let there be two equal numbers, 8 and 8, and let the number to be found be $1R$. From $1R$ and 8 comes the number $1R + 8$, which multiplied by $1R$ makes $13 + 8R$, a number equal to 64, the square of the number 8. Removing $8R$ from both sides, the equation stands between 13 and $64 - 8R$. Half the number of roots is 4; to its square, 16, adding 64, the number becomes 80; from its square root $\sqrt{3} 80$, if

the aforesaid half, 4, is taken away, there remains the number $\sqrt{3} 80 - 4$, which is sought. Added to 8, this makes the sum $\sqrt{3} 80 + 4$, which multiplied by the found (that is, the added) number $\sqrt{3} 80 - 4$ makes 64, the square of the number 8, as the following display shows. For 16, the square of the latter part 4, subtracted from 80, the square of the former part $\sqrt{3} 80$, leaves 64:

$$\begin{array}{r} \sqrt{3} 80 - 4 \\ \sqrt{3} 80 + 4 \\ \hline 64 \end{array}$$

Problem 18 [XVIII]. *To find a number midway between a number greater by any number of units and another number smaller by any number of units, such that the two extremes, multiplied together, make any given number.*

Let a middle number be found between a number that exceeds it by nine and another number that it exceeds by three, so that the extremes, multiplied together, make 133. Let the middle number sought be $1R$, standing between $1R + 9$ and $1R - 3$. For in this way it is exceeded by the former by nine, and exceeds the latter by three. These two extremes, multiplied together, make the number $13 + 6R - 27$, equal to 133. Adding 27 to both sides, the equality stands between $13 + 6R$ and 160. Removing $6R$ from both sides: between 13 and $160 - 6R$. Half the number of roots is 3; to its square, 9, adding 160, the number becomes 169; from its square root, 13, if the aforesaid half, 3, is taken away, there remains the value of $1R$, namely 10 — that is, the middle number sought. And the extremes will be 19 and 7, which multiplied together make 133.

Again, let a middle number be sought between two others, as before, so that the extremes multiplied together make 40. Once more the extremes will be found to be $1R + 9$ and $1R - 3$, which multiplied together make the number $13 + 6R - 27$, equal to 40. Adding 27 to both sides, the equation stands between $13 + 6R$ and 67. Removing $6R$ from both sides: between 13 and $67 - 6R$. Half the number of roots is 3; to its square, 9, adding 67, the number becomes 76; from its square root $\sqrt{3} 76$, if the aforesaid half, 3, is taken away (on account of the minus sign), there remains the value of the root, $\sqrt{3} 76 - 3$, the middle number sought. If 3 is subtracted from it, the smaller extreme becomes $\sqrt{3} 76 - 6$; and if 9 is added, the larger extreme becomes $\sqrt{3} 76 + 6$. These extremes, multiplied together, make $76 - 36$, that is, 40. For $+\sqrt{3} 2736$ and $-\sqrt{3} 2736$ cancel each other out, as appears in the adjoining display. Or else, because 36, the square of the latter part 6, subtracted from 76, the square of the former part, leaves 40, as we taught in chapter 23:

$$\begin{array}{r}
\sqrt{3} \ 76 - 6 \\
\sqrt{3} \ 76 + 6 \\
\hline
+ \sqrt{3} \ 2736 - 36 \\
76 - \sqrt{3} \ 2736 \\
\hline
76 - 36
\end{array}$$

We also solved this problem in the preceding chapter, riddle 146, by a simple equation.

Problem 19 [XIX]. *To find a number which exceeds two others by given numbers, such that those two, multiplied together, produce a number which exceeds the found number by a given number.*

Let a number be sought which exceeds two others by 8 and by 6, so that those two, multiplied together, produce a number greater by four than the found number. Let the number sought be $1R$. Those two smaller numbers, then, will be $1R - 8$ and $1R - 6$, which multiplied together make the number $13 - 14R + 48$, equal to $1R + 4$ — since the assumed number $1R$ is to be exceeded by four. Removing 4 from both sides, the equation stands between $13 - 14R + 44$ and $1R$. Adding $14R$ to both sides: between $13 + 44$ and $15R$. And removing 44 from both sides: between 13 and $15R - 44$. Half the number of roots is $15/2$; from its square, $225/4$, taking away 44 — that is, $176/4$ — there remains (on account of the minus sign) the number $49/4$; to its square root, $7/2$, if the aforesaid half, $15/2$, is added, the value of the root becomes $22/2$, that is, 11, the number sought. So the two smaller numbers will be 3 and 5, which multiplied together make 15, that is, $11 + 4$.

But since this quantity $15R - 44$ has a double root, if the square root found last, $7/2$, is subtracted from the aforesaid half, $15/2$, the remainder gives the smaller root, $8/2$, that is, 4, the number sought. If you take 8 and 6 away from it, the other two smaller numbers will be -4 and -2 , which multiplied together make $+8$, a number which exceeds the found number, 4, by four. Here you see that even absurd or fictitious numbers — that is, numbers less than nothing — satisfy the proposed riddle, which may perhaps seem remarkable to some.

SCHOLIUM.

In riddles similar to this one it sometimes happens that the equation found is simple — namely, whenever, in the number produced by multiplying the two cossic numbers, there is found an absolute number equal to the given excess; for then that absolute number is removed, and there remains an equation between 3

[the census] and R [the res]. Of this I wish the studious Reader to be warned, so that he is not disturbed when he sees a double equation sometimes occurring, and sometimes a simple one, in the various examples of the same riddle.

For let a number be sought which exceeds two others by 8 and by 6, so that those two, multiplied together, produce a number that exceeds the found number by 48. Let the number sought be $1R$. Those two smaller numbers, then, will be $1R - 8$ and $1R - 6$, which multiplied together make the number $13 - 14R + 48$, equal to $1R + 48$. Removing 48 from both sides, the equality stands between $13 - 14R$ and $1R$. Adding $14R$ to both sides: between 13 and $15R$. This equation is simple. So, dividing 15 by 1, the result is $1R = 15$ (since the cossic quantities 3 and R are collateral in degree) — namely the number sought. Therefore the other two smaller numbers will be 7 and 9, which multiplied together produce the number 63, which exceeds the found number 15 by 48.

Problem 20 [XX]. *To find a number which two others exceed by given numbers, such that those two, multiplied together, produce a number which exceeds a given multiple of the square generated from the found number by a given number.*

Let a number be found which two others exceed by 7 and by 19, and which, multiplied together, exceed the triple of the square generated from the found number by 73. Let the number sought be $1R$. The other two, larger, numbers will therefore be $1R + 7$ and $1R + 19$. These, multiplied together, make the number $13 + 26R + 133$, equal to $33 + 73$; for in this way it will exceed the triple of the square described from $1R$ by 73. Removing 73 from both sides, the equation stands between $13 + 26R + 60$ and 33. Taking away 13 from both sides: between $26R + 60$ and 23. Dividing all the terms by the number of censuses — that is, by 2 — the equation stands between $13R + 30$ and 13. Half the number of roots is $13/2$; to its...

[Conclusion of Problem 20]

to its square, $169/4$, add 30 — that is, $120/4$ — and the number $289/4$ results. If to its square root, $17/2$, the half found earlier, $13/2$, is added, it becomes $1R$ [res, the unknown] equal to $30/2$, that is, 15, the number sought. Therefore the two greater numbers will be 22 and 34, which multiplied together make the number 748. This exceeds by 73 the triple of the square 225 generated from the number 15 that was found — that is, it exceeds the number 675 by 73.

Problem 21 [XXI]

To find a number such that any multiple of it, together with the square of that same number, makes a given number.

Let a number be sought whose quadruple, together with its own square, makes 400. Posit the number we are looking for as $1R$. Its quadruple, $4R$, together with its own square, $1\bar{3}$ [census, x^2], makes the number $1\bar{3} + 4R$, equal to 400.

Subtracting $4R$ from both sides gives an equation between $1\bar{3}$ and $400 - 4R$. The half of the number of roots is 2; adding its square, 4, to 400 gives the number 404. From the square root of this, $\sqrt{3} 404$ [the square-root surd], subtract the aforesaid half, 2, and it becomes $1R = \sqrt{3} 404 - 2$, the number sought.

Its quadruple is $\sqrt{3} 6464 - 8$. Together with its square — which is $404 - \sqrt{3} 6464 + 4$, as the appended formula makes plain — it makes the number $404 - 4$, that is, 400; for $-\sqrt{3} 6464$ and $+\sqrt{3} 6464$ cancel each other, as the other formula shows.

$$\begin{array}{r}
 \sqrt{3} 404 - 2 \\
 \sqrt{3} 404 - 2 \\
 \hline
 - \sqrt{3} 1616 + 4 \\
 404 - \sqrt{3} 1616 \\
 \hline
 404 - \sqrt{3} 6464 + 4 \\
 \\
 404 - \sqrt{3} 6464 + 4 \\
 + \sqrt{3} 6464 - 8 \\
 \hline
 404 - 4
 \end{array}$$

Likewise, let a number be sought whose quadruple, together with its own square, makes 780. Positing $1R$, its quadruple $4R$ together with its square $1\bar{3}$ is the number $1\bar{3} + 4R$, equal to 780. Subtracting $4R$ from both sides gives an equation between $1\bar{3}$ and $780 - 4R$. The half of the number of roots is 2; adding its square, 4, to 780 gives the number 784. From the square root of this, 28, remove the aforesaid half, 2, and it becomes $1R = 26$, the number we are seeking. Its quadruple, 104, together with the square of the same, 676, makes 780.

Scholium

By the same route a number will be found which, together with its own square, makes any given number. Let a number be sought which together with its square makes 400. Positing $1R$ for the number sought, it will make, together with its square — that is, with $1\bar{3}$ — the number $1\bar{3} + 1R$, equal to the given number 400. Removing $1R$ from both sides gives an equation between $1\bar{3}$ and $400 - 1R$. The half of the number of roots is $1/2$; add its square, $1/4$, to 400 — that is, to $1600/4$ — and $1601/4$ results. From the square root of this, that is, from $\sqrt{3} 1601/4$, remove the half of the number of roots, namely $1/2$, and the value of the root becomes $\sqrt{3} 1601/4 - 1/2$, the number sought.

For its square is $1602/4 - \sqrt{3} 6404/16$, as the appended formula teaches. Together with this square, the number found, $\sqrt{3} 1601/4 - 1/2$, makes 400, the given number. For $1602/4$ and $-1/2$ (that is, $-2/4$) make $1600/4$, that is, 400; while $-\sqrt{3} 6404/16$ and $+\sqrt{3} 1601/4$ cancel each other, since they are equal — as is clear by cross-multiplication. For either numerator, multiplied into the denominator of the other fraction, produces 25616.

$$\sqrt{3} 1601/4 - 1/2$$

$$\sqrt{3} 1601/4 - 1/2$$

$$- \sqrt{3} 1601/16 + 1/4$$

$$+ 1601/4 - \sqrt{3} 1601/16$$

$$\text{Sum. } 1602/4 - \sqrt{3} 6404/16$$

Problem 22 [XXII]

Given any number, to find two others which, multiplied together, produce that same given number, and such that the squares of all three make any given sum.

Let the given number be 8, and let two others be found which, multiplied together, make that same number 8, and such that the squares of all three make $101 \frac{7}{9}$. The squares of the two numbers to be found must make $37 \frac{7}{9}$, as is clear: if the square 64 of the given number 8 is subtracted from $101 \frac{7}{9}$ (the sum of all three squares), the result is $37 \frac{7}{9}$. Posit one of the sought numbers as $1R$. Since the other, multiplied into this, must make 8, if 8 is divided by $1R$ the quotient will be $8/1R$, the other number; for, multiplied into the divisor $1R$, it produces 8.

The squares of these two numbers are 13 and $64/13$, which make the sum $(133 + 64)/13$ [census-of-census, x^4 , in the numerator], equal to $37 \frac{7}{9}$, or $340/9$. By cross-multiplication this equation is reduced to one between $133 + 576$ and 3403 .

Removing 576 from both sides gives an equation between 133 and $3403 - 576$. Dividing all the numbers by 9 (the number 9 being the coefficient marked with the highest character, 33), the equation becomes one between 133 and $(340/9)3 - 64$.

The half of the number of censuses is $340/18$; from its square, $115600/324$ — that is, from $356 \frac{256}{324}$ — if 64 is taken away, $292 \frac{256}{324}$ remains, that is, $94864/324$. To the square root of this, $308/18$, add the aforesaid half of the number of censuses, $340/18$, and 13 becomes $648/18$ (since among the consist numbers the middle terms are taken singly) — that is, 36. Its square root, 6, will be one of the numbers, which we posited as $1R$; and accordingly the other number, which we posited as $8/1R$, that is, 8 divided by $1R$ (namely by 6), will be $1 \frac{1}{3}$.

This may also be found in another way. Since the number $(340/9)\sqrt{3} - 64$ has a double root, if the square root found in the last place, $308/18$, is removed from the aforesaid half, $340/18$, the remainder will be the other census, $32/18$, or $16/9$, whose square root, $4/3$, or $1\frac{1}{3}$, gives the other number. So the two numbers sought are 6 and $1\frac{1}{3}$, which multiplied together make 8; and their squares, 36 and $16/9$ (or $1\frac{7}{9}$), make $37\frac{7}{9}$; and accordingly the squares of all three numbers — 8, 6, and $1\frac{1}{3}$ — make the sum $101\frac{7}{9}$.

Problem 23 [XXIII]

To find two numbers producing a given sum, and which multiplied together produce that same given number — the given product being not greater than the square described on the half of the given sum.

Or:

To distribute a given number into two, so that from one multiplied into the other any given number whatever is generated — that number being not greater than the square of the half of the given number.

Let two numbers producing the sum 100 be sought. Or (which is the same thing) let the given number 100 be cut into two which, multiplied together, make 10. Posit one number as $1R$, and thus the other as $100 - 1R$. These, multiplied together, make the number $100R - 1\bar{3}$, equal to 10. Adding $1\bar{3}$ to both sides, there is an equality between $100R$ and $10 + 1\bar{3}$. And removing 10 from both, there is an equation between $100R - 10$ and $1\bar{3}$. The half of the number of roots is 50; from its square, 2500, remove 10, and the number 2490 remains. To the square root of this, $\sqrt{3} 2490$, if the aforesaid half, 50, is added, $1R$ becomes $\sqrt{3} 2490 + 50$ — one of the numbers, which we posited as $1R$. The other, therefore, will be $100 - (\sqrt{3} 2490 + 50)$, that is, 100 with $\sqrt{3} 2490 + 50$ taken away, which number is $50 - \sqrt{3} 2490$. For, taking $\sqrt{3} 2490 + 50$ away from $\sqrt{3} 0 + 100$, there remain $-\sqrt{3} 2490 + 50$, that is, $50 - \sqrt{3} 2490$, as this formula indicates.

$$\begin{array}{r} \sqrt{3} 0 + 100 \\ \sqrt{3} 2490 + 50 \\ \hline - \sqrt{3} 2490 + 50 \end{array}$$

Thus the two numbers sought are $\sqrt{3} 2490 + 50$ and $50 - \sqrt{3} 2490$, which make the sum 100 (since $+\sqrt{3} 2490$ and $-\sqrt{3} 2490$ destroy each other).

$$\begin{array}{r} 50 + \sqrt{3} 2490 \\ 50 - \sqrt{3} 2490 \end{array}$$

And from one multiplied into the other — that is, from $50 + \sqrt{3} 2490$ into $50 - \sqrt{3} 2490$ — the number 10 is generated, as this formula indicates. This multiplication is carried out by taking the square 2490 of the latter part away from 2500, the square of the former part, as was said in Chapter 23.

Now since the equation found has a double root, we shall find the same numbers by way of the other, smaller root. For if the square root $\sqrt{3} 2490$, found in the last place, is taken from the aforesaid half 50, it becomes $1R = 50 - \sqrt{3} 2490$ — namely one of the numbers we are looking for. The other will be $50 + \sqrt{3} 2490$, which of course remains if the number $50 - \sqrt{3} 2490$ is subtracted from 100, as you see in this formula. And so there are found the same numbers, $50 - \sqrt{3} 2490$ and $50 + \sqrt{3} 2490$, but in reversed order.

$$\begin{array}{r} 100 + \sqrt{3} 0 \\ 50 - \sqrt{3} 2490 \\ \hline 50 + \sqrt{3} 2490 \end{array}$$

The given product, however, must be not greater than the square which arises from the half of the given sum. [Margin: ^a Scholium 14 of the ninth (book).] For the square of the half is greater ^a than the number produced from unequal parts, since this latter, together with the square of the intermediate section, is equal to that square. Hence the number 12 cannot be divided into two parts such that their multiplication produces the number 40. This the equation nicely teaches us: for positing one part as $1R$ and the other as $12 - 1R$, there arises an equation between 13 and $12R - 40$. And since the absolute number 40 cannot be taken away from 36, the square of the half of the number of roots, you will pronounce the problem to be impossible.

Problem 24 [XXIV]

To find a number whose square is exceeded by a given amount by another number, and which itself exceeds a second number by another given amount — so that the two extremes, multiplied together, produce any assigned number.

Let a number be sought whose square is exceeded by 5 by another number, and which itself exceeds a certain other by 2, and such that the two extremes, multiplied together, make 2538. Posit that number as $1R$, whose square is 13. The greater, then, will be $13 + 5$, and the lesser $13 - 2$. These two, multiplied together, generate the number $133 + 33 - 10$, equal to 2538. Adding 10 to both sides, the

equation is between $133 + 33$ and 2548. And removing 33 from both, it is between 133 and $2548 - 33$.

The census-of-census root must therefore be sought, in this way. The half of the number of censuses is $3/2$; to its square, $9/4$, add 2548 (that is, $10192/4$) and the number $10201/4$ results. From the square root of this, $101/2$, if the aforesaid half $3/2$ is removed, the number $98/2$ remains, that is, 49; and its square root, 7, will give the number sought. For its square, 49, lies between 54 (greater by 5) and 47 (less by 2), and 47 into 54 makes the given number 2538.

Problem 25 [XXV]

To find a number such that, when as many squares of that same number as you please are subtracted from its square-of-square, they leave any given number.

Let a number be sought such that four of its squares, subtracted from the square-of-square of the same, leave 2205. Posit the number as 1R, whose square is 13; and its four squares, 43, subtracted from 133 — that is, from the square-of-square of the posited number 1R — leave the number $133 - 43$, equal to 2205. Adding 43 to both sides, the equation is between 133 and $43 + 2205$. The census-of-census root must therefore be extracted, in this way. The half of the number of censuses is 2; to its square, 4, add 2205, and the number 2209 results. To the square root of this, 47, add the aforesaid half, 2, and the number 49 results, whose square root, 7, will give the number sought. For its four squares, 196, taken from the census-of-census of the same, 2401, leave 2205, the given number.

Problem 26 [XXVI]

To find two numbers which, multiplied together, generate a given number, and such that the sum of the squares produced from those same numbers is also given.

Let two numbers be found which, multiplied together, make 78, and whose squares make the sum 205. Since — as we shall prove in the following Lemma — two numbers multiplying each other produce a number that is the mean proportional between the squares of those same numbers, if the smaller square is posited as 13, the number 78 will be the mean proportional between the squares 13 and $205 - 13$. The product of 13 into $205 - 13$ will be equal to the square of the middle number 78, that is, to the number 6084. Adding 133 to both sides, the equation is between $133 + 6084$ and 2053. And removing 6084 from both, it is between 133 and $2053 - 6084$.

The census-of-census root must therefore be extracted, which is done thus. The half of the number of censuses is $205/2$; from its square, $42025/4$, subtract 6084 (that is, $24336/4$) and $17689/4$ remains. To the square root of this number, $133/2$, if the aforesaid half $205/2$ is added, the number becomes $338/2$, that is, 169 — one square, whose root, 13 , will give one of the numbers, namely the greater. And since the number $2053 - 6084$ has a double root, the other root will be found if we take the root found in the last place, $133/2$, away from the aforesaid half, $205/2$. The remaining number, $72/2$ — that is, 36 — gives the smaller square, whose root, 6 , gives the other of the numbers. So the two numbers found are 6 and 13 , which multiplied together make 78 ; and their squares, 36 and 169 , make the sum 205 .

The same problem will be worked out in irrational numbers, if two numbers are to be found which, multiplied together, make 100 , and the sum of whose squares is 300 . For positing one square as 13 , the other will be $300 - 13$; and between these the number 100 will be the mean proportional, as will be established from the following Lemma. From 13 into $300 - 13$ arises the number $3003 - 133$, [Margin: ^b 20 of the seventh (VII.20).] equal ^b to the number 10000 , which comes from the middle number 100 multiplied into itself. Adding 133 to both sides, the equation is between $133 + 10000$ and 3003 . And removing 10000 from both, it is between 133 and $3003 - 10000$.

Here too a double census-of-census root must be extracted, in this way. The half of the number of censuses is 150 ; from its square, 22500 , take 10000 , and the number 12500 remains. To the square root of this, $\sqrt{3} 12500$, if the aforesaid half 150 is added, the greater square becomes $\sqrt{3} 12500 + 150$, whose square root, namely $\sqrt{3} (\sqrt{3} 12500 + 150)$, will give the greater number. And if that root $\sqrt{3} 12500$ is taken from the aforesaid half 150 , the smaller square will remain, $150 - \sqrt{3} 12500$, whose square root, $\sqrt{3} (150 - \sqrt{3} 12500)$, will give the smaller number sought.

For from the greater root $\sqrt{3} (\sqrt{3} 12500 + 150)$ into the smaller root $\sqrt{3} (150 - \sqrt{3} 12500)$ — if, that is, their squares are multiplied together, as was said in Chapter 24 and appears in this formula — comes the number $\sqrt{3} 10000$, that is, the given number 100 . Likewise the squares $\sqrt{3} 12500 + 150$ and $150 - \sqrt{3} 12500$ make the sum 300 , as is manifest.

$$\begin{array}{r}
 \sqrt{3} (\sqrt{3} 12500 + 150) \\
 \sqrt{3} (150 - \sqrt{3} 12500) \\
 \hline
 - 12500 - \sqrt{3} 281250000 \\
 + \sqrt{3} 281250000 + 22500
 \end{array}$$

Product. $\sqrt{3} 10000$

The aforesaid multiplication, moreover, is more easily carried out by what was said in Chapter 23, if the numbers to be multiplied are so arranged. For if 12500, the square of the latter part, is taken from 22500, the square of the former part 150, the number 10000 will remain; and with the sign $\sqrt{3}$ set before it, the product number becomes $\sqrt{3} 10000$.

$\sqrt{3} (150 + \sqrt{3} 12500)$

$\sqrt{3} (150 - \sqrt{3} 12500)$

$\sqrt{3} 10000$

Lemma

Two numbers multiplied together produce a number that is the mean proportional between their squares, in the ratio of the numbers themselves.

Let there be two numbers, A and B. From A into B let C be made; from A into itself let the square D be made; and from B into itself the square E. I say that C is the mean proportional between D and E — that is, that E, C, D are in continued proportion, in the ratio of B to A. For since A, multiplying B and A, made C and D, [Margin: ^c 17 of the seventh (VII.17).] then ^c C will be to D as B to A. Likewise, since B, multiplying B and A, made E and C, [Margin: ^d 17 of the seventh (VII.17).] then ^d E also will be to C as B to A. Therefore E, C, D are in continued proportion, in the ratio of B to A — which was proposed.

A.	3.	B.	4.		
D.	9.	C.	12.	E.	16.

Corollary

Hence it follows that, if two numbers multiplying each other produce some number, then the squares of those numbers, multiplying each other, produce the square of the product-number. For since the number 12, produced from 3 into 4, is the mean proportional between their squares 9 and 16, [Margin: ^e 20 of the seventh (VII.20).] it follows ^e that from 9 into 16 comes the square of the number 12.

Scholium

This same Problem 26 will be solved without the preceding Lemma, by this reasoning. In the first example, posit one of the numbers as $1R$, so that the other will be $78/1R$, so that from their mutual multiplication the number 78 is produced. Their squares, 13 and $6084/13$, make the sum $(133 + 6084)/13$, equal to 205. By cross-multiplication this equation is reduced to one between $133 + 6084$ and 2053 . Removing 6084 from both sides, the equation is between 133 and $2053 - 6084$, as before.

Again, in the latter example, posit one of the numbers as $1R$, and the other will be $100/1R$, so that their multiplication makes 100. Their squares...

Throughout this section the cossist (symbolic) tokens are rendered as printed and glossed on first occurrence: N = *numerus* (a pure number/units), R = *res* (the unknown, x), 3 = *census* (x^2), $\mathfrak{C}$ = *cubus* (x^3), 33 = *census of census*, the zenzizencic (x^4), and $\sqrt{3}$ = the surd (irrational) square root. "Number of census" / "number of roots" mean the coefficient of x^2 and of x respectively.

**

Their squares are 13 [census, x^2] and $10000/13$, which make the sum $(133 + 10000)/13$ [where 133 is the zenzizencic, x^4], equal to 300. This equation, by cross-multiplication, is reduced to this one: between $133 + 10000$ and 3003 . And when 10000 is removed from both sides, there will be an equation between 133 and $3003 - 10000$, as above.

Problem 27

[Margin: XXVII.] **27. To find two numbers making a given sum, whose squares also make a given sum.**

Let the sum of the numbers be 19 and the sum of their squares 205. Put the first number = $1R$ [res, the unknown x], hence the second = $19 - 1R$. Their squares are 13 and $361 - 38R + 13$, which make the sum $361 - 38R + 23$, equal to 205. Adding $38R$ to both sides, there will be an equation between $361 + 23$ and $38R + 205$. Removing 205 from both, between $156 + 23$ and $38R$. And again removing 156 from both, between 23 and $38R - 156$. Dividing everything by 2, the number of census, there will be an equation between 13 and $19R - 78$. You resolve it thus. Half the number of roots is $19/2$; from its square $361/4$, subtract 78 — that is, $312/4$ — and there remains the number $49/4$. To the square root of this, $7/2$, if the aforesaid half $19/2$ is added, the greater root becomes $26/2$, that is, 13, the first

number sought (since the cossic number has two roots). The second, then, will be 6 — namely $19 - 1R$; for 6 will also be the lesser root, which is obtained if the last root found, $7/2$, is subtracted from the aforesaid half $19/2$, leaving the number $12/2$, or 6. The numbers sought, therefore, are 13 and 6, whose sum is 19 and whose sum of squares is 205.

Again, let the sum of the numbers be 30 and of their squares 600. Putting the first number = $1R$ and the second = $30 - 1R$, their squares make the sum $900 - 60R + 23$, equal to 600. Adding $60R$ to both, there will be an equation between $900 + 23$ and $60R + 600$. Removing 600 from both, between $300 + 23$ and $60R$. And again removing 300 from both, between 23 and $60R - 300$. Dividing everything by 2, the number of census, the equation will be between 13 and $30R - 150$. Its root is found thus. Half the number of roots is 15; from its square 225, subtract 150, and there remains 75. To the square root of this, $\sqrt{375}$ [the surd root $\sqrt{75}$], if the aforesaid half 15 is added, the greater root becomes $\sqrt{375} + 15$ — the prior number, which was put as $1R$. And if the same square root $\sqrt{375}$ is taken from the aforesaid half, the remaining lesser root becomes $15 - \sqrt{375}$, which number equals the posterior number $30 - 1R$. For if the found root $\sqrt{375} + 15$ is subtracted from 30, there remains the number $15 - \sqrt{375}$, as this formula shows. So the two numbers sought will be $\sqrt{375} + 15$ and $15 - \sqrt{375}$, whose sum is 30 and sum of squares 600, as these two formulas indicate.

$$\begin{array}{r}
 \sqrt{3} \ 75 + 15 \ 15 - \sqrt{3} \ 75 \\
 \sqrt{3} \ 75 + 15 \ 15 - \sqrt{3} \ 75 \\
 \hline
 + \sqrt{3} \ 16875 + 225 - \sqrt{3} \ 16875 + 75 \\
 75 + \sqrt{3} \ 16875 \ 225 - \sqrt{3} \ 16875 \\
 \hline
 300 + \sqrt{3} \ 67500 \ 300 - \sqrt{3} \ 67500
 \end{array}$$

The sum of the squares, however, must be greater than half the square generated from the given sum of the numbers, and less than the square of that same sum. Thus if the sum of the numbers were proposed as 30 and of the squares as 400, the problem would be impossible. For the sum of the squares [400] is not greater than half of the square 900 produced from the given sum 30. And so the final equation would come out between 13 and $30R - 250$, which cannot be resolved; for from 225, the square of half the number of roots, one cannot take away 250, and so on.

But if the sum of the squares is greater than the square of the sum of the proposed numbers, the problem still cannot be solved unless we grant that one number is greater than the sum itself and the other less than nothing. For

example, if the sum is given as 40 and the sum of the squares as 1602: we shall put one number = $1R$ and the other = $40 - 1R$. Their squares, 13 and $1600 - 80R + 13$, make the sum $1600 - 80R + 23$, equal to 1602. Adding $80R$ to both, there will be an equation between $1600 + 23$ and $80R + 1602$. Removing 1600 from both, between 23 and $80R + 2$. Dividing everything by 2, the number of census, between 13 and $40R + 1$. Now half the number of roots is 20; to its square 400, if 1 is added, they make 401. And if to $\sqrt{3401}$ the aforesaid half 20 is added, it becomes $1R$, that is, $20 + \sqrt{3401}$ — the prior number, which we put as $1R$; and this, taken from 40, leaves the posterior number $20 - \sqrt{3401}$, as may be seen in this formula. Here you see the prior number $20 + \sqrt{3401}$ to be greater than 40 (since $\sqrt{3401}$ is greater than 20), while the posterior, $20 - \sqrt{3401}$, is less than nothing — namely $20 - 20 \frac{1}{41}$.

Both of these are unsuitable [as actual numbers], and yet they satisfy the question. For together they make the sum 40 (since $+\sqrt{3401}$ and $-\sqrt{3401}$ cancel each other out), and their squares — $801 + \sqrt{3641600}$ and $801 - \sqrt{3641600}$ — make the sum 1602. You had something similar in Problem 149 of the preceding chapter. You see, then, that numbers less than nothing were not devised in vain, since by means of them problems too can be solved.

Problem 28

[Margin: XXVIII.] **28. To divide a given number into two, so that their squares make a given sum.**

Here nothing else is sought than that two numbers be found making a given number, such that their squares also make a given sum — which was done in the preceding problem.

Problem 29

[Margin: XXIX.] **29. To divide a given number into three continued-proportional numbers, of which the middle one is given — but its square must not be greater than the square of half of that number which remains after the given middle is subtracted from the given number.**

Let the number 283 be divided into three continued proportionals whose middle is 78. Put the first number = $1R$, and hence the third = $205 - 1R$, so that these two together make 205, the number which remains after the given middle 78 is taken from the given number 283. The first, $1R$, into the third, $205 - 1R$, gives the number $205R - 13$, [Margin a: "20. septi." — i.e. Prop. 20 of the seventh [book] (of Euclid) [?]] which will equal the square of the middle, namely the number 6084.

Adding 13 therefore to both, there will be an equation between $205R$ and $13 + 6084$. And removing 6084 from both, between $205R - 6084$ and 13. This equation is resolved thus. Half the number of roots is $205/2$; from its square $42025/4$, if 6084 is subtracted — that is, $24336/4$ — there remains the number $17689/4$. To its square root $133/2$, if the aforesaid half $205/2$ is added, the greater root becomes $338/2$, that is, 169. And if the said root $133/2$ is taken from the aforesaid half $205/2$, there remains the lesser root $72/2$, that is, 36. There are therefore three continued proportionals: 36, 78, 169.

Again, let the given number 100 be divided into three continued proportionals whose middle is 30. Put the first = $1R$, and hence the third = $70 - 1R$, so that these two together make 70, the number which remains after the given middle 30 is taken from the given number 100. The first, $1R$, into the third, $70 - 1R$, gives the number $70R - 13$, [Margin b: "20. septi." — i.e. Prop. 20 of the seventh [book] (of Euclid) [?]] which will equal the square of the middle, namely the number 900. Adding 13 therefore to both, [there will be an equation between $70R$ and 13] + 900. And removing 900 from both, between $70R - 900$ and 13. This equation you settle thus. Half the number of roots is 35; from its square 1225, subtract 900, and there remains the number 325. To its square root, if the aforesaid half 35 is added, the greater root becomes $\sqrt{3325} + 35$. And if the same root $\sqrt{3325}$ is taken from the aforesaid half 35, there remains the lesser root $35 - \sqrt{3325}$. And so the three numbers sought are $35 - \sqrt{3325}$, 30, $\sqrt{3325} + 35$ — continued proportionals making the sum 100. And what results from the first into the third equals the square of the middle, namely the number 900, as this formula indicates:

$$\begin{array}{r}
 \sqrt{3} \ 325 + 35 \\
 35 - \sqrt{3} \ 325 \\
 \hline
 - \ 325 - \sqrt{3} \ 398125 \\
 + \sqrt{3} \ 398125 + 1225 \\
 \hline
 900
 \end{array}$$

This multiplication will be quicker by what was said in chapter 23, if the numbers to be multiplied are arranged as this other formula shows. For if 325, the square of the latter part [$\sqrt{3325}$], is removed from 1225, the square of the former part 35, the remaining number is 900:

$$\begin{array}{r}
 35 + \sqrt{3} \ 325 \\
 35 - \sqrt{3} \ 325 \\
 \hline
 900
 \end{array}$$

If, in the same example, the middle number is given as 40, the question will be impossible. For putting the first number = $1R$ and the third = $60 - 1R$, the first into the third gives the number $60R - 13$, equal to the square of the middle, namely the number 1600. And when the equation is reduced, there will be an equality between 13 and $60R - 1600$, which cannot be. For half the number of roots is 30, and from its square 900 one cannot take away 1600, and so on. The reason is that the number 1600, the square of the given middle 40, is greater than the square 900 of half of 60 — the number which remains after the given middle 40 is taken from the given number 100.

Problem 30

[Margin: XXX.] **30. To divide a given number into two parts so that their cubes make any given sum whatever, provided it is greater than a quarter of the cube described from the given number.**

Let the given number 10 be divided into two parts whose cubes make 370 — a number greater than a quarter of the cube of 10, that is, greater than the number 250. Put the first part = $1R$ and the second = $10 - 1R$. The cubes of these parts are $1C$ [cubus, x^3] and $1000 - 300R + 303 - 1C$, whose sum is $1000 - 300R + 303$, and this must equal the number 370. Adding $300R$ to both, there will be an equation between $1000 + 303$ and $300R + 370$. And removing 370 from both, between $630 + 303$ and $300R$. And again removing 630 from both, between 303 and $300R - 630$. Dividing each of these numbers by the number of census — that is, by 30 — there will be an equation between 13 and $10R - 21$. This you resolve thus. Half the number of roots, 5, makes the square 25; from this, subtracting 21, there remains the number 4; to its square root 2, if the aforesaid half 5 is added, the greater root becomes 7. And the same root 2, taken from the aforesaid half 5, makes the lesser root 3. The parts, therefore, of the given number 10 are 7 and 3, whose cubes 343 and 27 make the sum 370.

This same problem was Problem 141 of the preceding chapter 29, but solved otherwise than here — namely, by a simple equation, which arises because of the different positioning of the numbers. There too we showed how this problem can be solved without Algebra.

Problem 31

[Margin: XXXI.] **31. To distribute a given number into two numbers, so that the excess between the square of the greater and the square of the lesser is**

the same as [the excess] between the square of the given number and the square of the greater part.

Let the given number be 30. Put its greater part = $1R$ and the lesser = $30 - 1R$. The three squares are 900, 13 , and $900 - 60R + 13$. The excess of 900 over 13 is $900 - 13$. The excess of 13 over $900 - 60R + 13$ is $60R - 900$, as is evident in this subtraction formula (the columns are: units | R | 3 ; the lower row is subtracted from the upper):

$$\begin{array}{r}
 + 0 + 0 R + 1 3 \\
 + 900 - 60 R + 1 3 \\
 \hline
 - 900 + 60 R + 0 3
 \end{array}$$

There is therefore an equation between these excesses — that is, between $900 - 13$ and $60R - 900$. Adding 900 to both, between $1800 - 13$ and $60R$. And again adding 13 to both, between 1800 and $13 + 60R$. And removing $60R$ from both, between $1800 - 60R$ and 13 . The root is found thus. Half the number of roots is 30; to its square 900, adding 1800, makes the number 2700; from its square root $\sqrt{32700}$, if the aforesaid half 30 is subtracted, there will remain the value of the one root, $\sqrt{32700} - 30$ — namely the greater part. And this, taken from the given number 30, leaves the lesser part $60 - \sqrt{32700}$.

— as appears in this subtraction formula:

$$\begin{array}{r}
 + \sqrt{3} 0 + 30 \\
 + \sqrt{3} 2700 - 30 \\
 \hline
 - \sqrt{3} 2700 + 60
 \end{array}$$

Now the three squares are 900; $3600 - \sqrt{39720000}$; and $6300 - \sqrt{338880000}$; which mutually exceed one another by this same number, $\sqrt{39720000} - 2700$, as is clear in these subtraction formulas:

$$\begin{array}{r}
 + 900 + \sqrt{3} 0 + 3600 - \sqrt{3} 9720000 \\
 + 3600 - \sqrt{3} 9720000 + 6300 - \sqrt{3} 38880000 \\
 \hline
 - 2700 + \sqrt{3} 9720000 - 2700 + \sqrt{3} 9720000
 \end{array}$$

In the latter formula, since $\sqrt{39720000}$ and $\sqrt{338880000}$ are commensurable — for when the greater is divided by the lesser, the quotient comes out rational, $\sqrt{34}$, that is, 2 — if the lesser is subtracted from the greater, the remainder is the number $\sqrt{39720000} - 2700$. All this is clear if one considers what was said about the subtraction of the signs + and -, and about the subtraction of surd roots.

Problem 32

[Margin: XXXII.] 32. To distribute a given number into two parts, so that the square of the greater part together with the lesser part makes any given number — one, however, less than the square of the given number.

Let the given number 10 be divided into two parts, so that the square of the greater part together with the lesser part makes $7\frac{3}{4}$. Put the greater part = $1R$ and the lesser = $10 - 1R$. The square of the greater part is $1\bar{3}$, which together with the lesser part makes the sum $1\bar{3} + 10 - 1R$, equal to the number $7\frac{3}{4}$. Removing 10 from both, there will be an equation between $1\bar{3} - 1R$ and $6\frac{3}{4}$. And adding $1R$ to both, between $1\bar{3}$ and $1R + 6\frac{3}{4}$. The root is found thus. Half the number of roots is $\frac{1}{2}$; to its square $\frac{1}{4}$, adding $6\frac{3}{4}$, makes the number 64. To its square root 8, if the aforesaid half $\frac{1}{2}$ is added, it makes the value of one root, $8\frac{1}{2}$, the greater part; the lesser, then, will be $1\frac{1}{2}$. For the square of the greater part, $8\frac{1}{2}$, is $72\frac{1}{4}$; and with the lesser part $1\frac{1}{2}$ added, it makes the number $73\frac{3}{4}$.

But if the lesser part is set as $1R$ and the greater as $10 - 1R$, the square of the greater part will be $100 - 20R + 1\bar{3}$; and with the lesser part $1R$ added, the sum becomes $100 - 19R + 1\bar{3}$, equal to the number $7\frac{3}{4}$. Adding therefore $19R$ to both, there will be an equation between $1\bar{3} + 100$ and $19R + 7\frac{3}{4}$. And removing $7\frac{3}{4}$ from both, between $1\bar{3} + 26\frac{1}{4}$ and $19R$. Removing again $26\frac{1}{4}$ from both, between $1\bar{3}$ and $19R - 26\frac{1}{4}$. Half the number of roots is $19/2$; from its square $[361/4]$, subtracting $26\frac{1}{4}$, that is $105/4$, there remains] the number $256/4$, whose square root is $16/2$; and this, added to the aforesaid half $19/2$, makes the greater root $35/2$, that is, $17\frac{1}{2}$ — which is useless, since it cannot be a part of the given number 10. The lesser root, therefore (since the cossic number $19R - 26\frac{1}{4}$ has a double root), will solve the problem. It is found if the square root last obtained, $16/2$, is taken from the aforesaid half $19/2$. The remaining number, $1\frac{1}{2}$ [$3/2$], will be the lesser root, giving the lesser part of the number 10; and the greater will therefore be $8\frac{1}{2}$, just as before.

You see, then, that not always can both roots which the cossic number has be taken for the solution of the proposed problem. Nevertheless the greater root found, $35/2$, although it does not pertain to the posed problem, answers perfectly to this equation between $1\bar{3}$ and $19R - 26\frac{1}{4}$. For the square of that root is $1225/4$, equal to $19R - 26\frac{1}{4}$: for $19R$ [at this root] make $665/2$, or $1330/4$, from which, if $26\frac{1}{4}$ — that is, $105/4$ — is subtracted, there remains the number $1225/4$.

Problem 33

[Margin: XXXIII.] **33. To cut a given number in extreme and mean ratio.**

Let the given number 10 be cut as proposed — that is, so that the lesser part, the greater part, and the given number 10 are continued proportionals; or (which is the same) so that the square of the greater part equals the number produced from the lesser part into the given number. Put the greater part = $1R$ and the lesser = $10 - 1R$. The square of the greater part is $1\bar{3}$, equal to the number $100 - 10R$, which results from the lesser part $10 - 1R$ into 10. Half the number of roots is 5; to its square 25, adding 100, makes the number 125; from its square root $\sqrt{3}125$, if the aforesaid half 5 is taken away, there remains the root $\sqrt{3}125 - 5$, the greater part. And when this is taken from 10, the remainder becomes the lesser part $15 - \sqrt{3}125$, as appears in this formula. For the square of the greater part is $150 - \sqrt{3}12500$, and just as much results from the lesser part into 10, as these two multiplication formulas demonstrate:

$$\begin{array}{r}
 \sqrt{3} \ 0 + 10 \\
 \sqrt{3} \ 125 - 5 \\
 \hline
 - \sqrt{3} \ 125 + 15 \\
 \\
 \sqrt{3} \ 125 - 5 \ 15 - \sqrt{3} \ 125 \\
 \sqrt{3} \ 125 - 5 + 10 \\
 \hline
 - \sqrt{3} \ 3125 + 25 \ 150 - \sqrt{3} \ 12500 \\
 125 - \sqrt{3} \ 3125 \\
 \hline
 150 - \sqrt{3} \ 12500
 \end{array}$$

Scholium

Without Algebra the given number will be divided in extreme and mean ratio in this way. To the square of the number add a fourth part of that same square, and from the square root of the number so composed subtract half the given number. The remaining number will be the greater part; and when this is taken from the given number, the remainder will be the lesser part. This procedure is gathered both from the solution of this problem and from the figure of Proposition 11, Book 2, of Euclid.

Cossist tokens used below (glossed once here): R = res, the unknown (x); $\bar{3}$ = census, its square (x^2); $\sqrt{3}$ = the square-root surd (radix quadrata) of the quantity that follows. Worked equations are transcribed as printed.

34. To divide a given number into two numbers so that the square of the greater has a given ratio to the square of the smaller.

[Margin: XXXIIII]

Let the number be 30, to be cut into two so that the square of the greater has to the square of the lesser the ratio two-and-a-quarter ($9/4$). Set the greater equal to $1R$ [res, the unknown] and the lesser to $30 - 1R$. The square of the greater is $1\bar{3}$ [census, its square]; and this is two-and-a-quarter times the square of the lesser — that is, of $900 - 60R + 1\bar{3}$. So if this [latter] square is multiplied by $2\frac{1}{4}$, there is an equation between $1\bar{3}$ and

$$(8100 - 540R + 9\bar{3}) / 4$$

which, reduced by cross-multiplication [in crucem], becomes this: between $4\bar{3}$ and $8100 - 540R + 9\bar{3}$. Adding $540R$ to both sides, the equation is between $4\bar{3} + 540R$ and $8100 + 9\bar{3}$. Taking away $4\bar{3}$ from each side, it is between $540R$ and $8100 + 5\bar{3}$. And again removing 8100 from both sides, it is between $5\bar{3}$ and $540R - 8100$. Dividing everything by 5, there is an equality between $1\bar{3}$ and $108R - 1620$.

Half the number of the roots is 54; from its square, 2916, take away 1620, and there remains 1296, whose square root is 36. Added to the aforesaid half, 54, this gives the greater root, 90 — which is not suitable for solving the puzzle, since it cannot be a part of the given number 30. If, on the other hand, that same square root, 36, is subtracted from the aforesaid half, 54, the remainder is the smaller root, 18, which is indeed the greater number sought. The lesser will then be 12; for 324, the square of the former, has to 144, the square of the latter, the ratio two-and-a-quarter.

Here you also see that, although the cossic number has a double root, one cannot always take both — even though the greater, too, answers exactly to the equation. For 8100, the square of the greater root 90, is equal to $108R - 1620$; since $108R$ [taken 90 times] make 9720, from which, if 1620 be taken, there remain 8100.

35. To divide a given number into two numbers so that their squares make up a given sum — greater, however, than half the square generated from the given number, and smaller than the square of that same number.

[Margin: XXXV]

Let the number 40 be divided into two whose squares make 928 — a number greater than 800, which is half the square generated from 40, and smaller than 1600, the square of that same number 40. Set one of the numbers equal to $1R$ and the other to $40 - 1R$. Their squares are $1\bar{3}$ and $1600 - 80R + 1\bar{3}$, which make the sum $2\bar{3} + 1600 - 80R$, equal to 928. Adding $80R$ to both sides, the equation is between $2\bar{3} + 1600$ and $80R + 928$. Taking away 928 from each side, it is between $2\bar{3} + 672$ and $80R$. Again removing 672 from both sides, it is between $2\bar{3}$ and $80R - 672$. And dividing everything by 2, between $1\bar{3}$ and $40R - 336$.

Half the number of the roots is 20; from its square, 400, take away 336, and there remains 64, whose square root is 8. Added to the aforesaid half, 20, it gives the greater root, 28 — that is, the greater number. The lesser will be 12, which the smaller root also exhibits: for if the square root found, 8, be taken from the aforesaid half, 20, there remain 12. Now the square of the greater number 28 is 784, and with 144, the square of the smaller number 12, it makes the given sum, 928.

This puzzle was also solved in puzzle 170 of the preceding chapter, by a simple equation.

36. To find two numbers with a given difference, whose squares make a given sum — the sum being, however, greater than the square of the given difference.

[Margin: XXXVI]

Let the given difference be 11, and the sum of the squares 281 — which is greater than 121, the square of the given difference 11. Set the smaller number equal to $1R$, and hence the greater to $1R + 11$. Their squares are $1\bar{3}$ and $1\bar{3} + 22R + 121$, which make the sum $2\bar{3} + 22R + 121$, equal to 281. Taking away 121 from both sides, there is an equality between $2\bar{3} + 22R$ and 160. Dividing everything by 2, between $1\bar{3}$ and $80 - 11R$.

Half the number of the roots is $11/2$; to its square, $121/4$, add 80 — that is, $320/4$ — and the result is $441/4$, whose square root is $21/2$. From this take away the aforesaid half, $11/2$, and there remains the value of one root, $10/2$, that is, 5, the smaller number. The greater will be 16. And the squares of these two numbers, 25 and 256, make 281.

Again, let the given difference be 11 and the sum of the squares 122. Set the smaller number, once more, equal to $1R$, and the greater to $1R + 11$. The squares of these numbers make, as before, the sum $2\bar{3} + 22R + 121$, equal to 122. Taking

away 121 from both sides, the equation is between $23 + 22R$ and 1. And removing $22R$ again, between 23 and $1 - 22R$. Dividing everything by 2, between 13 and $\frac{1}{2} - 11R$.

Half the number of the roots is $11/2$; to its square, $121/4$, add $\frac{1}{2}$, that is, $2/4$, and the result is $123/4$. Its square root is $\sqrt{3} 123/4$; from this take away the aforesaid half, $11/2$, and there remains the value of one root, $\sqrt{3} 123/4 - 11/2$, namely the smaller number; and adding 11, that is $22/2$, the greater number results, $\sqrt{3} 123/4 + 11/2$. The squares of these numbers make the sum 122, as these multiplication schemes show:

$$\sqrt{3} 123/4 + 11/2 \quad \sqrt{3} 123/4 - 11/2 \quad \sqrt{3} 123/4 + 11/2 \quad \sqrt{3} 123/4 - 11/2$$

$$\begin{aligned} & \bullet \quad \sqrt{3} 14883/16 + 121/4 - \sqrt{3} 14883/16 + 121/4 \quad 123/4 + \sqrt{3} 14883/16 \quad 123/4 - \sqrt{3} \\ & \quad 14883/16 \quad \text{—————} \quad 244/4 + \sqrt{3} 59532/16 \quad 244/4 - \sqrt{3} 59532/16 \end{aligned}$$

For $244/4$ and $244/4$ make $488/4$, that is, 122; and the surd roots destroy one another.

37. Given a number, to find another such that, when the given number is divided by it, the quotient exceeds the number found by the given number.

[Margin: XXXVII]

Let the given number be 60, and let another be sought by which, if 60 is divided, the quotient exceeds the found divisor by the number 7. Set the number sought equal to $1R$. The quotient will then be $60/1R$, which exceeds the divisor $1R$ by the number

$$(60 - 13) / 1R$$

(as is plain, if $1R$ is subtracted from $60/1R$); and this ought to be equal to the given number 7. So there is an equation between $(60 - 13)/1R$ and 7, which, by cross-multiplication, is reduced to this: between $60 - 13$ and $7R$. Adding 13, then, [to both sides,] it is between 60 and $13 + 7R$. And taking away $7R$ from each side, between 13 and $60 - 7R$.

Half the number of the roots is $7/2$; to its square, $49/4$, add 60, that is, $240/4$, and the result is $289/4$, whose square root is $17/2$. If the aforesaid half, $7/2$, is taken away, there remains the value of one root, $10/2$, that is, 5, the number sought. For

dividing 60 by 5 gives the quotient 12, which exceeds the divisor 5 by the number 7.

38. To divide a given number into two numbers so that the product of one into the other makes another given number, smaller than a quarter of the square generated from the first given number: that is, so that the square root of the given second (smaller) number is the mean proportional between the two numbers found.

[Margin: XXXVIII]

Let the given number 10 be divided into two, between which $\sqrt{3} 10$ is the mean proportional — that is, so that the number 10 is produced from their multiplication. Set one of the numbers equal to $1R$ and the other to $10 - 1R$. From their multiplication comes the number $10R - 13$, equal to 10. Adding 13 to both sides, the equation becomes: between $10R$ and $13 + 10$. And taking away 10 from each side, between $10R - 10$ and 13.

Half the number of the roots is 5; from its square, 25, take away 10, and the remaining number is 15. To its square root, $\sqrt{3} 15$, add the aforesaid half, 5, and this gives the greater root, $\sqrt{3} 15 + 5$. And that same square root, subtracted from the aforesaid half, 5, leaves the smaller root, $5 - \sqrt{3} 15$. These two numbers, $\sqrt{3} 15 + 5$ and $5 - \sqrt{3} 15$, make the sum 10; and from one into the other the product is likewise 10, as is manifest in this multiplication scheme:

$$\begin{array}{r}
 5 - \sqrt{3} 15 \quad \sqrt{3} 15 + 5 \quad \text{—————} \\
 \bullet \quad 25 - \sqrt{3} 375 \\
 \bullet \quad \sqrt{3} 375 - 15 \quad \text{—————} \quad 10
 \end{array}$$

For $-\sqrt{3} 375$ and $+\sqrt{3} 375$ cancel one another, and from the addition of +25 and -15 comes the number 10.

We solve this same puzzle by a simple equation in the preceding chapter, puzzle 172:

$$5 - \sqrt{3} 15 \quad 5 + \sqrt{3} 15 \quad \text{—————} \quad 10$$

The aforesaid multiplication of the number $5 - \sqrt{3} 15$ into $5 + \sqrt{3} 15$ is likewise carried out if 15, the square of the latter part, is subtracted from 25, the square of the former part, as was declared in Chapter 23. This same puzzle was solved in puzzle 23, though with another example proposed.

39. To divide a given number into two, so that the sum of their squares exceeds the product of their multiplication by a given number.

[Margin: XXXIX]

Let the given number be 30 and the given excess 228. Set one of the numbers equal to $1R$ and the other to $30 - 1R$. Their squares are 13 and $13 + 900 - 60R$, which make the sum $23 + 900 - 60R$; and this ought to exceed the number produced from $1R$ into $30 - 1R$ — that is, the number $30R - 13$ — by the excess 228. So there is an equality between $23 + 900 - 60R$ and $30R - 13 + 228$. Adding 13 to each side, between $33 + 900 - 60R$ and $30R + 228$. Adding $60R$ to both sides, between $33 + 900$ and $90R + 228$. Taking away 228 from each side, between $33 + 672$ and $90R$. And again removing 672 from both sides, between 33 and $90R - 672$. Dividing everything by 3, between 13 and $30R - 224$.

Half the number of the roots is 15; from its square, 225, take away 224, and there remains 1, whose square root, 1, added to the aforesaid half, 15, gives the greater root, 16. That same [root], subtracted from the aforesaid half, 15, leaves the smaller root, 14. So the sought parts of the number 30 are 16 and 14; for their squares, 256 and 196, make the sum 452, which exceeds 224, the product of 16 into 14, by the number 228.

We solve this same problem by a simple equation in puzzle 173 of the preceding chapter, though with another example proposed. If the number to be divided were proposed as 20 and the excess as 30, the problem would become impossible, as the operation itself will show.

40. To divide a given number into two, whose square roots have a given difference — the square of which [difference], however, is smaller than the given number.

[Margin: XL]

Let the given number 30 be divided into two whose square roots have the difference 5. Set the root of the smaller equal to $1R$, and hence the root of the greater to $1R + 5$, so that they exceed one another by five. The squares of these roots, 13 and $13 + 10R + 25$, make the sum $23 + 10R + 25$, equal to 30. Taking away 25 from both sides, there remains an equality between $23 + 10R$ and 5. Removing $10R$ again from each side, between 23 and $5 - 10R$. And dividing everything by 2, between 13 and $5/2 - 5R$.

Half the number of the roots is $5/2$; to its square, $25/4$, add $5/2$, that is, $10/4$, and the result is $35/4$. From its square root, $\sqrt{35/4}$, take away the aforesaid half, $5/2$, and there results the value of the root, $\sqrt{35/4} - 5/2$ — namely the root of the smaller number. Adding $5/2$, then, gives the root of the greater number, $\sqrt{35/4} + 5/2$. These two roots exceed one another by five. And squared, they make the numbers sought: $15 - \sqrt{3500/16}$ and $15 + \sqrt{3500/16}$, which make 30, the given number.

The same puzzle can also be solved by a simple equation, in this way. Set the root of the smaller number equal to $1R - 5/2$, and that of the greater to $1R + 5/2$, so that the latter exceeds the former by five. Both, multiplied into themselves and squared, make $13 - 5R + 25/4$ and $13 + 5R + 25/4$, whose sum, $23 + 50/4$, is equal to 30. Taking away $50/4$ — that is, $12\frac{1}{2}$ — from both sides, there remains an equation between 23 and $17\frac{1}{2}$. Dividing $17\frac{1}{2}$ by 2, there results $13 [= 35/4]$, and hence $1R = \sqrt{35/4}$. Since, then, the root of the smaller number was set as $1R - 5/2$, the root of the smaller number itself will be $\sqrt{35/4} - 5/2$; and the root of the greater number $\sqrt{35/4} + 5/2$, since it was set as $1R + 5/2$. So the roots found are the same as before, and so on.

41. To divide a given number into two, whose square roots multiplied together make any given number.

[Margin: XLI]

Let the given number 12 be cut into two, whose square roots multiplied together make 5. Set one of the numbers equal to $1R$ and the other to $12 - 1R$. Their roots will be $\sqrt{1R}$ and $\sqrt{12 - 1R}$. These, multiplied together, make $\sqrt{12R - 13}$, which number ought to be equal to the number 5. Therefore their squares, too — $12R - 13$ and 25 — will be equal. Adding 13 to both sides, the equation is between $12R$ and $13 + 25$. And taking away 25 from each side, between $12R - 25$ and 13.

Half the number of the roots is 6; from its square, 36, take away 25, and there remains the number 11. To its square root, $\sqrt{11}$, add the aforesaid half, 6, and there results the greater root, $\sqrt{11} + 6$, or $6 + \sqrt{11}$ — that is, the greater number sought. And if that same $\sqrt{11}$ be taken from the aforesaid half, 6, the remainder is the smaller root, $6 - \sqrt{11}$, namely the smaller number sought. And these two numbers, $6 + \sqrt{11}$ and $6 - \sqrt{11}$, make the proposed number 12; and their roots are $\sqrt{6 + \sqrt{11}}$ and $\sqrt{6 - \sqrt{11}}$, which, multiplied together, make $\sqrt{25}$, that is, 5, as this scheme demonstrates:

$$\sqrt{6 + \sqrt{11}} \sqrt{6 - \sqrt{11}} \text{ ————— } \sqrt{25}$$

For if their squares, $6 + \sqrt{3} 11$ and $6 - \sqrt{3} 11$, are multiplied together — which is done if 11, the square of the latter part, is subtracted from 36, the square of the former part, as was stated in Chapter 23 — the product will be $\sqrt{3} 25$, that is, 5.

By a simple equation we shall solve the same puzzle in this way. Set one number equal to $6 + 1R$ and the other to $6 - 1R$, whose roots, $\sqrt{3} (6 + 1R)$ and $\sqrt{3} (6 - 1R)$, multiplied together, make the number $\sqrt{3} (36 - 13)$, as this scheme indicates, equal to the number 5. Therefore their squares, too — $36 - 13$ and 25 — will be equal.

$$\begin{array}{r} \sqrt{3} (6 + 1R) \sqrt{3} (6 - 1R) \text{ ————— } - 6R - 13 \quad 36 + 6R \\ \text{—————} \quad \sqrt{3} (36 - 13) \end{array}$$

Adding 13 to each side, the equation is between 36 and $13 + 25$. And taking away 25 from both sides, between 11 and 13. So, dividing 11 by 1, there results $13 = 11$, and $1R = \sqrt{3} 11$. And since one of the numbers was set as $6 + 1R$, that number itself will be $6 + \sqrt{3} 11$; while the other, which we set as $6 - 1R$, will be $6 - \sqrt{3} 11$ — just as above.

42. Given a number, to find another which, together with the given number, makes the square of the number found.

[Margin: XLII]

Let the given number be 40. Set the number found equal to $1R$, whose square is 13. Therefore $1R + 40$ is equal to 13. Half the number of the roots is $\frac{1}{2}$; to its square, $\frac{1}{4}$, add 40, and the result is $40\frac{1}{4}$, that is, $\frac{161}{4}$. To its square root, $\sqrt{3} \frac{161}{4}$, add the aforesaid half, $\frac{1}{2}$, and there results the value of $1R$: $\sqrt{3} \frac{161}{4} + \frac{1}{2}$, namely the number sought, which together with the given number 40 makes $\sqrt{3} \frac{161}{4} + 40\frac{1}{2}$. This number is equal to the square of the number found, $\sqrt{3} \frac{161}{4} + \frac{1}{2}$; for this [number], multiplied into itself (as the appended scheme shows), produces the number $\frac{161}{2} + \sqrt{3} \frac{644}{16}$, that is, $40\frac{1}{2} + \sqrt{3} \frac{161}{4}$ — because these small quantities, $\sqrt{3} \frac{644}{16}$ and $\sqrt{3} \frac{161}{4}$, are equal.

$$\sqrt{3} \frac{161}{4} + \frac{1}{2} \quad \sqrt{3} \frac{161}{4} + \frac{1}{2} \text{ ————— }$$

$$\bullet \quad \sqrt{3} \frac{161}{16} + \frac{1}{4} \frac{161}{4} + \sqrt{3} \frac{161}{16} \text{ ————— } \frac{162}{4} (= 40\frac{1}{2}) + \sqrt{3} \frac{644}{16}$$

Again, let the given number be 100 [recte: 110]. Set the number sought equal to $1R$. Therefore $1R + 110$ is equal to 13. Half the number of the roots is $\frac{1}{2}$; to its square, $\frac{1}{4}$, add 110, and the result is $110\frac{1}{4}$, that is, $\frac{441}{4}$. To its square root, $\sqrt{3} \frac{441}{4}$, add the aforesaid half, $\frac{1}{2}$, and there results the value of $1R$: $\frac{22}{2}$, that is, 11,

the number sought. For its square, 121, is equal to the number formed from the given 110 and the found 11.

This problem was also proposed in puzzle 3 of this chapter, though in other numbers.

43. To divide a given number into two, so that the sum of the squares together with the product of one into the other makes any given number.

[Margin: XLIII]

Let the given number to be divided be 20, and let the squares together with the product of one part into the other make 309. Set one of the numbers equal to $1R$ and the other to $20 - 1R$. The squares make the sum $2\bar{3} + 400 - 40R$; and this, together with the product of one into the other — that is, with $20R - 1\bar{3}$ — makes the number $1\bar{3} + 400 - 20R$, equal to 309. Adding $20R$ to both sides, the equation is between $1\bar{3} + 400$ and $20R + 309$. And taking away 309 from each side, between $1\bar{3} + 91$ and $20R$. Again removing 91 from both sides, between $1\bar{3}$ and $20R - 91$.

Half the number of the roots is 10; from its square, 100, take away 91, and there remains the number 9. To its root, 3, add the aforesaid half, 10, and the greater root is 13. And that same root, 3, subtracted from the aforesaid half, 10, leaves the smaller root, 7. So the two parts of the number 20 are 13 and 7; for their squares, 169 and 49, together with 91, the product of one into the other, make 309.

We shall carry out the same by a simple equation, in this way. Set the greater part equal to $10 + 1R$ and the smaller to $10 - 1R$, so that together they make up 20. Their squares are $1\bar{3} + 20R + 100$ and $1\bar{3} - 20R + 100$, making the sum $2\bar{3} + 200$; and this, together with the product of one part into the other — namely with $100 - 1\bar{3}$ (which product is obtained if $1\bar{3}$, that is, the square of the latter part, is subtracted from 100, the square of the former part, as appears here) — makes the number $1\bar{3} + 300$, equal to 309.

Throughout this batch the cossist (symbolic) tokens are rendered as printed and glossed once here: R = *res* (the unknown, x); $\bar{3}$ = *census* (the square, x^2); $\bar{3}\bar{3}$ = *census of census* (x^4); $\sqrt{}$ = the printed square-root surd (the *radix* sign combined with a *census* mark, $\sqrt{\bar{3}}$). Coefficients precede the sign (e.g. $10R = 10x$, $1\bar{3}\bar{3} = x^4$).

(Conclusion of Problem 43, carried over from the foot of the previous page.)

$$\begin{array}{l} 10 + 1R \\ 10 - 1R \end{array}$$

100 - 13

equal to 309. So, subtracting 300 from both sides, there remains the equation between 13 and 9. And dividing 9 by 1, we get $13 = 9$, and hence $1R = 3$. Therefore the greater part, set as $10 + 1R$, will be 13; and the lesser, set as $10 - 1R$, will be 7, as before.

[Margin: XXXXIIII.] **44.** *To find a number which, multiplied into a given number, produces a number equal to that which the square of the number found makes together with any number that is less than the square produced from half the given number.*

Let the number to be found be one which, multiplied into 10, makes just as much as its square together with 20 — 20 being, of course, a number smaller than the square of half of 10. Let the sought number be set as $1R$; multiplied into 10 it makes $10R$, equal to $13 + 20$ (namely the square produced from $1R$, together with 20). Taking 20 from both sides, the equation between $10R - 20$ and 13 remains. Half the number of the roots is 5; from its square, 25, subtract 20, and the number 5 remains. Its root, $\sqrt{5}$, added to the said half 5, makes $\sqrt{5} + 5$, the greater root. And the same root subtracted from the aforesaid half 5 gives the lesser root, $5 - \sqrt{5}$. Each of the two numbers, $\sqrt{5} + 5$ and $5 - \sqrt{5}$, solves the problem. For the multiplication of $\sqrt{5} + 5$ into 10 makes $50 + \sqrt{500}$ — exactly what arises from $30 + \sqrt{500}$ (the square of the number $\sqrt{5} + 5$) together with 20. Likewise, $5 - \sqrt{5}$ into 10 gives the number $50 - \sqrt{500}$, equal to the square $30 - \sqrt{500}$ of the number $5 - \sqrt{5}$, together with 20.

Again, let the number to be found be one which, multiplied into 20, makes just as much as its square together with 84. Let the sought number be set as $1R$. From $1R$ into 20 comes the number $20R$, equal to $13 + 84$ (namely the square produced from $1R$, together with 84). Taking 84 from both sides, the equation between $20R - 84$ and 13 remains. Half the number of the roots is 10; from its square, 100, subtract 84, and the number 16 is left. Its root, 4, added to the aforesaid half 10, makes the greater root 14; and subtracted from the same half, it gives the lesser root 6. Each number, 14 and 6, solves the problem. For 14 into 20 gives the number 280, equal to that which its square, 196, makes together with 84. Likewise, 6 into 20 gives the number 120, equal to that which its square, 36, makes together with 84.

[Margin: XXXXV.] **45.** *To find a number which, added to any given number, makes a sum that, multiplied into the number found, produces any given number.*

Let the number to be found be one which, added to 12 and the sum multiplied into the number found, produces 189. Let the sought number be set as $1R$. Adding 12 gives the number $1R + 12$, which, multiplied into the number found (namely into $1R$), makes the number $13 + 12R$ equal to 189. Taking $12R$ from both sides, the equation will be between 13 and $189 - 12R$. Half the number of the roots is 6; to its square, 36, add 189, and the number 225 results. From its root, 15, subtract the aforesaid half 6, and the value of $1R$ remains: 9 — namely the sought number. For this, added to 12, makes 21; and from 21 into 9 comes the given number 189.

Again, let the number to be found be one which, added to 12 and the sum multiplied into the number found, makes the number 400. Let it be set as $1R$. Adding 12 gives the number $1R + 12$, which multiplied into $1R$ makes the number $13 + 12R$ equal to 400. Taking $12R$ from both sides, the equality between 13 and $400 - 12R$ remains. Half the number of the roots is 6; to its square, 36, add 400, and the number 436 results. If from its root, $\sqrt{436}$, the aforesaid half 6 is subtracted, the remainder becomes the value of one root, $\sqrt{436} - 6$ — namely the number we seek. For if it is added to 12, the number $\sqrt{436} + 6$ results, as you see in this formula:

$$\begin{array}{r} \sqrt{436} - 6 \\ + 12 \\ \hline \sqrt{436} + 6 \end{array}$$

And this number, $\sqrt{436} + 6$, multiplied by the number found, $\sqrt{436} - 6$, makes the given number 400, as this other formula shows:

$$\begin{array}{r} \sqrt{436} + 6 \\ \sqrt{436} - 6 \\ \hline 400 \end{array}$$

This multiplication comes about if 36, the square of the latter part 6, is subtracted from 436, the square of the former part [$\sqrt{436}$], as was explained in chapter 23.

[Margin: XXXXVI.] **46.** *To cut a given number into two parts, so that the number produced from one multiplied into the other, [and then] multiplied into the square of the given number, makes any given number.*

Let the given number 20 be the one to be cut into two, so that the number produced from their multiplication, if multiplied into 400 (the square of the given number), yields 300. Let one number be set as $1R$ and the other as $20 - 1R$. From $1R$ into $20 - 1R$ comes the number $20R - 13$. And this, multiplied into 400, makes $8000R - 4003$, which must equal 300. Adding 4003 to both sides, the equation will

be between $8000R$ and $300 + 4003$. And taking 300 from both sides, [it will be] between 4003 and $8000R - 300$. Dividing everything by 400, the equation will be between 13 and $20R - 3/4$. Half the roots is 10; from its square, 100 (or $400/4$), subtract $3/4$, and $397/4$ remains. Its root $\sqrt{(397/4)}$, added to the aforesaid half 10, makes the greater root $10 + \sqrt{(397/4)}$; and subtracted from the same half 10, it leaves the lesser root $10 - \sqrt{(397/4)}$. These are the sought parts. For from one into the other comes the number $3/4$ — if, that is, the square of the latter part, $397/4$, is subtracted from 100 (or from $400/4$), the square of the former part. And $3/4$ into 400 (the square of the given number 20) gives the number $1200/4$, that is, 300.

[Margin: XLVII.] 47. *To find a number whose square, increased by a given number, when multiplied into the same square diminished by another given number, produces any given number whatever.*

Let there be found a number whose square, increased by eight, when multiplied into the same square diminished by three, produces 6942. Let the sought number be set as $1R$. Its square, 13 , increased by eight, will be $13 + 8$; and diminished by three, $13 - 3$. These two numbers multiplied together make $133 + 53 - 24$, as you see in this formula:

$$\begin{array}{r}
 13 + 8 \\
 13 - 3 \\
 \hline
 - 33 - 24 \\
 133 + 83 \\
 \hline
 133 + 53 - 24
 \end{array}$$

This number must equal the number 6942. So, adding 24 to both sides, the equality between $133 + 53$ and 6966 arises. And taking 53 from both sides, [the equation is] between 133 and $6966 - 53$. Now proceed thus. Half the number of censuses is $5/2$; to its square, $25/4$, add 6966, that is, $27864/4$, and the sum $27889/4$ results. From its square root, $167/2$, subtract the aforesaid half $5/2$, and $162/2$ remains, that is, 81 — one census, whose root, 9, will give the value of one root for the sought number. For its square, 81, increased by eight, makes 89; and diminished by three, 78; and from 89 into 78 comes the given number 6942.

If, with the same things posited, the number 10 must be produced instead, the equation between 133 and $34 - 53$ is found. Now thus: half the number of censuses is $5/2$; to its square, $25/4$, add 34, that is, $136/4$, and the sum $161/4$ results. From its square root, $\sqrt{(161/4)}$, subtract the aforesaid half $5/2$, and it becomes $13 = \sqrt{(161/4)} - 5/2$; and the square root of this, $\sqrt{(\sqrt{(161/4)} - 5/2)}$, is the sought number. For its square, $\sqrt{(161/4)} - 5/2$, increased by eight (or $16/2$), makes $\sqrt{(161/4)}$

+ 11/2; and diminished by three (or 6/2), makes $\sqrt{(161/4)} - 11/2$. These two numbers multiplied together produce the number 10, as this formula indicates:

$$\begin{array}{r} \sqrt{(161/4)} + 11/2 \\ \sqrt{(161/4)} - 11/2 \\ \hline 40/4, \text{ that is, } 10 \end{array}$$

For if 121/4, the square of the latter part, is subtracted from 161/4, the square of the former part, there remains 40/4, that is, 10.

ÆNIGMATA VARIA NVMERORVM — Various Riddles of Numbers, applied to the concrete matters of business contracts, with some examples pertaining to second roots [square roots].

CAPVT XXXI. — Chapter 31

In this chapter, besides riddles, we shall set out some other questions, not particularly difficult, some of which can be dispatched not only by the rule of False [Position] but even by the rule of Three alone — so that the excellence of the rule of Algebra may shine forth all the more, seeing that it appears to embrace all the rules of Arithmetic. Let us begin here, then.

[Margin: I.] 1. *Someone standing by estimates the sum of money I have in my purse to be worth 600 gold pieces [aurei]. I correct his error thus: if to my money there were added the parts 1/2, 1/3, 1/4, and from the sum there were subtracted the part 1/12 of that same money of mine, then I would have 600 gold pieces. The question is: how much is my money?*

Let it be set as 1R gold pieces. If the parts 1/2 R, 1/3 R, 1/4 R are added — which make the sum 1 1/12 R — the whole money becomes 2 1/12 R. But taking away 1/12 R, the remainder becomes 2R, equal to 600. So, dividing 600 by 2, 1R becomes 300. And this is the money in my purse. For if the parts 1/2, 1/3, 1/4 are added — namely 150, 100, and 75 gold pieces — the sum becomes 625 gold pieces; and taking away 1/12 (namely 25 gold pieces), the remainder will be the number 600.

[Margin: II.] 2. *A certain traveler covers 9 miles each day. Another, after the tenth day has elapsed, sets out on the same journey, from the same place, and covers 14 miles each day. It is asked: on what day will the later traveler overtake the earlier one?*

Let it be set that both will be together after $1R$ days. Therefore the earlier one — beyond the 90 miles which he completed in the first ten days — will cover in $1R$ days a further $9R$ miles, since he covers 9 miles each day. The later one, covering 14 miles daily, will cover $14R$ miles in $1R$ days. And since in $1R$ days the earlier one will necessarily have covered just as many miles (together with the 90 he completed in 10 days) as the later covers in $1R$ days — seeing that they are then to be together — the equation will be between $9R + 90$ and $14R$. And taking $9R$ from both sides, [the equation is] between 90 and $5R$. So, dividing 90 by 5, $1R$ becomes 18. Therefore they will meet after 18 days. For the earlier one in 18 days completes 162 miles; and adding the 90 which he completed in the preceding 10 days, they become 252 miles — exactly as many as the later one completes in 18 days.

Scholium

This question, moreover, might be put in abstract numbers as follows. Let a number be sought which, multiplied into 9, and 90 added to the product, produces a number equal to the one generated from the same number multiplied into 14. For if the sought number is set as $1R$, there will be $9R + 90$ equal to the number $14R$. So, taking $9R$ from both sides, the equality between 90 and $5R$ is found. And dividing 90 by 5, $1R$ becomes 18 — the number we seek. For this, multiplied into 9, makes 162; and adding 90, the number 252 results, equal to that which comes from 18 into 14.

[Margin: III.] 3. *A traveler covers 9 miles each day. Another, after the tenth day has elapsed, begins the same journey. It is asked: how many miles must the latter cover [per day] in order to overtake the former in 18 days?*

Let it be set as $1R$ miles. Therefore in 18 days he will cover $18R$ miles. Now, since the earlier one, covering 9 miles each day, completes 162 miles in 18 days — and, adding the 90 he walked in the earlier 10 days, it is clear he has covered 252 miles — the equation is therefore between $18R$ and 252. And dividing 252 by 18, $1R$ becomes 14. So the later traveler must cover that many miles [per day] in order to overtake the earlier one in 18 days.

[Margin: IIII.] 4. *Someone, on being asked what hour of the day it was, answered thus: Half of the hours elapsed from midnight up to this moment, added to three-quarters of the hours remaining until the following midnight, will indicate the number of the hour you seek. It is asked, then: what hour is it?*

Let $1R$ be the hours elapsed since midnight. Since, from one midnight to another, 24 hours flow away, there will remain until the following midnight the hours $24 -$

1R. Three-quarters of this number make $(72 - 3R)/4$; and if to these is added $1/2 R$ (namely half of the elapsed hours, or $2R/4$), the number $(72 - 1R)/4$ results, equal to 1R. This equation, by cross-multiplication, is reduced to this one, between $4R$ and $72 - 1R$. Adding 1R to both sides, the equation will be between 72 and $5R$. So, dividing 72 by 5, 1R becomes $14 \frac{2}{5}$ — the number of hours that elapsed from midnight up to the hour of the question. And accordingly it will then be 2 o'clock and 24 minutes past noon, since one hour contains 60 minutes. This is proved as follows: half of the $14 \frac{2}{5}$ hours elapsed since midnight makes $7 \frac{1}{5}$ hours; and if to these are added three-quarters of the $9 \frac{3}{5}$ hours that remain until midnight (namely $7 \frac{1}{5}$), the hours become $14 \frac{2}{5}$.

We shall proceed in the same way if the discussion concerns the hours reckoned from sunset, in the Italian manner. For if someone answers that the hour then at hand is the one made up from three-quarters of the hours elapsed since the last sunset, added to half of the hours that remain until the next sunset — let us set 1R as the hours elapsed since the last sunset. Since, from one sunset to another, 24 hours flow, there remain until the next sunset the hours $24 - 1R$. Half of this number makes $(24 - 1R)/2$, that is, $(48 - 2R)/4$; and this, added to $3/4 R$ (three-quarters of the elapsed hours), makes $(48 + 1R)/4$, which number is equal to 1R. This equation, by cross-multiplication, is reduced to this one, between $4R$ and $48 + 1R$. And taking 1R from both sides, the equation will be between 48 and $3R$. So, dividing 48 by 3, 1R becomes 16 — the number of hours elapsed since sunset. This is proved thus: three-quarters of the 16 hours elapsed make 12 hours; and if to these is added half of the 8 hours that remain until the next sunset (namely 4), the hours become 16.

[Margin: V.] 5. *Someone has two measures of wine, one of which is worth 12 coins [nummi] and the other 15. He wishes to mix from both wines a single measure whose price is 13 coins. It is asked: how much wine is to be taken from each measure?*

Set 1R as the parts of a measure of the cheaper wine, and $1 - 1R$ as the measure of the dearer wine. Then set up the rule of Three twice, in this manner:

	measur e	coin s	measure	coins
Of the former wine	1	12	1R ?	makes 12R
Of the latter wine	1	15	$1 - 1R$?	makes $15 - 15R$

For if 1 measure of the former wine is worth 12 coins, 1R of a measure of the same wine will be worth 12R coins. And if 1 measure of the latter wine is worth

15 coins, 1 – 1R of a measure of the same wine will be worth 15 – 15R coins. Therefore 1R of a measure of the former wine, and 1 – 1R of a measure of the latter, will be worth 15 – 3R coins, which price shall be equivalent to 13 coins. So, adding 3R to both sides, the equation will be between 3R + 13 and 15. And taking 13 from both sides, [the equation is] between 3R and 2. So, dividing 2 by 3, 1R becomes 2/3. Therefore of the cheaper wine 2/3 of one measure is to be taken, and of the finer 1/3 of one measure — which is proved thus:

	measure	coins	measure	coins
Of the former wine	1	12	2/3 ?	are worth 8
Of the latter wine	1	15	1/3 ?	are worth 5

For 2/3 of one measure of the former wine will be worth 8 coins, and 1/3 of a measure of the latter wine, 5 coins; and therefore one mixed measure [is worth] 13 coins.

[Margin: VI.] 6. *A measure of wine is worth 10 coins. How much water, then, must be mixed into one measure so that one measure of the mixture is worth 7 coins?*

Set 1R as a measure of water. Then set up the rule of Three in this manner:

meas. of wine	meas. of water	coins	measure of mixture	coins
1	+ 1R	10	1 ?	is worth 10/(1 + 1R)

(Problem 6 is only begun here: its solution continues beyond this page.)

Algebra — Cap. XXXI (continued)

Cossist notation in this batch: R = res (the unknown, x); a pure absolute number carries a small superscript numerus/dragma circle in the printed fractions, rendered here as a plain numeral. Coefficients are written before the sign, e.g. 8R = 8x. All worked equations are transcribed from the tiles and check out arithmetically.

Problem 6 (conclusion of the proof)

For if 1 measure of wine together with 1R [res, x] measures of water is worth 10 nummi, then 1 measure of the wine-and-water mixture will be worth $10 / (1 + 1R)$. And so the equation will stand between $10 / (1 + 1R)$ and 7, which by cross-multiplication is reduced to this one, between 10 and $7 + 7R$. Removing 7 from

both sides, the equation will stand between 3 and 7℞. Dividing 3 by 7 gives 1℞ = 3/7. And that much water — 3/7 of one measure — must be mixed with 1 measure of wine, so that 1 mixed measure is worth 7 nummi. This is proved as follows.

mens. ex vino & aqua (measures of wine + water)	numm.	mens.	numm.
1 3/7	10	1 ?	worth 7

For if 1 measure of wine with 3/7 of one measure of water is worth 10 nummi, then 1 measure of the mixture alone will be worth 7 nummi.

Problem 7 [Margin: VII.]

There are 20 measures of wine in a certain vessel, each of which is worth 12 nummi. Water is then poured into that vessel until the vessel is filled with that wine and the water. And then 1 mixed measure is worth 10 nummi. It is asked, how great is the capacity of that vessel.

POSIT the measures to be 20 + 1℞ [i.e. 20 of wine plus 1℞ of water], of one measure [each]. Then set up the rule of three thus.

mens. vini	mens. aquæ	num m.	mens. mixta	numm.
20 +	1℞	240	1 ?	worth 240 / (20 + 1℞)

For if 1 measure of wine is worth 12 nummi, then the 20 + 1℞ measures of wine [together with the water] will be worth 240 nummi. Therefore 1 mixed measure will be worth 240 / (20 + 1℞) nummi. Thus the equation is found, between 240 / (20 + 1℞) and 10, which by cross-multiplication is reduced to this one, between 240 and 200 + 10℞. Removing 200 from each side, an equality will remain between 40 and 10℞. Dividing 40 by 10 gives 1℞ = 4; and that many measures of water were poured into the vessel, so that the whole vessel holds 24 measures. This is proved thus.

mens. vini & aquæ	numm.	mens.	numm.
24	240	1 ?	worth 10

For if 24 measures of wine and water are worth 240 nummi — since 1 measure of wine is worth 12 nummi — then 1 mixed measure will be worth 10 nummi.

Problem 8 [Margin: VIII.]

Two couriers from two cities, between which 140 leagues [Leucæ] intervene, set out on the same day, one toward the other. One covers 8 leagues each day, the other 6 leagues. The question is, when will they meet and encounter each other.

POSIT [the meeting] to be within 1R days, and set up the rule of three thus.

Dies	Leucæ	Dies	Leucæ
1	8	1R ?	8R
1	6	1R ?	6R

The former, then, in 1R days will cover 8R leagues, and the latter 6R leagues — that is [together], 14R leagues. So the equation stands between 14R and 140. Dividing 140 by 14 therefore gives 1R = 10. And when the tenth day is completed they will meet. This I prove as follows.

Dies	Leucæ	Dies	Leucæ
1	8	10 ?	80
1	6	10 ?	60

For the former in 10 days will cover 80 leagues, and the latter 60, which together make up the whole intervening span of 140 leagues.

Problem 9 [Margin: IX.]

Someone exchanging 568 gold pieces receives for them four kinds of coin. Seven coins of the first kind make 1 gold piece; of the second kind, 18; of the third, 21; and of the fourth, 28. He receives the same number of coins of each kind. It is asked, how many coins of each kind he gets.

POSIT 1R coins of each kind, and set up the rule of three four times thus.

Mon.	aur.	Mon.	aur.
7	1	1R ?	(1/7)R
18	1	1R ?	(1/18)R
21	1	1R ?	(1/21)R
28	1	1R ?	(1/28)R

For if 7 coins make 1 gold piece, then 1R coins will make (1/7)R gold pieces, and likewise for the rest. Now the fractions written in the fourth place make the sum

(71/252)℞ gold pieces, equal to the number 568 gold pieces. Dividing 568, then, by 71/252 gives 1℞ = 2016; and he receives that many coins of each kind. This I prove as follows.

Mon.	aur.	Mon.	aur.
7	1	2016 ?	288
18	1	2016 ?	112
21	1	2016 ?	96
28	1	2016 ?	72

For 2016 coins of the first kind make 288 gold pieces; of the second kind, 112 gold pieces; of the third, 96; and of the fourth, 72 — all of which make the sum of 568 gold pieces.

Problem 10 [Margin: X.]

A certain merchant buys wool and wax for 124 gold. Now 100 pounds of wool cost 7 gold, and 100 pounds of wax cost 14 gold; and he buys twice as many pounds of wool as of wax. The question is, how many pounds of each thing he bought.

POSIT 1℞ pounds of wax, and 2℞ pounds of wool; and set up the rule of three twice, in this way.

lib.	aur.	lib.	aur.
100 lanæ (wool)	7	2℞ lib. lanæ ?	(14/100)℞
100 ceræ (wax)	14	1℞ lib. ceræ ?	(14/100)℞

The equation will therefore be found between (28/100)℞ gold and 124 gold. Dividing 124 by 28/100 gives 1℞ = 442 6/7; and he bought that many pounds of wax, and 885 5/7 pounds of wool. This is proved thus.

lib.	aur.	lib.	aur.
100 lanæ	7	885 5/7 ?	62
100 ceræ	14	442 6/7 ?	62

For thus he spends 124 gold on the wool and the wax.

Problem 11 [Margin: XI.]

A certain man buys some ells [vlnæ] of cloth, which he again sold. Now he buys 5 ells for 7 gold, and sold 7 ells for 11 gold, and on this trade he gained 100 gold. It is asked, how many those ells were.

POSIT 1℞ ells, and set up the rule of three twice.

vlnæ	aur.	vlnæ	aur.
5	7	1℞ ?	(7/5)℞
7	11	1℞ ?	(11/7)℞

You see that if (7/5)℞ gold, which he spent, is subtracted from (11/7)℞ gold, which he received, there is left (6/35)℞ gold as profit. So the equation stands between (6/35)℞ gold and 100 gold. Dividing 100 by 6/35 gives 1℞ = 583 1/3 ells; and that many ells were bought and sold. This I prove thus.

vlnæ	aur.	vlnæ	aur.
5	7	583 1/3 ?	816 2/3
7	11	583 1/3 ?	916 2/3

Here you see that he gained 100 gold pieces.

Problem 12 [Margin: XII.]

A certain man buys some ells of cloth, spending 11 gold for 7 ells. And he again sold the whole cloth, being forced to give 5 ells for 7 gold; and on this trade he lost 100 gold. It is asked, how many ells he bought and sold.

Posit 1℞ ells, and so on, as before; and set up the rule of three twice, in this way.

vln.	aur.	vln.	aur.
7	11	1℞ ?	(11/7)℞
5	7	1℞ ?	(7/5)℞

Therefore, if (7/5)℞ gold, which he received, is subtracted from (11/7)℞ gold, which he spent, a loss of (6/35)℞ gold emerges; and the equation will stand between (6/35)℞ gold and 100 gold. Dividing 100, then, by 6/35 gives 1℞ = 583 1/3 ells; and that many ells were bought and sold. This I prove thus.

vln.	aur.	vln.	aur.
7	11	583 $\frac{1}{3}$?	916 $\frac{2}{3}$
5	7	583 $\frac{1}{3}$?	816 $\frac{2}{3}$

You see, then, that he made a loss of 100 gold pieces.

Problem 13 [Margin: XIII.]

One hundred pounds of wax are bought for 17 gold. It is asked, how many pounds must be sold for 1 gold so that, on 102 gold, one may gain 18 gold.

Posit 1R pounds, and set up the rule of three twice.

aur.	lib.	aur.	lib.
17	100	102 ?	600
1	1R	102 + 18 ?	120R

For 102 gold will give 600 pounds. And if 1R pounds are given for 1 gold, then 120R pounds will be given for 102 + 18 gold; and these 120R are equal to 600 pounds, since all the wax is sold. So the equation stands between 120R pounds and 600 pounds. Dividing 600, then, by 120 gives 1R = 5 pounds. And that many pounds must be sold for 1 gold, so that on 600 pounds a profit of 18 gold is made on the 102 gold. This I prove as follows.

lib.	aur.	lib.	aur.
100	17	600 ?	102
5	1	600 ?	120

Here, in the first example, you see 600 pounds yield 102 gold; in the second, however, 120 gold — that is, 102 + 18.

Problem 14 [Margin: XIII.]

A certain man buys 100 pounds of wax for 17 gold, and in selling them makes a loss of 18 gold on 102 gold. It is asked, how many pounds he gave for 1 gold.

Posit 1R pounds, and so on.

aur.	lib.	aur.	lib.
17	100	102 ?	600
1	1R	102 - 18 ?	84R

And the equation will stand between 84℞ pounds and 600 pounds. Dividing 600, then, by 84 gives 1℞ = 7 1/7 pounds. And that many pounds he gave for 1 gold, so as to make a loss of 18 gold on the 102 gold and the 600 pounds. This I prove thus.

lib.	aur.	lib.	aur.
100	17	600 ?	102
7 1/7	1	600 ?	84

Here you see that 102 gold were spent for 600 pounds, and that for the same the sum received was 84 gold — that is, 102 – 18.

Problem 15 [Margin: XV.]

Three merchants enter a partnership. The first puts in 40 gold for 2 months. The second, 20 gold for 5 months. The third, a certain sum of gold for 3 months. Now they gained 3276 gold, of which the first obtained 1040 gold, the second 1300, and the third 936. It is asked, how many gold pieces the third put in.

Let 1℞ gold be posited [for the third's stake].

aur.	mens.	lucrum aur. (profit)	aur.
40	2	3276	1040 (first)
20	5	3276	1300 (second)
1℞	3	3276	936 (third)

By multiplying the gold by the time, the numbers are reduced to the rule of three, in this way:

252 {	80	3276 ?	1040 (first)
	100	3276 ?	1300 (second)
	3℞, or 72	3276 ?	936 (third)

Now, in order to find the equation and the value of 1℞: the first number to be found in the rule of three is the sum of the three numbers 80, 100, and 3℞, in this way. Since that first number to be found stands to 80 as 3276 stands to 1040 [Margin: a — 19. sept.], the same number will result from [multiplying] the first into the fourth as from the second into the third. If, therefore, 262080 — the product of the second, 80, into the third, 3276 — is divided by the fourth, 1040, the first term, 252, will come out. And if from this the sum collected from 80 and 100

— namely 180 — is subtracted, the value 3R will remain, namely 72. Dividing 72, then, by 3 gives 1R = 24, the number of gold the third put in; which money, multiplied into 3 months, makes 72, and so on.

The value of 1R is also found in this way. Once the first number, 252, has been found [Margin: b — 19. sept.], the number 235872 — made from the first, 252, into the fourth, 936 — will be equal to the number 9828R — made from the second, 3R, into the third, 3276. Dividing 235872, then, by 9828 gives 1R = 24, as before.

Problem 16 [Margin: XVI.]

Three partners wish to distribute 455 gold among themselves on this condition: that as often as the first receives 2, the second receives 3; and as often as the second receives 4, the third receives 5. It is asked, how many gold each receives out of that sum of 455 gold.

POSIT that the first takes 1R; and the second, $1\frac{1}{2}R$, so that he has a number one-and-a-half times [sesquialter] the first; and the third, $1\frac{7}{8}R$, so that he has a number one-and-a-quarter times [sesquiquartus] the second.

1	R
$1\frac{1}{2}$	R
$1\frac{7}{8}$	R

The equation will therefore stand between $4\frac{3}{8}R$ and 455. Dividing 455, then, by $4\frac{3}{8}$ gives 1R = 104, the number for the first. The second, then, will get 156 — namely $1\frac{1}{2}R$ — and the third 195 — namely $1\frac{7}{8}R$. This is proved as follows. The three numbers 104, 156, 195 make the sum 455 to be divided. And as often as 2 is in 104, so often is 3 in 156; and as often as 4 is in 156, so often is 5 in 195, since the same quotient results: dividing 104 by 2 and 156 by 3; likewise dividing 156 by 4 and 195 by 5 [gives the same quotient].

Note on notation: the cossist sign **R** renders the *res* (the unknown, x); a coefficient stands before it (e.g. "1 R", "24 R"). Fractions of the unknown are written as coefficient-fractions of R (e.g. " $\frac{3}{20}R$ "). The small printed query mark ("?",) that appears beside one number in several proportion tables is a source-printed marker flagging the given term whose counterpart is being sought; it is transcribed verbatim and is **not** an editorial uncertainty flag (those are [?]). See parsing notes.

[continuation of Problem 16 / XVI]

The same procedure is to be followed if there are more partners. Suppose, for instance, there are four, and that as often as the third receives 6, so often the fourth receives 7: then $2\frac{3}{16}$ R [res, the unknown x] must be set down for the fourth, so that he may hold a ratio of one-and-a-sixth (*sesquisextum*) to the third. If, then, the sum to be divided is 21840 gold pieces, the equation will be between $6\frac{9}{16}$ R and 21840. Dividing 21840 by $6\frac{9}{16}$ gives $1\text{ R} = 3328$, the share of the first, whom we assumed to receive 1 R. The share of the second will be $1\frac{1}{2}$ R, that is, 4992. The share of the third will be $1\frac{7}{8}$ R, namely 6240. And the share of the fourth, finally, will be $2\frac{3}{16}$ R, that is, 7280 — which you may verify as before. For all four shares together make up the sum to be divided, 21840; and the same quotient results from dividing 3328 by 2 and 4992 by 3. Likewise again from dividing 4992 by 4 and 6240 by 5; and finally the same too from dividing 6240 by 6 and 7280 by 7.

The four shares set as multiples of the unknown:

Share	(in R)
first	1 R
second	$1\frac{1}{2}$ R
third	$1\frac{7}{8}$ R
fourth	$2\frac{3}{16}$ R

[The fraction of the sum-coefficient reads $6\frac{9}{16}$ R; the tens-digit-over-16 glyph is faint, but the arithmetic ($21840 \div 3328 = 6\frac{9}{16}$) fixes it. See QA.]

Problem 17 [Margin: XVII] — 17

I have four fragments of gold which together are worth 164 gold pieces. The first in value is double the second; the second triple the third; and the third quadruple the fourth. The value of each fragment is sought.

Set the fourth, or smallest, at 1 R gold pieces; the third at 4 R; the second at 12 R; and the first at 24 R. These are therefore 41 R, equal to 164 gold pieces. Dividing 164 by 41 gives $1\text{ R} = 4$; and to just so many gold pieces is the smallest fragment, the fourth, equivalent. The third accordingly will be worth 16 gold pieces, the second 48, and the first 96 — which gold pieces indeed make up the sum of 164 and preserve the proportions expressed in the problem.

Problem 18 [Margin: XVIII] — 18

Three partners contribute all their money and so gather 100 gold pieces. The money of the first is half that of the second, and the money of the second is one-third that of the third. It is asked how much each contributed.

Set the first to have contributed 1 R, the second 2 R, and the third 6 R. These will be 9 R, equal to 100 gold pieces. Dividing 100 by 9 therefore gives $1\text{ R} = 11\frac{1}{9}$, the money of the first. The money of the second will accordingly be $22\frac{2}{9}$, and that of the third $66\frac{6}{9}$. These three sums make up 100 gold pieces and preserve the proportions expressed in the problem.

Problem 19 [Margin: XIX] — 19

A dying man makes a will and leaves 3000 gold pieces to be distributed among his wife, his son, and two daughters, on this condition: that the son's share be double the mother's share, and the mother's share double the share of one daughter. It is asked how large each one's share is.

Set the share of one daughter at 1 R, of the mother at 2 R, and of the son at 4 R. These will then be 8 R, equal to 3000 gold pieces.

Share of	(in R)		(in gold)
one daughter	1 R		375
the [other] daughter	1 R	that is	375
the mother	2 R		750
the son	4 R		1500

(all in gold pieces)

Dividing 3000 by 8 gives $1\text{ R} = 375$, the share of one daughter; and accordingly the mother's share is 750 and the son's 1500.

Problem 20 [Margin: XX] — 20

A certain merchant, with each third part of his money, gains $\frac{1}{20}$ of that same money. Then, with each third part of the same money, he gains $\frac{1}{3}$ of the prior gain. And so, when everything is reckoned, he finds 864 gold pieces. It is asked how large the money was, and how large each gain.

Set the money at 1 R. Then the first gain was $\frac{3}{20}\text{ R}$, since each third part ($\frac{1}{3}\text{ R}$) earned $\frac{1}{20}[\text{R}]$. Next, the second gain was one-third of the prior gain, $\frac{3}{20}\text{ R}$ —

namely $[3/60 \text{ R}]$, that is, $1/20 \text{ R}$ — since each third part earned a third of its prior gain. Now 1 R , $3/20 \text{ R}$, and $1/20 \text{ R}$ make the sum $1 \frac{1}{5} \text{ R}$, equal to 864 gold pieces. Dividing 864 by $1 \frac{1}{5}$ therefore gives $1 \text{ R} = 720$, and just so many gold pieces did that merchant have. So the first gain, $3/20 \text{ R}$, was 108 gold pieces, and the second gain, $1/20 \text{ R}$, was 36 gold pieces; which two gains, together with 720 gold pieces, make 864 gold pieces.

[The statement's first fraction and the intermediate "namely ..." fraction are set in very small type; the surrounding integer results (108, 36, 720, 864) fix the working coefficients $3/20 \text{ R}$ and $1/20 \text{ R}$ with certainty. See QA.]

Problem 21 [Margin: XXI] — 21

A certain merchant, with each fourth part of his money, gains $1/20$ of that same money. Then, with each third part of the same money, he gains $1/10$ of the sum collected from that same money together with the gain. And so, when each step is reckoned, that merchant finds 400 gold pieces. It is asked how large the money was, and how large each gain.

Set the money at 1 R . Then the first gain was $4/20 \text{ R}$, that is, $1/5 \text{ R}$. Next, the second gain was $1/10$ of the sum collected — $6/5 \text{ R}$, from the prior gain and 1 R together — namely $9/25 \text{ R}$. Now 1 R , $1/5 \text{ R}$, and $9/25 \text{ R}$ make the sum $1 \frac{14}{25} \text{ R}$, equal to 400 gold pieces. Dividing 400 by $1 \frac{14}{25}$ therefore gives $1 \text{ R} = 256 \frac{16}{39}$ gold pieces, and just so many did that merchant have. The first gain, $1/5 \text{ R}$, was $51 \frac{11}{39}$ gold pieces; the second gain, $9/25 \text{ R}$, was $92 \frac{4}{13}$; which two gains, together with $256 \frac{16}{39}$, make 400 gold pieces.

Problem 22 [Margin: XXII] — 22

A certain merchant, with each fourth part of his money, gains $1/30$ of that same money. Then, with each of the fifth parts of the same prior money, he gains $1/27$ of that same money. And so, reckoning each step, he finds 2136 gold pieces. It is asked how large the money was, and how large each gain.

Set the money at 1 R . Then the first gain was $4/30 \text{ R}$, or $2/15 \text{ R}$; and the second gain $5/27 \text{ R}$. Now 1 R , $2/15 \text{ R}$, and $5/27 \text{ R}$ make the sum $1 \frac{43}{135} \text{ R}$, equal to 2136 gold pieces, the merchant's money. Dividing 2136 by $1 \frac{43}{135}$ gives $1 \text{ R} = 1620$. So the first gain, $2/15 \text{ R}$ [printed here as $4/15 \text{ R}$], was 216 gold pieces, and the second, $5/27 \text{ R}$, was 300 gold pieces; which two gains, together with 1620 gold pieces, make the sum 2136 gold pieces.

[The first-gain coefficient is printed once as "4/30 R or 2/15 R" and once, in the recapitulation, as an ambiguous "4/15 R"; the value $216 = 2/15 \times 1620$ fixes it at 2/15 R. See QA — possible compositor slip in the recapitulation.]

Problem 23 [Margin: XXIII] — 23

Three golden apples are sold for 11 pennies (denarioli). It is asked how many apples are sold for 572 pennies.

This problem is worked by the rule of three in this way:

Pennies	golden apples	Pennies	golden apples
11	3	572?	156

By algebra, however, it will be solved if, for the fourth number to be found in the rule of three, 1 R of apples is set down. [Margin: ^a "Et 19. Sept." [?]] Since the product of the first into the fourth is as much as the product of the second into the third, the number 11 R produced from the first into the fourth will be equal to the number 1716 produced from the second into the third.

Pennies	golden apples	Pennies	golden apples
11	3	572?	1 R

Dividing 1716 by 11 gives 1 R = 156, the number of apples, which cost 572 pennies. You see, then, the excellence of algebra even in problems of the rule of three — which will also appear in the following example.

Problem 24 [Margin: XXIV] — 24

47 pounds (librae) of certain goods cost 30 gold pieces. It is asked, what will 100 pounds cost?

Set 1 R, as you see here:

lb.	gold	lb.	gold
47	30	100?	1 R

There is therefore an equation between 47 R and 3000. Dividing 3000 by 47 gives 1 R = $63 \frac{39}{47}$ gold pieces.

[The answer's fractional numerator is faint (reads 39 or 19 over 47); $3000 \div 47 = 63 \frac{39}{47}$ fixes it at 39/47. See QA.]

Problem 25 [Margin: XXV] — 25

There is a certain mass of silver weighing 7 marks, each of which contains 5 half-ounces (semiunciae) of pure silver and 11 half-ounces of copper. There is then mixed with that molten mass another mass of pure copper, weighing 21 marks. It is asked how much pure silver each mark of that new mass of 28 marks will contain.

Here it is certain that fewer half-ounces of pure silver are contained in the 28 marks than in the 7, since the half-ounces contained in the latter are distributed equally among the 28 marks. And because in those 7 marks are placed 35 half-ounces of pure silver, dividing these by 28 gives, in each mark, $1\frac{1}{4}$ half-ounces of pure silver. If, then, one mark weighs 16 half-ounces, there will be in each mark $14\frac{3}{4}$ half-ounces of copper. That this is so is easily proved. For the weight of 21 marks of pure copper will yield 336 half-ounces of copper; and if there are added the 77 half-ounces contained in the earlier 7 marks, there come to be 413 half-ounces of copper — which number is likewise produced from $14\frac{3}{4}$ (the number of half-ounces of copper we said were in each mark) taken into the 28 marks. But that this problem may be solved by algebra, the example will be set up according to the inverse rule of three.

mixed silver	pure silver	mixed silver	pure silver
7 marks	5 half-ounces	28 marks?	1 R half-ounces

And since, in the inverse rule of three, the number made from the first into the second is equal to the number made from the third into the fourth — because the third number ought to hold the first place in the ordinary rule of three, as we said in the *Arithmetica Practica* — there will be an equation between 35 and 28 R. Dividing 35 by 28 gives $1\text{ R} = 1\frac{1}{4}$ half-ounces of pure silver, as before.

Problem 26 [Margin: XXVI] — 26

Five messmates (convictores) pay 11 gold pieces each week. How much, then, will 18 messmates pay in 40 days?

This problem belongs to the compound rule of three, which is reduced to the simple rule of three by multiplying the messmates into the time, as here:

Messmates	Days	gold	Messmates	Days	gold
5	7	11	18	40?	1 R

reduced by multiplying the messmates into the time, as here:

Messmates	gold	Messmates	gold
35	11	720?	1 R

For one man in two days pays as much as two men in one; and one man in 35 days as much as five in seven. And this is the reason why the said multiplication ought to be made. Setting, then, 1 R as the fourth number, there will be an equation between 35 R and 7920. Dividing 7920 by 35 gives 1 R = 226 $\frac{2}{7}$ gold pieces, which is exactly what 18 messmates will pay in 40 days.

Problem 27 [Margin: XXVII] — 27

For carrying three equal weights over 7 leagues (leucae), 11 groats (grossi) are owed to the carrier, each of which is worth 25 pennies. I now ask, how many similar weights are to be carried over 40 leagues for 400 groats?

This problem too belongs to the compound rule of three, as you see here:

Groats	Weights	Leagues	Groats	Weights	Leagues
11	3	7	400?	1 R	40

Set 1 R for the weights to be carried, and reduce the rule to the ordinary rule of three by multiplying the weights into the distance travelled: for as much is to be paid for 1 weight over 2 leagues as for 2 weights over 1 league; and as much for 1 weight over 21 leagues as for 3 weights over 7 leagues. You see this reduction here:

Groats	Weights	Groats	Weights
11	21	400?	40 R

There will therefore be an equation between 440 R and 8400. Dividing 8400 by 440 gives 1 R = 19 $\frac{1}{11}$ weights to be carried over 40 leagues for 400 groats. For if for 3 weights over 7 leagues 11 groats are paid, then for 19 $\frac{1}{11}$ weights over 7 leagues 70 groats will be paid, as you see here:

Weights	Groats	Weights	Groats
3	11	19 $\frac{1}{11}$?	70

Then, if over 7 leagues 70 groats are paid, over 40 leagues 400 groats will be paid, as you see here:

Leagues	Groats	Leagues	Groats
7	70	40?	400

Problem 28 [Margin: XXVIII] — 28

Three tailors (sartores) finish 7 tunics in 14 days. In how many days, then, will two tailors finish 8 tunics?

Set 1 R for the days.

Tailors	Days	Tunics	Tailors	Days	Tunics
3	14	7	2	1 R	8

The numbers reduced by multiplication to the ordinary rule of three, as here (the products shown above: $3 \times 14 = 42$, and $2 \times 1 \text{ R} = 2 \text{ R}$):

42	7	2 R	8

There will be an equation between 336 and 14 R. Dividing 336 by 14 gives $1 \text{ R} = 24$ days, in which two tailors will finish 8 tunics. For if 3 tailors in 14 days finish 7 tunics, two tailors in 14 days — that is, in the same time — will finish $4 \frac{2}{3}$ tunics, as is seen here:

Tailors	Tunics	Tailors	Tunics
3	7	2?	$4 \frac{2}{3}$

Then, if $4 \frac{2}{3}$ tunics are completed in 14 days, 8 tunics will be completed in 24 days, as is here established:

Tunics	Days	Tunics	Days
$4 \frac{2}{3}$	14	8?	24

Problem 29 [Margin: XXIX] — 29

A certain citizen promised a servant, for 12 months, a wage of 10 gold pieces and 1 tunic. But after 7 months he gave him the tunic and 2 gold pieces. At what value, then, was that tunic reckoned?

Set its price to be 1 R gold pieces. And say: if 12 months demand $1 \text{ R} + 10$ gold pieces, how much does 1 month require? You will find $(1 \text{ R} + 10)/12$ gold pieces.

Again say: if 7 months demand 1 R + 2 gold pieces, how much does 1 month require? You will find $(1 \text{ R} + 2)/7$ gold pieces, as you see here:

Months	Gold pieces	Month	Gold pieces
12	1 R + 10	1?	$(1 \text{ R} + 10)/12$
7	1 R + 2	1?	$(1 \text{ R} + 2)/7$

There will therefore be an equation between $(1 \text{ R} + 10)/12$ and $(1 \text{ R} + 2)/7$, since each is the wage of 1 month. This equation, by cross-multiplication, is reduced to this one: between $7 \text{ R} + 70$ and $12 \text{ R} + 24$. Taking away 24 from both sides, the equation will be between $7 \text{ R} + 46$ and 12 R . And again taking away 7 R from both, between 46 and 5 R . Dividing 46 by 5 gives $1 \text{ R} = 9 \frac{1}{5}$ gold pieces, for the price of the tunic. For by this reckoning the wage of 12 months will be $19 \frac{1}{5}$ gold pieces — namely 10 gold pieces and the tunic. Therefore the wage of 7 months will be $11 \frac{1}{5}$ gold pieces, as you see here — namely 2 gold pieces and the price of the tunic, $9 \frac{1}{5}$ gold pieces.

Months	Gold	Months	Gold
12	$19 \frac{1}{5}$	7?	$11 \frac{1}{5}$

Algebra — Chapter XXXI (continued)

Riddles applied to material things

30. [Margin: XXX.] *A certain citizen fixes a wage for 30 days with his lazy servant in this way: he will give the man 7 groats for each day he works, but fine him 5 groats for each day he idles. When the reckoning is made after the 30 days, the servant neither receives anything nor pays anything back to his master. On how many days, then, did he work, and on how many did he stop working?*

Let 1R [res — the cossist unknown, x] be the number of days of labour, and therefore $30 - 1\text{R}$ the number of days of idleness. Set up the rule of three twice, in this way:

	Dies	grossi	Dies	grossi
Laboris	1	7	$1\text{R}?$	7R
Otiij	1	5	$30 - 1\text{R}?$	$150 - 5\text{R}$

Since, then, he earned exactly as much as he forfeited, there will be an equation between 7R and $150 - 5\text{R}$. Adding 5R to both sides, between 12R and 150. Dividing

150 by 12, then, 1R becomes $12\frac{1}{2}$ — the days of labour — and the days of idleness $17\frac{1}{2}$. This is proved as follows:

	Dies	grossos	Dies	grossos
Laboris	1	7	$12\frac{1}{2}?$	$87\frac{1}{2}$
Otij	1	5	$17\frac{1}{2}?$	$87\frac{1}{2}$

Here you see that the wage is exactly as great as the fine.

31. [Margin: XXXI.] *A certain merchant sold 20 pounds, partly of saffron and partly of ginger, for 45 gold pieces. He sold 1 pound of saffron for 3 gold pieces, and 1 pound of ginger for $\frac{1}{2}$ gold piece. The question is: how many pounds of saffron were sold, and how many of ginger?*

Let 1R be the number of pounds of saffron, and therefore $20 - 1R$ the number of pounds of ginger; and set up the rule of three twice, in this way:

	lib.	aur.	lib.	aur.
Croci	1	3	1R?	3R
Zinzib.	1	$\frac{1}{2}$	$20 - 1R?$	$(20 - 1R)/2$

The equation, therefore, is between the sum collected from 3R gold pieces and $(20 - 1R)/2$ gold pieces, and 45 gold pieces. Now that sum is $10 + 5/2R$ gold pieces. (For $(20 - 1R)/2$ is equivalent to the number $10 - \frac{1}{2}R$; and if 3R gold pieces be added to it, the sum becomes $10 + 5/2R$.) The equation, then, is between $10 + 5/2R$ gold pieces and 45 gold pieces. Taking away 10 gold pieces from both sides, between $5/2R$ and 35. Dividing 35, therefore, by $5/2$, 1R becomes 14. And that many pounds of saffron were sold, and consequently 6 pounds of ginger. This is proved thus:

	lib.	aur.	lib.	aur.
Croci	1	3	14?	42
Zinzib.	1	$\frac{1}{2}$	6?	3

Here you see that from the price of 14 pounds of saffron and 6 pounds of ginger, 45 gold pieces are gathered.

32. [Margin: XXXII.] *A certain trader has two kinds of coin, 560 in number, which are worth 160 gold pieces. Some of those coins are worth $\frac{1}{3}$ gold piece each, and each of the rest $\frac{1}{4}$ gold piece. The number of the former and of the latter is sought.*

Let 1R be the number of the former, and $560 - 1R$ of the latter; and set up the rule of three twice, in this way:

Monet.	aur.	Monet.	aur.
1	$\frac{1}{3}$	$1R?$	$\frac{1}{3}R$
1	$\frac{1}{4}$	$560 - 1R?$	$(560 - 1R)/4$

The numbers found in the fourth place are equivalent to 160 gold pieces. The sum of those numbers comes to $140 + \frac{1}{12}R$ gold pieces. (For $(560 - 1R)/4$ is equivalent to the number $140 - \frac{1}{4}R$; and if $\frac{1}{3}R$ be added, the sum becomes $140 + \frac{1}{12}R$.)

There is therefore an equation between $140 + \frac{1}{12}R$ gold pieces and 160 gold pieces. Taking away 140 from each side, between $\frac{1}{12}R$ and 20. Dividing 20, then, by $\frac{1}{12}$, $1R$ becomes 240. And that many coins are of the first kind, each worth $\frac{1}{3}$ gold piece; while of the second kind, each worth $\frac{1}{4}$ gold piece, there are 320. This is proved thus:

Monet.	aur.	Monet.	aur.
1	$\frac{1}{3}$	240?	80
1	$\frac{1}{4}$	320?	80

Here you discern that the numbers placed in the fourth position make 160 gold pieces.

33. [Margin: XXXIII.] *A certain butcher buys oxen. Asked how much he paid for one, he answered: "By as much as I bought 10 oxen too dear by 40 gold pieces, by that same amount I would have bought 18 oxen too dear by 96 gold pieces." The price is sought.*

This problem will be solved by the method of the rule of Algebra, without cossist numbers and without positing $1R$ — which may seem quite remarkable. We shall do the same in the next three problems that follow. [In what follows, "oxen" stands for the price of one ox, treated as the unknown.] Since, then, "10 oxen - 40 gold pieces" is the excess of the price of 10 oxen above 40 gold pieces (seeing that the butcher says he spent more than 40 gold pieces, which he ought not to have spent), and likewise "18 oxen - 96 gold pieces" is the excess of the price of 18 oxen above 96 gold pieces, which he ought not to have spent — and these two excesses are posited as equal — there will be an equation between "10 oxen - 40 gold pieces" and "18 oxen - 96 gold pieces." Adding 96 gold pieces to both sides, between "10 oxen + 56 gold pieces" and "18 oxen." (For from 96 added to -40 the sum +56 arises.) And removing 10 oxen from each side, between 56 gold pieces and 8 oxen. Dividing 56 by 8, then, the price of 1 ox becomes 7 gold pieces. This is proved thus: 10 oxen will be worth 70 gold pieces; and taking away the 40 he spent beyond the just price, 30 gold pieces remain for the price of 10 oxen.

Likewise 18 oxen will be worth 126 gold pieces; and deducting the 96 he ought not to have spent, there remain similarly 30 gold pieces for the price of 18 oxen.

If that butcher had said instead that he bought 10 oxen too dear by 40 gold pieces by as much as he could have bought 18 oxen too cheap by 96 gold pieces, there would arise an equation between "10 oxen – 40 gold pieces" and "96 gold pieces – 18 oxen" — because the former would be the excess of the price of 10 oxen above 40 gold pieces, while the latter is the excess of 96 gold pieces above the price of 18 oxen. Adding 18 oxen to both sides, then, there will be an equation between "28 oxen – 40 gold pieces" and 96 gold pieces. And adding a further 40 gold pieces to both sides, between 28 oxen and 136 gold pieces. Dividing 136 by 28, then, the price of 1 ox becomes $4\frac{6}{7}$ gold pieces. This is proved thus: 10 oxen will be worth $48\frac{4}{7}$ gold pieces; and taking away 40 gold pieces, there remain $8\frac{4}{7}$ gold pieces for the price of 10 oxen. Likewise 18 oxen will be worth $87\frac{3}{7}$ gold pieces; and these being taken from 96, there remain similarly $8\frac{4}{7}$ gold pieces for the price of 18 oxen.

Likewise, if you say: "For 4 groats – 3 eggs, [I bought] 41 eggs," there will be an equation between "4 groats – 3 eggs" and "41 eggs." Adding 3 eggs to both sides, between 4 groats and 44 eggs. Dividing 44 by 4, then, the quotient becomes 11; and that many eggs are bought for 1 groat.

Likewise, if you say: "4 groats – 3 denarii make 97 pennies," you will find that 1 groat makes 25 denarii.

Likewise, if someone says: "4 gold pieces + 125 groats make $10\frac{1}{4}$ gold pieces," then, taking away 4 gold pieces from both sides, there will be an equation between 125 groats and $6\frac{1}{4}$ gold pieces. Dividing $6\frac{1}{4}$ gold pieces by 125 groats, then, the quotient becomes $\frac{1}{20}$ gold piece — the value of 1 groat; that is, 1 groat makes $\frac{1}{20}$ of one gold piece. But if you divide 125 groats by $6\frac{1}{4}$ gold pieces, the quotient becomes 20; and that many groats make 1 gold piece.

34. [Margin: XXXIV.] *When 7 ells of a certain cloth are bought for $4\frac{1}{2}$ gold pieces, at the same price 17 ells are bought for 10 gold pieces and $18\frac{4}{7}$ groats. The question is: how many groats make 1 gold piece?*

Since the proportion of 7 ells to $4\frac{1}{2}$ gold pieces is the same as that of 17 ells to 10 gold pieces [and] $18\frac{4}{7}$ groats, we shall arrange these four proportional numbers in this order:

vlnæ	aur.	vlnæ	aur. groff.
7	$4\frac{1}{2}$	17?	10. $18\frac{4}{7}$

[Margin: a. 19. septimi (Euclid, *Elements* VII.19 — the products of the extremes and of the means of a proportion are equal).]

a Since as much arises from the first multiplied into the fourth as from the second into the third, there will be an equation between 70 gold pieces + 130 groats and $153\frac{1}{2}$ gold pieces, or $76\frac{1}{2}$ gold pieces. Taking away 70 gold pieces from both sides, between 130 groats and $6\frac{1}{2}$ gold pieces. Dividing 130, then, by $6\frac{1}{2}$, the quotient becomes 20; and that many groats are equivalent to 1 gold piece.

35. [Margin: XXXV.] *Someone receives saffron from a merchant for 10 gold pieces. Then again he received from the same merchant 24 pounds of saffron. Finally he returns to the merchant 30 pounds of saffron, and the merchant, having reckoned the price of the saffron, gives him back 14 gold pieces. The price of 1 pound of saffron is sought.*

Here you see that 10 gold pieces + 24 pounds is the whole debt which the buyer owes the merchant. Likewise 30 pounds – 14 gold pieces. There is therefore an equation between 10 gold pieces + 24 pounds and 30 pounds – 14 gold pieces. Adding 14 gold pieces to both sides, between 24 gold pieces + 24 pounds and 30 pounds. Taking away 24 pounds from each side, between 24 gold pieces and 6 pounds. Dividing 24 by 6, then, the quotient becomes 4; and that many gold pieces make the price of 1 pound. This is proved thus: for 10 gold pieces, $2\frac{1}{2}$ pounds are bought; and so the buyer received from the merchant $26\frac{1}{2}$ pounds, which are worth 106 gold pieces. If, therefore, 30 pounds be restored to the merchant, the merchant will owe the buyer $3\frac{1}{2}$ pounds, since the buyer has received only $26\frac{1}{2}$ pounds. And $3\frac{1}{2}$ pounds are worth 14 gold pieces, which the merchant gives back to the buyer.

36. [Margin: XXXVI.] *A certain citizen finds — he knows not how many — poor people before the door of his house. To them, out of the pennies he has in his hand, he gives seven pennies each. This done, 24 pennies remain in his hand. But if he had wished to give each 9, he would have been short by 32 pennies. It is asked how many poor people there were, and how many pennies the citizen had in his hand.*

Here you see that 7 sums of pennies — each of which is equal to the number of the poor — + 24 pennies, are equal to the whole number of pennies which the citizen had in his hand; because as many pennies as there are poor make one sum, and accordingly 7 pennies taken as many times as there are poor will constitute 7 sums. In the same way 9 sums of pennies, each equal to the same number of the poor, – 32 pennies, will be equal to the whole number of pennies which the citizen had in his hand. Therefore there will be an equation between "7

sums + 24" and "9 sums – 32." Adding 32 to both sides, between "7 sums + 56" and "9 sums." Taking away 7 sums from the 9, between 56 and 2 sums. Dividing 56 by 2, then, the quotient becomes 28 — the number of the poor. For if 7 pennies be given to each, 196 pennies will have been paid out; and since 24 pennies were then left over, the citizen had 220 pennies in his hand. But if he had wished to give each 9 pennies, he would have needed to have 252 pennies in his hand. He would therefore have been short by 32 pennies.

37. [Margin: XXXVII.] *For 70 parcels [summae] of pounds of certain goods a duty is paid of "1 parcel - 32 gold pieces." And for 200 parcels there is paid "1 parcel + 20 gold pieces." It is asked what 1 parcel is worth.*

Let 1R gold pieces be [the value of one parcel], and set up the rule of three twice, in this way:

Summæ	aur.	Summa	aur.
70	1R – 32	1?	(1R – 32)/70
200	1R + 20	1?	(1R + 20)/200

For if for 70 parcels a duty of 1R – 32 gold pieces is paid, for 1 parcel there will be paid (1R – 32)/70 gold pieces, and so on. And thus there will be an equality between (1R – 32)/70 gold pieces and (1R + 20)/200 gold pieces, since each number is the duty on 1 parcel. This equation, by cross-multiplication, is reduced to this one: between 200R – 6400 gold pieces and 70R + 1400 gold pieces. Adding 6400 to both sides, then, there will be an equation between 200R and 70R + 7800. And taking away 70R from each side, between 130R and 7800. Dividing 7800 by 130, then, 1R becomes 60 gold pieces, the price of one parcel. This is proved thus: for 70 parcels a duty of 28 gold pieces will be paid — namely 1R – 32 gold pieces; and for 200 parcels a duty of 80 gold pieces — namely 1R + 20 gold pieces — as appears here:

Summæ	aur.	Summæ	aur.
70	28	200?	80

38. [Margin: XXXVIII.] *Ten ells of red cloth together with 4 ells of black cloth cost 88 gold pieces; and at the same price 2 ells of red cloth together with 4 ells of black cloth cost 32 gold pieces. The question concerns the price of 1 ell.*

Let 1R be the price of 1 ell of red cloth. Then 10 ells [of red] will cost 10R; and accordingly 4 ells of black cloth will cost 88 gold pieces – 10R. But 2 ells of red cloth will cost 2R; and therefore 4 ells of black cloth [will cost] 32 gold pieces – 2R.

There will therefore be an equation between 88 gold pieces – 10R and 32 gold pieces – 2R, since each number is the price of 4 ells of black cloth. Adding 10R to both sides, then, there will be an equation between 88 gold pieces and 32 gold pieces + 8R. And taking away 32 gold pieces from each side, between 56 gold pieces and 8R. Dividing 56 by 8, then, 1R becomes 7 gold pieces, the price of one ell of red cloth. This is proved: for 10 ells [of red] will be worth 70 gold pieces, and therefore 4 ells of black cloth 18 gold pieces, which together with 70 make 88. Therefore 1 ell of black cloth will be worth $4\frac{1}{2}$ gold pieces. And at this price 2 ells of red cloth together with 4 ells of black cloth will be worth 32 gold pieces.

But if 10 ells of red cloth together with 4 ells of black cloth cost 88 gold pieces, and at the same price 2 ells of red cloth together with 6 ells of black cloth cost 41 gold pieces, then 4 ells of black cloth will cost 88 gold pieces – 10R, and 6 ells of black cloth 41 gold pieces – 2R — supposing, of course, that 1R is set for the price of 1 ell of red cloth, as before. But there will not be an equation between 88 gold pieces – 10R and 41 gold pieces – 2R, since the former number is the price of 4 ells of black cloth, but the latter of 6 ells. One must therefore first investigate the price of 4 ells from the price of 6 ells, by the rule of three, in this way:

Vlnæ	aur.	Vlnæ	aur.
6	41 – 2R	4?	$(164 - 8R)/6$

And the price of 4 ells will be found to be $(164 - 8R)/6$ gold pieces. And so, once this is found, there will be an equation between 88 gold pieces – 10R and $(164 - 8R)/6$ gold pieces. This, by cross-multiplication, is reduced to this one: between 528 gold pieces – 60R and 164 gold pieces – 8R. Adding 60R to both sides, between 528 gold pieces and 164 gold pieces + 52R. And taking away 164 gold pieces from each side, between 364 gold pieces and 52R. Dividing 364 by 52, then, 1R becomes 7 gold pieces, for the price of 1 ell of red cloth, just as before.

And if 10 ells of red cloth together with 4 ells of black cloth cost 88 gold pieces, as before, but 2 ells of red cloth together with 4 ells of black cloth were worth only 20 gold pieces, you would find $1R = 8\frac{1}{2}$ for the price of 1 ell of red cloth — which you can prove. For 10 ells of red cloth will be worth 85 gold pieces, and 4 ells of black cloth 3 gold pieces; and therefore 1 ell [of black cloth is worth] $\frac{3}{4}$ gold piece. And so 2 ells of red cloth will be worth 17 gold pieces, which together with 3 gold pieces — namely with the price of 4 ells [of black cloth] — make 20 gold pieces.

39. [Margin: XXXIX.] *A merchant, staking the same sum of gold pieces at three fairs, gained at each of them 1/8 of his sum of gold pieces. Then, betaking himself to other fairs, he gained 3/10 of that same sum of his, now increased by the earlier*

gain. And so he found that he had 1287 gold pieces. The sum of gold pieces which he had at the start is sought.

Let $1R$ be posited. And since at each fair he gained $1/8R$, he will have, after those three fairs, $1\frac{3}{8}R$ of gold — that is, $11/8R$. Then, since with $11/8R$ he gained $3/10R$ [that is, $3/10$ of it], namely $33/80R$, if this gain be added to $11/8R$, the sum becomes $1144/640R$, which he had in the last place. This is equal to 1287 gold pieces.

Dividing 1287, then, by $1144/640$, $1R$ becomes 720 — the sum of gold pieces of that merchant. For its $3/8$, which he gained at the three fairs, are 270 gold pieces, which together with 720 make 990 gold pieces; with which he finally gained $3/10$ — namely $2970/10$, that is, 297 gold pieces — which together with 990 make 1287 gold pieces.

If this problem were proposed in abstract numbers, it would run in this way: to find a number which, together with $3/8$ of itself, and with $3/10$ of this sum, makes 1287. For this number will be found to be 720, and so on.

40. [Margin: XL.] *In the Emperor's army the number of foot-soldiers is eight times greater than the number of cavalry. Among them 392000 gold pieces are distributed, so that 5 are given to each foot-soldier and 16 to each cavalryman. Find how many foot-soldiers there are, and how many cavalry.*

Let the cavalry be $1R$ [res, the unknown, = x], and therefore the foot-soldiers $8R$, and set up the double rule of three, as follows:

Eques	aur.	Equitum	aur.
1	16	$1R ?$	$16R$
Pedes: 1	5	$8R ?$	$40R$

(That is: one cavalryman [*Eques*] gets 16 gold pieces, so $1R$ cavalry [*Equitum*] get $16R$; one foot-soldier [*Pedes*] gets 5, so $8R$ foot-soldiers get $40R$.)

Therefore $56R$ gold pieces equal 392000 gold pieces. So, dividing 392000 by 56, $1R = 7000$, the number of cavalry; and so the foot-soldiers were 56000, eight times as many. For distributed in this way, 112000 gold pieces go to the cavalry and 280000 to the foot-soldiers, which together make 392000 gold pieces.

41. [Margin: XLI.] *Three soldiers wanted to divide some booty of gold pieces equally among themselves. But a quarrel arose, as happens, they came to blows, and each seized as much as he could by force. Then, once the quarrel was settled, each counted his money, and at length they found: the first, on receiving 5 from the second, had a number equal to the number of gold pieces left over to the second; the*

second, on receiving 7 from the third, had a number equal to the third's remainder; and finally the third, on receiving 9 from the first, had a number three times what was left to the first. Find the total of gold pieces found in that booty, and how much each seized.

Suppose the first seized $1R$. On receiving 5 from the second, then, he had $1R + 5$, and this number is said to equal the second's remainder. So before the second gave 5 to the first, he had $1R + 10$. On receiving 7 from the third, he had $1R + 17$, and this was just as much as the third's remainder. Therefore before the third gave 7 to the second, he had $1R + 24$. And on receiving 9 from the first, he had $1R + 33$, which number is said to be three times what remains to the first — that is, $1R - 9$. Therefore three times this, $3R - 27$, equals $1R + 33$. Adding 27 to both sides, the equation is between $1R + 60$ and $3R$. Removing $1R$ from both sides, between 60 and $2R$. Dividing 60 by 2, then, $1R = 30$, the number of the first.

Now the number of the second, which was found to be $1R + 10$, was 40. And the third, which was found to be $1R + 24$, had 54; and the total of gold pieces was 124 — which you can verify. For the first, on receiving 5 from the second, will have 35, as many as are left over to the second. And the second, on receiving 7 from the third, will have 47, as many as are left over to the third. Finally the third, on receiving 9 from the first, will have 63, which number is three times the first's remainder, namely 21. Now if the 124 gold pieces are divided into three equal parts, each soldier will get $41 \frac{1}{3}$ gold pieces.

42. [Margin: XLII.] *Four companions found a purse full of gold pieces, of which each took a part by chance. But when each counted his own gold pieces, they discovered: the first, on receiving 25 from the second, had a number equal to the second's remainder; the second, on receiving 30 from the third, had three times the third's remainder; the third, on receiving 40 from the fourth, had a number twice the fourth's remainder; and finally the fourth, on receiving 50 from the first, had a number three times the first's remainder, plus 5 more. Find the total of gold pieces, and how much each took.*

Suppose the first took $1R$. On receiving 25 from the second, then, he will have $1R + 25$; and just as much will be left over to the second, so that the second, before he gave 25 to the first, had $1R + 50$. On receiving 30 from the third, [the second] will have $1R + 80$. There remained, therefore, to the third $\frac{1}{3}R + 26 \frac{2}{3}$, so that that number [$1R + 80$] is three times this; and accordingly the third, before he gave 30 to the second, had $\frac{1}{3}R + 56 \frac{2}{3}$. On receiving 40 from the fourth, [the third] will have $\frac{1}{6}R + 96 \frac{2}{3}$, which number must be twice the fourth's remainder; and accordingly the fourth then had left $\frac{1}{6}R + 48 \frac{1}{3}$. And before he gave 40 to the

third, he had $\frac{1}{6}R + 88\frac{1}{3}$. This one, finally, on receiving 50 from the first, will have $\frac{1}{6}R + 138\frac{1}{3}$, which number, less 5, must be three times the number $1R - 50$ that is left over to the first. Therefore three times this, $3R - 150$, equals $\frac{1}{6}R + 133\frac{1}{3}$. Adding 150 to both sides, then, there is an equality between $3R$ and $\frac{1}{6}R + 283\frac{1}{3}$. Removing $\frac{1}{6}R$ from both sides, between $2\frac{5}{6}R$ and $283\frac{1}{3}$. Dividing $283\frac{1}{3}$ by $2\frac{5}{6}$, then, $1R = 100$, the number of the first.

Amount	Expression
100	$1R$
150	$1R + 50$
90	$\frac{1}{3}R + 56\frac{2}{3}$
105	$\frac{1}{6}R + 88\frac{1}{3}$

Therefore the second, $1R + 50$, will be 150; the third, $\frac{1}{3}R + 56\frac{2}{3}$, will be 90; and finally the fourth, $\frac{1}{6}R + 88\frac{1}{3}$, will be 105. For in this way the first, with the 25 of the second, will have 125, and just that much also remains to the second; the second, with the 30 of the third, will have 180, namely three times the number 60 that is left over to the third; the third, in turn, with the 40 of the fourth, will have 130, which is twice the number 65 that is left over to the fourth; and finally the fourth, with the 50 of the first, will have 155, which is three times the number 50 that is left over to the first [plus the 5].

43. [Margin: XLIII.] *In an imperial army made up of Germans, Hungarians, and Italians, there are 25000 Germans. The Hungarians make up $\frac{1}{2}$ of the Germans and the Italians together, and finally the Italians make up $\frac{1}{8}$ of the Germans and the Hungarians together. Find the number of Hungarians and of Italians, and how large the whole army is.*

Let the Hungarians be $1R$. The Germans together with the Italians will therefore be $2R$, and the whole army $3R$. Now since the Italians make up $\frac{1}{8}$ of the Germans together with the Hungarians, and the Hungarians with the Germans are $1R + 25000$, the Italians will be $\frac{1}{8}R + 3125$.

Since, then, the Germans are 25000, the Hungarians $1R$, and the Italians $\frac{1}{8}R + 3125$, the whole army will be $1\frac{1}{8}R + 28125$, equal to $3R$. Removing $1\frac{1}{8}R$ from both sides, the equation will be between 28125 and $1\frac{7}{8}R$. Dividing 28125 by $1\frac{7}{8}$, $1R = 15000$, the number of Hungarians. The Italians, for their part, will be 5000 — namely $\frac{1}{8}$ of the Germans and the Hungarians, which together make 40000 — and the whole army will be 45000.

44. [Margin: XLIV.] *There are two commanders, one of whom has 40 fewer soldiers than the other. Each distributes 1200 gold pieces among his own soldiers, and it turns out that each soldier of the first commander gets 5 gold pieces more than each soldier of the second commander. The question is how many soldiers each commander has, and how many gold pieces each soldier receives.*

Let $1R$ be the smaller number of soldiers, and $1R + 40$ the larger. If 1200 gold pieces are divided among these soldiers, the quotients will be $1200 / 1R$ and $1200 / (1R + 40)$ — namely the numbers of gold pieces that fell to each. And since each of the former group gets 5 more than each of the latter, if 5 is added to the latter quotient, the sum becomes $(5R + 1400) / (1R + 40)$ — as you see in this formula — equal to the former quotient, $1200 / 1R$:

$$\frac{1200}{1R + 40} + \frac{5}{1} \rightarrow \frac{1200 + (5R + 200)}{1R + 40} = \frac{5R + 1400}{1R + 40}$$

(the printed formula shows the numerators 1200 and $5R + 200 [= 5 \times (1R + 40)]$ over the common denominator $1R + 40$).

This equation, by cross-multiplication, is reduced to one between $1200R + 48000$ and 53 [census, the square, $= x^2$] + $1400R$. Removing $1200R$ from both sides gives one between 48000 and $53 + 200R$. Again removing $200R$ from both sides, between $48000 - 200R$ and 53 ; and dividing by 5, between $9600 - 40R$ and 13 . Half the number of roots is 20; to its square, 400, add 9600, giving the number 10000; from its square root, 100, subtract the said half, 20, and 80 remains as the value of $1R$ — namely the smaller number of soldiers. Therefore the larger will be 120, that is, 40 more than 80 — which is verified. For dividing 1200 by 80 gives quotient 15, and each of the 80 soldiers receives that many gold pieces; and dividing 1200 by 120 gives quotient 10, the number of gold pieces that each of the 120 soldiers receives. Here you see that each of the former group has 5 more than each of the latter.

45. [Margin: XLV.] *Two companies, one of which exceeds the other by 4 men, divide an equal sum of gold pieces man by man. To each member of the smaller company there fall 8 gold pieces more than to each member of the larger. And the total of gold pieces exceeds the number of members of both companies together by 172 gold pieces. Find the total of gold pieces and of members.*

Let there be $1R$ men in the smaller company, and therefore $1R + 4$ in the larger. The total of gold pieces will be $2R + 176$ [= the combined membership $2R + 4$, plus 172]. This total, divided by $1R$ and by $1R + 4$, gives the quotients $(2R + 176) / 1R$ and $(2R + 176) / (1R + 4)$ — namely the numbers of gold pieces that fall to each

man. And since the former quotient exceeds the latter by 8 gold pieces, if 8 gold pieces are added to the latter, the sum becomes $(10R + 208) / (1R + 4)$ — as you see in this formula — equal to the former quotient, $(2R + 176) / 1R$:

$$\frac{2R + 176}{1R + 4} + \frac{8}{1} \rightarrow \frac{(2R + 176) + (8R + 32)}{1R + 4} = \frac{10R + 208}{1R + 4}$$

(the printed formula shows the numerators $2R + 176$ and $8R + 32 [= 8 \times (1R + 4)]$ over the common denominator $1R + 4$).

This equation, by cross-multiplication, is reduced to one between $103 + 208R$ and $23 + 184R + 704$. Removing 23 from both sides, the equation will be between $83 + 208R$ and $184R + 704$. Again removing $184R$ from both sides, between $83 + 24R$ and 704 . Again removing $24R$ from both sides, between 83 and $704 - 24R$. And dividing everything by 8 , between 13 and $88 - 3R$. Half the number of roots is $3/2$; to its square, $9/4$, add 88 (that is, $352/4$), giving the number $361/4$; from its square root, $19/2$, subtract the said half, $3/2$, and there remains as the value of $1R = 16/2$, that is, 8 , the number of the smaller company. The number of members of the larger will be 12 , namely $1R + 4$. And finally the total of gold pieces will be 192 , namely $2R + 176$ — which is verified. For dividing the 192 gold pieces by 8 and by 12 gives the quotients 24 and 16 , of which the former exceeds the latter by 8 gold pieces.

46. [Margin: XLVI.] *A certain merchant, going to the fair three times, gained in his first dealing a sum equal to his own money, so that after the first dealing he had double his money. Then in the second dealing he gained the square root of that double, and 2 gold pieces besides. Finally in the third dealing he gained the square of the whole previous sum, and 4 gold pieces on top. And so he found that he had 510 gold pieces. Find the sum of money he had at the start.*

Suppose that after the second dealing he had $1R$, with which, in the third dealing, he gained its square and 4 gold pieces besides, so that this gain was $13 + 4$. Therefore after the third dealing he had the sum $13 + 1R + 4$, equal to 510 gold pieces. Removing 4 from both sides, the equation will be between $13 + 1R$ and 506 . Removing $1R$ again from both sides, between 13 and $506 - 1R$. Half the number of roots is $1/2$; to its square, $1/4$, add 506 , giving the number $506 \frac{1}{4}$, that is, $2025/4$; from its square root, $45/2$, take away the said half, $1/2$, and there remains as the value of $1R = 44/2$, that is, 22 . And that many gold pieces he had after the second dealing.

And since this [total] was made up of the square root of the money he had after the first dealing, plus 2 gold pieces, subtracting the 2 leaves 20 — namely the

money together with the root of the money he had after the first dealing. Therefore let that money be set as 13, whose square root is 1R. And after the second dealing [less the 2] the amount will be $13 + 1R$, equal to 20. Removing 1R from both sides, the equation will be between 13 and $20 - 1R$. Half the number of roots is $1/2$; to its square, $1/4$, add 20 (that is, $80/4$), giving the number $81/4$; from its square root, $9/2$, take away the said half, $1/2$, and there remains as the value of $1R = 8/2$, that is, 4. And that much was the gain of the second dealing, plus 2. Removing 4 and 2, then, from 22 (namely from the money he had after the second dealing), 16 remains; and that much he had after the first dealing — which money is said to be double what he brought to the fair. And accordingly he had 8 gold pieces at the start — which is verified. For after the first dealing he had 16 gold pieces, and after the second 22, namely the root of the number 16 (and 2 besides), that is, 6, which with 16 makes 22. With these he gained the square, namely 484, and 4 besides; and from 22, 484, and 4 the total of 510 gold pieces is gathered.

47. [Margin: XLVII.] *It seemed good at this point to explain, by the rule of algebra, the device by which Archimedes detected the quantity of silver mixed into the golden crown.*

Take, then, two masses — one of gold, the other of silver — of equal weight with the crown. Then let a bronze vessel, skilfully made, be filled with water; and when the crown of 100 pounds is put in, suppose 65 pounds of water overflow. Then, having refilled the vessel with water, suppose that a mass of pure gold of 100 pounds, put in, drives out 60 pounds of water. Finally, having again refilled the vessel with water, suppose that a mass of pure silver, put in, drives out 90 pounds of water. Let 1R pounds of silver be assumed to be contained in the crown, and therefore $100 - 1R$ pounds of gold; and set up the double rule of three, as follows:

	lib.	aqu æ	lib.	lib.
Auri (gold)	100 0	60	$100 - 1R ?$	$(6000 - 60R) / 100$
Argenti (silver)	100 0	90	$1R ?$	$90R / 100$

For you will find that $100 - 1R$ pounds of gold drive out $(6000 - 60R) / 100$ pounds of water, since 100 pounds of gold drive out 60; and that silver drives out $90R / 100$ pounds of water, since 100 pounds of silver drive out 90 pounds of water. (For since the fractions $(6000 - 60R) / 100$ and $90R / 100$ have the same

denominator, it is enough to add the numerators and give the resulting sum, 6000 + 30R, the same denominator 100.) These equal the 65 pounds which the crown drives out. This equation, between $(6000 + 30R) / 100$ and 65, is reduced by cross-multiplication to one between $6000 + 30R$ and 6500. Removing 6000 from both sides, the equation will be between 30R and 500. Dividing 500 by 30, then, $1R = 16 \frac{2}{3}$; and that many pounds of silver were mixed into the crown. Therefore only $83 \frac{1}{3}$ pounds of gold are found in it — which is verified, as you see here:

	lib.	aquæ	lib.	aquæ
Argenti (silver)	100	90	$16 \frac{2}{3} ?$	15
Auri (gold)	100	60	$83 \frac{1}{3} ?$	50

For you will find that $16 \frac{2}{3}$ pounds of silver drive out 15 pounds of water, and $83 \frac{1}{3}$ pounds of gold drive out 50 pounds of water, which with the 15 make 65 — as many pounds of water as flow out when the crown is put into the water.

SCHOLIUM

This same problem will be solved in the same way. There are three masses of the same weight, namely 10 pounds each: one of pure gold, another of pure silver, and a third mixed of gold and silver. The first drives out $\frac{1}{30}$ pound of water, the second $\frac{3}{4}$ pound of water, and the third $\frac{1}{8}$ pound of water. [numeral: ?] For the double rule of three will be set up as follows, if 1R pounds of silver be assumed in the third mass, and therefore $10 - 1R$ pounds of gold:

		aquæ		aquæ
Auri (gold)	lib. 10	lib. $\frac{1}{30}$	10 – 1R lib. of gold ?	lib. $(10 - 1R) / 300$
Argenti (silver)	lib. 10	lib. $\frac{3}{4}$	1R lib. of silver	lib. $3R / 40$

For the two numbers in the fourth place will make the sum $(400 + 860R) / 12000$ pounds of water, equal to $\frac{1}{8}$ pound of water; and this equation, by cross-multiplication, will be reduced to one between 12000 and $3200 + 6880R$. Removing 3200 from both sides, the equation will be between 8800 and 6880R. Dividing 8800 by 6880, then, $1R = 1 \frac{12}{43}$; and that many pounds of silver are in the third mass, and accordingly there will be in it $8 \frac{31}{43}$ pounds of gold — which is verified, as you see here:

		aq.		aq.
Argenti (silver)	lib. 10	lib. 3/4	arg. lib. 1 12/43 ?	lib. 165 / 1720
Auri (gold)	lib. 10	lib. 1/30	aur. lib. 8 31/43 ?	lib. 375 / 12900

You will find that 1 12/43 pounds of silver drive out 165/1720 pound of water, and 8 31/43 pounds of gold drive out 375/12900 pound of water, which together make 1/8 pound of water.

But if anyone should wish for a more accurate method of determining the quantity of gold and silver in that crown, or in the third mass just mentioned, let him read the very learned little book of Marino Ghetaldi [Marin Getaldić], a patrician of Ragusa, which is entitled *Promotus Archimedes*. There, in Proposition 17, he lists the various causes of the errors that can be committed in Archimedes' aforesaid device; and then, in Proposition 18, he demonstrates a more accurate device.

48. [Margin: XLVIII.] *Three merchants gained 700 gold pieces, which they distributed among themselves in proportion to the money each had brought to the venture, so that the second's share exceeded the first's by 12 gold pieces, and the third's share exceeded the second's by 16 gold pieces. Find how large each one's share was.*

Let the first's share be $1R$; the second's share will therefore be $1R + 12$, and the third's $1R + 28$. These three shares together make $3R + 40$, equal to 700. Removing 40 from both sides, the equation will be between $3R$ and 660. Dividing, then, [the solution continues]

(Cossist notation in this section: the res sign — the unknown quantity, x — is rendered R ; the census sign — the square of the unknown, x^2 — is rendered $\mathfrak{z}$. Constants are given as plain numerals, as printed.)

[Conclusion of Problem 48]

and so 660 divided by 3 gives $1R$ [res, the unknown] = 220, the share of the first. The share of the second will be 232, and of the third 248 — which all together make 700.

49 [Margin: XLIX]

Three gamblers, each of whom had brought a third part of the sum of gold coins, had this run of luck: right away the first won $\frac{1}{2}$ of the second's; then the second $\frac{1}{3}$ of

the third's; and finally the third $\frac{1}{4}$ of the first's. And so it turned out that each ended with 700 gold. It is asked how many gold coins each brought to the game.

Let the first's money be $1R$. If he gives $\frac{1}{4}R$ to the third, $\frac{3}{4}R$ is left to him; this, together with $\frac{1}{2}$ of the second's, must make 700. Therefore $\frac{1}{2}$ of the second is $700 - \frac{3}{4}R$, and hence he brought double that, namely $1400 - 1\frac{1}{2}R$. For after the second loses $\frac{1}{2}$, there remains to him $700 - \frac{3}{4}R$; and this number, together with $\frac{1}{3}$ of the third's, must make 700. Therefore $\frac{1}{3}$ of the third is $\frac{3}{4}R$, and so he brought to the game $2\frac{1}{4}R$, or $\frac{9}{4}R$. If the third loses $\frac{1}{3}$ of this, $\frac{6}{4}R$ remains to him, which together with $\frac{1}{4}$ of the first's — namely with $\frac{1}{4}R$ — will give him $\frac{7}{4}R$, equal to 700.

So dividing 700 by $\frac{7}{4}$ gives $1R = 400$: this is the money of the first. Accordingly the second, whom we found to have brought $1400 - 1\frac{1}{2}R$, brought to the game 800 gold; and the third 900 gold, namely $2\frac{1}{4}R$.

This is verified. For if the first loses $\frac{1}{4}$, 300 gold remain, which together with $\frac{1}{2}$ of the 800 gold the second brought make 700 gold. Likewise $\frac{1}{2}$ of the second's, namely 400 gold, together with $\frac{1}{3}$ of the third's — that is, with 300 — likewise makes 700 gold. And finally $\frac{2}{3}$ of the third's — that is, 600 gold, after it has lost $\frac{1}{3}$ — together with $\frac{1}{4}$ of the first's, that is 100, again makes 700 gold.

50 [Margin: L]

Someone has two golden vessels and one lid, whose price is 150 gold. When this lid is added to the price of the first vessel, it makes a price triple the price of the second vessel. But the same lid added to the second vessel makes a price equal to the price of the first vessel. The question concerns the price of each vessel.

Let the price of the first be $1R$. Then $1R + 150$ is triple the price of the second vessel, and so the second's value is $\frac{1}{3}R + 50$ gold. If the lid of 150 gold is added to it, its price will then be $\frac{1}{3}R + 200$ gold, equal to $1R$, namely the price of the first vessel. Taking away $\frac{1}{3}R$ from each side, the equation is between $\frac{2}{3}R$ and 200. So dividing 200 by $\frac{2}{3}$ gives $1R = 300$ gold, that is, the price of the first vessel. The price of the second vessel, which we found to be $\frac{1}{3}R + 50$, will be 150 gold.

This is verified. For 300 gold — that is, the price of the first vessel — together with the price of the lid, that is, with 150, make the sum 450, triple the price of the second vessel, namely 150 gold. And 150 gold, namely the price of the second vessel, together with the lid, or with 150 gold, make the sum 300 gold, equal to the price of the first vessel.

51 [Margin: LI]

Someone buys — I do not know how many — partridges, such that if he had bought separately $\frac{1}{3}$ and $\frac{1}{4}$ and $\frac{1}{5}$ of them, and 6 more besides, he would have 100 partridges. The number of partridges is sought.

Let there be $1R$, whose $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$ make $\frac{47}{60}R$; these together with 6 make $\frac{47}{60}R + 6$, equal to 100. Taking away 6 from both sides, the equation is between $\frac{47}{60}R$ and 94. So dividing 94 by $\frac{47}{60}$ gives $1R = 120$, namely the number of partridges bought. For its $\frac{1}{3}$ is 40, and $\frac{1}{4}$ is 30, and $\frac{1}{5}$ is 24, which parts together with 6 make the sum of 100 partridges.

52 [Margin: LII]

Alexander the Great [Alexander Magnus], one day conversing familiarly with the philosopher Callisthenes [Calisthenes] about his own age and that of his friends Hephaestion [Ephestio] and Clitus [Clytus], argued as follows. "I," he said, "am two years older than my Hephaestion. But Clitus takes in, within his own age, the years of both, and four years besides." To which Callisthenes replied: "Since my father lived 96 years, this account has been pleasing to me, O King. For his age held precisely the years of the three of you." It is asked at what age Alexander held this conversation, and how many years both Hephaestion and Clitus then had.

Let Hephaestion's years be $1R$. Then Alexander's years were $1R + 2$, and Clitus's $2R + 6$; and all of these make $4R + 8$, equal to 96. Taking away 8 from both sides, the equation is between $4R$ and 88. So dividing 88 by 4 gives $1R = 22$, namely Hephaestion's years. Alexander therefore had 24 years, and Clitus 50 — which years, gathered into one sum, make 96 years.

53 [Margin: LIII]

Nicanor, in the battle in which he perished, met Judas Maccabeus [Iuda Machabaeo] with an army drawn up in a square, made up of Syrian auxiliary soldiers and of those he had brought with him. Now 35,000 were slain from that army. The rest escaped by flight, and their number was 156, beyond the number of the Syrian auxiliary soldiers. The question is: with how many soldiers did Nicanor meet Maccabeus, and how many auxiliary soldiers did he have, and how many likewise did he bring with him.

Let that square army be 13 [census, the square of the unknown], in which the slain soldiers, 35,000, are counted. Likewise the number of those saved by flight, 156, and the number of the auxiliary soldiers, is unknown. Let this number be $1R$.

If to it are added the slain soldiers, 35,000, and the 156 saved by flight, the sum $1R + 35156$ is formed, equal to 13, namely the whole square army. Half of the number of the roots is $\frac{1}{2}$; the square of this is $\frac{1}{4}$; adding 35156 — that is, $140624/4$ — the number $140625/4$ is produced, whose square root is $375/2$. If the aforesaid half of the number of the roots, namely $\frac{1}{2}$, is added, $1R$ becomes $376/2$, that is, 188, namely the number of the auxiliary soldiers. He himself brought with him 35156; and so the whole square army comprises 35344, which number is a square having 188 as its root.

54 [Margin: LIV]

Someone has 4 masses mixed from silver and copper. The weight of the first contains 11 marks, each of which is made up of 9 half-ounces of pure silver and 7 half-ounces of pure copper — since we take 1 mark to comprise 16 half-ounces. The weight of the second mass is 15 marks, each of which contains 7 half-ounces of silver and 9 half-ounces of copper. The weight of the third mass has 24 marks, each of which consists of 10 half-ounces of silver and 6 half-ounces of copper. The weight of the fourth mass, finally, is 136 marks, each of which contains 14 half-ounces of silver and 2 half-ounces of copper. He wishes to make a single mass out of these, by adding some marks of silver, in such a way that each mark contains 15 half-ounces of pure silver and 1 half-ounce of pure copper. The question now is how much pure silver must be mixed into those masses, and how many marks the new mass will have.

Let there be $1R$ marks of pure silver to be added. Since all the mixed marks in the 4 masses are 186, and all the half-ounces of pure silver are 2348, and of copper 628 — as you see here:

Marcae mixtae [mixed marks]	Semiunciae puri arg. [half-oz. pure silver]	Semiunc. puri cupri [half-oz. pure copper]
11	99	77
15	105	135
24	240	144
136	1904	272
186	2348	628

For since the first mass is of 11 marks, each mark containing 9 half-ounces of pure silver and 7 half-ounces of copper, there will be in that whole mass 99 half-ounces of pure silver and 77 of pure copper, and so on. So the rule of three must be set up in this way:

Marcae mixtae +

Mar. pu. arg.

[mixed marks +
marks pure silver]

Semiunc. cupri
[half-oz. copper]

Mar. mix. [mixed
mark]

Semiun. cupri
[half-oz. copper]

186 + 1R

628

1?

628 / (186 + 1R)

That is: the mixed marks, 186 + 1R, contain 628 half-ounces of copper (as many as are contained in the 4 composed masses, since no new half-ounce of copper is added); how many half-ounces of copper, then, will 1 new mixed mark contain? You will find $628/(186 + 1R)$ half-ounces of copper, which fraction is equivalent to 1 half-ounce of copper — because by hypothesis just 1 half-ounce of copper is contained in one new mixed mark. This equation, reduced by cross-multiplication, comes to this: between 628 and 186 + 1R. Taking away 186 from each side, the equation will be between 442 and 1R. Therefore $1R = 442$; and that many marks of pure silver must be mixed in, so that each mark comprises 15 half-ounces of pure silver and 1 half-ounce of copper.

This is proved thus. Since in the 4 given masses there are 186 mixed marks, if the 442 just found are added, there come to be 628 mixed marks; and so each will have 1 half-ounce of copper, since in all of them together there are 628 half-ounces of copper.

And since one mark is set at 16 half-ounces, 442 marks of pure silver will comprise 7072 half-ounces. If to these are added the 2348 half-ounces of pure silver found in the 4 given masses, there come to be 9420 half-ounces of pure silver; and these, distributed among the 628 marks of the newly made mass, give each mark 15 half-ounces of pure silver.

This same question can also be solved easily without the aid of Algebra. For since in the 4 given masses there are 628 half-ounces of copper, the new mass to be made must be of just that many marks, so that 1 half-ounce of copper can be assigned to each. So if the 186 mixed marks that the 4 masses contain are subtracted from 628, there remain 442 marks of pure silver to be added to the 186 marks, so as to make the number of marks 628 comprised in the newly made mass.

But if the mass is to be made from the 4 given ones with silver added, such that each mark contains 13 half-ounces of pure silver and 3 half-ounces of copper: dividing the 628 half-ounces of copper by 3 gives the number of marks in the new mass, $209\frac{1}{3}$. For in this way each mark will contain 3 half-ounces of copper. So if from $209\frac{1}{3}$ the 186 marks contained in the first 4 masses are taken away, the

remaining number will be $23\frac{1}{3}$ marks of pure silver to be added; and there will be formed the number $209\frac{1}{3}$ of marks comprised in the new mass, each of which will comprise 13 half-ounces of pure silver. For in $209\frac{1}{3}$ marks there are $10048/3$ half-ounces; if from these 628 are taken away — that is, $1884/3$ — half-ounces of copper, there remain $8164/3$ half-ounces of pure silver; and these, distributed among the $209\frac{1}{3}$ marks, give each mark 13 half-ounces of pure silver.

But if the mass is to be made from the same 4 masses so that each mark contains 12 half-ounces of pure silver and 4 half-ounces of pure copper: then copper must be added, not silver. For if the 2348 half-ounces of silver that are in those 4 masses are distributed among 186 marks, there will be found in each mark more half-ounces of silver than 12 — namely $12\frac{116}{186}$. So dividing the 2348 half-ounces of pure silver by 12 gives in the quotient the number of marks in the new mass, $195\frac{2}{3}$. For thus each mark will contain 12 half-ounces of pure silver. And when the 186 marks that the 4 masses embrace are taken away, there remain $9\frac{2}{3}$ marks of copper to be added, so that a new mass of $195\frac{2}{3}$ marks may be formed, each of which will contain 4 half-ounces of copper. For in $195\frac{2}{3}$ marks there are $9392/3$ half-ounces; from these, if 2348 half-ounces of silver are taken away — that is, $7044/3$ — there are left $2348/3$ half-ounces of copper, which, distributed among the $195\frac{2}{3}$ marks, give each mark 4 half-ounces of copper.

Now if it is asked how much copper must be added to those 4 masses, so that in each mark of the newly made mass there are found 15 half-ounces of pure copper and 1 of silver: the rule of three must be set up in this way, letting 1R be the number of marks of pure copper.

Marcae mix. +

Marcae pu. cup.

[mixed marks +

marks pure
copper]

Semiun. arg. [half-
oz. silver]

Mar. mix. [mixed
mark]

Sem. arg. pu. [half-
oz. pure silver]

186 + 1R

2348

1?

2348 / (186 + 1R)

And the fourth number will be equal to 1 half-ounce of pure silver. This equality, reduced by cross-multiplication, comes to this: between 2348 and $186 + 1R$.

Taking away 186 from each side, the equation is between 2162 and 1R. Therefore $1R = 2162$, namely the number of marks of copper to be mixed in, so that each mark contains 15 half-ounces of copper and 1 of silver. For since in those 4 masses there are 186 marks, adding the 2162 just found makes the number of marks 2348; and so each will have 1 half-ounce of silver, since in all of them

together there are 2348 half-ounces of silver. And since 2162 marks of pure copper comprise 34592 half-ounces of copper, if the 628 half-ounces of copper contained in the four earlier masses are added, there come to be 35220 half-ounces; and these, distributed among the 2348 marks of the newly made mass, give each mark 15 half-ounces of copper.

But if the question is how much silver 1 mark contains, if the 4 masses are made into one and nothing is added to them: the rule of three will stand thus.

Marcae [marks]	Semiunc. arg. [half-oz. silver]	Marca [mark]
186	2348	1?

Therefore 1 mark will contain $12 \frac{58}{93}$ half-ounces of silver, and accordingly $3 \frac{35}{93}$ half-ounces of copper.

Likewise, if it is asked how much copper 1 mark contains, the rule will stand thus.

Marcae [marks]	Semiunc. cupri [half-oz. copper]	Marca [mark]
186	628	1?

For 1 mark will contain $3 \frac{35}{93}$ half-ounces of pure copper, and therefore $12 \frac{58}{93}$ half-ounces of pure silver.

And if the question is: a mass, of what weight, containing in each mark 3 half-ounces of silver and 13 of copper, is to be mixed with those 4 masses, so that each mark of the newly made mass contains 5 half-ounces of silver and 11 of copper — let 1R be put for the number of marks to be added, and 3R for the number of half-ounces of silver contained in that mass, since the number of half-ounces in that mass is triple the number of marks, seeing that each mark is said to comprise 3 half-ounces of silver. So the rule of three will stand thus:

Mar. mix. + Mar. mix. novae [mixed marks + new mixed marks]	Semiunc. arg. + Sem. arg. novae [half-oz. silver + new half-oz. silver]	Marca nova [new mark]	Sem. arg. [half-oz. silver]
186 + 1R	2348 + 3R	1?	$(2348 + 3R) / (186 + 1R)$

For the fourth number will be equal to 5 half-ounces of silver — as many as 1 mark of the new mass must include. This equality, reduced by cross-multiplication, comes to this: between $2348 + 3R$ and $930 + 5R$. Taking away 930 from each side, the equation is between $1418 + 3R$ and $5R$. Taking away $3R$ again

from both sides, it is between 1418 and 2R. So dividing 1418 by 2 gives 1R = 709, the number of marks that the fifth mass to be mixed in will contain. Adding the 186 marks of the four earlier masses, there comes to be a new mass of 895 marks, each of which contains 5 half-ounces of silver and 11 of copper.

This is proved thus. Multiply the 895 marks by 16 half-ounces, and you make 14320 half-ounces, partly of silver and partly of copper, contained in the new mass of 895 marks. Then multiply the same 895 marks by the 5 half-ounces of silver that each mark must contain, and you make 4475 half-ounces of silver contained in that new mass. Finally multiply the same 895 marks by the 11 half-ounces of copper that each mark must contain, and you produce 9845 half-ounces of copper, which together with the 4475 half-ounces of silver make 14320 — namely, as many half-ounces as are in all contained in the 895 marks.

55 [Margin: LV]

Two enter into a partnership: the second brings with him twice as much money as the first, and 5 gold besides. They made a profit of 960 gold, of which 300 gold fell to the first and 660 to the second. So how much did each invest?

Put 1R for the first, and accordingly 2R + 5 for the second. The sum of both is 3R + 5, on which they made a profit of 960 gold. So set up the rule of three thus:

3R + 5	960	1R?	960R / (3R + 5)

You will find that the first, who invested 1R, gains 960R/(3R + 5); and this number is equal to the 300 gold that fell to the first. This equation, by cross-multiplication, is reduced to this: between 960R and 900R + 1500. So taking away 900R from both sides, there remains the equation between 60R and 1500. So dividing 1500 by 60 gives 1R = 25, for the gold the first invested. Therefore the second invested 55 gold.

This is proved. For the sum of both is 80 gold. So the rule of three will stand thus, twice:

80	960	25?	300
80	960	55?	660

56 [Margin: LVI]

Two have money — namely 200 gold together — and the money of the second divided by the money of the first gives a quotient of $1\frac{1}{2}$. The question is how much money each has.

This question, together with the ones that follow, pertains to second powers [ad secundas radices]. Let $1R$ be put for the first, and for the second [?]. [Margin: 1 A.] So the second root must be reduced to the first, in this way. Since both together have 200 gold, the equation will be

(Continuation of Problem 56, begun on the previous page.)

the equation stands between $1R + 1A$ [$R = res$, the unknown x ; $A =$ a second, lettered unknown] and 200. Take $1R$ from both sides, and it lies between $1A$ and $200 - 1R$. So $1A$ resolves into $200 - 1R$. I therefore set out afresh the first man's number as $1R$ and the second's as $200 - 1R$, which together make 200 gold pieces [aur., aurei]. Now dividing the second man's number by the first man's gives the quotient $(200 - 1R) / 1R$, equal to $1\frac{1}{2}$ — that is, $3/2$. By cross-multiplication this proportion reduces to an equation between $400 - 2R$ and $3R$. Adding $2R$ to both sides gives the equation between 400 and $5R$. Dividing 400 by 5 yields the quotient 80, the first man's number. The second man will therefore have 120, namely $200 - 1R$; and dividing that number by 80 gives the quotient $1\frac{1}{2}$.

Alternatively. Let the second man's number be set as $1R$ and the first man's as $1A$ gold pieces. Then again the equation will be between $1R + 1A$ and 200. Taking $1R$ from both sides, it lies between $1A$ and $200 - 1R$; and thus $1A$ resolves into $200 - 1R$. So once more I set $1R$ for the second and $200 - 1R$ for the first. Dividing the former by the latter gives the quotient $1R / (200 - 1R)$, equal to $3/2$. By cross-multiplication this equality reduces to an equation between $2R$ and $600 - 3R$. Adding $3R$ to both sides gives the equation between $5R$ and 600. Dividing 600 by 5 yields $1R = 120$ for the second man's number. The first man therefore has 80, namely $200 - 1R$, as before.

57 [Margin: LVII.]

Seven merchants owe a certain creditor money in this way. Six of them, the seventh excluded, together owe 994 gold pieces. Next six, the first set aside, owe 882 gold pieces. Six, the second set aside, owe 952 gold pieces. Six, the third removed, owe 896 gold pieces. Six, the fourth excluded, owe 910 gold pieces. Six, the fifth set aside, owe 840 gold pieces. And finally six, the sixth excepted, owe 1036 gold pieces. It is now asked what the total sum of the debt is, and how much each man owes.

First, because the seventh man's debt is excluded from the whole sum: if the seventh man's debt is set as 1R, the total sum of the debt will be $994 + 1R$ gold pieces.

Next, because the first man's debt is excluded: if 1A is set for it, the total sum of the debt will be $882 + 1A$ gold pieces.

Third, because the second man's debt is excluded: put 1B for it, and the total sum of the debt will be $952 + 1B$ gold pieces.

Fourth, because the third man's debt is taken out: if 1C is set for it, the total sum of the debt will be $896 + 1C$ gold pieces.

Fifth, because the fourth man's debt is subtracted: if 1D is set for it, the total sum of the debt will be $910 + 1D$ gold pieces.

Sixth, because the fifth man's debt is suppressed: if 1E is set for it, the total sum of the debt will be $840 + 1E$ gold pieces.

Seventh and last, because the sixth man's debt is not expressed: if 1F is set for it, the total sum of the debt will be $1036 + 1F$ gold pieces.

These seven expressions of the whole debt are:

994	+	1R
882	+	1A
952	+	1B
896	+	1C
910	+	1D
840	+	1E
1036	+	1F

Now since the sum of all, that is, the whole debt, is here set out seven times, as you see, we shall have six equations if we compare the top expression with each of the ones below it — for the reason that any whole is equal to itself. Namely:

The first: $994 + 1R$ is equal to $882 + 1A$. So, taking 882 from both, there will be an equation between $112 + 1R$ and 1A. Since, then, 1A has been set for the first man's debt, the first man's debt will be $112 + 1R$ gold pieces.

Next, in the same way, the equation will be between $994 + 1R$ and $952 + 1B$. Taking 952 from both sides, between $42 + 1R$ and $1B$, and hence the second man's debt is $42 + 1R$.

Third, likewise, the equation will be between $994 + 1R$ and $896 + 1C$. Taking 896 from both sides, between $98 + 1R$ and $1C$; and so the third man's debt will be $98 + 1R$.

Fourth, by the same reasoning, the equation will be between $994 + 1R$ and $910 + 1D$. Taking 910 from both, between $84 + 1R$ and $1D$. And the fourth man's debt will be $84 + 1R$ gold pieces.

Fifth, no differently, the equation will be between $994 + 1R$ and $840 + 1E$; and therefore the fifth man's debt will be $154 + 1R$ gold pieces.

Sixth, in no other way, the equation will be between $994 + 1R$ and $1036 + 1F$. Taking 994 from both, between $1R$ and $42 + 1F$. And taking 42 again from both, between $1R - 42$ and $1F$; and so the sixth man's debt will be $1R - 42$ gold pieces.

Debt of the first	112	+	1R
of the second	42	+	1R
of the third	98	+	1R
of the fourth	84	+	1R
of the fifth	154	+	1R
of the sixth	1R	-	42
of the seventh	1R		

And because $1R$ was set for the seventh man's debt, the debts of all will be these, which make the sum $7R + 448$ gold pieces, equal to $994 + 1R$ — since this number too is equal to the whole debt, as we said just above. So taking $1R$ from both sides, there will be an equation between $6R + 448$ and 994 . And taking 448 again from both, between $6R$ and 546 . Dividing 546 by 6, then, gives $1R = 91$, the seventh man's debt, which was $1R$. From this you will easily draw out the debts of the others. For since the first man's debt was $112 + 1R$, that debt will be 203 —

— the total being made up of 91 and 112. So too the sixth man's debt will be 49, namely $1R - 42$, and likewise for the rest, as you see here:

Debt of	
the first	203
	3

Debt of			
the second	13		
	3		
the third	18	<i>gold pieces.</i>	Sum of all, or the whole debt:
	9		
the fourth	17		1085 gold pieces.
	5		
the fifth	24		
	5		
the sixth	49		
the seventh	91		

Proof. Taking the seventh man's debt from the whole debt, the other six will owe 994 gold pieces. Again, taking the first man's debt from the same whole debt, the other six will owe 882 gold pieces. And so for the rest.

58 [Margin: LVIII.]

Two partners have two sums of gold pieces. Their sum, subtracted from the sum of the squares produced from them, leaves 78. But their sum, added to the number produced by their multiplication, makes 39. It is asked what these numbers are.

We shall explain this problem by means of the following figure, in which the square on the line AC is divided into two squares and two complements, each of which is a mean proportional between the two squares — as we proved in the Lemma to Proposition 54 of Book 10 of Euclid, and showed above also in numbers, in the Lemma to Problem 26 of Chapter 30. Let the smaller number be set as 1R, namely the line AB, and the larger as 1A, namely the line BC. Their combined sum, namely the line AC, is set as 1B. The square on the line AB will then be 13 [3 = *census*, the square of the *res*, i.e. x^2]. And since the sum of the numbers, 1B, subtracted from the two squares on the lines AB and BC leaves 78, those two squares will make $78 + 1B$. For if 1B is taken from them, 78 remains. Therefore the square on the line BC will be $78 + 1B - 13$. For if the two squares make $78 + 1B$, as we said, then the square on BC alone will be $78 + 1B - 13$.

Next, because the sum of the numbers AB and BC, namely 1B, added to the product of their multiplication — that is, to the complement AD — makes 39, this complement will be $39 - 1B$. For, likewise, if 1B is added, the complement itself becomes 39.

Diagram. A large square with its left side marked, from top to bottom, by the points A (top-left corner), B (midpoint of the left side), and C (bottom-left corner). A horizontal line drawn through B and a vertical line divide the square into four regions. The upper strip (of height $AB = 1R$, the smaller number) is split into a wide left rectangle and a narrow right square; the lower strip (of height $BC = 1A$, the larger number) is likewise split.

- A (top-left corner of the square)
- B (point on the left side, below A; marks off AB above from BC below)
- C (bottom-left corner)
- Upper-left region (complement): $39 - 1B$
- Upper-right region (small square, = square on AB): 13
- Label above the upper-right square, marking its width: $1R$
- Lower-left region (large square, = square on BC): $78 + 1B - 13$
- Lower-right region (complement, printed vertically): $39 - 1B$
- Label at the right side, marking the height BC: $1A$

Now, since the square on the line AC — which is worth $1B$ — consists of the two squares on the lines AB and BC (that is, of $78 + 1B$) and of the two complements (that is, of $78 - 2B$), and since the square on the line AC, formed from $1B$, is $1B3$ [the *census* of B, i.e. B^2], there will be an equation between $1B3$ and $156 - 1B$. This latter number is put together from $78 + 1B$ (that is, the two squares) and from $78 - 2B$ (that is, the two complements). The root of this number $156 - 1B$ will be investigated in no other way than from the number $156 - 1R$, as if this were equal to 13 , in this manner. The half of the number of roots is $\frac{1}{2}$; its square is $\frac{1}{4}$; adding 156 — that is, $624/4$ — makes the number $625/4$; the square root of this is $25/2$; and if the aforesaid half, $\frac{1}{2}$, is taken away, there remains as the value $1R = 24/2$, that is, 12 . And this is the worth of the line AC, that is, of $1B$, the sum of the numbers $1R$ and $1A$.

Since, then, $1B$ is 12 , the complement $39 - 1B$ will be 27 , and the larger square $78 + 1B - 13$ will be $90 - 13$. And the complement 27 is the mean proportional between this larger square, $90 - 13$, and the smaller square, 13 , as we said at the outset. So if $90 - 13$ is multiplied into 13 , the result is $903 - 133$ [$33 = \text{census of census}$, the *zenzizencic*, i.e. x^4], namely the number 729 , [Margin: (a) 20. Sept. [?]] equal to the square described on 27 . Adding 133 to both sides gives an equation between 903 and $729 + 133$; and taking 729 from both, between $903 - 729$ and 133 .

We shall find the zenzizencic root of this number $903 - 729$ as follows. The half of the number of *censuses* is 45; if from its square, 2025, we take 729, there remains the number 1296, whose root is 36. If the aforesaid half, 45, is added, it becomes 81, the larger; but the smaller will be 9, if from the aforesaid half, 45, the root found, 36, is subtracted. The *censuses* are therefore 81 and 9, whose square roots are 9 and 3, the numbers sought. For their squares, 81 and 9, have the sum 90; if the sum of the numbers themselves, 12, is subtracted, the remainder is the number 78. And if the same sum, 12, is added to 27, the number produced from their multiplication, the sum 39 is made.

Moreover, once 1B — the sum of the numbers 1R and 1A — was found to be 12, we shall more easily find each of the numbers 1R and 1A separately, by Problem 172 of Chapter 29. For since the complement, which is made from AB into CB, was found to be 27: if the number 12 is divided into two parts so that from one into the other is made the number 27 (which is not greater than the fourth part of the square made from the number 12), then, as we taught in that Problem, one part 3 and the other 9 will be found. The same will also be achieved by Problem 38 of Chapter 30.

Or else, if the number 12 is divided, by Problem 170 of Chapter 29 or Problem 35 of Chapter 30, into two numbers whose squares together make 90 — namely the sum of the two squares on the lines AB and CB, which was found to be 90 — the two parts 3 and 9 will again be found.

SCHOLIUM.

From this problem it is easily understood that anyone who wishes to solve questions by algebra ought to be thoroughly practised in the science of geometry, as we said in Chapter 1. For this problem would scarcely, or in no way, be solved by one who is ignorant of geometry, as is plain.

59 [Margin: LIX.]

Two partners have two sums of gold pieces. The squares of the numbers produced from the sums make 340; but the sums themselves, multiplied together, make $\frac{6}{7}$ of the larger square. The question is, how large these sums are.

Let the larger sum be set as 1R and the smaller as 1A. The squares are 13 and 1A3 [A², the *census* of A], which together must make 340.

Therefore 13 will be $340 - 1A3$, and 1A3 will be $340 - 13$. But the two numbers multiplied together make the number 1RA, equal to $\frac{6}{7} 3$ — that is, $\frac{6}{7}$ of the larger square, which is 13. If, then, there is an equation between 1RA and $\frac{6}{7} 3$,

there will also be an equation between the squares of these numbers, namely between $13A^3 [x^2 \cdot A^2]$ and $36/49 \cdot 33 [(6/7)^2 \cdot x^4]$. By cross-multiplication this equation reduces to an equation between $493A^3$ and 3633 . And by lowering the sign 3, of the first power, it will be an equation between $49A^3$ and 363 . Then by the Rule of Three: if 363 are equivalent to $49A^3$, 13 will be equivalent to the number $49A^3/36$, as you see here, the sign 3 being cast off from the first and third numbers:

36	49A ³	1[3]	→	49A ³ /36

And since $1A^3$ was equal to the number $340 - 13$, there will also be an equation between $1A^3$ and $340 - 49A^3/36$ — seeing that 13 was found equal to the number $49A^3/36$. Adding $49A^3/36$ to both sides, then, there will be an equation between $1A^3 + 49A^3/36$ and 340 . Since, then, $1A^3$ makes $36/36 A^3$ (because $36/36$ and 1 are equal numbers), there will be an equation between $85/36 A^3$ and 340 . Dividing 340 by $85/36$, then, gives $1A^3 = 144$, whose root, 12 , will be one of the numbers, namely the smaller sum of money, for which we set $1A$, whose square is $1A^3$.

But since the square of the larger sum was $340 - 1A^3$: if the value of $1A^3$, namely 144 , is taken from 340 , the remainder makes the square of the larger sum, 196 , whose root is 14 , the other of the numbers, namely the larger sum, which we set as $1R$. So the first has 14 gold pieces and the second 12 . For the squares of these numbers, 196 and 144 , make 340 ; and 14 into 12 makes the number 168 , that is, $6/7$ of the larger square, 196 .

60 [Margin: LX.]

Three partners have sums of money. The first says to the second: if you were to give me $\frac{1}{2}$ of your money, I should have 100 gold pieces. The second says to the third: if you were to give me $\frac{1}{3}$ of your money, I should have 100 gold pieces. The third, finally, says to the first: if you were to give me $\frac{1}{4}$ of your money, I should have 100 gold pieces. It is asked how much each of them has.

Let the first man's money be set as $1R$ gold pieces, the second's as $1A$, and the third's as $1B$. The first, then, together with $\frac{1}{2}$ of the second's, will have $1R + \frac{1}{2}A$, which number is equal to 100 gold pieces. So taking $1R$ from both sides, there will be an equation between $\frac{1}{2}A$ and $100 - 1R$; and hence also between their doubles, namely between $1A$ and $200 - 2R$. And so, for the second man, in place of $1A$, I put $200 - 2R$. This second man, together with $\frac{1}{3}$ of the third's, will have $200 - 2R + \frac{1}{3}B$, which number too is equal to 100 gold pieces. Adding $2R$ to both sides, then, there will be an equality between $200 + \frac{1}{3}B$ and $100 + 2R$. And taking 100 again from both, between $\frac{1}{3}B$ and $2R - 100$; and hence also between their triples, namely

between $1B$ and $6R - 300$ gold pieces. And so, for the third man, in place of $1B$, I put $6R - 300$ gold pieces. This third man, together with $\frac{1}{4}$ of the first's — that is, with $\frac{1}{4}R$ — will have $6\frac{1}{4}R - 300$, a number likewise equal to 100 gold pieces. Adding 300 to both sides, then, there will be an equation between $6\frac{1}{4}R$ and 400. Dividing 400 by $6\frac{1}{4}$, then, gives $1R = 64$; and the first man has that many gold pieces. The second, who was found to have $200 - 2R$, has 72 gold pieces. And finally the third, who was found to have $6R - 300$, has 84 gold pieces. For thus the first, 64, together with $\frac{1}{2}$ of the second's, 72 — that is, with 36 — as well as the second, 72, together with $\frac{1}{3}$ of the third's, 84 — that is, with 28 — and the third, 84, together with $\frac{1}{4}$ of the first's, 64 — that is, with 16 — will each have 100 gold pieces.

61 [Margin: LXI.]

Three men have money. The first says to the rest: if you two still had 100 gold pieces more, the sum of yours would be double my sum. The second says to the rest: if you still had 100 gold pieces more, the sum of yours would be triple my sum. The third says to the rest: if you still had 100 gold pieces more, the sum of yours would be quadruple my sum. It is asked how much each has.

Let the first man's money be set as $1R$, and the other two together as $1A$. These two, with 100 gold pieces, will have a sum double the first man's money. So the equation will be between $2R$ and $1A + 100$. Taking 100 from both sides, between $2R - 100$ and $1A$; and hence the second and third, who are supposed to have $1A$, will have $2R - 100$. And if the first man's sum, which is $1R$, is added, the sum of all three will be $3R - 100$.

Next, let the second man's sum be set as $1B$. The first and third will accordingly have $3R - 100 - 1B$, seeing that all three together have $3R - 100$. And because the sum of the first and third, together with 100 gold pieces, is triple the second man's sum, there will be an equation between $3R - 1B$ (for if 100 is added to $3R - 100 - 1B$, the sum becomes $3R - 1B$) and $3B$. Adding $1B$ to both sides, then, there will be an equality between $3R$ and $4B$; and therefore also between their sub-quadruples, that is, between $\frac{3}{4}R$ and $1B$. And hence the second man, who has $1B$, will have $\frac{3}{4}R$.

Finally, let the third man's sum be set as $1C$. The remaining two will accordingly have $3R - 100 - 1C$, seeing that all three together have $3R - 100$ gold pieces. And since the sum of the first and second, together with 100 gold pieces, is quadruple the third man's sum, there will be an equation between $3R - 1C$ (for if 100 is added to $3R - 100 - 1C$, the sum becomes $3R - 1C$) and $4C$. Adding $1C$ to both sides,

there will be an equation between $3R$ and $5C$; and therefore also between their sub-quintuples, that is, between $\frac{3}{5}R$ and $1C$. And hence the third man will have $\frac{3}{5}R$.

So, since the first has $1R$, the second $\frac{3}{4}R$, and the third $\frac{3}{5}R$, all three together will have $2\frac{7}{20}R$, or $\frac{47}{20}R$, which equals $3R - 100$. There is therefore an equation between $\frac{47}{20}R$ (or $2\frac{7}{20}R$) and $3R - 100$. Taking $2\frac{7}{20}R$ from both sides, between 100 and $\frac{13}{20}R$. Dividing 100 by $\frac{13}{20}$, then, gives $1R = 153\frac{11}{13}$ gold pieces, the first man's money. The second, having $\frac{3}{4}R$, will have $115\frac{5}{13}$. And the third, having $\frac{3}{5}R$, will have $92\frac{4}{13}$. For the second and third, together with 100 , will have $307\frac{9}{13}$, a number that is double the number $153\frac{11}{13}$, which the first has. Likewise the first and third, together with 100 , will have $346\frac{2}{13}$, a number that is triple the number $115\frac{5}{13}$, which the second has. And finally the second and first, together with 100 , will have $369\frac{3}{13}$ gold pieces, a number that is quadruple the number $92\frac{4}{13}$, which the third has.

62 [Margin: LXII.]

Three men have money. The first says to the rest: if I still had 100 gold pieces, my sum would become equal to your two sums. The second says to the rest: if I still had 100 gold pieces, my sum would be double yours. The third, finally, says to the rest: if I still had 100 gold pieces, my sum would become triple yours. It is asked how much each has.

Let the first man's sum be set as $1R$. Then, with 100 , he will have $1R + 100$, and just so large will be the sum of the second and third; and all —

Throughout this batch the cossist unknown is printed as the *res* sign, a "2-with-a-tail" glyph; I render it **R** and place its coefficient before it. So " $1R$ " = one *res* (our x), " $2R + 100$ " = $2x + 100$, and so on. R = *res* [the unknown quantity, i.e. x]. Where a second or third unknown is needed, Clavius uses the plain capitals **A** and **B**; these are auxiliary unknowns, not gold coins. "aur." = *aureus* (gold coin).

and so all three together will have $2R + 100$.

Let the second man's money be posited as $1A$. Therefore with 100 [added] he will have $1A + 100$, and this quantity is double the sum of the first and third, which is $2R + 100 - 1A$. (For since all three together have $2R + 100$, if $1A$ — namely the second man's sum — is taken away, the sum of the first and third that remains is $2R + 100 - 1A$.) Hence the equation is between $1A + 100$ and $4R + 200 - 2A$. Adding $2A$ to both sides, then removing 100 from each, we get $3A$ and $4R + 100$. Dividing

everything by 3 gives the equation between $1A$ and $\frac{4}{3}R + \frac{100}{3}$. And so, since the second man was posited to have $1A$, his sum is $\frac{4}{3}R + \frac{100}{3}$.

Finally, let the third man's sum be posited as $1B$. Then with 100 [added] he will have $1B + 100$, and this quantity is triple the sum of the first and second — that is, triple $(1R + \frac{4}{3}R + \frac{100}{3}) = \frac{7}{3}R + \frac{100}{3}$. So the equation is between $1B + 100$ and three times $\frac{7}{3}R + \frac{100}{3}$, namely $7R + 100$; that is, between $1B$ and $7R$. And so the third man's sum, which was posited as $1B$, is $7R$.

Accordingly, since the first has $1R$, the second $\frac{4}{3}R + \frac{100}{3}$, and the third $7R$, all three together will have $9\frac{1}{3}R + \frac{100}{3}$. But all three together also had $2R + 100$. The equation is therefore between $9\frac{1}{3}R + \frac{100}{3}$ and $2R + 100$. Removing $2R$ from both, and then again removing $\frac{100}{3}$ (that is, $33\frac{1}{3}$) from both sides, we get $7\frac{1}{3}R$ and $66\frac{2}{3}$. Dividing $66\frac{2}{3}$ by $7\frac{1}{3}$ gives $1R = 9\frac{1}{11}$, the first man's money. The second, having $1\frac{1}{3}R + 33\frac{1}{3}$, will have $45\frac{5}{11}$. The third, finally, having $7R$, will have $63\frac{7}{11}$.

For in this way the first, with 100, will make the sum $109\frac{1}{11}$, equal to the sum of the second and third. The second, with 100, will make the sum $145\frac{5}{11}$, double the sum of the first and third. And the third, with 100, will make $163\frac{7}{11}$, a sum triple the sum of the first and second.

63. [Margin: LXIII.]

Three men have money. The first says to the other two: if you took 100 gold from your [combined] sum, I would have a sum equal to what remains of your sum. The second says to them: if you took 100 gold from your sum, my sum would be double the remainder of your sum. Finally the third says to them: if you took 100 gold from your sum, my sum would be triple the remainder of your sum. What is each one's sum?

Let the first man's money be posited as $1R$. Then the other two together will have $1R + 100$, so that, once 100 is removed, they have a remaining sum equal to the first man's sum. And all three together will have $2R + 100$.

Let the second man's money be posited as $1A$, which is double the money of the first and third if they remove 100 gold. Now the first and third have $2R + 100 - 1A$ (for since all three have $2R + 100$, if $1A$ — the second man's sum — is taken away, the remainder, the sum of the first and third, becomes $2R + 100 - 1A$); and once 100 is removed they will have $2R - 1A$, of which double is $1A$, the second man's sum. So the equation is between $4R - 2A$ and $1A$. Adding $2A$ to both sides gives $4R$

and $3A$; and dividing everything by 3 gives $\frac{4}{3}R$ and $1A$. Hence the second man's sum, which we posited as $1A$, is $\frac{4}{3}R$.

Finally, let the third man's sum be posited as $1B$, which is triple the sum of the first and second if they remove 100 gold. Now the first and second have $\frac{7}{3}R$, and once 100 is removed they will have $\frac{7}{3}R - 100$; of this quantity the triple is $7R - 300$, which will equal the third man's sum $1B$. Therefore the third has $7R - 300$.

Accordingly, since the first has $1R$, the second $\frac{4}{3}R$, and the third $7R - 300$, all three together will have $9\frac{1}{3}R - 300$. But all three together also had $2R + 100$. The equation is therefore between $9\frac{1}{3}R - 300$ and $2R + 100$. Adding 300 to both sides gives $9\frac{1}{3}R$ and $2R + 400$; removing $2R$ from both gives $7\frac{1}{3}R$ and 400. Dividing 400 by $7\frac{1}{3}$ gives $1R = 54\frac{6}{11}$, the first man's money. The second, having $\frac{4}{3}R$, will have $72\frac{8}{11}$. And the third, having $7R - 300$, will have $81\frac{9}{11}$. This is proved as follows.

If from $154\frac{6}{11}$, the sum of the second and third, 100 be thrown out, the remaining sum $54\frac{6}{11}$ equals the first man's sum. If from $136\frac{4}{11}$, the sum of the first and third, 100 be thrown out, the remaining sum $36\frac{4}{11}$ is half the second man's sum, $72\frac{8}{11}$. And finally, if from $127\frac{3}{11}$, the sum of the first and second, 100 be removed, the remaining sum $27\frac{3}{11}$ is one-third the third man's sum, $81\frac{9}{11}$.

64. [Margin: LXIII.]

Three men have money. The first says to the other two: if I removed 100 gold from my sum, your sum would be quadruple my remaining sum. The second says to them: if I threw away 100 gold from my sum, your sum would be triple my remaining sum. Finally the third says to them: if I took 100 gold from my sum, your sum would be double my remaining sum. What is each one's sum?

Let the first man's money be posited as $1R + 100$. Then the second and third will have $4R$, so that this sum is quadruple the first man's sum once he has thrown away 100 gold; and all three will have $5R + 100$.

Next, let the second man's money be posited as $1A + 100$, so that, once 100 is thrown away, the remaining sum $1A$ is one-third the sum of the first and third — which is $5R + 100 - 1A + 100$ [i.e. $(5R + 100) - (1A + 100) = 5R - 1A$] [?]. Thus the equation is between $3A$ and $5R - 1A$. Adding $1A$ to both sides gives $4A$ and $5R$; and dividing both by 4 gives $1A$ and $\frac{5}{4}R$. Since the second man was posited to have $1A + 100$, he will have $\frac{5}{4}R + 100$.

Finally, let the third man's sum be posited as $1B + 100$, so that, once 100 is thrown away, the remaining sum $1B$ is half the sum of the first and second, which is $2\frac{1}{4}R + 200$. Thus the equation is between $2B$ and $2\frac{1}{4}R + 200$. Dividing everything by 2 gives $1B$ equal to $\frac{9}{8}R + 100$. Since the third man was posited to have $1B + 100$, he will have $\frac{9}{8}R + 200$.

Accordingly, since the first has $1R + 100$, the second $\frac{5}{4}R + 100$, and the third $\frac{9}{8}R + 200$, all three together will have $3\frac{3}{8}R + 400$. But all three together also had $5R + 100$. The equation is therefore between $3\frac{3}{8}R + 400$ and $5R + 100$. Again removing $3\frac{3}{8}R$ from both sides gives 300 and $1\frac{5}{8}R$. Dividing 300 by $1\frac{5}{8}$ gives $1R = 184\frac{8}{13}$. The first, then, who was posited to have $1R + 100$, will have $284\frac{8}{13}$ gold. The second, having $\frac{5}{4}R + 100$, will have $330\frac{10}{13}$ gold. And the third, finally, having $\frac{9}{8}R + 200$, will have $407\frac{9}{13}$ gold. This is proved as follows.

For if the first removes 100 from his sum, the remaining quantity, $184\frac{8}{13}$, is one-quarter the sum of the second and third, which is $738\frac{6}{13}$. If the second throws away 100 from his sum, the remaining quantity, $230\frac{10}{13}$, is one-third the sum of the first and third, which is $692\frac{4}{13}$. Finally, if the third removes 100 from his sum, the remaining quantity, $307\frac{9}{13}$, is half the sum of the first and second, which is $615\frac{5}{13}$.

65. [Margin: LXV.]

Three men have money. The first says to the other two: if I gave you 100 gold, you would make a sum quintuple my remaining sum. The second says to them: if I gave you 100 gold, your sum would be sextuple my remaining sum. Finally the third says to them: if I gave you 100 gold, you would make a sum septuple my remaining sum. What is each one's sum?

Let the first man's money be posited as $1R + 100$. Then the other two will have $5R - 100$, so that if the first gives them 100 they have $5R$, a quantity five times the first man's remaining quantity [$1R$]. And all three together will have $6R$.

Let the second man's money be posited as $1A + 100$, so that after he has given 100 to the other two, the remainder $1A$ is one-sixth the sum of the others together with the 100 gold they received from the second. Now the first and third have $6R - 1A - 100$; and with the second man's 100 they will have $6R - 1A$, which quantity is six times $1A$, that is, equal to $6A$. Adding $1A + 100$ to both sides gives $6R + 100$ and $7A + 100$; removing 100 from both gives $6R$ and $7A$. Dividing both quantities by 7, $\frac{6}{7}R$ equals $1A$. Therefore the second, whom we posited to have $1A + 100$, will have $\frac{6}{7}R + 100$.

Finally, let the third man's money be posited as $1B + 100$, so that after he has given 100 to the other two, the remainder $1B$ is one-seventh the sum of the first and second together with the 100 gold they received from the third. Now the first and second have $13/7R$; and with the third man's 100 they will have $13/7R + 100$, which quantity is seven times $1B$, that is, equal to $7B$. Dividing both quantities by 7 gives the equation between $13/49R + 100/7$ and $1B$. Therefore the third, whom we posited to have $1B + 100$, will have $13/49R + 1000/7$.

Accordingly, since the first has $1R + 100$, the second $6/7R + 100$, and the third $13/49R + 1000/7$, all three together will have $2\ 6/49R + 2400/7$. But all three together also had $6R$. The equality is therefore between $2\ 6/49R + 2400/7$ and $6R$. Removing $2\ 6/49R$ from both sides gives $2400/7$ and $3\ 43/49R$. Dividing $2400/7$ by $3\ 43/49$ gives $1R = 88\ 8/19$. The first, then, to whom we gave $1R + 100$, will have $188\ 8/19$. The second, who was found to have $6/7R + 100$, will have $175\ 15/19$. And the third, finally, having $13/49R + 1000/7$, will have $166\ 6/19$. This is proved as follows.

For if the first gives 100 to the other two, they will have $442\ 2/19$, a quantity five times the first man's remaining sum, $88\ 8/19$. If the second gives 100 to the other two, they will have $454\ 14/19$, a quantity six times the second man's remaining sum, $75\ 15/19$. And finally, if the third gives 100 to the other two, they will have $464\ 4/19$, a quantity seven times the third man's remaining sum, $66\ 6/19$.

66. [Margin: LXVI.]

Three men have money. The first says to the other two: if you gave me 100 gold, my sum would equal your remaining sum. The second says to them: if you gave me 100 gold, my sum would be double your remaining sum. Finally the third says to them: if you gave me 100 gold, my sum would be triple your remaining sum. What is each one's sum?

Let the first man's money be posited as $1R$. If he receives 100 gold from the other two, he will have $1R + 100$; and the other two will then have just as much, and therefore before they gave the 100 gold they had $1R + 200$. And all three will have $2R + 200$.

Let the second man's money be posited as $1A$. If he receives 100 gold from the other two, he will have $1A + 100$, and this quantity is double the remaining sum of the first and third, which, before they gave 100 gold to the other two, is $2R + 200 - 1A$, and after they have given 100 gold to the other two, is $2R + 100 - 1A$. Of this, then, the double, $4R + 200 - 2A$, will equal $1A + 100$. Adding $2A$ to both sides gives the equation between $4R + 200$ and $3A + 100$; and removing 100 from both gives $4R + 100$ and $3A$. Dividing both quantities by 3 gives the equation between $4/3R +$

$100/3$ and $1A$. Therefore the second has $4/3R + 100/3$, since we posited him to have $1A$.

Finally, let the third man's money be posited as $1B$. If he receives 100 gold from the other two, he will have $1B + 100$, and this quantity is triple the remaining sum of the first and second, which, before they gave 100 gold to the other two, is $7/3R + 100/3$, and after they have given 100 gold to the other two, is $7/3R - 200/3$. Of this, then, the triple, $7R - 200$, equals $1B + 100$. Adding 200 to both sides gives the equality between $7R$ and $1B + 300$; removing 300 from both gives $7R - 300$ and $1B$. The third, then, whom we posited to have $1B$, will have $7R - 300$.

Accordingly, since the first has $1R$, the second $4/3R + 100/3$, and the third $7R - 300$, all three together will have $9\frac{1}{3}R - 800/3$. But all three together also had $2R + 200$. The equality is therefore between $9\frac{1}{3}R - 800/3$ and $2R + 200$. Adding $800/3$ to both sides gives $9\frac{1}{3}R$ and $2R + 1400/3$; removing $2R$ from both gives $7\frac{1}{3}R$ and $1400/3$. Dividing $1400/3$ by $7\frac{1}{3}$ gives $1R = 63\frac{7}{11}$, the first man's money. The second, having $4/3R + 100/3$, will have $118\frac{2}{11}$. And the third, whom we found to have $7R - 300$, will have $145\frac{5}{11}$. This is proved as follows.

For the first, with 100, will have the sum $163\frac{7}{11}$, equal to the sum of the others once they have given 100 gold. The second, with 100, will make the sum $218\frac{2}{11}$, double the sum of the others after they have given 100 gold. And the third, finally, with 100, will make the sum $245\frac{5}{11}$, triple the sum of the others after they have given 100 gold.

67. [Margin: LXVII.]

Three men have money. The first says to the other two: if you gave me one-half of your sum, I would have 100 gold. The second says to them: if you gave me one-third of your sum, I would have 100 gold. Finally the third says to them: if you gave me one-quarter of your sum, I would have 100 gold. What is each one's money?

Let the first man's money be posited as $100 - 1R$. Then the other two will have $2R$. For in this way, if they give one-half — namely $1R$ — to the first, he will have 100 gold. And all three together will have $100 + 1R$.

Let the second man's money be posited as $1A$. Then the first and third will have $100 + 1R - 1A$; and if they give one-third of this to the second — namely $100/3 + 1/3R - 1/3A$ — the second will have $1A + 100/3 + 1/3R - 1/3A$, that is, $2/3A + 100/3 + 1/3R$, which quantity equals 100 gold. If, then, $100/3$ is taken from both sides, the equation is between $2/3A + 1/3R$ and $200/3$. Rearranging and dividing gives $1A = 100 - 1/2R$. Therefore the second will have $100 - 1/2R$.

Let the third man's money be posited as $1B$. And since the first and second have $200 - \frac{3}{2}R$, if they give one-quarter of this to the third — namely $50 - \frac{3}{8}R$ — the third will have $1B + 50 - \frac{3}{8}R$, which quantity equals 100 gold. Removing 50 from both sides gives the equation between $1B - \frac{3}{8}R$ and 50; adding $\frac{3}{8}R$ to both gives $1B$ and $50 + \frac{3}{8}R$. Therefore the third will have $50 + \frac{3}{8}R$.

Accordingly, since the first has $100 - 1R$, the second $100 - \frac{1}{2}R$, and the third $50 + \frac{3}{8}R$, and all three together had $100 + 1R$, the equation is between $250 - \frac{9}{8}R$ and $100 + 1R$. Adding $\frac{9}{8}R$ to both sides gives 250 and $100 + 2\frac{1}{8}R$; removing 100 from both gives 150 and $2\frac{1}{8}R$. Dividing 150 by $2\frac{1}{8}$ gives $1R = 70 \frac{10}{17}$. The first, then, having $100 - 1R$, will have $29 \frac{7}{17}$. The second, having $100 - \frac{1}{2}R$, will have $64 \frac{12}{17}$. And the third, finally, having $50 + \frac{3}{8}R$, will have $76 \frac{8}{17}$.

For the first with one-half the sum of the other two, the second with one-third the sum of the other two, and the third with one-quarter the sum of the other two, each has 100 gold.

68. [Margin: LXVIII.]

Three men have money. The first says to the other two: if I gave you one-half of my sum, you would have 100 gold. The second says to them: if I gave you one-third of my sum, you would have 100 gold. Finally the third says to them: if I gave you one-quarter of my money, you would have 100 gold.

VEL [Or, equivalently:]

The second and third say to the first: if you gave us one-half of your money, we would have 100 gold. The first and third say to the second: if you gave us one-third of your money, we would have 100 gold. Finally the first and second say to the third: if you gave us one-quarter of your money, we would have 100 gold. What is each one's sum?

Let the first man's money be posited as $1R$. Then the other two will have $100 - \frac{1}{2}R$, so that, receiving one-half of the first's money — namely $\frac{1}{2}R$ — they have 100. And all three will have $\frac{1}{2}R + 100$.

Let the second man's money be posited as $1A$. Then the sum of the first and third is $\frac{1}{2}R + 100 - 1A$; and if they receive one-third of the second — namely $\frac{1}{3}A$ — they will have $\frac{1}{2}R + 100 - 1A + \frac{1}{3}A$, which quantity equals 100. Adding $1A$ to both sides and removing 100 from both gives $\frac{1}{2}R + \frac{1}{3}A$ and $1A$. Again removing $\frac{1}{3}A$ from both gives $\frac{1}{2}R$ and $\frac{2}{3}A$. Dividing both by $\frac{2}{3}$ gives $\frac{3}{4}R$ and $1A$. Therefore the second has $\frac{3}{4}R$.

Let the third man's money be posited as 1B. And since the first and second have $7/4R$, and if they receive one-quarter of the third — namely $1/4B$ — they have $7/4R + 1/4B$, a sum equal to 100 gold. Removing $7/4R$ from both sides, and dividing both quantities by $1/4$, gives 1B equal to $400 - 7R$. Therefore the third has $400 - 7R$.

Accordingly, since the first has $1R$, the second $3/4R$, and the third $400 - 7R$, and all three together had $1/2R + 100$, if $5\frac{1}{4}R$ is added to both sides the equality is between 400 and $100 + 23/4R$ [that is, $100 + 5\frac{3}{4}R$], and between 300 and $23/4R$. Dividing 300 by $23/4$ gives $1R = 52\frac{4}{23}$, the first man's money. The second, having $3/4R$, will have $39\frac{3}{23}$ gold. And the third, having $400 - 7R$, will have $34\frac{18}{23}$ gold. This is proved as follows.

For the second and third with one-half the first's money, the first and third with one-third the second's money, and the first and second with one-quarter the third's money, each pair has 100 gold.

69. [Margin: LXIX.]

Three merchants enter a partnership for 12 months. The first invests 100 gold, the second 200 gold, and the third 300 gold. After two months the first adds some pounds of pepper, of which 3 pounds are worth 1 gold. And after four months the second adds a mass of silver, of which 1 mark is worth 7 gold. When the 12 months are over, the first receives from the profit 50 gold, the second 110 gold, and the third 90 gold, so that the total profit was 250 gold. It is asked how many pounds of pepper the first invested, and how many marks of silver the second.

Suppose the first invested $1R$ pounds of pepper, and the second $1A$ marks of silver. Since, then, 3 pounds of pepper are worth 1 gold, $1R$ pounds will be worth $1/3R$ gold; and since 1 mark of silver is worth 7 gold, $1A$ marks will be worth $7A$ gold, as you see here:

Lib. [pounds]	aur. [gold]	Lib. [pounds]	aur. [gold]
3	1	$1R ?$	$1/3R$
Marca [mark]	aur.	Marc. [marks]	aur.
1	7	$1A ?$	$7A$

And since each man's money must be multiplied by the time: if the first man's 100 gold is multiplied by 12 months, the number 1200 results; from the second's 200 gold in 12 months, the number 2400; and from the third's 300 gold in 12 months, the number 3600. Again, from the first's $1/3R$ over the 10 months during which it

stood in the partnership (for after two months he invested pepper worth $1/3R$ gold, that is, $1R$ pounds of pepper), the number $10/3R$ gold results [?]; and from the 7A over the 8 months during which it stood in the trade (for after 4 months he invested 1A marks, worth 7A), the number 56A marks results. Therefore the first man's number is $1200 + 10/3R$, the second's $2400 + 56A$ marks, and the third's 3600 — which three numbers make $7200 + 10/3R + 56A$. From this the example will be set up according to the Rule of Three, applied three times over:

Capital $\times$ time	Profit (aur.)
$1200 + 10/3R$ [?]	50
$2400 + 56A$ [?]	110
3600 [?]	90
total $7200 + 10/3R + 56A$	250

That is: if $7200 + 10/3R + 56A$ earn 250 gold, what will $1200 + 10/3R$ earn? And what will $2400 + [56A]$...

[Note on cossist symbols used below: R = *res* (the unknown, x; printed as a "2-with-tail" glyph); 3 = *census* (x^2); $\mathfrak{C}$ = *cubus* (x^3); 33 = *census of census* / *zensizensus* (x^4); $\sqrt{3}$ = *radix quadrata* (square root); $\sqrt{\mathfrak{C}}$ = *radix cubica* (cube root). *aur.* = *aurei* (gold pieces); *marc.* = *marcae* (marks of silver); *lib.* = *librae* (pounds); *palm.* = *palmi* (palms, a unit of length). A, B are auxiliary cossic letters.]

[Algebra, Cap. XXXI, cont.] — Problem 69 (concluded)

[The rule-of-three arrangement stands:] $2400 + 56A$, and what will the 3600 gold pieces give? The profits come out as 50, 110, and 90 gold pieces, just as was posited in the problem.

[Margin: Euclid VII.19] Therefore the product of the first number into the fourth terms equals the product of the second into the third. Multiplying the first into the fourth, 90, yields the number $648000 + 300R + 5040A$, which will equal the number 900000 produced from the second, 250, into the third, 3600. Taking 648000 from both sides, there is an equation between $300R + 5040A$ and 252000. And taking $300R$ from both sides, between $5040A$ and $252000 - 300R$. Dividing both these numbers by 5040, 1A becomes $50 - (5/84)R$. And accordingly 56A make $2800 - 3\frac{1}{3}R$. These, added to $7200 + (10/3)R$ — that is, $+ 3\frac{1}{3}R$ — make the number 10000, which will occupy the first place in the rule of three, in place of the cossic number $7200 + (10/3)R + 56A$, so that you may resolve other cossic numbers as well, as you see here:

$$\begin{array}{l} | \quad 1200 + (10/3)R \quad ? \quad 50 \\ 10000 \quad 250 \quad | \quad 2400 + 56A \quad ? \quad 110 \\ | \quad 3600 \quad ? \quad 90 \end{array}$$

For 10000 into 110 makes the number 1100000, equal to the number 600000 + 14000A produced from 250 into 2400 + 56A. So, taking 600000 from both sides, there is an equality between 500000 and 14000A. Dividing both these numbers by 14000, 1A becomes 35 5/7 marks of silver. Finally, 10000 into 50 makes the number 500000, equal to the number 300000 + (2500/3)R produced from 250 into 1200 + (10/3)R. Taking 300000 from both sides, there is an equality between 200000 and (2500/3)R. Dividing 200000 by (2500/3), 1R becomes 240 pounds of pepper. This is proved as follows: 35 5/7 marks of silver will be worth 250 gold pieces, and 240 pounds of pepper 80 gold pieces, as you see here:

Marca	aur.	Marc.	aur.
1	7	35 5/7 ?	250

Lib.	aur.	Lib.	aur.
3	1	240 ?	80

And since the 250 gold pieces stood in trade for 8 months, multiplying 250 by 8 gives the number 2000, which must be added to 2400 in place of 56A, so that the middle third-number becomes 4400. Likewise, since the 80 gold pieces stood in trade for 10 months, multiplying 80 by 10 gives the number 800, which must be added to 1200 in place of (10/3)R, so that the topmost third-number becomes 2000. Hence the worked example will stand anew thus:

$$\begin{array}{l} | \quad 2000 \quad ? \quad 50 \\ 10000 \quad 250 \quad | \quad 4400 \quad ? \quad 110 \\ | \quad 3600 \quad ? \quad 90 \end{array}$$

Here you see the three third-numbers make 10000, the first number of the rule of three, and these same [three shares] gain 50, 110, and 90, and so on.

Problem 70

[Margin: LXX]

70. *Three merchants enter into a partnership. The first invests 35 gold pieces more than the second. The second and the third together invest 84 gold pieces. The common profit is 66 gold pieces, of which the third receives 21 gold pieces as his*

share. The question is: how much did each invest, and how much did both the first and the second receive of the profit?

Suppose the first invested $1R + 35$ gold pieces, and the second $1R$ gold pieces, so that the first invested 35 gold pieces beyond the second's sum. And the third $84 - 1R$ gold pieces, so that the sum of the second and the third makes 84 gold pieces. Then let the first's profit be posited as $1A$, and the second's as $1B$. And since the third's profit is 21 gold pieces, the example will stand thus:

$$\begin{array}{l} \{ 1R + 35 \ ? \ 1 \ A \\ 66 \ \{ \ 1R \ ? \ 1 \ B \\ | \ 84 - 1R \ ? \ 21 \end{array}$$

[Margin: Euclid VII.19] And since the product of the first number into the fourth equals the product of the second into the third: the number $2499 + 21R$ is produced from the first, $119 + 1R$, into the lowest fourth term [21]; and from the second, 66, into the lowest third term, $84 - 1R$, is produced the number $5544 - 66R$, which equals it. So, adding $66R$ to both sides, there comes an equation between $2499 + 87R$ and 5544. And taking 2499 from both sides, between $87R$ and 3045. Dividing 3045 by 87, $1R$ becomes 35. And accordingly the first invested 70 gold pieces, namely $1R + 35$; the second 35, namely $1R$; and the third 49, namely $84 - 1R$. So the first number of the rule of three will be 154. Hence the example will now stand thus:

$$\begin{array}{l} \{ 70 \ ? \ 1 \ A \\ 154 \ \{ \ 35 \ ? \ 1 \ B \\ | \ 49 \ ? \ 21 \end{array}$$

From 154 into $1A$ comes the number 154 A , equal to the number 4620 produced from 66 into 70. So, dividing both numbers by 154, $1A$ becomes 30. Likewise, from 154 into $1B$ comes the number 154 B , equal to the number 2310 produced from 66 into 35. So, dividing both numbers by 154, $1B$ becomes 15. Therefore the first received 30 gold pieces of the profit, and the second 15 gold pieces, which is proved thus:

$$\begin{array}{l} \{ 70 \ ? \ 30 \\ 154 \ \{ \ 35 \ ? \ 15 \\ | \ 49 \ ? \ 21 \end{array}$$

ÆNIGMATA VARIA AD FIGVRAS Geometricas pertinentia. — CAPVT XXXII.

[Various Riddles Pertaining to Geometrical Figures. — Chapter 32.]

In this final chapter, nearly everything that has been taught concerning irrational numbers will be brought to practical use; for the riddles to be set out will be varied ones about geometrical figures, in which for the most part the sides, diameters, and areas are wont to be expressed by surd numbers. The first of these riddles is this.

1. *There is a certain rectangle, longer on one side, ABCD, whose area contains 120 palms, and whose diameter is $\sqrt{3} 2600$ palms [i.e. $\sqrt{2600}$]. The sides are sought.*

[Margin: I.]

[Margin: Euclid I.47] Since the squares of the sides AB and BC together equal the square of the diameter AC — which is 2600 — if the square of the side AB is set as $1R$, the square of the side BC will be $2600 - 1R$. Therefore the sides themselves will be $\sqrt{3} 1R$ and $\sqrt{3} (2600 - 1R)$, which multiplied together will produce the area BD, namely 120 palms, as we said in Cap. 1, Book 3 of the *Geometria Practica*. And for that reason the squares of the sides, $1R$ and $2600 - 1R$, multiplied together will produce the square of the area, namely 14400, by the corollary of the Lemma of Riddle 26, Cap. 30. Since, then, from $1R$ into $2600 - 1R$ there comes $2600R - 13$, there will be an equation between 14400 and $2600R - 13$. Adding 13 to both sides, between $13 + 14400$ and $2600R$. And taking 14400 from both sides, between 13 and $2600R - 14400$.

Now proceed thus. Half the number of roots is 1300; from its square, 1690000, subtracting 14400, there remain 1675600. To the square root of this number, $\sqrt{3} 1675600$, if the aforesaid half, 1300, be added, $1R$ becomes $1300 + \sqrt{3} 1675600$. And if the same root be taken from the aforesaid half, the remainder becomes the smaller root, $1300 - \sqrt{3} 1675600$ (because this equation has a double root, on account of the minus sign by which the absolute number is affected). Therefore the square of the smaller side AB will be $1300 - \sqrt{3} 1675600$, and the square of the larger side BC will be $1300 + \sqrt{3} 1675600$ — which is also what remains if the square of the side AB, $1300 - \sqrt{3} 1675600$, be subtracted from 2600, the square of the diameter AC, as appears in this formula. You see, then, that the two squares $1300 - \sqrt{3} 1675600$ and $1300 + \sqrt{3} 1675600$ together make 2600, the square of the diameter:

$$\begin{array}{r}
 2600 + \sqrt{3} \ 0 \\
 1300 - \sqrt{3} \ 1675600 \\
 \hline
 1300 + \sqrt{3} \ 1675600
 \end{array}$$

Diagram. A rectangle, longer on the horizontal side, with vertices lettered A (top-left), D (top-right), B (bottom-left), C (bottom-right). The diagonal is drawn from A down to C and is labelled with the surd for the diameter. The sides carry the cossic values of the first solution.

- A (top-left corner)
- D (top-right corner)
- B (bottom-left corner)
- C (bottom-right corner)
- Diagonal A–C: $\sqrt{3} \ 2600$ [= $\sqrt{2600}$, the diameter]
- Left side A–B: $1R$ [the square of the shorter side]
- Bottom side B–C: $2600 - 1R$ [the square of the longer side]

And if their sides $\sqrt{3} (1300 - \sqrt{3} \ 1675600)$ and $\sqrt{3} (1300 + \sqrt{3} \ 1675600)$ be multiplied together, the area of the rectangle $\sqrt{3} \ 14400$ is produced — that is, 120 palms. This multiplication is done by subtracting the square of the latter part, 1675600, from 1690000, the square of the former part, and prefixing the sign $\sqrt{3}$ to the remaining number, as is settled from what we wrote in Cap. 23 and 24, and as appears in this formula:

$$\begin{array}{r}
 \sqrt{3} \ (1300 - \sqrt{3} \ 1675600) \\
 \sqrt{3} \ (1300 + \sqrt{3} \ 1675600) \\
 \hline
 \sqrt{3} \ 14400
 \end{array}$$

ANOTHER WAY. If the square of the smaller side AB is set as $1R$, and thereby the square of the larger side as $2600 - 1R$, the sides themselves will be $\sqrt{3} \ 1R$ and $\sqrt{3} (2600 - 1R)$, which multiplied together will make the area 120 palms. They make $\sqrt{3} (2600R - 13)$, as you see here. This multiplication is done by multiplying the squares themselves, $2600 - 1R$ and $1R$, together and prefixing the sign $\sqrt{3}$ to the product. There is therefore an equation between 120 and $\sqrt{3} (2600R - 13)$, and consequently also between their squares, 14400 and $2600R - 13$, as above, etc.

$$\begin{array}{r}
 \sqrt{3} \ (2600 - 1R) \\
 1R \\
 \hline
 \sqrt{3} \ (2600R - 13)
 \end{array}$$

ANOTHER WAY. Let the smaller side AB be set as 1R, and thereby its square as 13. The square of the larger side BC will then be 2600 – 13, and the side BC itself $\sqrt{3} (2600 - 13)$. Let AB, 1R, be multiplied into BC, $\sqrt{3} (2600 - 13)$ — which is done, as was said in Cap. 26, by reducing 1R to the square 13 and prefixing the sign $\sqrt{3}$. For if $\sqrt{3} 13$ be multiplied into $\sqrt{3} (2600 - 13)$ — which is done by multiplying their squares 13 and 2600 – 13 together, as was said in Cap. 26 — the number $\sqrt{3} (26003 - 133)$ will be produced for the area of the rectangle, 120. Therefore the equation will be between $\sqrt{3} (26003 - 133)$ and 120, and accordingly also between their squares 26003 – 133 and 14400. And adding 133 to both sides, between 26003 and 133 + 14400. And taking 14400 from both sides, between 26003 – 14400 and 133. Now we shall extract the zensizensic root [i.e. solve for x^4] thus:

$$\begin{array}{r} \sqrt{3} (2600 - 13) \\ 13 \\ \hline \sqrt{3} (26003 - 133) \end{array}$$

Half the number of censuses is 1300; from its square, 1690000, taking away 14400, there remain 1675600. To the square root of this number, $\sqrt{3} 1675600$, if the aforesaid half, 1300, be added, 13 becomes $1300 + \sqrt{3} 1675600$. And if the same root be taken from the aforesaid half, there remains the smaller square, $1300 - \sqrt{3} 1675600$. Therefore the square of the larger side BC will be $1300 + \sqrt{3} 1675600$, and the square of the smaller side AB will be $1300 - \sqrt{3} 1675600$; and the larger root will be $\sqrt{3} (1300 + \sqrt{3} 1675600)$, and the smaller $\sqrt{3} (1300 - \sqrt{3} 1675600)$, as above, etc.

ANOTHER WAY. [Margin: Euclid I.47] Since the squares of the sides AB and BC make 2600, and from one side into the other comes the area 120, two numbers must be found — by Riddle 26, Cap. 30 — which multiplied together make 120, and whose squares make 2600. This is done as follows. Let the smaller square number be set as 13, and the larger as 2600 – 13. And since, by the Lemma of Riddle 26, Cap. 30, the number 120 — which the sides of the squares 13 and 2600 – 13 produce — is the mean proportional between the squares themselves, [Margin: Euclid VII.20] the number 26003 – 133, produced from 13 into 2600 – 13, will equal the square of the number 120, that is, the number 14400. There is therefore an equation between 26003 – 133 and 14400. Adding 133 to both sides, between 26003 and 133 + 14400. And taking 14400 from both sides, between 26003 – 14400 and 133, as above, etc.

2. *A field longer on one side has its sides in septuple proportion [7 : 1], and their squares taken together have to their sum a hundredfold proportion [100 : 1]. The sides, area, and diameter are sought.* [Margin: II.]

Let the smaller side be set as $1R$ and the larger as $7R$, whose sum is $8R$; and their squares, $1\bar{3}$ and $49\bar{3}$, make a sum of $50\bar{3}$, a hundredfold of the sum $8R$. Therefore the equation is between $50\bar{3}$ and $800R$. So, dividing 800 by 50 , $1R$ becomes 16 , the smaller side. The larger will be 112 , in septuple proportion. Therefore the area will be 1792 , produced from one side into the other. And the diameter will be $\sqrt{3} 12800$. That is: the squares of the sides, 256 and 12544 , will make the sum 12800 , a hundredfold of the sum of the sides, 16 and 112 , which is 128 ; and the square of the diameter will be 12800 , the sum of the two squares of the sides; and accordingly the diameter is $\sqrt{3} 12800$.

3. There is a rectangle of 4500 palms [in area], and its length is triple its breadth. The sides and the diameter are sought. [Margin: III.]

Let the breadth be set as $1R$ and the length as $3R$. These sides multiplied together produce the area $3\bar{3}$, equal to 4500 . So, dividing 4500 by 3 , $1\bar{3}$ becomes 1500 , and $1R$ becomes $\sqrt{3} 1500$, the breadth of the rectangle; and its triple, $\sqrt{3} 13500$, will be its length. From the breadth into the length comes the area $\sqrt{3} 20250000$ — that is, 4500 . And the square of the diameter will be 15000 , equal to the two squares of the sides, and the diameter itself will be $\sqrt{3} 15000$.

4. There is a parallelepiped — that is, a quadrilateral column — of 3375 palms [in volume]. The height has to the length of the base, and this length to the breadth of the base, a sesquialteral proportion [3 : 2]. The measurements are sought. [Margin: IIII.]

Let the height be set as $9R$, the length as $6R$, and the breadth as $4R$, in continued sesquialteral proportion. The breadth into the length makes the base $24\bar{3}$, and this into the height makes the volume of the parallelepiped $216\mathfrak{C}$, equal to 3375 . So, dividing 3375 by 216 , $1\mathfrak{C}$ becomes $15 \frac{135}{216}$, and $1R$ becomes the cube root, $2\frac{1}{2}$. Therefore $9R$ make $22\frac{1}{2}$, the height; $6R$ make 15 , the length of the base; and $4R$ make 10 , the breadth of the base — which measurements, multiplied by one another, make 3375 .

5. There is a parallelepiped having a square base, whose side is one-tenth [subdecuple] of the height of the parallelepiped, whose volume contains 6780 palms. The base and the height are sought. [Margin: V.]

Let the side of the base be set as $1R$, and thereby the height of the parallelepiped as $10R$. From $1R$ into itself comes $1\bar{3}$, namely the square base, which multiplied into the height — that is, into $10R$ — makes $10\mathfrak{C}$, namely the volume of the parallelepiped, equal to 6780 . So, dividing 6780 by 10 , $1\mathfrak{C}$ becomes 678 , and $1R$ becomes $\sqrt{\mathfrak{C}} 678$, the side of the base; which multiplied into itself makes $\sqrt{\mathfrak{C}}$

459684, the area of the base. And the height, having a tenfold proportion to the side, will be $\sqrt{678000}$; which multiplied into the area of the base — that is, into $\sqrt{459684}$ — makes $\sqrt{311665752000}$ [numeral: ? — printed long integer at the legibility limit of the scan; equals $6780^3 = 311665752000$], the volume of the whole parallelepiped, namely 6780.

6. *There is a rectangular surface having its length quadruple its breadth, and an area of 576. The sides are sought.* [Margin: VI.]

Let the smaller [side] be set as $1R$ and the larger as $4R$. These multiplied together produce $4\bar{3}$, the area of the rectangle, equal to 576. So, dividing 576 by 4, $1\bar{3}$ becomes 144, and $1R$ becomes 12, the smaller side. The larger will therefore be 48.

7. *There is a rectangular quadrangular column whose base-sides have a sesquitertian proportion $[4 : 3]$, and whose height has to the greater side of the base a proportion "double-superbipartient-thirds" $[8 : 3]$, i.e. $2\frac{2}{3}$. Finally, the solid content of the column contains 93312 palms. Each dimension is sought.* [Margin: VII.]

Let the smaller side of the base be set as $3R$ and the larger as $4R$, and the height as $10\frac{2}{3}R$, so that it has to the greater side $4R$ the proportion double-superbipartient-thirds, and the smaller side has to the greater the sesquitertian [proportion]. These three numbers multiplied together produce $128\bar{c}$, namely the solid content of the column, equal to 93312. So, dividing 93312 by 128, $1\bar{c}$ becomes 729, and $1R$ becomes 9. Therefore the smaller side of the base, which we set as $3R$, will be 27; the larger, $4R$, will be 36; and the height, $10\frac{2}{3}R$, will be 96. And these three dimensions multiplied by one another make 93312.

8. *There are two unequal towers erected upon two square bases. The sides of the bases have a sesquitertian proportion $[4 : 3]$, which the heights and the solid contents also have; and the heights themselves, taken inversely [permutatim], are double the sides of the bases. Finally, the solid contents of both towers together comprise 21000 palms. Each dimension is sought.* [Margin: VIII.]

Let the side of the smaller base be set as $3R$ and that of the larger as $4R$, so that the one has to the other the sesquitertian proportion without a fraction. The smaller base will be $9\bar{3}$ and the larger $16\bar{3}$. And since the heights, taken inversely, are double the sides of the bases — that is, the height of the tower on the larger base is double the side of the smaller base, and the height of the tower on the smaller base is double the side of the larger base — the former will be $6R$ and the latter $8R$; and they will have the sesquitertian proportion, [Margin: Euclid V.15]

the very same, namely, that the sides of the bases have; and the tower on the smaller base will be taller than the tower on the larger base. Since...

(Continuation of Problem 8 — the enigma of the two proportional towers, begun on the preceding leaf.)

Now, as we shall demonstrate in the following Lemma, when two numbers cross-multiply two others that stand in the duplicate ratio of the first pair, they yield two numbers in the same ratio as the first pair; [Margin: (Euclid) VI.20] and the bases have the duplicate ratio of the sides of the bases, and so too do the heights. It therefore comes about that the greater height, 8 R [res — the first cossist power, the unknown x], multiplied into the lesser base, 9 3 [census — the second power, x^2], produces 72 Ć [cubus — the third power, x^3], the tower on the lesser base; and the lesser height, 6 R, multiplied into the greater base, 16 3, produces 96 Ć, the tower on the greater base.

$$\begin{aligned} 8 \text{ R} \times 16 \text{ 3} &\rightarrow 96 \text{ Ć} \\ 6 \text{ R} \times 9 \text{ 3} &\rightarrow 72 \text{ Ć} \\ (\text{cross-multiplication: } 8 \cdot 9 &= 72, 6 \cdot 16 = 96) \end{aligned}$$

By the Lemma that follows, these products will have the same sesquitertian ratio [4 : 3] as the heights themselves (or the sides of the bases). It follows that the tower on the greater base and of lesser height is the major tower, while the tower on the lesser base and of greater height is the minor tower. Both towers together make 168 Ć, a sum equal to 21000. Dividing 21000 by 168, then, gives 1 Ć = 125, and 1 R = 5. Accordingly the side of the major tower, which was set as 4 R, will be 20 palms; and the side of the minor tower, set as 3 R, will be 15 palms. The remaining dimensions will be as set out in this table. Here you see that the sides of the bases, the heights, and the solidities of the towers all have the sesquitertian ratio [4 : 3], while the bases themselves have the ratio super-seven-parts-of-nine [16 : 9], that is, the duplicate of the sesquitertian ratio. Finally, the solidities of both towers make 21000 palms, as the enigma proposes.

Dimension	Value
Side of the base of the minor tower [Latus basis turris minoris]	15
Side of the base of the major tower [Latus basis turris maioris]	20
Base of the minor tower [Basis minoris turris]	225
Base of the major tower [Basis maioris turris]	400
Height of the minor tower [Altitudo minoris turris]	40
Height of the major tower [Altitudo maioris turris]	30

Dimension	Value
Solidity of the minor tower [Soliditas minoris turris]	9000
Solidity of the major tower [Soliditas maioris turris]	12000

LEMMA

If two numbers are cross-multiplied into two others that have the duplicate ratio of them, the products will have the same ratio as the numbers themselves.

Let there be any two numbers, 8 and 6, having the sesquitertian ratio [4 : 3]; and two others, 16 and 9, having the duplicate of the sesquitertian ratio, namely the super-seven-parts-of-nine ratio [16 : 9], as appears here: 16, 12, 9. Let 72 be formed from 8 into 9, and 96 from 6 into 16. I say that 96 is to 72 as 8 is to 6.

$$8 \times 16 \rightarrow 96$$

$$6 \times 9 \rightarrow 72$$

$$12 \text{ (mean proportional: } 16 : 12 : 9 \text{)}$$

For since 16 and 9 have the duplicate of the ratio of 8 to 6, a mean proportional, 12, falls between them, in the ratio of 8 to 6. Hence, since as the first (8) is to the second (6), so the third (16) is to the fourth (12), [Margin: (Euclid) VII.19] the same number is produced from the first (8) into the fourth (12) as from the second (6) into the third (16), namely 96. Likewise, since as the first (8) is to the second (6), so the third (12) is to the fourth (9), [Margin: (Euclid) VII.19] the same number, 72, is produced from the first (8) into the fourth (9) as from the second (6) into the third (12). And so, since 96 is made from 12 into 8, and 72 from 12 into 6, [Margin: (Euclid) VII.17] the ratio of 96 to 72 will be the same as that of 8 to 6.

Hence it follows that the lesser height must be multiplied into the greater base, and the greater height into the lesser base, so that the solidities produced have the same ratio as the heights; and therefore that the greater tower has the lesser height, while the lesser tower has the greater height. For this reason it was said in the enigma that the heights are, cross-wise, double the sides of the bases.

9. [Margin: IX] *There is a right triangle whose base (I call "base" the side opposite the right angle — i.e. the hypotenuse) contains 52 palms. The sides [legs] have the double-and-two-fifths ratio [dupla superbipartiens quintas = 12 : 5]. The two sides are sought.*

Let the lesser be set as 5 R and the greater as 12 R, so that without a fraction they may have the double-and-two-fifths ratio. The squares of the sides are 25 3 and 144 3, which together make the sum 169 3, [Margin: (Euclid) I.47] equal to the

square of the base — the number 2704. Dividing 2704 by 169 gives $1\ 3 = 16$, and so $1\ R = 4$. The lesser side, then, $5\ R$, will be 20 palms, and the greater, $12\ R$, will be 48 palms — which is proved: for the two squares of the sides, 400 and 2304, make 2704, the square of the base.

10. [Margin: X] *There is an oblong [rectangle longer on one side] whose area is 500 palms, and whose two sides together make 100 palms. The sides are sought.*

Let one side be set as $1\ R$, and the other as $100 - 1\ R$. Multiplied together, these make $100\ R - 1\ 3$, which is the area of the rectangle, equal to 500. Adding $1\ 3$ to each side, then, there will be an equation between $100\ R$ and $1\ 3 + 500$. And removing 500 from both, between $100\ R - 500$ and $1\ 3$. The Zensic [census] root is extracted thus: half the number of roots is 50; from its square, 2500, subtracting 500, there remains the number 2000. To the root of this, $\sqrt{2000}$ [$\sqrt{}$ = the square-root sign, printed as the radix-census mark $\sqrt{3}$], if the aforesaid half, 50, is added, the greater root becomes $50 + \sqrt{2000}$. And if the same root $\sqrt{2000}$ is subtracted from the aforesaid half, the remainder becomes the lesser root, $50 - \sqrt{2000}$. The greater side, therefore, is $50 + \sqrt{2000}$, and the lesser $50 - \sqrt{2000}$ — which is also obtained if the greater is subtracted from 100, the sum of the sides, as you see here. This is proved, because the two sides $50 + \sqrt{2000}$ and $50 - \sqrt{2000}$ make 100, and multiplied together produce the area 500. This multiplication is carried out as taught in Chapter 23: the 2000, the square of the latter part, is subtracted from 2500, the square of the former part, as the formula set beside it shows.

$$\begin{array}{r}
 100 + \sqrt{0} \\
 50 + \sqrt{2000} \\
 \hline
 50 - \sqrt{2000} \text{ (} 100 - \text{larger side} = \text{smaller side)} \\
 \\
 50 + \sqrt{2000} \\
 50 - \sqrt{2000} \\
 \hline
 500 \text{ (product of the two sides} = 2500 - 2000)
 \end{array}$$

11. [Margin: XI] *There is a rectangle whose diagonal [diameter] is $\sqrt{180}$, and whose greater side has to the lesser a triple ratio $[3 : 1]$. The sides and the area are sought.*

Let the lesser side be set as $1\ R$ and the greater as $3\ R$. Their squares, $1\ 3$ and $9\ 3$, make the sum $10\ 3$, [Margin: (Euclid) I.47] equal to the square of the diagonal — that is, the number 180. Dividing 180 by 10, then, gives $1\ 3 = 18$, and so $1\ R = \sqrt{18}$, the lesser side. The greater, therefore, being its triple, will be $\sqrt{162}$. The area will be $\sqrt{2916}$, that is, 54.

12. [Margin: XII] *There is a right triangle, one of whose sides [legs] is $\sqrt{18 + 3}$. The other side and the base [hypotenuse] together make $\sqrt{162 + 9}$. The individual sides are sought.*

(First, the square of the given leg $\sqrt{18 + 3}$ is formed:)

$$\begin{array}{r}
 \sqrt{18 + 3} \\
 \sqrt{18 + 3} \\
 \hline
 \sqrt{162 + 9} (= 3 \cdot (\sqrt{18 + 3})) \\
 18 + \sqrt{162} (= \sqrt{18} \cdot (\sqrt{18 + 3})) \\
 \hline
 27 + \sqrt{648} (= (\sqrt{18 + 3})^2)
 \end{array}$$

Let the other, unknown, leg be set as 1 R ; the base will therefore be $\sqrt{162 + 9} - 1 \text{ R}$. The squares of the [two] legs are $1 \text{ } \bar{3}$ and $27 + \sqrt{648}$, [Margin: (Euclid) I.47] which taken together are equal to the square of the base, namely the number $243 - 18 \text{ R} + 1 \text{ } \bar{3} + \sqrt{52488} - \sqrt{(648 \text{ } \bar{3})}$ [the last term = $\sqrt{648} \cdot \text{R}$, i.e. the square root of 648 census], as the two formulae of the squaring set out below show.

(Square of the base $\sqrt{162 + 9} - 1 \text{ R}$:)

$$\begin{array}{r}
 \sqrt{162 + 9} - 1 \text{ R} \\
 \sqrt{162 + 9} - 1 \text{ R} \\
 \hline
 - \sqrt{(162 \text{ } \bar{3})} - 9 \text{ R} + 1 \text{ } \bar{3} (= -1 \text{ R} \cdot (\sqrt{162 + 9} - 1 \text{ R})) \\
 \sqrt{13122 + 81} - 9 \text{ R} (= 9 \cdot (\sqrt{162 + 9} - 1 \text{ R})) \\
 + 162 + \sqrt{13122} - \sqrt{(162 \text{ } \bar{3})} (= \sqrt{162} \cdot (\sqrt{162 + 9} - 1 \text{ R})) \\
 \hline
 + 243 - 18 \text{ R} + 1 \text{ } \bar{3} + \sqrt{52488} - \sqrt{(648 \text{ } \bar{3})}
 \end{array}$$

If, then, $\sqrt{648}$ is subtracted from each side, there will be an equation between $1 \text{ } \bar{3} + 27$ and $243 - 18 \text{ R} + 1 \text{ } \bar{3} + \sqrt{41472} - \sqrt{(648 \text{ } \bar{3})}$. For if $\sqrt{648}$ is subtracted from $\sqrt{52488}$ — which are commensurable, standing in a ninefold ratio, since if $\sqrt{52488}$ is divided by $\sqrt{648}$ the quotient is $\sqrt{81}$, that is, 9 — the remainder becomes $\sqrt{41472}$. And removing $1 \text{ } \bar{3}$ from each, between 27 and $243 - 18 \text{ R} + \sqrt{41472} - \sqrt{(648 \text{ } \bar{3})}$; and again removing 27 from each, between 0 and $216 - 18 \text{ R} + \sqrt{41472} - \sqrt{(648 \text{ } \bar{3})}$. Adding 18 R and $\sqrt{(648 \text{ } \bar{3})}$ to both sides, between $18 \text{ R} + \sqrt{(648 \text{ } \bar{3})}$ and $216 + \sqrt{41472}$ — that is, between $\sqrt{(648 \text{ } \bar{3})} + 18 \text{ R}$ and $216 + \sqrt{41472}$.

Dividing, then, this number by that one (the cossist signs R and $\bar{3}$ being cast off from the divisor — because this whole aggregate, $\sqrt{(648 \text{ } \bar{3})} + 18 \text{ R}$, is the number of roots, since in forming the square of the base the term $\sqrt{(648 \text{ } \bar{3})}$ arose from 1 R into $\sqrt{162}$ taken twice, and therefore $\sqrt{(648 \text{ } \bar{3})}$ must be reckoned as a number of roots; consult what we wrote near the end of Chapter 26). The cossist signs R and

3 are cast off from the divisor just as, if an equation is between 20 and 5 R, so that we divide 20 by 5, the sign R is cast off, because 5 R is the number of roots. So (dividing by $\sqrt{(648 \text{ } 3) + 18 \text{ R}}$) the value 1 R will result — namely the other side of the triangle, which was set as 1 R.

But in order for this division to be carried out, both the divisor $\sqrt{648 + 18}$ and the dividend must be multiplied by $\sqrt{648 - 18}$, as we taught in Chapter 23. For thus the new divisor becomes 324, and the new dividend $\sqrt{3359232 + 1296}$, which multiplications are shown here.

$$\begin{array}{r}
 \sqrt{648 + 18} \\
 \sqrt{648 - 18} \\
 \hline
 324 \text{ (= } 648 - 324 \text{)} \\
 \\
 216 + \sqrt{41472} \\
 \sqrt{648 - 18} \\
 \hline
 - 3888 - \sqrt{13436928} \text{ (= } -18 \cdot (216 + \sqrt{41472})) \\
 + \sqrt{30233088} + \sqrt{26873856} \text{ (= } \sqrt{648} \cdot (216 + \sqrt{41472})) \\
 \hline
 \sqrt{3359232 + 1296}
 \end{array}$$

In the first multiplication, the square 324 of the latter part, 18, is subtracted from 648, the square of the former part, as we taught in Chapter 23. In the second, we subtracted $\sqrt{13436928}$ from $\sqrt{30233088}$ (because they are commensurable, the latter number having to the former the sesquialterate ratio [3 : 2]), and the residual number $\sqrt{3359232}$ was produced. Likewise from $\sqrt{26873856}$ — that is, from 5184 — we subtracted 3888, and the whole product $\sqrt{3359232 + 1296}$ was produced.

Now if $\sqrt{3359232}$ is divided by 324 — that is, by $\sqrt{104976}$ — the quotient becomes $\sqrt{32}$; and dividing 1296 by 324 gives the quotient 4. The value of 1 R, therefore, is $\sqrt{32 + 4}$, and this is the other side of the triangle sought, which we set as 1 R. The base opposite the right angle, which we posited as $\sqrt{162 + 9 - 1 \text{ R}}$, will be $\sqrt{50 + 5}$ — the number that remains if the root found, $\sqrt{32 + 4}$, is subtracted from $\sqrt{162 + 9}$, namely from the sum of the side sought and the base. This is also evident because the squares of the [two] legs taken together make $75 + \sqrt{5000}$, the square of the base found, $\sqrt{50 + 5}$. For those two squares are $27 + \sqrt{648}$ and $48 + \sqrt{2048}$.

Another, easier example. Let one side [leg] be 4, and let the sum of the other side and the base be 8. Let the other, unknown, leg be set as 1 R, and hence the base as $8 - 1 \text{ R}$. The squares of the two legs, 16 and 1 3, [Margin: (Euclid) I.47] taken together are equal to the square of the base, namely the number $64 - 16 \text{ R} +$

1 3. Removing 1 3 from each side, then, there will be an equation between 16 and $64 - 16 R$. Adding $16 R$ to both, between $16 + 16 R$ and 64. And removing 16 from each, between $16 R$ and 48. Dividing 48 by 16, then, gives $1 R = 3$, for the other leg. Hence the base, $8 - 1 R$, will be 5 — which is proved, since the squares of the legs, 16 and 9, are equal to the square of the base, 25. The area of the triangle will be 6, namely half the number 12 produced from one leg into the other.

13. *There is a circle whose diameter is 120, and from a certain point of the circumference a perpendicular dropped to the diameter is $\sqrt{(2925 - \sqrt{405000})}$. The parts of the diameter are sought.*

Diagram. A circle with a horizontal diameter drawn from B (left end) through the center to C (right end). A point A lies on the upper-left arc of the circumference. From A a perpendicular AD is dropped to the diameter, meeting it at D (between B and the center). The center of the circle is marked E, lying to the right of D on the diameter. The chords AB and AC are drawn, forming the two right triangles ABD and ACD. BD is the lesser segment of the diameter, DC the greater.

- A (upper-left of circumference): point from which the perpendicular is dropped
- B (left end of diameter): endpoint of the diameter
- C (right end of diameter): endpoint of the diameter
- D (on the diameter, below A): foot of the perpendicular AD
- E (on the diameter, right of D): center of the circle

Let the lesser part BD be set as $1 R$, and hence the greater, CD, as $120 - 1 R$. And since, by the scholium to Proposition 13 of Book 6 of Euclid, AD is the mean proportional between BD and CD, [Margin: (Euclid) VI.17] the rectangle under BD and CD will be equal to the square of AD. Now the product of BD ($1 R$) into CD ($120 - 1 R$) is the number $120 R - 1 3$. This, therefore, is equal to the square of the line AD, namely the number $2925 - \sqrt{405000}$. Adding $1 3$ to each side, then, there will be an equation between $120 R$ and $1 3 + 2925 - \sqrt{405000}$. And removing $2925 - \sqrt{405000}$ from both, between $1 3$ and $120 R - 2925 - \sqrt{405000}$ [i.e. $120 R$ less the binomial $(2925 - \sqrt{405000})$]. From this number, then, we shall extract the square root thus. Half the number of roots is 60; from its square, 3600, subtracting $2925 - \sqrt{405000}$, there remains the number $675 + \sqrt{405000}$, as you see in this formula.

$$\begin{array}{r} 3600 + \sqrt{0} \\ 2925 - \sqrt{405000} \\ \hline \end{array}$$

$$675 + \sqrt{405000}$$

To the square root of this, which is $\sqrt{675 + \sqrt{405000}}$, if the aforesaid half, 60, is added, the greater root becomes $60 + \sqrt{675 + \sqrt{405000}}$, and such is the portion CD. And if the same $\sqrt{675 + \sqrt{405000}}$ is subtracted from the same half, 60, the lesser root remains, $60 - \sqrt{675 + \sqrt{405000}}$, for the lesser portion BD — which is also obtained if from the whole diameter, 120, you subtract the portion CD, namely $60 + \sqrt{675 + \sqrt{405000}}$, as you see here.

$$\begin{array}{r} 120 + \sqrt{0} \\ 60 + \sqrt{675 + \sqrt{405000}} \\ \hline 60 - \sqrt{675 + \sqrt{405000}} \end{array}$$

Now, in order to find the value of each portion, $60 + \sqrt{675 + \sqrt{405000}}$ and $60 - \sqrt{675 + \sqrt{405000}}$, the root must be extracted from this binomial, $675 + \sqrt{405000}$, as we taught at the end of Chapter 28 — namely in this way. Let the greater part, 675, be divided (by the Lemma of Chapter 28) into two parts which, multiplied together, produce a fourth of the square of the lesser part, $\sqrt{405000}$. This is done thus. Half of the number to be divided, 675, is $675/2$; from its square, $455625/4$, subtracting a fourth of the square of the lesser part — which is $405000/4$ — there remains the number $50625/4$, whose square root is $225/2$. Added to the aforesaid half, $675/2$, it makes $900/2$, that is, 450, the greater part. And the same root, $225/2$, subtracted from the same half, $675/2$, leaves $450/2$, that is, 225, the lesser part. And these two parts, 450 and 225, multiplied together produce 101250, a fourth of the square 405000 of the lesser part. Now the roots of the parts 225 and 450, joined by the sign + — thus $\sqrt{225} + \sqrt{450}$, that is, $15 + \sqrt{450}$ — will give the root of the binomial $675 + \sqrt{405000}$. To this root, if the number 60 (namely half the number of roots) is added, the greater root becomes $75 + \sqrt{450}$, for the greater portion CD. And if the same root is subtracted from the same half, 60, the lesser portion BD remains, $45 - \sqrt{450}$, as this first formula shows — which is also obtained if the greater portion, $75 + \sqrt{450}$, is subtracted from the whole diameter, 120, as this second formula shows.

$$\begin{array}{r} 60 + \sqrt{0} \\ 15 + \sqrt{450} \\ \hline 45 - \sqrt{450} \text{ (BD)} \end{array}$$

$$\begin{array}{r} 120 + \sqrt{0} \\ 75 + \sqrt{450} \\ \hline 45 - \sqrt{450} \text{ (BD)} \end{array}$$

You see, then, how laborious was the extraction of the root from this cossist number, $120 R - 2925 - \sqrt{405000}$ — a difficulty that arose from the composite

irrational number ($-2925 - \sqrt{405000}$). We shall therefore propose another, easier example, in the matter of sines.

Let, then, in the same figure, the diameter BC be 2000, so that the whole sine [radius] is 1000; and let AD be the sine of 20 degrees, namely 342. The versed sines of the arcs AB and AC, and the sine of the complement DE, are sought. For BD let $1R$ be set, and hence CD will be $2000 - 1R$. And since AD is the mean proportional between BD and CD (from the scholium to Proposition 13 of Book 6 of Euclid), [Margin: (Euclid) VI.17] the number $2000R - 1\bar{3}$, produced from BD into CD, will be equal to the square of AD, namely the number 116964. Adding $1\bar{3}$ to each side, there will be an equation between $2000R$ and $1\bar{3} + 116964$. And removing 116964 from both, between $1\bar{3}$ and $2000R - 116964$.

Now thus: half the number of roots is 1000; from its square, 1000000, subtracting 116964, there remains the number 883036. To the square root of this, $\sqrt{883036}$, if the aforesaid half, 1000, is added, the greater root becomes $1000 + \sqrt{883036}$, for the greater portion CD. And if the same root is subtracted from the same half, 1000, the remainder becomes the lesser root, $1000 - \sqrt{883036}$, for the lesser part BD — which is also obtained if the greater portion is subtracted from the whole diameter, 2000, as you see in this formula.

$$\begin{array}{r} 2000 + \sqrt{} \\ 1000 + \sqrt{883036} \\ \hline 1000 - \sqrt{883036} \end{array}$$

And so, if all these are wanted in whole numbers: since the greater root of the number 883036 is 940 and the lesser 939, CD will be 1940 (greater than the true value) or 1939 (less than the true value). Likewise BD will be 60 (less than the true value) or 61 (greater than the true value). And if the semidiameter CE, 1000, is subtracted from 1940, there remains the sine of the complement, 940 (greater than the true value) or 939 (less than the true value), and so on.

Furthermore, once the segments BD and CD have been found, if the straight lines AB and AC are drawn, the chords themselves will be found from Proposition 47 of Book 1 of Euclid, since the triangles ABD, ACD, etc. become right triangles. In the prior example the chord AB will be found to be $\sqrt{(5400 - \sqrt{6480000})}$, and the chord AC $\sqrt{(9000 + \sqrt{6480000})}$, and so on.

[Cossist notation used throughout, rendered as printed: R = res (the unknown, x); $\bar{3}$ = census (x^2); $\bar{3}\bar{3}$ = census-of-census (x^4); $\sqrt{}$ = the universal square-root sign (radix); $\sqrt[4]{}$ = fourth root (radix of the census-of-census). Coefficients precede the sign, e.g. $150R = 150 \text{ res.}$]

14. There is a circle whose diameter is divided, by a line drawn perpendicular to the diameter, in extreme and mean ratio, and one of the two smaller chords is $150 - \sqrt{4500}$. The question is: how great is the diameter, how great are its portions, and what are the remaining lines?

Diagram. A circle with two perpendicular diameters meeting at the center, and a square (rhombus) inscribed on their four endpoints.

- A (left end of the horizontal diameter)
- B (right end of the horizontal diameter)
- C (top end of the vertical diameter)
- D (bottom end of the vertical diameter)
- E (center, where the two diameters cross)
- Sides AC, CB, BD, DA are drawn as chords, forming the inscribed square ACBD; AB and CD are its diagonals (the two diameters).

Let EB, the smaller part of the diameter AB, be posited as $1R$. Since AB is cut at E in extreme and mean ratio, and AE is the greater segment, AE will be the mean proportional between AB and EB. [Margin: a. Corollary to VI.8] Moreover BC too is the mean proportional between these same lines AB and EB; therefore AE and BC are equal. But BC is posited as $150 - \sqrt{4500}$; therefore AE too is $150 - \sqrt{4500}$.

[Margin: b. Corollary to VI.8] Since, then, CE is the mean proportional between AE and EB, [Margin: c. VI.17] the number $150R - \sqrt{(45003)}$, produced from AE into EB, will be equal to the square on CE; and therefore the line CE will be $\sqrt{(150R - \sqrt{(45003)})}$. But this same CE is $\sqrt{(27000 - \sqrt{405000000} - 13)}$, because 13, the square of the line EB, subtracted from $27000 - \sqrt{405000000}$, the square of the line BC, [Margin: d. I.47] leaves as remainder $27000 - \sqrt{405000000} - 13$, the square of the line CE.

Therefore the two quantities $150R - \sqrt{(45003)}$ and $27000 - \sqrt{405000000} - 13$ are equal, since each has been shown equal to the square of the line CE. Adding 13 to both sides, then, there is an equation between $13 + 150R - \sqrt{(45003)}$ and $27000 - \sqrt{405000000}$. And when $150R - \sqrt{(45003)}$ is removed from both sides, [it stands] between 13 and $27000 - \sqrt{405000000} - 150R - \sqrt{(45003)}$ [recte: $27000 - \sqrt{405000000} - 150R + \sqrt{(45003)}$].

From this cossic number we shall extract the value of $1R$ as follows. The whole is like a diminished cossic number, of which the portion $27000 - \sqrt{405000000}$ is taken as the absolute number, and the other portion, $- 150R - \sqrt{(45003)}$, as the number of roots. For $\sqrt{(45003)}$ too is a number of roots, as we said in Problem 12

and taught at the end of chapter 26. Accordingly, half the number of roots will be $75 - \sqrt{1125}$; to its square, $6750 - \sqrt{25312500}$, if there is added $27000 - \sqrt{405000000}$, the number becomes $33750 - \sqrt{632812500}$; and from its square root the half-number-of-roots, which was $75 - \sqrt{1125}$, must be subtracted, so that the value of $1R$ may be obtained.

Now from this residue, $33750 - \sqrt{632812500}$, we shall extract the square root thus. Let the greater term, 33750 , be cut into two parts which, multiplied together, produce a fourth part of the square of the lesser term — this fourth part being 158203125 . This will be done, according to the doctrine of the Lemma of chapter 28, thus: from 284765625 , the square of half the greater term 33750 , let the said fourth part, 158203125 , be subtracted; and let the square root, 11250 , of the remaining number, 126562500 , be added to and subtracted from 16875 , the half of the number 33750 . For the numbers produced, 28125 and 5625 , will be the parts sought, which multiplied together make 158203125 . And these parts, coupled by the sign of subtraction, will give the root of the proposed residue — with the square-root sign, however, prefixed to each part, in this manner: $\sqrt{28125} - \sqrt{5625}$, as we handed down at the end of chapter 28; that is, $\sqrt{28125} - 75$. From this root there must be subtracted the half-number-of-roots, $75 - \sqrt{1125}$, on account of the $-$ sign by which the number of roots was affected. And the remaining number will be $\sqrt{40500} - 150$, as the value of $1R$, that is, for the line EB .

Since, then, AE , equal to CB , is $150 - \sqrt{4500}$, the whole diameter will be $\sqrt{18000}$; for $\sqrt{40500}$ and $\sqrt{4500}$ are commensurable, having a triple ratio, and when the smaller is subtracted from the greater, $\sqrt{18000}$ is left. But CE will be $\sqrt{(\sqrt{1620000000} - 36000)}$, the number produced from AE into EB . Finally AC will be $\sqrt{(\sqrt{405000000} - 9000)}$, since its square is equal to the two squares of the lines AE and CE . And since AC equals AD , and CE equals ED , all the lines of this figure will be known.

And thus you see that the two squares of the lines AC and BC — namely $\sqrt{405000000} - 9000$ and $27000 - \sqrt{405000000}$ — make 18000 , the square of the diameter AB , as I.47 of Euclid requires, [Margin: e. III.31] since the angle ACB , being in a semicircle, is right.

Another way. Since the greater segment AE , equal to CB , is $150 - \sqrt{4500}$, if EB is posited as $1R$, the whole diameter AB will be $150 - \sqrt{4500} + 1R$. And thus the square of the greater segment AE , $27000 - \sqrt{405000000}$, will be equal to the number $150R - \sqrt{(45003)} + 13$, produced from EB (that is, from $1R$) into the whole AB (that is, into $150 - \sqrt{4500} + 1R$). And when the parts of this equation are

transposed, as above, there will again be found an equation between 13 and $27000 - \sqrt{405000000} - 150R + \sqrt{(45003)}$, and so on.

Another way. Let the whole diameter AB be posited as $1R$; then the smaller segment EB will be $1R - 150 - \sqrt{4500}$ [recte: $1R - 150 + \sqrt{4500}$]. [Margin: f. VI.17] And thus the number $13 - 150R - \sqrt{(45003)}$ [recte: $13 - 150R + \sqrt{(45003)}$], produced from $1R$ (that is, from AB) into $1R - 150 - \sqrt{4500}$ [recte: $1R - 150 + \sqrt{4500}$] (that is, into EB), will be equal to the square $27000 - \sqrt{405000000}$, [the square] of the line AE. And when $150R - \sqrt{(45003)}$ is added to both sides, the equation will be between 13 and $27000 - \sqrt{405000000} + 150R - \sqrt{(45003)}$. From this cosmic number, then, we shall extract the root thus. Let the whole number be understood as composed of two parts, of which the former, $27000 - \sqrt{405000000}$, stands for the absolute number, and the latter, $+ 150R - \sqrt{(45003)}$, stands for the number of roots. [Half the number of roots is 75 -] $\sqrt{1125}$; to its square, 6750 - $\sqrt{25312500}$, if there is added $27000 - \sqrt{405000000}$, [the number becomes 33750] - $\sqrt{632812500}$; from whose square root, $\sqrt{28125} - 75$ (which will be found as above), with $75 - \sqrt{1125}$, $1R$ becomes $\sqrt{18000}$, for the diameter AB — exactly as we found above, and so on.

15. There is a square whose side is 7. The diagonal is sought.

[Margin: g. Scholium to I.47] The square of the side is 49, whose double is the square of the diagonal. Therefore the square of the diagonal will be 98, and the diagonal itself $\sqrt{98}$.

16. There is a square whose diagonal is $\sqrt{98}$. The side is sought.

[Margin: a. Scholium to I.47] Since the square of the diagonal is double the square of the side, and the square of the diagonal is 98, the square of the side will be 49, and the side itself $\sqrt{49}$ — that is, 7.

17. A square whose diagonal and side make a sum of 6. The diagonal and side are sought.

Let the side be $1R$, and hence the diagonal $6 - 1R$. [Margin: b. Scholium to I.47] And since the square of the diagonal is double the square of the side, and the square of the side is 13 while the square of the diagonal is $36 - 12R + 13$, there will be an equation between 23 and $36 - 12R + 13$. Removing 13 from both sides: between 13 and $36 - 12R$.

Now thus: half the number of roots is 6; to its square, 36, if 36 is added, the number becomes 72; and from its root $\sqrt{72}$, if the aforesaid half 6 is subtracted, the remaining value of $1R$ becomes $\sqrt{72} - 6$, and so much will be the side. Which,

subtracted from 6, makes the remaining diagonal $12 - \sqrt{72}$, as you see in this formula:

$$\frac{\sqrt{0} + 6}{\sqrt{72} - 6}$$

$$12 - \sqrt{72}$$

Which is proved, because the square of the diagonal, $216 - \sqrt{41472}$, is double the square of the side, $108 - \sqrt{10368}$. And the diagonal and side make the sum 6.

18. There is a square whose side multiplied into the diagonal makes 10. The side and diagonal are sought.

Let the side be $1R$; hence the diagonal $10/1R$. For $1R$ into $10/1R$ makes 10; and this number, $10/1R$, is the quotient when 10 is divided by $1R$. For thus the quotient, multiplied into the divisor $1R$, produces the dividend 10. [Margin: c. Scholium to I.47] And since the square of the diagonal is double the square of the side, and the square of the side is 13 while the square of the diagonal is $100/13$, there will be an equality between 23 and $100/13$, which by cross-multiplication is reduced to this: between 233 and 100.

Dividing 100 by 2, then, 133 becomes 50, and $1R = \sqrt[4]{50}$, for the side. But since the diagonal was posited as $10/1R$ — that is, 10 divided by $1R$ — if 10 is divided by $\sqrt[4]{50}$ (namely by the value of $1R$), the quotient $\sqrt[4]{200}$ results, for the diagonal. Which is proved: for the side $\sqrt[4]{50}$ into the diagonal $\sqrt[4]{200}$ makes $\sqrt[4]{10000}$, that is 10. And $\sqrt[4]{40000}$, the square of the diagonal, is double the square of the side, $\sqrt[4]{2500}$.

19. There is a square whose diagonal exceeds the side by the number 3. The side and diagonal are sought.

Let the side be $1R$; hence the diagonal $1R + 3$. [Margin: d. Scholium to I.47] And since the square of the diagonal is double the square of the side, and the square of the side is 13 while the square of the diagonal is $13 + 6R + 9$, there will be an equation between 23 and $13 + 6R + 9$. Removing 13 from both sides: between 13 and $6R + 9$. Half the number of roots is 3; to its square 9, adding 9, the number becomes 18; to its root $\sqrt{18}$, if the aforesaid half 3 is added, $1R$ becomes $\sqrt{18} + 3$, for the side. The diagonal, moreover, will be $\sqrt{18} + 6$. Which is proved, because the square of the diagonal, $54 + \sqrt{2592}$, is double the square of the side, $27 + \sqrt{648}$. And the diagonal exceeds the side by the number 3.

20. There is a square whose side, multiplied into the difference between the diagonal and the side, makes 15. The diagonal and side are sought.

Let the side be $1R$, and therefore the difference between the diagonal and the side $15/1R$, obtained from dividing 15 by $1R$, so that the quotient multiplied into the divisor $1R$ may produce the dividend 15. Since, then, the diagonal exceeds the side by this difference, the diagonal will be $1R + 15/1R$. [Margin: e. Scholium to I.47] And since the square of the diagonal is double the square of the side, and the square of the side is 13 while the square of the diagonal is $(133 + 303 + 225)/13$, there will be an equation between 23 and $(133 + 303 + 225)/13$, which by cross-multiplication is reduced to an equation between 233 and $133 + 303 + 225$.

Removing 133 from both sides, then, the equation is between 133 and $303 + 225$. From this number we shall extract the census-of-census root thus. Half the number of censuses is 15; to its square, 225, adding 225, makes the number 450; to whose square root $\sqrt{450}$, if the aforesaid half 15 is added, 13 becomes $\sqrt{450} + 15$, and $1R$ becomes $\sqrt{(\sqrt{450} + 15)}$, as was taught at the end of chapter 12; and so much is the side. The double of this square, $\sqrt{450} + 15$, gives the square of the diagonal, $\sqrt{1800 + 30}$; and hence the diagonal will be $\sqrt{(\sqrt{1800 + 30})}$. Which is proved: for if the side is subtracted from the diagonal, the remaining difference $\sqrt{(\sqrt{1800 + 30})} - \sqrt{(\sqrt{450} + 15)}$, multiplied into the side $\sqrt{(\sqrt{450} + 15)}$, produces the number 15. Consult chapter 24, where this multiplication is set out explicitly.

21. There is a rectangle whose area is 30 and the ratio of whose sides is sesquialter [3:2]. The sides and diagonal are sought.

Let the smaller side be $2R$, and hence the greater $3R$, sesquialter to it. From the multiplication of the sides together comes the number 63 , the area of the rectangle, equal to the given area 30. Dividing 30 by 6, then, 13 becomes 5, and $1R = \sqrt{5}$. And since we posited the smaller side to be $2R$, the smaller side will be $\sqrt{20}$ — namely the number double the root, that is, double the number $\sqrt{5}$. Since, then, the greater side was posited as $3R$, the greater side will be $\sqrt{45}$ — namely the number triple of one root, which was $\sqrt{5}$.

[Margin: f. Scholium to I.47] Since the squares of the sides, 20 and 45, together equal the square of the diagonal, the square of the diagonal will be 65, and the diagonal itself $\sqrt{65}$. Which is proved, because the side $\sqrt{20}$ into the side $\sqrt{45}$ makes $\sqrt{900}$ — that is, 30, exactly the area we posited for the rectangle.

22. There is a rectangle whose area is 80 and difference of sides is 2. The sides and diagonal are sought.

Let the smaller side be $1R$, and hence the greater $1R + 2$. These sides, multiplied together, make the area $13 + 2R$, equal to the given area 80. Removing $2R$ from both sides, then, there will be an equality between 13 and $80 - 2R$. Half the

number of roots is 1; to its square 1, adding 80, makes the number 81; from its root 9, if the aforesaid half 1 is subtracted, the remaining value of $1R$ becomes 8, for the smaller side. The greater, then, will be 10, exceeding it by two. The side 8 into the side 10 makes the proposed area 80. The two squares of the sides, 64 and 100, equal the square of the diagonal; and hence the square of the diagonal will be 164, and the diagonal itself $\sqrt{164}$.

23. There is a rectangle whose area is 80 and sum of the two sides 20. The sides and diagonal are sought.

Let one side be $1R$, and hence the other $20 - 1R$. These, multiplied together, make the area $20R - 13$, equal to the given area 80. Adding 13 to both sides, then, there will be an equation between $13 + 80$ and $20R$. And removing 80 from both, between 13 and $20R - 80$. Half the number of roots is 10; from its square 100, subtracting 80, the number 20 remains; to its root $\sqrt{20}$, adding the aforesaid half 10, $1R$ becomes $\sqrt{20} + 10$, the greater, for one side. And the same root $\sqrt{20}$, subtracted from the aforesaid half 10, makes the remaining smaller root $10 - \sqrt{20}$ — which is also obtained if the found side $\sqrt{20} + 10$ is subtracted from 20, the sum of the sides, as you see here:

$$\begin{array}{r} \sqrt{0} + 20 \\ \sqrt{20} + 10 \\ \hline - \sqrt{20} + 10 \end{array}$$

[Margin: a. I.47] The found sides, $10 + \sqrt{20}$ and $10 - \sqrt{20}$, multiplied together, produce the proposed area 80. This multiplication is easily done if (by the teaching of chapter 23) the square of the latter part is subtracted from 100, the square of the greater part, as you see in this second formula:

$$\begin{array}{r} 10 + \sqrt{20} \\ 10 - \sqrt{20} \\ \hline 80 \end{array}$$

Now the squares of the sides, $120 + \sqrt{8000}$ and $120 - \sqrt{8000}$, [Margin: a. I.47] equal the square of the diagonal; and hence the square of the diagonal will be 240, and the diagonal itself $\sqrt{240}$.

24. There is a rectangle whose diagonal is 30 and sum of sides 42. The sides and area are sought.

Let one side be $1R$, and hence the other $42 - 1R$. [Margin: b. I.47] Their squares, 13 and $1764 - 84R + 13$, equal the square of the diagonal; and therefore there will

be an equation between $23 + 1764 - 84R$ and 900. And when $84R$ is added to both sides, between $84R + 900$ and $23 + 1764$. Again, when 864 is removed from both, between 23 and $84R - 864$. And dividing all by 2, between 13 and $42R - 432$.

Half the number of roots is 21; from its square 441, subtracting 432, the number 9 remains; to its root 3, adding the aforesaid half 21, the greater root becomes 24, for the greater side. And the same root 3, subtracted from the aforesaid half 21, leaves the smaller root 18, for the smaller side — which is also obtained if the found greater side 24 is subtracted from 42, the sum of the sides. Now 24 into 18 makes the area 432. Here too you see that the squares of the sides, 576 and 324, make 900, the square of the proposed diagonal 30.

25. There is a rectangle whose area together with the diagonal makes 15, and the difference of the sides is $\sqrt{5}$. The sides, diagonal, and area are sought.

Let the smaller side be $1R - \sqrt{(1\frac{1}{4})}$ (this number $\sqrt{(1\frac{1}{4})}$ being half the given difference $\sqrt{5}$), and the greater $1R + \sqrt{(1\frac{1}{4})}$. For thus the latter will exceed the former by the difference $\sqrt{5}$, since the smaller subtracted from the greater leaves $\sqrt{5}$, as you see in this formula:

$$\begin{array}{r} 1R + \sqrt{(1\frac{1}{4})} \\ 1R - \sqrt{(1\frac{1}{4})} \\ \hline \sqrt{5} \end{array}$$

For $\sqrt{(1\frac{1}{4})}$ and $\sqrt{(1\frac{1}{4})}$ must be added together, on account of the different signs + and -; and this addition is done thus: if $\sqrt{(1\frac{1}{4})}$ is doubled, and to the doubled number $\sqrt{(20/4)}$ — that is, $\sqrt{5}$ — the + sign of the upper number is understood to be prefixed, as we said in chapter 4.

The side $1R - \sqrt{(1\frac{1}{4})}$ into the side $1R + \sqrt{(1\frac{1}{4})}$ makes the area $13 - 1\frac{1}{4}$. And since the area together with the diagonal makes 15, if from this sum the said area $13 - 1\frac{1}{4}$ is subtracted, there will remain the diagonal $16\frac{1}{4} - 13$, whose square will be $133 + 264 \frac{1}{16} - 32\frac{1}{2}3$ — which keep. [Margin: c. I.47] Since the square of the smaller side, $1R - \sqrt{(1\frac{1}{4})}$, is $13 - \sqrt{(53)} + 1\frac{1}{4}$, and the square of the greater side, $1R + \sqrt{(1\frac{1}{4})}$, is $13 + \sqrt{(53)} + 1\frac{1}{4}$, these two squares make up $23 + 2\frac{1}{2}$. This number will equal the square of the diagonal kept above, namely $133 + 264 \frac{1}{16} - 32\frac{1}{2}3$. Adding $32\frac{1}{2}3$ to both sides, then, there will be an equality between $34\frac{1}{2}3 + 2\frac{1}{2}$ and $133 + 264 \frac{1}{16}$. And again, removing $261 \frac{9}{16}$ from both sides, between $34\frac{1}{2}3 - 261 \frac{9}{16}$ and 133 .

Now thus: half the number of censuses is $17\frac{1}{4}$; from its square, $297\frac{9}{16}$, subtracting $261\frac{9}{16}$, the number 36 remains; whose root 6, subtracted from the aforesaid half $17\frac{1}{4}$, leaves the smaller census ...

(Cossist notation used on these pages, glossed once here: **1R** = one res (the unknown, x); **13** = one census (x^2); **133** = census of a census (x^4); $\sqrt{3} N$ = the radix, i.e. the square root, of the quantity N ; $\sqrt{33} N$ = the radix of a zenzizensus, i.e. the fourth root, of N . Throughout, the printed res sign renders in the scan as a "2-with-a-tail.")

[Continuation of Problem 25 — rectangle: sides sought from area + diagonal]

[yields] the census $11\frac{1}{4}$, and the smaller root $\sqrt{3} 11\frac{1}{4}$. Therefore the smaller side, which was set as $1R - \sqrt{3} \frac{5}{4}$, will be $\sqrt{3} 11\frac{1}{4} - \sqrt{3} \frac{5}{4}$; while the larger, which we set as $1R + \sqrt{3} \frac{5}{4}$, will be $\sqrt{3} 11\frac{1}{4} + \sqrt{3} \frac{5}{4}$. These two sides differ from each other by the number $\sqrt{3} 5$, as may be seen in this subtraction formula — that is, $\sqrt{3} 5$. And the same two sides, multiplied together, make 10 for the area, as you see here. For $+$ $\sqrt{3} \frac{225}{16}$ and $- \sqrt{3} \frac{225}{16}$ cancel each other, while $\frac{45}{4} - \frac{5}{4}$ make $\frac{40}{4}$, that is, 10.

$$\sqrt{3} 11\frac{1}{4} + \sqrt{3} \frac{5}{4} \quad \sqrt{3} 11\frac{1}{4} - \sqrt{3} \frac{5}{4} \quad \text{—————} \quad 0 + \sqrt{3} \frac{20}{4}$$

$$\sqrt{3} \frac{45}{4} - \sqrt{3} \frac{5}{4} \quad \sqrt{3} \frac{45}{4} + \sqrt{3} \frac{5}{4} \quad \text{—————}$$

$$\bullet \quad \sqrt{3} \frac{225}{16} - \frac{5}{4} \frac{45}{4} - \sqrt{3} \frac{225}{16} \quad \text{—————} \quad \frac{40}{4} \text{ id est, } 10$$

[Margin: Euclid I.47]

Next, the squares of the sides are $\frac{50}{4} - \sqrt{3} \frac{900}{4}$ [recte: $\frac{900}{16}$] and $\frac{50}{4} + \sqrt{3} \frac{900}{4}$, which make the sum 25, equal to the square of the diameter (because $- \sqrt{3} \frac{900}{4}$ and $+ \sqrt{3} \frac{900}{4}$ cancel each other, and from $\frac{50}{4}$ and $\frac{50}{4}$ come $\frac{100}{4}$, that is, 25). Hence $\sqrt{3} 25$ — namely 5 — will be the diameter, which together with the area found, 10, makes the sum 15 that was proposed.

Now although the cossic expression $34\frac{1}{2}3 - 261 \frac{9}{16}$ has a double root — as we showed in chapter 12 — nevertheless the larger root, which is $\sqrt{3} 23\frac{1}{4}$ (and which is obtained if the root 6 found above is added to the aforesaid half, $17\frac{1}{4}$; for the number so composed, $23\frac{1}{4}$, will be the larger census, and $\sqrt{3} 23\frac{1}{4}$ the larger root), accomplishes nothing here. For the sides would then be $\sqrt{3} 23\frac{1}{4} - \sqrt{3} \frac{5}{4}$ and $\sqrt{3} 23\frac{1}{4} + \sqrt{3} \frac{5}{4}$, which multiplied together will make a larger area than 10, and a larger diameter than 5 will be found; and so the sum composed from the area and the diameter would be larger than the proposed sum, 15. You see here, then, the remarkable fact that one cannot always take either root of the equation —

which indeed rarely happens. And so, if the examination made through the one root does not answer to the problem, the other root must be taken. As here in the proposed problem, since the examination through the larger root does not succeed, the smaller root was therefore taken.

Another, similar example. Let the difference of the sides be 2, and let the area together with the diameter make 58. Again the sides, the diameter, and the area are sought.

Let the smaller side be $1R - 1$ (1 being half the difference of the sides), and the larger $1R + 1$. For thus the one exceeds the other by two, since the smaller taken from the larger leaves 2, as you see here.

$$1R + 1 \quad 1R - 1 \quad \text{—————} \quad 0 + 2$$

The squares of the sides are $13 - 2R + 1$ and $13 + 2R + 1$, which make the sum $23 + 2$, equal to the square of the diameter. [Margin: Euclid I.47] And $(1R - 1)$ times $(1R + 1)$ makes the area $13 - 1$, which together with the diameter must make 58. If, therefore, the said area is subtracted from 58, the remaining diameter becomes $59 - 13$, whose square is $133 + 3481 = 1183$. And since a little earlier the same square [of the diameter] was found to be $23 + 2$, these two squares will be equal to each other.

Adding, therefore, 1183 to both sides, the equation will be between $133 + 3481$ and $1203 + 2$. Removing 2 from each side, it is between $133 + 3479$ and 1203; and again removing 3479 from both, between 133 and $1203 - 3479$. Half the number of census-terms is 60; from its square, 3600, subtract 3479, and there remains the number 121, whose root is 11; then, subtracted from the aforesaid half 60, it leaves 49, the smaller census — and hence the smaller root will be 7. Since, therefore, the smaller side was set as $1R - 1$ and the larger as $1R + 1$, the former will be 6 and the latter 8, whose difference is 2. From 6 times 8 the area becomes 48, which together with the diameter must make 58. Subtracting, then, 48 from 58, the remaining diameter becomes 10 — which is also found if the squares of the sides, 36 and 64, are gathered into one sum, 100, which will equal the square of the diameter, and so the diameter will be $\sqrt{100}$, that is, 10. [Margin: Euclid I.47]

Here too, with the larger root — which is $\sqrt{71}$ (because the larger root, 11, added to the aforesaid half, 60, makes 71, the larger census, and therefore the larger root will be $\sqrt{71}$) — you will accomplish nothing. For the sides will be $\sqrt{71} - 1$ and $\sqrt{71} + 1$, larger than 6 and 8, and so will contain a larger area than 48; and their squares will make up a square of the diameter larger than 100; and hence

the diameter will be larger than 10; and therefore the area together with the diameter will produce a larger sum than 58. Here again you see that one cannot always take both roots for the solution of the problem, as we said a little earlier.

26. [Margin: XXVI] There is a rectangle one of whose sides is 6, and what is made from the other side times the diameter is 80. The other side, the diameter, and the area are sought.

Let the other side be $1R$. The squares of the sides, 6 and $1R$, are 36 and 13, whose sum, $13 + 36$, will equal the square of the diameter. [Margin: Euclid I.47] And since from the unknown side times the diameter comes the number 80, if 80 is divided by $1R$, the unknown side, the quotient $\frac{80}{1R}$ will result, and the diameter will be just that; for since from this quotient, $\frac{80}{1R}$, times $1R$, the divisor, the dividend 80 is produced, the square of the diameter will be $\frac{6400}{13}$, equal to the earlier square of the same diameter, $13 + 36$. This equation, by cross-multiplication, is reduced to this: between $133 + 363$ and 6400. Removing, then, 363 from both sides, the equation is between 133 and $6400 - 363$. Half the number of census-terms is 18; to its square, 324, add 6400, and it makes the number 6724, whose root is 82; if the aforesaid half, 18, is taken away, the value of 13 remains, 64. Therefore $1R$ will be 8, the side sought. For from the side 6 times the side 8 the area becomes 48; and the squares of the sides, 36 and 64, make the square of the diameter, 100, and so the diameter will be 10. Here you see that from the side 8, found, times the diameter 10, comes 80.

27. There is an equilateral triangle ABC whose area is 60. The side BC and the perpendicular AD are sought.

Diagram. An equilateral triangle with apex A at the top and base BC at the bottom, the perpendicular AD drawn from A to the base, meeting it at D, the midpoint of BC.

- A (apex): vertex A
- B (lower left): base endpoint
- D (base midpoint): foot of the perpendicular AD
- C (lower right): base endpoint

[Margin: Euclid, scholium to prop. 26 of Book I] Since AD cuts BC in half, let BD be $1R$ and each of the sides $2R$; and the square of BD will be 13, and the square of the side AB, 43. [Margin: Euclid I.47] Since, then, the two squares of BD and AD equal the square of AB, the square of AD will be 33, and AD itself $\sqrt{33}$. From BD

(1R) times AD ($\sqrt{3} 33$) comes the area of the triangle ABC, as we demonstrated in chapter 2 [of Book 4 of the *Geometria Practica*] —

— namely $\sqrt{3} 333$, as appears here: because 1R must be squared, and the sign $\sqrt{3}$ prefixed to the square, 13; and thus $\sqrt{3} 33$ must be multiplied by $\sqrt{3} 13$, as we wrote at the end of chapter 26.

$$\sqrt{3} 33 \text{ 1R} \text{ ————— } \sqrt{3} 333$$

And since the area is set as 60, the equation will be between 60 and $\sqrt{3} 333$, and therefore also between their squares, 3600 and 333. Dividing, then, 3600 by 3 gives $133 = 1200$, and $\sqrt{33} 1200$ will be the value of 1R, that is, of the line BD; and hence the whole side BC will be $\sqrt{33} 19200$, namely the double of $\sqrt{33} 1200$. For in order to double $\sqrt{33} 1200$, it must be multiplied by the zenzizensus [fourth power] of two, namely by 16. Now since the area is produced from BD, $\sqrt{33} 1200$, times AD, [and is] 60 — that is, the root $\sqrt{33} 12960000$ — [when this] is divided by $\sqrt{33} 1200$, the quotient $\sqrt{33} 10800$ results for the perpendicular AD; which is proved. For from $\sqrt{33} 1200$ times $\sqrt{33} 10800$ comes $\sqrt{33} 12960000$, that is, 60, the proposed area.

Another example. Let the area of the triangle be $\sqrt{3} 243$. With the same things posited, an equality will be found between $\sqrt{3} 243$ and $\sqrt{3} 333$, and hence also between their squares, 243 and 333. Dividing, then, 243 by 3 gives $133 = 81$, whose $\sqrt{33}$ [fourth root] is 3, the quantity of the line BD; and hence the whole side BC will be 6. But if the area $\sqrt{3} 243$ is divided by BD, which is 3 — that is, by $\sqrt{3} 9$ — the quotient becomes $\sqrt{3} 27$ for the perpendicular AD; which is proved, because from BD, 3 (that is, $\sqrt{3} 9$), times AD, $\sqrt{3} 27$, comes $\sqrt{3} 243$, the proposed area.

XXVIII. 28. [Margin: XXVIII] There is an equilateral triangle ABC whose perpendicular AD is 6. The side BC and the area are sought.

Diagram. The same equilateral triangle redrawn: apex A at the top, base BC at the bottom, and the perpendicular AD from A to the midpoint D of BC.

- A (apex): vertex A
- B (lower left): base endpoint
- D (base midpoint): foot of the perpendicular AD
- C (lower right): base endpoint

Let BD be $1R$, and AD will be found to be $\sqrt{3} 33$, as in the preceding problem. And since the same AD is set at 6, the equation will be between 6 and $\sqrt{3} 33$. If, therefore, 6 is divided by $\sqrt{3} 3$ — the number of the greater character, 3 — then $1R$ becomes $\sqrt{3} 12$ for the line BD ; because, as we said at the end of chapter 26, the number $\sqrt{3} 33$ is a number of roots [*res*], and not of censuses.

Or thus: since the equation is between 6 and $\sqrt{3} 33$, there will also be an equation between their squares, 36 and 33. Dividing, then, 36 by 3 gives $13 = 12$, and $1R = \sqrt{3} 12$ for the line BD . And therefore the whole side BC will be $\sqrt{3} 48$, namely double $\sqrt{3} 12$. The area, moreover, will be $\sqrt{3} 432$ — the number produced from BD , $\sqrt{3} 12$, times AD , 6, that is, $\sqrt{3} 36$.

Another example. Let the perpendicular AD be 3. With the same things posited, an equation will be found between 3 and $\sqrt{3} 33$. Dividing, then, 3 by $\sqrt{3} 3$ — which is the number of roots, as was said — gives $1R = \sqrt{3} 3$ for the line BD , and the whole side BC will be $\sqrt{3} 12$.

Or thus: since the equation is between 3 and $\sqrt{3} 33$, there will also be an equation between their squares, 9 and 33. Dividing, then, 9 by 3 gives $13 = 3$, and $1R = \sqrt{3} 3$ for the line BD , and the whole side BC will be $\sqrt{3} 12$. The area, moreover, will be $\sqrt{3} 27$ — the number produced from BD , $\sqrt{3} 3$, times AD , 3, that is, $\sqrt{3} 9$. That this is true is plainly evident, since the square of the side, namely 12, has to the square of the perpendicular, that is, to 9, the sesquitertian ratio [4 : 3], as Euclid requires in proposition 12 of Book 13.

XXIX. 29. [Margin: XXIX] There is a rhombus $ABCD$ whose area is 60, and the ratio of the diagonals AC , BD is sesquiquartan [5 : 4]. The side and the diagonals are sought.

Diagram. A rhombus standing on a vertex: A at the top, C at the bottom, B at the left, D at the right. The two diagonals AC (vertical) and BD (horizontal) cross at the centre E , dividing each other into halves.

- A (top): vertex
- B (left): vertex
- D (right): vertex
- C (bottom): vertex
- E (centre): intersection of the diagonals AC and BD

Let the greater diagonal AC be $5R$, and hence the smaller, BD , $4R$. [Margin: Euclid, scholium to prop. 34 of Book I] Since, then, the diagonals in a rhombus bisect

each other, and bisect the angles as well, [Margin: Euclid, scholium to prop. 26 of Book I] AE — which in the isosceles triangle ABD bisects the angle A — will be perpendicular to BD. Therefore, since the area of triangle ABD is produced from AE times BE (as we showed in chapter 2 of Book 4 of the *Geometria Practica*), [Margin: Euclid II.1] and the area of triangle CBD from CE — that is, from AE — times DE, the whole area of the rhombus will be produced from AE times BD. And in the same way the same area will be produced from BE times AC. Now from BE, 2R, times AC, 5R, comes the number 10 $\frac{3}{4}$. Therefore the equation will be between 60 and 10 $\frac{3}{4}$, since each number is the area of the rhombus. Dividing, then, 60 by 10 gives 1 $\frac{3}{4}$ = 6, and 1R = $\sqrt{3}$ 6. Therefore AC, set as 5R, will be $\sqrt{3}$ 150; and BD, set as 4R, will be $\sqrt{3}$ 96. This is proved, because from BE, $\sqrt{3}$ 24 (half of the diagonal BD, $\sqrt{3}$ 96), times AC, $\sqrt{3}$ 150, comes the area of the rhombus, $\sqrt{3}$ 3600, that is, 60. And thus both diagonals are known.

Then, since AE and EB are $\sqrt{3}$ 37 $\frac{1}{2}$ and $\sqrt{3}$ 24, and their squares, 37 $\frac{1}{2}$ and 24, together make 61 $\frac{1}{2}$ — a sum equal to the square of the side AB — [Margin: Euclid I.47] the side AB will be $\sqrt{3}$ 61 $\frac{1}{2}$.

XXX. 30. [Margin: XXX] There is a rhombus ABCD whose area is 60, and when the greater diagonal AC is divided by the smaller, BD, the quotient is 1 $\frac{1}{4}$. The diagonals and the side are sought.

Since from BE times AC comes the area of the rhombus, from BD times AC will come double the area, namely 120. Let, then, the smaller diagonal BD be 1R, and hence the greater diagonal AC 1 $\frac{1}{4}$ R, so that, divided by BD, it gives the quotient 1 $\frac{1}{4}$. From BD, 1R, times AC, 1 $\frac{1}{4}$ R, comes the number $\frac{5}{4}$ $\frac{3}{4}$, which will equal 120. By cross-multiplication, then, there will be an equality between 5 $\frac{3}{4}$ and 480. Dividing, therefore, 480 by 5 gives 1 $\frac{3}{4}$ = 96, and 1R = $\sqrt{3}$ 96 for the diagonal BD, which was set as 1R. But AC, set as 1 $\frac{1}{4}$ R, will be $\sqrt{3}$ 150; because $\sqrt{3}$ 96, multiplied by $\frac{5}{4}$ — that is, by $\sqrt{3}$ $\frac{25}{16}$ — makes $\sqrt{3}$ 150. Or also: if to $\sqrt{3}$ 96 is added a quarter [of it], namely $\sqrt{3}$ 6, the sum becomes $\sqrt{3}$ 150. And thus the diagonals BD, $\sqrt{3}$ 96, and AC, $\sqrt{3}$ —

— $\sqrt{3}$ 150 have been found; which is proved, because from $\sqrt{3}$ 24, half of the diagonal BD, times $\sqrt{3}$ 150, the diagonal AC, comes the area of the rhombus, $\sqrt{3}$ 3600, that is, 60. Then, since AE and EB are $\sqrt{3}$ 37 $\frac{1}{2}$ and $\sqrt{3}$ 24, [Margin: Euclid I.47] their squares, 37 $\frac{1}{2}$ and 24, will make a sum, 61 $\frac{1}{2}$, equal to the square of the side AB; and therefore the side AB will be $\sqrt{3}$ 61 $\frac{1}{2}$.

APPENDIX.

These, diligent reader, are the things which seemed to me worth writing about the rule of Algebra and its explanation. Whoever requires more, let him seek it from the other writers on this rule. Nevertheless I will not be reluctant to add, by way of appendix, four riddles of Ptolemy, taken from the first book of the Greek Epigrams, and a fifth, a different one, of Euclid — with the Greek epigrams themselves appended, together with their interpretation, not indeed word for word, but so that the matter might become clearer, rendering only their sense. The riddles of Ptolemy are these.

1. On the statue of Pallas — how many talents of gold it weighs.

[Greek: Πρῶτον. — "First."]

[Greek epigram (Palatine Anthology XIV.123), transcribed as printed: Παλλὰς ἐγὼ τελέθω σφυρήλατος, αὐτὰρ ὁ χρυσὸς Αἰζηῶν πέλεται δῶρον ἀοιδόπων, Ἥμισυ μὲν χρυσοῖο Χαείσιος, ὀγδοάτην δὲ Θέσπις, καὶ δεκάτην μοῖραν ἔθηκε Σόλων. Αὐτὰρ ἑικοστήν Θεμίσων, τὰ δὲ λοιπὰ τάλαντα Ἐννέα, καὶ τέχνη, δῶρον Λειστοδίκου. (A few accented letters are ligatured/abraded in the scan; this fount also sets ε where the received text has ρ, and the page's Λ in Λειστοδίκου is crossbar-less — both are reproduced as set. See QA.)]

Clavius's Latin paraphrase: I am Pallas, wrought with the hammer in this fashion; but the gold is the gift of young men who devote themselves to the study of poetry. Charisius contributed half of the gold; Thespis an eighth; Solon next a tenth; and Themison a twentieth; but the remaining nine talents, and likewise the fee owed to the craftsman, Aristodicus contributed. — The question is the total weight of the statue, and how many talents each contributed.

Let the weight of the statue be $1R$ of talents of gold. Then Charisius contributed $\frac{1}{2}R$ talents; Thespis $\frac{1}{8}R$; Solon $\frac{1}{10}R$; Themison $\frac{1}{20}R$; and Aristodicus 9 talents — all which parts make up $\frac{31}{40}R + 9$, a sum equal to $1R$. Removing, then, $\frac{31}{40}R$ from each side, the equality will be between 9 and $\frac{9}{40}R$. Dividing, then, 9 by $\frac{9}{40}$ gives $1R = 40$; and of so many talents was the statue. Charisius, then, contributed 20 talents, namely half the total weight; Thespis 5 talents, that is, an eighth part; Solon 4 talents, namely a tenth part; and Themison, finally, 2 talents, namely a twentieth part — all which parts, together with the 9 talents that Aristodicus contributed, make up 40 talents.

2. On the herds of Augeas — how many oxen there were.

[Greek: Δεύτερον. — "Second."]

[Greek epigram (Palatine Anthology XIV.4), transcribed as printed: Αὐγεῖην ἐρέεινε μέγα σθένος Ἀλκίδαο, Πληθὺν βουκολίων διζήμενος, ὃς δ' ἀπάμειπτο· Ἀμφὶ μὲν Ἀλφειοῖο ῥοὰς φίλος ἥμισυ τῶνδε, Μοῖρῃ δ' ὀγδοάτῃ ὄχθον Κρόνου ἀμφινέμονται. Δωδεκάτῃ δ' ἀπάνευθε Ταραξίπποιο παρ' οὔρον, Ἀμφὶ δ' ἄρ' Ἥλιν δῖαν ἔεικοστή νεμέδονται. Αὐτὰρ ἐν Ἀρκαδίῃ τειηκοστήν προλέλοιπα, Λοιπὰς δ' αὖ λ[?][?]εις ἀγέλας τόδε πεντήκοντα. (A few word-endings and accents are ligatured/abraded in the scan — see QA.)]

Clavius's Latin paraphrase: The noble Hercules asked Augeas about the multitude of his herds, and he replied: Half of these, my friend, graze around the river Alpheus; an eighth around the hill of Saturn [Cronus]; a twelfth, far from here, beside the outermost regions of Taraxippus; a twentieth of them graze around Elis; a thirtieth I have left in Arcadia; and the remaining herds, fifty in number, you may see for yourself. — The question is the number of the oxen, and how many were in each place.

Let the number of oxen be $1R$. Then beside the river Alpheus there were $\frac{1}{2}R$ oxen; around the hill of Saturn $\frac{1}{8}R$; beside the outermost regions of Taraxippus $\frac{1}{12}R$; around the mountain of Elis $\frac{1}{20}R$; in Arcadia $\frac{1}{30}R$; and with Augeas 50 oxen — all which parts make up $\frac{19}{24}R + 50$, a sum equal to $1R$. Removing, then, $\frac{19}{24}R$ from each side, the equality will be between 50 and $\frac{5}{24}R$. Dividing, therefore, 50 by $\frac{5}{24}$ gives $1R = 240$, for the number of all the oxen; of which...

(Continuation of Epigram 2 — the herds of Augeas, resumed at the catchword "quorum".)

of which the half — 120 — were around the river Alpheus; and 30 oxen, that is $\frac{1}{8}$, around the hill of Saturn [Cronus]; and $\frac{1}{12}$, namely 20, by the far end of the Taraxippus; but by the mountain of Elis, 12, namely $\frac{1}{20}$; and finally in Arcadia, 8 oxen — the $\frac{1}{30}$ part. All these oxen, together with the 50 that were near Augeas, make 240.

[Margin: III.]

3. THE CHANNELS OF THE BRONZE LION [LEONIS ÆNEI CANALES]

Τρίτον. [Greek: "Third."]

Χάλκεός εἰμι λέων, κρουνοὶ [?] δέ μοι ὄμματα δοιὰ Καὶ στόμα, σὺν δὲ θέναρ
 δεξιτεροῖο ποδός. Πλήθει δὲ κρατῆρα δι' ἡμασι [?] δεξιὸν ὄμμα, Καὶ λαὶον
 τελιοῖς [?], κ(αὶ) πισύρεσσι θέναρ. Ἄρκιον ἐξ ὥρέων [?] πληῆσαι στόμα· ἐν δ'
 ἅμα πάντα, Καὶ στόμα; κ(αὶ) γλῆναι [?], κ(αὶ) θέναρ, εἰπὲ πόσον;

(Greek verse; the sense is given in Clavius's Latin paraphrase that follows.)

I am a bronze lion. My channels [spouts] are two eyes, and a mouth together with the sole of my right foot. Now they fill one and the same mixing-bowl [crater]: the right eye in two days, the left in three, and the sole in four days; while the mouth can fill it in six hours. Tell me, then, in how much time all of these together — mouth, eyes, and sole — will fill that same bowl.

Let 1R [one res, i.e. the unknown number of hours] be posited for the time, in hours, and let the argument run thus:

Hor.	Crater.	Hor.	Crater.
48		1R	1/48 R
72	1		1/72 R
96			1/96 R
6			1/6 R

If 1 bowl is filled in 48 hours, then in 1R hours 1/48 R of a bowl will be filled; and likewise for the rest, as you see by applying the Rule of Three four times over. All these parts together make 61/288 R of one bowl, equal to 1 bowl. Therefore, if 1 is divided by 61/288, the result is 1R = 288/61, that is, 4 44/61 hours, in which the bowl will be filled — which is proved as you see here:

Hor.	Crater.	Hor.	Crater.
48		4 44/61	6/61
72	1		4/61
96			3/61
6			48/61

For if the right eye fills 1 bowl in 48 hours, then in 4 44/61 hours it will fill 6/61 of one bowl, and so on; and all these parts of the bowl together make up one whole bowl.

[Margin: IV.]

4. ON THE STATUES OF ZETHUS, AMPHION, AND THEIR MOTHER ANTIOPE [DE STATVIS. ZETHI, AMPHIONIS, ac matris ipsorum Antiopes]

Τέταρτον. [Greek: "Fourth."]

Ἄμφω μὲν ἡμεῖς εἴκοσι μναῖ ἔλκομεν, Ζῆθος τε χ' ὦ ξυνώαιμ[ος] [?], ἰὼ δέ μου λάβη [?] Τεῖτον [?], τὸ τέταρτόν τε τοῦ δ' Ἀμφίον[ος] [?] Ἐξ πάντ' ἀνδρῶν [?], μητρὸς [?]ρήσεις σταθμόν.

(Greek verse; the sense follows in the Latin paraphrase.)

We two together weigh twenty minas — I, Zethus, and my kinsman equally. But if you take a third part of my minas and a fourth part of Amphion's minas, you will find six in all: and that is our mother's weight. The question is: how many minas did Zethus weigh, and how many his kinsman Amphion?

Set $1R$ minas for Zethus, and hence $20 - 1R$ minas for Amphion. Take a third part of Zethus, $\frac{1}{3}R$, and a fourth part of Amphion, $5 - \frac{1}{4}R$; together these make $5 + \frac{1}{12}R$, equal to 6. Removing 5 from each side, the equation stands between $\frac{1}{12}R$ and 1. Dividing 1 by $\frac{1}{12}$ therefore gives $1R = 12$; and Zethus's statue weighed that many minas. Amphion's weighed 8, and Antiope's 6 — which minas all together make 26.

[Margin: V.]

5. A GEOMETRICAL PROBLEM OF EUCLID [EVCLIDIS GEOMETRICVM]

Εὐκλείδου Γεωμετρικόν. [Greek: "Euclid's Geometrical (problem)."]

Ἡμίον[ος] καὶ ὄν[ος] φορέε[?][?] ὄϊνον ἔβαινον, Αὐτὰρ ὄν[ος] στενάχιζεν ἐπ' ἄχθεϊ φόρτῳ ἑοῖο [?]. Τὴν δὲ βαρυστενάχουσιν ἰδοῦσ' ἐρέεινεν ἐκείνη [?].

(Continuation of the Greek verse of Epigram 5.)

Μῆτερ τί κλαίεις' [?] ὀλοφύρεαι ἢ ὕτε κούρη; Εἰ μέτρον ἔμοι δοίης· διπλάσιον σέθεν ἦρα [?]. Εἰ δὲ ἐν ἀντιλάβοις, πάντως ἰσότητα φυλάξεις. Εἰπέ τὸ μέτρον ἄεισε [?] Γεωμετρὴς ὅτι ἴ[?][?].

(Greek verse; the sense follows in the Latin paraphrase and the two verse renderings below.)

A mule and a she-ass were going along carrying wine, and the ass was groaning under the pain of her burden. Seeing this, the mule questioned the heavily-groaning ass thus: "Mother, why do you lament so? Why do you pour out tears like a girl? If you give me one measure, I shall carry twice as much as you; but if instead you take one from me, we shall carry exactly the same weight." And so I should like you, most skilled geometer, to tell me the measure.

Another rendering [Alia versio]

A servant loads onto the she-mule and the ass two skins filled with wine; and when he saw the little ass sluggish, worn out by the weight, planting her steps slowly, the mule asks: "Why, dear mother, do you hold back and groan? If you would hand over to me one measure from your skin, I would then carry double the load; but if I hand you one measure, the loads will become equal for us both. Tell these measures, learned geometer."

Another rendering [Alia Versio]

A mule was carrying wine, accompanied by a she-ass, who complains of the vast weight of her load. He marvels at his mother's heavy groans, and says: "Why do your mournful eyes stream so with tears? Such tenderness, mother, befits delicate girls, whom unaccustomed toil oppresses and wears down. If you pour one measure into my skins, I myself shall carry twice the weight of your wine; but if instead you lift off one part from my pack, then each of us will bear an equal weight. Tell me those measures, O learned geometer — otherwise you will not be worthy of Phoebus's name."

Set $1R$ for the mule's measures, and accordingly $\frac{1}{2}R + 1\frac{1}{2}$ for the ass's measures. For in this way, if the ass gives the mule one measure, the mule will have $1R + 1$ — a weight that will be double the ass's remaining weight, namely $\frac{1}{2}R + \frac{1}{2}$. But if the mule gives the ass one measure, the mule's remaining weight will be $1R - 1$, equal to the ass's weight, which will then be $\frac{1}{2}R + 2\frac{1}{2}$. Add 1 to each side, and the equation stands between $1R$ and $\frac{1}{2}R + 3\frac{1}{2}$. Take $\frac{1}{2}R$ away from each side, and the equality is between $\frac{1}{2}R$ and $3\frac{1}{2}$. Dividing $3\frac{1}{2}$ by $\frac{1}{2}$ therefore gives $1R = 14/2$, that is, 7 for the mule's measures. The ass, then, having $\frac{1}{2}R + 1\frac{1}{2}$, will carry $3\frac{1}{2} + 1\frac{1}{2}$ measures — that is, 5. Which is proved: for if the ass gives the mule 1 measure, the mule will have 8 measures, which are double the four measures left to the ass. But if the mule gives the ass 1 measure, each will carry 6 measures.

INDEX OF THE CHAPTERS CONTAINED IN THE ALGEBRA [INDEX CAPITVM QVÆ IN ALGEBRA CONTINENTVR]

Chapter	Title	Page
PROOEMIV M	On the excellence of Algebra [De Algebrae præstantia].	3
CAP. I.	On the inventor of Algebra, and its name.	4
II.	On cossic, or denominate, numbers.	5
III.	On the numeration of cossic numbers.	9
IV.	On the addition and subtraction of cossic numbers.	9
V.	On the multiplication and division of cossic numbers.	13
VI.	On feigned numbers, or numbers less than nothing.	16
VII.	On fractions of cossic numbers.	17
VIII.	On the Rule of Algebra.	20
IX.	On the variety of equations.	21
X.	On the reduction of equations.	22
XI.	On the division prescribed by the Rule of Algebra.	24
XII.	On the extraction of roots, which the Rule of Algebra mentions.	24
XIII.	On the multitude of Rules of Algebra which other authors introduce.	33
XIV.	A question proposed: whether it is possible or not, or rather inept and trifling; and by what means this may be recognized from the	34

Chapter	Title	Page
	very operation performed by the Rule of Algebra.	
XV.	On second roots.	36
XVI.	On irrational numbers, or surds.	38
XVII.	On the reduction of simple roots to the same denomination.	39
XVIII.	On the multiplication and division of simple roots.	41
XIX.	On the addition of simple roots.	44
XX.	On the subtraction of simple roots.	47
XXI.	To place any number of mean proportionals between two numbers.	48
XXII.	On the addition and subtraction of compound irrational numbers.	49
XXIII.	On the multiplication and division of compound and diminished irrational numbers.	[?]
XXIV.	On the roots of compound or diminished irrational numbers, which others call Universals; and on their algorithm.	58
XXV.	On the minutiae [fractions] of irrational numbers, and on their algorithm.	65
XXVI.	On irrational, or denominate, numbers, together with their algorithm.	66
XXVII.	On binomials and apotomes, or residuals.	65 [?]
XXVIII.	On the extraction of roots from binomials and apotomes; where, in passing, [he treats] of the	72

Chapter	Title	Page
	other irrational lines about which Euclid disputes in Book 10.	
XXIX.	Various riddles of abstract numbers to be unraveled by Algebra, in which an equation between two numbers occurs — which is called a simple equation.	79
XXX.	Various riddles of abstract numbers to be resolved by Algebra, in which an equation occurs between three numbers, one of which is equal to the other two — which is called a compound equation.	129
XXXI.	Various riddles of numbers applied to material things, with several examples pertaining to second roots.	145
XXXII.	Various riddles pertaining to geometrical figures.	from p. 169 to the end.

FINIS.

(End of the Index of the Algebra, and the close of the Algebra and of Tomus II.)

— terminal blank leaf. Cream/unprinted parchment with light foxing and a horizontal crease; no printed text, ruling, or manuscript annotation.

The leaf is unprinted. Faint show-through of a woodcut diagram (a grid/net figure) from the preceding printed page is visible, but there is no text on this leaf itself. At the top there is a modern pencil conservation/collation note recording the gatherings:

[Margin: (modern pencil) "1–5 = XXXX XXXX XXX XXX III.84"]

(A binder's/librarian's collation tally counting the gatherings of the five tomes and a shelf/inventory mark; not part of the printed book.) A metal photographic reference tab appears at the lower right of the scan.

[Back cover / binding.] The rear board of the volume: a blind-tooled (impressed, un-inked) leather-over-board binding. The design is a series of concentric rectangular frames formed by rolled foliate and interlace borders, enclosing a central panel with a small oval medallion/centerpiece. Two metal clasp-catches (with green verdigris) survive on the fore-edge board. The board is worn and abraded. No lettering or title is legible on the cover. Black-and-white checkered photographic scale bars appear at the top of the scan. This closes the volume.

[End of volume.]